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Energy Loss in Collisions of Icy Spheres: Loss Mechanism and Size–Mass Dependence

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The coefficient of restitution of collisions of icy spheres at low impact velocities has been measured by A. P. Hatzes, F. G. Bridges, and D. N. C. Lin (1988, *Mon. Not. R. Astron. Soc.* 231, 1091–1115) and by F. G. Bridges, A. Hatzes, and D. N. C. Lin (1984, *Nature* 309, 333–335). However, the measurements were made only for spheres against massive ice bricks and must be extended or extrapolated if they are to describe collisions of ring particles of arbitrary size and mass. Physical models based upon a perfectly elastic low velocity region, i.e., plastic deformation or brittle fracture, do not accurately describe either the velocity dependence or the size dependence of the data. This failure suggests that the cancellation of size dependence and mass dependence predicted by plastic deformation/brittle fracture models may not occur for icy rings. This means that in ring systems, where sizes vary by orders of magnitude, that the coefficient of restitution for most collisions may be quite different from the measured values. I propose an alternative model, based upon energy loss through viscous dissipation, which matches both the velocity and size dependence of the data quite well. The model should predict the coefficient of restitution for collisions of particles having arbitrary size and mass once the mass dependence is determined by further experiments.

1. INTRODUCTION

The dynamics of particles within ring systems represent an overall balance of competing mechanisms. On one hand, gravitational stirring and collisions driven by Keplarian shear tend to decircularize orbits, increase random motion of the particles, and increase collision rates. On the other, inelasticity in collisions tends to regularize orbits and decrease collision rates. If the collisions become increasingly inelastic as impact speeds increase, then a rough balance can be achieved where energy going into random, dispersive, motion equals that being lost during collisions (e.g., Stewart *et al.* 1984). In any event, this energy loss plays an important role in determining the mean free path between collisions, kinematic viscosity, ring thickness, and spreading rates (Cook and Franklin 1964, Goldreich and Tremaine 1978, Stewart *et al.* 1984). Also, near the Roche limit, energy loss should enhance prospects for formation of "dynamic ephemeral objects" (Weidenschilling *et al.* 1984) or reaccretion after disruptive collisions of satellites (Esposito and Colwell 1989).

The parameter traditionally used to describe the loss of kinetic energy is the coefficient of restitution, ε. Here, we will consider only head-on collisions, neglect rotational effects, and define ε by

$$v_f = -\varepsilon v, \tag{1.1}$$

where $v_f = \dot{X}_f$ is the relative velocity after the collision, $v = \dot{X}_i$ is the relative velocity before the collision, and X is the relative coordinate. We also assume that the coordinates are chosen such that X is positive and is, therefore, the distance between the centers of the particles. For ring systems where particles move in more or less circular orbits, relative velocities are generally quite small, typically less than 1 cm/sec (Esposito 1986). It is often assumed, perhaps tacitly, that ε is mainly a property of the particles' composition (ice), more or less independent of particle size and mass. However, this assumption has not been tested for collisions of icy particles.

A second factor to consider in ring dynamics is that the constituent particles have a broad range of sizes (and masses), with small ones being far more abundant than larger ones (Marouf *et al.* 1983, Cuzzi *et al.* 1984, Zebker *et al.* 1985). It is now generally accepted that the diversity of sizes affects ring viscosity and the spatial distribution of ring particles via gravitational encounters and variation of collisional cross sections with size (Cuzzi *et al.* 1979, Ward 1981, Tyler *et al.* 1983, Petit and Hénon 1987, Brophy *et al.* 1990, Ohtsuki 1992, Salo 1992). Variations of

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ε with size could possibly alter these conclusions and introduce new size–mass effects.

When discussing collisions of ring particles, one great uncertainty is the shape of the particles and the nature of their surfaces. For the purpose of this paper, ring particles are considered to be spherical and to have rough surfaces (Morfill *et al.* 1983, Durisen 1984, Borderies *et al.* 1984).

In Section 2, we discuss the existing, published experimental measurements of the coefficient of restitution on which this paper is based. Following this, some formalism for spherical collisions, together with some results for the elastic case, are presented. In the remainder of the paper, various physical models which might be used to generalize the experimental measurements are discussed. In Section 4, we discuss the difficulties presented by models based upon plastic deformation and brittle fracture. In Section 5, we present an alternative model, based upon viscous dissipation, which describes the experimental data much more successfully. Our summary and conclusions are given in Section 6.

2. EXPERIMENTAL RESULTS

Recently, a series of experiments (Hatzes *et al.* 1988) was performed to determine the coefficient of restitution of frosted, solid, water ice spheres at low impact velocities. More specifically, the coefficient of restitution was measured for collisions of a 2.5-cm-radius ice sphere with the face of a massive ice brick, a proxy for a sphere of infinite radius, for different impact velocities. The methods were similar to those used in an earlier study (Bridges *et al.* 1984) in which the frosts of the particles were thicker and the collisions less elastic. In this paper, we concentrate mainly on the 1988 work, partly because the surface conditions seem somewhat more appropriate for most ring particles, but more importantly, because the more recent study included "size" dependence. Specifically, the experimenters examined collisions of spheres whose local radii of curvatures were 5, 10, and 20 cm at the impact point, thereby simulating impacts of spheres having identical elastic properties, but a larger size. Strictly speaking, these are radius of curvature effects; here, for simplicity, we will call them size effects. When considering size effects, R will denote the radius of curvature at the impact point; when discussing true spheres, it will denote particle radius. The two are, of course, the same for spherical particles.

