PDE Seminar: Day 5
ABP Estimates

Yeryang Kang

February 13, 2026

1 Introduction: The Need for Strong Solutions

In previous sessions, we discussed Schauder theory, and the following contrast holds:

- **Classical Solutions ($C^{2,\alpha}$):** Require coefficients a^{ij} to be Hölder continuous (Schauder Theory).
- **Weak Solutions (H^1):** Require divergence form structure $D_i(a^{ij}D_ju) = f$.

What if the operator is in **non-divergence form** ($a^{ij}D_{ij}u = f$) but the coefficients are **only continuous (C^0)** or merely bounded (L^∞)? In this case, classical derivatives may not exist everywhere. We seek a solution $u \in W^{2,p}_{loc}(\Omega)$ that satisfies the equation "almost everywhere (a.e.)". This is called a **Strong Solution**.

2 Geometric Tools for ABP Estimate

To bound strong solutions without C^2 regularity, we replace the classical "Maximum Principle" with a geometric measure theory approach.

Definition 2.1 (Upper Contact Set)**.** *The upper contact set Γ^+ of u is the set of points where the tangent plane lies above the graph of u:*

$$\Gamma^+ = \{x \in \Omega \mid u(y) \le u(x) + p \cdot (y - x) \text{ for some } p \in \mathbb{R}^n, \forall y \in \Omega\}.$$

Basically, Γ^+ is where u is "concave" (touched by a supporting hyperplane from above).

Definition 2.2 (Normal Mapping)**.** *The normal mapping (or sub-gradient) $\chi(x)$ is the set of slopes p of the supporting hyperplanes at x. For $u \in C^2$, this is simply the gradient map:*

$$\chi(E) = \{\nabla u(x) \mid x \in E \cap \Gamma^+\}.$$

3 Theorem 3.6: The ABP Estimate

The Alexandrov-Bakelman-Pucci (ABP) estimate is the fundamental L^∞ bound for non-divergence operators with bounded measurable coefficients.

Theorem 3.1 (9.1. ABP Estimate)**.** *Let $Lu \ge f$ in a bounded domain Ω and $u \in C^0(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$. Then*

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u^+ + C\|f/\mathcal{D}^*\|_{L^n(\Omega)}. \tag{1}$$

Sketch of Proof. This estimate links the maximum of u to the measure of the "upper contact set" Γ^+, where u is concave and lies below its tangent plane. By considering the image of the gradient map ∇u on Γ^+, one relates the volume of a ball (the range of the gradient) to the integral of the determinant of the Hessian (the Jacobian), which is bounded by f due to the PDE. □

Proof. Assume, without loss of generality, that $u \leq 0$ on $\partial\Omega$ (otherwise, replace u with $u - \sup_{\partial\Omega} u^+$). Let $M = \sup_\Omega u > 0$.

1. The Upper Contact Set

Define the **upper contact set** Γ^+ as the set where u is concave and stayed "above" its supporting planes:

$$\Gamma^+ = \{x \in \Omega : u(y) \leq u(x) + \nabla u(x) \cdot (y - x) \text{ for all } y \in \Omega\}.$$

For $x \in \Gamma^+$, the Hessian matrix $D^2u(x)$ is negative semi-definite, implying $-D^2u \geq 0$. From the PDE $Lu \geq f$, and assuming L is the Laplacian Δ for simplicity (or a general elliptic operator with determinant $\mathcal{D}^*$), we have the following inequalities, mainly owing to arithmetic mean - geometric mean inequality :

$$\frac{(-Lu)^n}{n^n \det(A)} \leq \det(-D^2u) \leq \left(\frac{-\Delta u}{n}\right)^n \leq C\left|\frac{f}{\mathcal{D}^*}\right|^n.$$

