Testing for mean reversion based on Gibbs sampling

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Abstract

We employ a Bayesian approach to test for mean reversion in the American stock market on S&P 500 daily data 2004-2020, which manifests the heteroscedasticity in its volatility. We examine the data with a two-regimes Markov Switching model with mixture Gaussian, as we capture the data at tranquil and volatile regimes These jumps between states are determined by the latent two-state indicator with unknown transition probabilities. A Bayesian framework is designed to draw from the posterior of the unobserved states and model parameters via Gibbs sampler. The approach is straightforward and generates posterior Variance Ratio(VR) statistics of standardized observation and randomized realizations for the hypothesis test. We characterize two obvious volatility regimes in the data and find the support of mean reversion by virtue of the VR test.

Keywords: Mean Reversion; Markov Switching Model; Hidden Markov Model; Gibbs Sampler; Variance Ratio Test

1 Introduction

Mean reversion, as an indicator of predictability, has drawn lots of attention and has been identified to be conspicuous in financial markets. Mean reversion in finance suggests that asset prices or market factors will eventually revert to a long-run equilibrium mean or average level. If validated, the theory can be employed in trading strategies. For example, statistical arbitrage utilizes the concept to invest in broadly diversified portfolios over short periods.

This market behaviour has been recognized by researchers and investors for a long period of time. It has been found in assets other than stocks, including commodities [Lubnau and Todorova, 2015, Schwartz [1997]] and foreign exchange rates [Larsen and Sørensen, 2007]. In the industry, fund managers and individual investors often take pairs trading, a market-neutral strategy, to "arbitrage" profit virtually under any market conditions. Many contributors have taken minor details into consideration, such as transaction costs [Leung and LI, 2015] and multiple assets [d'Aspremont, 2007]. However, from our previous work experiences in systematic options trading, we noticed even a simple setting of a Z-Score Entry/Exit thresholds would be an effective trading signal. Besides, mean-reverting in the implied volatility could also be advantageous. Traders can make an inference from implied volatilities and make strategic decisions of taking long or short positions. For example, when the implied volatility is at its lower tail, an options trader might look to take advantage of buying calendar or diagonal spreads.

On the statisticians' side, much financial time series often present heteroskedastic in both the second moments, which cannot be catched by the homoscedastic data generating process. To assess this kind of data, one possible solution is to embrace the mean-reverting character into the model set-up as a parameter. For example, the Heston model[1], a stochastic volatility model, does not only allow heteroscedasticity but also convey mean-reverting volatility. A Bayesian framework can be adapted to make parameter inference on the posterior sampling distribution. This method can be implemented properly to study the mean reversion in implied volatility mentioned above.

Alternatively, it can also be deduced from the Variance Ratio(VR) test, which was originally introduced to test the random walk hypothesis. Poterba and Summers [1988] has explored its potential for testing mean reversion in stock data and identified that a considerable fraction of variance can be explained by a transitory component. However, they found no meaningful difference in comparing results from homoscedastic and heteroscedastic data, which conflicts with the transitory component. Albert and Chib [1993] took advantage of the Bayesian Gibbs sampling approach to study the autoregressive time series with regime switching, where the transitory component is explicitly defined by the regimes of Markov Chain. The strategy has been further embodied by Kim et al. [1998] and it allowed the Gibbs sampler to make appropriate use of the information contained in data. Both revealed statistically satisfying results in simulation and empirical studies.

The plan of the paper is as follows. Section 2 provides a brief description of the Markov Switching model and streamlines how Gibbs sampler can be applied to derive the model parameters and calculate the p-value of the VR test. Section 3 eyeballs the S&P 500(SPX) daily data first and presents the empirical results. Section 4 concludes

[1]We proposed to analyze based on stochastic volatility model as well, but due to size limitation to this project, we will be focused on the Markov Switching model in this report, a brief methodology for the Heston model can be found in Appendix.

remarks and directions for future research. The appendix includes the Heston model, step-wise proofs of Bayesian analysis and the brief inference correctness check as long as the main R code outside the TMS package.

2 Methodology

We will define the two-state Markov Switching model firstly, and streamline how to implement the Gibbs sampler. In the last subsection, it will emphasize the reasoning and methodology of VR test.

2.1 Markov Switching Model

In this paper, we've chosen a Markov Switching model with two regimes. It can be embodied as a K-state Hidden Markov Model(HMM) with observation y_t follows a mixture Gaussian.

$$y_t \sim N(0,\ \sigma_t^2),\ \text{where}\ \ \sigma_t^2 = \sum_{k=1}^{2} \sigma_k^2\ \mathbb{I}\{S_t = k\}$$

Where the parameters are all unobserved and to be estimated:

- $\sigma_1 < \sigma_2$
- The state variable, S_t, follows a Markov process with transition probability $p_{ij} = \Pr[S_t = j | S_{t-1} = i], \sum_{j=1}^{K} p_{ij} = 1,\ i, j = 1, 2.$

2.2 Gibbs Sampler

The Gibbs sampling method will be implemented to estimation. Let $\vec{y_T} = \{y_t, t = 1, \ldots, T\}$, $\vec{S_T} = \{S_t, t = 1, \ldots, T\}$, $\theta = \{\sigma_1^2, \sigma_2^2, \vec{p}\}$, $\vec{p} = (p_{12}, p_{21})^T$. Besides, let θ_{-i} denote the value of θ without the i^{th} element.

