20311298_Assignment 2 (2 a,b,c)

Monday, 19 December, 2022 7:48 PM

2. Budget = £ 120, 000, 000
   \# of Vehicles $\leq$ 12
   \# of seats $\geq$ 800
   \# of type A vehicle $\geq$ 2
   \# of type B vehicle $\geq$ 2

max $150{,}000\, x_1 + 175{,}000\, x_2 + 218{,}000\, x_3$

a) min $-150 x_1 - 175 x_2 - 218 x_3$ (in k)

subject to

$$6.5 x_1 + 7.8 x_2 + 9.2 x_3 \leq 120$$
$$x_1 + x_2 + x_3 \leq 12$$
$$45 x_1 + 60 x_2 + 75 x_3 \geq 800$$
$$x_1 \geq 2$$
$$x_2 \geq 2$$

① rewrite constraints

$$6.5 x_1 + 7.8 x_2 + 9.2 x_3 - 120 \leq 0$$
$$x_1 + x_2 + x_3 - 12 \leq 0$$
$$-45 x_1 - 60 x_2 - 75 x_3 + 800 \leq 0$$
$$-x_1 + 2 \leq 0$$
$$-x_2 + 2 \leq 0$$

② $L = -150 x_1 - 175 x_2 - 218 x_3 +$
$\lambda_0 (6.5 x_1 + 7.8 x_2 + 9.2 x_3 - 120) +$
$\lambda_1 (x_1 + x_2 + x_3 - 12) +$
$\lambda_2 (-45 x_1 - 60 x_2 - 75 x_3 + 800)$
$\lambda_3 (-x_1 + 2) +$
$\lambda_4 (-x_2 + 2)$

b) ③ $\nabla L = 0$:

$$-150 + 6.5 \lambda_0 + \lambda_1 - 45 \lambda_2 - \lambda_3 = 0 \quad \text{— (1)}$$
$$-175 + 7.8 \lambda_0 + \lambda_1 - 60 \lambda_2 - \lambda_4 = 0 \quad \text{— (2)}$$
$$-218 + 9.2 \lambda_0 + \lambda_1 - 75 \lambda_2 = 0 \quad \text{— (3)}$$

optimality conditions.

$$\lambda_0 (6.5 x_1 + 7.8 x_2 + 9.2 x_3 - 120) = 0 \quad \text{— (4)}$$
$$\lambda_1 (x_1 + x_2 + x_3 - 12) = 0 \quad \text{— (5)}$$
$$\lambda_2 (-45 x_1 - 60 x_2 - 75 x_3 + 800) = 0 \quad \text{— (6)}$$
$$\lambda_3 (-x_1 + 2) = 0 \quad \text{— (7)}$$
$$\lambda_4 (-x_2 + 2) = 0 \quad \text{— (8)}$$

complementary slackness

④ (i) If $\lambda_0 = 0$, $\lambda_1 \neq 0$, $\lambda_2 = 0$, $\lambda_3 \neq 0$, $\lambda_4 \neq 0$

(7): $x_1 = 2$
(8): $x_2 = 2$
(5): $x_1 + x_2 + x_3 = 12$
$(2) + (2) + x_3 = 12$
$x_3 = 8$

Let $x^* = (2, 2, 8)$ be a feasible solution and $f(x) = -150{,}000\, x_1 - 175{,}000\, x_2 - 218{,}000\, x_3$

$f(x)$ is a convex continuously differentiable function over $\mathbb{R}^3$
($\because$ objective function is an affine function and constraints are hyperplanes that form a convex polytope)

By the KKT conditions for Convex Linearly Constrained Problems Theorem,

$$\left[\begin{array}{l} (1): \ -150 + 6.5(0) + \lambda_1 - 45(0) - \lambda_3 = 0 \Rightarrow \lambda_1 - \lambda_3 = 150 \Rightarrow \lambda_3 = 218 - 150 = 68 \\ (2): \ -175 + 7.8(0) + \lambda_1 - 60(0) - \lambda_4 = 0 \Rightarrow \lambda_1 - \lambda_4 = 175 \Rightarrow \lambda_4 = 218 - 175 = 43 \end{array}\right.$$

$$\left[\begin{array}{l}
(1):\ -150 + 6.5(0) + \lambda_1 - 45(0) - \lambda_3 = 0 \Rightarrow \lambda_1 - \lambda_3 = 150 \Rightarrow \lambda_3 = 218 - 150 = 68 \\
(2):\ -175 + 7.8(0) + \lambda_1 - 60(0) - \lambda_4 = 0 \Rightarrow \lambda_1 - \lambda_4 = 175 \Rightarrow \lambda_4 = 218 - 175 = 43 \\
(3):\ -218 + 9.2(0) + \lambda_1 - 75(0) = 0 \Rightarrow \lambda_1 = 218
\end{array}\right.$$

