PDE Seminar: Day 2
Energy Methods

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Abstract

This document serves as an extended reference for the second day of the 25W PDE Seminar. It provides a descriptive analysis of the four pillars of elliptic theory: (1) The structural definition of Ellipticity, (2) Variational techniques via Energy Methods, (3) Geometric control via Maximum Principles, and (4) Fine regularity via Hölder Spaces. We integrate perspectives from standard texts—Evans, Gilbarg-Trudinger (GT), and Han-Lin—to bridge the gap between weak and classical solutions.

Contents

1 Elliptic Operators and Ellipticity

Primary Reference: L.C. Evans, Partial Differential Equations, Chapter 6.1.

The study of elliptic partial differential equations (PDEs) is essentially the generalization of the Laplace equation, $\Delta u = 0$, to operators with variable coefficients. In this section, we define the operators rigorously and discuss the physical and mathematical implications of "ellipticity."

1.1 Divergence vs. Non-Divergence Form

Second-order linear operators typically appear in two distinct forms, each suited to different analytical techniques.

Definition 1.1 (General Linear Operator (negative signature representation))**.** Let $\Omega \subset \mathbb{R}^n$ be an open, connected set.

1. **Divergence Form:** Useful for integration by parts and energy methods (weak solutions).

$$Lu = -\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u. \tag{1}$$

2. **Non-Divergence Form:** Useful for maximum principles and classical regularity (strong solutions).

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u. \tag{2}$$

We assume the symmetry condition $a^{ij} = a^{ji}$.

Remark 1.1 (Equivalence)**.** If the coefficients a^{ij} are differentiable (C^1), one can convert between forms using the product rule: $(a^{ij}u_{x_i})_{x_j} = a^{ij}u_{x_i x_j} + (a^{ij})_{x_j}u_{x_i}$. However, if $a^{ij} \in L^\infty$ (potentially discontinuous), the divergence form is the only meaningful formulation, as the classical non-divergence operator is ill-defined.

1.2 Uniform Ellipticity

The core structural assumption is that the operator "looks like" the Laplacian at every point.

Definition 1.2 (Uniform Ellipticity)**.** The operator L is **uniformly elliptic** if there exists a constant $\theta > 0$ such that for almost every $x \in \Omega$ and all vectors $\xi \in \mathbb{R}^n$,

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2. \tag{3}$$

1.2.1 Physical Interpretation

Consider the heat diffusion or electrostatic potential. The matrix $A = (a^{ij})$ represents the **conductivity** of the medium.

- Condition (3) ensures that the medium is conductive in *all* directions. There is no direction ξ in which diffusion is blocked (which would lead to a degenerate parabolic equation) or reversed (which would lead to a hyperbolic equation).
- The constant θ represents the minimum conductivity. "Uniform" means this minimum does not vanish as we move across the domain Ω.

2 Energy Methods and Functional Analysis

Primary Reference: L.C. Evans, Partial Differential Equations, Chapter 6.2.

Classical methods fail when coefficients are not smooth (e.g., composite materials where conductivity jumps). Energy methods replace pointwise derivatives with integral averages, leveraging the structure of Hilbert spaces (L^2, H^1).

2.1 Derivation of the Weak Formulation

Suppose we want to solve the Dirichlet problem:

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Multiplying by a test function $v \in C_c^\infty(\Omega)$ and integrating by parts (moving one derivative from u to v) leads to the equation:

$$\int_\Omega \sum_{i,j} a^{ij} u_{x_i} v_{x_j} + \sum_i b^i u_{x_i} v + cuv\, dx = \int_\Omega fv\, dx. \tag{4}$$

This integral makes sense even if u is only once differentiable. This motivates the definition of the Sobolev space $H_0^1(\Omega)$, the closure of C_c^∞ under the norm $\|u\|_{H^1}^2 = \int(|Du|^2 + |u|^2)$.

2.2 The Bilinear Form and Coercivity

We define the bilinear form $B : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ corresponding to L:

$$B[u, v] := \int_\Omega (a^{ij} D_i u D_j v + b^i D_i uv + cuv) dx.$$

The crux of the energy method is **Gårding's Inequality**, which states that elliptic operators are "positive definite up to a shift."

