Introduction to Finite Element Methods

Gauss quadrature rules

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Objective

- Revisit numerical intergration methods
- Gauss integration(optimal)

Set-up

The notebook relies on Python code. To initialize the notebook, select **Cell->Run All Below**

Introduction

In general, case a mapping between elements and reference(master) elements is used to simplify the integration domain.

We need to compute the definite integral a lot in the FEM computations.

$$I := \int_a^b f(x)\, dx$$

The integral can be viewed graphically as the area between the x-axis and the curve y = f (x) in the region of the limits of integration. Thus, we can interpret numerical integration as an approximation of that area.

In [8]:

```
## Do not change the text in this box
f_box = Text(value="exp(-x**2)", description=r'$f(x)=$')
n_slider = widgets.IntSlider(min=2,max=20, step=2, value=4, description=r'$n_{start
}$')
method_type = Dropdown(options=['left', 'right', 'midpoint','trapezoid','simpson'],
                        value='left', description='Method:')
interact(AT.plot3Areas,f=f_box,a="-1.0",b="1.0",n=n_slider,method=method_type);
```

Some quick explanations of how the notebook works:

The meaning of the variables are as follows:

- "a": the left endpoint of the integration interval
- "b": the right endpoint of the integration interval
- "n" the number of sub-intervals to use for approximating the integral
- "f(x)": the function to integrate.

How to define the function:

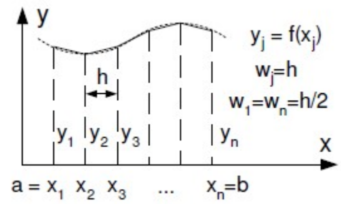
- x**p means x^p. For example, x**0.5 means $\sqrt{x}$
- for a constant function $f(x) = c$ use $c * (x * *0)$ (every function definition needs an x in it!)
- for e^p use exp(p) (so for e use exp(1))
- for π use pi
- type trig functions the way you would on your calculator(sin(x), cos(x), etc)

Some functions you might want to try:

- 4-x**2
- cos(x)
- sin(x**2)
- exp(x**2)
- exp(-x**2/2)/sqrt(2.*pi) (this is the famous "normal curve")
- exp(x)/x (make sure that 0 is not part of your interval!)

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

The trapezoidal rule of numerical integration simply approximates the area by the sum of several equally spaced trapezoids under the curve between the limits of a and b.



The height of a trapezoid is found from the integrand, $y_j = y(x_j)$, evaluated at equally spaced points, x_j and x_{j+1}. Thus, a typical contribution is $Area = h\frac{y_j+y_{j+1}}{2}$, where $h = x_{j+1} - x_j$ is the spacing. Thus, for n points (and $n-1$ spaces), the well-known approximation is

$$I \approx h\left(\frac{1}{2}y_1 + y_2 + y_3 + \ldots + y_{n-1} + \frac{1}{2}y_n\right)$$

where $w_j = h$, except $w_1 = w_n = \frac{h}{2}$.

In [9]:
```
## Do not change the text in this box
# -3*x**2+2*x+6
f_box = Text(value="exp(-x**2)", description=r'$f(x)=$')
n_slider = widgets.IntSlider(min=1,max=20, step=2, value=2, description=r'$n_{start}$')
method_type = Dropdown(options=['trapezoid'],
                        value='trapezoid', description='Method:')
interact(AT.plot3Areas,f=f_box,a="-1.0",b="1.0",n=n_slider,method=method_type);
```

Gauss integration

The famous mathematician Gauss posed this question:

- What is the minimum number of points, n, required to exactly integrate a polynomial?
- What are the corresponding function values and weights?

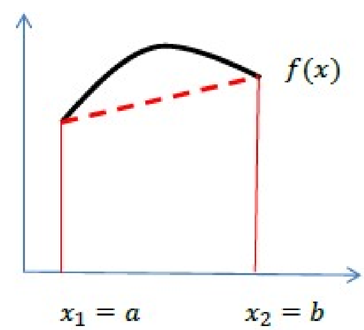


Figure 1 Trapezoidal rule

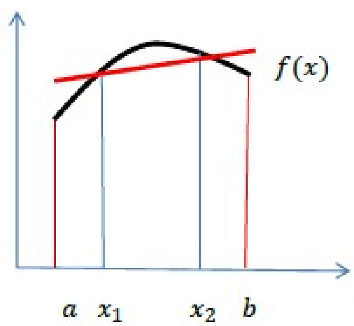


Figure 2 Gaussian quadrature

Example:

Let $[a, b] = [-1, 1]$ and $n = 2$

We want to choose x_1, x_2, w_1 and w_2 so that

$$\int_{-1}^{1} f(x)\,dx \approx w_1 f(x_1) + w_2 f(x_2)$$

The approximation should be exact for any polynomial of degree 3 or less.

