Minimal HW

Problem 1:

Prove that e is irrational.

The definition of $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... \to \sum_{n=0}^{\infty} \frac{1}{n!}$.

Answer:

Proof by contradiction $p \wedge \neg q$ to derive q.

Then "prove that e is irrational" becomes "prove that e is rational".

if $e = \frac{a}{b}$ where a, b are positive integers.

Define $x = b!(e - \sum_{n=0}^{b} \frac{1}{n!})$

Substitute e where $x = b!(\sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^{b} \frac{1}{n!})$

Distribute $b!$ to get the form $\sum_{n=0}^{\infty} \frac{b!}{n!} - \sum_{n=0}^{b} \frac{b!}{n!} = \sum_{n=b+1}^{\infty} \frac{b!}{n!}$ where all of the terms in the series are positive and that derives that $x > 0$.

And if we expand $\sum_{n=b+1}^{\infty} \frac{b!}{n!}$ then $\frac{b(b-1)(b-2)(b-3)...}{n(n-1)(n-2)...(b+1)b(b-1)(b-2)...}$

After canceling out terms both in numerator and denominator we get:

$\frac{1}{n(n-1)(n-2)...(b+1)}$ where we have $n - b$ terms in the denominator so the bound is $x \leq \frac{1}{(b+1)^{n-b}}$ where $b + 1$ is the smallest term in the series.

$0 < \sum_{n=b+1}^{\infty} \frac{b!}{n!} < \sum_{b+1}^{\infty} \frac{1}{(b+1)^{n-b}}$ where the last part can be rewritten as

$\sum_{k=1}^{\infty} \frac{1}{(b+1)^k}$ is a geometric series in the form of $\frac{1/(b+1)}{1-\frac{1}{b+1}} = \frac{1}{b}$.

Since b is a positive integer $\frac{1}{b} < 1$ where $x < 1$ and $x > 0$.

There is no integer between 0 and 1 hence it's a contradiction.

Problem 2:

Use a direct proof to show that every odd integer is the difference between two squares.

Answer:

Assume that there is a difference as $a^2 - b^2$ where $a = 2i + 1$ and $b = 2i$ for some integer i.

So $a^2 - b^2 = (a - b)(a + b)$

where $(2i+1)^2 - (2i)^2 = 4i + 1 = 2(2i) + 1$ and assume replace $2i$ with some integer k. So the $2k + 1$ is odd.

Problem 3:

Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.

Answer:

Assume that r is a rational number and ir is an irrational number then $s = r + ir$ is irrational.

Proof by contradiction assumes that s is rational.

If $s = \frac{a}{b}$ and $r = \frac{c}{d}$ where a, b, c, d are integers while $b \neq 0$ and $d \neq 0$.

Then $ir = a/b - c/d = (ad - bc)/(bd)$ must be a rational number that contradicts the hypothesis of ir as the irrational number.

The assumption that s is rational is false, by contradiction s is irrational.

Problem 4:

Prove or disprove that the product of two irrational numbers is irrational.

Answer:

Assume that $ir1$ and $ir2$ are two irrational numbers and p is the product where $p = ir1 \times ir2$ which is irrational.

Let's take $ir1 = ir2 = \sqrt{2}$, since $\sqrt{2}$ is irrational then $p = 2$ is a rational number which disproves the hypothesis.

Problem 5:

Use a proof by contraposition to show that if $x + y \geq 2$ where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.

Answer:

Proof by contraposition assumes that if $x < 1 \land y < 1$ then $x + y < 2$. The assumption is a negation of $x + y \geq 2$ hence the proof is complete.

Problem 6:

Show that if n is an integer and $n^3 + 5$ is odd, then n is even using

a. Proof by contraposition assumes that if n is odd then n^3+5 is even. Let's $n=2k+1$, then $(2k+1)^3+5=8k^3+12k^2+4k+6 \rightarrow 2(4k^2+6k+2k+3)$ which is even.

b. Proof by contradiction assumes that if n is odd ($\neg q$) and n^3+5 is odd as well (p). If $n=2k+1$ then $(2k+1)^3+5=2(4k^2+6k+2k+3)$ is even which is wrong and justifies the contradiction.

Problem 7:

The barber is the one who shaves all those men who do not shave themselves. The question is, does the barber shave himself?

Answer:

Let denote U - all those men

Let S be a propositional function for shaving himself which is either T or F.

Let Q be a propositional function for shaving x where $Q(x)$ means x is shaved by a barber.

The premise is $\forall x \in U(\neg S(x)) \iff Q(x)$ and $Q(b) \iff S(b)$

So $S(b) \iff Q(b) \iff (\neg S(b)) \equiv S(b) \iff \neg S(b)$ which is the contradiction, so the initial premise is contradictory.

Problem 8:

Show that if x and y are integers and both xy and $x+y$ are even, then both x and y are even.

Answer:

Proof by contraposition where if x or y are odd then xy or $x+y$ are odd (applying De Morgan's law).

1st case is when $x=2k+1$(odd) and $y=2j+1$(odd)

Then $xy=(2k+1)(2j+1)=2(2jk+k+j)+1$ is odd

and $x+y=2k+1+2j+1=2k+2j+2=2(k+j+1)$ is even

2nd case is when $x=2k+1$(odd) and $y=2j$(even)

then $xy=(2k+1)2j=4jk+2j=2(jk+j)$ is even

and $x+y=2k+1+2j=2(k+j)+1$ is odd

3rd case is when $x = 2k$ and $y = 2j + 1$ which is the same as the 2nd case where x and y are interchangeable.