Exercise 1

1. **Vectors**

1. consider the following vector:

$$u = \begin{bmatrix} 1 \\ -3 \end{bmatrix} ; \; v = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$

(a) calculate the norm of u and of v

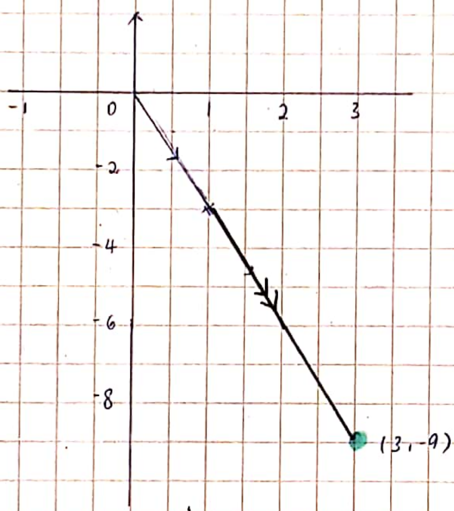
$$\|u\| = \sqrt{u_1^2 + u_2^2 + \dots u_N^2}$$

$$\|u\| = \sqrt{1^2 + (-3)^2} = \sqrt{10}$$
$$\|v\| = \sqrt{2^2 + (-6)^2} = \sqrt{40}$$

(b) calculate the sum of the two vectors.

$$\vec{u} + \vec{v} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ -6 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix}$$

(c) Show the result geometrically.



(d) calculate the difference v - u.

$$\vec{v} - \vec{u} = \begin{bmatrix} 2 \\ -6 \end{bmatrix} - \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

(e) $7\vec{v} - 5\vec{u} = 7\begin{bmatrix} 2 \\ -6 \end{bmatrix} - 5\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 14 \\ -42 \end{bmatrix} - \begin{bmatrix} 5 \\ -15 \end{bmatrix}$

$$= \begin{bmatrix} 9 \\ -27 \end{bmatrix}$$

(f) $u \cdot v = \begin{bmatrix} 1 \\ -3 \end{bmatrix}\begin{bmatrix} 2 \\ -6 \end{bmatrix} = 2 + 18 = 20$

(g) $u \times v = \begin{vmatrix} i & j \\ 1 & 2 \\ -3 & -6 \end{vmatrix} = -6 - (-6) = 0$

(h) u and v are linearly dependent, because v can be expressed as a linear combination of u.

$$v = \begin{bmatrix} 2 \\ -6 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -3 \end{bmatrix} = 2u$$

2. (a) $\|u \times v\| = \|u\| \, \|v\| \sin\theta$

$= \|u\| \, \|v\| \sin 90$

$= \|u\| \, \|v\|$

$= 3 \times 5$

$= 15$.

~~(b) The x-coordinate of u x v is~~

~~$\|u \times v\| = \begin{vmatrix} i & j & k \\ x_1 & x_2 & 0 \\ 0 & 0 & x_3 \end{vmatrix}$~~

~~$= i($~~

$\|u \times v\| = \begin{vmatrix} x & y & z \\ u_1 & u_2 & 0 \\ 0 & 0 & v_1 \end{vmatrix}$

$= x(u_2 v_1) - y(u_1 v_1) + 0z$

(b) The x-coordinate of $u \times v$ is > 0.
$u_2 v_1$ is a positive constant since $u_2 > 0$ and $v_1 > 0$
(both lie in positive axis)

(c) The y-coordinate of $u \times v$ is < 0.
$-(u_1 v_1)$ is a negative constant. since $u_1 > 0$ and $v_1 > 0$
(both lie in positive axis.

(d) The z-coordinate of $u \times v$ is $= 0$.