Let $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

then

$$\int_{-1}^{1} f(x)\,dx = a_0 \int_{-1}^{1} 1\,dx + a_1 \int_{-1}^{1} x\,dx + a_2 \int_{-1}^{1} x^2\,dx + a_3 \int_{-1}^{1} x^3\,dx$$

Each of the definite integrals on the right-hand side has an integrand of degree 3 or less. Gaussian Quadrature should be exact for each of these.

Applying Gaussian Quadrature to each remaining integral yields:

$$\int_{-1}^{1} 1\,dx = 2 = w_1 + w_2$$

$$\int_{-1}^{1} x\,dx = 0 = w_1 x_1 + w_2 x_2$$

$$\int_{-1}^{1} x^2\,dx = \frac{2}{3} = w_1 x_1^2 + w_2 x_2^2$$

$$\int_{-1}^{1} x^3\,dx = 0 = w_1 x_1^3 + w_2 x_2^3$$

$w_1 + w_2 = 2$

$w_1 x_1 + w_2 x_2 = 0$ we obtain $w_1 = \frac{2x_2}{x_2 - x_1}$ and $w_2 = \frac{2x_1}{x_1 - x_2}$

Using these values for 3^{rd} equation $x_1 x_2 = -\frac{1}{3}$ and 4^{th} equation $x_1 = -x_2$.

Solving the nonlinear system gives us

$$w_1 = 1, w_2 = 1, x_1 = -\frac{\sqrt{3}}{3}, x_2 = \frac{\sqrt{3}}{3}$$

Therefore,

$$\int_{-1}^{1} f(x)\, dx \approx f\left(\frac{-\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$$

Above Gauss quadrature integral has degree of precision 3. Trapezoidal rule has degree of precision 1.

Number of points, n	**Weights, w_i**	**Points, x_i**
1	2	0
2	1	$\pm\frac{1}{\sqrt{3}}$
3	0	$\frac{8}{9}$
	$\pm\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
4	$\frac{18+\sqrt{30}}{36}$	$\pm\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}$
	$\frac{18-\sqrt{30}}{36}$	$\pm\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}$

Example

$\int_{-1}^{1} \left(x^2 + \cos x\right)\ dx$ = $\frac{x^3}{3}\Big|_{-1}^{1} + \sin x\Big|_{-1}^{1} = \frac{2}{3} + 2\sin(1)$

Trapezoidal rule :

$\int_{-1}^{1} \left(x^2 + \cos x\right)\ dx \approx \frac{2}{2}(f\,(-1) + f\,(1))$

Gauss quadrature: $\int_{-1}^{1} \left(x^2 + \cos x\right)\ dx \approx f\left(\frac{-\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$

In [5]:
```
from scipy.integrate import quad
from numpy import sqrt, sin, cos, pi

def f(x):
    return x**2 + cos(x) #integrant
```

In [10]:
```
I_actual =2/3 + 2*sin(1) #computed integral

print("Analytical computation of integral       is %1.10f" % I_actual)

#Trapezoidal rule for integral computation
h=2
I_trapezoidal = h*(f(-1)+f(1))/2
error_trapezoidal = abs(I_actual - I_trapezoidal)
print("Trapezoidal rule computation of integral is %1.10f and error is %1.10f" % (I
_trapezoidal, error_trapezoidal))
```

```
Analytical computation of integral       is 2.3496086363
Trapezoidal rule computation of integral is 3.0806046117 and error is 0.730995
9755
```

In [11]:
```
#Gauss quadrature rule for integral computation
I_Gauss_Quadrature = f(-sqrt(3)/3)+f(sqrt(3)/3)
error_gauss = abs(I_actual - I_Gauss_Quadrature)
print("Analytical computation of integral       is %1.10f" % I_actual)
print("Gauss quadrature computation of integral is %1.10f and error is %1.10f" % (I
_Gauss_Quadrature, error_gauss) )
```

```
Analytical computation of integral       is 2.3496086363
Gauss quadrature computation of integral is 2.3424903221 and error is 0.007118
3142
```

Gauss quadrature on arbitrary intervals

Use substition or transformation to transform $\int_a^b f(x)\,dx$ into an integral defined over [-1, 1]

Let $x = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)t$, with $t \in [-1, 1]$ Then

$$\int_a^b f(x)\,dx = \int_{-1}^1 f\left(\frac{1}{2}(a+b) + \frac{1}{2}(b-a)t\right)\frac{b-a}{2}\,dt$$

Summary

- Numerical experiements

- Gauss quadrature concepts:

  - using n quadrature points, a polynomial P(x) of degree (2n – 1) or less will be integrated exactly.
  - Compared to trapezoidal rule