Theorem 2.1 (Energy Estimate). *There exist constants $\beta > 0$ and $\gamma \geq 0$ such that*

$$B[u, u] \geq \beta \|u\|_{H_0^1}^2 - \gamma \|u\|_{L^2}^2 \quad \textit{for all } u \in H_0^1(\Omega). \tag{5}$$

Idea of Proof: The principal term $a^{ij} D_i u D_j u$ gives $\theta \int |Du|^2$ by ellipticity. The lower order terms (b^i, c) can be bounded by $\epsilon \int |Du|^2 + C_\epsilon \int u^2$ using Cauchy-Schwarz and Young's inequality. Choosing ϵ small enough allows the gradient term to dominate.

Detailed Proof. By the uniform ellipticity condition, we have $\sum a^{ij} \xi_i \xi_j \geq \theta |\xi|^2$. Thus,

$$\int_\Omega \sum a^{ij} u_{x_i} u_{x_j}\, dx \geq \theta \int_\Omega |Du|^2\, dx.$$

Now we handle the lower order terms. Let $M = \max_i \|b^i\|_{L^\infty}$ and $K = \|c\|_{L^\infty}$. Using Cauchy-Schwarz and Young's inequality with ϵ ($|ab| \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2$):

$$\begin{aligned} \left| \int_\Omega \sum b^i u_{x_i} u\, dx \right| &\leq \int_\Omega \sum |b^i||Du||u|\, dx \\ &\leq M \int_\Omega \left(\epsilon |Du|^2 + \frac{1}{4\epsilon} u^2 \right) dx. \end{aligned}$$

Similarly, $|\int cu^2| \leq K \int u^2$. Combining these estimates into $B[u,u]$:

$$B[u,u] \geq \theta\|Du\|_{L^2}^2 - M\epsilon\|Du\|_{L^2}^2 - \left(\frac{M}{4\epsilon} + K\right)\|u\|_{L^2}^2.$$

We choose $\epsilon = \frac{\theta}{2M}$ to absorb the gradient term. Then $\theta - M\epsilon = \frac{\theta}{2}$.

$$B[u,u] \geq \frac{\theta}{2}\|Du\|_{L^2}^2 - C\|u\|_{L^2}^2.$$

Since $u \in H_0^1(\Omega)$, the norm is $\|u\|_{H^1}^2 = \|Du\|_{L^2}^2 + \|u\|_{L^2}^2$. Thus $\|Du\|_{L^2}^2 = \|u\|_{H^1}^2 - \|u\|_{L^2}^2$. Substituting this yields:

$$B[u,u] \geq \frac{\theta}{2}\|u\|_{H^1}^2 - \left(\frac{\theta}{2} + C\right)\|u\|_{L^2}^2.$$

Setting $\beta = \theta/2$ and $\gamma = \theta/2 + C$ completes the proof. □

2.3 Lax-Milgram Theorem: Existence

The Lax-Milgram theorem is the "Fundamental Theorem of Linear Algebra" for infinite-dimensional Hilbert spaces.

Theorem 2.2 (Lax-Milgram)**.** *Let H be a real Hilbert space and $B : H \times H \to \mathbb{R}$ be a bilinear form that is:*

1. ***Bounded:*** $|B[u,v]| \leq \alpha\|u\|\|v\|$,
2. ***Coercive:*** $B[u,u] \geq \beta\|u\|^2$.

Then for any $f \in H^*$, there exists a unique $u \in H$ such that $B[u,v] = \langle f, v\rangle$ for all $v \in H$.

Detailed Proof. Since B is not necessarily symmetric, we cannot use the Riesz Representation Theorem directly on B. Instead, we use Riesz to define an operator.

1. Construction of Operator A**:** For a fixed $u \in H$, the map $v \mapsto B[u,v]$ is a bounded linear functional on H. By the Riesz Representation Theorem, there exists a unique element $w \in H$ such that $(w,v)_H = B[u,v]$ for all v. We define $Au = w$. This defines a linear operator $A : H \to H$ satisfying:

$$(Au, v)_H = B[u,v].$$

From the boundedness of B, $\|Au\| \leq \alpha\|u\|$, so A is bounded. Similarly, let w_f be the Riesz representative of the functional f (i.e., $(w_f, v) = \langle f, v\rangle$). The problem $B[u,v] = \langle f, v\rangle$ becomes equivalent to solving the operator equation $Au = w_f$.

