Mathematical Modeling and Consulting

JOHNS HOPKINS UNIVERSITY

Sponsor

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Final Report

How Much Ice Do You Need?

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Abstract

McDonald's Corporation is the world's largest chain of hamburger fastfood restaurants, and selling soft drinks is a significant portion of McDonald's business. The server is not accustomed to putting much thought in measuring the amount of ice put in the cup. This often results in a overly diluted, overly concentrated or warmer drink for the customer. Our task is to provide a suggestion for the optimal amount of ice for soda, such that the average consumer will be most satisfied. We approach this problem by first creating an experiment that measures consumer preferences to the amount of ice in a large McDonald's cup, and different points in time after the ice is initially mixed with the soda. The collected data is statistically analyzed to give us an idea of the optimal amount of ice to be added. Secondly, we calculate the different temperatures and the amount of dilution of the resulting drink, using specific heat capacities of soda and ice. This can be used complementary to our first experiment to provide more theoretical reasonings behind any trends or conclusions.

Acknowledgments

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Introduction

McDonald's Corporation is the world's largest chain of hamburger fastfood restaurants, serving around 68 million customers daily in 119 countries. Mcdonald's primarily sells hamburgers, cheeseburgers, chicken, French fries, breakfast items, soft drinks, milkshakes and desserts. No meal is complete without a drink; and from Diet Coke to low-fat milk to fresh-brewed, hot coffee, McDonald's serves many different varieties of beverages.

Given that soft drinks are normally assumed to be the perfect accompaniment to a fast food meal, their temperature is also essential to the overall satisfaction of the consumer. We have been tasked by McDonald's to find the optimal amount of ice to put into their standard large size cups, and the provide them with data on consumer satisfaction as time elapses.

Problem Statement

Selling soft drinks is a significant portion of McDonald's business, be it as a thirst quencher, or as part of the extra value meal. The server is not accustomed to putting much thought in measuring the amount of ice put in the cup. This often results in a overly diluted, overly concentrated or overly cold drink for the customer. This is likely to lower overall customer satisfaction, since a drink is a significant complement to a meal. Thus, customers are likely to appreciate if the right amount of ice was added for optimal satisfaction.

To further define this problem, the exogenous variables are the proportion of ice to put in a drink. The endogenous variable would be the resulting temperature and concentration of the drink, as we are assuming that a customer's satisfaction is affected only by the temperature and concentration of the drink.

Technical Background

To aid us in our analysis of the experiment that we plan to do, we will be using the Pearson's chi-squared test to test how significant our experiment results are. It tests a null hypothesis stating that the frequency distribution of certain events observe in a sample is consistent with a particular theoretical distribution. The events must be mutually exclusive and have total probability one.
The Pearson's chi-squared test is used to assess two types of comparisons: tests of goodness of fit and tests of independence. Ths first step is to calculate the chi-squared statistic X^2, which resembles a normalized sum of squared deviations between observed and theoretical frequencies, as shown in Equation (1)

$$X^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} \tag{1}$$

where O_i is the observed frequency of category i being chosen,
E_i is the expected theoretical frequency of category i being chosen,
n is the number of different categories,

The second step is to determine the degrees of freedom, d, of that statistic, which is essentialy the number of frequencies reduced by the number of parameters of the fitted distribution. In the third step, X^2 is compared to the critical value of no significaned from the χ^2_d distribution, which usually gives a good approximation fo the distribution of X^2.
By comparing the X^2 statistic calculated in Equation (1) to the appropiate χ^2distribution with d degrees of freedom, we will obtain a p-value. The p-value is the probability of obtaining a test statistic at least as extrememas the one that is actually observed, assuming the null hypothesis is true.

H_0: All n categories are chosen with equal probability

H_a: All n ncateogires are chosen with unequal probability.

If the p-value is greater than a given α, it means that the result is not that extremem assuming the null hypothesis is true. Thus, there is insufficient evidence to reject the null hypothesis that all n categories are chosen with equal probability. However, if the p-value is greater than α, it means that observed result is too extreme assuming the null hypothesis is true. Thus, we have sufficient evidence to reject the null hypothesis and accept the alternative hypothesis.

While carrying out this test, we have to be wary of 2 statistical errors : Type I and Type II errors. Type I error is the incorrect rejection of a true null hypothesis (a false positive). The probaility of a Type I error is equal to the α that we use. Most statistical test usually use 0.05 for its α. Type II error is seen as a false negative, where we accept the null hypothesis when it is not true. The probability of a Type II error is usually denoted by β. The power of a test is $1 - \beta$.

There are a few assumptions in place when using the chi-squared test.

