2.2 Postulate

.1 state space

(1) Postulate 1

The system is completely described by its state vector; unit vector in the system's state space

- qubit has a two-dimensional state space
- any linear combination is a superposition of the states $|\psi>$ with amplitude ai(계수) for the state $|\psi>$

.2 Evolution

(2) Postulate 2

The evolution of a closed quantum system is described by a unitary transformation

- X matrix == NOT gate
- X, Z == bit flip and phase flip matrix
- every open system can be described as part of a larger closed system (Universe)

(2-1) Postulate 2'

The time evolution of the state of a closed quantum system is described by the Schrodinger equation

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle.$$

: h is a physical constant(Planck's constant), H = Hermitian operator(Hamiltomian)

$$H = \sum_E E|E\rangle\langle E|,$$

- E : eigenvalues == energy of the state |E>
- The state |E> == energy eigenstates(or stationary states)
- ground state energy, ground state

$$|E\rangle \rightarrow \exp(-iEt/\hbar)|E\rangle.$$

: only change in time

.3 Quantum measurement

(3) Postulate 3

Quantum measurements are described by a collection {Mm} of measurement operators

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$$

: the prob. result m occurs

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$$

: the state of the system after the measurement

.4 Distinguishing quantum states

– non-orthogonal quantum states cannot be distinguished

-> no quantum measurement capable of distinguishing the states

– 양자 상태 간 내적은 두 상태가 얼마나 유사한지를 나타냄

-> 직교할 경우 내적값은 0 I.e. 두 상태가 완전히 구별 가능

-> non-orthogonal한 경우 내적 0이 아님 I.e. 두 상태가 어느 정도 겹침

※

– 내적의 절댓값의 제곱은 한 상태에서 다른 상태로의 전이 확률을 의미

– 내적은 두 상태 벡터 사이의 각도와 관련 -> 상태 간의 유사성을 의미

– 내적이 0이면 두 상태는 서로 독립적, 유사하지 않음

.5 Projective measurements

: 관측가능량의 고윳값과 고유벡터를 기반으로 양자 상태를 특정 고유 상태로 투영하는 과정

: 측정 결과는 확률적으로 결정됨, 측정 후 상태는 해당 결과에 대응하는 고유 상태로 변화

※ Observable 은 Hermitian 연산자로 표현됨

– Hermitian 연산자의 고유값과 고유벡터는 항상 실수 -> 측정 가능한 물리적 양의 가능한 결과를 나타냄

– 허미션 연산자를 spectral decomposition

$$\hat{A} = \sum_i a_i \hat{P}_i \qquad \hat{P}_i = |\phi_i\rangle\langle\phi_i|$$

– ai는 observable의 고유값, pi는 고유값에 대응하는 고유벡터의 투영 연산자

– 양자 상태를 observable로 측정할 때, 측정 결과는 고유값 ai중 하나

- 측정 결과가 a_i가 나올 확률은 다음과 같습니다:
  $P(a_i) = \langle\psi|\hat{P}_i|\psi\rangle = |\langle\phi_i|\psi\rangle|^2$
- 측정 후 양자 상태는 해당 고유벡터로 투영됩니다:
  $|\psi'\rangle = \frac{\hat{P}_i|\psi\rangle}{\sqrt{\langle\psi|\hat{P}_i|\psi\rangle}} = \frac{|\phi_i\rangle\langle\phi_i|\psi\rangle}{\sqrt{|\langle\phi_i|\psi\rangle|^2}}$

- The average value of the observable M

$$\langle M \rangle \equiv \langle \psi | M | \psi \rangle$$

- Heisenberg's Uncertainly Principle

$$\Delta(C)\Delta(D) \geq \frac{|\langle \psi | [C, D] | \psi \rangle|}{2}$$

: 각각 C와 D의 표준편차

$$\Delta(C) = \sqrt{\langle \psi | (C - \langle C \rangle)^2 | \psi \rangle}$$

.6 POVM measurements(Positive Operator-Valued Measure)

: Project 측정보다 더 유연하고, 현실적 측정 장치의 한계를 더 잘 반영

(1) 수학적 정의

1. **양의 준위 연산자**:
   $E_i \geq 0 \quad \text{for all } i$
   이는 모든 E_i의 고유값이 0 이상임을 의미합니다.