The paramters $\vec{S_t}, \theta$ is to be updated by Gibbs sampler in the following steps.

- Step 1 Draw $\vec{S_T} \mid \theta$;
- Step 2 Draw $\sigma_1^2 \mid \vec{S_T}$;
- Step 3 Draw $\sigma_2^2 \mid \sigma_1^2, \vec{S_T}$;
- Step 4 Draw $\vec{p} \mid \vec{S_T}$

We took burn-in samples of 1,000 and samples of 10,000 in emprical study.

2.2.1 Step 1 Hidden State Values Simulation - Viterbi Algorithm

As indicated by the name of HMM, the state values are latent. We will apply the Viterbi algorithm to find the most likely state values given observations y_t.

Mathematically, the Viterbi algorithm is an optimization problem over posterior estimation.

$$\vec{S_T} = \arg\max_{\vec{S_T}} \Pr(\vec{S_T} \mid \vec{y_T})$$

Let $\delta_t(k)$ be the probability of arriving at state k at time t assuming that the most likely path was taken.

$$\delta_t(k) \triangleq \max_{\vec{S_{t-1}}} \Pr(\vec{S_{t-1}}, S_t = k \mid \vec{y_T})$$

Intuitively, $\delta_t(k)$ defines the most probable path to state j at step t by taking the most probable path to state i at t-1, followed by a transition from i to k.

Let $a_t(j)$ denote the most likely previous state on the most likely path to j at step t:

$$a_t(j) = \max_i \delta_{t-1}(i) \cdot p_{ik} \cdot \psi_t(k)$$

Where $\psi_t(k) = \Pr(y_t \mid S_t = k)$ is the local evidence at step t.

In Gibbs sampler, it will initialize $\delta_1(k) = f_y(y_1 \mid S_1 = k)$ and terminating with the most probable final state $S_T^* = \arg\max_i \delta_T(i)$, the most probable sequence of states is estimated backwards.

$$S_t^* \mid (S_{t+1}^* = j) = a_{t+1}(j)$$

Finally, we'll get $(S_1^*, \ldots, S_T^*)$ estimation to be used in later steps of the Gibbs Sampler.

2.2.2 Step 2: Bayesian Generator for σ_1^2 for each state

To generate σ^2's such that $\sigma_1^2 < \sigma_2^2$, we proposed a scalar parameter $\gamma > 0$ and define $\sigma_2^2 = (1+\gamma)\sigma_2^2$. Note that the subscript of σ^2 here represent the state 1 and 2 respectively, rather than the time.

The Bayesian framework[2] can be documented as follows.

$$\sigma_1^2 \sim IG(\frac{1}{2}, \frac{1}{2}) \qquad \text{(Prior)}$$

$$Y_{1t} = \frac{y_t}{\sqrt{(1 + \mathbb{I}\{S_t = 2\}\gamma)}} \sim N(0, \sigma_1^2)$$

$$\sigma_1^2 \mid \{Y_{1t}\}_{t=1,\ldots,T} \sim IG(\frac{1+T}{2}, \frac{1 + \sum_{t=1}^{T} Y_{1t}^2}{2}) \qquad \text{(Posterior)}$$

[2]Proof will be drafted in the Appendix

2.2.3 Step 3: Bayesian Generator for σ_2^2 for each state

Let $\bar{\gamma} = 1 + \gamma$, $\sigma_2^2 = (1+\gamma)\sigma_1^2 = \bar{\gamma}\sigma_1^2$.

$N_2 = \{t : S_t = 2, t = 1, \ldots, T\}, T_2 =$ sum of elements in N_2

The Bayesian framework[3] can be documented as follows.

$$\bar{\gamma} \sim IG(\frac{1}{2}, \frac{1}{2}) \cdot \mathbb{I}\{\bar{\gamma} > 1\} \qquad \text{(Prior)}$$

$$Y_{2t} = \frac{y_t}{\sqrt{\sigma_1^2}}, t \in T_2 \sim N(0, \bar{\gamma})$$

$$\bar{\gamma} \mid \{Y_{2t}\}_{t \in T_2}, \theta_{-2} \sim IG(\frac{1+T_2}{2}, \frac{1+\sum_{t=1}^{N_2} Y_{2t}^2}{2}) \cdot \mathbb{I}\{\bar{\gamma} > 1\} \quad \text{(Posterior)}$$

Then, at each iteration, $\sigma_2^2 = \sigma_1^2 \cdot \bar{\gamma}$ with updated σ_1^2 and $\bar{\gamma}$ at current iteration.

