Hydrodynamics of the Polydisperse Granular Disk

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I. THE MODEL

We consider a system of polydisperse granular gas in the Saturnian environment. The polydispersity means that the gas consists of various species of constituents with different masses. Distribution of species can be assumed to be continuous and having a distribution function $\eta(m)$, where m is the mass of a constituent in a specy. Here we consider the function $\eta(m)$ to be stationary and normalized to unity

$$\int \eta(m)\, dm = 1 \; . \tag{1}$$

In the Saturnian environment, the gas itself forms a very thin disk around the central gravitational force. Therefore, we will consider the granular gas to be two dimensional and with azimuthal symmetry. We will also neglect the self gravity of disk particles. The task we are investigating in this work is the effect of disbalance in granular temperatures among species and its possible influence to the dynamics of the ring particles. So, the main macroscopic parameters we are analyzing are the mass density of species $\rho(t, m, \boldsymbol{r})$, the mean momentum of species $\rho\boldsymbol{u}(t, m, \boldsymbol{r})$ and the granular temperature of species $T(t, m, \boldsymbol{r})$. The next assumption we make is that the granular gas is rarified, hence all the macroscopic functions can be defined by one particle distribution function $f(t, m, \boldsymbol{r}, \boldsymbol{v})$. The distribution function itslef obeys the kinetic Boltzmann equation.

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\frac{\partial f}{\partial \boldsymbol{r}} + \boldsymbol{w}\frac{\partial f}{\partial \boldsymbol{v}} = I_c(t, m, \boldsymbol{r}, \boldsymbol{v}) \; , \tag{2}$$

where $I_c(t, m, \boldsymbol{r}, \boldsymbol{v})$ is the binary collision integral integrated over the masses of species

$$I_c(t, m, \boldsymbol{r}, \boldsymbol{v}) = \int \eta(m') I_c(t, m', m, \boldsymbol{r}, \boldsymbol{v})\, dm' \; . \tag{3}$$

The exact form of the collision integral will be given further.

II. GENERALIZED TRANSPORT EQUATIONS

Let us first write down the transport equations for macroscopic paramteres without the specific assumptions of our considered model. To do that, we should make a hydrodynamic approximation. It is possible if in the system there exist a distance ℓ in the next scale scale $\sigma \ll \ell \ll L$, where σ is the size of a constituent, and L is the global size of the system. At the scale of ℓ the macroscopic parameters can be defined using the distribution function, and the transport processes of these parameters can be extended to the scale of the system L. Obviously, such scales can be easily defined in the Saturnian ring system, hence our hydrodynamic approximation is justified. Assuming that we have solved the Boltzmann equation (2) and know the function $f(t, m, \boldsymbol{r}, \boldsymbol{v})$, the macroscopic parameters can be obtained as the velocity moments of this function.

The zeros moment gives us the number density

$$n(t, m, \boldsymbol{r}) = \int f(t, m, \boldsymbol{r}, \boldsymbol{v}) \, d\boldsymbol{v} \ , \tag{4}$$

or the mass density

$$\rho(t, m, \boldsymbol{r}) = mn(t, m, \boldsymbol{r}) = \int mf(t, m, \boldsymbol{r}, \boldsymbol{v}) \, d\boldsymbol{v} \ . \tag{5}$$

First moment gives us the momentum density

$$\rho\boldsymbol{u}(t, m, \boldsymbol{r}) = \int m\boldsymbol{v}f(t, m, \boldsymbol{r}, \boldsymbol{v}) \, d\boldsymbol{v} \ . \tag{6}$$

Now we can introduce the local velocity variable

$$\boldsymbol{c} = \boldsymbol{v} - \boldsymbol{u} \ , \tag{7}$$

and the granular temperature

$$\frac{D}{2} nT(t, m, \boldsymbol{r}) = \int \frac{m\boldsymbol{c}^2}{2} f(t, m, \boldsymbol{r}, \boldsymbol{v}) \, d\boldsymbol{v} \ , \tag{8}$$

where D is the dimension of the system. Usually it is equal to 3, but in our case we will use $D = 2$. Further we will also need the next moments, which are in fact tensors. First of all it is the *stress* tensor

$$\Pi_{ij}(t, m, \boldsymbol{r}) = m \int v_i v_j f(t, m, \boldsymbol{r}, \boldsymbol{v}) \, d\boldsymbol{v} \ , \tag{9}$$

which can be split into two parts using (7)

$$\Pi_{ij}(t, m, \boldsymbol{r}) = \rho u_i u_j + P_{ij} \ . \tag{10}$$

The first term is the dynamic part of the stress tensor. The second part is hence the *internal stress* tensor

$$P_{ij}(t, m, \boldsymbol{r}) = m \int c_i c_j f(t, m, \boldsymbol{r}, \boldsymbol{v}) \, d\boldsymbol{v} \ . \tag{11}$$

Splitting it further into a traceless tensor, we can write

$$P_{ij} = \delta_{ij} p_{id} + \pi_{ij} \ , \quad \pi_{ii} = 0 \ , \tag{12}$$

where

$$p_{id} = \frac{1}{D} \int m\boldsymbol{c}^2 f(t, m, \boldsymbol{r}, \boldsymbol{v})\, d\boldsymbol{v} = nT \ , \tag{13}$$

is the pressure of an ideal gas. The traceless tensor π_{ij} is the so called *viscous stress* tensor.

Finally, we will also need the tensor

$$Q_{ijk} = m \int c_i c_j c_k f(t, m, \boldsymbol{r}, \boldsymbol{v})\, d\boldsymbol{v} \ , \tag{14}$$

or more specifically its trace over two indices

$$q_i = \boldsymbol{q} = \frac{1}{2} Q_{ijj} = \int \frac{m\boldsymbol{c}^2}{2} c_i f(t, m, \boldsymbol{r}, \boldsymbol{v})\, d\boldsymbol{v} \ , \tag{15}$$

which is the *heat flow* vector.

Using all the abovementioned definitions, we can now write the transport equations for mass macroscopic parameters. To do that, we can use the Boltzmann equation (2), or to be more specific, multiply it with a certain function of the dynamic variables $A(t, m, \boldsymbol{r}, \boldsymbol{v})$ and integrate over the velocity space.

$$\int A \frac{\partial f}{\partial t}\, d\boldsymbol{v} + \int A \boldsymbol{v} \frac{\partial f}{\partial \boldsymbol{r}}\, d\boldsymbol{v} + \int A \boldsymbol{w} \frac{\partial f}{\partial \boldsymbol{v}}\, d\boldsymbol{v} = \int A I_c(f, f')\, d\boldsymbol{v} \ = \left\langle \frac{\partial A}{\partial t} \right\rangle_c , \tag{16}$$

where the right hand side is the average change of the dynamic function over time due to collisions. Further we can write

$$\int \left(\frac{\partial}{\partial t}(Af) + \frac{\partial}{\partial \boldsymbol{r}}(Af\boldsymbol{v}) + \frac{\partial}{\partial \boldsymbol{v}}(Af\boldsymbol{w}) - f \left[\frac{\partial A}{\partial t} + \boldsymbol{v} \frac{\partial A}{\partial \boldsymbol{r}} + \boldsymbol{w} \frac{\partial A}{\partial \boldsymbol{v}} \right] \right) d\boldsymbol{v} = \left\langle \frac{\partial A}{\partial t} \right\rangle_c . \tag{17}$$

The third term here can be transformed using Gauss divergence theorem into

$$\int \frac{\partial}{\partial \boldsymbol{v}}(Af\boldsymbol{w})\, d\boldsymbol{v} = \oint Af\boldsymbol{w} \cdot d\boldsymbol{\sigma} = 0 \ , \tag{18}$$

where the contour integration is performed over the boundaries of the dynamic variables, here namely at $v \to \pm\infty$, and vanishes there. This is due to the distribution function f, which always vanishes at this limits. So, finally we are left with the next equation

$$\frac{\partial}{\partial t} \int Af\, d\boldsymbol{v} + \frac{\partial}{\partial \boldsymbol{r}} \int Af\boldsymbol{v}\, d\boldsymbol{v} - \int f \left[\frac{\partial A}{\partial t} + \boldsymbol{v} \frac{\partial A}{\partial \boldsymbol{r}} + \boldsymbol{w} \frac{\partial A}{\partial \boldsymbol{v}} \right] d\boldsymbol{v} = \left\langle \frac{\partial A}{\partial t} \right\rangle_c . \tag{19}$$

Now we can start writing the specific transport equations.