2. **완전성**:
   $\sum_i E_i = I$
   여기서 I는 단위 연산자입니다. 이는 측정 결과가 항상 어떤 하나의 결과를 산출함을 의미합니다.

.7 Phase

(1) global phase factor $e^{i\theta}$

: 파동함수에 일정하게 추가되는 위상 요소. 물리적 관측값이나 측정 결과에 영향을 미치지 않음

: statistics of measurement 관점에서, global phase factor를 곱한 상태와 그렇지 않은 상태는 정확히 일치함

$$\langle \psi | M_m^\dagger M_m | \psi \rangle \text{ and } \langle \psi | e^{-i\theta} M_m^\dagger M_m e^{i\theta} | \psi \rangle = \langle \psi | M_m^\dagger M_m | \psi \rangle.$$

.8 Composite systems

(4) Postulate 4

: The state space of a composite physical system is the tensor product of the state spaces of the component physical systems.

$$|\psi_1\rangle = \alpha|0\rangle + \beta|1\rangle$$
$$|\psi_2\rangle = \gamma|0\rangle + \delta|1\rangle$$

이 두 큐비트로 이루어진 복합 시스템의 상태는 다음과 같이 텐서곱으로 표현됩니다:

$$|\psi_{\text{total}}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$
$$= (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$$

이를 전개하면:

$$|\psi_{\text{total}}\rangle = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$$

– Entangled state (얽힘 상태)

두 개의 양자 시스템 A와 B가 있을 때, 이들의 전체 상태를 $|\psi_{AB}\rangle$로 나타냅니다. 만약 이 상태가 다음과 같이 각각의 시스템의 상태 $|\psi_A\rangle$와 $|\psi_B\rangle$의 텐서곱으로 표현될 수 있다면:

$$|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$$

이 상태는 얽혀 있지 않은 상태, 즉 분리 가능한 상태(separable state)입니다.

하지만 얽힘 상태는 이런 형태로 분리할 수 없습니다. 즉, 전체 시스템의 상태가 두 시스템의 독립적인 상태의 곱으로 표현되지 않는 경우입니다. 예를 들어, 다음과 같은 상태는 얽힘 상태입니다:

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

1. **비국소성 (Non-locality)**:
   얽힘은 비국소성을 특징으로 합니다. 얽힌 입자들이 서로 멀리 떨어져 있어도, 한 입자의 상태를 측정하면 다른 입자의 상태가 즉시 결정됩니다. 이는 아인슈타인이 "유령같은 원격 작용 (spooky action at a distance)"이라 부른 현상입니다.

2. **양자 중첩 (Quantum Superposition)**:
   얽힌 상태는 여러 상태의 중첩으로 표현됩니다. 예를 들어, $|\Phi^+\rangle$ 상태는 $|00\rangle$와 $|11\rangle$의 중첩입니다.

3. **양자 상관관계 (Quantum Correlation)**:
   얽힌 입자들은 강한 상관관계를 가집니다. 한 입자의 측정 결과가 다른 입자의 결과와 상관관계를 가집니다.

2.4 The density operator
: 혼합상태(ensemble of pure states)와 순수상태를 모두 기술할 수 있는 수학적 도구

$$\rho \equiv \sum_i p_i |\psi_i\rangle\langle\psi_i|.$$

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \xrightarrow{U} \sum_i p_i U|\psi_i\rangle\langle\psi_i|U^\dagger = U\rho U^\dagger$$

1. **추적(Trace)**: 밀도 연산자의 추적은 항상 1입니다.
   $\mathrm{Tr}(\rho) = 1$
2. **양의 정부호(Positive Semi-definiteness)**: 모든 상태 $|\phi\rangle$에 대해,
   $\langle\phi|\rho|\phi\rangle \geq 0$
3. **에르미트 연산자(Hermitian Operator)**: 밀도 연산자는 에르미트 행렬입니다.
   $\rho^\dagger = \rho$