2.2.4 Step 4 Draw $p \mid \theta_{-3}, \vec{S_T}$

Given the data $\vec{S_T}$, we can get the number of transitions from state i to state j, i.e. $n_{ij}, i, j = 1, 2$. Then the likelihood function conditioned on the initial state is given by

$$L(p) = \prod_{\forall i,j} p_{ij}^{n_{ij}}$$

From the likelihood, it's clear that the Beta family of distributions is a conjugate prior for each of the transition probabilities. We choose the prior $p_{12}, p_{21} \overset{iid}{\sim} Beta(1,1)$, equivalently $U(0,1)$. The Bayesian framework[4] can be documented as follows.

$$\begin{aligned} \pi(p_{12}, p_{21}) &\propto Beta(1,1) \cdot Beta(1,1) && \text{(Prior)} \\ p_{12} \mid \vec{S_T} &\sim Beta(1 + n_{12}, 1 + n_{11}) && \text{(Posterior)} \\ p_{21} \mid \vec{S_T} &\sim Beta(1 + n_{21}, 1 + n_{22}) && \text{(Posterior)} \\ p_{12} \mid \vec{S_T} &\perp p_{21} \mid \vec{S_T} \end{aligned}$$

2.3 Variance Ratio Test

At each Gibbs run, one more step to act is the VR calculation. The objective is to simulate from the posterior distribution of the variance of mixture Gaussian distribution given its observed values. Based on the observed series and simulated variance series, VR statistics can be obtained. After enough iterations, we could simulate the posterior sampling distribution of VR values to make inference in terms of mean reversion. However, as this step is not considered to be parameter estimation, but it will proceed at each Gibbs sweep following Step 1 to 4. Define and calculate q period standardized return $r_{q,t} = \tilde{y_t} - \tilde{y_{Q,t}}$.

[3]Proof will be drafted in the Appendix

[4]Proof will be drafted in the Appendix

Then, the Variance Ratio(VR) is defined as following.

$$VR(q) = \frac{Var[r_{Q,t}]}{Q \cdot Var[r_{1,t}]}$$

The reasoning behind the test of random walk is: If the observation series follows a random walk, then $r_{1,t}$ is serially random and the Q-period return,$r_{Q,t}$, is simply the accumulation of Q successive $r_{1,t}$. As a result, the VR will be unity for all periods Q.

The two VR statistics of interest and to convey the VR test is given by:

1. Standardized observations: $\{y_t^* = y_t/\sigma_t, t = 1, \ldots, T\}$. The σ_t is from the last iteration of Gibbs. Define and calculate Q period standardized return $r_{Q,t} = \tilde{y_t} - \tilde{y_{Q,t}}$. The VR based on $r_{Q,t}$ is denoted by $VR(Q)$

2. Randomized returns: Generate the randomize the return from $z_t \sim N(0, 1)$, and get the randomized returns $r^*_{Q,t} = z_t \cdot \sigma_t$ in which way it will keep the information contained in σ_t. Calculate the VR statistics based on $\{r^*_{Q,t}, t = 1, \ldots, T\}$, denoted by $VR^*(Q)$.

The reason why the randomized returns are generated in this way is to maintain the information contained in the variance series, especially dynamics in the state space. In comparing $VR(Q)$ and $VR^*(Q)$, we're actually comparing the VR test statistics for standardized historical returns and the test statistic under the null hypothesis

After M Gibbs sampler iterations, it will produce the posterior distribution of the VR for standardized historical returns, $VR(Q)$, and the empirical distribution of the VR under the null of no mean reversion, $VR^*(Q)$. Then the p-values can be calculated as:

$$P(H_0) = \frac{\#(VR^*(Q) < VR(Q))}{M}$$

Then conclusion can be drawn based on the p-value. We'll inference at the significant level of 0.05. At a p-value < 0.05, we will reject the null hypothesis of no mean reversion and conclude mean-reversion.

3 Empirical Results

3.1 Data

The SPX index is one of the most important indices in finance. Unlike some equities, such as Tesla, or commodities, such as natural gas, the high trading volume of large market cap companies under the SPX pronounces the difficulty of it being manipulated. Also, it can indicate the world's market greed and fear and has different behaviour over a different time period which could be a point of interest in testing our model.