Therefore, optimality conditions $\nabla f(x^*) + \sum_{i=0}^{2} \lambda_i a_i = 0$ is satisfied

Since $\lambda_0 = 0$ and $\lambda_2 = 0$, (4) and (6) $= 0$

Since $x_1 + x_2 + x_3 = 12$, $(5) = 0$

$x_1 = 2$, $(7) = 0$

$x_2 = 2$, $(8) = 0$

Therefore, complementary slackness $\lambda_i(a_i^T x^* - b_i) = 0$, $i = 0, 1, 2, 3, 4$ is satisfied.

By the KKT Conditions for Convex Linearly Constrained Problems Theorem,

$x^* = (2, 2, 8)$ is an optimal solution.

Primal solution:

$$-150{,}000\, x_1 - 175{,}000\, x_2 - 18{,}000\, x_3$$
$$= -150{,}000(2) - 175{,}000(2) - 18{,}000(8)$$
$$= -2{,}394{,}000$$

c)

$$\left.\begin{array}{l}
\min\ -150x_1 - 175x_2 - 218x_3 \\
\text{subject to} \\
6.5x_1 + 7.8x_2 + 9.2x_3 - 120 \leq 0 \\
x_1 + x_2 + x_3 - 12 \leq 0 \\
-45x_1 - 60x_2 - 75x_3 + 800 \leq 0 \\
-x_1 + 2 \leq 0 \\
-x_2 + 2 \leq 0
\end{array}\right\}\ \text{Primal problem.}$$

Linear programming problem can be written as

$$\begin{array}{ll} \min & c^T \underline{x} \\ \text{s.t} & A\underline{x} \leq b \end{array}$$

$$c^T = (-150 \quad -175 \quad -218)$$

$$A = \begin{pmatrix} 6.5 & 7.8 & 9.2 \\ 1 & 1 & 1 \\ -45 & -60 & -75 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 120 \\ 12 \\ -800 \\ -2 \\ -2 \end{pmatrix}$$

The Lagrangian is $L(\underline{x}, \lambda) = c^T \underline{x} + \underline{\lambda}^T (A\underline{x} - b)$

$$= (c^T + \lambda^T A)\underline{x} - \underline{\lambda}^T \underline{b}$$

$$= (c + A^T \lambda)^T \underline{x} - b^T \underline{\lambda}$$

The dual objective function

$$q(\lambda) = \min_{\underline{x} \in \mathbb{R}^3_+} L(\underline{x}, \lambda) = \min_{\underline{x} \in \mathbb{R}^3_+} (c + A^T\lambda)^T \underline{x} - b^T \underline{\lambda} = \begin{cases} -b^T\lambda, & \underline{c} + A^T\underline{\lambda} = 0 \\ -\infty, & \text{otherwise} \end{cases}$$

The dual problem is

$$\begin{cases} \text{maximise} & -\underline{b}^T \underline{\lambda} \\ \text{s.t.} & A^T \underline{\lambda} = -c \\ & \underline{\lambda} \in \mathbb{R}^5_+ \end{cases}$$

equivalent to

$$\begin{cases} \text{maximize} & -120\lambda_0 - 12\lambda_1 + 800\lambda_2 + 2\lambda_3 + 2\lambda_4 \\ \text{s.t.} & 6.5\lambda_0 + \lambda_1 - 45\lambda_2 - \lambda_3 = 150 \\ & 7.8\lambda_0 + \lambda_1 - 60\lambda_2 - \lambda_4 = 175 \\ & 9.2\lambda_0 + \lambda_1 - 75\lambda_2 = 218 \\ & \lambda \in \mathbb{R}^5_+ \end{cases}$$