혼합 상태의 예

혼합 상태는 여러 순수 상태의 확률적 혼합으로 표현됩니다. 예를 들어, 시스템이 상태 $|\psi_1\rangle$에 확률 p_1, 상태 $|\psi_2\rangle$에 확률 p_2로 있을 때, 밀도 연산자는 다음과 같이 표현됩니다:
$\rho = p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2|$

측정 결과의 기대값

양자 상태 ρ에서 측정 연산자 O의 기대값은 다음과 같이 계산됩니다:
$\langle O\rangle = \mathrm{Tr}(\rho O)$

(1) 혼합 상태에서의 밀도 연산자 표현

- 상태 $|\psi_1\rangle$: $|0\rangle$ (기저 상태)
- 상태 $|\psi_2\rangle$: $|1\rangle$ (여기 상태)
- 확률 p_1: 0.6
- 확률 p_2: 0.4

이 혼합 상태의 밀도 연산자는 다음과 같이 계산됩니다:

$$\rho = p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2|$$

여기서 $|\psi_1\rangle = |0\rangle$이고 $|\psi_2\rangle = |1\rangle$이므로:

$$\rho = 0.6|0\rangle\langle 0| + 0.4|1\rangle\langle 1|$$

밀도 연산자를 행렬 형태로 표현하면 다음과 같습니다:

$$\rho = 0.6\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0.4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.4 \end{pmatrix}$$

(2) 얽힘상태에서의 밀도연산자

$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

이 상태의 밀도 연산자는 다음과 같습니다:

$\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$

이를 계산하면:

$|\psi_{AB}\rangle\langle\psi_{AB}| = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$

$\rho_{AB} = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$

행렬 형태로 표현하면:

$$\rho_{AB} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \sum_i \langle i_B|\rho_{AB}|i_B\rangle$$

부분 추적 결과는 다음과 같습니다:

$$\rho_A = \frac{1}{2}\begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0+1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$$

이 결과는 시스템 A가 최대 혼합 상태(maximally mixed state)에 있음을 나타냅니다.

※ 부분 추적 하는 방법

먼저 $|0\rangle_B$를 사용하여 추적을 계산해 봅시다:

$$\langle 0_B|\rho_{AB}|0_B\rangle = \frac{1}{2}\begin{pmatrix} \langle 0_B|00\rangle\langle 00|0_B\rangle & \langle 0_B|00\rangle\langle 01|0_B\rangle \\ \langle 0_B|10\rangle\langle 00|0_B\rangle & \langle 0_B|10\rangle\langle 01|0_B\rangle \end{pmatrix}$$

이는 다음과 같습니다:

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

다음으로 $|1\rangle_B$를 사용하여 추적을 계산해 봅시다:

$$\langle 1_B|\rho_{AB}|1_B\rangle = \frac{1}{2}\begin{pmatrix} \langle 1_B|10\rangle\langle 10|1_B\rangle & \langle 1_B|10\rangle\langle 11|1_B\rangle \\ \langle 1_B|11\rangle\langle 10|1_B\rangle & \langle 1_B|11\rangle\langle 11|1_B\rangle \end{pmatrix}$$

이는 다음과 같습니다:

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(1) Z-Y decomposition

$$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$$

- $R_z(\theta)$는 Z축을 기준으로 θ만큼 회전하는 연산입니다:

$$R_z(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

- $R_y(\theta)$는 Y축을 기준으로 θ만큼 회전하는 연산입니다:

$$R_y(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$

(2) X-Y decomposition

$$U = e^{i\alpha} R_x(\beta) R_y(\gamma) R_x(\delta)$$

- $R_x(\theta)$: X축을 기준으로 θ만큼 회전하는 연산입니다.

$$R_x(\theta) = \begin{pmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$

- $R_y(\theta)$: Y축을 기준으로 θ만큼 회전하는 연산입니다.

