Section 1.1 : Functions

In this section we're going to make sure that you're familiar with functions and function notation. Both will appear in almost every section in a Calculus class so you will need to be able to deal with them.

First, what exactly is a function? The simplest definition is an equation will be a function if, for any x in the domain of the equation (the domain is all the x's that can be plugged into the equation), the equation will yield exactly one value of y when we evaluate the equation at a specific x.

This is usually easier to understand with an example.

Example 1 Determine if each of the following are functions.

(a) $y = x^2 + 1$

(b) $y^2 = x + 1$

(a) $y = x^2 + 1$ ***Hide Solution*** ▾

This first one is a function. Given an x, there is only one way to square it and then add 1 to the result. So, no matter what value of x you put into the equation, there is only one possible value of y when we evaluate the equation at that value of x.

(b) $y^2 = x + 1$ ***Hide Solution*** ▾

The only difference between this equation and the first is that we moved the exponent off the x and onto the y. This small change is all that is required, in this case, to change the equation from a function to something that isn't a function.

To see that this isn't a function is fairly simple. Choose a value of x, say $x = 3$ and plug this into the equation.

$$y^2 = 3 + 1 = 4$$

Now, there are two possible values of y that we could use here. We could use $y = 2$ or $y = -2$. Since there are two possible values of y that we get from a single x this equation isn't a function.

Note that this only needs to be the case for a single value of x to make an equation not be a function. For instance, we could have used $x = -1$ and in this case, we would get a single y ($y = 0$). However, because of what happens at $x = 3$ this equation will not be a function.

Next, we need to take a quick look at function notation. Function notation is nothing more than a fancy way of writing the y in a function that will allow us to simplify notation and some of our work a little.

Let's take a look at the following function.

$$y = 2x^2 - 5x + 3$$

Using function notation, we can write this as any of the following.

$$\begin{aligned} f\left(x\right) &= 2x^2 - 5x + 3 & \qquad g\left(x\right) &= 2x^2 - 5x + 3 \\ h\left(x\right) &= 2x^2 - 5x + 3 & \qquad R\left(x\right) &= 2x^2 - 5x + 3 \\ w\left(x\right) &= 2x^2 - 5x + 3 & \qquad y\left(x\right) &= 2x^2 - 5x + 3 \\ & & \vdots & \end{aligned}$$

Recall that this is NOT a letter times x, this is just a fancy way of writing y.

So, why is this useful? Well let's take the function above and let's get the value of the function at $x = -3$. Using function notation we represent the value of the function at $x = -3$ as $f\left(-3\right)$. Function notation gives us a nice compact way of representing function values.

Now, how do we actually evaluate the function? That's really simple. Everywhere we see an x on the right side we will substitute whatever is in the parenthesis on the left side. For our function this gives,

$$\begin{aligned} f\left(-3\right) &= 2{\left(-3\right)}^2 - 5\left(-3\right) + 3 \\ &= 2\left(9\right) + 15 + 3 \\ &= 36 \end{aligned}$$

Let's take a look at some more function evaluation.

Example 2 Given $f\left(x\right) = -x^2 + 6x - 11$ find each of the following.

(a) $f\left(2\right)$

(b) $f\left(-10\right)$

(c) $f\left(t\right)$

(d) $f\left(t-3\right)$

(e) $f\left(x-3\right)$

(f) $f\left(4x-1\right)$

(a) $f\left(2\right)$ ***Hide Solution*** ▾

$$f\left(2\right)=-{\left(2\right)}^{2}+6(2)-11=-3$$

(b) $f\left(-10\right)$***Hide Solution*** ▾

$$f\left(-10\right)=-{\left(-10\right)}^{2}+6\left(-10\right)-11=-100-60-11=-171$$

Be careful when squaring negative numbers!

(c) $f\left(t\right)$ ***Hide Solution*** ▾

$$f\left(t\right)=-t^{2}+6t-11$$

Remember that we substitute for the x's WHATEVER is in the parenthesis on the left. Often this will be something other than a number. So, in this case we put t's in for all the x's on the left.

