PDE Seminar: Day 4
Schauder Estimates

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1 Overview: Classical Solvability

In the previous section, we established the uniqueness of solutions using Maximum Principles. Now, we address the question of "existence" and "regularity" for the Dirichlet problem:

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{1}$$

The space C^k is not complete, which implies that the limit of a sequence of C^k functions may not belong to the same class. To resolve this issue, we introduce Hölder spaces. Specifically, since $C^{k,\alpha}$ lies between C^k and C^{k+1}, the additional regularity requirement is strictly less than one full derivative; nevertheless, the functional analytic properties of the space are significantly superior to those of C^k. This is precisely the reason why we introduce Hölder spaces. In particular, $C^{k,\alpha}$ is a Banach space (a complete normed space).

2 Theorem 3.4: Schauder Estimates

The Schauder estimates are *a priori* estimates that bound the $C^{2,\alpha}$ norm of the solution by the $C^{0,\alpha}$ norm of the source term f.

Theorem 2.1 (6.2. Schauder Estimates)**.** *Let $u \in C^{2,\alpha}(\Omega)$ satisfy $Lu = f$ with $f \in C^\alpha$. Under uniform ellipticity and C^α bounds on coefficients:*

$$|u|^*_{2,\alpha;\Omega} \leq C(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}) \tag{2}$$

where $C = C(n, \alpha, \lambda, \Lambda)$.

Sketch of Proof. The proof generalizes the regularity theory of the Laplacian to variable coefficient operators via the **"Freezing of Coefficients"** technique.

1. **Base Case (Constant Coefficients):** We assume the estimate holds for operators with constant coefficients (e.g., $L_0 = \sum a_{ij}(x_0)D_{ij}$). This is derived from the Newtonian potential theory.

2. **Localization:** We focus on a small ball $B_r(x_0)$ inside the domain using a smooth cutoff function η.

3. **Perturbation:** Inside this small ball, the variable coefficients $a_{ij}(x)$ deviate very little from the constant value $a_{ij}(x_0)$. We treat the difference $(L - L_0)u$ as a small perturbation error.

4. **Absorption:** We show that the "error" terms can be absorbed into the left-hand side of the inequality because their norm is proportional to the small radius r^α.

5. **Interpolation:** Finally, we use interpolation inequalities to control intermediate derivatives (like ∇u) using only the highest derivatives ($D^2 u$) and the function value (u).

□

Detailed Proof (Simplified). We proceed in four clear steps: Localization, Freezing, Estimation, and Absorption.

Step 1: Localization. Fix a point $x_0 \in \Omega'$ and choose a radius r small enough so that the ball $B_r(x_0) \subset \Omega$. Let $\eta \in C_c^\infty(B_r)$ be a cutoff function such that $\eta \equiv 1$ in $B_{r/2}$ and $0 \leq \eta \leq 1$. Define $v = \eta u$. Note that $v = u$ in the smaller ball $B_{r/2}$.

Step 2: Freezing the Coefficients. We compare the operator L (with variable coefficients $a_{ij}(x)$) to the "frozen" operator L_0 (with constant coefficients $a_{ij}(x_0)$):

$$L_0 = \sum_{i,j} a_{ij}(x_0) D_{ij}.$$

We want to apply the known Schauder estimate for L_0 to v. Let's compute $L_0 v$:

$$\begin{aligned} L_0 v &= L_0(\eta u) \\ &= \eta L_0 u + \text{Commutators involving } \nabla \eta, \nabla u \dots \\ &= \eta(L_0 - L)u + \eta L u + \text{Lower Order Terms.} \end{aligned}$$

Since $Lu = f$, we can rearrange this as:

$$L_0 v = \eta f + \eta \sum_{i,j} (a_{ij}(x_0) - a_{ij}(x)) D_{ij} u + \text{LOT}, \tag{3}$$

where LOT (Lower Order Terms) contains terms with Du and u, which are "easier" to bound.

Step 3: Applying the Constant Coefficient Estimate. We apply the standard Schauder estimate for constant coefficients to equation (3):

$$\|v\|_{C^{2,\alpha}} \leq C \left(\|\text{RHS of (3)}\|_{C^{0,\alpha}} \right).$$

Let's analyze the critical term on the RHS: $E = \eta \sum (a_{ij}(x_0) - a_{ij}(x)) D_{ij} u$. The $C^{0,\alpha}$ norm of a product roughly satisfies $\|gh\|_\alpha \leq \|g\|_\alpha \|h\|_\infty + \|g\|_\infty \|h\|_\alpha$. The key insight is that on the support of η (which is B_r), the difference in coefficients is small:

$$|a_{ij}(x_0) - a_{ij}(x)| \leq [a_{ij}]_\alpha |x_0 - x|^\alpha \leq [a_{ij}]_\alpha r^\alpha.$$

Thus, the norm of the error term is bounded by:

$$\|E\|_{C^{0,\alpha}} \leq C r^\alpha \|u\|_{C^{2,\alpha}(B_r)} + C_r \|u\|_{C^2(B_r)}.$$

Step 4: Absorption and Conclusion. Substituting the bound for E back into the main estimate:

$$\|u\|_{C^{2,\alpha}(B_{r/2})} \leq \|v\|_{C^{2,\alpha}} \leq C \left(\|f\|_{C^{0,\alpha}} + r^\alpha \|u\|_{C^{2,\alpha}(B_r)} + C_r \|u\|_{C^1} \right).$$