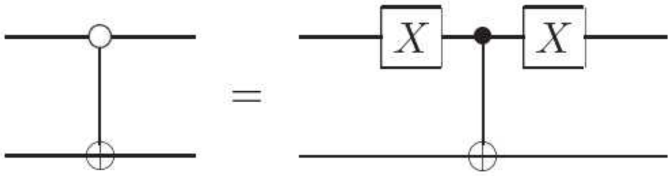
$$R_y(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$

(3) Corollary. Unitary gate on a single qubit

Corollary 4.2: Suppose U is a unitary gate on a single qubit. Then there exist unitary operators A, B, C on a single qubit such that $ABC = I$ and $U = e^{i\alpha} AXBXC$, where α is some overall phase factor.

(4) Circuit Identitiy
a) HH = I
b) XX = YY = ZZ = I
c) CNOT*CNOT = I
d) SS = Z
e) TT = S
f) TTTT = I
g) H = (X + Z) / root(2)

(5) Controlled operation with a NOT gate
(6) measurement : irreversible operation -> destroying quantum information and replacing it with classical information



(7) Observable(관측 가능한 양자 연산자)
: 고윳값이 실수 -> 물리적 관측 가능한 값이 실수여야 하기 때문 -> 자동적으로 허미션 행렬
: 허미션 행렬
-> 하고싶다 : obtain a measurement result indicating one of the two eigenvalues, leaving a post-measurement state which is the corresponding eigenvector

.5 universal quantum gate
: 모든 양자 연산을 구현하는 데 필요한 기본적인 빌딩 블록
(1) 단일 큐비트 게이트 : H, X 등
(2) 다중 큐비트 게이트 : CNOT 게이트

: H게이트와 CNOT 게이트의 조합은 universal set 형성 가능 -> 모든 양자 연산 구현 가능
: 임의의 unitary matrix on d dimensional Hilbert space 는 two-level unitary matrices의 곱으로 표현 가능함
=> single qubit과 CNOT gate 함께 사용 시 임의의 two-level unitary operation 대체 가능
: 일반적으로 단일 큐비트 게이트 + CNOT 게이트 조합으로 형성 가능

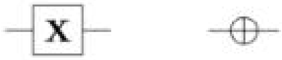
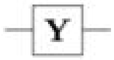
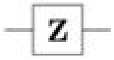
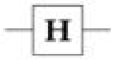
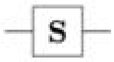
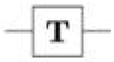
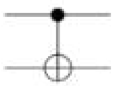
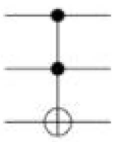
.6
(1) Approximating unitary operators
: 일반적으로, unitary operations의 집합은 연속적이기 때문에, discrete set of gates는 임의의 unitary operation을 대체하기 위해 사용 x
-> any unitary operation을 approximate하는데 사용됨!

$$E(U, V) \equiv \max_{|\psi\rangle} \|(U - V)|\psi\rangle\|,$$

:= error, U는 implement 하고자 하는 unitary op, V는 사전에 이미 implement된 unitary op

(2) Universality of H + phase + CNOT + S gate

※ 참고

Operator	Gate(s)	Matrix
Pauli-X (X)		$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Pauli-Y (Y)		$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
Pauli-Z (Z)		$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Hadamard (H)		$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
Phase (S, P)		$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
$\pi/8$ (T)		$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$
Controlled Not (CNOT, CX)		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
Controlled Z (CZ)		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
SWAP		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
Toffoli (CCNOT, CCX, TOFF)		$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

※ 참고

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}; \quad T = \begin{bmatrix} 1 & 0 \\ 0 & \exp(i\pi/4) \end{bmatrix}$$

The phase shift gates are related to each other as follows:

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = P(\pi)$$

$$S = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = P\left(\frac{\pi}{2}\right) = \sqrt{Z}$$

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix} = P\left(\frac{\pi}{4}\right) = \sqrt{S} = \sqrt[4]{Z}$$

$$|0\rangle \mapsto |0\rangle \text{ and } |1\rangle \mapsto e^{i\varphi}|1\rangle$$

※ 오일러 관계

- $e^{ix} = \cos x + i \sin x$

● Postulate 1.