In the impact experiments, as was the case for Bridges *et al.* (1984), the spheres were attached to a disc pendulum, so the "mass" of the particles was actually an effective mass which includes the moment of inertia of the pendulum. (For further details, see Hatzes *et al.* 1988, also Bridges *et al.* 1984.) This becomes an important consideration when discussing the combined size–mass dependence of ε for actual ring particles.

Experimental results from Hatzes *et al.* (1988) are shown in Fig. 1, together with their empirical fits to the data (curve a). To get these fits, a function the form $Ce^{-\gamma v} + Dv^{-b}$ was employed for each particle radius. The exponential term works well for $v > 0.2$ cm/sec, whereas the power law term provides the more rapid increase at smaller v as ε climbs to unity. The parameters were chosen, in all cases, to make $\varepsilon = 1$ at some small value of v, a choice consistent with earlier work made under somewhat different conditions (Bridges *et al.* 1984), and with the expectation of energy dissipation by plastic deformation or brittle fracture (see later discussion). The data, however, particularly for larger radii (Figs. 1b–1d), are also clearly consistent with ε going to unity as v approaches zero.

The question, then, is how to generalize the experimental results of Hatzes *et al.* (1988) to the physically interesting case of collisions by spheres having arbitrary size and mass. Our goal here is to find a physically based model, describing collisions of such spheres, adjust parameters so as to describe both the velocity and size dependence in the Hatzes *et al.* (1988) data, then use this model to gain insight on the general case.

3. COLLISIONS: ELASTIC RESULTS FOR SPHERES

Before considering specific models for energy loss, we present some relevant formalism and briefly discuss some results for elastic scattering. Consider two colliding spheres, having masses M_1, M_2 and radii R_1, R_2, as shown in Fig. 2. Since the distance between the spheres' centers, X, is less than the sum of the radii, it can be written as $X = R_1 + R_2 - \alpha$, $\alpha > 0$, where α is the total deformation in the system. Generally, it is more convenient to describe the collision by α than by X, with $d\alpha = -dX$. During a collision, α increases from $\alpha = 0$ (when $\dot{\alpha}_1 = v$) to $\alpha_{\max}$ (when $\dot{\alpha}_{\max} = 0$), then returns to $\alpha_f > 0$ (when $\dot{\alpha}_f = v_f = -\varepsilon v$), after which the spheres separate. During impact, the particles are acted upon by a mutual force $F(\alpha)$, and their motion is described by the equation (for $\alpha > 0$),

$$F(\alpha) = -\overline{M}\ddot{\alpha}, \tag{3.1}$$

where

$$\overline{M} = M_1 M_2/(M_1 + M_2) \tag{3.2}$$

is the reduced mass. Although we have written $F(\alpha)$ in (3.1), it is understood that F can also depend upon particle sizes, masses, and elastic properties as well as $\dot{\alpha}$, time, and possibly other variables. Also, in the present circumstances, frosts will alter the surface by producing small-

