25W PDE Seminar: Themes and Methods in PDEs

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1 Introduction

Under the overarching title **Themes and Methods in PDEs**, this seminar draft investigates the qualitative and quantitative frameworks governing second-order elliptic equations. Our discussion is structured around three core methodological pillars:

- **The Method of Comparison (Ch. 3):** We establish the Weak and Strong Maximum Principles, which utilize barrier functions and the Hopf Lemma to demonstrate the geometric rigidity inherent in elliptic operators.
- **Classical Perturbation Theory (Ch. 6):** We examine the Schauder estimates, where the method of "freezing coefficients" allows us to bootstrap local regularity into global $C^{2,\alpha}$ solvability via the Method of Continuity.
- **Geometric Measure Methods (Ch. 9):** We transition to the study of strong solutions ($W^{2,n}$) in non-divergence form. Here, the primary method is the Alexandrov-Bakelman-Pucci (ABP) estimate, which relates the supremum of a solution to the measure of its contact set through the normal mapping.

By comparing these chapters, we illustrate how the definition of a "solution" evolves alongside the methods used to bound it, providing a comprehensive overview of modern elliptic theory.

2 Prerequisites (Minimal Background)

In this section, we establish the foundational functional analytic framework required to discuss the existence and regularity of solutions to second-order elliptic partial differential equations.

(*) Lebesgue Spaces L^p

Definition 2.1. For $1 \le p < \infty$, the Lebesgue space on a domain $\Omega \subseteq \mathbb{R}^n$ is defined as:

$$L^p(\Omega) := \left\{ u \text{ measurable} : \int_\Omega |u|^p \, dx < \infty \right\}, \quad \|u\|_{L^p} := \left(\int_\Omega |u|^p \, dx \right)^{1/p}.$$

For $p = \infty$, we use the essential supremum norm.

Remark 2.1. L^p spaces provide the *ambient scale* for defining weak derivatives and establishing energy estimates. They allow us to treat functions as elements of a Banach space.

Theorem 2.2 (Hölder Inequality)**.** Let $1 \leq p, q \leq \infty$ be conjugate exponents such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then:

$$\int_\Omega |fg| \, dx \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Sketch of Proof. The proof relies on Young's inequality: $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ for $a, b \geq 0$.

1. Normalize f and g such that $\|f\|_{L^p} = 1$ and $\|g\|_{L^q} = 1$.
2. Apply Young's inequality pointwise: $|f(x)g(x)| \leq \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q}$.
3. Integrate over Ω to obtain $\int |fg| \leq \frac{1}{p} + \frac{1}{q} = 1$.
4. Rescale by the original norms to conclude the general case.

□

(*) Weak Derivatives and Sobolev Spaces

Definition 2.3 (Weak Derivative)**.** A function $u \in L^1_{\text{loc}}(\Omega)$ is said to have a weak derivative $v = \partial_i u$ if for every test function $\varphi \in C_0^\infty(\Omega)$, we have:

$$\int_\Omega u \, \partial_i \varphi \, dx = -\int_\Omega v \varphi \, dx.$$

Remark 2.2. This definition generalizes the classical integration-by-parts formula, allowing us to differentiate functions with "kinks" or jumps that are not classically differentiable.

Definition 2.4 (Sobolev Space)**.** The Sobolev space $W^{1,p}(\Omega)$ consists of functions whose weak derivatives are also in L^p:

$$W^{1,p}(\Omega) := \{u \in L^p(\Omega) : \partial_i u \in L^p(\Omega) \text{ for } i = 1, \ldots, n\}.$$

For the Hilbert space case $p = 2$, we denote $W^{1,2}(\Omega)$ as $H^1(\Omega)$.

Theorem 2.5 (Poincaré Inequality)**.** Let Ω be a bounded domain. If $u \in W_0^{1,p}(\Omega)$, there exists a constant C depending only on Ω and p such that:

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}.$$

Sketch of Proof. The proof utilizes the fundamental theorem of calculus and the boundary conditions.

1. Since $u \in W_0^{1,p}(\Omega)$, we can approximate u by $C_0^\infty(\Omega)$ functions.
2. For $u \in C_0^\infty(\Omega)$, represent $u(x)$ as the integral of its derivative along a line segment starting from the boundary.
3. Apply Jensen's or Hölder's inequality to the integral representation.
4. Integrate the resulting inequality over the entire domain to bound the L^p norm by the gradient's L^p norm.

□

(*) Boundary Traces and $W_0^{1,p}$

Definition 2.6. $W_0^{1,p}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the $W^{1,p}$ norm.

Remark 2.3. In the context of PDEs, $u \in W_0^{1,p}(\Omega)$ is the rigorous way to say $u = 0$ on $\partial\Omega$. Note that boundary values are defined via a *trace operator* because functions in L^p are not defined pointwise on sets of measure zero (like the boundary).

(*) Distributions and Weak Solutions

Definition 2.7 (Distribution)**.** A distribution T is a continuous linear functional on the space of test functions $C_0^\infty(\Omega)$.