(d) $f\left(t-3\right)$ ***Hide Solution*** ▾

$$f\left(t-3\right)=-{\left(t-3\right)}^{2}+6\left(t-3\right)-11=-t^{2}+12t-38$$

Often instead of evaluating functions at numbers or single letters we will have some fairly complex evaluations so make sure that you can do these kinds of evaluations.

(e) $f\left(x-3\right)$ ***Hide Solution*** ▾

$$f\left(x-3\right)=-{\left(x-3\right)}^{2}+6\left(x-3\right)-11=-x^{2}+12x-38$$

The only difference between this one and the previous one is that we changed the t to an x. Other than that, there is absolutely no difference between the two! Don't get excited if an x appears inside the parenthesis on the left.

(f) $f(4x-1)$ ***Hide Solution*** ▾

$$f(4x-1) = -(4x-1)^2 + 6(4x-1) - 11 = -16x^2 + 32x - 18$$

This one is not much different from the previous part. All we did was change the equation that we were plugging into the function.

All throughout a calculus course we will be finding roots of functions. A root of a function is nothing more than a number for which the function is zero. In other words, finding the roots of a function, $g(x)$, is equivalent to solving

$$g(x) = 0$$

Example 3 Determine all the roots of $f(t) = 9t^3 - 18t^2 + 6t$

Hide Solution ▾

So, we will need to solve,

$$9t^3 - 18t^2 + 6t = 0$$

First, we should factor the equation as much as possible. Doing this gives,

$$3t\left(3t^2 - 6t + 2\right) = 0$$

Next recall that if a product of two things are zero then one (or both) of them had to be zero. This means that,

$$3t = 0 \qquad \text{OR,} \qquad 3t^2 - 6t + 2 = 0$$

From the first it's clear that one of the roots must then be $t = 0$. To get the remaining roots we will need to use the quadratic formula on the second equation. Doing this gives,

$$
\begin{aligned}
t &= \frac{-\left(-6\right) \pm \sqrt{\left(-6\right)^{2} - 4\left(3\right)\left(2\right)}}{2\left(3\right)} \\
&= \frac{6 \pm \sqrt{12}}{6} \\
&= \frac{6 \pm \sqrt{\left(4\right)\left(3\right)}}{6} \\
&= \frac{6 \pm 2\sqrt{3}}{6} \\
&= \frac{3 \pm \sqrt{3}}{3} \\
&= 1 \pm \frac{1}{3}\sqrt{3} \\
&= 1 \pm \frac{1}{\sqrt{3}}
\end{aligned}
$$

In order to remind you how to simplify radicals we gave several forms of the answer.

To complete the problem, here is a complete list of all the roots of this function.

$$t = 0, \ t = \frac{3 + \sqrt{3}}{3}, \ \ t = \frac{3 - \sqrt{3}}{3}$$

Note we didn't use the final form for the roots from the quadratic. This is usually where we'll stop with the simplification for these kinds of roots. Also note that, for the sake of the practice, we broke up the compact form for the two roots of the quadratic. You will need to be able to do this so make sure that you can.

This example had a couple of points other than finding roots of functions.

The first was to remind you of the quadratic formula. This won't be the last time that you'll need it in this class.

The second was to get you used to seeing "messy" answers. In fact, the answers in the above example are not really all that messy. However, most students come out of an Algebra class very used to seeing only integers and the occasional "nice" fraction as answers.

So, here is fair warning. In this class I often will intentionally make the answers look "messy" just to get you out of the habit of always expecting "nice" answers. In "real life" (whatever that is) the answer is rarely a simple integer such as two. In most problems the answer will be a decimal that came about from a messy fraction and/or an answer that involved radicals.