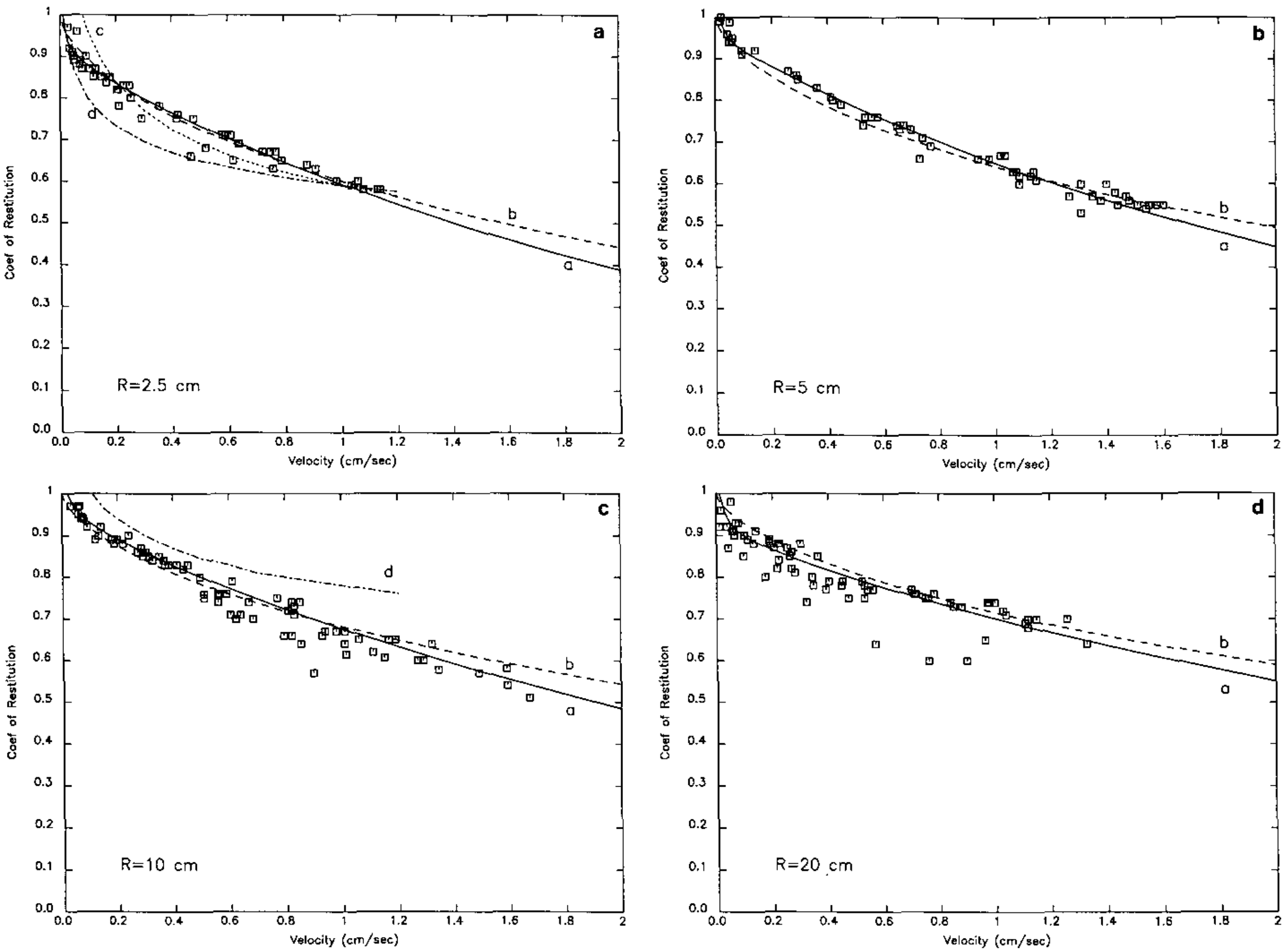

FIG. 1. (a)–(d) Fits to the coefficient of restitution for spheres having different radii of curvature. (curve a) —— Empirical, Hatzes *et al.* (1988). (curve b) ---, Viscous dissipation, $\xi_0 = 0.16$, $p = 0.65$. (curve c) · · · , Tabor, $v_c = 0.0809$, $q = 0.371$. (curve d) ·–·–·– , Tabor, $v_c = 0.0314$, $q = 0.42$.

scale irregularities: some account of this, in an average way, is included in Section 5. Generally, only collisional forces along the contact surface, a disk of radius a (Fig. 2), need be considered, and $F(\alpha)$ can be expressed as an integral of a pressure $p(\alpha, r)$ over the disk. If $p(\alpha, r)$ is

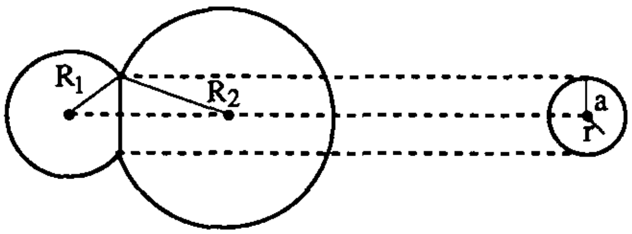

FIG. 2. Geometry for colliding spheres. The small circle to the right shows the cross section of the collisional surface.

specified, in principle, (3.1) can be used to calculate v_f, and hence ε, for any v.

The case of elastic collisions of spheres was first considered by Hertz (1886), with later developments being made by Timoshenko (Timoshenko and Goodier 1951) and others. In Hertz theory, the material is taken to be completely elastic, obeying Hook's law, and there is no dissipation. The resulting force can be written as (Timoshenko and Goodier, 1951)

$$F_H(\alpha) = k\sqrt{\overline{R}}\,\alpha^{3/2}, \qquad (3.3)$$

where $\overline{R}$ is the reduced radius

$$\overline{R} = R_1R_2/(R_1 + R_2) \qquad (3.4)$$

and k depends upon the elastic properties of the spheres. For a collision using the apparatus of Hatzes *et al.* (1988),

R_2 is infinite and $R = R_1$, the radius of the impacting sphere.

When (3.3) is used in (3.1), there is no general analytic solution for α as a function of time. However, one can find expressions for the value of maximum deformation

$$\alpha_{\max} = \left(\frac{5\overline{M}v^2}{4k\sqrt{\overline{R}}}\right)^{2/5} \tag{3.5}$$

and the time of maximum deformation

$$t_{\max} = 1.609\left(\frac{\overline{M}}{k\sqrt{\overline{R}}}\right)^{2/5} v^{-1/5} \tag{3.6}$$

in terms of the impact velocity v. These expressions have been checked experimentally and have been found to apply for a variety of materials as long as velocity is sufficiently small (Raman 1918, Goldsmith 1960). As we will see, Hertz theory plays an important role in our discussion of inelastic collisons.

4. PLASTIC DEFORMATION AND BRITTLE FRACTURE

Under static conditions, many materials deform elastically, with stress being proportional to strain (Hook's law), as long as stress is less than some critical value p_c. For $p < p_c$, internal pressure increases steadily with deformation and the process is reversible. Once the critical stress is reached, however, processes like plastic flow and brittle fracture occur which leave the material permanently deformed. For objects made of such materials, one expects elastic collisions at speeds less than a critical velocity, v_c, and permanent deformation only for speeds greater than this value; indeed, many materials are found to collide elastically at sufficiently low speeds (Raman 1918, Andrews 1931, Barkan 1972, Walton 1993). For ice, the favored energy loss mechanism is through brittle fracture (Borderies *et al.* 1984), but this mechanism has important features in common with plastic deformation: each has a velocity region, $v < v_c$, within which collisions are elastic; and each initially occurs at the point of contact, propagating further into the medium as impact velocity increases. Because of these similarities, it is plausible that fracture and plastic deformation may be describable by the same models, particularly if constants such as yield stresses and elastic moduli are incorporated within adjustable parameters determined by fitting scattering data.

Our approach is similar to that employed by Borderies *et al.* (1984) in that we employ plastic deformation as a proxy for brittle fracture in models for collisions of icy spheres. (There are none for brittle fracture.) In particular, we use plastic deformation models when trying to fit the scattering data of Hatzes *et al.* (1988).

