Quantum-computation-and-quantum-information

1.4 Quantum algorithm ~ 2.1 Linear algebra

1.4 Quantum algorithms

- Can we simulate a classical logic circuit using a quantum circuit?
- If yes, how to?

- Can we find a task which a quantum computer may perform better than a classical computer?

1.4.1 classical computations on a quantum computer

- Quantum circuits cannot be used to directly simulate classical circuits

- Can reversible gate simulate irreversible gate in classical circuits?

1.4.1 classical computations on a quantum computer

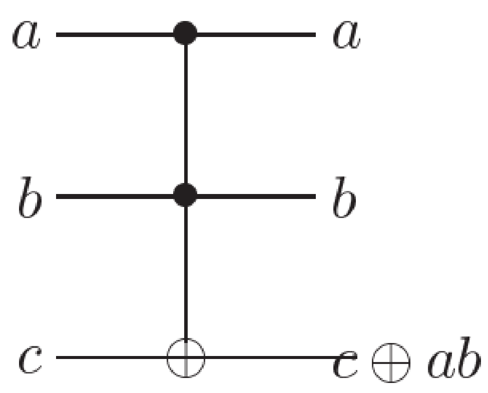
- Any classical circuit can be replaced by an equivalent circuit containing only reversible elements

- By making use of a reversible gate; Toffoli gate

1.4.1 classical computations on a quantum computer

- Toffoli gate = three input bits & three output bits

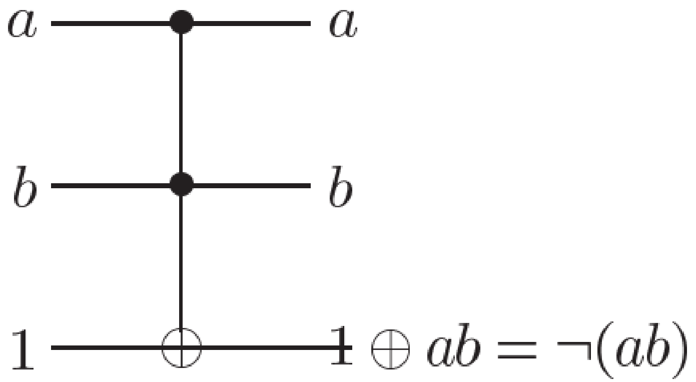
Inputs			Outputs		
a	b	c	a'	b'	c'
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0

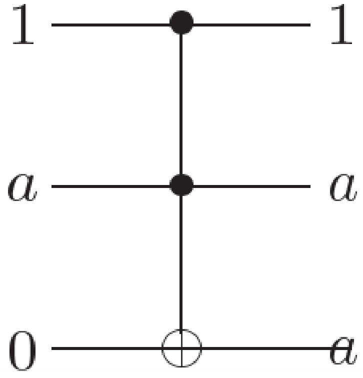


- Two control bits, one target bit

1.4.1 classical computations on a quantum computer

- Can be used to simulate NAND gates and FANOUT





- All other elements in a classical circuit

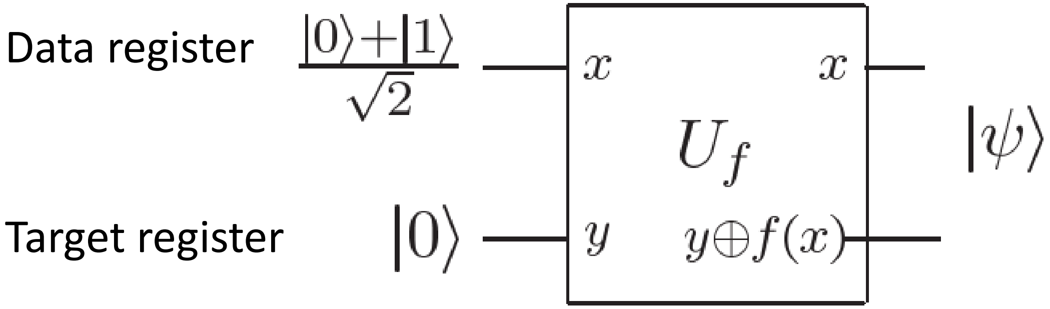
1.4.1 classical computations on a quantum computer

- Can it be a quantum logic gate?