One of the more important ideas about functions is that of the **domain** and **range** of a function. In simplest terms the domain of a function is the set of all values that can be plugged into a function and have the function exist and have a real number for a value. So, for the domain we need to avoid division by zero, square roots of negative numbers, logarithms of zero and logarithms of negative numbers (if not familiar with logarithms we'll take a look at them a little **later**), *etc*. The range of a function is simply the set of all possible values that a function can take.

Let's find the domain and range of a few functions.

Example 4 Find the domain and range of each of the following functions.

(a) $f(x) = 5x - 3$

(b) $g(t) = \sqrt{4 - 7t}$

(c) $h(x) = -2x^2 + 12x + 5$

(d) $f(z) = |z - 6| - 3$

(e) $g(x) = 8$

(a) $f(x) = 5x - 3$ ***Hide Solution*** ▾

We know that this is a line and that it's not a horizontal line (because the slope is 5 and not zero…). This means that this function can take on any value and so the range is all real numbers. Using "mathematical" notation this is,

$$\text{Range}: \quad (-\infty, \infty)$$

This is more generally a polynomial and we know that we can plug any value into a polynomial and so the domain in this case is also all real numbers or,

$$\text{Domain}: \quad -\infty < x < \infty \qquad \text{or} \qquad (-\infty, \infty)$$

(b) $g(t) = \sqrt{4 - 7t}$ ***Hide Solution*** ▾

This is a square root and we know that square roots are always positive or zero. We know then that the range will be,

$$\text{Range}: \quad [0, \infty)$$

For the domain we have a little bit of work to do, but not much. We need to make sure that we don't take square roots of any negative numbers, so we need to require that,

$$\begin{aligned} 4 - 7t &\ge 0 \\ 4 &\ge 7t \\ \frac{4}{7} &\ge t \qquad \Rightarrow \qquad t \le \frac{4}{7} \end{aligned}$$

The domain is then,

$$\text{Domain : } \; t \le \frac{4}{7} \qquad \text{or} \qquad \left(-\infty, \frac{4}{7}\right]$$

(c) $h\left(x\right) = -2x^2 + 12x + 5$ ***Hide Solution*** ▾

Here we have a quadratic, which is a polynomial, so we again know that the domain is all real numbers or,

$$\text{Domain : } \; -\infty < x < \infty \qquad \text{or} \qquad \left(-\infty, \infty\right)$$

In this case the range requires a little bit of work. From an Algebra class we know that the graph of this will be a **parabola** that opens down (because the coefficient of the x^2 is negative) and so the vertex will be the highest point on the graph. If we know the vertex we can then get the range. The vertex is then,

$$x = -\frac{12}{2\left(-2\right)} = 3 \qquad y = h\left(3\right) = -2(3)^2 + 12\left(3\right) + 5 = 23 \qquad \Rightarrow \qquad \left(3, 23\right)$$

So, as discussed, we know that this will be the highest point on the graph or the largest value of the function and the parabola will take all values less than this, so the range is then,

$$\text{Range : } \; \left(-\infty, 23\right]$$

(d) $f\left(z\right) = \left|z - 6\right| - 3$ ***Hide Solution*** ▾

This function contains an absolute value and we know that absolute value will be either positive or zero. In this case the absolute value will be zero if $z = 6$ and so the absolute value portion of this function will always be greater than or equal to zero. We are subtracting 3 from the absolute value portion and so we then know that the range will be,

$$\text{Range : } \; \left[-3, \infty\right)$$

We can plug any value into an absolute value and so the domain is once again all real numbers or,

$$\text{Domain : } \; -\infty < z < \infty \qquad \text{or} \qquad \left(-\infty, \infty\right)$$

(e) $g(x) = 8$ ***Hide Solution*** ▾

This function may seem a little tricky at first but is actually the easiest one in this set of examples. This is a constant function and so any value of x that we plug into the function will yield a value of 8. This means that the range is a single value or,

$$\text{Range}: \quad 8$$

The domain is all real numbers,

$$\text{Domain}: \quad -\infty < x < \infty \qquad \text{or} \qquad (-\infty, \infty)$$

In general, determining the range of a function can be somewhat difficult. As long as we restrict ourselves down to “simple” functions, some of which we looked at in the previous example, finding the range is not too bad, but for most functions it can be a difficult process.