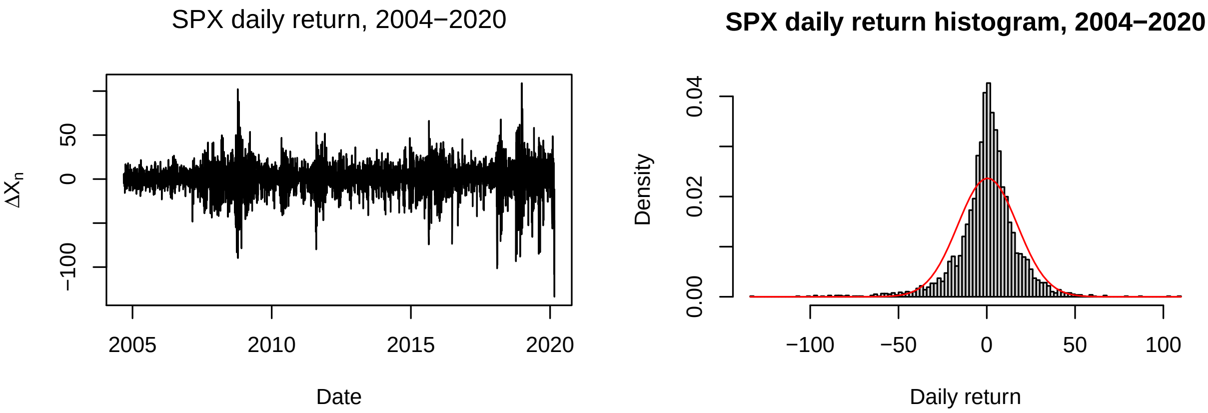


Figure 1: Plot and distribution of SPX daily returns

Figure 1 plots the SPX daily returns and shows that it's not stationary due to non-constant the volatility. The histogram reveals that the returns are much more peaked and have fatter tails compared to the tails of a normal distribution. It's likely to signature a Gaussian model subject to heteroskedasticity.

On the other hand, it's clear that the SPX returns can be characterized by two regimes, a tranquil and a volatile. For example, the first 3˜4 years of daily return behaves like stationary series, while the market seems to fluctuate more starting in 2017, the beginning of the 2008 financial crisis. Besides, the volatility of SPX return kind of converges within the two regimes respectively. For example, the volatilities during the crisis and the most recent two months do not differ significantly on average.

3.2 Results

3.2.1 Latent State Series

Table 1: Transition Counts in the State Space

	Period Q						
	5	10	30	40	50	60	100
n11	0	0	0	0	524	0	0
n12	0	0	1928	0	5	0	0
n21	0	0	1928	0	4	0	0
n22	3903	3903	47	3903	3370	3903	3903

From the plot in Figure 1, the SPX returns are persistent over time, i.e. a large return on one trading day is often followed by a series of large returns in subsequent trading days. Such clustering of returns will result in a relatively quiet transition path, i.e. at the current state in the Markov Chian, it's far more likely to stay in the state than transiting to the other state. This can be diagnosed from Table 1, where a lot of zero entries are. Even at Q =50, the most floating path reports giant n_{11} and n_{22} compared to single-digit jumps between the two states.

3.2.2 Variance Ratio Test

In the empirical study over SPX daily returns, the Gibbs sampler simulates posterior of model parameters and the VR statistics for the chosen periods. The lags we evaluated at has been chosen to be within one quarter of a year, apart from the biggest of 5 months.

Table 2: Posterior VR for the standardized SP500 Daily Returns

	Period Q						
	5	10	30	40	50	60	100
Posterior VR for the standardized SPX Returns							
Mean	0.1972	0.0949	0.0317	0.0232	0.0195	0.0159	0.0093
Median	0.1972	0.0949	0.0317	0.0232	0.0195	0.0159	0.0093
SD	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000
VR for Randomized SPX Returns							
Mean	0.2000	0.1000	0.0333	0.0250	0.0201	0.0167	0.0100
Median	0.2000	0.1000	0.0333	0.0250	0.0201	0.0167	0.0100
SD	0.0045	0.0023	0.0007	0.0006	0.0005	0.0004	0.0002
p-value of Null Hypothesis of No Mean Reversion							
p_Val	0.2657	0.0115	0.0148	0.0005	0.1232	0.0237	0.0021

Table 2 summarized the VR based on the standardized returns and randomized de-standardized, and the p-value for the null hypothesis of no mean reversion. Except for the 5-day and 50-day, all tests support statistically significant evidence of the mean reversion in return series. At the lag of 40 trading days, 2 months, the p-value is the smallest and suggests the strongest evidence of mean reversion in daily returns.

We noticed the zero standard deviation of the VR values based on standardized returns, which is considered as a plausible sign. It indicates that the posterior variance series manages to detect the heteroscedasticity by the two defined regimes. Overall, we'd conclude that the SPX daily returns present mean reversion over various periods/horizons.

4 Discussion

The variance ratio statistics has become a popular analysis tool of time series and widely spread in the test of mean reversion. Recent research has found that the pattern of heteroscedasticity considerably affects the probability of the variance ratio test to reject the null hypothesis of random walk, or no mean reversion under certain arrangements.

This paper models the heteroscedastic SPX 500 daily return with a two-regime Markov switching model that capture the heteroscedasticity by a tranquil and a volatile regimes. We embrace the Bayesian approach to estimate the latent state values and model parameters by the Gibbs sampler. The methodology is clearly specified step by step and the prior choices are based on full conditions distributions. At each iteration, we compute variance ratios on a standardized scale as well as a randomized de-standardized scale to test for mean reversion.

The persistence of heteroskedasticity will encourage the return series to retain at the current state. It will propose unbalanced transition tables as well as the posterior transition probabilities. In the empirical results, the biggest p-value at the lag of 50 trading days gives the strong evidence of no mean reversion. In the meantime, the transition occurs more frequently at a lag of 50 than at other lags, which can be revealed from Table 1. Potential analysis can be disclosed with observation series relatively wandering frequently in the state space.