A. Continuity equation

Putting $A(t, m, \boldsymbol{r}, \boldsymbol{v}) = m$ in (19) we have

$$\frac{\partial}{\partial t}\int mf\, d\boldsymbol{v} + \frac{\partial}{\partial \boldsymbol{r}}\int mf\boldsymbol{v}\, d\boldsymbol{v} = 0\ , \tag{20}$$

and from (5), (6) we have the well known continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}}(\rho \boldsymbol{u}) = 0\ . \tag{21}$$

B. Momentum balance equation

Now, putting $A(t, m, \boldsymbol{r}, \boldsymbol{v}) = m\boldsymbol{v} = mv_i$ we have

$$\frac{\partial}{\partial t}\int m\boldsymbol{v}f\, d\boldsymbol{v} + \frac{\partial}{\partial r_j}\int mfv_iv_j\, d\boldsymbol{v} - \boldsymbol{w}\int mf\, d\boldsymbol{v} = \left\langle \frac{\partial(mv_i)}{\partial t}\right\rangle_c\ , \tag{22}$$

and using (9) we have

$$\frac{\partial(\rho\boldsymbol{u})}{\partial t} + \frac{\partial \Pi_{ij}}{\partial r_j} = \rho\boldsymbol{w}\ . \tag{23}$$

The collisional term vanishes due to the momentum conservation law. The right hand side $\rho\boldsymbol{w} = \boldsymbol{F}_{ext}$ is the density of the external forces acting on our system. In this work we take the external force to be the central gravitational field force $\boldsymbol{w} = -\dfrac{\partial U(r)}{\partial \boldsymbol{r}}$, where r is the distance to the field source. Splitting the stress tensor as we did before we have the equation for the momentum density transport

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial r_j}(\rho u_i u_j) = -\frac{\partial(nT)}{\partial r_i} - \frac{\partial \pi_{ij}}{\partial r_j} + \rho w_i\ . \tag{24}$$

C. Kinetic energy balance equation

Now we put $A(t, m, \boldsymbol{r}, \boldsymbol{v}) = \dfrac{m\boldsymbol{v}^2}{2}$ and have

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\int m(\boldsymbol{c}^2 + 2u_ic_i + \boldsymbol{u}^2)f\, d\boldsymbol{v} + \frac{1}{2}\frac{\partial}{\partial \boldsymbol{r}}\int m(\boldsymbol{c}^2 + 2u_ic_i + \boldsymbol{u}^2)v_jf\, d\boldsymbol{v} -\\ &- w_j\int mfv_i\frac{\partial v_i}{\partial v_j}\, d\boldsymbol{v} = \left\langle \frac{\partial}{\partial t}\left(\frac{m\boldsymbol{v}^2}{2}\right)\right\rangle_c = -T\xi(t, m, \boldsymbol{r}, T)\ .\end{aligned} \tag{25}$$

Due to the dissipative nature of the granular gases, the energy conservation law does not hold in this case. Hence, the average change of the kinetic energy does not vanish and can be written as shown in the right hand side of the above equation. The term $\xi(t, m, \boldsymbol{r}, T)$ is the cooling term. Proceeding further we have

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int \frac{m\boldsymbol{c}^2}{2} f\, d\boldsymbol{v} + \frac{\partial}{\partial t}\int \frac{m\boldsymbol{u}^2}{2} f\, d\boldsymbol{v} + \frac{\partial}{\partial r_j}\int \frac{m\boldsymbol{c}^2}{2} v_j f\, d\boldsymbol{v} + \frac{\partial}{\partial r_j}\int m f u_i c_i v_j\, d\boldsymbol{v} + \\
&+ \frac{\partial}{\partial r_j}\int \frac{m\boldsymbol{u}^2}{2} v_j f\, d\boldsymbol{v} = \delta_{ij} w_j \rho u_i - T\xi = u_i \cdot \rho w_i - T\xi \; .
\end{aligned} \tag{26}
$$

Here we used the fact that $\int c_i f\, d\boldsymbol{v} = 0$. Further we have

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{D}{2} nT + \frac{\rho \boldsymbol{u}^2}{2}\right) + \frac{\partial}{\partial r_j}\int \frac{m\boldsymbol{c}^2}{2}(c_j + u_j)\, d\boldsymbol{v} + \frac{\partial}{\partial r_j}\int m f u_i c_i (c_j + u_j)\, d\boldsymbol{v} + \\
&+ \frac{\partial}{\partial r_j}\left(\frac{\rho \boldsymbol{u}^2}{2} u_j\right) = \rho \boldsymbol{w}\cdot\boldsymbol{u} - T\xi \; ,
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{D}{2} nT + \frac{\rho \boldsymbol{u}^2}{2}\right) + \frac{\partial q_i}{\partial r_i} + \frac{\partial}{\partial r_i}\left(\frac{D}{2} u_i nT\right) + \frac{\partial}{\partial r_i} u_j \int m f c_i c_j\, d\boldsymbol{v} + \\
&+ \frac{\partial}{\partial r_i}\left(\frac{\rho \boldsymbol{u}^2}{2} u_i\right) = \rho \boldsymbol{w}\cdot\boldsymbol{u} - T\xi \; ,
\end{aligned} \tag{28}
$$

and using the definition of the internal stress tensor we end up with

$$
\frac{\partial}{\partial t}\left(\frac{D}{2} nT + \frac{\rho \boldsymbol{u}^2}{2}\right) + \frac{\partial}{\partial r_i}\left(\frac{D}{2} u_i nT + \frac{\rho \boldsymbol{u}^2}{2} u_i + \delta_{ij} u_j nT + \pi_{ij} u_j + q_i\right) = \rho \boldsymbol{w}\cdot\boldsymbol{u} - T\xi \; , \tag{29}
$$

and finally we have the energy balance equation

$$
\frac{\partial}{\partial t}\left(\frac{D}{2} nT + \frac{\rho \boldsymbol{u}^2}{2}\right) + \frac{\partial}{\partial r_i} u_i \left(\frac{D+2}{2} nT + \frac{\rho \boldsymbol{u}^2}{2}\right) + \frac{\partial}{\partial r_i}(\pi_{ij} u_j) + \frac{\partial q_i}{\partial r_i} = \rho \boldsymbol{w}\cdot\boldsymbol{u} - T\xi \; . \tag{30}
$$

D. Heat flow vector and Viscous stress tensor

In the balance equations we have written above, there are also the unknown terms as q_i the heat flow vector and π_{ij} the viscous stress tensor. Before we have introduced the hydrodynamic scale ℓ at which we are investigating the system. Let us now introduce a small parameter $\varepsilon = \ell/L$ around which we can use series expansion of all the macroscopic parameters. Note that at zeros order expansion around ε both q_i and π_{ij} vanish, leaving a system of an ideal gas without internal molecular interaction. One can also show that at this level of approximation, the distribution function is the well known Maxwell function

$$
f^{(0)}(t, m, \boldsymbol{r}, \boldsymbol{v}) = \frac{n}{(2\pi m T)^{D/2}} \exp\left\{-\frac{m(\boldsymbol{v} - \boldsymbol{u})^2}{2T}\right\} \; , \tag{31}
$$

where the superscript (0) denotes the zeros order approximation. The heat flow vector and the viscous stress tensor in the first order approximation $\pi_{ij}, q_i \sim \varepsilon$ can be written as

$$\pi_{ij} = -\nu \left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i} - \frac{2}{D}\delta_{ij}\frac{\partial u_k}{\partial r_k} \right) , \tag{32}$$

$$q_i = -\lambda \, \mathrm{grad}\, T , \tag{33}$$

where ν is the coeffitient of viscosity and λ is the coeffitient of heat conductivity.

