One and Multi-Period Asset Pricing Models

Chapters 2, 3, 7 - Lecture Notes by Markus Brunnermeier
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Today's talk

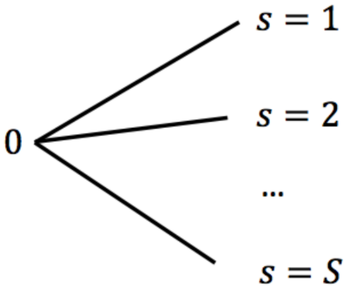
- State-space representation of security structures
- One period setting
- No arbitrage
- State prices and the Fundamental Theorem of Asset Pricing
- Pricing formulas for one period
  - Stochastic discount factors (SDFs)
  - Beta models
- Extension to multiple periods

One-Period Asset Model

Setup

- **Two dates:** present ($t = 0$) and future ($t = 1$)
- World is fixed at $t = 0$ but has S states at $t = 1$
  - e.g. flipping a coin, prices go up or down, S=2
- p_j: price of a security j at $t = 0$
- x^j_s: payoff at $t = 1$ for state s
- Payoff vector for asset j

$$\mathbf{x}^j = \begin{bmatrix} x^j_1 \\ x^j_2 \\ \vdots \\ x^j_S \end{bmatrix} \in \mathbb{R}^S$$



Security Structure

- Market consists of securities $j = 1, \ldots, J$
- A *security structure* $\mathbf{X} \in \mathbb{R}^{S \times J}$ is the matrix formed by concatenating payoff vectors

$$\mathbf{X} \;=\; [\mathbf{x}^1, \ldots, \mathbf{x}^J]$$

- A *portfolio* $\mathbf{h} = [h_1, \ldots, h^j] \in \mathbb{R}^J$ denotes amount held for each of the J securities
- The price of the portfolio is $\mathbf{p}^T\mathbf{h}$ and its payoff is $\mathbf{Xh}$
- The *asset span* of $\mathbf{X}$ is the set of achievable payoffs

$$\langle \mathbf{X} \rangle = \{\mathbf{Xh} : \mathbf{h} \in \mathbb{R}^J\} \tag{1}$$

- We say *the market is complete* if $\langle \mathbf{X} \rangle = \mathbb{R}^S$, or $\mathrm{rank}(\mathbf{X}) = S$

Particular examples

- *Arrow-Debreu securities:* $\mathbf{e}_s = [0, \dots, 1, \dots, 0]^T$, only pays off at state s.
- *Risk-free bonds:* $\mathbf{x}^b = [1, \dots, 1]$, pays off at every state.
- *Calls and puts:* Let $\mathbf{x}^1 = [1, 2, \dots, S]^T$. Then

$$x_s^k = \max(s + 1 - k, 0) \quad \text{and} \quad x_s^k = \max(k - s, 0), \quad k = 2, \dots, S \tag{2}$$

specify payoffs for call and put options on $\mathbf{x}^1$, respectively. Then with $\mathbf{X} = [\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^S]$

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ S & S-1 & \dots & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{X} = \begin{bmatrix} 1 & 1 & 2 & \dots & S-1 \\ 2 & 0 & 1 & \dots & S-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S-1 & 0 & 0 & \dots & 1 \\ S & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{3}$$

so both call and put options can be used to construct complete markets.

No Arbitrage

- **Notation:**
  - $y \geq x \implies y_i \geq x_i \quad \forall i$
  - $y > x \implies y_i \geq x_i \quad \forall i$ and $y \neq x$ (at least 1 strict inequality)
  - $y \gg x \implies y_i > x_i \quad \forall i$

- **Arbitrage:** $\exists \mathbf{h}$ s.t. $\mathbf{p}^T\mathbf{h} \leq 0$ and $\mathbf{Xh} \geq \mathbf{0}$, with at least 1 strict inequality
- **Strong arbitrage:** $\exists \mathbf{h}$ s.t. $\mathbf{p}^T\mathbf{h} < 0$ and $\mathbf{Xh} \geq \mathbf{0}$
- The converse can be stated as follows:
  - **No arbitrage (NA):** $\mathbf{Xh} > 0 \implies \mathbf{p}^T\mathbf{h} > 0 \quad \forall \mathbf{h}$
  - **No strong arbitrage (NSA):** $\mathbf{Xh} \geq 0 \implies \mathbf{p}^T\mathbf{h} \geq 0 \quad \forall \mathbf{h}$