- The sample data is a simple random sample from a fixed distribution or population where each member of the population has an equal probability of selection. We have tried to achieve this assumption as much as possible by ensuring our sample group is as representative of the population by choosing test subjects from different backgrounds.
- Sample size is sufficiently big. This is to avoid committing a Type II error.
- Observations are assumed to be independent of each other. There is no correlation between the test subject's preferences in any way.

Methods

Firstly, we have a few assumptions on hand in order to simplify this problem. We assume that the consumer's taste depends entirely on the dilution and temperature of the drink. Since both the dilution and temperature of the drink rely entirely on the ice proportion, these two values would come hand-in-hand. Also, any sample group that we use represents the population's preferences accurately. Only a similar note, the 4 different time parameterrs which we perform the experiment is sufficient to represent the overall satisfaction the customer has with the drink.
We are interested in approaching this problem using 2 different methods. The first method would be experimenting with the 4 most popular types of soda at McDonald's, namely Coca Cola, Fanta Orange, Sprite and Diet Coke. Then we would experiment and gauge the satisfaction of the test subjects by experimenting with different amounts of ice to find out the optimal proportion of ice to soda.
This experiment will take place over the course of 4 days, where we will each drink will be tested on each day. For each time parameter, we will give the test subject 3 different cups (labelled A, B and C) with different ice proportions in them (40%, 60%, 75%). The ice will be left in the drink for time parameter t, and then the test subject will drink the 3 different cups, and indicate their preference by ranking the cups (3 is their favorite cup, and 1 is their least favorite cup). There will be 4 similar tests done on the same day but with a different time parameter t. These 4 experiments will be scheduled an hour apart from each other, so there will be no aftereffect or bias from the previous experiment. The 4 different time parameters t will be t=0.5 minutes, t=2 minutes, t=5 minutes, t=30 minutes, which indicates the time elapsed after the ice is mixed with the drink. The test subject is allowed to sip the drink at time=t, assuming the ice is placed in the drink at t=0. We will tabulate the preferences of the entire sample group, and provide a conclusion about the consumer's preferred ice proportion in each soda drink. This will be a blind test and the subject will not know which ice proportions cups A, B and C will have (it will be random every round).

Ice Proportion	A	B	C
t=0.5mins			
t=2mins			
t=5mins			
t=30mins			

Table 1: Sample form each test subject will need to fill out for each drink

Ice Proportion	A	B	C
t=0.5mins	3	2	1
t=2mins	1	3	2
t=5mins	2	3	1
t=30mins	1	2	3

Table 2: Example of a response by a test subject

We would also be looking to approach this problem from an alternative, and supplementary, method. The second method would be using physics-based modeling. Utilizing the specific heat capacities of soda and ice (already found as specific values), we can calculate the different temperatures and dilution that the resulting drink will be. This would not really tell us anything about preference of the population, but if provides more of a perspective on how different ice proportions affect temperatures and dilutions. However, this is purely theoretical since it does not take into account heat loss to the environment and cup. This will be mainly a supporting tool and not used in place of the first approach.

Results

	40%	60%	75%
t=0.5 mins	15	25	32
t=2 mins	14	24	34
t=5 mins	14	27	31
t=30 mins	18	36	18

Table 3: Experiment results for Coke

	40%	60%	75%
t=0.5 mins	15	27	30
t=2 mins	20	19	33
t=5 mins	14	29	29
t=30 mins	17	30	25

Table 4: Experiment results for Sprite

	40%	60%	75%
t=0.5 mins	15	23	34
t=2 mins	19	23	30
t=5 mins	18	27	27
t=30 mins	12	35	25

Table 5: Experiment results for Fanta Orange

	40%	60%	75%
t=0.5 mins	15	24	33
t=2 mins	21	19	32
t=5 mins	16	24	32
t=30 mins	18	22	32

Table 6: Experiment results for Diet Coke

Volume of ice to volume of soda	Dilution	Temperature (Celsius)
1/10	0.09	16.2
1/8	0.11	14.3
1/6	0.15	11.2
1/5	0.18	8.8
1/4	0.23	5.5

Table 7: Calculated dilution and temperature for difference ice volumes

Analysis

We run the Pearson's chi-squared test on the results that we have obtain with the following null hypothesis.

H_0: The 3 different proportions are desired to an equal extent by the population.

H_a: The 3 different proportions are not desired to an equal extent by the population.

Since there are 72 points to be awarded (12 test subjects can give 6 points each), the expected score of each cup should be $72/3 = 24$. These will be the E_i for each of the 3 categories. Using Equation (1) and comparing it to a χ^2 distribution with 2 degress of freedom, we can obtain a p-value. We choose $l\alpha$ to be 0.05. For a certain category, if the p-value is smaller than 0.05, the result is significant and we can reject the null hypothesis. Otherwise, the result is not significant and we do not reject the null hypothesis for that category.

