Definitions and Proofs

Axiom 1: (Sets are objects).
If A is a set, then A is also an object. In particular, given two sets A and B, it is meaningful to ask whether A is also an element of B.
(Not all objects are sets, unless in pure set theory, where all objects are sets).

Axiom 2: (Empty set)
There exists a set $\varnothing$, known as the empty set, which contains no elements, i.e. For every object x we have $x \notin \varnothing$.

Definition: (Equality of sets).
Two sets A and B are equal, A = B, iff every element of A is an element of B and vice versa. To put it another way, A = B if and only if every element x of A belongs also to B, and every element y of B belongs also to A.
(if $x \in A$ and $A = B$, then $x \in B$)

Thus the "is an element of" relation $\in$ obeys the axiom of substitution.
This notion of equality is reflexive, symmetric, and transitive.

Proof.
Let A, B and C be sets.
Reflexive:
For any object x, $x \in A$ if and only if $x \in A$. By the definition of equality of sets, A = A.

Symmetric:
Suppose A = B
By definition, if $x \in B$ then $x \in A$. If $y \in A$ then $y \in B$.
$\forall x (x \in A \text{ iff } x \in B)$ and $\forall x (x \in B \text{ iff } x \in A)$.

Transitive:
Suppose A = B and B = C
By definition, $\forall x (x \in A \text{ iff } x \in B)$ and $\forall x (x \in B \text{ iff } x \in C)$.
Also, $\forall y (y \in C \text{ iff } y \in B)$ and $\forall y (y \in B \text{ iff } y \in A)$.
Since any element of A is also in the set B, then any element of A is also in C and vice versa.

Axiom 3 (Singleton sets and pair sets)
If a is an object, then there exists a set {a} whose only element is a, i.e., for every object y, we have $y \in \{a\}$ if and only if y = a; we refer to {a} as the singleton set whose element is a. Furthermore, if a and b are objects, then there exists a set {a, b} whose only elements are a and b; i.e., for every object y, we have $y \in \{a, b\}$ if and only if y = a or y = b; we refer to this set as the pair set formed by a and b.

Axiom 4 (Pairwise union).
Given any two sets A, B, there exists a set $A \cup B$, called the union $A \cup B$ of A and B, whose elements consists of all the elements which belong to A or B or both. In other words, for any object x, $x \in A \cup B \Longleftrightarrow (x \in A \text{ or } x \in B)$. Recall that "or" refers by default in mathematics to inclusive or: "X or Y is true" means that "either X is true, or Y is true, or both are true".

Definition: (Subsets)

Let A, B be sets. We say that A is a subset of B, denoted $A \subseteq B$, iff every element of A is also an element of B, i.e. For any object x, $x \in A \Longrightarrow x \in B$. We say that A is a proper subset of B, denoted A B, if $A \subseteq B$ and A = B.

Because these definitions involve only the notions of equality and the "is an element of" relation, both of which already obey the axiom of substitution, the notion of subset also automatically obeys the axiom of substitution. Thus for instance if $A \subseteq B$ and A = A' , then A; $\subseteq$ B.

Axiom 5 (Axiom of specification)

Let A be a set, and for each $x \in A$, let P(x) be a property pertaining to x (i.e., P(x) is either a true statement or a false statement). Then there exists a set, called $\{x \in A : P(x) \text{ is true}\}$ (or simply $\{x \in A : P(x)\}$ for short), whose elements are precisely the elements x in A for which P(x) is true. In other words, for any object y, $y \in \{x \in A : P(x) \text{ is true}\} \Longleftrightarrow (y \in A \text{ and } P(y) \text{ is true})$. This axiom is also known as the axiom of separation.

Definition: (Intersections).

The intersection $S1 \cap S2$ of two sets is defined to be the set $S1 \cap S2 := \{x \in S1 : x \in S2\}$. In other words, $S1 \cap S2$ consists of all the elements which belong to both S1 and S2. Thus, for all objects x, $x \in S1 \cap S2 \Longleftrightarrow x \in S1$ and $x \in S2$.

Two sets A, B are said to be disjoint if $A \cap B = \varnothing$.

Definition: (Difference sets).

Given two sets A and B, we define the set A − B or A\B to be the set A with any elements of B removed: $A \setminus B := \{x \in A : x \notin B\}$;

Axiom 6 (Replacement).

