1 Time Complexity

Master's theorem for $T(n) = aT(\frac{n}{b}) + f(n)$ where $a \geq 1$ and $b > 1$
Let $c_{crit} = log_b(a)$ and if $f(n) = \theta(n^c)$

1. If $c < c_{crit}$ then $T(n) = \theta(n^{c_{crit}})$
2. If $c = c_{crit}$ then $T(n) = \theta(n^c log(n))$
3. If $c > c_{crit}$ then $T(n) = \theta(f(n))$
4. If $f(n) = \theta(n^{c_{crit}} log^k(n))$, then $T(n) = \theta(n^{c_{crit}} log^{k+1}(n))$

2 Sorting

2.1 Bubblesort

Time complexity: $\Omega(n)$ $\theta(n^2)$ $O(n^2)$
Invariant: At the end of iteration `j`, the biggest `j` items are correctly sorted in the final `j` positions of the array.

2.2 SelectionSort

Time complexity: $\Omega(n^2)$ $\theta(n^2)$ $O(n^2)$
Invariant: At the end of iteration `j`: the smallest `j` items are correctly sorted in the first `j` positions of the array.

2.3 InsertionSort

Time complexity: $\Omega(n)$ $\theta(n^2)$ $O(n^2)$
Invariant: At the end of iteration `j`: the first `j` items of the array are sorted in order.

2.4 QuickSort

Time complexity: $\Omega(nlog(n))$ $\theta(nlog(n))$ $O(n^2)$ $O(nlog(n))$ **(for paranoid QuickSort)**
Invariant: For every `i < low`: `B[i] < pivot` and for every `j > high`: `B[j] > pivot`. Duplicates: Use three-way partition to store duplicates.

2.5 MergeSort

Time complexity: $\Omega(nlog(n))$ $\theta(nlog(n))$ $O(nlog(n))$
Invariant: At the end of each loop, the subarrays are sorted.

2.6 QuickSelect

Time complexity: $\Omega(n)$ $\theta(n)$ $O(n^2)$
Invariant: For every `i < low`: `B[i] < pivot` and for every `j > high`: `B[j] > pivot`.

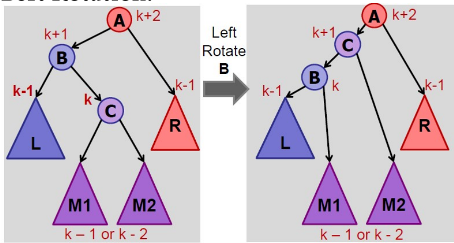
3 Trees

3.1 Binary Search Tree (BST)

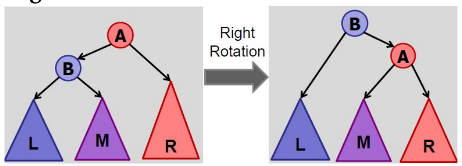
Only has 2 children per node. Left child is smaller than parent. Right child is larger than parent.
Operations: $O(log(n))$ / $O(n)$ (balanced)

3.2 AVL Tree

Maximum height between two children tree is 1.
Operations: $O(log(n))$
Left Rotation:



Right Rotation:



Insertion Max Rotations: 2
Deletion Max Rotations: $log(n)$
Invariant: `|height(u) - height(v)| < 2` if `v` and `u` are sibling nodes. `|height(u) - height(v)| > 0` if `u` is the parent node of `v`.

3.3 Order Statistics (Rank finding)

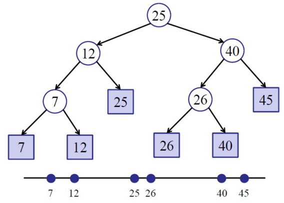
AVL Tree augmented with a `weight` property in each node.
Updating weights on rotate: $O(1)$

3.4 Interval Trees

Store the maximum value of entire subtree under a node as a parameter. Update `max` during rotations by taking `Math.max` of all children under the two nodes being swapped.
Time complexity: $O(log(n))$

3.5 1D Range Finding

All leaves hold the value, each internal node `v` stores the `MAX` of any leaf in the left sub-tree.



Operations:

`findSplit`	Finds the node to start searching.
`traverseLeft`	Traverse the left subtree after findSplit.
`traverseRight`	Traverse the right subtree after findSplit.

Complexity:

`Query`	$O(k + log(n))$
`Build`	$O(nlog(n))$
`Space`	$O(n)$

3.6 (a, b) trees

An (a, b) tree has a bound of $2 \leq a \leq \frac{b+1}{2}$.

Rule 1:

Node Type	#Keys		#Children	
	Min	Max	Min	Max
Root	1	b - 1	2	b
Internal	a - 1	b - 1	a	b
Leaf	a - 1	b - 1	0	0

Rule 2: A **non-leaf node** must have one more child than its number of keys.
Rule 3: All **leaf nodes** must be at the same depth (from the root).

`search`	Traverse from the root node and binary search the node array. ($O(log(n))$).
`insert`	Traverse from the root to find point of insertion. Split if needed ($O(log(n))$).
`delete`	Traverse from the root to find node and delete. Sacrifice/merge/split if needed ($O(log(n))$).
`split`	Find the median element in the node array and sacrifice to the parent. The split halves are now the child of the promoted node.
`merge`	Merges two siblings. Delete the parent and add the parent to the keylist of one child. Merge the two children.
`share`	Merge and split combined.