Notice that $\|u\|_{C^{2,\alpha}}$ appears on the Right-Hand Side. However, it is multiplied by r^α. By choosing the radius r **sufficiently small** (depending on the ellipticity and coefficients), we can ensure $Cr^\alpha \le 1/2$. We can then subtract this term to the Left-Hand Side ("absorbing" the highest order term):

$$\frac{1}{2}\|u\|_{C^{2,\alpha}(B_{r/2})} \le C\left(\|f\|_{C^{0,\alpha}} + \|u\|_{C^1(B_r)}\right).$$

Finally, using the interpolation inequality $\|u\|_{C^1} \le \epsilon\|u\|_{C^{2,\alpha}} + C_\epsilon\|u\|_{C^0}$ to handle the lower order terms, and covering the compact set Ω' with finitely many such small balls, we arrive at the desired result:

$$\|u\|_{C^{2,\alpha}(\Omega')} \le C\left(\|f\|_{C^{0,\alpha}(\Omega)} + \|u\|_{C^0(\Omega)}\right).$$

□

3 Theorem 3.5: Method of Continuity

Schauder estimates provide a bound "if" a solution exists. The Method of Continuity uses this bound to prove that a solution "actually" exists.

Theorem 3.1 (Method of Continuity). *Let Ω be $C^{2,\alpha}$ and L be strictly elliptic with $c \le 0$. If the Dirichlet problem for Δ is solvable for all C^α data, then $Lu = f, u = \varphi$ is uniquely solvable in $C^{2,\alpha}(\overline{\Omega})$.*

Sketch of Proof. Define a family of operators $L_t = (1-t)\Delta + tL$.

1. Let $E = \{t \in [0,1] : L_t u = f \text{ is solvable}\}$.
2. $0 \in E$ by assumption. E is shown to be open via the Inverse Function Theorem (or Banach Fixed Point Theorem).
3. E is shown to be closed using the a priori Schauder estimates to pass to the limit. Thus $1 \in E$.

□

Detailed Proof. The proof relies on establishing estimates for the Laplacian (Model Case) and extending the result to general operators using the Method of Continuity.

Step 1: The Model Case ($\Delta u = f$). The solution is represented by the Newtonian potential $u = N * f$. The crucial difficulty is that the kernel for the second derivatives, $D^2N(z)$, behaves like $|z|^{-n}$, which is not integrable near the origin. However, we exploit the Hölder continuity of f. We can express the second derivative roughly as:

$$D^2u(x) \sim \int_{\mathbb{R}^n} D^2N(x-y)\,(f(y) - f(x))\,dy.$$

Since $f \in C^{0,\alpha}$, the difference term provides a decay factor: $|f(y)-f(x)| \le C|x-y|^\alpha$. Multiplying the singularity $|x-y|^{-n}$ by this factor yields $|x-y|^{-n+\alpha}$. Since $-n+\alpha > -n$, this new kernel *is* locally integrable. This "cancellation of singularity" ensures that D^2u exists and belongs to $C^{0,\alpha}$.

Step 2: The Method of Continuity. We connect the Laplacian to the general operator L using the family $L_t = (1-t)\Delta + tL$ for $t \in [0,1]$. Let E be the set of t for which $L_t u = f$ is solvable.

- **Non-empty ($0 \in E$):** We know the Laplacian ($t = 0$) is solvable from Step 1.
- **Openness:** If L_{t_0} is invertible, standard operator theory implies that small perturbations (nearby t) are also invertible. Thus, E is open.

- **Closedness:** This is where the Schauder estimate is vital. If a sequence $t_k \in E$ converges to t^*, the corresponding solutions u_k satisfy the uniform bound $\|u_k\|_{C^{2,\alpha}} \leq C\|f\|_{C^{0,\alpha}}$. By compactness (Arzelà-Ascoli), the sequence u_k converges to a limit solution u^*, proving that $t^* \in E$.

Since E is non-empty, open, and closed in $[0,1]$, we must have $E = [0,1]$. Therefore, the problem is solvable for L (at $t = 1$). □

3.1 Why Hölder Spaces? (Proof Idea)

The proof is built upon the Newtonian potential for the Laplacian.

1. **Constant Coefficients (Model Case):** Consider $\Delta u = f$. The solution is given by convolution with the fundamental solution $N(x)$:

$$u(x) = \int_{\mathbb{R}^n} N(x-y)f(y)dy$$

   Taking two derivatives yields a singular integral:

$$D_{ij}u(x) = \int_{\mathbb{R}^n} D_{ij}N(x-y)f(y)dy$$

   The kernel $D_{ij}N$ is not integrable near the singularity. However, using the Hölder condition $|f(y) - f(x)| \leq K|x-y|^\alpha$, we can utilize the "cancellation of singularities" property. This ensures that D^2u exists and is Hölder continuous.

2. **Variable Coefficients (Freezing Coefficients):** For the general operator Lu, we fix a point x_0 and rewrite the equation as:

$$L_{x_0}u = f + (L_{x_0} - L)u$$

   where L_{x_0} has constant coefficients frozen at x_0. Since the coefficients are continuous, the error term $(L_{x_0} - L)$ is small in a small neighborhood of x_0. We can then view this as a perturbation of the constant coefficient case.

4 Summary of Chapter 6

- Schauder Estimates (Thm 3.4): Provide the necessary *a priori* bounds ($C^{2,\alpha}$ control).
- Method of Continuity (Thm 3.5): Converts these bounds into an existence proof by deforming a known operator (Laplacian) into the target operator.
- This establishes the "Classical Existence Theory" for elliptic PDEs.