Before looking at detailed fits to data, however, it is of interest to look at some general features involving size–mass dependence of the models, arising from the existence of the elastic region, $v < v_c$. In this region, Hertz theory, hence Eq. (3.5), should apply. In addition, the models considered here (Andrews 1930, Tabor 1948) have a simple relation between the critical deformation, α_c, and the reduced radius,

$$\alpha_c \propto \overline{R}. \tag{4.1}$$

Taking $v = v_c$ and $\alpha_{\max} = \alpha_c$ in (3.5), one obtains

$$\frac{\overline{M}v_c^2}{\sqrt{\overline{R}}} \propto \alpha_c^{5/2} \propto \overline{R}^{5/2}$$

or

$$v_c \propto \sqrt{\overline{R}^3/\overline{M}}. \tag{4.2}$$

According to (4.2), size and mass dependence tends to cancel for spheres of constant density, $M \propto R^3$, leaving only a weak size–mass dependence for v_c. This being the case, one might hope that the ε obtained by the Hatzes *et al.* (1988) collisional data might apply more or less as is for collisions in rings. Unfortunately, the pure-size dependence found by Hatzes *et al.* (1988) does not support this hope. Since the experiments were performed with M_2 fixed and R_2 very large, (4.2) implies $v_c \propto R_1^{3/2}$. Thus, the region of elasticity should grow by a factor of about 23 as R_1 increases from 2.5 to 20 cm, and no such increase is apparent in Fig. 1. The only escape from this difficulty is to have a v_c so small in the 2.5-cm case that the elastic region's increase remains unobservable for larger radii. To see if this is possible, one must look at the actual models.

Andrews' Model

The only analytical formulation of a plastic deformation model is that of Andrews (1930). In particular, a force in (3.1) is specified throughout a collision, both for elastic and for plastic deformation. Andrews assumed that pressure is of a Hertzian form and is a maximum on the collisional axis (Fig. 2), decreasing to zero when $r = a$. The resulting force is, therefore, completely elastic until pressure at $r = 0$ reaches p_c. As deformation continues, a central, plastic region with a reduced pressure $p = p_c$ for $r < r_c$ expands, while the pressure remains elastic for $r_c < r \leq a$. One can, of course, envision a similar, central region of reduced pressure for a collisionally fractured region. The maximum value of r_c, r_c (max), occurs when $\dot{\alpha} = 0$ ($\alpha = \alpha_{\max}$), after which rebound begins. Thereafter, p is Hertzian for r_c (max) $< r \leq a$ and $p = p_c$ for $0 \leq r$

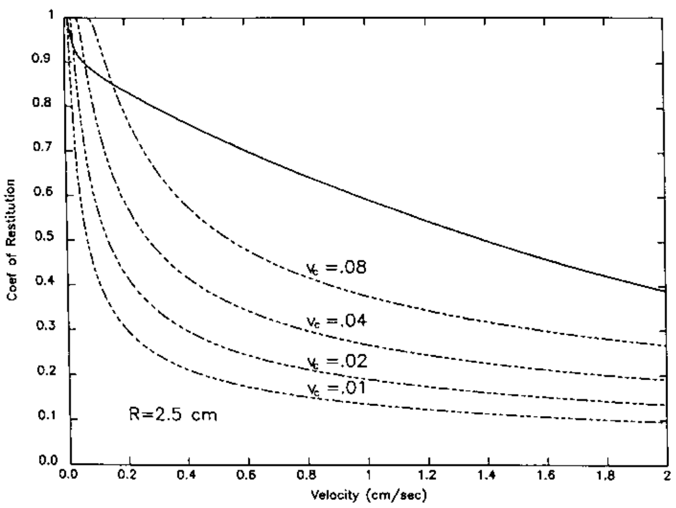

FIG. 3. Coefficient of restitution for $R = 2.5$-cm spheres as found by Hatzes *et al.* (1988) and as predicted by Andrews' model. ——, Empirical fit of Hatzes *et al.* (1988). ---, Andrews' model.

$< r$ (max), with rebound ending when $a = r_c$ (max). The resulting force gives (see also Borderies *et al.* 1984)

$$\varepsilon^2 = \frac{1}{3(v/v_c)^2}\left[-2 + \sqrt{30\left(\frac{v}{v_c}\right)^2 - 5}\right] \tag{4.3}$$

for $v > v_c$, where v_c is given by (4.2). All size and mass dependence, as well as that upon elastic constants, is contained in v_c and is, therefore, automatically included if v_c is treated as an adjustable parameter.

Figure 3 shows the predicted ε for several values of v_c, together with the fit of Hatzes *et al.* (1988) for $R = 2.5$-cm particles. As can be seen, Andrews' model does not come close to describing the experimental data for any value of v_c. The reason is that the only way to make collisions for $v > 0.2$ cm/sec more elastic is to increase v_c, but in doing so, the fit at small v gets worse and worse.

Tabor's Model

A quite different, more phenomenological approach has been made by Tabor (1948) (see also Bowden and Tabor 1954, Barkan 1972) based upon observations made in the laboratory under quasi-static conditions. First, as predicted by plastic deformation theory, a metallic sphere pressed into a flat anvil of softer metal with a sufficiently small force makes no permanent indention. Furthermore, when the force is increased, the sphere makes an indention (crater) of diameter d whose radius of curvature, R', is always greater than that of the sphere, R (Fig. 4). This greater radius of curvature for the crater suggests that the sphere first penetrates to a depth corresponding to R and that the crater than recovers to position R', the diameter d of the indention remaining fixed.