Let A be a set. For any object $x \in A$, and any object y, suppose we have a statement P(x, y) pertaining to x and y, such that for each $x \in A$ there is at most one y for which P(x, y) is true. Then there exists a set $\{y : P(x, y) \text{ is true for some } x \in A\}$, such that for any object z, $z \in \{y : P(x, y) \text{ is true for some } x \in A\} \Longleftrightarrow P(x, z)$ is true for some $x \in A$.

Axiom 3.7 (Infinity).

There exists a set N, whose elements are called natural numbers, as well as an object 0 in N, and an object n++ assigned to every natural number $n \in N$, such that the Peano axioms (Axioms 2.1 - 2.5) hold.

Lemma 1: (Single choice). Let A be a non-empty set. Then there exists an object x such that $x \in A$.

Proof by Contradiction.

Suppose there does not exist any object x such that $x \in A$.
Then for all objects x, we have $x \notin A$.
Also, by we have $x \in \varnothing$. (Axiom 2)
Thus $x \in A \Leftrightarrow x \in \varnothing$ (both statements are equally false), and so
$A = \varnothing$ (Definition of Equality of Sets)
Contradiction.

Lemma 2: If a and b are objects, then $\{a, b\} = \{a\} \cup \{b\}$. If A, B, C are sets, then the union operation is commutative (i.e., $A \cup B = B \cup A$) and associative (i.e., $(A \cup B) \cup C = A \cup (B \cup C)$). Also, we have $A \cup A = A \cup \varnothing = \varnothing \cup A = A$.

Proof. (Associative)
By Definition of Equality of Sets, we need to show that every element x of $(A \cup B) \cup C$ is an element of $A \cup (B \cup C)$, and vice versa.
So suppose first that $x \in (A \cup B) \cup C$.
At least one of $x \in A \cup B$ or $x \in C$ is true. (Axiom 4)
We now divide into two cases.
If $x \in C$, $x \in B \cup C$
$x \in A \cup (B \cup C)$.
Suppose instead $x \in A \cup B$
$x \in A$ or $x \in B$.
If $x \in A$ then $x \in A \cup (B \cup C)$ (Axiom 4)
If $x \in B$ then we have $x \in B \cup C$ and hence $x \in A \cup (B \cup C)$.
Thus in all cases we see that every element of $(A \cup B) \cup C$ lies in $A \cup (B \cup C)$. A similar argument shows that every element of $A \cup (B \cup C)$ lies in $(A \cup B) \cup C$, and so $(A \cup B) \cup C = A \cup (B \cup C)$ as desired.

Proof. $\{a, b\} = \{a\} \cup \{b\}$
$x \in \{a, b\}$ therefore $x = a$ or $x = b$. (Axiom 3)
If $x = a$ then $x \in \{a\}$, if $x = b$ then $x \in \{b\}$.
Thus $x \in \{a\} \cup \{b\}$ (Axiom 4)
Since $\forall x\, (x \in \{a\} \cup \{b\}$ and $x \in \{a, b\})$, $\{a, b\} = \{a\} \cup \{b\}$ (Definition of Equality)

Proof. (Commutative)
If $x \in A \cup B$, then $x \in A$ or $x \in B$.
Suppose $x \in A$, then $x \in B \cup A$ (Axiom 4)
Suppose $x \in B$, then $x \in B \cup A$ (Axiom 4)
So every element $x \in A \cup B$ is also an element of $B \cup A$. Thus $A \cup B = B \cup A$ (Definition of Equality)

Proof. $A \cup A = A \cup \varnothing = \varnothing \cup A = A$
$x \in A \cup A$ iff $x \in A$ or $x \in A$, which is the same as $x \in A$ since $A = A$. (Proven earlier)
Therefore, $x \in A \cup A$ iff $x \in A$.
$x \in A \cup \varnothing$ iff $x \in A$ or $x \in \varnothing$. But $x \in \varnothing$ is always false. (Axiom 2)

So, $x \in A \cup \varnothing$ iff $x \in A$ and $x \in \varnothing \cup A$ iff $x \in A$.
So $\forall x)= (x \in A \cup \varnothing$ and $x \in \varnothing \cup A$ and $x \in A$ and $x \in A \cup A)$ Hence, all the sets are equal.