: the state of the particle is represented by a vector $|\psi>$ in a Hilbert space

: state space sholud be found by experiment

● Postulate 2.

: the evolution of a “closed” quantum system is described by a unitary transformation

$$|\psi\rangle\ at\ t_1 \xrightarrow{\text{unitary transformation}} |\psi'\rangle\ at\ t_2$$

● Postulate 2’; continuous ver.

: the time evolution of the state of a “closed” quantum system is described by Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \mathcal{H}|\psi\rangle$$

- $\hbar$ is Planck's constant, 1.054 1.054×10^{-34} $(J \cdot s)$
- $\mathcal{H}$ is called *Hamiltonian*. Hamiltonian describes how the system should evolve.

- $\frac{d}{dt}|\psi\rangle = -i\frac{\mathcal{H}}{\hbar}|\psi\rangle \rightarrow |\psi(t)\rangle = e^{-i\mathcal{H}(t-t_0)/\hbar}|\psi(t_0)\rangle$
- $U(t-t_0) = e^{-i\mathcal{H}(t-t_0)/\hbar}$ is an unitary operator

● Postulate 3; Copenhagen interpretation.

: If the particle is in a state $|\psi>$, measurement of the variable Ω will yield one of the eigenvalues ω_i with probability $p(\omega_I) = |<\omega_I|\psi>|^2$

: [1]the state of the system will change from $|\psi>$ to $|\omega_i>$ as a result of measurement

: we can constructed the measurement operator

<=> generally, measurement operators are guessed from the corresponding classical models.

: $<\psi|\Omega|\psi>$ = expectation value of the measurement.

※ 정리

- Assumption
  - The particle is in a state $|\psi\rangle$ (normalized vector)
  - We want to measure the variable corresponding to Ω (Hermitian operator)
- Measured value: one of the eigenvalues ω_i (real)
- Probability of measuring ω_i: $|\langle\omega_i|\psi\rangle|^2$
- The state right after the measurement: $|\omega_i\rangle$ (collapse)
- Expected value of measurement: $\langle\psi|\Omega|\psi\rangle$

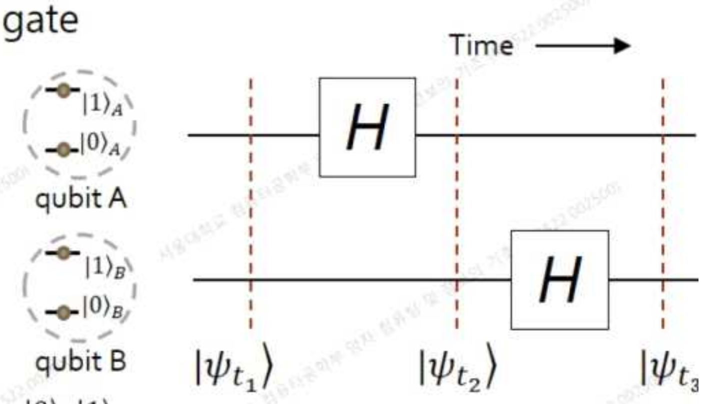
---

1) called “collapse”

● Ex) tensor product on quantum qubits

- Example: Hadamard gate

Input	Output
$\|0\rangle$	$(\|0\rangle + \|1\rangle)/\sqrt{2}$
$\|1\rangle$	$(\|0\rangle - \|1\rangle)/\sqrt{2}$

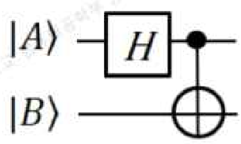


  - Assume that at t_1, $|\psi_{t_1}\rangle = |0\rangle_A|1\rangle_B$.
  - At t_2, only qubit A should be changed by Hadamard gate. How can we write such kind of situation in equation?
  - $|\psi_{t_2}\rangle = (H_A \otimes I_B)(|0\rangle_A \otimes |1\rangle_B) = (H_A|0\rangle_A) \otimes (I_B|1\rangle_B) = \frac{|0\rangle_A+|1\rangle_A}{\sqrt{2}} \otimes |1\rangle_B$
  - $|\psi_{t_3}\rangle = (I_A \otimes H_B)|\psi_{t_2}\rangle = \left(I_A \frac{|0\rangle_A+|1\rangle_A}{\sqrt{2}}\right) \otimes (H_B|1\rangle_B) = \frac{|0\rangle_A+|1\rangle_A}{\sqrt{2}} \otimes \frac{|0\rangle_B-|1\rangle_B}{\sqrt{2}}$
  - Of course, this operation can be written as $H_A \otimes H_B$:

    $|\psi_{t_3}\rangle = (H_A \otimes H_B)(|0\rangle_A \otimes |1\rangle_B) = (H|0\rangle_A) \otimes (H|1\rangle_B) = \frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}} \otimes \frac{|0\rangle_B - |1\rangle_B}{\sqrt{2}}$

● Ex) Bell basis

- How to create Bell state?
  - Entangling circuit



Input		Output
$\|A\rangle$	$\|B\rangle$	
$\|0\rangle$	$\|0\rangle$	$\|\phi^+\rangle = (\|00\rangle + \|11\rangle)/\sqrt{2}$
$\|0\rangle$	$\|1\rangle$	$\|\psi^+\rangle = (\|01\rangle + \|10\rangle)/\sqrt{2}$
$\|1\rangle$	$\|0\rangle$	$\|\phi^-\rangle = (\|00\rangle - \|11\rangle)/\sqrt{2}$
$\|1\rangle$	$\|1\rangle$	$\|\psi^-\rangle = (\|01\rangle - \|10\rangle)/\sqrt{2}$

● How the open system will evolve? (closed 아니라서 postulate로 못 알아냄)
: Density matrix 로 알아냄~

- The density operator (or density matrix) is defined as

$$\rho \equiv \sum_i p_i |\psi_i\rangle\langle\psi_i|.$$

- Example
  - 90% of $|0\rangle$ and 10% of $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$
  - $\rho = \frac{9}{10}|0\rangle\langle 0| + \frac{1}{10}|+\rangle\langle +| = \frac{19|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|}{20} = \begin{bmatrix} \frac{19}{20} & \frac{1}{20} \\ \frac{1}{20} & \frac{1}{20} \end{bmatrix}$

- Postulate 3
  - Example: 90% of $|0\rangle$ and 10% of $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, $M_0 = |0\rangle\langle 0|$ and $M_1 = |1\rangle\langle 1|$. What is the probability of measuring 0?
    - $p(0|0) = \langle 0|M_0^\dagger M_0|0\rangle = 1$, $p(0|+) = \langle +|M_0^\dagger M_0|+\rangle = \frac{1}{2}$.
    - Total probability is $0.9 \times 1 + 0.1 \times \frac{1}{2} = 0.95$
  - If initial state was $|\psi_i\rangle$, the probability of getting result m is $p(m|i) = \langle \psi_i|M_m^\dagger M_m|\psi_i\rangle = \mathrm{tr}(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|)$
    - Note $\langle\beta|\alpha\rangle = \langle\beta|(\sum_{j=1}^{n}|j\rangle\langle j|)|\alpha\rangle = \sum_{j=1}^{n}\langle j|\alpha\rangle\langle\beta|j\rangle = \mathrm{tr}(|\alpha\rangle\langle\beta|)$

$$p(m) = \sum_i p_i \cdot p(m|i) = \sum_i p_i \mathrm{tr}(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|) = \mathrm{tr}(M_m^\dagger M_m \rho)$$

  - Example: from previous page, $\rho = \begin{bmatrix} \frac{19}{20} & \frac{1}{20} \\ \frac{1}{20} & \frac{1}{20} \end{bmatrix}$.

$$M_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow p(m) = \mathrm{tr}(M_m^\dagger M_m \rho) = \mathrm{tr}\left(\begin{bmatrix} \frac{19}{20} & 0 \\ 0 & 0 \end{bmatrix}\right) = 0.95$$

● Reformulation of Postulate 1

: The system is completely described by its density operator, which is a positive operator p with trace one, acting on the state space of the system.