If the same force is applied a second time, the sphere is pushed to its original depth and returns to its original position, suggesting that recovery is made through residual elastic forces. By assuming that the total work done by dissipative (nonelastic) forces is p_c times the crater volume corresponding to R', where p_c is the mean pressure where plastic deformation occurs, and that the recovery from R to R' is done by Hertzian forces, Tabor finds a relation between v_f and v which can be written as (Barkan 1972)

$$v_f = K\left(v^2 - \frac{3}{8}v_f^2\right)^{3/8} \tag{4.4}$$

or, written $v_f = \varepsilon v$ (in this context, v is the speed, not velocity, of the particle),

$$\varepsilon = \frac{K}{v^{1/4}}\left(1 - \frac{3}{8}\varepsilon^2\right)^{3/8}, \tag{4.5}$$

where

$$K \propto p_c^{5/8}R^{3/8}/M^{1/8}, \tag{4.6}$$

R and M being the radius and mass of the sphere, respectively. An analogous situation for fracturing would be for the cracks to form and then partially close. It is easy to see that K, which is a free parameter, is closely related to v_c, the value of v in (4.5) when $\varepsilon = 1$. Indeed, inserting $v = v_c$ and $\varepsilon = 1$ in (4.5) gives

$$K = v_c^{1/4}/(5/8)^{3/8}, \tag{4.7}$$

allowing (4.5) to be rewritten as

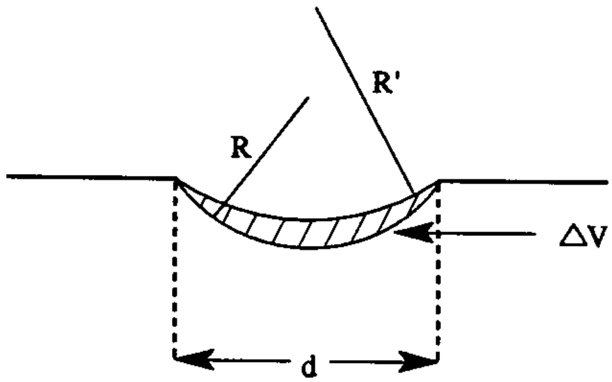

FIG. 4. Geometry of original crater and its subsequent recovery as used in Tabor's model. R is the radius of curvature of the original crater and R' that of the final crater.

$$\varepsilon = \left(\frac{v_c}{v}\right)^{1/4} \left(\frac{1 - 3\varepsilon^2/8}{1 - 3/8}\right)^{3/8}, \tag{4.8}$$

which has solutions $\varepsilon < 1$ for $v > v_c$. Note from (4.7) and (4.6) that

$$v_c \propto \sqrt{R^3/M}, \tag{4.9}$$

which is just (4.2) written for a single sphere and a flat, massive surface. Finally, the $\frac{3}{8}$ in (4.8) can be generalized to be a second parameter q

$$\varepsilon = \left(\frac{v_c}{v}\right)^{1-2q} \left(\frac{1 - q\varepsilon^2}{1 - q}\right)^{q}, \tag{4.10}$$

which reduces to (4.8) when $q = \frac{1}{2}$ and has solutions $\varepsilon < 1$ for $v > v_c$ when $q < \frac{1}{2}$. This new parameter, q, which in Tabor's (1948) original work arises from allowing p_c to vary as some power of the crater diameter, d, gives considerable additional flexibility in fitting experimental data. Nevertheless, (4.8) still has great difficulty in fitting the Hatzes *et al.* (1988) data for ice over the entire observational range of v. Thus, with $v_c = 0.0809$ cm/sec, $q = 0.371$ (curve c, Fig. 1a), ε fits the $R = 2.5$ cm data reasonably well for $v > 0.2$ cm/sec. However, v_c is obviously too large, and reducing it to a value required by the low energy data produces too much inelasticity at larger v (curve d with $v = 0.0134$ cm/sec and $q = 0.42$). This exercise shows that the difficulty which Hatzes *et al.* (1988) had in fitting the velocity data over the entire range is a real one.

The biggest difficulty with (4.8), though, is that $v_c \propto R^{3/2}$ and ε increases far too rapidly with size. Thus, $v_c = 0.0134$ for $R = 2.5$ cm, which gives a too-inelastic ε, becomes $v_c = 0.107$ for $R = 10$ cm, which gives a too-elastic ε (curve d, Fig. 1c). A still smaller v_c is evidently required to satisfy size dependence constraints, but this reduction would make the fit for the 2.5-cm data still worse. The size problem appears to be a very fundamental one for plastic deformation theory when applied to the observed collisions of icy spheres. Whether Andrews' and Tabor's models apply also for brittle fracturing is, of course, unknown. It is quite clear, though, that existing models based upon the existence of a critical velocity v_c do not work in the present circumstances. As a corollary, there is no particular reason for believing that size and mass dependences of the coefficient of restitution should substantially cancel. In any event, we must turn to other models to adequately describe the data of Hatzes *et al.* (1988).

5. A VISCOUS DISSIPATION MODEL

If models employing a critical velocity do not work, one is drawn to alternatives in which energy dissipation goes to zero ($\varepsilon = 1$) only as the impact velocity goes to zero. Of the various possibilities, the one focused on here is velocity-dependent dissipation, such as in viscous damping. Hatzes *et al.* (1988) pointed out a number of reasons for thinking that a significant amount of energy is dissipated within surface frosts. It seems quite conceivable that such frosts, as contrasted to a crystalline solid, might be well described by an effective viscosity. Here, we will show that a model using a linear, velocity (i.e., strain rate) dependent force, such as in a Kelvin–Voigt, viscoelastic solid (Goldsmith 1960), matches both the velocity and the size dependences of the experimental data quite well. The model is phenomenological, with input both from experiment and from Hertz theory, and accommodates both size and mass effects. At present, unfortunately, there is no experimental data on mass effects to determine the mass dependence.

