> Please show **all** your work! Answers without supporting work will not be given credit.
> You have 3 hours to complete this exam.

Name:_____

1. Find a simple closed formula for $f(x)$ with proof, if $f(x)$ is defined as

$$f(x) = \sum_{n=0}^{2019} \frac{1}{x + x^{\frac{2n}{2019}}}$$

   for $x > 0$.

2. Let P be a $n \times n$ matrix with real entries. Prove that $P^2 = P$ if and only if $\operatorname{rk} P = \operatorname{tr} P$ and $\operatorname{rk}(I - P) = \operatorname{tr}(I - P)$, where I denotes the $n \times n$ identity matrix.

3. We define a positive interger as "birturli" if its decimal representative does not contain all ten digits 0, 1, 2, .. 9. Hence, for example, the number is 9853562851431 is the birturli(since there is no 0 and 7). However, the number 4828302638470560031639314 is not the birturli (all digits 0, ...9 present). Let $\mathcal{B}$ be the set of all birturli numbers. Determine whether series

$$\sum_{n \in \mathcal{B}} \frac{1}{n}$$

   converges or diverges.

4. Let G be a group. Suppose that n and m are relatively prime integers such that $x^n y^n = y^n x^n$ and $x^m y^m = y^m x^m$ for all $x, y \in G$. Prove that G is abelian, i.e. $xy = yx$ for all $x, y \in G$.