3. $\|u\| = \|v\| = \|u - v\| = 2\sqrt{2}$

(a)

$\|u\| = 2\sqrt{2}$

$\sqrt{u_1^2 + u_2^2} = \sqrt{8}$

$u_1^2 + u_2^2 = 8$

$\|v\| = 2\sqrt{2}$

$\sqrt{v_1^2 + v_2^2} = \sqrt{8}$

$v_1^2 + v_2^2 = 8$

$\|u - v\| = \sqrt{8}$

$\cancel{\sqrt{(u_1^2 + u_2^2) - (v_1^2 + v_2^2)}}$

$\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2} = \sqrt{8}$

$u_1^2 - 2u_1 v_1 + v_1^2 + u_2^2 - 2u_2 v_2 + v_2^2 = 8$

$(u_1^2 + u_2^2) + (v_1^2 + v_2)^2 - 2(u_1 v_1 + u_2 v_2) = 8$

$-2(u_1 v_1 + u_2 v_2) = -8$

$u_1 v_1 + u_2 v_2 = 4$

$\|u + v\| = \sqrt{(u_1 + v_1)^2 + (u_2 + v_2)^2}$

$= \sqrt{u_1^2 + 2u_1 v_1 + v_1^2 + u_2^2 + 2u_2 v_2 + v_2^2}$

$= \sqrt{(u_1^2 + u_2^2) + (v_1^2 + v_2^2) + 2(u_1 v_1 + u_2 v_2)}$

$= \sqrt{8 + 8 + 8}$

$= \sqrt{24} = 2\sqrt{6}$ ~~units~~ #

(b) $\cancel{\|u \times v\| = \|u\|\,\|v\| \sin\theta}$

$\cancel{\theta = \arcsin \frac{\|u \times v\|}{\|u\|\,\|v\|}}$

$u \cdot v = \|u\|\,\|v\| \cos\theta$

$\cos\theta = \frac{u \cdot v}{\|u\|\,\|v\|}$

$\theta = \arccos \frac{u \cdot v}{\|u\|\,\|v\|}$

$= \arccos \frac{(u_1 v_1 + u_2 v_2)}{\|u\|\,\|v\|}$

$= \arccos \frac{4}{(2\sqrt{2})^2}$

$= \arccos \frac{4}{8}$

$= 60°$ #

Scanned with CamScanner

2. Matrices

① $A = \begin{bmatrix} 3 & -1 & 0 \\ 2 & -3 & 2 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 4 & 1 \end{bmatrix}$

$C = \begin{bmatrix} 4 & -1 \\ 1 & -2 \end{bmatrix}$

(a) $A + B = \begin{bmatrix} 3 & -1 & 0 \\ 2 & -3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ -2 & 4 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

(b) $A + C$ (Not Applicable)

(c) $2C + 3I_2$

$2\begin{bmatrix} 4 & -1 \\ 1 & -2 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{pmatrix} 8 & -2 \\ 2 & -4 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

$= \begin{pmatrix} 11 & -2 \\ 2 & -1 \end{pmatrix}$

② $A = \begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & -2 & 2 & 1 \end{bmatrix}$

(a) $A^T = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & -2 & 1 & 0 \end{bmatrix}$

(b) $A + B$ = Not Applicable

(c) $A^T + B = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & -2 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & -2 & 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 2 & 6 & -1 \\ 5 & -4 & 3 & 1 \end{bmatrix}$

(d) $A \times B = \begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & -2 & 2 & 1 \end{bmatrix}$

$4 \times 2 \quad 2 \times 4$

$= \begin{bmatrix} 7 & -6 & 9 & 3 \\ -2 & 4 & 2 & -2 \\ 5 & -2 & 11 & 1 \\ -1 & 0 & -3 & 0 \end{bmatrix}$

(f) $B \times A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \\ -1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 10 & 6 \\ 3 & 12 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & -1 & 3 \end{bmatrix}$

(a) A^2

Not Applicable

(b) $A^T A = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \end{bmatrix}$

$= 1 + 1 + 9$

$= 11$ #

(c) $AA^T = \begin{bmatrix} 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

$= 1 + 1 + 9$

$= 11$ #