III. AZIMUTHALLY SYMMETRIC FLAT DISK

After obtaining general transport equations for macroscopic parameters, let us now apply them into our problem of two dimensional azimuthally symmetric disk with a central gravitational forcing. It is convenient to switch into polar coordinates $x, y \to r, \theta$, where r is the distance to the potential field center and θ is the azimuthal position. Now our distribution function and hence all the macroscopic variable are the functions of these two variables and their time derivatives. However, the azimuthal symmetry of the disk means that the distribution function and all macroscopic variables do not depend on θ, and we can write

$$f = f(t, m, r, \dot{r}, \dot{\theta}) . \tag{34}$$

Next thing to consider is that the potential function has the form

$$U(r) = -\frac{GM_P}{r} , \tag{35}$$

where M_P is the mass of the planet and G is the gravitational constant. The acceleration in this field is given by

$$\boldsymbol{w} = -\frac{\partial U(r)}{\partial r}\frac{\boldsymbol{r}}{r} = -\frac{GM_P}{r^2}\frac{\boldsymbol{r}}{r} . \tag{36}$$

Introducing Keplerian angular velocity

$$\Omega(r) = \sqrt{\frac{GM_P}{r^3}} , \tag{37}$$

we can write

$$\boldsymbol{w} = -\Omega^2 r \cdot \boldsymbol{e}_r , \tag{38}$$

and

$$\boldsymbol{u} = \Omega r \cdot \boldsymbol{e}_\theta , \tag{39}$$

where $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are the unit vectors of the new non-inertial polar coordinate system. Also, the ∇ vector in polar coordinates reads

$$\nabla = \frac{\partial}{\partial r_i} = \frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\boldsymbol{e}_\theta \; . \tag{40}$$

Now, let us rewrite the continuity equation (21)

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho\Omega r)}{\partial \theta} = 0 \; , \tag{41}$$

and this leads to the obvious result for our system

$$\frac{\partial \rho}{\partial t} = 0 \; , \tag{42}$$

that the mass density is stationary. To proceed further, let us first write the tensor values in details. First tensor is

$$\rho u_i u_j = \rho \begin{pmatrix} 0 \\ \Omega r \end{pmatrix} \circ \begin{pmatrix} 0 & \Omega r \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \rho\Omega^2 r^2 \end{pmatrix} \; , \tag{43}$$

where $\circ$ denotes the outer product of two vectors. Now we have

$$\frac{\partial}{\partial r_i}(\rho u_i u_j) = \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \rho\Omega^2 r^2 \end{pmatrix} = 0 \; . \tag{44}$$

The viscous-stress tensor (32) reads

$$\pi_{ij} = -\nu \begin{pmatrix} 0 & \frac{\partial(\Omega r)}{\partial r} \\ \frac{\partial(\Omega r)}{\partial r} & 0 \end{pmatrix} \; , \tag{45}$$

and since

$$\frac{\partial(\Omega r)}{\partial r} = -\frac{\Omega}{2} \; , \tag{46}$$

we have

$$\pi_{ij} = \frac{1}{2} \begin{pmatrix} 0 & \nu\Omega \\ \nu\Omega & 0 \end{pmatrix} \; . \tag{47}$$

Its spacial derivative reads

$$\frac{\partial \pi_{ij}}{\partial r_i} = \frac{1}{2} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \end{pmatrix} \begin{pmatrix} 0 & \nu\Omega \\ \nu\Omega & 0 \end{pmatrix} \; , \tag{48}$$

and since

$$\frac{\partial\Omega}{\partial r} = -\frac{3}{2}\frac{\Omega}{r} , \tag{49}$$

we have

$$\frac{\partial\pi_{ij}}{\partial r_i} = -\frac{3}{4}\frac{\nu\Omega}{r}\boldsymbol{e}_\theta . \tag{50}$$

Now we can rewrite the momentum transport equation (24) as

$$\rho\frac{\partial\boldsymbol{u}}{\partial t} = -\nabla(nT) + \frac{3}{4}\frac{\nu}{r^2}\boldsymbol{u} - \rho\nabla U . \tag{51}$$

Let us focus on the kinetic energy balance equation (30). In the two dimensional case $D = 2$, and the corresponding tensors read

$$\pi_{ij}u_j = \frac{1}{2}\begin{pmatrix} 0 & \nu\Omega \\ \nu\Omega & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \Omega r \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \nu\Omega^2 r & 0 \end{pmatrix} , \tag{52}$$

and therefore

$$\frac{\partial(u_j\pi_{ij})}{\partial r_i} = -\nu\Omega^2 . \tag{53}$$

The spacial derivatives of the second bracket terms vanish and the last term in the left hand side reads

$$\frac{\partial q_i}{\partial r_i} = -\lambda\Delta T . \tag{54}$$

Since the Laplace operator in polar coordiantes is

$$\Delta T = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2} , \tag{55}$$

finally we can write the temperature balance equation as

$$n\frac{\partial T}{\partial t} + \rho\boldsymbol{u}\frac{\partial\boldsymbol{u}}{\partial t} = \nu\Omega^2 + \lambda\frac{\partial^2 T}{\partial r^2} + \frac{\lambda}{r}\frac{\partial T}{\partial r} - T\xi . \tag{56}$$

Multiplying equation (51) by $\boldsymbol{u}$ we can replace the second term and arrive at

$$n\frac{\partial T}{\partial t} = \boldsymbol{u}\nabla(nT) - \frac{3}{4}\frac{\nu u^2}{r^2} + \rho\boldsymbol{u}\nabla U + \nu\Omega^2 + \lambda\frac{\partial^2 T}{\partial r^2} + \frac{\lambda}{r}\frac{\partial T}{\partial r} - T\xi . \tag{57}$$

Since the vector $\boldsymbol{u}$ is always azimuthally directed, its scalar product by the radial field gradients vanish, so we have

$$n\frac{\partial T}{\partial t} = \frac{\nu\Omega^2}{4} + \lambda\frac{\partial^2 T}{\partial r^2} + \frac{\lambda}{r}\frac{\partial T}{\partial r} - T\xi . \tag{58}$$

We see that the temperature evolution depends on the viscous shear energy gain $\propto \nu\Omega^2$ which is always positive and pumps energy into the system. Also, it depends on the dissipative energy loss term $-T\xi$ which will be discussed in details later. And is also depends on the radial heat flow terms if we assume that temperature is not homogenious in radial direction. Looking at equation (58) we can already tell that the stationalry solution $\dot{T} = 0$ can be obtained at nonzero values of temperature, since the shear gain does not depend on temperature.

IV. TEMPERATURE COOLING RATE

Let us now analyze the cooling term ξ. In order to do that, we need to calculate the collisional part of the energy transfer equation. Since

$$-T\xi(t, m, \boldsymbol{r}, T) = \int_0^\infty dm' \int \frac{m'\boldsymbol{v}^2}{2} I_c(t, m', m, \boldsymbol{r}, \boldsymbol{v})\, d\boldsymbol{v} \ , \tag{59}$$

we need to know the collision integral $I_c(t, m', m, \boldsymbol{r}, \boldsymbol{v})$. The exact form of the collision integral depends on the microscopic details of the collision mechanism. Therefore, let us first establish the collision laws of granular particles in our system.

