PDE Seminar: Day 1
Sobolev Spaces and Distributions

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1 σ-Algebras and Measurable Sets

A σ-algebra provides the formal framework for defining sets that are "measurable" in the sense of measure theory.

Definition 1.1 (σ-algebra)**.** Let X be a set. A σ-algebra on X is a non-empty collection Σ of subsets of X satisfying:

1. $\emptyset \in \Sigma$ and $X \in \Sigma$.
2. **Closed under complement:** If $E \in \Sigma$, then $E^c = X \setminus E \in \Sigma$.
3. **Closed under countable unions:** If $\{E_k\}_{k=1}^{\infty} \subseteq \Sigma$, then $\bigcup_{k=1}^{\infty} E_k \in \Sigma$.

By De Morgan's laws, these conditions imply Σ is also closed under countable intersections.

2 Set Functions and Measures

Definition 2.1 (Set Function)**.** A set function is a function whose domain is a family of subsets (usually a σ-algebra) of a given set X, mapping to the extended real line $[-\infty, \infty]$.

Definition 2.2 (Measure)**.** Let X be a set and Σ be a σ-algebra over X. A set function $\mu : \Sigma \to [0, \infty]$ is called a **positive measure** if:

1. $\mu(\emptyset) = 0$.
2. **Countable Additivity:** For any sequence $\{E_k\}_{k=1}^{\infty}$ of pairwise disjoint sets in Σ:

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k).$$

The triple (X, Σ, μ) is called a **measure space**.

3 Lebesgue Measure on $\mathbb{R}^n$

The Lebesgue measure extends the notions of length, area, and volume to Euclidean spaces $\mathbb{R}^n$.

Definition 3.1 (Lebesgue Outer Measure)**.** For any subset $E \subset \mathbb{R}^n$, the Lebesgue outer measure $\lambda^*(E)$ is defined as:

$$\lambda^*(E) := \inf \left\{ \sum_{k=1}^{\infty} \operatorname{vol}(C_k) \right\},$$

where the infimum is taken over all sequences of open rectangular cuboids $\{C_k\}$ such that $E \subset \bigcup_{k=1}^{\infty} C_k$.

- In 1D, $\mathrm{vol}(C_k)$ is the length $l(I) = b - a$.
- In n-dimensions, for $C = I_1 \times \cdots \times I_n$, $\mathrm{vol}(C) = \prod_{i=1}^{n} l(I_i)$.

Proposition 3.1 (Carathéodory Criterion)**.** *A set $E \subset \mathbb{R}^n$ is **Lebesgue measurable** if for every test set $A \subset \mathbb{R}^n$:*

$$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c).$$

The collection $\mathcal{L}$ of all such measurable sets forms a σ-algebra. Restricting λ^* to $\mathcal{L}$ gives the Lebesgue measure m, i.e., $m(E) = \lambda^*(E)$ for $E \in \mathcal{L}$. The triple $(\mathbb{R}^n, \mathcal{L}, m)$ is the Lebesgue measure space.

Exercise 3.1 (Properties of Lebesgue Measure)**.** *For $n \geq 1$:*

1. *Verify that $\mathcal{L}$ satisfies the axioms of a σ-algebra.*
2. *Show that the restriction $\lambda^*|_{\mathcal{L}}$ satisfies countable additivity, hence is a measure.*

Solution 3.1.

1. *We only show closure under countable unions. Let $\{E_k\} \subset \mathcal{L}$ be pairwise disjoint and set $E = \bigcup_k E_k$. For any A, iterating the Carathéodory condition gives*

$$\lambda^*(A) = \sum_{k=1}^{N} \lambda^*(A \cap E_k) + \lambda^*\left(A \cap \bigcap_{k=1}^{N} E_k^c\right).$$

   Letting $N \to \infty$ yields

$$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c),$$

   so $E \in \mathcal{L}$.

2. *For countable additivity, let $\{E_k\} \subset \mathcal{L}$ be disjoint and $E = \bigcup_k E_k$. Apply Carathéodory with $A = E$:*

$$\lambda^*(E) = \sum_{k=1}^{N} \lambda^*(E_k) + \lambda^*\left(E \cap \bigcap_{k=1}^{N} E_k^c\right).$$

   The last term decreases to 0 as $N \to \infty$, hence

$$\lambda^*(E) = \sum_{k=1}^{\infty} \lambda^*(E_k).$$

   Thus $\lambda^*|_{\mathcal{L}}$ is a measure.