4 Hashing

4.1 Hashing with Chaining

If a current bucket is occupied, add to the linked list in the bucket.

- **Insert:** $O(1 + cost(f))$
- **Search/Delete:** $O(n + cost(f))$ where n is the number of nodes.
- Expected Number of items per bucket = $\frac{n}{m}$.
- Under Simple Uniform Hashing Assumption (every key is likedly to be mapped to every permutation), $E(\text{search}) = O(1)$.
- Can still add items when $m == n$ and search efficiently.

4.2 Hashing with Open Addressing

If the current bucket is full, find the next available bucket (Linear Probing).

- **Insert/Delete/Search:** $O(1)$ if $\alpha < 1$ where $\alpha = \frac{n}{m}$ where n is the #items and m is #buckets.
- When deleting a key, replace the bucket with `DELETED`. Functions will probe past `DELETED`.
- Clusters of size $\Theta(log(n))$ form when table is $\frac{1}{4}$ full. Caching of arrays make this fast.
- Unable to insert and search efficiently when $m == n$.
- Expected cost of operation is $\leq \frac{1}{1-a}$.

4.3 Double Hashing

Two ordinary hash functions $f(k)$ and $g(k)$ with new hash function:

$$h(k,i) = f(k) + i \cdot g(k) \bmod m$$

$g(k)$ must be relatively prime to m for $h(k,i)$ to hit all buckets.

4.4 Hashcode

`Integer`	The int value itself.
`Long`	Split the long into 32 bits, XOR.
`String`	Iterate through each char, sum the value with an offset.

G

4.5 Resizing

If $(n == m)$, then $m = 2m$ and if $(n < \frac{m}{4})$, then $m = \frac{m}{2}$.

Definition 4.1 *Operation has amortized cost $T(n)$ if for every integer k, the cost of k operations is $\leq kT(N)$.*

4.6 HashSet

- Stores a bit instead of key.
- $P(\text{false positive}) = 1 - (1 - \frac{1}{n})^n \approx 1 - (\frac{1}{e})^{\frac{n}{m}}$.

4.7 Bloom Filter

- Use 2 hash functions $f(k)$ and $g(k)$. Both buckets must return `True` for a positive match.
- $P(\text{false positive}) = (1 - \frac{1}{e}^{\frac{2n}{n}})^2$.
- $P(\text{bit is } 0) = (1 - \frac{1}{m})^{kn} \approx e^{-kn/m}$.
- $P(\text{collision at 1 spot}) = 1 - e^{-kn/m}$.
- $P(\text{collision at 1 spot}) = (1 - e^{-kn/m})^k$

5 Graphs

Definition 5.1 *A graph consists of at least a node, and can consist of edges that connect nodes in the graph.*

Definition 5.2 *A connected graph has all nodes connected by a path.*

Definition 5.3 *The degree of a node is the number of adjacent edges on that node.*

Definition 5.4 *The degree of a graph is the maximum number of adjacent edges in the graph.*

Definition 5.5 *Diameter is the maximum distance between two nodes following the shortest path.*

Definition 5.6 *Star graph has all nodes connected to a central node.*

Definition 5.7 *Clique is a complete graph with diameter* 1 *and degree* $n-1$.

Definition 5.8 *A cycle is a connected graph with a loop. It has a diameter* $\frac{n}{2}$ *or* $\frac{n}{2} - 1$ *and degree* 2.

Definition 5.9 *A bipartite graph is a graph with nodes divided into two sets. There are no edges between nodes of the same set.*

5.1 Representation

- Adjacency List. Each node has a linked list of edges to other nodes. ($O(V+E)$ memory).
- Adjacency Matrix. 2D array that stores path from a node to another. ($O(V^2)$ memory).

5.2 Traversal

- BFS. Use a queue to explore neighbours level by level.
- DFS. Use a stack to explore the maximum depth, then neighbours.

5.3 Directed Acyclic Graph

A topological sorted graph can find shortest paths in $O(E)$ by walking through the graph once.

5.3.1 Post-order DFS

- Use post-order DFS to add nodes to the set.
- Time Complexity: $O(V+E)$.

5.3.2 Kahn's Algorithm

- Maintain a set of nodes with no incoming edges. Remove edges and add them to the set when a node has no incoming edge.
- Time Complexity: $O(Elog(V))$ or $O(E+V)$.

5.4 Shortest Paths

5.4.1 Bellman Ford

- Relax all edges V times. Can detect negative weight cycles.
- Time Complexity: $O(EV)$ (Can optimise to terminate early when no weight changes detected).
- Negative weight cycles can be detected with $|V|+1$ iterations. To label all negative weight cycles, run for $2|V|$.

5.4.2 Dijkstra's

- Relax shortest edge using a priority queue.
- Time Complexity: $O(Elog(V))$ (Assumes PQ operations are $log(v)$).
- Each edge is relaxed once and each node is added to the priority queue once.

6 Heaps

Binary heap stores inserts nodes from left to right and ensures the maximum height is $O(floor(log(n)))$.

`insert`	$O(log(n))$
`delete`	Swap with min element and `bubbleDown` min element. $O(log(n))$
`decreaseKey`	$O(log(n))$
`extractMax`	$O(1)$

For a lookup array (swap positions when we bubble):

`leftChild`	$2x+1$
`rightChild`	$2x+2$
`parent`	$floor((x-1)/2)$

7 Dynamic Programming