2. Injectivity and Closed Range: By coercivity, $\beta\|u\|^2 \leq B[u,u] = (Au,u) \leq \|Au\|\|u\|$. Thus:

$$\beta\|u\| \leq \|Au\|.$$

This implies A is injective ($Au = 0 \implies u = 0$). Furthermore, the range $R(A)$ is closed. If $Au_n \to y$, then $\{Au_n\}$ is Cauchy. By the inequality, $\{u_n\}$ is Cauchy and converges to some u. Thus $Au_n \to Au = y \in R(A)$.

3. Surjectivity: We show $R(A) = H$. Since $R(A)$ is closed, we can decompose $H = R(A) \oplus R(A)^\perp$. Let $z \in R(A)^\perp$. Then $(Az, z) = 0$. By coercivity, $\beta\|z\|^2 \leq (Az,z) = 0$, implying $z = 0$. Hence $R(A)^\perp = \{0\}$, and A is surjective.

Since A is bijective, there exists a unique u such that $Au = w_f$. □

Remark 2.1 (From Estimates to Existence)**.** If $\gamma = 0$ in Gårding's inequality (e.g., if $c \geq 0$ and $b^i = 0$), then B is strictly coercive. Lax-Milgram immediately guarantees a unique weak solution $u \in H_0^1(\Omega)$. If $\gamma > 0$, we rely on the *Fredholm Alternative*: either a unique solution exists, or the homogeneous problem has a non-trivial solution (eigenfunctions).

3 Maximum Principles

Primary Reference: Q. Han & F. Lin, Elliptic PDEs, Chapter 2.

While Energy Methods operate globally (integrals), Maximum Principles operate locally (pointwise values). They exploit the convexity of solutions to elliptic equations. Here we focus on operators in non-divergence form.

3.1 The Weak Maximum Principle (positive signature representation)

The intuition comes from 1D calculus: if $u''(x) > 0$, the function is convex and cannot have an interior maximum.

Theorem 3.1 (Weak Maximum Principle). *Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy $Lu \geq 0$ (subsolution) in Ω, with $c(x) \leq 0$. Then:*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+. \tag{6}$$

Proof Strategy (Barrier Method):

1. First, assume strict inequality $Lu > 0$. At an interior maximum x_0, we must have D^2u negative semi-definite ($a^{ij}D_{ij}u \leq 0$) and $Du = 0$. Thus $Lu \leq c(x_0)u(x_0)$. If $c = 0$, this contradicts $Lu > 0$.
2. For the general case $Lu \geq 0$, we introduce a perturbation $v_\epsilon(x) = u(x) + \epsilon e^{\lambda x_1}$ for sufficiently large λ. One shows $Lv_\epsilon > 0$, applies step 1, and lets $\epsilon \to 0$.

Detailed Proof. **Case 1: The Strict Case ($Lu > 0$).** Suppose for contradiction that u attains a maximum at an interior point $x_0 \in \Omega$ such that $u(x_0) = \max_{\bar{\Omega}} u > \max_{\partial\Omega} u^+ \geq 0$. At this maximum point x_0:

- $Du(x_0) = 0$.
- The Hessian matrix $D^2u(x_0)$ is negative semi-definite. Since $A = (a^{ij})$ is positive definite, the trace $\mathrm{tr}(AD^2u) = \sum a^{ij}D_{ij}u \leq 0$.

Substituting these into the operator:

$$Lu(x_0) = \sum a^{ij}D_{ij}u(x_0) + \sum b^i D_i u(x_0) + c(x_0)u(x_0) \leq 0 + 0 + c(x_0)u(x_0).$$

Since $c \leq 0$ and $u(x_0) > 0$, we have $c(x_0)u(x_0) \leq 0$. Thus $Lu(x_0) \leq 0$, which contradicts the assumption $Lu > 0$. Therefore, the maximum cannot be interior.

Case 2: The General Case ($Lu \geq 0$). Let $u_\epsilon(x) = u(x) + \epsilon e^{\lambda x_1}$ for $\epsilon > 0, \lambda > 0$. We compute $L(e^{\lambda x_1})$. Since $D_1(e^{\lambda x_1}) = \lambda e^{\lambda x_1}$ and $D_{11}(e^{\lambda x_1}) = \lambda^2 e^{\lambda x_1}$:

$$L(e^{\lambda x_1}) = e^{\lambda x_1}(a^{11}\lambda^2 + b^1\lambda + c).$$

By ellipticity, $a^{11} \geq \theta$. We can choose λ large enough so that $\theta\lambda^2$ dominates the $b^1\lambda$ and c terms, making $L(e^{\lambda x_1}) > 0$. Then $Lu_\epsilon = Lu + \epsilon L(e^{\lambda x_1}) > 0$. By Case 1, $\sup_\Omega u_\epsilon = \sup_{\partial\Omega} u_\epsilon$. Letting $\epsilon \to 0$, we conclude $\sup_\Omega u \leq \sup_{\partial\Omega} u^+$. □

3.2 Hopf Lemma and Strong Maximum Principle

The Weak Maximum Principle does not rule out the possibility that the maximum is achieved *both* inside and on the boundary (e.g., a constant function). The Strong Principle says non-constant solutions *never* achieve an interior maximum.