To compute the p-value we use the following R code in Listing 1

Listing 1: R code for chi-squared test

```
x=c(a,b,c) #a,b,c represents the 3 data points for each time parameter and drink
chisq.test(x)
```

	40%	60%	75%	p-value	significance?
t=0.5 mins	15	25	32	0.047	significant
t=2 mins	14	24	34	0.016	significant
t=5 mins	14	27	31	0.037	significant
t=30 mins	18	36	18	0.011	significant
Sum of significant rows	61	112	115		

Table 8: Experiment results for Coke

Tables 8, 9 ,10 and 11 shows the poll results for the four different drinks of Coke, Sprite, Fanta Orange, and Diet Coke respectively, but now with two added columns: the p-value, and the significance. As before, the rows represent the poll results taken after 0.5 minutes, 2 minutes, 5 minutes and 30 minutes; the columns represent the percentage of the cup filled (not by volume but by height of the cup). Each subject is asked to rank their preference of the amount of ice from 1 - 3 at each point in time, 3 being the most enjoyable cup and 1 being the least enjoyable cup. These preferences are collected and summed, and shown in the tables. For each row, the higher the number, the more satisfactory the subject is at that point in time.

	40%	60%	75%	p-value	significance?
t=0.5 mins	15	27	30	0.072	not significant
t=2 mins	20	19	33	0.079	not significant
t=5 mins	14	29	29	0.044	significant
t=30 mins	17	30	25	0.011	significant
Sum of significant rows	31	59	54		

Table 9: Experiment results for Sprite

	40%	60%	75%	p-value	significance?
t=0.5 mins	15	23	34	0.022	significant
t=2 mins	19	23	30	0.275	not significant
t=5 mins	18	27	27	0.325	not significant
t=30 mins	12	35	25	0.004	significant
Sum of significant rows	27	58	59		

Table 10: Experiment results for Fanta Orange

The p-value is calculated using the Chi-Squared Test. The Chi-squared test is a statistical hypothesis test in which the sampling distribution of the test statistic is considered significant when the null hypothesis is true. The alternative hypothesis in this case is that the respondents (our test subjects) are *indifferent* between the percentages of the cup being filled with ice. This means that 1/3 of respondents choose 40%, 1/3 of the responsdents choose 60% and the final 1/3 choose 75% - as a whole subject population, they do not care how much ice is put in the cup. The null hypothesis is that there exists a bias, or a tendency towards one percentage than another. This null hypothesis would mean that we can observe a clear trend in the data being shown, and thus we can use the data presented.

The significance threshold being used here is 0.05%. This means that if the p-value calculated is less than 0.05, we can accept null hypothesis, and the data is considered significant. If the p-value calculated is greater than 0.05, we can reject the null hypothesis, and the data is considered less significant.

Based on our experimental results collected, we can see a relatively clear trend across all four drinks at each point in time.

At t = 0.5 minutes, it is obvious that the cup with the most amount of ice, at 75% of the cup filled (not by volume but by height of the cup), gave the subjects the most satisfaction. This is likely because, at 30 seconds after the soda is mixed with the ice, the soda is chilled the fastest with the most ice, and little dilution occurs. Given that the sodas are not watered down by dilution yet, subjects enjoy the coldest of drinks, which in this case is the one with the most ice.

At t = 2 minutes, there is a less clear trend, but generally the subjects still prefer 75% as compared to 40% or 60%. This is reasonable, given that the ice still has not diluted much, and the most ice would still chill the drink the most and provide the most satisfaction.