A. Collision mechanism

In order to be able to do any plausible analysis, we assume all the consituents to be perfectly spherical, or disks in two dimensions, with various radii $\sigma_\alpha \propto m_\alpha^{1/3}$, where by lowercase greek letters $\alpha, \beta, \ldots$ we denote the species. The momentum conservation law states

$$m_\alpha \boldsymbol{v}_\alpha + m_\beta \boldsymbol{v}_\beta = m_\alpha \boldsymbol{v}'_\alpha + m_\beta \boldsymbol{v}'_\beta \ , \tag{60}$$

where primed velocities are the postcollisional velocities. The energy conservation law does not hold in a granular system, and the change of the postcollisional velocities is given by the restitution coeffitient

$$\boldsymbol{g}' = -\varepsilon \boldsymbol{g} \ . \tag{61}$$

We assumed here that the tangential part of the relative velocity $\boldsymbol{g} = \boldsymbol{g}_{\alpha\beta} = \boldsymbol{v}_\alpha - \boldsymbol{v}_\beta$ does not change, and the normal part changes its sign and decreases according to the restitution coeffitient $0 < \varepsilon < 1$. We do not consider the case of complete elasticity $\varepsilon = 1$ and the case of sticky spheres $\varepsilon = 0$. Defining the normal direction of the collision as the line connecting the centers of two spheres, its unit vector reads

$$\boldsymbol{n} = \boldsymbol{n}_{\alpha\beta} = \frac{\boldsymbol{r}_\alpha - \boldsymbol{r}_\beta}{\sigma_\alpha + \sigma_\beta} \ , \tag{62}$$

where $\boldsymbol{r}_\alpha$ and $\boldsymbol{r}_\beta$ are the positions of the constituents at the moment of collision. Now the postcollisional velocities can be written as

$$\begin{aligned} \boldsymbol{v}'_\alpha &= \boldsymbol{v}_\alpha - \frac{\mu}{m_\alpha}(1+\varepsilon)(\boldsymbol{g}\cdot\boldsymbol{n})\boldsymbol{n} \ , \\ \boldsymbol{v}'_\beta &= \boldsymbol{v}_\beta + \frac{\mu}{m_\beta}(1+\varepsilon)(\boldsymbol{g}\cdot\boldsymbol{n})\boldsymbol{n} \ , \end{aligned} \tag{63}$$

where $\mu = \mu_{\alpha\beta} = \dfrac{m_\alpha m_\beta}{m_\alpha + m_\beta}$ is the reduced mass of the collision $\alpha\beta$. Here note that the vector $\boldsymbol{n}$ can be arbitrary and independent of the values of velocities.

Let us now estimate the changes of momentum and kinetic energy of a single particle after a collision.

$$\delta\boldsymbol{p}_\alpha = -\delta\boldsymbol{p}_\beta = \pm\mu(1+\varepsilon)(\boldsymbol{g}\cdot\boldsymbol{n})\boldsymbol{n} \ . \tag{64}$$

The actual sign of momentum transport depends on the configuration of $\boldsymbol{g}$ and $\boldsymbol{n}$, but the total change of momentum is obviously zero $\delta\boldsymbol{p}_\alpha + \delta\boldsymbol{p}_\beta = 0$.

Now let us look at the change of kinetic energy

$$\begin{aligned} \delta E_\alpha &= \frac{m_\alpha \boldsymbol{v}'^2_\alpha}{2} - \frac{m_\alpha \boldsymbol{v}^2_\alpha}{2} = -\mu(1+\varepsilon)(\boldsymbol{g}\cdot\boldsymbol{n})(\boldsymbol{v}_\alpha\cdot\boldsymbol{n}) + \frac{\mu^2}{2m_\alpha}(1+\varepsilon)^2(\boldsymbol{g}\cdot\boldsymbol{n})^2 \ , \\ \delta E_\beta &= \frac{m_\beta \boldsymbol{v}'^2_\beta}{2} - \frac{m_\beta \boldsymbol{v}^2_\beta}{2} = \mu(1+\varepsilon)(\boldsymbol{g}\cdot\boldsymbol{n})(\boldsymbol{v}_\beta\cdot\boldsymbol{n}) + \frac{\mu^2}{2m_\beta}(1+\varepsilon)^2(\boldsymbol{g}\cdot\boldsymbol{n})^2 \ . \end{aligned} \tag{65}$$

Introducing the center of mass reference with velocity $M\boldsymbol{v}_C = m_\alpha\boldsymbol{v}_\alpha + m_\beta\boldsymbol{v}_\beta$, where $M = M_{\alpha\beta} = m_\alpha + m_\beta$, we have

$$\begin{aligned} \boldsymbol{v}_\alpha &= \boldsymbol{v}_C + \frac{\mu}{m_\alpha}\boldsymbol{g} \ , \\ \boldsymbol{v}_\beta &= \boldsymbol{v}_C - \frac{\mu}{m_\beta}\boldsymbol{g} \ , \end{aligned} \tag{66}$$

and the kinetic energy change can be written in the next form

$$\begin{aligned} \delta E_\alpha &= -\mu(1+\varepsilon)(\boldsymbol{g}\cdot\boldsymbol{n})(\boldsymbol{v}_C\cdot\boldsymbol{n}) - \frac{1-\varepsilon^2}{2}\frac{\mu^2}{m_\alpha}(\boldsymbol{g}\cdot\boldsymbol{n})^2 \ , \\ \delta E_\beta &= +\mu(1+\varepsilon)(\boldsymbol{g}\cdot\boldsymbol{n})(\boldsymbol{v}_C\cdot\boldsymbol{n}) - \frac{1-\varepsilon^2}{2}\frac{\mu^2}{m_\beta}(\boldsymbol{g}\cdot\boldsymbol{n})^2 \ . \end{aligned} \tag{67}$$

We can see now that the total energy change is not zero and reads

$$\delta E_\alpha + \delta E_\beta = -\frac{1-\varepsilon^2}{2}\mu(\boldsymbol{g}\cdot\boldsymbol{n})^2 \ . \tag{68}$$

This is the amount of energy dissipated out of the system after a single collision. The higher the impact velocity, the more energy is dissipated. Looking at (67) we can say that the first terms are identical for both particles and have opposite signs. This is the portion of kinetic energy transported from one particle to another, and that amount of energy stays in the system and does not dissipate. The exact direction of energy transport depends on the configuration of vectors $\boldsymbol{g}$, $\boldsymbol{v}_C$ and $\boldsymbol{n}$. However the second terms are always negative and generally not identical. We see that these terms are responsible for the energy dissipation and for different masses of particles, the dissipated amounts of energy are not equal. Namely, the *higher* the mass of the colliding particle compared to its opponent, the *less* amount of energy it loses during the collision. This effect leads to the breakage of energy equipartition among different species of particles, and leads to non-equal granular temperatures among all various species.