The bridge between interior and boundary behavior is the **Hopf Lemma**.

Lemma 3.2 (Hopf Boundary Point Lemma)**.** *Suppose $Lu \geq 0$ and $x_0 \in \partial\Omega$ is a strict maximum point (i.e., $u(x_0) > u(x)$ for interior x). Assume Ω satisfies the interior sphere condition at x_0. Then:*

$$\frac{\partial u}{\partial \nu}(x_0) > 0. \tag{7}$$

Interpretation: The solution must approach the maximum on the boundary with a strictly positive slope. It cannot "flatten out" tangentially.

Detailed Proof. Assume without loss of generality that the interior sphere B is centered at the origin with radius R, touching $\partial\Omega$ at x_0. Let A be the annulus $B_R(0) \setminus B_{R/2}(0)$. We construct a barrier function $h(x) = e^{-\alpha|x|^2} - e^{-\alpha R^2}$. Note that $h(x) = 0$ on $|x| = R$ (the boundary containing x_0) and $h(x) > 0$ inside. A calculation similar to the Weak Maximum Principle proof shows that for α sufficiently large, $Lh > 0$ in the annulus A.

Consider $v(x) = u(x) - u(x_0) + \epsilon h(x)$.

- On the outer boundary $|x| = R$: $v(x) = u(x_0) - u(x_0) + 0 = 0$.
- On the inner boundary $|x| = R/2$: $u(x) - u(x_0) \leq -\delta < 0$ (since x_0 is a strict max). We choose ϵ small enough so $\epsilon h \leq \delta$. Then $v \leq 0$.
- In the interior of A: $Lv = Lu - cu(x_0) + \epsilon Lh \geq 0 - 0 + \text{positive} > 0$.

By the Weak Maximum Principle applied to v, we have $v \leq 0$ in A. Since $v(x_0) = 0$, the outward normal derivative at x_0 must be non-negative:

$$\frac{\partial v}{\partial \nu}(x_0) \geq 0 \implies \frac{\partial u}{\partial \nu}(x_0) + \epsilon \frac{\partial h}{\partial \nu}(x_0) \geq 0.$$

Calculation shows $\frac{\partial h}{\partial \nu}(x_0) = -2\alpha R e^{-\alpha R^2} < 0$. Thus $\frac{\partial u}{\partial \nu}(x_0) \geq -\epsilon \frac{\partial h}{\partial \nu}(x_0) > 0$. □

Theorem 3.3 (Strong Maximum Principle (positive signature representation))**.** *If $Lu \geq 0$ in a connected domain Ω and u attains a global maximum at an interior point, then u is constant.*

Proof Strategy: If the set where u attains its maximum is neither empty nor the whole domain, we can find a point on the boundary of this set strictly inside Ω. Applying the Hopf Lemma on a small ball touching this point leads to a contradiction.

Detailed Proof. Let $M = \sup_\Omega u$. Define the set $\Sigma = \{x \in \Omega : u(x) = M\}$. Since u is continuous, Σ is closed in Ω. Suppose $\Sigma \neq \Omega$. Since Ω is connected, the boundary $\partial\Sigma \cap \Omega$ is non-empty. Let $y \in \Omega \setminus \Sigma$ be a point closer to Σ than to $\partial\Omega$. We can expand a ball centered at y until it just touches Σ at some point $x_0 \in \Sigma$. Let B be this ball. Then $u(x) < M$ for all $x \in B$ and $u(x_0) = M$. This satisfies the conditions for the Hopf Lemma at x_0 (with B as the interior sphere). Therefore, $\frac{\partial u}{\partial \nu}(x_0) > 0$. However, since x_0 is an interior maximum point of Ω, we must have $Du(x_0) = 0$, which implies $\frac{\partial u}{\partial \nu}(x_0) = 0$. This is a contradiction. Thus, $\Sigma = \Omega$, meaning $u \equiv M$ everywhere. □

4 Hölder Spaces ($C^{k,\alpha}$)

Primary Reference: D. Gilbarg & N.S. Trudinger, Elliptic PDEs of Second Order, Chapter 4.