Because of the difficulty in finding the range for a lot of functions we had to keep those in the previous set somewhat simple, which also meant that we couldn’t really look at some of the more complicated domain examples that are liable to be important in a Calculus course. So, let’s take a look at another set of functions only this time we’ll just look for the domain.

Example 5 Find the domain of each of the following functions.

(a) $f(x) = \dfrac{x-4}{x^2-2x-15}$

(b) $g(t) = \sqrt{6+t-t^2}$

(c) $h(x) = \dfrac{x}{\sqrt{x^2-9}}$

(a) $f(x) = \dfrac{x-4}{x^2-2x-15}$ ***Hide Solution*** ▾

Okay, with this problem we need to avoid division by zero, so we need to determine where the denominator is zero which means solving,

$$x^2 - 2x - 15 = (x-5)(x+3) = 0 \qquad \Rightarrow \qquad x = -3, \; x = 5$$

So, these are the only values of x that we need to avoid and so the domain is,

$$\text{Domain}: \quad \text{All real numbers except } x = -3 \;\; \& \;\; x = 5$$

(b) $g(t) = \sqrt{6 + t - t^2}$ **Hide Solution ▾**

In this case we need to avoid square roots of negative numbers and so need to require that,

$$6 + t - t^2 \geq 0 \qquad \Rightarrow \qquad t^2 - t - 6 \leq 0$$

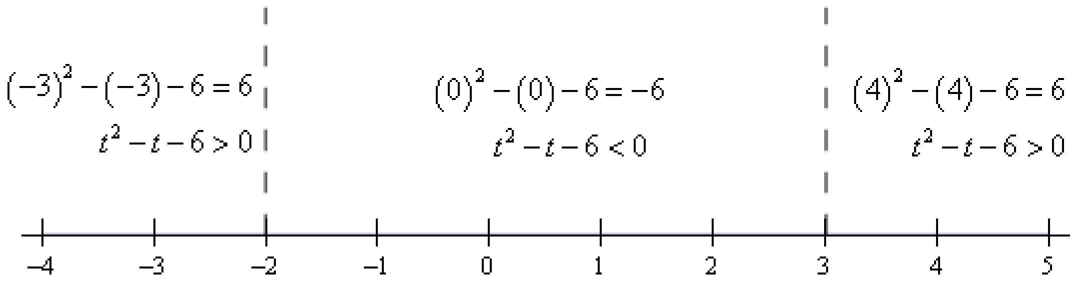
Note that we multiplied the whole inequality by -1 (and remembered to switch the direction of the inequality) to make this easier to deal with. You'll need to be able to solve inequalities like this more than a few times in a Calculus course so let's make sure you can solve these.

The first thing that we need to do is determine where the function is zero and that's not too difficult in this case.

$$t^2 - t - 6 = (t - 3)(t + 2) = 0$$

So, the function will be zero at $t = -2$ and $t = 3$. Recall that these points will be the only place where the function *may* change sign. It's not required to change sign at these points, but these will be the only points where the function can change sign. This means that all we need to do is break up a number line into the three regions that avoid these two points and test the sign of the function at a single point in each of the regions. If the function is positive at a single point in the region it will be positive at all points in that region because it doesn't contain the any of the points where the function may change sign. We'll have a similar situation if the function is negative for the test point.

So, here is a number line showing these computations.