- NA $\implies$ NSA

No Arbitrage

- **Law of One Price (LOOP):** Any two portfolios $\mathbf{h}, \mathbf{k} \in \mathbb{R}^J$ with equal payoffs must have equal price
$$\mathbf{X}\mathbf{h} = \mathbf{X}\mathbf{k} \implies \mathbf{p}^T\mathbf{h} = \mathbf{p}^T\mathbf{k}$$
- NA $\implies$ NSA $\implies$ LOOP $\implies$ every portfolio with zero payoff has zero price
- NSA $\implies$ LOOP:
  - Contrapositive: suppose $\exists\, \mathbf{h}, \mathbf{k}$ s.t. $\mathbf{X}\mathbf{h} = \mathbf{X}\mathbf{k}$, but $\mathbf{p}^T\mathbf{h} \neq \mathbf{p}^T\mathbf{k}$
  - Then $\mathbf{X}(\mathbf{h} - \mathbf{k}) = 0$ but $\min\left(\mathbf{p}^T(\mathbf{h} - \mathbf{k}), \mathbf{p}^T(\mathbf{k} - \mathbf{h})\right) < 0$
  - !LOOP $\implies$!NSA □
- LOOP $\implies$ every portfolio with zero payoff has zero price:
  - Similar to above: let $\mathbf{X}\mathbf{h} = \mathbf{0}$ but $\mathbf{p}^T\mathbf{h} \neq 0$
  - Setting $\mathbf{k} = \mathbf{0}$ for LOOP leads to contrapositive □

State Prices

- A vector of *state prices*: $\mathbf{q} \in \mathbb{R}^S$ s.t.

$$\mathbf{p} = \mathbf{X}^T \mathbf{q} = \sum_{s=1}^{S} q_s \cdot \mathbf{x}_s, \qquad \text{or} \qquad p_j = \sum_{s=1}^{S} q_s \cdot x_s^j \quad \forall j$$

- Implies all assets are discounted *exactly the same way* from payoffs $\mathbf{x}^j$. Each state has an associated price.

- When does such a $\mathbf{q}$ exist?

Fundamental Theorem of Asset Pricing (Stiemke's Theorem)

A security structure $\mathbf{X} \in \mathbb{R}^{S \times J}$ provides no arbitrage opportunities iff it yields a positive state price vector.

- Equivalent statement: either $\exists\, \mathbf{h} \in \mathbb{R}^J$ s.t.

$$\mathbf{p}^T \mathbf{h} \leq 0 \quad \text{and} \quad \mathbf{X}\mathbf{h} \geq \mathbf{0}$$

  with at least 1 nonzero, or $\exists\, \mathbf{q} \in \mathbb{R}^S$ s.t.

$$\mathbf{p} = \mathbf{X}^T \mathbf{q} \quad \text{and} \quad \mathbf{q} \gg \mathbf{0}$$

- An example of a *theorem of the alternative*. Important class of theorems in convex optimization.

- *Proof:* First we restate the alternatives as

$$\text{I.} \quad \exists\, \mathbf{h} \in \mathbb{R}^n : \quad \mathbf{A}\mathbf{h} > \mathbf{0} \tag{4}$$

$$\text{II.} \quad \exists\, \mathbf{z} \in \mathbb{R}^m : \quad \mathbf{A}^T\mathbf{z} = \mathbf{0} \quad \text{and} \quad \mathbf{z} \gg \mathbf{0} \tag{5}$$

  where

$$\mathbf{A} = \begin{bmatrix} -\mathbf{p}^T \\ \mathbf{X} \end{bmatrix} \in \mathbb{R}^{m \times n} \quad \text{and} \quad \mathbf{z} \propto \begin{bmatrix} 1 \\ \mathbf{q} \end{bmatrix}. \tag{6}$$