Previous studies with different strategies in mean reversion concludes that the longer the investment horizon, the stronger the evidence of mean reversion is. However, we don't identify such a trend in this paper. This may come from the assumption of the homogeneous first moment of returns in our model. On the other hand, the different evaluation may come from the data. We tried intra-day data before the current empirical results, and the behaviours of the p-values do differ from what we have in this paper. Due to limitations of access to dataset, we only study 16 years daily data. Potential evaluation over a bi-weekly or monthly excess return may produce other viewpoints.

In estimating the latent activities in the state space, we employ the Viterbi algorithm, which maximizes the posterior probabilities. It's known to be globally consistent, but the result may differ from a maximizer of the posterior marginals. Whether they will lead to seriously distinct estimates is to be uncovered. Besides, the bias of the variance ratio test based on the model design is not discussed in this paper. Through Kim et al. [1998] concludes unbiases in the test in a slightly different model setting, we cannot affirm the bias in our model.

A Heston Stochastic Volatility Model

Let S_t denote the value of GSPC at time t and then $X_t = \log(S_t)$ and its time-dependent volatility V_t can be modeled by the two-dimensional process.

$$
\begin{aligned}
\mathrm{d}X_t &= (\alpha - \frac{1}{2}X_t^2)\ \mathrm{d}t + V_t\ \mathrm{d}B_{1t} \\
\mathrm{d}V_t &= -\gamma(V_t - \mu)\ \mathrm{d}t + \sigma V_t^{1/2}\mathrm{d}B_{2t} \\
\mathrm{d}B_{1t}\ \mathrm{d}B_{2t} &= \rho \mathrm{d}t
\end{aligned}
$$

Where:

- α is the drift of the asset representing continuously compounded interest rate.
- V_t is the stock price variance, a latent stochastic volatility process.
- μ is the expected value of V_t, the long-run average. The Feller condition $\mu > \frac{1}{4}\sigma^2$ is imposed to have a stationary volatility process.
- $\gamma > 0$ indicates the rate of mean reversion of volatility to its long-run mean, μ.
- $\sigma > 0$ is the volatility of V_t, a scale parameter.
- B_1 and B_2 are Brownian motion with correlation ρ, $|\rho| < 1$.

B Conjugate Inverse Gamma Prior - σ_1^2

The posterior of the conjugate Inverse Gamma prior can be derived as follows:

$$
\begin{aligned}
\sigma_1^2 &\sim IG(\frac{1}{2}, \frac{1}{2}) \\
\Rightarrow f_{\sigma_1^2}(x) &\propto x^{-\frac{3}{2}} \exp(-\frac{1}{2x}) && \text{(Prior)} \\
Y_{1t} \mid \sigma_1^2 &= \frac{y_t}{\sqrt{1 + \gamma \cdot \mathbb{I}\{S_t = 2\}}} \sim N(0, \sigma_1^2) \\
\Rightarrow L(Y_{1t}) &\propto \prod_t \frac{1}{\sqrt{\sigma_1^2}} \exp(-\frac{Y_{1t}^2}{2\sigma_1^2}) \\
&= (\sigma_1^2)^{\frac{T}{2}} \exp(-\frac{\sum_{t=1}^T Y_{1t}^2}{2\sigma_1^2}) \\
\Rightarrow \Pr(\sigma_1^2 = x \mid Y_{1t}) &\propto x^{-\frac{3}{2}} \exp(-\frac{1}{2x}) \cdot x^{\frac{T}{2}} \exp(-\frac{\sum_{t=1}^T Y_{1t}^2}{2x}) \\
&= x^{-(\frac{1+T}{2}+1)} \cdot \exp(-\frac{1 + \sum_{t=1}^T Y_{1t}^2}{2x}) \\
&\sim IG(\frac{1+T}{2}, \frac{1 + \sum_{t=1}^T Y_{1t}^2}{2}) && \text{(Posterior)}
\end{aligned}
$$

C Conjugate Truncated Inverse Gamma Prior - $\bar{\gamma}$

We choose the prior as $\bar{\gamma} \sim IG(\frac{1}{2}, \frac{1}{2}) \cdot \mathbb{I}\{\bar{\gamma} > 1\}$, a truncated Inverse Gamma distribution. Then, we define $Y_{2t} = \frac{y_t}{\sqrt{\sigma_1^2}}, N_2 = \{t : S_t = 2, t = 1, \ldots, T\}, T_2 =$ sum of elements in N_2.

Alternatively, we use Y_{2t} to illustrate observations at state 2 only. In this way, the posterior has been derived similarly to the previous one, except that we add the indicator to the posterior distribution in the end.