From this we can see that the only region in which the quadratic (in its modified form) will be negative is in the middle region. Recalling that we got to the modified region by multiplying the quadratic by a -1 this means that the quadratic under the root will only be positive in the middle region and so the domain for this function is then,

$$\text{Domain}: \quad -2 \leq t \leq 3 \quad \text{or} \quad [-2, 3]$$

(c) $h\left(x\right)=\dfrac{x}{\sqrt{x^2-9}}$ ***Hide Solution*** ▾

In this case we have a mixture of the two previous parts. We have to worry about division by zero and square roots of negative numbers. We can cover both issues by requiring that,

$$x^2-9>0$$

Note that we need the inequality here to be strictly greater than zero to avoid the division by zero issues. We can either solve this by the method from the previous example or, in this case, it is easy enough to solve by inspection. The domain is this case is,

$$\text{Domain}: \quad x<-3 \ \& \ x>3 \qquad \text{or} \qquad \left(-\infty,-3\right) \ \& \ \left(3,\infty\right)$$

The next topic that we need to discuss here is that of **function composition**. The composition of $f(x)$ and $g(x)$ is

$$\left(f\circ g\right)\left(x\right)=f\left(g\left(x\right)\right)$$

In other words, compositions are evaluated by plugging the second function listed into the first function listed. Note as well that order is important here. Interchanging the order will more often than not result in a different answer.

Example 6 Given $f\left(x\right)=3x^2-x+10$ and $g\left(x\right)=1-20x$ find each of the following.

(a) $\left(f\circ g\right)\left(5\right)$

(b) $\left(f\circ g\right)\left(x\right)$

(c) $\left(g\circ f\right)\left(x\right)$

(d) $\left(g\circ g\right)\left(x\right)$

(a) $\left(f\circ g\right)\left(5\right)$ ***Hide Solution*** ▾

In this case we've got a number instead of an x but it works in exactly the same way.

$$\begin{aligned}\left(f\circ g\right)\left(5\right)&=f\left(g\left(5\right)\right)\\&=f\left(-99\right)=29512\end{aligned}$$

(b) $\left(f\circ g\right)\left(x\right)$ ***Hide Solution*** ▾

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f(1-20x) \\
&= 3(1-20x)^2 - (1-20x) + 10 \\
&= 3\left(1 - 40x + 400x^2\right) - 1 + 20x + 10 \\
&= 1200x^2 - 100x + 12
\end{aligned}
$$

Compare this answer to the next part and notice that answers are NOT the same. The order in which the functions are listed is important!

(c) $(g \circ f)(x)$ ***Hide Solution*** ▾

$$
\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= g\left(3x^2 - x + 10\right) \\
&= 1 - 20\left(3x^2 - x + 10\right) \\
&= -60x^2 + 20x - 199
\end{aligned}
$$

And just to make the point one more time. This answer is different from the previous part. Order is important in composition.

(d) $(g \circ g)(x)$ ***Hide Solution*** ▾

In this case do not get excited about the fact that it's the same function. Composition still works the same way.

$$
\begin{aligned}
(g \circ g)(x) &= g(g(x)) \\
&= g(1-20x) \\
&= 1 - 20(1-20x) \\
&= 400x - 19
\end{aligned}
$$

Let's work one more example that will lead us into the next section.

Example 7 Given $f(x) = 3x - 2$ and $g(x) = \frac{1}{3}x + \frac{2}{3}$ find each of the following.

(a) $(f \circ g)(x)$

(b) $(g \circ f)(x)$

(a) $(f \circ g)(x)$ ***Hide Solution*** ▾

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f\left(\frac{1}{3}x + \frac{2}{3}\right) \\
&= 3\left(\frac{1}{3}x + \frac{2}{3}\right) - 2 \\
&= x + 2 - 2 \\
&= x
\end{aligned}
$$

(b) $(g \circ f)(x)$ ***Hide Solution*** ▾

$$
\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= g(3x - 2) \\
&= \frac{1}{3}(3x - 2) + \frac{2}{3} \\
&= x - \frac{2}{3} + \frac{2}{3} \\
&= x
\end{aligned}
$$

In this case the two compositions were the same and in fact the answer was very simple.

$$(f \circ g)(x) = (g \circ f)(x) = x$$

This will usually not happen. However, when the two compositions are both x there is a very nice relationship between the two functions. We will take a look at that relationship in the next section.