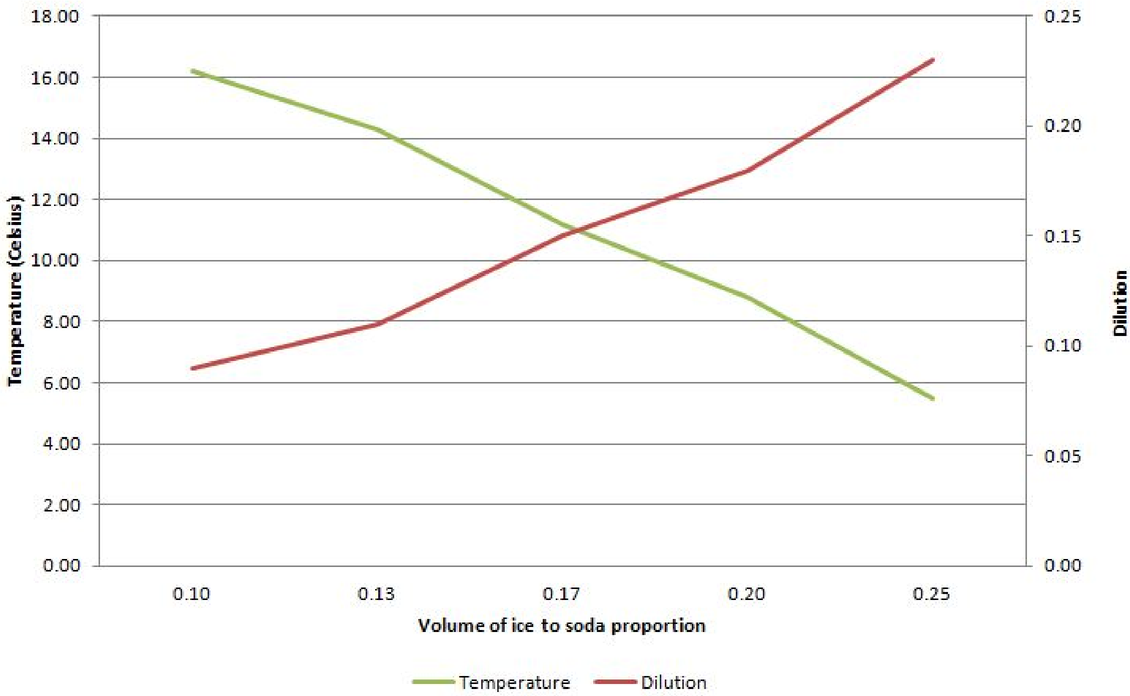
	40%	60%	75%	p-value	significance?
t=0.5 mins	15	24	33	0.034	significant
t=2 mins	21	19	32	0.130	not significant
t=5 mins	16	24	32	0.069	not significant
t=30 mins	18	22	32	0.115	not significant
Sum of significant rows	15	24	33		

Table 11: Experiment results for Diet Coke

At t = 5 minutes, we see a split preference between 60% and 75%. At 5 minutes, more of the ice has dissolved, and dilution in both cups are likely the same given that their temperatures are also roughly equal. The continued low satisfaction for 40% is likely due to the lack of chilling effect of this low quantity of ice.

At t = 30 minutes, we can assume that much of the ice have dissolved, and the resulting dilution from the ice is very high. The more ice there is, the more diluted the soda is. Thus we see a decline in satisfaction for the 75% cup. It is still more satisfying than the 40% cup, probably because it still manages to maintain its coldness, whereas the 40% cup is likely to be less chilled.

Figure 1: Graph of Dilution and Temperature against different ice volumes



Theoretical Analysis

Volume of ice to volume of soda	Dilution	Temperature (Celsius)
1/10	0.09	16.2
1/8	0.11	14.3
1/6	0.15	11.2
1/5	0.18	8.8
1/4	0.23	5.5

Table 12: Calculated dilution and temperature for difference ice volumes

Using the lemmas under Appendix A, we are able to calculate the final temperature and dilution. This is assuming we have the ratio of the volume of ice to soda. This is a different measuring scale from the experimental-based approach. In the experimental-based approach, the percentage is based on the height of a standard regular McDonald's cup. However, in this approach, we are using the ratio as exact absolute volume. Thus, we first calculate how much energy it would take to melt the ice, and see how many degrees it would lower the temperature of the drink by. Then, we calculate what is the temperature that the soda and melted ice would eventually converge to. Also, the dilution is calculated by finding the resulting volume of the melted ice (since ice has a lower density than water), and dividing that by the amount of soda.

Figure 1 is generated by plotting the data from table 12. These basically show a more theoretical approach, which asserts that as more ice is put in the soda, the temperature is further decreased, but the dilution is increased. This take-away from this theoretical result is congruent to those of our experimental results. This analysis, and the generated trendline, is important because if our sponsor decides on a certain volume of ice to soda proportion and would like to find out the dilution and temperature effects, we can extropolate and further provide the desired calculations.

Conclusion

	40%	60%	75%
Coke	61	112	115
Sprite	31	59	54
Fanta Orange	27	58	59
Diet Coke	15	24	33
Total	134	253	261

Table 13: Experimental Totals

Table 13 shows the compiled set of preferences for each soda. To get this table, we summed up the preferences of each soda at each point in time ($t = 0.5$, 2, 5, 30 minutes), but taking into account only those data points that are considered significant. This allows us to measure only *clear* trends in the data, and hence scrutinize those specific instances that exhibit an evident preference towards one amount of ice over another.