- The alternatives are mutually exclusive: (II) implies $\mathbf{h}^T(\mathbf{A}^T\mathbf{z}) = 0$, but (I) implies

$$\mathbf{h}^T\mathbf{A}^T\mathbf{z} \;=\; (\mathbf{A}\mathbf{h})^T\mathbf{z} \;>\; 0.$$

Fundamental Theorem of Asset Pricing (Stiemke's Theorem)

- Next assume (I) does not hold, which implies that

$$\mathcal{S}_1 = \{\mathbf{u} = \mathbf{Ah} : \ \mathbf{h} \in \mathbb{R}^n\} \quad \text{and} \quad \mathcal{S}_2 = \{\mathbf{v} \in \mathbb{R}^m : \ \mathbf{v} > \mathbf{0}\} \tag{7}$$

are disjoint. By hyperplane separation there is a unit vector $\mathbf{z} \in \mathbb{R}^m$ s.t.

$$\mathbf{z}^T\mathbf{v} \ > \ 0, \ \ \forall\, \mathbf{v} \in \mathcal{S}_2 \qquad \text{and} \qquad \mathbf{z}^T\mathbf{Ah} \ = \ 0, \ \ \forall\, \mathbf{h} \in \mathbb{R}^n. \tag{8}$$

- Then $\mathbf{A}^\mathbf{T}\mathbf{z} = \mathbf{0}$, otherwise

$$\|\mathbf{A}^T\mathbf{z}\|^2 \ = \ \mathbf{z}^T\mathbf{AA}^T\mathbf{z} \ > \ 0 \tag{9}$$

but this contradictions the hyperplane condition for $\mathcal{S}_1$ (set $\mathbf{h} = \mathbf{A}^T\mathbf{z}$).

- Finally show $\mathbf{z} \gg 0$: All the basis vectors of $\mathbb{R}^m$ are in $\mathcal{S}_2$.
  The hyperplane condition then implies all coordinates of $\mathbf{z}$ must be positive. $\square$

State Prices in Incomplete Markets

- In general there can be multiple $\mathbf{q}$s, e.g. if $J < S$.
- However there is a unique $\mathbf{q}^*$ in $\langle \mathbf{X} \rangle$, known as the *pricing kernel*. For instance,

$$\mathbf{X}^T(\mathbf{X}\mathbf{u}) = \mathbf{p} \tag{10}$$

provides a unique solution for $\mathbf{q}^* = \mathbf{X}\mathbf{u}$ as long as the $\mathbf{x}^j$s are linearly independent. Furthermore, for any valid $\mathbf{q}$,

$$\mathbf{q}^* = \text{proj}\,(\mathbf{q} \,|\, \langle \mathbf{X} \rangle). \tag{11}$$

- Finally, there exists a unique $\mathbf{q} \gg 0$ iff markets are complete and there is no arbitrage.
  - $\Longleftarrow$ Just solve $\mathbf{X}^T\mathbf{q} = \mathbf{p}$.
  - $\Longrightarrow$ If markets are not complete but there is no arbitrage, then $\exists\, \mathbf{v} \neq \mathbf{0}$ s.t. $\mathbf{X}^T\mathbf{v} = \mathbf{0}$. Moreover we can find $\mathbf{q} \gg \mathbf{0}$ and some $\alpha \in \mathbb{R}$ s.t.

$$\mathbf{q} + \alpha\mathbf{v} \;\gg\; 0.$$

So there are infinitely many solutions to $\mathbf{X}^T\mathbf{q} = \mathbf{p}$. Contrapositive. $\square$

One-Period Asset Pricing: Stochastic Discount Factor (SDF)

- Discounted price if no arbitrage: $p_j = \mathbf{q}^T \mathbf{x}^j$
- Introduce state probabilities π_s and **stochastic discount factor (SDF)** $m_s = \frac{q_s}{\pi_s}$ (state price per unit probability)

$$p_j = \sum_{s=1}^{S} \pi_s \frac{q_s}{\pi_s} x_s^j = \sum_s \pi_s m_s x_s^j = \mathbb{E}[\mathbf{m} \odot \mathbf{x}^j] \quad (12)$$

- Consider a risk-free bond with $\mathbf{x}^b = \mathbf{1}$. Since $\mathbb{E}[\mathbf{m} \odot \mathbf{x}^j] = \mathbb{E}[\mathbf{m}]\,\mathbb{E}[\mathbf{x}^j] + Cov(\mathbf{m}, \mathbf{x}^j)$,

$$p_b = \mathbb{E}[\mathbf{m}] = 1/R^f \quad \Longrightarrow \quad p_j = \frac{\mathbb{E}[\mathbf{x}^j]}{R^f} + Cov[\mathbf{m}, \mathbf{x}^j] \quad (13)$$

  and for any asset j.