A formal proof is given by

$$
\begin{aligned}
\bar{\gamma} &\sim IG(\frac{1}{2}, \frac{1}{2}) \cdot \mathbb{I}\{\bar{\gamma} > 1\} \\
\Rightarrow f_{\bar{\gamma}}(x) &\propto x^{-\frac{3}{2}} \exp(-\frac{1}{2x}), \ x > 1 \\
Y_{2t} \mid \bar{\gamma} &\sim N(0, \bar{\gamma}) \\
\Rightarrow L(Y_{2t}) &\propto \prod_{T_2} \frac{1}{\sqrt{\sigma_1^2}} \exp(-\frac{Y_{1t}^2}{2\sigma_1^2}) \\
&= (\bar{\gamma})^{\frac{N_2}{2}} \exp(-\frac{\sum_{T_2} Y_{1t}^2}{2\bar{\gamma}}) \\
\Rightarrow \Pr(\bar{\gamma} = x \mid Y_{2t}) &\propto x^{-\frac{3}{2}} \exp(-\frac{1}{2x}) \cdot x^{\frac{N_2}{2}} \exp(-\frac{\sum_{T_2} Y_{2t}^2}{2x}) \\
&= x^{-(\frac{1+N_2}{2}+1)} \cdot \exp(-\frac{1 + \sum_{T_2} Y_{2t}^2}{2x}) \\
&\sim IG(\frac{1+N_2}{2}, \frac{1 + \sum_{T_2} Y_{2t}^2}{2}) \cdot \mathbb{I}\{\bar{\gamma} > 1\}
\end{aligned}
$$

Then, the σ_2^2 can be updated by the definition $\sigma_2^2 = \sigma_1^2 \cdot \bar{\gamma}$.

D Left Truncated Inverse Gamma Generator

To draw from the left truncated Inverse Gamma, it's equivalent to generate from a right truncated Gamma distribution.

For a continuous random variable with pdf and cdf specified by $h(x)$ and $H(x)$ respectively, the distribution of the truncated random variable, denoted by X, over interval (a, b) is given by:

$$
f_X(x) = \begin{cases} \frac{h(x)}{H(b)-H(a)} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} ; F_X(x) = \frac{H(\max(\min(x, b), a)) - H(a)}{H(b) - H(a)}
$$

In over case, $h(\cdot), H(\cdot)$ are pdf and cdf of Gamma distribution, and the truncated random variable has a range of $(0, 1)$.

Then the algorithm can be summerized as following:

1) Generate $u \sim Uniform(0, 1)$

2) Set $x = F_X^{-1}(u) = H^{-1}(u \cdot H(1))$

3) Output $\frac{1}{x}$, which is a random draw from left truncated Inverse Gamma.

E Conjugate Beta Prior - Transition Probabilities

As we take $Beta(1, 1)$ as the prior, which is equivalent to Uniform(0,1), the posterior can be derived from the likelihood function.

$$\begin{aligned} L(p) &= \prod_{\forall i,j} p_{ij}^{n_{ij}} \\ &= p_{12}^{n_{12}}(1 - p_{12})^{n_{11}} \cdot p_{21}^{n_{212}}(1 - p_{21})^{n_{22}} \\ &\sim Beta(1 + n_{12}, 1 + n_{11}) \cdot Beta(1 + n_{21}, 1 + n_{22}) \quad \text{(Posterior)}. \end{aligned}$$

F Simulation Study

We're struggled in simulating a good heteroscdedastic series and tried with the following example to check the correctness of Gibbs sampler. Both transition probabilities behave relatively well in and most of posterior samples located within the credible interval.