Definition 2.8 (Weak Formulation)**.** Consider the Poisson equation $-\Delta u = f$ with $u = 0$ on $\partial\Omega$. We say $u \in H_0^1(\Omega)$ is a weak solution if:

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx \quad \forall \varphi \in C_0^\infty(\Omega).$$

Remark 2.4. This formulation "shifts" the derivatives from the unknown u onto the smooth test function φ, allowing us to find solutions in H^1 even if f is only in L^2 or H^{-1}.

(*) Elliptic Operators and Ellipticity

Definition 2.9. A second-order differential operator L in divergence form is given by:

$$Lu = -\sum_{i,j=1}^n \partial_i (a^{ij}(x) \partial_j u).$$

Definition 2.10 (Uniform Ellipticity)**.** The operator L is uniformly elliptic if there exist constants $0 < \lambda \le \Lambda$ such that:

$$\lambda |\xi|^2 \le \sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j \le \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \forall x \in \Omega.$$

Remark 2.5. Uniform ellipticity ensures that the quadratic form associated with L is positive definite, which is essential for the coercivity required in existence theorems.

(*) Energy Methods and Functional Analysis

Theorem 2.11 (Lax–Milgram Theorem)**.** Let H be a Hilbert space and $a(\cdot, \cdot)$ be a bilinear form that is:

1. Bounded: $|a(u, v)| \le M \|u\| \|v\|$
2. Coercive: $a(u, u) \ge \alpha \|u\|^2$ for some $\alpha > 0$.

Then for any linear functional $\ell \in H^*$, there exists a unique $u \in H$ such that $a(u, v) = \ell(v)$ for all $v \in H$.

Sketch of Proof.

1. Use the Riesz Representation Theorem to represent the bilinear form $a(u, v)$ as (Au, v) for some linear operator $A : H \to H$.
2. Coercivity implies that A is injective and has a closed range.
3. Show that the range of A is the entire space H (otherwise, there exists a non-zero element in the orthogonal complement, contradicting coercivity).
4. Apply the Riesz Representation Theorem to the functional ℓ to find $f \in H$ such that $\ell(v) = (f, v)$. The solution is $u = A^{-1}f$.

□

(*) Maximum Principles

Theorem 2.12 (Weak Maximum Principle)**.** Suppose $Lu \leq 0$ in Ω (where L is elliptic). Then:

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+.$$

Remark 2.6. This reflects the physical intuition that for diffusion processes, the maximum value cannot be attained in the interior unless the solution is constant.

(*) Hölder Spaces

Definition 2.13. For $\alpha \in (0, 1)$, the Hölder semi-norm is defined as:

$$[u]_{C^{0,\alpha}(\Omega)} := \sup_{x,y\in\Omega, x\neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

The space $C^{k,\alpha}(\Omega)$ consists of functions whose k-th derivatives are α-Hölder continuous.

Remark 2.7. L^p and Sobolev spaces are sufficient for existence, but Hölder spaces are necessary for the classical Schauder estimates and pointwise regularity theory.

Conceptual Flow

The logical progression of the seminar is as follows:

$$L^p \xrightarrow{\text{Weak Deriv.}} W^{1,p} \xrightarrow{\text{Var. Form.}} \text{Weak solution} \xrightarrow{\text{Ellipticity}} \text{Energy Estimates} \xrightarrow{\text{Iteration/Embedding}} \text{Regularity.}$$

3 Main Theorems

Theorem 3.1 (3.1. Weak Maximum Principle)**.** Let L be elliptic in the bounded domain Ω. Suppose that

$$(3.4) \quad Lu \geq 0 \text{ (resp. } \leq 0) \text{ in } \Omega, \quad c = 0 \text{ in } \Omega, \tag{1}$$

with $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. Then the maximum (resp. minimum) of u in $\overline{\Omega}$ is achieved on $\partial\Omega$:

$$(3.5) \quad \sup_{\Omega} u = \sup_{\partial\Omega} u \quad (\inf_{\Omega} u = \inf_{\partial\Omega} u). \tag{2}$$

Sketch of Proof. The proof utilizes a perturbation argument to handle the case where the maximum might be interior.

1. Assume first that $Lu > 0$. If u had an interior maximum at x_0, then $\nabla u(x_0) = 0$ and the Hessian $D^2u(x_0)$ would be negative semi-definite. By ellipticity, $Lu(x_0) = -\mathrm{tr}(AD^2u) \leq 0$, contradicting $Lu > 0$.
2. For the general case $Lu \geq 0$, consider $u_\epsilon = u + \epsilon e^{\gamma x_1}$.
3. Choose γ large enough so that $L(e^{\gamma x_1}) > 0$. Then $L(u_\epsilon) > 0$ for all $\epsilon > 0$.
4. Apply step 1 to u_ϵ and let $\epsilon \to 0$.