B. Collision integral

After establishing the mechanical rules of a single collision, let us write the collision integral which appears in the right hand side of the Boltzmann equation (2). The collision integral accounts for the change in the one particle distribution function due to collisions. The actual form of the collision integral of inelastic gases reads

$$\begin{aligned} I_c(t, m_\alpha, \boldsymbol{r}, \boldsymbol{v}_\alpha) = &\int dm_\beta g_2(\sigma_{\alpha\beta})\eta(m_\beta)\sigma_{\alpha\beta}^{D-1}\int d\boldsymbol{v}_\beta \int d\boldsymbol{n}\Theta(-\boldsymbol{g}\cdot\boldsymbol{n})|\boldsymbol{g}\cdot\boldsymbol{n}|\times \\ &\times\left(\frac{1}{\varepsilon^2}f(t, m_\alpha, \boldsymbol{r}, \boldsymbol{v}_\alpha'')f(t, m_\beta, \boldsymbol{r}, \boldsymbol{v}_\beta'') - f(t, m_\alpha, \boldsymbol{r}, \boldsymbol{v}_\alpha)f(t, m_\beta, \boldsymbol{r}, \boldsymbol{v}_\beta)\right) \ , \end{aligned} \tag{69}$$

where $\sigma_{\alpha\beta} = \sigma_\alpha + \sigma_\beta$ is the distance between center of colliding particles, $g_2(\sigma_{\alpha\beta})$ is the Enskog factor which accounts for the difference in the locations of centers of particles at the collision which will be set to unity further $g_2(\sigma_{\alpha\beta}) = 1$, $\Theta(x)$ is the Heaviside step function which is included in order to account only the approaching particles and $\boldsymbol{v}_\alpha'', \boldsymbol{v}_\beta''$ are the velocities of the inverse collision. There is an important property of the collision integral, which states that given a function $\psi_\alpha(\boldsymbol{v}_\alpha)$, and its change due to a direct collision

$\Delta\psi_\alpha(\boldsymbol{v}_\alpha) = \psi_\alpha(\boldsymbol{v}'_\alpha) - \psi_\alpha(\boldsymbol{v}_\alpha)$ we can write

$$\begin{aligned}\frac{d}{dt}\langle\psi_\alpha(\boldsymbol{v}_\alpha)\rangle_c = \int \psi_\alpha(\boldsymbol{v}_\alpha)\frac{\partial f_\alpha}{\partial t}d\boldsymbol{v}_\alpha = \int \psi_\alpha(\boldsymbol{v}_\alpha) I_c(t, m_\alpha, \boldsymbol{r}, \boldsymbol{v}_\alpha)d\boldsymbol{v}_\alpha = \\ \int dm_\beta \eta(m_\beta)\sigma_{\alpha\beta}^{D-1}\int d\boldsymbol{v}_\alpha d\boldsymbol{v}_\beta \int d\boldsymbol{n}\Theta(-\boldsymbol{g}\cdot\boldsymbol{n})|\boldsymbol{g}\cdot\boldsymbol{n}|\times \\ \times f(t, m_\alpha, \boldsymbol{r}, \boldsymbol{v}_\alpha) f(t, m_\beta, \boldsymbol{r}, \boldsymbol{v}_\beta)\Delta\psi_\alpha(\boldsymbol{v}_\alpha)\ .\end{aligned} \tag{70}$$

C. Energy change rate

In order to estimate the change of kinetic energy due to collisions, we can put (67) into (70) as $\Delta\psi_\alpha(\boldsymbol{v}_\alpha)$ and write

$$\begin{aligned}\int \frac{m_\alpha v_\alpha^2}{2} I_c(t, m_\alpha, \boldsymbol{r}, \boldsymbol{v}_\alpha)d\boldsymbol{v}_\alpha = \int dm_\beta \eta(m_\beta)\sigma_{\alpha\beta}^{D-1}\int d\boldsymbol{v}_\alpha d\boldsymbol{g}\int d\boldsymbol{n}\Theta(-\boldsymbol{g}\cdot\boldsymbol{n})|\boldsymbol{g}\cdot\boldsymbol{n}|\times \\ \times f(t, m_\alpha, \boldsymbol{r}, \boldsymbol{v}_\alpha) f(t, m_\beta, \boldsymbol{r}, \boldsymbol{v}_\beta)\delta E_\alpha(\boldsymbol{v}_\alpha, \boldsymbol{g})\ ,\end{aligned} \tag{71}$$

where we used the fact that $d\boldsymbol{v}_\alpha d\boldsymbol{v}_\beta = d\boldsymbol{v}_\alpha d\boldsymbol{g}$. However, in order to proceed further, we need to have the knowledge about the distribution functions. In what follows we will use Maxwellian velocity distribution functions (31). Obviously, one could take it only as zeros order approximation of a more complex polynomial series expansion, however for the sake of brevity we assume only the Maxwellian approximation. In the case of not very dense granular gases this choice is justified. The distribution function itself reads

$$f(t, m_\alpha, \boldsymbol{r}, \boldsymbol{v}_\alpha) = n_\alpha\left(\frac{m_\alpha}{2\pi T_\alpha}\right)^{D/2}\cdot\exp\left\{-\frac{m_\alpha(\boldsymbol{v}_\alpha - \boldsymbol{u}_\alpha)^2}{2T_\alpha}\right\}\ , \tag{72}$$

but for now we set the average flow velocity $u = \Omega r$ to be zero and the system to be spacially homogeneous, therefore

$$f(t, m_\alpha, \boldsymbol{v}_\alpha) = n_\alpha\left(\frac{\kappa_\alpha}{\pi}\right)^{D/2}\cdot\exp\left(-\kappa_\alpha v_\alpha^2\right)\ , \tag{73}$$

where $\kappa_\alpha = \dfrac{m_\alpha}{2T_\alpha}$, and now

$$f(t, m_\alpha, \boldsymbol{v}_\alpha) f(t, m_\beta, \boldsymbol{v}_\beta) = n_\alpha n_\beta\left(\frac{\kappa_\alpha\kappa_\beta}{\pi^2}\right)^{D/2}\cdot\exp\left(-\kappa_\alpha v_\alpha^2 - \kappa_\beta v_\beta^2\right)\ , \tag{74}$$

using $\boldsymbol{v}_\beta = \boldsymbol{g} - \boldsymbol{v}_\alpha$ we can write the exponential terms as

$$\begin{aligned}\kappa_\alpha v_\alpha^2 + \kappa_\beta v_\beta^2 = \kappa_\alpha v_\alpha^2 + \kappa_\beta(\boldsymbol{g} - \boldsymbol{v}_\alpha)^2 = \kappa_\alpha v_\alpha^2 + \kappa_\beta(g^2 + v_\alpha^2 - 2\boldsymbol{g}\cdot\boldsymbol{v}_\alpha) = \\ = (\kappa_\alpha + \kappa_\beta)v_\alpha^2 + \kappa_\beta g^2 - 2\kappa_\beta g v_\alpha\cos\gamma\ ,\end{aligned} \tag{75}$$

where γ is the angle between the impact velocity $\boldsymbol{g}$ and the velocity $\boldsymbol{v}_\alpha$. Now the average rate of change of a function $\psi_\alpha(\boldsymbol{v}_\alpha)$ can be written as

$$\begin{aligned}\frac{d}{dt}\langle\psi_\alpha(\boldsymbol{v}_\alpha)\rangle = \int dm_\beta n_\alpha n_\beta \eta(m_\beta)\sigma_{\alpha\beta}^{D-1}\left(\frac{\kappa_\alpha\kappa_\beta}{\pi^2}\right)^{D/2}\int d\boldsymbol{v}_\alpha d\boldsymbol{g}\int d\boldsymbol{n}\Theta(-\boldsymbol{g}\cdot\boldsymbol{n})|\boldsymbol{g}\cdot\boldsymbol{n}|\times \\ \times\exp\left(-(\kappa_\alpha+\kappa_\beta)v_\alpha^2-\kappa_\beta g^2+2\kappa_\beta g v_\alpha\cos\gamma\right)\Delta\psi_\alpha(\boldsymbol{g},\boldsymbol{v}_\alpha)\ .\end{aligned} \tag{76}$$