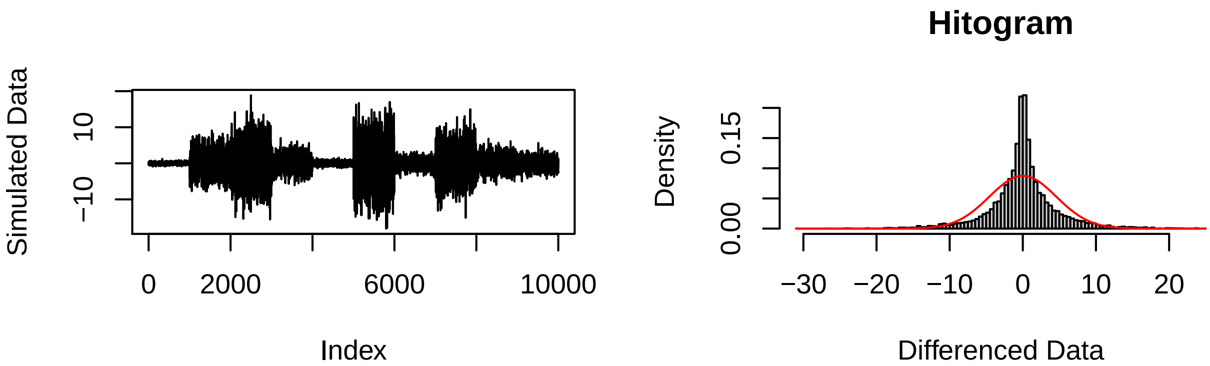


Figure 2: Plots and Hitogram of simulated heteroscdedasticity Data

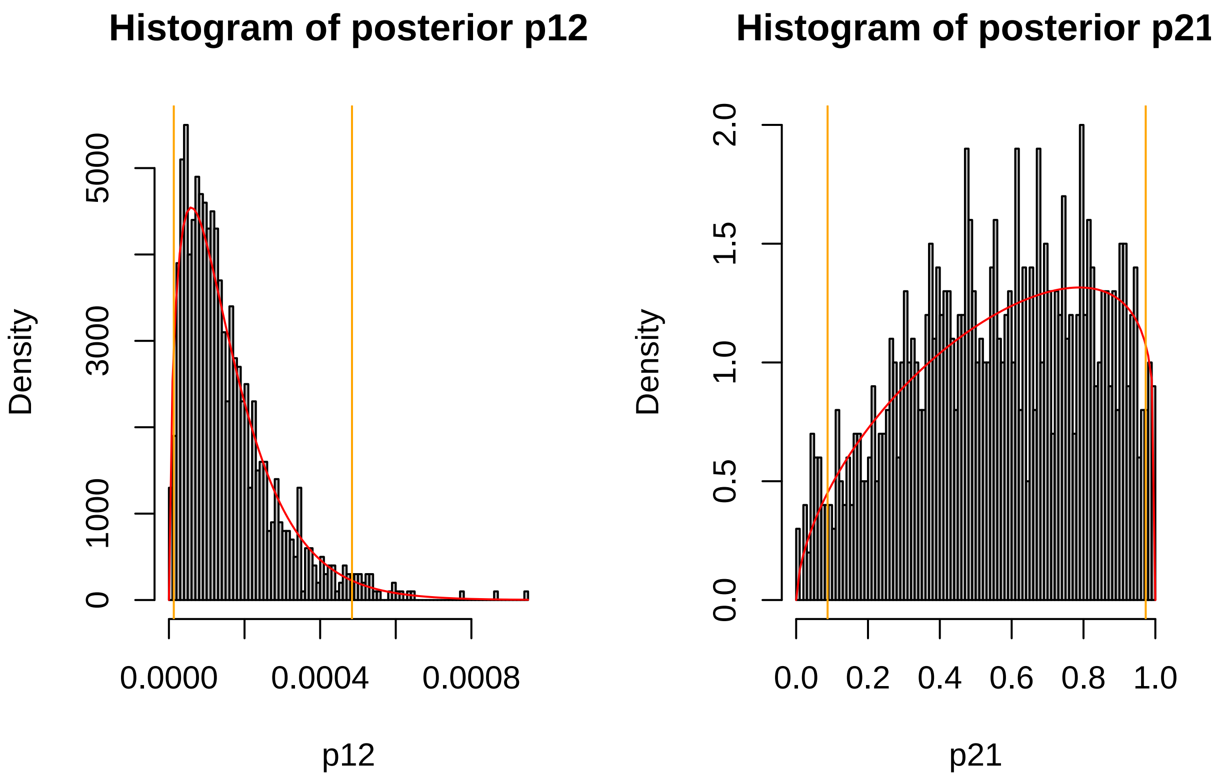
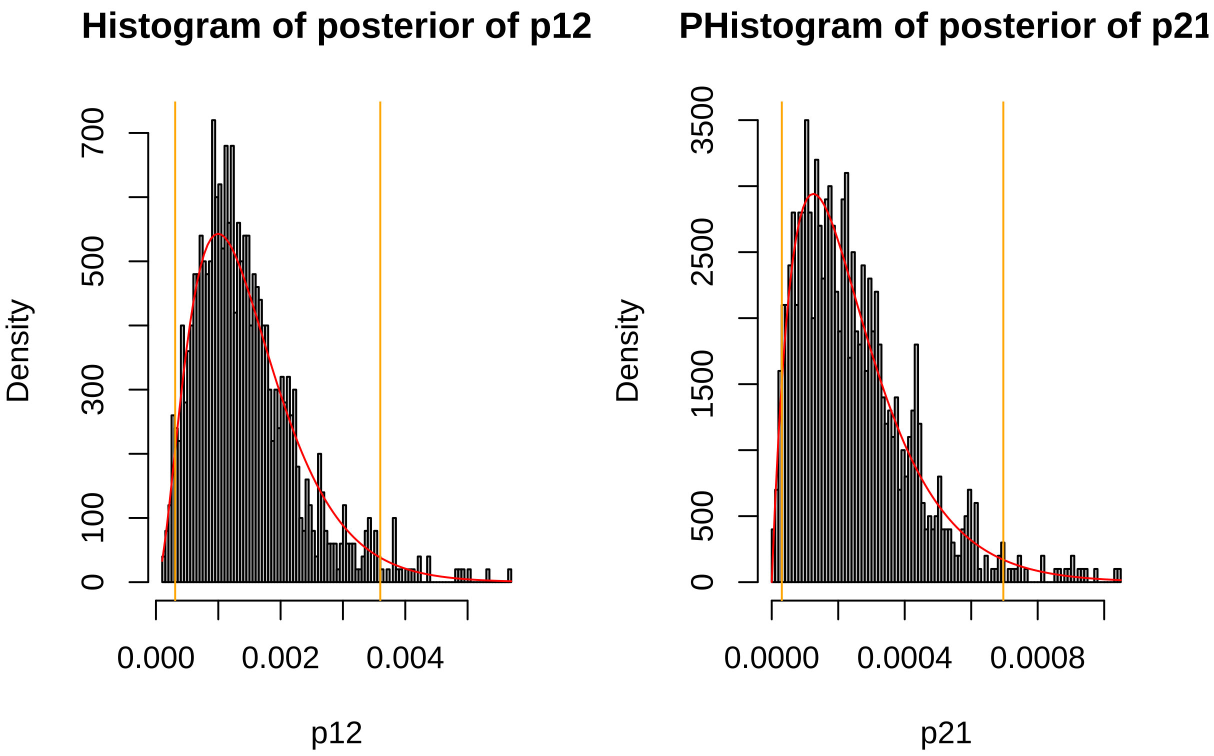