Example 3.1 (Dirichlet Function in 1D)**.** Define $f : \mathbb{R} \to \mathbb{R}$ as the indicator of the rationals:

$$f(x) = \mathbf{1}_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

On $[0, 1]$:

- $\mathbb{Q} \cap [0, 1]$ is countable, so $m(\mathbb{Q} \cap [0, 1]) = 0$.
- $m([0, 1] \setminus \mathbb{Q}) = 1$.

The Lebesgue integral is

$$\int_{[0,1]} f \, dm = 1 \cdot m(\mathbb{Q}) + 0 \cdot m(\mathbb{R} \setminus \mathbb{Q}) = 0.$$

4 Lebesgue Spaces L^p

Let (X, Σ, μ) be a measure space and f a measurable function.

4.1 L^p spaces for $0 < p < \infty$

Define the L^p (semi-)norm by

$$\|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{1/p}.$$

The space $L^p(X, \mu)$ consists of all measurable f with $\|f\|_p < \infty$ (identifying functions equal a.e.).

4.2 L^∞ space

For a measurable $g : X \to [0, \infty]$, define the set of essential bounds

$$S = \{\alpha \in [0, \infty] \mid \mu(\{x : g(x) > \alpha\}) = 0\}.$$

If $S = \emptyset$, set $\|g\|_\infty = \infty$; otherwise, the **essential supremum** is

$$\|g\|_\infty := \inf S.$$

Functions in $L^\infty(X, \mu)$ are essentially bounded.

5 Key Inequalities and Convexity

5.1 Conjugate Exponents

Definition 5.1 (Conjugate Exponents)**.** Two positive real numbers p and q are called **conjugate exponents** if

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Common pairs: $p = q = 2$ (self-conjugate, Hilbert space case) and $p = 1, q = \infty$ (limiting case).

5.2 Hölder's Inequality

Proposition 5.1 (Basic Hölder Inequality)**.** *Let p, q be conjugate exponents with $1 < p < \infty$, and let f, g be non-negative measurable functions on (X, μ). Then*

$$\int_X fg \, d\mu \leq \left(\int_X f^p \, d\mu \right)^{1/p} \left(\int_X g^q \, d\mu \right)^{1/q}.$$

Proof. Set $A = \|f\|_p$, $B = \|g\|_q$.

1. If $A = 0$, then $f = 0$ a.e., so $fg = 0$ a.e., and the inequality holds.
2. If $A > 0$ and $B = \infty$, the inequality is trivial.
3. Assume $0 < A, B < \infty$. Define normalized functions $F = f/A$, $G = g/B$, so that $\|F\|_p = \|G\|_q = 1$.

   For any x with $0 < F(x), G(x) < \infty$, write $F(x) = e^{s/p}$, $G(x) = e^{t/q}$. By convexity of the exponential,

   $$e^{\frac{s}{p} + \frac{t}{q}} \leq \frac{1}{p} e^s + \frac{1}{q} e^t,$$

which gives the pointwise Young inequality

$$F(x)G(x) \le \frac{F(x)^p}{p} + \frac{G(x)^q}{q} \qquad \text{for a.e. } x. \tag{*}$$

Integrating (*) over X yields

$$\int_X FG\,d\mu \le \frac{1}{p}\int_X F^p\,d\mu + \frac{1}{q}\int_X G^q\,d\mu = \frac{1}{p} + \frac{1}{q} = 1.$$

Substituting back $F = f/A$, $G = g/B$ gives $\frac{1}{AB}\int_X fg\,d\mu \le 1$, i.e., $\int_X fg\,d\mu \le AB$.

□

5.3 Convex Functions

Definition 5.2 (Convex Function)**.** A function $\phi : (a,b) \to \mathbb{R}$ (with $-\infty \le a < b \le \infty$) is **convex** if

$$\phi((1-\lambda)x + \lambda y) \le (1-\lambda)\phi(x) + \lambda\phi(y)$$

for all $x, y \in (a,b)$ and all $\lambda \in [0,1]$.

5.4 General Hölder Inequality

Proposition 5.2 (Hölder in L^p)**.** *Let $1 \le p \le \infty$ and q be its conjugate exponent. If $f \in L^p(\mu)$ and $g \in L^q(\mu)$, then $fg \in L^1(\mu)$ and*

$$\|fg\|_1 \le \|f\|_p\,\|g\|_q.$$

Proof. The case $1 < p < \infty$ follows from the basic Hölder inequality applied to $|f|$ and $|g|$.