● Reformulation of Postulate 2

- The evolution of a *closed* quantum system is described by a *unitary transformation*. That is, the state ρ of the system at time t_1 is related to the state ρ' of the system at time t_2 by a unitary operator U which depends only on the times t_1 and t_2,

$$\rho' = U\rho U^\dagger$$

● Reformulation of Postulate 3

- Quantum measurements are described by a collection $\{M_m\}$ of *measurement operators*. These are operators acting on the state space of the system being measured. The index m refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is ρ immediately before the measurement then the probability that result m occurs is given by

$$p(m) = tr(M_m^\dagger M_m \rho)$$

- The state of the system after the measurement is

$$\frac{M_m \rho M_m^\dagger}{\mathrm{tr}(M_m^\dagger M_m \rho)}$$

- The measurement operators satisfy the *completeness equation*,

$$\sum_m M_m^\dagger M_m = I$$

● Postulate 4

- The state space of a composite physical system is the tensor product of the state spaces of component physical systems. Moreover, if we have systems numbered 1 through n, and system number i is prepared in the state ρ_i, then the joint state of the total system is $\rho_1 \otimes \rho_2 \otimes \cdots \rho_n$.

● Reduced density matirx

- Suppose we have physical systems A and B, whose state is described by a density operator ρ^{AB}.
  - Example: $|0_A\rangle|0_B\rangle$ with probability of ¾, $|0_A\rangle|1_B\rangle$ with probability of ¼.
  - $\rho^{AB} = \frac{3}{4}|0_A\rangle\langle 0_A| \otimes |0_B\rangle\langle 0_B| + \frac{1}{4}|0_A\rangle\langle 0_A| \otimes |1_B\rangle\langle 1_B|$
- Reduced density operator for system A is defined by

$$\rho^A \equiv \mathrm{tr}_B(\rho^{AB})$$

where tr_B is a map of operators known as the *partial trace* over system B.

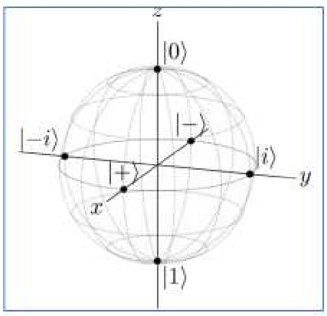
  - Example (not mathematically sound, but gives intuition):

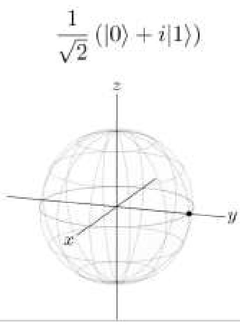
$$\rho^A = \mathrm{tr}_B(\rho^{AB}) = \sum_{j=1}^{2} \langle j_B|\rho^{AB}|j_B\rangle$$
$$= \langle 0_B|\rho^{AB}|0_B\rangle + \langle 1_B|\rho^{AB}|1_B\rangle = \frac{3}{4}|0_A\rangle\langle 0_A| + \frac{1}{4}|0_A\rangle\langle 0_A| = |0_A\rangle\langle 0_A|$$

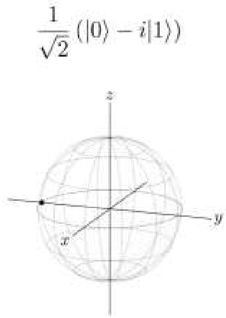
- Formally, the partial trace is defined by