We now turn to the regularity of solutions. A fundamental issue in PDE theory is that the space of continuous functions C^0 is not well-adapted to elliptic operators. If $\Delta u = f$ with $f \in C^0$, it is *not* guaranteed that $u \in C^2$. The second derivatives may verify the equation almost everywhere but fail to be continuous.

To fix this, we need slightly more regularity: Hölder continuity.

4.1 Definitions and Norms

Definition 4.1 (Hölder Continuity)**.** A function u is Hölder continuous with exponent $\alpha \in (0, 1]$ at x_0 if:

$$\sup_{x\in\Omega} \frac{|u(x) - u(x_0)|}{|x - x_0|^\alpha} < \infty.$$

It is uniformly Hölder continuous if the constant is independent of x_0.

We define the Hölder seminorm and the full norm for the space $C^{k,\alpha}(\bar{\Omega})$:

$$[u]_{k,\alpha} = \sum_{|\beta|=k} \sup_{x\neq y} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^\alpha}, \tag{8}$$

$$\|u\|_{C^{k,\alpha}} = \|u\|_{C^k} + [u]_{k,\alpha}. \tag{9}$$

4.2 Why Hölder Spaces?

1. **Optimal Regularity:** If $f \in C^{0,\alpha}$, then $u \in C^{2,\alpha}$. The fractional order α is preserved from the data to the second derivatives of the solution.

2. **Compactness:** By the Arzelà-Ascoli theorem, a bounded sequence in $C^{k,\alpha}$ has a convergent subsequence in C^k (and in $C^{k,\beta}$ for $\beta < \alpha$). This is crucial for existence proofs.

4.3 Interpolation Inequalities

A powerful technical tool in GT (Chapter 6) is the ability to bound intermediate derivatives.

Proposition 4.1 (Global Interpolation)**.** *For any $\epsilon > 0$, there exists a constant $C(\epsilon)$ such that:*

$$\|u\|_{C^1} \leq \epsilon \|u\|_{C^2} + C(\epsilon)\|u\|_{C^0}.$$

More generally, intermediate norms ($C^{1,\alpha}$) can be interpolated between higher ($C^{2,\alpha}$) and lower (C^0) norms.

This allows us to treat lower-order terms ($b^i D_i u + cu$) as "perturbations." Since we can make their contribution arbitrarily small (by choosing ϵ), they do not destroy the invertability (solvability of the PDE) governed by the principal part $a^{ij}D_{ij}u$.

4.4 The Schauder Estimates

The Schauder estimates are a priori inequalities that bound the solution's norm by the data's norm.

Theorem 4.2 (Interior Schauder Estimate)**.** *Let $u \in C^{2,\alpha}(\Omega)$ be a solution to $Lu = f$, with coefficients in $C^{0,\alpha}$. Then for any subdomain $\Omega' \subset\subset \Omega$:*

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq C \left(\|f\|_{C^{0,\alpha}(\Omega)} + \|u\|_{C^0(\Omega)} \right). \tag{10}$$

The constant C depends only on the ellipticity constant θ, the $C^{0,\alpha}$ norms of the coefficients, and the distance from Ω' to $\partial\Omega$.

Sketch of Proof. The proof generalizes the regularity theory of the Laplacian to variable coefficient operators via the **"Freezing of Coefficients"** technique.

1. **Base Case (Constant Coefficients):** We assume the estimate holds for operators with constant coefficients (e.g., $L_0 = \sum a_{ij}(x_0)D_{ij}$). This is derived from the Newtonian potential theory.

2. **Localization:** We focus on a small ball $B_r(x_0)$ inside the domain using a smooth cutoff function η.
3. **Perturbation:** Inside this small ball, the variable coefficients $a_{ij}(x)$ deviate very little from the constant value $a_{ij}(x_0)$. We treat the difference $(L - L_0)u$ as a small perturbation error.
4. **Absorption:** We show that the "error" terms can be absorbed into the left-hand side of the inequality because their norm is proportional to the small radius r^α.
5. **Interpolation:** Finally, we use interpolation inequalities to control intermediate derivatives (like ∇u) using only the highest derivatives ($D^2 u$) and the function value (u).

$\square$