Damped Harmonic Oscillator

To begin the development, we assume, in (3.1), a force of the form

$$\begin{aligned} F(\alpha) &= k\alpha + \beta\dot{\alpha} \quad (\alpha > 0) \\ &= 0 \quad (\alpha < 0). \end{aligned} \tag{5.1}$$

In the dissipative case, $\beta > 0$, $F(\alpha)$ becomes attractive during a part of the return cycle. Although this seems unphysical, ice frosts frequently show just such (sticking) forces (Hatzes *et al.* 1991).

With $\beta = 0$, the first term leads to a half-cycle of simple harmonic motion with frequency ω_0,

$$\omega_0 = \sqrt{k/\overline{M}}. \tag{5.2}$$

It would be better, of course, to use the Hertz force (3.3) instead of $k\alpha$, but this leads to an equation having no analytic solution. However, some results from Hertz theory can be inserted by hand, as will be seen later.

The second, velocity-dependent term, gives rise to energy loss through viscous dissipation at a rate controlled by the constant β. Again, although a more general form of velocity dependence might be preferable, the linear form should be adequate for small $\dot{\alpha}$. Using (5.1), (3.1) has an exact solution (Brach 1991),

$$\alpha = \frac{v}{\omega} e^{-t/2\tau} \sin(\omega t), \tag{5.3}$$

which gives an exponentially damped motion characterized by

$$\tau = \overline{M}/\beta \tag{5.4}$$

and a modified frequency

$$\omega = \omega_0 \sqrt{1 - 1/(2\omega_0\tau)^2}. \quad (5.5)$$

The solution has been chosen such that $\alpha(0) = 0$ and $\dot{\alpha}(0) = v$. An impact begins when $t = 0$ and ends when $t = \pi/\omega$. Using $v_f = \dot{\alpha}(\pi/\omega) = -\varepsilon v$, we find

$$\varepsilon = e^{-\pi/2\omega\tau}. \quad (5.6)$$

As it stands, however, (5.6) does not reflect the fact that an impact has taken place. That is, as far as (5.3) is concerned, a spring could have been stretched and released, accelerating a mass $\overline{M}$ to a speed v at that time $t = 0$. To simulate an impact, we make β depend upon the impact velocity v, $\beta = \beta_0 v^p$, equivalent to

$$\tau = \tau_0 v^{-p}, \quad (5.7)$$

which makes ε approach unity as v goes to zero. As will be seen, this simple extension of the model makes it account remarkably well for the unusual velocity dependence of the coefficient of restitution.

Two features of (5.6) are worth noting. First, although energy dissipation is initiated through β via τ, the amount of dissipation is strongly influenced by the undamped frequency ω_0. The reason is that ω_0 influences the duration of the impact, and hence the time during which energy is lost. Second, the constants ω_0 and τ_0 appear only as a product, so that ε depends only on the parameters $\omega_0\tau_0$ and the power law index of the velocity dependence, p. Thus, it is not possible to determine individually either ω_0 or τ_0 by fitting (5.6) to experimental data. Conversely, obtaining ω_0, say, from some model is of no help in fitting the data unless τ_0 is also determined. On the other hand, inserting a change in ω_0 from a model will automatically cause a corresponding change in $\omega_0\tau_0$, whatever the original value of that constant might be. Since ε depends only on $1/(2\omega_0\tau)$, it is convenient to introduce a new variable

$$\xi = 1/(2\omega_0\tau) = \xi_0 v^p, \quad (5.8)$$

where $\xi_0 = 1/(2\omega_0\tau_0)$. Inserting (5.8) into (5.6) gives

$$\varepsilon = e^{-\pi\xi/\sqrt{1-\xi^2}}. \quad (5.9)$$

From (5.9), one can also write the inverse relation

$$\xi = \sqrt{\frac{(\ln \varepsilon)^2}{\pi^2 + (\ln \varepsilon)}}. \quad (5.10)$$

The physical range of ξ is $0 \leq \xi < 1$, with ε going to unity as ξ approaches zero and ε going to zero and ξ approaches unity. From (5.8), we note that $\xi = 1$, hence $\varepsilon = 0$, at some finite v. This prediction may well indicate a breakdown of the model at sufficiently large v—but there is nothing in the existing data to exclude this interesting prediction either. Whatever the physical behavior of ε at large v turns out to be, our model should have no difficulties for the range of velocities being considered here.

Using (5.9) and (5.10), it is easy to determine ξ_0 and p for any set of experimental data from two data points. Thus, specifying ε, hence ξ, at $v = 1$ cm/sec determines ξ_0 in (5.8) at once, so that p is determined specifying by ε at any other v. For definiteness, we have fit the $R = 2.5$ cm data of Fig. 1a by choosing ε at $v = 0.2$ cm/sec and $v = 1$ cm/sec to obtain

$$\xi_0 = 0.16, \quad p = 0.65. \quad (5.11)$$

Results are shown in curve b. Agreement with the data is, overall, quite good, and there is no real problem reconciling the regions $v > 0.2$ cm/sec and $v < 0.2$ cm/sec.