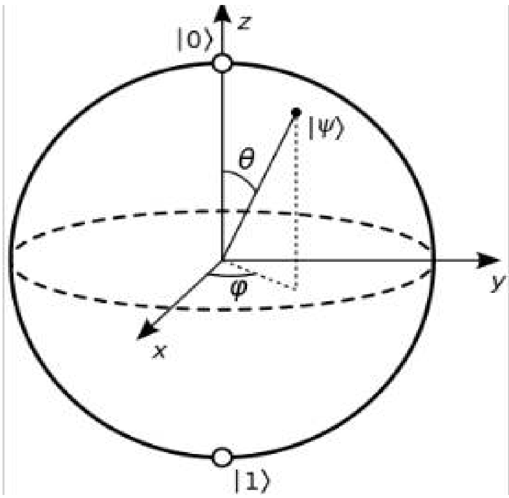
$$\mathrm{tr}_B\left(|a_{1A}\rangle\langle a_{2A}| \otimes |b_{1B}\rangle\langle b_{2B}|\right) \equiv |a_{1A}\rangle\langle a_{2A}| \otimes \mathrm{tr}\left(|b_{1B}\rangle\langle b_{2B}|\right)$$
$$= |a_{1A}\rangle\langle a_{2A}| \left(\langle b_{2B}|b_{1B}\rangle\right)$$

and it should be linear in its input.









: Given an orthonormal basis, any pure state $|\psi>$ of a two−level quantum system can be written as a superposition of the basis $|0>$ and $|1>$.

$$|\psi\rangle = \cos(\theta/2)|0\rangle \ + e^{i\phi}\sin(\theta/2)|1\rangle = \cos(\theta/2)|0\rangle \ + (\cos\phi + i\sin\phi)\ \sin(\theta/2)|1\rangle,$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$.

스핀 1/2 시스템에서는 각운동량 연산자가 파울리 행렬로 표현되며, 회전 연산자는 다음과 같이 표현됩니다.

$$U(\theta, \hat{n}) = e^{-i\theta(\hat{n}_x\sigma_x + \hat{n}_y\sigma_y + \hat{n}_z\sigma_z)/2}$$

여기서 σ_x, σ_y, σ_z는 파울리 행렬이며, $\hat{n} = (\hat{n}_x, \hat{n}_y, \hat{n}_z)$는 회전 축의 단위 벡터입니다.

※ 임의의 회전 연산자는 다음과 같다.(임의의 축을 기준으로 θ 각)

$U = \cos(\theta/2)I - isin(\theta/2)\sigma_{x,y,z}$

$$
R_z(\theta) = e^{i\frac{\theta}{2}\sigma_z} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{\left(\frac{i\theta}{2}\right)^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{\left(\frac{-i\theta}{2}\right)^n}{n!} \end{pmatrix} = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}
$$

$$
R_y(\theta) = \begin{pmatrix} \sum_{n=0,2,4\ldots}^{\infty} \frac{\left(\frac{i\theta}{2}\right)^n}{n!} & -i \sum_{n=1,3,5\ldots}^{\infty} \frac{\left(\frac{i\theta}{2}\right)^n}{n!} \\ i \sum_{n=1,3,5\ldots}^{\infty} \frac{\left(\frac{i\theta}{2}\right)^n}{n!} & \sum_{n=0,2,4\ldots}^{\infty} \frac{\left(\frac{i\theta}{2}\right)^n}{n!} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}
$$

$$
R_x(\theta) = \begin{pmatrix} \sum_{n=0,2,4\ldots}^{\infty} \frac{\left(\frac{i\theta}{2}\right)^n}{n!} & \sum_{n=1,3,5\ldots}^{\infty} \frac{\left(\frac{i\theta}{2}\right)^n}{n!} \\ \sum_{n=1,3,5\ldots}^{\infty} \frac{\left(\frac{i\theta}{2}\right)^n}{n!} & \sum_{n=0,2,4\ldots}^{\infty} \frac{\left(\frac{i\theta}{2}\right)^n}{n!} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}
$$

※ 테일러 급수 참고

$$
e^{i\frac{\theta}{2}\sigma_j} = \sum_{n=0}^{\infty} \frac{\left(\frac{i\theta}{2}\right)^n}{n!} \sigma_j^n
$$

- A general quantum state is characterized by its density operator which is a positive operator w/ trace = 1 on H.
- The density of a pure state |ψ> is the projector |ψ><ψ|.
- Any set of orthogonal product pure states can be perfectly distinguishable by some PPT(positive-transpose-preserving) operation.
- if not, no way to discriminate each other.