For $p = \infty$ (so $q = 1$), the definition of $\|\cdot\|_\infty$ gives $|f(x)g(x)| \le \|f\|_\infty |g(x)|$ a.e. Integrating yields the result. The case $p = 1, q = \infty$ is symmetric. □

6 Weak Derivatives

6.1 Motivation and Definition

We begin by substantially weakening the notion of partial derivatives.

Notation. Let $C_c^\infty(U)$ denote the space of infinitely differentiable functions $\phi : U \to \mathbb{R}$ with compact support in U. Functions $\phi \in C_c^\infty(U)$ are called *test functions*.

Motivation. Assume $u \in C^1(U)$. For any $\phi \in C_c^\infty(U)$, integration by parts gives

$$\int_U u\phi_{x_i}\,dx = -\int_U u_{x_i}\phi\,dx \qquad (i = 1, \ldots, n). \tag{1}$$

There are no boundary terms because ϕ has compact support. More generally, if $u \in C^k(U)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index of order $|\alpha| = k$, then

$$\int_U u\,D^\alpha\phi\,dx = (-1)^{|\alpha|}\int_U D^\alpha u\,\phi\,dx. \tag{2}$$

This follows by applying (1) repeatedly.

Observing that the left-hand side of (2) makes sense for any locally integrable u, we are led to the following definition.

Definition 6.1 (Weak Derivative)**.** Let $u, v \in L^1_{\text{loc}}(U)$ and let α be a multi-index. We say that v is the α^{th} **weak partial derivative** of u, written

$$D^\alpha u = v,$$

provided

$$\int_U u\, D^\alpha \phi\, dx = (-1)^{|\alpha|} \int_U v\phi\, dx \qquad \text{for all } \phi \in C_c^\infty(U). \tag{3}$$

Lemma 6.1 (Uniqueness)**.** *A weak α^{th}-partial derivative of u, if it exists, is unique up to sets of measure zero.*

Proof. Assume $v, \tilde{v} \in L^1_{\text{loc}}(U)$ both satisfy (3) for all test functions ϕ. Then

$$\int_U (v - \tilde{v})\phi\, dx = 0 \quad \forall \phi \in C_c^\infty(U),$$

which implies $v = \tilde{v}$ almost everywhere. ☐

6.2 Example

Example 6.1. Let $n = 1$, $U = (0, 2)$, and

$$u(x) = \begin{cases} x, & 0 < x \le 1, \\ 1, & 1 \le x < 2. \end{cases}$$

Define

$$v(x) = \begin{cases} 1, & 0 < x \le 1, \\ 0, & 1 < x < 2. \end{cases}$$

We claim $u' = v$ in the weak sense. Indeed, for any $\phi \in C_c^\infty(U)$,

$$\begin{aligned} \int_0^2 u\phi'\, dx &= \int_0^1 x\phi'\, dx + \int_1^2 \phi'\, dx \\ &= \big[x\phi(x)\big]_0^1 - \int_0^1 \phi\, dx + \phi(2) - \phi(1) \\ &= -\int_0^1 \phi\, dx = -\int_0^2 v\phi\, dx, \end{aligned}$$

which is exactly condition (3) with $\alpha = 1$.

7 Compact Sets in $\mathbb{R}^n$: A Brief Review

- Typical examples of compact sets in $\mathbb{R}^n$ (by Heine–Borel theorem):
  - Finite sets.
  - Closed hypercubes: $[a_1, b_1] \times \cdots \times [a_n, b_n]$.
  - Closed balls: $\overline{B}(x_0, r)$.
  - n-spheres: $\{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$.

8 Sobolev Spaces

Fix $1 \le p \le \infty$ and let k be a nonnegative integer. Sobolev spaces are function spaces whose members have weak derivatives up to order k lying in L^p.

8.1 Definition and Basic Properties

Definition 8.1 (Sobolev Space $W^{k,p}(U)$)**.** The Sobolev space $W^{k,p}(U)$ consists of all locally summable functions $u : U \to \mathbb{R}$ such that for each multi-index α with $|\alpha| \leq k$, the weak derivative $D^\alpha u$ exists and belongs to $L^p(U)$.

Remark 8.1.