Size Dependence

So far, the colliding particles have been treated as point masses, with no account being taken of a radially distributed force. To take account of size effects, we now assume that the size dependence of the undamped (elastic) frequency, ω_0, is given by the Hertz model. Note that the actual value of ω_0 is not needed, nor is there any advantage in using it. Specifically, we need only, from (3.6),

$$\omega_0 \propto (1/t_{\max}) \propto \overline{R}^{0.2}, \quad (5.12)$$

where, as before, $\overline{R}$ is given by (3.4). When discussing the Hatzes *et al.* (1988) data, $\overline{R}$ is the radius of the colliding sphere, but the expression (5.12) is quite general. According to (3.6), ω_0 also has a v dependence, but since this has already been incorporated into τ via the parameter p (Eq. (5.7)), this factor can be ignored. The R dependence, though, is new. Since $\xi \propto 1/\alpha_0\alpha\overline{R}^{-0.2}$, one has

$$\xi = \xi_0 r^{-0.2} v^p \quad (5.13)$$

where

$$r = \overline{R}/(2.5 \text{ cm}), \quad (5.14)$$

which agrees with (5.8) for $\overline{R} = 2.5$ cm.

Results, with $\overline{R} = R$ in (5.14), are shown in Figs. 1b–1d for $R = 5$ cm, $R = 10$ cm, and $R = 20$ cm. It is remarkable that the size dependence of the Hertz model, when applied to different models, is expressed so differently. For plastic deformation models, size enters through v_c as $R^{3/2}$ and increases elasticity much too rapidly. For our viscous dissipation model, it enters through ω_0 as $R^{-0.2}$ and accommodates the experimentally observed size dependence from $R = 2.5$ cm to $R = 20$ cm, a factor of eight. Instead of the 16 parameters used by Hatzes *et al.* (1988), both velocity and size dependence are accommodated by only 2.

Mass Dependence

The mass dependence of ε, unfortunately, is experimentally unconstrained since Hatzes *et al.* (1988) did not vary the mass in their experiments. It has been generally assumed that mass dependence is largely cancelled by size dependence, as in Eq. (4.9). However, the apparent discrepancy between the size dependence found by Hatzes *et al.* (1988) with that of (4.2), together with the utter failure of plastic deformation models to resolve the discrepancy, suggests that cancellation may not occur for icy spheres: whether this is actually the case, of course, must be determined by experiment. How this might be done will be discussed later, and for the moment we will consider the implications of noncanceling size and mass dependences, ignoring temporarily such complications as pendulums and effective vs actual particle mass.

In our model, ε is determined completely by $\xi = 1/(2\omega_0\tau)$. Since the Hertz model, Eqs. (5.12) and (3.6), gives a specific mass dependence to ω_0, $\omega_0 \propto \overline{M}^{-0.4}$, any further mass dependence must come from $\tau = \overline{M}/\beta$. Ostensibly, this gives $\tau \propto \overline{M}$ hence $(\omega_0\tau) \propto \overline{M}^{0.6}$, but this is unlikely to be correct. The reason is that the entire inertial mass $\overline{M}$ is unlikely to contribute to the energy dissipation process: more likely, dissipation is weakened to some extent, so that we have $\omega_0\tau \propto \overline{M}^{\kappa}$, hence $\xi \propto \overline{M}^{-\kappa}$, with $\kappa < 0.6$. This weakening can be parameterized by giving β a power law mass dependence in $\overline{M}$, analogous to that leading to a power law dependence $\tau \propto v^{-p}$ in (5.7). Using $\xi \propto \overline{M}^{-\kappa}$, (5.13) generalized to

$$\xi = \xi_0 r^{-0.2} m^{-\kappa} v^{p}, \tag{5.15}$$

where

$$m = \overline{M}/60\,g \tag{5.16}$$

and 60 g is the mass of an $R = 2.5$-cm ice sphere. In the experimental situation of Hatzes *et al.* (1988), $\overline{R} = R$, $\overline{M} = M$, size and mass dependences cancel only if $\kappa = -0.2/3$. This is not impossible, of course, but there is no reason to expect it.

To get a feel for the implications of (5.15), it is best to rewrite the equation for the usual physical situation where $M \propto R^3$ for both colliding particles so that (3.4) and (3.2) become

$$\overline{R} = R_1/(1 + \delta), \quad \overline{M} = r_1^3/(1 + \delta^3), \tag{5.17}$$

where

$$\delta = R_1/R_2 \leq 1. \tag{5.18}$$

Inserting these expressions in (5.15) then gives

$$\xi = \xi_0 f(\delta, \kappa) r_1^{-3\kappa - 0.2}, \tag{5.19}$$

where

$$f(\delta, \kappa) = (1 + \delta)^{0.2}(1 + \delta^3)^{\kappa}, \tag{5.20}$$

and

$$r_1 = R_1/2.5\text{ cm}, \tag{5.21}$$

where R_1 is the radius of the smaller mass. In this reformulation, $f(\delta, \kappa)$, a factor of order unity, depends only upon the mass ratio, like v_c (Eq. (4.2)) in plastic deformation and fracture theory. Now, however, there is an additional factor $r_1^{-3\kappa-0.2}$ which depends explicitly upon the radius of the smaller particle and, for particles in planetary rings, can vary by orders of magnitude. Because of this extra factor, the size of the smaller colliding particle tends to be much more important than that of the larger particle. In particular, for $\kappa > 0$, collisions in which one particle is small tend to be far less elastic than when neither is small. Some representative results are shown in Fig. 5, where it is assumed that ξ_0 and p are given by (5.11). In Fig. 5a, no mass dependence ($\kappa = 0$) is assumed, leaving *only the size dependence* $r^{-0.2}$ *in* (5.15), *i.e., the size* dependence seen in the experiments of Hatzes *et al.* (1988). Also, for each value of R_1, the range of ε corresponding to changes in δ (the mass ratio) is shown. Even for this case, collisions involving $R = 0.25$-cm particles are much less elastic than $R = 2.5$-cm ones, no matter what the size of the larger particle. When a small amount of mass dependence, $\kappa = 0.1$, is added (Fig. 5b), the same conclusion applies in a greatly amplified form. The lesson is that in systems where size (and mass) vary by orders of magnitude, unless size and mass dependence cancel almost completely, size–mass quickly becomes a very important factor in determining the outcome of collisions.

