Definition 1: (Functions)

Let X, Y be sets, and let $P(x, y)$ be a property pertaining to an object $x \in X$ and an object $y \in Y$, such that for every $x \in X$, there is exactly one $y \in Y$ for which $P(x, y)$ is true (this is sometimes known as the vertical line test). Then we define the function $f : X \to Y$ defined by P on the domain X and range Y to be the object which, given any input $x \in X$, assigns an output $f(x) \in Y$, defined to be the unique object $f(x)$ for which $P(x, f(x))$ is true. Thus, for any $x \in X$ and $y \in Y$, $y = f(x) \Longleftrightarrow P(x, y)$ is true.

Definition 2: (Equality of functions).
Two functions $f : X \to Y$, $g : X \to Y$ with the same domain and range are said to be equal, $f = g$, if and only if $f(x) = g(x)$ for all $x \in X$. (If $f(x)$ and $g(x)$ agree for some values of x, but not others, then we do not consider f and g to be equal.)

Definition 3 (Composition).
Let $f : X \to Y$ and $g : Y \to Z$ be two functions, such that the range of f is the same set as the domain of g. We then define the composition $g \circ f : X \to Z$ of the two functions g and f to be the function defined explicitly by the formula $(g \circ f)(x) := g(f(x))$. If the range of f does not match the domain of g, we leave the composition $g \circ f$ undefined.

Definition 4 : (One-to-one functions).
A function f is one-to-one (or injective) if different elements map to different elements: $x = x \Longrightarrow f(x) = f(x\ )$.
Equivalently, a function is one-to-one if $f(x) = f(x') \Longrightarrow x = x'$

Definition 5: (Onto functions).
A function f is onto (or surjective) if $f(X) = Y$, i.e., every element in Y comes from applying f to some element in X: For every $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

Definition 6: (Bijective Functions)
Functions $f : X \to Y$ which are both one-to-one and onto are also called bijective or invertible.

If f is bijective, then for every $y \in Y$, there is exactly one x such that $f(x) = y$ (there is at least one because of surjectivity, and at most one because of injectivity).

Lemma 1: (Composition is associative).
Let $f : Z \to W$, $g : Y \to Z$, and $h : X \to Y$ be functions. Then $f \circ (g \circ h)=(f \circ g) \circ h$.

Proof.
Since $g \circ h$ is a function from X to Z, $f \circ (g \circ h)$ is a function from X to W. Similarly $f \circ g$ is a function from Y to W, and hence $(f \circ g) \circ h$ is a function from X to W. Thus $f \circ (g \circ h)$ and $(f \circ g) \circ h$ have the same domain and range. In order to check that they are equal, we see from Definition 3.3.7 that we have to verify that $(f \circ (g \circ h))(x) = ((f \circ g) \circ h)(x)$ for all $x \in X$. But by Definition 3.3.10 $(f \circ (g \circ h))(x) = f((g \circ h)(x)) = f(g(h(x)) = (f \circ g)(h(x)) = ((f \circ g) \circ h)(x)$ as desired.