Figure 3: Posterior Check for transition proabilites with default prior



G R Code for Empirical Study

To proceed the code below, our well-orgnized R package **TMS** is required.

```r
#load and clean data
spx_data = lapply(c('../data/SPX_2004_2019.txt','../data/SPX_2020_2020.txt'),
                  function(x){
                    read.delim2(x, header = FALSE, sep = "\t")%>%
                      separate("V1",c("DateTime", "Open", "High",
                                      "Low", "Close"),sep = ",")
                  }) %>%  bind_rows()
spx_data$DateTime <- gsub('\\.', '-', spx_data$DateTime)
daily_data<- data.frame(Date = as.Date(spx_data$DateTime),
                        value = as.numeric(spx_data$Open))
daily_data<- subset(daily_data, !duplicated(daily_data$Date, fromLast=T))
```

```r
# Parallel computing utilize multi-core machine
# to test on different q value
parllel_proc<-function(vec,data){
  parallel::mclapply(vec, function(i){
       # Store the row to its own variable for ease
      out<-tms.fit(y = data, nsamples = 1e4, s.out = TRUE,q = i)
      pval <- VR.pval(out$VR)
      list(out=out,i=i, pval=pval)
    }, mc.cores = parallel::detectCores()-1, mc.allow.recursive = TRUE)
}
# Example data(2008~2009)
new_data <- filter(daily_data, Date <= as.Date('2009-10-01'))
new_data <- filter(new_data, Date >= as.Date('2008-04-01'))
vec <- c(seq(from=1,to=100,by=5))
crisis_res <- parllel_proc(vec,diff(new_data$value))
```

H R Code for Simulation study

The R code for inference correctness visual check is given in the following. Results have been presented in both default prior parameters and random prior parameters for variance and transition probabilities.

```r
require(TMS) # load the package
require(dplyr)
default.prior <- FALSE #Will check both: TRUE/FALSE
if(default.prior){
```

```r
  tran_prior<-list(shape1=1, shape2=1);var_prior<-list(shape=1/2, rate=1/2)
}else{
  x <- rexp(4)
  tran_prior<-list(shape1=x[1],shape2=x[2])
  var_prior<-list(shape=x[3], rate=x[4])
}
# Simulate heteroscdedasticity data
n <- 1e5
sd_l <- c(0.3,3,5,2,0.5,6, 1.3,4.2,2,1.5); #2 as a threshold
y <- NULL
for(ii in 1:10){
  sd <- sd_l[ii]
  y<- c(y, rnorm(1e3,sd=sd))
}
# Gibbs output
nsamples = 1e3
system.time({
  gibbs.fit <-tms.fit(nsamples=nsamples, y=y, q=20, tran_prior=tran_prior,
                      var_prior=var_prior, par.out = TRUE,s.out=TRUE)
})
gibbs.Theta <- gibbs.fit[["Theta"]]
gibbs.sigma1 <- gibbs.Theta[,"sigma1"]
gibbs.p12 <- gibbs.Theta[,"p12"]
gibbs.p21 <- gibbs.Theta[,"p21"]
# Vissual Check analytic posterior
# ---- Check for sigma1^2
plot(gibbs.sigma1, ylab = expression(sigma[1]*" | "*S[t]),
     main = expression("Scatter Plot of "*sigma[1]*" | "*S[t]))
abline(h=mean(gibbs.sigma1), col="red")
# ---- Check for transition probabilities
S <- gibbs.fit[["S"]][nsamples,]
tran.count <- trans.num(S)
n_12 <- tran.count$n12;n_11 <-tran.count$n11
n_21 <- tran.count$n21;n_22 <-tran.count$n22
par(mfrow=c(1,2))
# Example code for p12
p1_shape1<-tran_prior$shape1+n_12
p1_shape2 <- tran_prior$shape2+n_11
```

```r
hist(gibbs.p12, freq = FALSE, breaks = 1e2, xlab = "p12",
     main = "Posterior Check for p12", col = "grey")
curve(dbeta(x,p1_shape1, p1_shape2),
      col="red", add = TRUE)
abline(v=qbeta(0.025,p1_shape1, p1_shape2), col="orange")
abline(v=qbeta(0.975,p1_shape1, p1_shape2),col="orange")
```

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