Proposition 3: (Sets are partially ordered by set inclusion).
Let A, B, C be sets. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$. If $A \subseteq B$ and $B \subseteq A$, then $A = B$. Finally, if $A \subsetneq B$ and $B \subsetneq C$ then $A \subsetneq C$.

Proof.
Suppose that $A \subseteq B$ and $B \subseteq C$.
To prove that $A \subseteq C$, we have to prove that every element of A is an element of C. So, let us pick an arbitrary element x of A. Then, since $A \subseteq B$, x must then be an element of B. But then since $B \subseteq C$, x is an element of C. Thus every element of A is indeed an element of C.

Proposition 4 (Sets form a boolean algebra)
Let A, B, C be sets, and let X be a set containing A, B, C as subsets.
(a) (Minimal element) We have $A \cup \varnothing = A$ and $A \cap \varnothing = \varnothing$.
(b) (Maximal element) We have $A \cup X = X$ and $A \cap X = A$.
(c) (Identity) We have $A \cap A = A$ and $A \cup A = A$.
(d) (Commutativity) We have $A \cup B = B \cup A$ and $A \cap B = B \cap A$.
(e) (Associativity) We have $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.
(f) (Distributivity) We have $A \cap (B \cup C)=(A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C)=(A \cup B) \cap (A \cup C)$.
(g) (Partition) We have $A \cup (X \setminus A) = X$ and $A \cap (X \setminus A) = \varnothing$.
(h) (De Morgan laws) We have $X \setminus (A \cup B)=(X \setminus A) \cap (X \setminus B)$ and $X \setminus (A \cap B)=(X \setminus A) \cup (X \setminus B)$.

Exercise 1:
Using only Definition 3.1.4, Axiom 3.1, Axiom 3.2, and Axiom 3.3, prove that the sets $\varnothing$, $\{\varnothing\}$, $\{\{\varnothing\}\}$, and $\{\varnothing, \{\varnothing\}\}$ are all distinct (i.e., no two of them are equal to each other).

Proof.
Show that the empty set is distinct from other sets.

$\varnothing \in \{\varnothing\}$ (Axiom 3)
$\varnothing \notin \varnothing$ (Axiom 2)
Thus, not every element of $\{\varnothing\}$ is an element of $\varnothing$, hence $\varnothing \neq \{\varnothing\}$.

$\{\varnothing\} \in \{\{\varnothing\}\}$ (Axiom 3)
$\{\varnothing\} \notin \varnothing$ (Axiom 2)
Thus, not every element of $\{\{\varnothing\}\}$ is an element of $\varnothing$, hence $\varnothing \neq \{\{\varnothing\}\}$.

$\varnothing \in \{\varnothing, \{\varnothing\}\}$ (Axiom 3)
$\varnothing \notin \varnothing$ (Axiom 2)
Thus, not every element of $\{\varnothing, \{\varnothing\}\}$ is an element of $\varnothing$, hence $\varnothing \neq \{\varnothing, \{\varnothing\}\}$.

Show that $\{\varnothing\}$ and $\{\{\varnothing\}\}$ are distinct.

$\varnothing \in \{\varnothing\}$ (Axiom 3)
$\{\varnothing\} = \{\{\varnothing\}\}$ iff $\forall \varnothing$ ($\varnothing \in \{\varnothing\}$ and $\varnothing \in \{\{\varnothing\}\}$. However, $\varnothing \notin \{\{\varnothing\}\}$ because $\varnothing \neq \{\varnothing\}$ as proven earlier.
So $\{\varnothing\} \neq \{\{\varnothing\}\}$.

Show that $\{\{\varnothing\}\}$ and $\{\varnothing, \{\varnothing\}\}$ are distinct.
$\varnothing \in \{\varnothing, \{\varnothing\}\}$ and $\varnothing \notin \{\{\varnothing\}\}$ as proven earlier. Therefore, $\{\{\varnothing\}\} \neq \{\varnothing, \{\varnothing\}\}$

Show that $\{\varnothing\}$ and $\{\varnothing, \{\varnothing\}\}$ are distinct.
$\{\varnothing\} \in \{\varnothing, \{\varnothing\}\}$ (Axiom 3)
$\{\varnothing\} \in \{\varnothing\}$ iff $\{\varnothing\} = \varnothing$. However, proven earlier that $\varnothing \neq \{\varnothing\}$. Thus, $\varnothing\} \neq \{\varnothing, \{\varnothing\}\}$.