New Experiments

To clarify present ambiguities and make explicit predictions about the dynamics of the broad size distribution rings, mass-dependent experiments should be carried out. The cleanest way of doing this is surely through free (or nearly free) mass measurements, e.g., dropping an ice sphere on an ice plate, thereby avoiding problems associated with having an effective mass. Unfortunately, the Earth's gravity makes it very difficult to produce and measure impact and recoil velocities of 1 cm/sec or less.

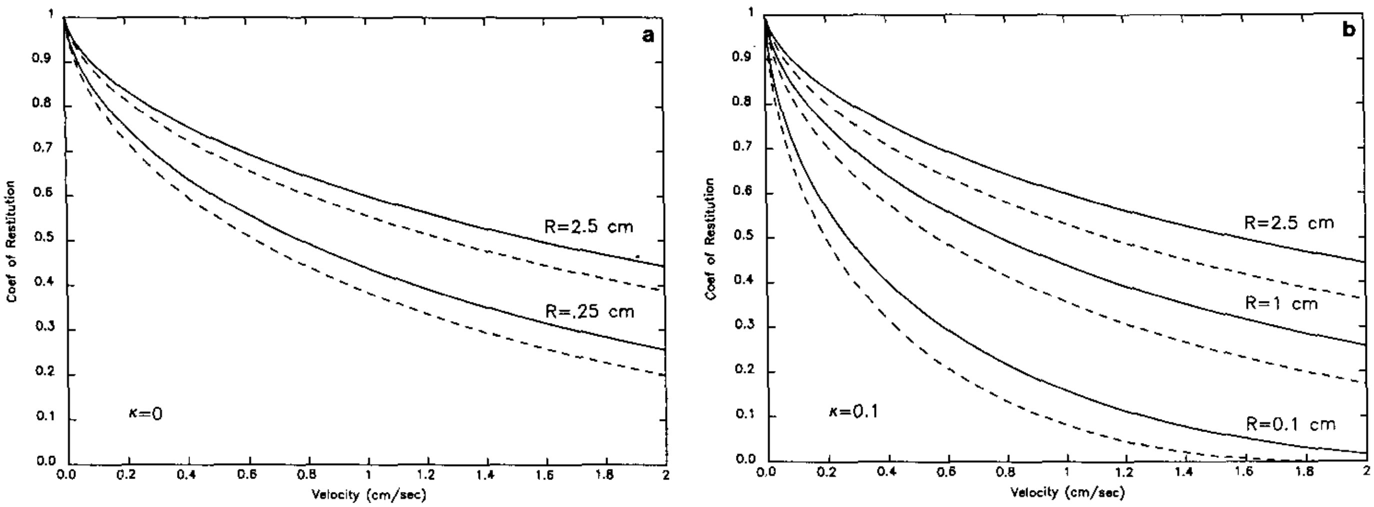

FIG. 5. The coefficient of restitution for spheres of different radii as predicted by the viscous dissipation model. ——, $\delta = 0$ (sphere and massive brick). ---, $\delta = 1$ (equal mass).

Thus, comprehensive data at very low velocities would be difficult to obtain except in a low-gravity environment, e.g., in a spacecraft or by using long, light pendulums.

The alternative is to do Hatzes *et al.* (1988) or Bridges *et al.* (1984) disc pendulum-type experiments. This greatly simplifies the low velocity problem, but only at the expense of always having an effective mass in a collision instead of the actual particle mass. Unfortunately, mass enters the picture in two different ways: through its inertial properties, as in (3.1), which includes all of the effective mass, and by being the media of energy dissipation, which, to at least some extent, is only the particle mass. To disentangle the effects, two types of experiment are probably necessary: the first, varying the moment of inertia of the pendulum while keeping particle mass fixed, to isolate and understand effective mass dependence; the second, keeping the moment of inertia fixed while varying the particle mass, to measure the dependence on the particle mass.

6. SUMMARY AND CONCLUSIONS

The experimental data of Hatzes *et al.* (1988), to be most useful, must be extended through experiments or extrapolated via models to include collisions of ring particles having arbitrary size and mass. In this paper, we have looked at some physically based models to see if they can describe the existing data and provide a plausible extrapolation.

We have examined two models which assume complete elasticity during impacts at less than some critical velocity, such as should occur if energy dissipation is by plastic deformation or brittle fracture. The size dependence of elastic scattering implies that the critical velocity should increase with the size of the colliding objects in a definite fashion. Neither model, however, even with all parameters taken as adjustable, can accommodate such an increase. An important corollary is that the cancellations of size and mass predicted by such models may well not occur for collisions of icy spheres.

As an alternative, we have developed a model based upon no finite region of elasticity and energy loss by visous dissipation. This model fits both the size and the velocity dependence of the coefficient of restitution found by Hatzes *et al.* (1988) using only two parameters. The model also accommodates a mass dependence, but further experiments must be done to determine the actual dependence, and thus the overall size–mass dependence, of physical particles. Lacking an efficient size–mass cancellation, the mass of the smaller colliding particle becomes very important in determining its coefficient of restitution during a collision.

So far, little attention seems to have been paid to the possibility that the coefficient of restitution has an appreciable size–mass dependence. Our work suggests that this may well not be the case, although further experiments are needed to settle the question. If there is an appreciable size–mass dependence, then the results of Hatzes *et al.* (1988) (or Bridges *et al.* 1984) may need to be modified significantly when applied to real ring systems.

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