- Since prices tend to grow over time, typically $Cov[\mathbf{m}, \mathbf{x}^j] < 0$.

One-Period Asset Pricing: Stochastic Discount Factor (SDF)

- Define the return of asset j as $\mathbf{R}^j \doteq \mathbf{x}^j / p_j$, then $\mathbb{E}\left[\mathbf{m} \odot \mathbf{R}^j\right] = 1$.
- Moreover, since $R^f = 1/\mathbb{E}[\mathbf{m}]$,

$$\mathbb{E}\left[\mathbf{m} \odot \left(\mathbf{R}^j - R^f\right)\right] \;=\; \mathbb{E}[\mathbf{m}]\left(\mathbb{E}\left[\mathbf{R}^j\right] - R^f\right) + Cov(\mathbf{m}, \mathbf{R}^j) \;=\; 0 \tag{14}$$

- Or

$$\mathbb{E}\left[\mathbf{R}^j\right] - R^f = -\frac{Cov(\mathbf{m}, \mathbf{R}^j)}{\mathbb{E}[\mathbf{m}]} \tag{15}$$

- The excess return for a generic asset j is determined solely by its covariance with the SDF. Investors are only compensated for holding *systematic risk*.

Relationship to security market line / CAPM

- Consider the unique SDF $\mathbf{m}^* = [q_1^*/\pi_1, \ \ldots, \ q_S^*/\pi_S]^T$
- Treat $\mathbf{m}^*$ as a payoff vector with price $p_{m^*} > 0$. We can define $\mathbf{R}^* = \mathbf{m}^*/p_{m^*}$ and write

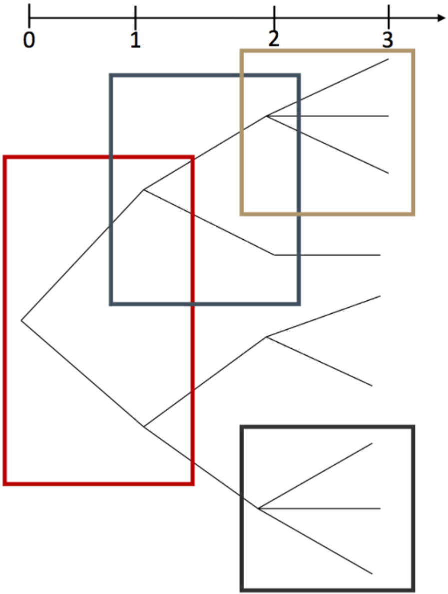
$$\mathbb{E}[\mathbf{R}^j] - R^f \ = \ -\frac{Cov(\mathbf{R}^*, \mathbf{R}^j)}{\mathbb{E}[\mathbf{R}^*]} \ = \ -\underbrace{\frac{Cov(\mathbf{R}^*, \mathbf{R}^j)}{Var(\mathbf{R}^*)}}_{\beta_j} \cdot \frac{Var(\mathbf{R}^*)}{\mathbb{E}[\mathbf{R}^*]} \tag{16}$$

- By plugging in $\mathbb{R}^* \to \mathbb{R}^j$ to (16), we get $\mathbb{E}[\mathbf{R}^*] - R^f = -Var(\mathbf{R}^*)/\mathbb{E}[\mathbf{R}^*]$, and

$$\mathbb{E}[\mathbf{R}^j] - R^f \ = \ \beta_j \left(\mathbb{E}[\mathbf{R}^*] - R^f\right). \tag{17}$$

- It remains to show that $\mathbf{m}^*$ corresponds to the payoff of the market portfolio.