From this table, we can see that across all sodas, there is a clear disinclination towards the cup being filled 40% with ice. Yet between 60% or 75%, there is no clear 'winner' either. However, we know that subjects prefer the cup with more ice if they are to finish their drink quickly, and less ice if they are going to keep the drink for a while. Based on the data and analysis performed *in this experiment alone*, we can conclude that in filling a soda cup, customers, on average, **prefer a cup filled with 75% of ice if they intend to drink it in the short term, but prefer a cup filled with 65% of ice if they intend to drink it in the long term.**

Appendix A

Lemmas

q=mCΔT,
where C = specific heat capacity (J/g C)
q = quantity of heat in joules
m = mass in grams
ΔT= change in temperature
so C= q/(m*Δ T)

Appendix B

Glossary

Specific heat capacity. Amount of heat per unit mass required to raise the temperature by one degree Celsius
Heat of fusion. Amount of heat needed to change its state from a solid to a liquid per unit mass

Appendix C

Matlab code

Listing C.1: Matlab code for calculating resulting temperature and dilution

```
function [dilution, temp] = SHC(vsoda,vice, w )
%vsoda=volume of soda in cm3, Assume density of soda is 1g/cm3
%vice=volume of ice in cm3, Assume density of ice is 0.9167g/cm3
%w=specific heat capacity of soda in J
%specfic heat capacity of water = 4.1813J/g/K
%specific heat of fusion = 334J/g
%Assume ice is 0degrees Celsius, soda is 25 degrees Celsius
%dilution is dilution of resulting solution
%temp is resulting temperature of solution in Celsius

watershc=4.1813;
fusion=334;
density=0.9167;
t=25; %room temperature

vice=vice*density; %find mass of ice
dilution=vice/vsoda; %find dilution
energymelt=334*vice; %find energy needed to melt ice

temp=(t*vsoda*w-energymelt)/(watershc*vice+vsoda*w); %resulting temperature

end
```

Appendix D

R code

Listing D.1: R code for implementing Pearson's chi-squared test

```
x=c(15,25,32)
chisq.test(x)
x=c(14,24,34)
chisq.test(x)
x=c(14,27,31)
chisq.test(x)
x=c(18,36,18)
chisq.test(x)
x=c(15,27,30)
chisq.test(x)
x=c(20, 19, 33)
chisq.test(x)
x=c(14, 29, 29)
chisq.test(x)
x=c(18, 36, 18)
chisq.test(x)
x=c(15,23,34)
chisq.test(x)
x=c(19,23,30)
chisq.test(x)
x=c(18,27,27)
chisq.test(x)
x=c(12,25,35)
chisq.test(x)
x=c(15,24,33)
chisq.test(x)
x=c(21,19,32)
chisq.test(x)
x=c(16,24,32)
chisq.test(x)
x=c(18,22,32)
chisq.test(x)
```

Selected Bibliography Including Cited Works

[1] American Mathematical Society. *MathSciNet: Mathematical Reviews on the Web.* ⟨http://www.ams.org/mathscinet/⟩. Accessed June 17, 2009.

Because an online reference may be changed at any time, it is conventional to tie the reference to the date when the resource was accessed.

[2] Roger R. Bate, Donald D. Mueller, and Jeremy E. While. *Fundamentals of Astrodynamics.* Dover, 1971.

A standard textbook on astrodynamics. It provided a reference for orbital mechanics and satellite propagation.

[3] Ingrid Carlbom and Joseph Paciorek. Planar Geometric Projections and Viewing Transformations. *Computing Surveys*, 1978.

Gives a thorough background to projective geometry and vertical perspective projection. This includes details about calculating projections using homogeneous coordinates and projection matrices.

[4] Gelfand and Fomin. *Calculus of Variations.* Prentice-Hall, 1963.

Discusses the essential principle of variational method for optimal path problems.

[5] George Grätzer. *More Math Into LaTeX.* Birkhäuser, Boston, MA, fourth edition, 2007.

[6] Jacob Kogan. *Introduction to Clustering Large and High-Dimensional Data.* Cambridge, 2007.

Focuses on a few of the most important clustering algorithms, providing also some useful optimization techniques for high-dimensional objective functions.

[7] David A. Vallado. ***Fundamentals of Astrodynamics and Applications.*** **Space Technology, 2007.**

A professional astrodynamics reference. It emphasizes the practical use of astrodynamics in space missions.

[8] **Emo Welzl. Smallest Enclosing Disks (Balls and Ellipsoids).** ***New Results and New Trends in Computer Science*****, 1991.**

Outlines a smallest circle algorithm that runs in linear time using recursion.