2. The Gradient Map

Consider the map $g : \Omega \to \mathbb{R}^n$ defined by $g(x) = \nabla u(x)$. If u attains its maximum M at $x_0 \in \Omega$ and $u \leq 0$ on $\partial\Omega$, the image of the contact set under the gradient map, $g(\Gamma^+)$, must cover a ball of radius $M/\mathrm{diam}(\Omega)$. Specifically, it can be shown that:

$$B_{M/d}(0) \subset \nabla u(\Gamma^+), \quad \text{where } d = \mathrm{diam}(\Omega).$$

3. Change of Variables (Integration)

We calculate the volume of the image $g(\Gamma^+)$ using the Jacobian of the gradient map, which is the determinant of the Hessian:

$$|B_{M/d}(0)| \leq \int_{g(\Gamma^+)} dp \leq \int_{\Gamma^+} |\det(D^2u)| dx.$$

Substituting the bound from the PDE in Step 1 (A:= coefficient matrix of the elliptic operator L):

$$\omega_n \left(\frac{M}{d}\right)^n \leq \int_{\Gamma^+} \left(\frac{f}{n(\det A)^{1/n}}\right)^n dx \leq \int_\Omega \frac{f^n}{n^n \mathcal{D}^*} dx.$$

Taking the n-th root of both sides yields:

$$M \leq Cd\left(\int_\Omega \left|\frac{f}{\mathcal{D}^*}\right|^n dx\right)^{1/n}.$$

This concludes the estimate: $\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + C\|f/\mathcal{D}^*\|_{L^n(\Omega)}$. □

4 Theorem 3.7: $W^{2,p}$ Solvability

Using the ABP estimate (for uniqueness/bounds) and Calderon-Zygmund estimates (for regularity), we establish the existence theory.

Theorem 4.1 ($W^{2,p}$ Existence and Uniqueness)**.** *Let Ω be a $C^{1,1}$ domain. Suppose $a^{ij} \in C^0(\overline{\Omega})$ (continuous coefficients) and L is uniformly elliptic. For any $f \in L^p(\Omega)$ with $p > n$, the Dirichlet problem:*

$$\begin{cases} a^{ij} D_{ij} u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

has a unique strong solution $u \in W^{2,p}(\Omega)$. Furthermore,

$$\|u\|_{W^{2,p}(\Omega)} \le C\|f\|_{L^p(\Omega)}.$$

4.1 Proof Strategy

1. **Uniqueness:** Follows immediately from the **ABP Estimate**. If $Lu = 0$ (with $f = 0$), then the L^n norm of f is 0, so $\sup |u| \le 0 \implies u \equiv 0$.

2. **A Priori Estimates (L^p estimates):** The core hard analysis relies on the **Calderon-Zygmund Theory**.

   - For the Laplacian $\Delta u = f$, singular integral theory proves $\|D^2 u\|_{L^p} \le C\|f\|_{L^p}$.
   - By perturbation (since a^{ij} are continuous), locally we can treat $a^{ij} \approx$ const and apply the Laplacian estimates.

3. **Existence (Method of Continuity):**

   - We connect L to Δ via $L_t = (1-t)\Delta + tL$.
   - Since we have the global $W^{2,p}$ a priori estimate (derived from step 2 and ABP), the set of solvable t is closed (similar to the Schauder proof).
   - Openness is shown via the contraction mapping principle using the smallness of perturbation in L^p.

5 Summary: Evolution of Elliptic Theory

Feature	**Classical (Ch. 6)**	**Weak (Ch. 8)**	**Strong (Ch. 9)**
Space	$C^{2,\alpha}$	$H^1 = W^{1,2}$	$W^{2,p}$ $(p > n)$
Coefficients	Hölder ($C^{0,\alpha}$)	Measurable (L^∞)	Continuous (C^0)
Data (f)	Hölder ($C^{0,\alpha}$)	Distributional (H^{-1})	L^p Space
Key Tool	Schauder Estimates	Energy / Lax-Milgram	ABP Estimate
Method	Potential Theory	Hilbert Space	Geometric Measure

Table 1: Comparison of Solution Concepts