(i) When $p = 2$, we denote $H^k(U) = W^{k,p}(U)$ for $k = 0, 1, \ldots$. The letter H is used because $H^k(U)$ is a Hilbert space. Note that $H^0(U) = L^2(U)$.

(ii) We identify functions in $W^{k,p}(U)$ that agree almost everywhere.

Definition 8.2 (Sobolev Norm)**.** For $u \in W^{k,p}(U)$, define its Sobolev norm by

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\displaystyle\sum_{|\alpha| \leq k} \int_U |D^\alpha u|^p dx \right)^{1/p} & (1 \leq p < \infty), \\ \displaystyle\sum_{|\alpha| \leq k} \operatorname*{ess\,sup}_U |D^\alpha u| & (p = \infty). \end{cases}$$

8.2 Convergence and Local Spaces

Definition 8.3 (Convergence in Sobolev Spaces)**.**

1. Let $\{u_m\}_{m=1}^\infty, u \in W^{k,p}(U)$. We say u_m **converges to** u **in** $W^{k,p}(U)$, written

$$u_m \to u \quad \text{in } W^{k,p}(U),$$

   provided $\lim\limits_{m \to \infty} \|u_m - u\|_{W^{k,p}(U)} = 0$.

2. We say $u_m \to u$ **in** $W^{k,p}_{\mathbf{loc}}(U)$ if $u_m \to u$ in $W^{k,p}(V)$ for every $V \subset\subset U$ (i.e., V is compactly contained in U).

8.3 Space $W_0^{k,p}(U)$: Functions with Zero Boundary Values

Definition 8.4 ($W_0^{k,p}(U)$)**.** We denote by $W_0^{k,p}(U)$ the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$.

Thus $u \in W_0^{k,p}(U)$ if and only if there exist functions $u_m \in C_c^\infty(U)$ such that $u_m \to u$ in $W^{k,p}(U)$. Intuitively, $W_0^{k,p}(U)$ consists of functions $u \in W^{k,p}(U)$ that satisfy

$$\text{“}D^\alpha u = 0 \text{ on } \partial U\text{”} \quad \text{for all } |\alpha| \leq k - 1,$$

in a weak sense. This interpretation will be made precise by the theory of traces.

9 Poincaré Inequality

The Poincaré inequality shows that for functions vanishing on the boundary, the L^p norm can be controlled by the norm of the gradient.

Theorem 9.1 (Poincaré Inequality for $W_0^{1,p}(\Omega)$)**.** *Let Ω be a bounded domain. If $u \in W_0^{1,p}(\Omega)$, there exists a constant $C = C(\Omega, p)$ such that*

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}.$$

Proof. We present the proof for $n = 1$; the generalization to $n \geq 2$ is straightforward.

First, prove the inequality for $u \in C_0^\infty(\Omega)$. Since Ω is bounded, enclose it in a cube $Q = [0,a]^n$. Extend u by zero to Q, so $u \in C_0^\infty(Q)$.

For $x = (x_1, \dots, x_n) \in Q$, we have

$$u(x) = \int_0^{x_1} \partial_1 u(t, x_2, \dots, x_n)\, dt,$$

because $u(0, x_2, \dots, x_n) = 0$. By Hölder's inequality,

$$|u(x)|^p \leq \left(\int_0^{x_1} |\partial_1 u(t, x_2, \dots, x_n)|\, dt \right)^p \leq x_1^{p-1} \int_0^{x_1} |\partial_1 u(t, x_2, \dots, x_n)|^p\, dt.$$

Integrate over Q:

$$\begin{aligned} \int_Q |u(x)|^p dx &\leq \int_Q x_1^{p-1} \left(\int_0^{x_1} |\partial_1 u(t, x')|^p\, dt \right) dx \\ &= \int_{[0,a]^{n-1}} \int_0^a x_1^{p-1} \left(\int_0^{x_1} |\partial_1 u(t, x')|^p\, dt \right) dx_1\, dx', \end{aligned}$$

where $x' = (x_2, \dots, x_n)$. Changing the order of integration,

$$\begin{aligned} &\int_0^a x_1^{p-1} \left(\int_0^{x_1} |\partial_1 u(t, x')|^p\, dt \right) dx_1 \\ &\qquad = \int_0^a |\partial_1 u(t, x')|^p \left(\int_t^a x_1^{p-1}\, dx_1 \right) dt \\ &\qquad = \int_0^a |\partial_1 u(t, x')|^p \cdot \frac{a^p - t^p}{p}\, dt \\ &\qquad \leq \frac{a^p}{p} \int_0^a |\partial_1 u(t, x')|^p\, dt. \end{aligned}$$