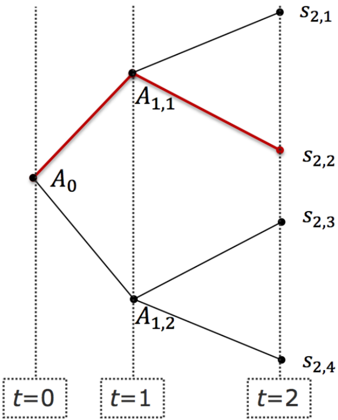
Multi-Period Asset Model



Dynamic Trading and Model Completeness

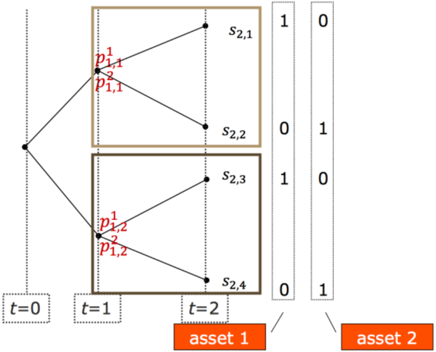
- T periods. Look at each branch as a one-period setting
- Even if with limited assets, we can trade at each time-period to achieve completeness
- *Short-lived assets* only pays out at the period after purchase. Need T assets for completeness.
  First replicate an asset bought at $t = 0$ and pays \$1 at $s_{2,2}$
  - Arrow security: costs \$1, pays \$1 for a specific next state
  - $q_{2,2}$: state price for $s_{2,2}$
  - $\implies$ $q_{2,2}$ Arrow securities are needed when $A_{1,1}$ occurs
  - To get \$1 at $A_{1,1}$, need $q_{1,1}$ Arrow securities at $t = 0$
  - $\implies$ Buy $q_{1,1}q_{2,2}$ Arrow securities at $t = 0$, use payoff to purchase $q_{2,2}$ Arrow securities at $t = 1$

  Repeat for other states to establish completeness



- *Long-lived assets* pay off over many time periods. Can be traded any time.
- Example: Two assets paying off at $t = 2$.
    - *Trading strategy* $[j, A_{t,i}]$: the cash flow of asset j purchased in event $A_{t,i}$ and sold one period later.
    - $p^j_{t,i}$: price of asset j at time and state t, i
    - Enumerating strategies creates payoff matrix

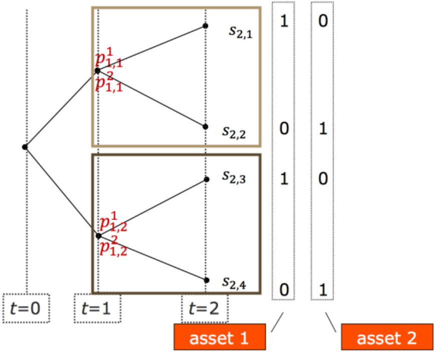
Strategy	$[1, A_0]$	$[2, A_0]$	$[1, A_{1,1}]$	$[2, A_{1,1}]$	$[1, A_{1,2}]$	$[2, A_{1,2}]$
Event A_0	$-p^1_0$	$-p^2_0$	0	0	0	0
Event $A_{1,1}$	$p^1_{1,1}$	$p^2_{1,1}$	$-p^1_{1,1}$	$-p^2_{1,1}$	0	0
Event $A_{1,2}$	$p^1_{1,2}$	$p^2_{1,2}$	0	0	$-p^1_{1,2}$	$-p^2_{1,2}$
State $s_{2,1}$	0	0	1	0	0	0
State $s_{2,2}$	0	0	0	1	0	0
State $s_{2,3}$	0	0	0	0	1	0
State $s_{2,4}$	0	0	0	0	0	1



- Example contd.