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

1.4.2 Quantum parallelism

- Consider the circuit



- Resulting in the state

$$\frac{|0, f(0)\rangle + |1, f(1)\rangle}{\sqrt{2}} .$$

1.4.2 Quantum parallelism

- Unlike classical parallelism, single f(x) circuit is employed to evaluate the function for multiple values of x simultaneously.
- Quantum parallelism enables all possible values of the function f to be evaluated simultaneously, even though we apparently only evaluated f once.

2.1 Linear algebra

- Review some basic concepts from linear algebra
- Describe the standard notations which are used for these concepts in the study of quantum mechanics

2.1 Linear algebra

- Vector

$$\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$

The space of all n-tuples

$$\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} + \begin{bmatrix} z'_1 \\ \vdots \\ z'_n \end{bmatrix} \equiv \begin{bmatrix} z_1 + z'_1 \\ \vdots \\ z_n + z'_n \end{bmatrix}$$

$$z \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \equiv \begin{bmatrix} z z_1 \\ \vdots \\ z z_n \end{bmatrix}$$

2.1 Linear algebra

Notation	Description
z^*	Complex conjugate of the complex number z. $(1+i)^* = 1 - i$
$\|\psi\rangle$	Vector. Also known as a *ket*.
$\langle\psi\|$	Vector dual to $\|\psi\rangle$. Also known as a *bra*.
$\langle\varphi\|\psi\rangle$	Inner product between the vectors $\|\varphi\rangle$ and $\|\psi\rangle$.
$\|\varphi\rangle \otimes \|\psi\rangle$	Tensor product of $\|\varphi\rangle$ and $\|\psi\rangle$.
$\|\varphi\rangle\|\psi\rangle$	Abbreviated notation for tensor product of $\|\varphi\rangle$ and $\|\psi\rangle$.
A^*	Complex conjugate of the A matrix.
A^T	Transpose of the A matrix.
$A^\dagger$	Hermitian conjugate or adjoint of the A matrix, $A^\dagger = (A^T)^*$. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\dagger = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}$.
$\langle\varphi\|A\|\psi\rangle$	Inner product between $\|\varphi\rangle$ and $A\|\psi\rangle$. Equivalently, inner product between $A^\dagger\|\varphi\rangle$ and $\|\psi\rangle$.

$$|A\rangle = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ \vdots \\ A_N \end{pmatrix}$$

$$\langle A| = \begin{pmatrix} A_1^*, & A_2^*, & A_3^*, & A_4^*, & \cdots, & A_N^* \end{pmatrix}$$

2.1.1 Bases and linear independence

- A spanning set for a vector space

$$|v\rangle = \sum_i a_i |v_i\rangle$$

- Linearly dependent

$$a_1|v_1\rangle + a_2|v_2\rangle + \cdots + a_n|v_n\rangle = 0.$$

- Linearly independent

2.1.1 Bases and linear independence

- Any two sets of linearly independent vectors which span a vector space V contain the same number of elements.

- Basis for V

- Dimension of V

2.1.2 Linear operators and matrices

- Linear operator

$$A\left(\sum_i a_i|v_i\rangle\right) = \sum_i a_i A\left(|v_i\rangle\right).$$

$$A|v\rangle$$

- Identity operator I_V $\quad I_V|v\rangle \equiv |v\rangle$
- Zero operator 0 $\quad 0|v\rangle \equiv 0.$
- Composition of B with A $\quad BA|v\rangle$

(A : V -> W and B : W -> X)

2.1.2 Linear operators and matrices

- Matrices can be regarded as linear operators

- Matrix representation of the operator A

$$A|v_j\rangle = \sum_i A_{ij}|w_i\rangle.$$

Basis for V

Basis for W

- Connection between matrices and linear operators

: specify a set of input and output basis state

2.1.2 Linear operators and matrices

$$A|v_j\rangle = \sum_i A_{ij}|w_i\rangle.$$

$\mathbb{R}^2$에서 $\mathbb{R}^3$로의 linear transformation

$$T(x,y) = (2x+y, 3y, 3x+2y)$$

$\mathbb{R}^3$의 basis를 $\{(1,0,0),(1,1,0),(1,1,1)\}$로 한다면

$$T(1,0) = (2,0,3) = 2(1,0,0) - 3(1,1,0) + 3(1,1,1)$$

$$T(0,1) = (1,3,2) = -2(1,0,0) + 1(1,1,0) + 2(1,1,1)$$

$$\begin{bmatrix} 2 & -2 \\ -3 & 1 \\ 3 & 2 \end{bmatrix}$$

2.1.3 The Pauli matrices

$$\sigma_0 \equiv I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \sigma_1 \equiv \sigma_x \equiv X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_2 \equiv \sigma_y \equiv Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_3 \equiv \sigma_z \equiv Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