To proceed further, let us use the assumption of two dimensional flat system. The generalization to the three dimensional system leads essentially to the same calculations, with some trivial tweaks. So, in the two dimensional case, the cross section term $d\boldsymbol{n}\Theta(-\boldsymbol{g}\cdot\boldsymbol{n})$ leads to the integration over a half circle and gives π. Introducing the angle θ between the impact velocity $\boldsymbol{g}$ and the collision vector $\boldsymbol{n}$, we can write $|\boldsymbol{g}\cdot\boldsymbol{n}| = g\cos\theta$ with θ varying from $-\pi/2$ to $+\pi/2$. Switching to polar coordinates we have $d\boldsymbol{v}_\alpha d\boldsymbol{g} = g v_\alpha dg dv_\alpha d\theta d\gamma$, with $\gamma\in[0\div 2\pi]$:

$$\begin{aligned}\frac{d}{dt}\langle\psi_\alpha(\boldsymbol{v}_\alpha)\rangle = \frac{n_\alpha\kappa_\alpha}{\pi}\int dm_\beta n_\beta\kappa_\beta\eta(m_\beta)\sigma_{\alpha\beta}\int dg dv_\alpha\int_{-\pi/2}^{\pi/2}d\theta\int_0^{2\pi}d\gamma\ g^2 v_\alpha\cos\theta\times \\ \times\exp\left(-(\kappa_\alpha+\kappa_\beta)v_\alpha^2-\kappa_\beta g^2+2\kappa_\beta g v_\alpha\cos\gamma\right)\Delta\psi_\alpha(\boldsymbol{g},\boldsymbol{v}_\alpha)\ .\end{aligned} \tag{77}$$

The energy change rate now reads

$$\begin{aligned}\left\langle\frac{dE_\alpha}{dt}\right\rangle = \frac{n_\alpha\kappa_\alpha}{\pi}\int dm_\beta n_\beta\kappa_\beta\eta(m_\beta)\sigma_{\alpha\beta}\int dg dv_\alpha\ g^2 v_\alpha\exp\left(-(\kappa_\alpha+\kappa_\beta)v_\alpha^2-\kappa_\beta g^2\right)\times \\ \times\int_{-\pi/2}^{\pi/2}d\theta\ \cos\theta\int_0^{2\pi}d\gamma\ \exp\left(2\kappa_\beta g v_\alpha\cos\gamma\right)\times \\ \times\left(-\mu(1+\varepsilon)g v_\alpha\cos\theta\cos(\gamma-\theta)+\frac{\mu^2}{2m_\alpha}(1+\varepsilon)^2 g^2\cos^2\theta\right)\ ,\end{aligned} \tag{78}$$

here we used $\boldsymbol{v}_\alpha\cdot\boldsymbol{n} = v_\alpha\cos(\gamma-\theta)$. Let us first concentrate on the angular integrals

$$\begin{aligned}S_{\theta\gamma,1} &= -\mu(1+\varepsilon)g v_\alpha\int_{-\pi/2}^{\pi/2}d\theta\ \cos^2\theta\int_0^{2\pi}d\gamma\ \cos(\theta-\gamma)\exp(R\cos\gamma)\ , \\ S_{\theta\gamma,2} &= \frac{\mu^2 g^2}{2m_\alpha}(1+\varepsilon)^2\int_{-\pi/2}^{\pi/2}d\theta\ \cos^3\theta\int_0^{2\pi}d\gamma\ \exp(R\cos\gamma)\ ,\end{aligned} \tag{79}$$

where $R = 2\kappa_\beta g v_\alpha \geq 0$. The first integral over γ reads

$$\begin{aligned}S_{\gamma,1} &= \cos\theta\int_0^{2\pi}\cos\gamma\exp(R\cos\gamma)d\gamma+\sin\theta\int_0^{2\pi}\sin\gamma\exp(R\cos\gamma)d\gamma = \\ &= \cos\theta\int_0^{2\pi}\cos\gamma\exp(R\cos\gamma)d\gamma = 2\cos\theta\int_0^{\pi}\cos\gamma\exp(R\cos\gamma)d\gamma\ .\end{aligned} \tag{80}$$

Here we used the fact that the function under the second integral is odd, hence vanishes, and the function under the first integral is even, hence can be split in half. Finally we end up with two integrals over γ that cannot be solved using standart functions and should be represented as modified Bessel functions. They are

$$
\begin{aligned}
S_{\gamma,1} &= \int_0^\pi \cos\gamma \exp(R\cos\gamma)d\gamma = \pi I_1(R) \ , \\
S_{\gamma,2} &= \int_0^\pi \exp(R\cos\gamma)d\gamma = \pi I_0(R) \ ,
\end{aligned}
\tag{81}
$$

where

$$
I_\nu(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos t}\cos(\nu t)dt - \frac{\sin(\pi\nu)}{\pi}\int_0^\infty e^{-x\cosh t - \nu t}dt \ , \tag{82}
$$

is the so called modified Bessel function. For the values of the parameter $\nu = 0, 1$ we obtain our integrals over γ. Now we can write

$$
\begin{aligned}
S_{\theta\gamma,1} &= -\frac{8}{3}\pi\mu(1+\varepsilon)gv_\alpha I_1(R) \ , \\
S_{\theta\gamma,2} &= \frac{4\pi\mu^2 g^2}{3m_\alpha}(1+\varepsilon)^2 I_0(R) \ ,
\end{aligned}
\tag{83}
$$

where the integration over θ is trivial

$$
\int_{-\pi/2}^{\pi/2} \cos^3\theta \ d\theta = \frac{4}{3} \ . \tag{84}
$$

The energy change rate now is written as

$$
\begin{aligned}
\left\langle \frac{dE_\alpha}{dt} \right\rangle = \frac{4n_\alpha\kappa_\alpha}{3}(1+\varepsilon)\int dm_\beta \mu n_\beta \kappa_\beta \eta(m_\beta)\sigma_{\alpha\beta}\int dg dv_\alpha \ g^3 v_\alpha \times \\
\times \exp(-(\kappa_\alpha+\kappa_\beta)v_\alpha^2 - \kappa_\beta g^2)\left((1+\varepsilon)\frac{\mu}{m_\alpha}gI_0(R) - 2v_\alpha I_1(R)\right) \ ,
\end{aligned}
\tag{85}
$$

or more specifically

$$
\begin{aligned}
\left\langle \frac{dE_\alpha}{dt} \right\rangle = \frac{4n_\alpha\kappa_\alpha}{3}(1+\varepsilon)\int dm_\beta \mu n_\beta \kappa_\beta \eta(m_\beta)\sigma_{\alpha\beta}\int_0^\infty dg \ g^3 e^{-\kappa_\beta g^2} \times \\
\times \int_0^\infty dv_\alpha \ v_\alpha \exp\left(-(\kappa_\alpha+\kappa_\beta)v_\alpha^2\right)\left((1+\varepsilon)\frac{\mu}{m_\alpha}gI_0(2\kappa_\beta g v_\alpha) - 2v_\alpha I_1(2\kappa_\beta g v_\alpha)\right) \ .
\end{aligned}
\tag{86}
$$

The integrals of Bessel functions are computable using the next formula

$$
\begin{aligned}
\int_0^\infty x^{\alpha-1}e^{-px^2}I_\nu(cx)dx = A_\nu^\alpha \ , \qquad [\Re(p), \ \Re(\alpha+\nu) > 0, \ |\arg c| < \pi] \ , \\
A_\nu^\alpha = 2^{-\nu-1}c^\nu p^{-(\alpha+\nu)/2}\cdot\frac{\Gamma((\alpha+\nu)/2)}{\Gamma(\nu+1)}\cdot {}_1F_1\left(\frac{\alpha+\nu}{2};\nu+1;\frac{c^2}{4p}\right) \ ,
\end{aligned}
\tag{87}
$$

where

$$
{}_1F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{a^{(n)} z^n}{b^{(n)} n!} \ , \tag{88}
$$

is the so called confluent hypergeometric function. In our case we need the next two specific cases

$$
\begin{aligned}
\int_0^\infty x e^{-px^2} I_0(cx) dx &= A_0^2 \ , \\
\int_0^\infty x^2 e^{-px^2} I_1(cx) dx &= A_1^3 \ ,
\end{aligned} \tag{89}
$$

with

$$
\begin{aligned}
p &= \kappa_\alpha + \kappa_\beta \ , \\
c &= 2\kappa_\beta g \ .
\end{aligned} \tag{90}
$$