□

Lemma 3.2 (3.4. Hopf Boundary Point Lemma)**.** Suppose L is uniformly elliptic, $c = 0$, and $Lu \geq 0$ in Ω. Let $x_0 \in \partial\Omega$ such that:

(i) u is continuous at x_0;

(ii) $u(x_0) > u(x)$ for all $x \in \Omega$;

(iii) $\partial\Omega$ satisfies an interior sphere condition at x_0.

Then the outer normal derivative $\frac{\partial u}{\partial \nu}(x_0) > 0$.

Sketch of Proof. The core idea is to construct a "barrier function" in the interior ball $B \subset \Omega$ that touches x_0.

1. Construct $v(x) = e^{-\alpha|x-x_C|^2} - e^{-\alpha R^2}$, where x_C is the center of the interior ball.
2. Show that for large α, $Lv > 0$ in an annular region within the ball.
3. Use the Weak Maximum Principle on $u + \epsilon v$ to show that u must increase as it approaches x_0 from the interior.

□

Theorem 3.3 (3.5. Strong Maximum Principle)**.** Let L be uniformly elliptic, $c = 0$, and $Lu \geq 0$ in a domain Ω. If u achieves its maximum in the interior of Ω, then u is constant.

Sketch of Proof. This follows from the Hopf Lemma. If u is not constant, we can find a point y near the interior maximum and a ball B where $u < \max u$, but the ball touches a point where $u = \max u$. Hopf's Lemma would then imply a non-zero gradient at the maximum, which contradicts the first-order condition $\nabla u = 0$. □

Theorem 3.4 (6.2. Schauder Estimates)**.** Let $u \in C^{2,\alpha}(\Omega)$ satisfy $Lu = f$ with $f \in C^\alpha$. Under uniform ellipticity and C^α bounds on coefficients:

$$|u|^*_{2,\alpha;\Omega} \leq C(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}) \tag{3}$$

where $C = C(n, \alpha, \lambda, \Lambda)$.

Sketch of Proof. This is a local-to-global perturbation argument.

1. Prove the estimate for the constant-coefficient operator (Laplacian) using Newtonian potentials.
2. Treat $Lu = f$ as a perturbation of a constant-coefficient operator by freezing coefficients at a point x_0.
3. Use a scaling argument and "cutoff" functions to handle the error terms.

□

Theorem 3.5 (6.8. Method of Continuity)**.** Let Ω be $C^{2,\alpha}$ and L be strictly elliptic with $c \leq 0$. If the Dirichlet problem for Δ is solvable for all C^α data, then $Lu = f, u = \varphi$ is uniquely solvable in $C^{2,\alpha}(\overline{\Omega})$.

Sketch of Proof. Define a family of operators $L_t = (1 - t)\Delta + tL$.

1. Let $I = \{t \in [0, 1] : L_t u = f \text{ is solvable}\}$.
2. $0 \in I$ by assumption. I is shown to be open via the Inverse Function Theorem (or Banach Fixed Point Theorem).
3. I is shown to be closed using the a priori Schauder estimates to pass to the limit. Thus $1 \in I$.

□

Theorem 3.6 (9.1. ABP Estimate)**.** Let $Lu \geq f$ in a bounded domain Ω and $u \in C^0(\overline{\Omega}) \cap W^{2,n}_{\text{loc}}(\Omega)$. Then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C\|f/\mathcal{D}^*\|_{L^n(\Omega)}. \tag{4}$$

Sketch of Proof. This estimate links the maximum of u to the measure of the "upper contact set" Γ^+, where u is concave and lies below its tangent plane. By considering the image of the gradient map ∇u on Γ^+, one relates the volume of a ball (the range of the gradient) to the integral of the determinant of the Hessian (the Jacobian), which is bounded by f due to the PDE. □

Theorem 3.7 (9.15. $W^{2,p}$ Solvability)**.** Under $C^{1,1}$ domain and C^0 coefficients, $Lu = f$ has a unique solution $u \in W^{2,p}(\Omega)$ for $f \in L^p$.

4 Summary of Frameworks

Aspect	**Chapter 6 (Schauder)**	**Chapter 7/8 (Sobolev)**
Solution Concept	Classical $C^{2,\alpha}$	Weak $W^{1,2}$
Existence Method	Method of Continuity	Lax-Milgram / Galerkin
Key Estimates	Schauder $C^{2,\alpha}$	Energy / Gagliardo-Nirenberg
Data Required	Hölder Continuous	L^p / H^{-1}
Core Intuition	Local Perturbation	Minimizing Energy

Table 1: Comparison: Classical vs. Variational Theories

Aspect	**Chapter 8 (Divergence)**	**Chapter 9 (Strong)**
Operator Form	$\mathrm{div}(A\nabla u) = f$	$a^{ij}D_{ij}u = f$
Solution Space	$W^{1,2}$ (Energy solutions)	$W^{2,n}$ (Strong solutions)
Max Principle	Weak (Integrable)	ABP (Pointwise/Contact)
Regularity	De Giorgi-Nash-Moser	Krylov-Safonov
Methods	Variational / Test Functions	Measure Theory / Geometry

Table 2: Comparison: Divergence vs. Non-Divergence Forms