2.1.4 Inner products

- A function which takes as input two vectors from a vector space, and produces a complex number as output.

$$\langle v|w\rangle$$

$$\begin{bmatrix} v_1^* \dots v_n^* \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

- Inner product space

(1) $(\cdot,\cdot)$ is linear in the second argument,

$$\left(|v\rangle, \sum_i \lambda_i |w_i\rangle\right) = \sum_i \lambda_i \left(|v\rangle, |w_i\rangle\right).$$

에르미트 성질

(2) $(|v\rangle, |w\rangle) = (|w\rangle, |v\rangle)^*$.

(3) $(|v\rangle, |v\rangle) \geq 0$ with equality if and only if $|v\rangle = 0$.

2.1.4 Inner products

- Hilbert space

- Orthogonal

- Norm $\| |v\rangle \| \equiv \sqrt{\langle v|v\rangle}$

- Unit vector
- Normalized form $|v\rangle / \| |v\rangle \|$

2.1.4 Inner products

- A set |i> of vectors with index I is Orthonormal = unit vector & orthogonal

- Gran-Schmidt procedure; used to produce an orthonormal basis set

$$|v_1\rangle \equiv |w_1\rangle / \| \, |w_1\rangle \, \| \quad |v_{k+1}\rangle \equiv \frac{|w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i | w_{k+1} \rangle |v_i\rangle}{\| |w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i | w_{k+1} \rangle |v_i\rangle \|}.$$

2.1.4 Inner products

$$|\phi\rangle\langle\psi| \doteq \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} \begin{pmatrix} \psi_1^* & \psi_2^* & \cdots & \psi_N^* \end{pmatrix} = \begin{pmatrix} \phi_1\psi_1^* & \phi_1\psi_2^* & \cdots & \phi_1\psi_N^* \\ \phi_2\psi_1^* & \phi_2\psi_2^* & \cdots & \phi_2\psi_N^* \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N\psi_1^* & \phi_N\psi_2^* & \cdots & \phi_N\psi_N^* \end{pmatrix}$$

- Outer product $|w\rangle\langle v|$
- Completeness relation for orthonormal vectors $\sum_i |i\rangle\langle i| = I.$
- For representing any operator in the outer product notation.

$$\begin{aligned} A &= I_W A I_V \\ &= \sum_{ij} |w_j\rangle\langle w_j|A|v_i\rangle\langle v_i| \\ &= \sum_{ij} \langle w_j|A|v_i\rangle |w_j\rangle\langle v_i|, \end{aligned}$$

2.1.4 Inner products

- Cauchy-Schwarz inequality

$$|\langle v|w\rangle|^2 \leq \langle v|v\rangle\langle w|w\rangle$$

$$\begin{aligned}
\langle v|v\rangle\langle w|w\rangle &= \sum_i \langle v|i\rangle\langle i|v\rangle\langle w|w\rangle \\
&\geq \frac{\langle v|w\rangle\langle w|v\rangle}{\langle w|w\rangle}\langle w|w\rangle \\
&= \langle v|w\rangle\langle w|v\rangle = |\langle v|w\rangle|^2
\end{aligned}$$

2.1.5 Eigenvectors and eigenvalues

- Eigenvector & Eigenvalue

$$A|v\rangle = v|v\rangle$$

- Characteristic function

$$c(\lambda) \equiv \det|A - \lambda I|$$

- Diagonal representation & Diagonalizable

$$A = \sum_i \lambda_i |i\rangle\langle i|$$

2.1.6 Adjoints and Hermitian operators

- Adjoint (or Hermitian conjugate)

$$(|v\rangle, A|w\rangle) = (A^\dagger|v\rangle, |w\rangle).$$

- Properties

$$(AB)^\dagger = B^\dagger A^\dagger$$

$$|v\rangle^\dagger \equiv \langle v|$$

$$(A^\dagger)^\dagger = A$$

$$\left(\sum_i a_i A_i\right)^\dagger = \sum_i a_i^* A_i^\dagger$$

2.1.6 Adjoints and Hermitian operators

- The action of the Hermitian conjugation operation on matrix

$$\begin{bmatrix} 1+3i & 2i \\ 1+i & 1-4i \end{bmatrix}^{\dagger} = \begin{bmatrix} 1-3i & 1-i \\ -2i & 1+4i \end{bmatrix}.$$