When $\alpha = \nu + 2$, the integral formula can be simplified

$$
A_\nu^{\nu+2} = \frac{c^\nu}{(2p)^{\nu+1}} \exp\left(\frac{c^2}{4p}\right) \ . \tag{91}
$$

The calculation of this formula is straightforward and will be omitted. Finally, we can write

$$
\begin{aligned}
\int_0^\infty x e^{-px^2} I_0(cx) dx &= \frac{1}{2p} \exp\left(\frac{c^2}{4p}\right) \ , \\
\int_0^\infty x^2 e^{-px^2} I_1(cx) dx &= \frac{c}{(2p)^2} \exp\left(\frac{c^2}{4p}\right) \ ,
\end{aligned} \tag{92}
$$

and the energy change rate becomes

$$
\begin{aligned}
\left\langle \frac{dE_\alpha}{dt} \right\rangle = \frac{4n_\alpha \kappa_\alpha}{3}(1+\varepsilon) \int dm_\beta \ \frac{\mu \kappa_\beta}{\kappa_\alpha + \kappa_\beta} n_\beta \eta(m_\beta) \sigma_{\alpha\beta} \left((1+\varepsilon)\frac{\mu}{2m_\alpha} - \frac{\kappa_\beta}{\kappa_\alpha + \kappa_\beta} \right) \times \\
\times \int_0^\infty dg \ g^4 \exp\left(-\frac{\kappa_\alpha \kappa_\beta}{\kappa_\alpha + \kappa_\beta} g^2 \right) \ ,
\end{aligned} \tag{93}
$$

where the integration over g is trivial

$$
\int_0^\infty x^4 e^{-cx^2} dx = \frac{3}{8c^2} \sqrt{\frac{\pi}{c}} \ , \quad c > 0 \ , \tag{94}
$$

or substituting the values of c we get

$$
\begin{aligned}
\left\langle \frac{dE_\alpha}{dt} \right\rangle = \frac{\sqrt{\pi}}{2} \int dm_\beta \ n_\alpha n_\beta \sigma_{\alpha\beta} \eta(m_\beta) \mu (1+\varepsilon) \frac{\kappa_\alpha + \kappa_\beta}{\kappa_\alpha \kappa_\beta} \sqrt{\frac{\kappa_\alpha + \kappa_\beta}{\kappa_\alpha \kappa_\beta}} \times \\
\times \left(\frac{\mu(1+\varepsilon)}{2m_\alpha} - \frac{\kappa_\beta}{\kappa_\alpha + \kappa_\beta} \right) \ ,
\end{aligned} \tag{95}
$$

and finally, after arithmetic manipulations, we end up with

$$\left\langle \frac{dE_\alpha}{dt} \right\rangle = \frac{\sqrt{2\pi}}{2} \int dm_\beta \; n_\alpha n_\beta \sigma_{\alpha\beta}\eta(m_\beta) \frac{\mu}{m_\alpha} \sqrt{\frac{m_\alpha T_\beta + m_\beta T_\alpha}{m_\alpha m_\beta}} \times \\ \times \left(-(1-\varepsilon^2) T_\alpha + \frac{\mu}{m_\beta}(1+\varepsilon)^2 (T_\beta - T_\alpha) \right) \; , \tag{96}$$

or in terms of the temperature evolution

$$\frac{dT_\alpha}{dt} = \frac{\sqrt{2\pi}}{2} \int dm_\beta \; n_\beta \sigma_{\alpha\beta}\eta(m_\beta) \frac{\mu}{m_\alpha} \sqrt{\frac{m_\alpha T_\beta + m_\beta T_\alpha}{m_\alpha m_\beta}} \times \\ \times \left(-(1-\varepsilon^2) T_\alpha + \frac{\mu}{m_\beta}(1+\varepsilon)^2 (T_\beta - T_\alpha) \right) \; . \tag{97}$$

We can see that the first term containing $-(1-\varepsilon^2)$ is the temperature decay due to inelasic collisions, and the second term is the heat flux among the species.

D. Dynamic Parameters of a Granular Gas Mixture

Here we will look at certain dynamic parameters that we use later. First of all, let us calculate the average impact velocity value for given $\alpha\beta$ pair of species. This value will be needed in order to calculate the average collision frequency. The general form of the average value is

$$\langle g_{\alpha\beta} \rangle = \frac{\int g_{\alpha\beta} \cdot f(t, m_\alpha, \boldsymbol{r}, \boldsymbol{v}_\alpha) f(t, m_\beta, \boldsymbol{r}, \boldsymbol{v}_\beta) d\boldsymbol{v}_\alpha d\boldsymbol{v}_\beta}{\int f(t, m_\alpha, \boldsymbol{r}, \boldsymbol{v}_\alpha) f(t, m_\beta, \boldsymbol{r}, \boldsymbol{v}_\beta) d\boldsymbol{v}_\alpha d\boldsymbol{v}_\beta} = \\ = \frac{1}{n_\alpha n_\beta} \int g_{\alpha\beta} \cdot f(t, m_\alpha, \boldsymbol{r}, \boldsymbol{v}_\alpha) f(t, m_\beta, \boldsymbol{r}, \boldsymbol{v}_\beta) d\boldsymbol{v}_\alpha d\boldsymbol{v}_\beta \; , \tag{98}$$

and following the scheme from the previous section we write

$$\langle g_{\alpha\beta} \rangle = \frac{\kappa_\alpha \kappa_\beta}{\pi^2} \int_0^\infty dg \; g^2 e^{-\kappa_\beta g^2} \int_0^\infty dv_\alpha \; v_\alpha e^{-(\kappa_\alpha+\kappa_\beta)v_\alpha^2} \int_0^{2\pi} d\phi \int_0^{2\pi} d\gamma \; \exp\left(2\kappa_\beta v_\alpha g \cos\gamma\right) = \\ = \frac{2\kappa_\alpha\kappa_\beta}{\pi} \int_0^\infty dg \; g^2 e^{-\kappa_\beta g^2} \int_0^\infty dv_\alpha v_\alpha I_0(2\kappa_\beta g \cdot v_\alpha) \exp\left(-(\kappa_\alpha+\kappa_\beta)v_\alpha^2\right) \; ,$$

and finally the average impact velocity of species $\alpha\beta$ in the 2-dimensional gas reads

$$\langle g_{\alpha\beta} \rangle = \sqrt{\frac{\pi}{2}} \cdot \sqrt{\frac{m_\alpha T_\beta + m_\beta T_\alpha}{m_\alpha m_\beta}} \; . \tag{99}$$