Therefore,

$$\int_Q |u|^p dx \leq \frac{a^p}{p} \int_Q |\partial_1 u|^p dx \leq \frac{a^p}{p} \int_Q |\nabla u|^p dx.$$

Hence,

$$\|u\|_{L^p(\Omega)} = \|u\|_{L^p(Q)} \leq \frac{a}{p^{1/p}} \|\nabla u\|_{L^p(\Omega)}.$$

Now for general $u \in W_0^{1,p}$, take a sequence $\{u_k\} \subset C_0^\infty(\Omega)$ with $u_k \to u$ in $W^{1,p}(\Omega)$. Applying the inequality to each u_k and passing to the limit yields the result. $\square$

Remark 9.1 (General Poincaré Inequality)**.** *For $u \in W_0^{n,p}(\Omega)$ with Ω bounded, there exists $C = C(\Omega, n, p)$ such that*

$$\|u\|_{L^p(\Omega)} \leq C \sum_{|\alpha|=n} \|D^\alpha u\|_{L^p(\Omega)}.$$

10 Boundary Values and Trace Operator

Remark 10.1. *Boundary values of Sobolev functions are defined via a* ***trace operator****. For $u \in W_0^{1,p}(\Omega)$, the trace Tu is a well-defined element of $L^p(\partial\Omega)$ (for sufficiently smooth $\partial\Omega$). This is not a pointwise restriction but a continuous linear operator*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega).$$

The space $W^{1,p}(\Omega)$ precisely consists of functions whose trace is zero.

11 Distributions and Weak Solutions

11.1 Distributions: Basic Idea

A **distribution** generalizes the concept of a function. Formally, a distribution T is a continuous linear functional

$$T : C_0^\infty(\Omega) \to \mathbb{R}.$$

Example 11.1 (Regular Distributions)**.** If $g \in L^1_{\text{loc}}(\Omega)$, then

$$T_g(\varphi) := \int_\Omega g(x)\varphi(x)dx$$

defines a distribution.

Example 11.2 (Dirac Delta)**.** The Dirac delta δ_{x_0} is defined by $\delta_{x_0}(\varphi) = \varphi(x_0)$.

Test functions $\varphi \in C_0^\infty(\Omega)$ are ideal because they are smooth and vanish near the boundary, so integration by parts produces no boundary terms.

11.2 Weak Formulation of Poisson's Equation

Consider the Dirichlet problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

If u were smooth, multiplying by a test function $\varphi \in C_0^\infty(\Omega)$ and integrating by parts gives

$$\int_\Omega \nabla u \cdot \nabla\varphi dx = \int_\Omega f\varphi dx.$$

This identity makes sense even when u is not twice differentiable.

Definition 11.1 (Weak Solution)**.** A function $u \in H_0^1(\Omega)$ is a **weak solution** of $-\Delta u = f$ with $u = 0$ on $\partial\Omega$ if

$$\int_\Omega \nabla u \cdot \nabla\varphi dx = \int_\Omega f\varphi dx \quad \forall\varphi \in C_0^\infty(\Omega).$$

Key idea: Derivatives are transferred from u to the test function φ. This allows solutions when u lacks classical second derivatives and when f is only in $L^2(\Omega)$ or even $H^{-1}(\Omega)$.

11.3 The Dual Space $H^{-1}(\Omega)$

Definition 11.2. The space $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$:

$$H^{-1}(\Omega) = \left(H_0^1(\Omega)\right)^*.$$

Its elements are continuous linear functionals on $H_0^1(\Omega)$.

Remark 11.1 (Interpretation)**.**

- *Elements of $H^{-1}(\Omega)$ are distributions that act continuously on $H_0^1(\Omega)$.*
- *The negative index indicates that H^{-1} functions are "one derivative less regular" than L^2.*
- $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ *is the natural mapping of the Laplacian.*
- *In the weak formulation, it is natural to allow $f \in H^{-1}(\Omega)$.*

Key takeaway: $H^{-1}(\Omega)$ is the natural space for the right-hand side f in the weak formulation of second-order elliptic equations, ensuring the formulation remains well-defined even for irregular data.