2.1.6 Adjoints and Hermitian operators

- Hermitian (or self-adjoint)
- Projector

Suppose W is a k-dimensional vector subspace of the d-dimensional vector space V.
By Gram-Schmidt, exists an orthonormal basis |1> , … , |d> for V, |1> , … , |k> for W

$$P \equiv \sum_{i=1}^{k} |i\rangle\langle i| \quad := \text{Vector space p}$$

2.1.6 Adjoints and Hermitian operators

- Normal ⇔ diagonalizable := spectral decomposition

$$AA^\dagger = A^\dagger A$$

- Unitary; square matrix whose columns are orthonormal

$$U^\dagger U = I \qquad (U|v\rangle, U|w\rangle) = \langle v|U^\dagger U|w\rangle = \langle v|I|w\rangle = \langle v|w\rangle$$

-> Outer product representation of any unitary U.

Let $|u_i\rangle$ be any orthonormal basis set. Then $|w_i\rangle = U|u_i\rangle$ is also orthonormal basis set.

$$U = \sum_i |w_i\rangle\langle v_i|$$

Conversely…?

2.1.6 Adjoints and Hermitian operators

- Positive operators; for any vector $|v\rangle$, $(|v\rangle, A|v\rangle)$ is a real, non-neg.

- Any positive operator is automatically Hermitian.

(A = B+iC where B, C are Hermitian)

Note that for any positive operator A, $A = B + iC$ where B, C are Hermitian.

i) For any $|v\rangle$, $\langle v|A|v\rangle$ is real, non-negative.
ii) $B = \sum\lambda|j\rangle\langle j|$, $C = \sum\lambda|k\rangle\langle k|$

$\langle v|B + iC|\, v\rangle = \sum\lambda\langle v|j\rangle\langle j|v\rangle + \sum i\lambda\langle v|k\rangle\langle k|v\rangle$ is non negative, real.
Since $\langle v|j\rangle\langle j|v\rangle$ is non-negative, real, A = B is Hermitian.

2.1.7 Tensor products

- Putting vector spaces together to form large vector spaces

$V \otimes W$ $\quad |v\rangle|w\rangle$, $|v, w\rangle$ or even $|vw\rangle$

- Properties

(1) For an arbitrary scalar z and elements $|v\rangle$ of V and $|w\rangle$ of W,

$$z\left(|v\rangle \otimes |w\rangle\right) = \left(z|v\rangle\right) \otimes |w\rangle = |v\rangle \otimes \left(z|w\rangle\right).$$

(2) For arbitrary $|v_1\rangle$ and $|v_2\rangle$ in V and $|w\rangle$ in W,

$$\left(|v_1\rangle + |v_2\rangle\right) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle.$$

(3) For arbitrary $|v\rangle$ in V and $|w_1\rangle$ and $|w_2\rangle$ in W,

$$|v\rangle \otimes \left(|w_1\rangle + |w_2\rangle\right) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle.$$

2.1.7 Tensor products

- How to act the linear operator on Tensor product space?

- How to act inner product on Tensor product space?

2.1.7 Tensor products

- Linear operator acting on the space V ⊗ W

  Suppose |u>, |v> are vectors in V and W, and A, B are linear operator on V and W.

$$(A \otimes B)(|v\rangle \otimes |w\rangle) \equiv A|v\rangle \otimes B|w\rangle.$$

- Extends in the obvious way

  Suppose A : V -> V’ and B : W -> W’ map. Then arbitrary linear operator C mapping V ⊗ W to V’ ⊗ W’ is $C = \sum c_i A_i \otimes B_i$
  , and so $$\left(\sum_i c_i A_i \otimes B_i\right) |v\rangle \otimes |w\rangle \equiv \sum_i c_i A_i |v\rangle \otimes B_i |w\rangle.$$