Let us estimate now the collision frequency of a particle of size α with the species of size β. To do so, imagine that a particle α has an infinitely small size and is moving among particles with sizes of $\sigma_\alpha + \sigma_\beta = \sigma_{\alpha\beta}$ and number density n_β. In this case the collision frequency reads

$$\omega_{\alpha\beta} = \sigma_{\alpha\beta} n_\beta \langle g_{\alpha\beta} \rangle = \sqrt{\frac{\pi}{2}} \cdot \sigma_{\alpha\beta} n_\beta \sqrt{\frac{m_\alpha T_\beta + m_\beta T_\alpha}{m_\alpha m_\beta}} \ , \tag{100}$$

and now the temperature evolution can be written in the next form

$$\frac{dT_\alpha}{dt} = \int d\eta(m_\beta) \omega_{\alpha\beta} \left(-A_{\alpha\beta} T_\alpha + B_{\alpha\beta}(T_\beta - T_\alpha) \right) \ , \tag{101}$$

where

$$\begin{aligned} A_{\alpha\beta} &= \frac{1-\varepsilon^2}{2} \frac{\mu}{m_\alpha} \ , \\ B_{\alpha\beta} &= \frac{(1+\varepsilon)^2}{2} \frac{\mu^2}{m_\alpha m_\beta} \ , \end{aligned} \tag{102}$$

where the coeffitient $A_{\alpha\beta}$ stands for the energy decay rate per single collision between $\alpha\beta$ species and the coeffitient $B_{\alpha\beta}$ stands for the energy exchange rate per single collision. Following the same procedure we can see that

$$\langle g^2 \rangle = \frac{1}{n_\alpha n_\beta} \int g^2 f(t, m_\alpha, \boldsymbol{r}, \boldsymbol{v}_\alpha) f(t, m_\alpha, \boldsymbol{r}, \boldsymbol{v}_\alpha) d\boldsymbol{v}_\alpha d\boldsymbol{v}_\beta = 2 \cdot \frac{m_\alpha T_\beta + m_\beta T_\alpha}{m_\alpha m_\beta} \ , \tag{103}$$

and the amount of energy dissipated from the system due to a single collision on average equals to

$$\langle \delta E_{dis,\alpha\beta} \rangle = -\frac{1-\varepsilon^2}{2} \mu \langle g^2_{\alpha\beta} \rangle = -(1-\varepsilon^2)\mu \left(\frac{T_\alpha}{m_\alpha} + \frac{T_\beta}{m_\beta} \right) \ , \tag{104}$$

hence qe can see that the amount of energy lost by partcile of specy α per collision on average is

$$\langle \delta E_{dis,\alpha} \rangle = -(1-\varepsilon^2) \frac{\mu}{m_\alpha} T_\alpha \propto A_{\alpha\beta} T_\alpha \ . \tag{105}$$

The same logic stands for the energy exchange rate.

E. Normal Solution

In the works of Garzo, Dufty etal it was shown that a system of granular mixture gases attains after certain relaxation time the so called Normal Solution state. In this case the time and spacial dependence of the distribution function is given only through the generalized hydrodynamic (macroscopic) variables such as number density, granular temperature etc. The most important outcome of this solution is the equality of cooling rates of each species. So, refrasing the above statement, by analogy with the non-dissipative and purely elastic gases, when system attains after certain relaxation time the state of equal temperatures of all the species, in the case of dissipative mixture, we attain the state of equal cooling rates of temperatures, or

$$\frac{1}{T_\alpha}\frac{dT_\alpha}{dt} = \frac{1}{T_\beta}\frac{dT_\beta}{dt} = -\xi(t) \ , \tag{106}$$

where the cooling rate function $\xi(t)$ equals among all the species.

F. Mean Field Approximation

To proceed further, let us introduce the so called mean field species. This is an imaginary species of particles which obey the next granular temperature and masses

$$\begin{aligned} \bar{T} &= \int d\eta(m_\alpha) T_\alpha \ , \\ \bar{m} &= \int d\eta(m_\alpha) m_\alpha \ . \end{aligned} \tag{107}$$

The idea is that the mixture can be replaced by an averaged monodisperse system which has identical averaged hydrodynamic parameters. Now, integrating the temperature evolution equation over all species we write

$$\begin{aligned} \int d\eta(m_\alpha)\frac{dT_\alpha}{dt} &= -\int d\eta(m_\alpha)\int d\eta(m_\beta)\omega_{\alpha\beta}\left(A_{\alpha\beta}T_\alpha - B_{\alpha\beta}(T_\beta - T_\alpha)\right) \ , \\ \frac{d}{dt}\int d\eta(m_\alpha)T_\alpha &= -\int d\eta(m_\alpha)T_\alpha\int d\eta(m_\beta)\omega_{\alpha\beta}A_{\alpha\beta} + \iint d\eta(m_\alpha)d\eta(m_\beta)\omega_{\alpha\beta}B_{\alpha\beta}(T_\beta - T_\alpha) \ , \end{aligned}$$

and using the fact that the term $\omega_{\alpha\beta}B_{\alpha\beta}(T_\beta - T_\alpha)$ is antisymmetric, and the integration over both dummy indices $\alpha\beta$ vanishes, we end up with

$$\frac{d\bar{T}}{dt} = -\int d\eta(m_\alpha)T_\alpha\int d\eta(m_\beta)\omega_{\alpha\beta}A_{\alpha\beta} \ , \tag{108}$$

On the other hand, the evolution equation in the case of equality of cooling rates is written

$$\frac{dT_\alpha}{dt} = -\xi(t)T_\alpha \ , \tag{109}$$

and integrating it over all the species we write

$$\begin{aligned} \int d\eta(m_\alpha)\frac{dT_\alpha}{dt} &= -\int d\eta(m_\alpha)\xi(t)T_\alpha \\ \frac{d\bar{T}}{dt} &= -\xi(t)\bar{T} \ , \end{aligned}$$

and substituting the left hand side with Eq. (108) we can write the cooling rate in the next form

$$\xi(t) = \frac{\int d\eta(m_\alpha) T_\alpha \int d\eta(m_\beta) \omega_{\alpha\beta} A_{\alpha\beta}}{\int d\eta(m_\alpha) T_\alpha} \; . \tag{110}$$

To proceed further, we make the next approximation which we call the mean field approximation. The term $\omega_{\alpha\beta} A_{\alpha\beta}$ is the average energy loss due to collision between species $\alpha\beta$. Integration over all species can be approximated my the energy loss of the single mean field species, with its apropriate values of ω and A. Hence, we can write

$$\xi(t) \approx \frac{\bar{T} \cdot \bar{\omega} \bar{A}}{\bar{T}} = \bar{\omega} \bar{A} \; . \tag{111}$$

The term $\bar{A}$ should be of the form

$$\bar{A} = \frac{1-\varepsilon^2}{2} \frac{\bar{\mu}}{\bar{m}} = \frac{1-\varepsilon^2}{4} \; , \tag{112}$$

and the collision frequency of the mean field can be replaced by its value from the planetary ring environment, which is according to Schmidt etal. equals to

$$\bar{\omega} \approx 3\Omega\tau \; , \tag{113}$$

where τ is the optical thickness of the rings. Finally, for the cooling rate we have

$$\xi(t) = \xi \approx \frac{3}{4}(1-\varepsilon^2)\Omega\tau \; . \tag{114}$$

V. STATIONARY SOLUTION

Let us now seek for stationary solution for the temperature evolution equations. For now, let us assume the radial independence of the temperatures $\frac{\partial T}{\partial r} = 0$. The stationar solution of Eq. (58) now reads

$$(k_1 \nu_{l,\alpha} + k_2 \nu_{nl,\alpha})\Omega^2 = k_3(1-\varepsilon^2)\Omega\tau \cdot T_\alpha \; , \tag{115}$$

where k_1, k_2, k_3 are dimensionless constants. The local and nonlocal parts of the viscosity can be written in the next form (Schmidt etal.)

$$\begin{aligned} \nu_{l,\alpha} &\approx \frac{T_\alpha}{\Omega m_\alpha} \frac{\tau}{1+\tau^2} \; , \\ \nu_{nl,\alpha} &\approx \Omega D_\alpha^2 \tau \; , \end{aligned} \tag{116}$$

where

$$D_\alpha = \int d\eta(m_\beta)\sigma_{\alpha\beta} \ , \tag{117}$$

is the average cross section. Finally, substituting all the parameter values, we end up with the next stationary temperatures

$$T_\alpha = \frac{k_2 \cdot m_\alpha \Omega^2 D_\alpha^2 (1+\tau^2)}{k_3 \cdot m_\alpha (1-\varepsilon^2)(1+\tau^2) - k_1} \ . \tag{118}$$