Strategy	$[1, A_0]$	$[2, A_0]$	$[1, A_{1,1}]$	$[2, A_{1,1}]$	$[1, A_{1,2}]$	$[2, A_{1,2}]$
Event A_0	$-p_0^1$	$-p_0^2$	0	0	0	0
Event $A_{1,1}$	$p_{1,1}^1$	$p_{1,1}^2$	$-p_{1,1}^1$	$-p_{1,1}^2$	0	0
Event $A_{1,2}$	$p_{1,2}^1$	$p_{1,2}^2$	0	0	$-p_{1,2}^1$	$-p_{1,2}^2$
State $s_{2,1}$	0	0	1	0	0	0
State $s_{2,2}$	0	0	0	1	0	0
State $s_{2,3}$	0	0	0	0	1	0
State $s_{2,4}$	0	0	0	0	0	1

  - The payoff matrix has full rank iff the highlighted submatrix has full rank
  - *Generically*, the markets are *dynamically* complete



- Generically the *branching number* of the tree determines assets needed for completeness
  - e.g. binomial tree $\implies$ Black-Scholes, 2 assets

Stochastic Discount Factors

Some changes to notation:

- For an asset, replace $\mathbf{x} \to p_{t+1} + x_{t+1}$ and $\mathbf{m} \to m_{t+1}$
  - need to keep track of the prices p_t at each time
  - x_t represents deviation from price
- Both $\{x_t\}_{t=0}^T$ and $\{m_t\}_{t=0}^T$ are *random processes*
  - x_t and m_t are random variables over some state space Ω (e.g. corresponding directly to stock price fluctuations)
- Each x_t and m_t measurable w.r.t. algebra $\mathcal{F}_t$ of the state space Ω
- Furthermore $\{\mathcal{F}_t\}_{t=0}^T$ is a filtration: $\mathcal{F}_u \subseteq \mathcal{F}_v$ if $u \leq v$.
  - Uncertainty increases over time; cannot know the future

Multi-period Stochastic Discount Factors

- One-period pricing becomes

$$p_t = \mathbb{E}_t \left[m_{t+1} \cdot (p_{t+1} + x_{t+1}) \right] \tag{18}$$

- Multi-period SDF: $M_{t+1} \doteq m_1 \cdot m_2 \cdot \ldots \cdot m_{t+1}$
- Multiplying both sides by M_t:

$$M_t p_t = \mathbb{E}_t \left[M_{t+1}(p_{t+1} + x_{t+1}) \right] \tag{19}$$

- Then for $t = 0$ and $t = 1$,

$$p_0 = \mathbb{E}_0 \left[M_1(p_1 + x_1) \right] \quad \text{and} \quad M_1 p_1 = \mathbb{E}_1 \left[M_2(p_2 + x_2) \right]$$

- Therefore

$$p_0 \;=\; \mathbb{E}_0 \left[\mathbb{E}_1 \left[M_2(p_2 + x_2) \right] + M_1 x_1 \right] \;=\; \mathbb{E}_0 \left[\mathbb{E}_1 \left[M_2 p_2 + M_2 x_2 + M_1 x_1 \right] \right] \tag{20}$$

Multi-period Stochastic Discount Factors

- Law of Iterated Expectations: $\mathbb{E}_u\left[\mathbb{E}_v[X]\right] = \mathbb{E}_u[X]$ for all $\mathcal{F}_u \subseteq \mathcal{F}_v$
- Therefore

$$p_0 = \mathbb{E}_0\left[\mathbb{E}_1\left[M_2 p_2 + M_2 x_2 + M_1 x_1\right]\right] = \mathbb{E}_0\left[M_2 p_2 + M_2 x_2 + M_1 x_1\right] \tag{21}$$

- Iterating k steps yields

$$p_0 = \mathbb{E}_0[M_k p_k] + \sum_{t=1}^{k} \mathbb{E}_0[M_t x_t] \tag{22}$$

- Assuming that $\lim_{k\to\infty} \mathbb{E}_0[M_k p_k] = 0$, we have

$$p_0 = \sum_{t=1}^{\infty} \mathbb{E}_0[M_t x_t]$$

Some Ideas for Next Steps

- Extension to continuous time / state space
- Options pricing
- Modern derivation of portfolio theory / Black-Scholes
- Clarification on the multi-period setting
  - Some of the notation / objects in the multi-period setting are not clearly explained
  - Seems to be a widespread problem in financial economics literature
- Multi-period SDF: crucial for pricing and represents systemic risk
  - Relationship to market variables?
  - Empirical models: Fama-French, factor models, PCA / ML models

Thanks!