2.1.7 Tensor products

- Extends to inner product on the spaces V ⊗ W

$$\left(\sum_i a_i|v_i\rangle \otimes |w_i\rangle, \sum_j b_j|v'_j\rangle \otimes |w'_j\rangle\right) \equiv \sum_{ij} a_i^* b_j \langle v_i|v'_j\rangle\langle w_i|w'_j\rangle.$$

2.1.7 Tensor products

- Matrix representation

$$A \otimes B \equiv \overbrace{\left.\begin{bmatrix} A_{11}B & A_{12}B & \dots & A_{1n}B \\ A_{21}B & A_{22}B & \dots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}B & A_{m2}B & \dots & A_{mn}B \end{bmatrix}\right\}}^{nq} mp\,.$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \times 2 \\ 1 \times 3 \\ 2 \times 2 \\ 2 \times 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 6 \end{bmatrix}$$

2.1.8 Operator functions

- F : complex number -> complex number
- Define a corresponding matrix function on normal matrices

$$A = \sum_a a|a\rangle\langle a| \qquad f(A) \equiv \sum_a f(a)|a\rangle\langle a|$$

- E.g)

$$Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \exp(\theta Z) = \begin{bmatrix} e^{\theta} & 0 \\ 0 & e^{-\theta} \end{bmatrix}$$

2.1.8 Operator functions

- Trace of a matrix

$$\mathrm{tr}(A) \equiv \sum_i A_{ii}.$$

- Properties

$$\mathrm{tr}(AB) \overset{\text{cyclic}}{=} \mathrm{tr}(BA) \qquad \mathrm{tr}(zA) \overset{\text{linear}}{=} z\,\mathrm{tr}(A)$$

- The trace of a matrix is invariant under the unitary Similarity transformation; $A \rightarrow UAU^\dagger$

$$\mathrm{tr}(UAU^\dagger) = \mathrm{tr}(U^\dagger UA) = \mathrm{tr}(A)$$

2.1.8 Operator functions

- Useful tech in evaluating the trace of an operator

Suppose |ψ> is a unit vector and A is arbitrary operator.
By using Gram-Schmidt, exists an orthonormal basis |i> including |ψ>.

$$\begin{aligned}\operatorname{tr}(A|\psi\rangle\langle\psi|) &= \sum_i \langle i|A|\psi\rangle\langle\psi|i\rangle \\ &= \langle\psi|A|\psi\rangle.\end{aligned}$$

The expectation value by
operator A in state |ψ>

2.1.9 The commutator and anti-commutator

- Commutator, commute between two operation A and B

$$[A, B] \equiv AB - BA.$$

- Anti-commutator, anti-commute

$$\{A, B\} \equiv AB + BA;$$

- Simultaneous diagonalization theorem

suppose A and B are Hermitian operators.

[A, B] = 0 if and only if there exists an orthonormal basis s.t. both A and B are diagonal with respect to that basis.

2.1.9 The commutator and anti-commutator

- Commutation relations for the Pauli matrices

$$[X, Y] = 2iZ; \quad [Y, Z] = 2iX; \quad [Z, X] = 2iY.$$

- Anti-commutation?

2.10 The polar and singular value decomposition

- Way of breaking linear operators up into simpler parts

i.e. general linear operator -> products of unitary operator and positive operators

2.10 The polar and singular value decomposition

- Polar decomposition

Let A be a linear operator on a vector space V.
Then there exists unitary U and positive operators J and K s.t.

A =UJ = KU

Where the unique positive operators J and K satisfying these equations are defined by $J \equiv \sqrt{A^\dagger A}$ and $K \equiv \sqrt{AA^\dagger}$.

- Singular value decomposition

Let A be a square matrix. Then there exist unitary matrices U and V, and a diagonal matrix D with non-negative entries s.t.

$$A = UDV$$

The diagonal elements of D are called the singular values of A.

2.10 The polar and singular value decomposition

- Proof

$$A = \underbrace{S}_{\text{unitary}}\ \underbrace{J}_{\text{positive operator}} \quad \text{by polar decomposition.}$$

Since positive operator is normal, by the spectral decomposition,

$$J = \underbrace{T}_{\text{unitary}}\ \underbrace{D}_{\text{diagonal, non-neg.}}\ \underbrace{T^{-1}}_{\text{unitary}}$$

$$\therefore A = S(TDT^{-1}) = VDV \quad \blacksquare$$

- Why do SVD?; If one of singular value is 0, then A is singular.

Thank you

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