T

$k_0, \quad k_1 = A k_0, \quad k_2 = A k_1$

$$\overbrace{A\left[k_0, k_1, k_2\right]}^{3}$$

$$= \left[A k_0, A k_1, A k_2\right]$$

$$= \left[k_1, k_2, k_3\right]$$

$$= \left[k_0, k_1, k_2\right] \begin{bmatrix} 0 & 0 & \alpha_0 \\ 1 & 0 & \alpha_1 \\ 0 & 1 & \alpha_2 \end{bmatrix}$$

$$+ \left[0, 0, k_3 - \alpha_0 k_0 - \alpha_1 k_1 - \alpha_2 k_2\right]$$

$$\boxed{A K_3 = K_3 Z + \hat{K}_3 e_2^T}$$

$$Z = \begin{bmatrix} 0 & 0 & \alpha_0 \\ 1 & 0 & \alpha_1 \\ 0 & 1 & \alpha_2 \end{bmatrix}$$

$$\hat{K}_3 = K_3 - \alpha_0 K_0 - \alpha_1 K_1 - \alpha_2 K_2$$

$$A K_3 = K_3 Z + \hat{K}_3 e_2^T$$

$$K_3 = Q_3 R_3$$

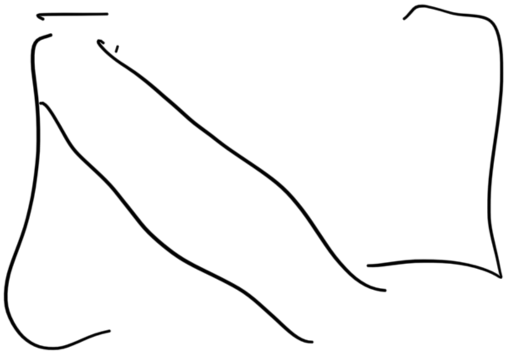
$$A Q_3 R_3 = Q_3 R_3 Z + \hat{K}_3 e_2^T$$

$$A Q_3 = Q_3 R_3 Z R_3^{-1} + \hat{K}_3 e_2^T R_3^{-1}$$

$$H_3 = R_3 Z R_3^{-1}$$

$$H_3 = \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & * & * \end{bmatrix}$$

H - upper Hessenberg

$H =$ 

$$e_2^T R_3^{-1}$$

$$R_3 = \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}$$

$$e_2^T R_3^{-1} = e_2^T \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} = \gamma e_2^T$$

$$e_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{AQ_k = Q_k H_k + \gamma q_{k+1} e_k^T}$$

columns of Q

are orthonorm

... orthonormal
q_{k+1} belong to the
next Krylov subspace
we can select α_s
To make $q_{k+1} \perp$
columns of Q

$$AQ_k = Q_k H_k + h_{k,k+1} q_{k+1}$$

$$Q_k^* q_{k+1} = 0$$

$$\|q_{k+1}\| = 1 \qquad Q_k^* Q_k = I$$

H_k – upper Hessenberg
Arnoldi relation

$$AQ_k = Q_k H_k + h_{k,k+1} q_{k+1} e_k^T$$

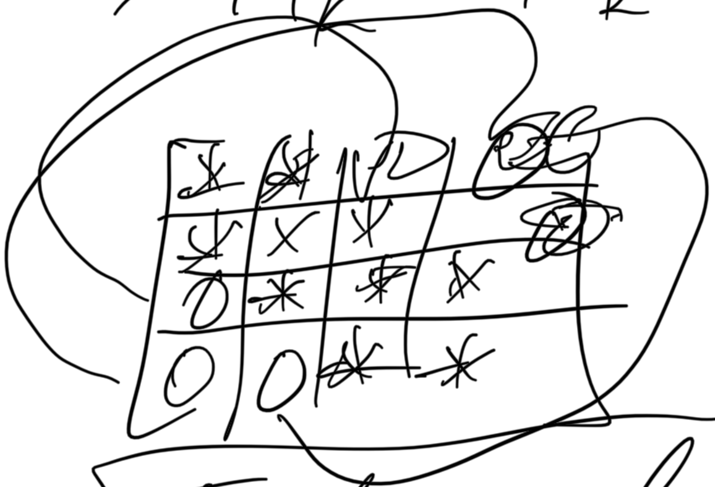
Q_k^* ? Q_k^*

$$Q_k^* A Q_k = H_k + 0$$

$$(H = Q^* A Q)$$

$$A = A^*$$

$$\Rightarrow H_k = H_k^*$$



Tridiagonal

$$A Q_k = Q_k T_k + h_{k,k+1} q_{k+1} e_k^T$$

$$\boxed{x = x_0 + Q_j c_j}$$

c_j $\quad \underbrace{Q_j^* A Q_j}\, c_j = Q_j^* (f - A$

$$r_j = Q_j^* r_0 = \gamma e_0$$

$$Q_j = \left[\frac{r_0}{\|r_0\|}, \dots \right.$$

$$A = A^* > 0$$

$$\Longrightarrow \quad T = Q_j^* A Q_j > 0$$

$$x_j = P_j z_j$$

$$P_j = [P_{j-1}, p_j]$$

$$z_j = \begin{bmatrix} z_{j-1} \\ \zeta_j \end{bmatrix}$$

$$P_j z_j = [P_{j-1} \; p_j] \begin{bmatrix} z_{j-1} \\ \zeta_j \end{bmatrix}$$

$$= P_{j-1} z_{j-1} + p_j \zeta_j$$

$$x_{j-1}$$

$$(Ax,x) - 2(f,x)$$

$$Ax_* = f$$

$$(Ax,x) - 2(Ax_*,x)$$

$$(A(x-x_*),(x-x_*))$$

7

$$(A(x-x_*), x-x_*)$$

Problem: Too many vectors to store

Solution: RESTART

iterations | restart

x_j – current approximati[on]

Look for correction:

$A(x_j+e_j)=f$

$Ae_j = f - Ax_j = r_j$

Generate Krylov subspac[e] $\boxed{\text{from scratch}}$

$$q = \frac{\sqrt{\mathrm{cond}(A^*A)} - 1}{\sqrt{\mathrm{cond}(A^*A)} + 1}$$

cond(A)

$$q = \frac{\mathrm{cond}(A)-1}{\mathrm{cond}(A)+1}$$

$Ax=f$

$x = Vc'$

$$AVc = f$$

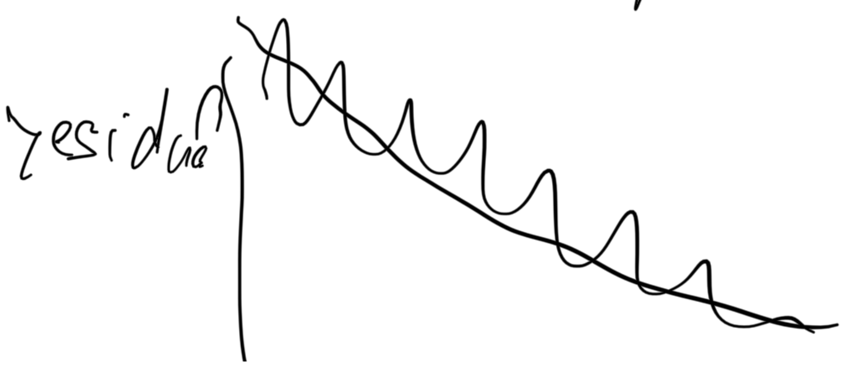
N equations for m unknowns

$m \ll N$

$$\underbrace{W^* A V}_{\text{square}} c = W^* f$$

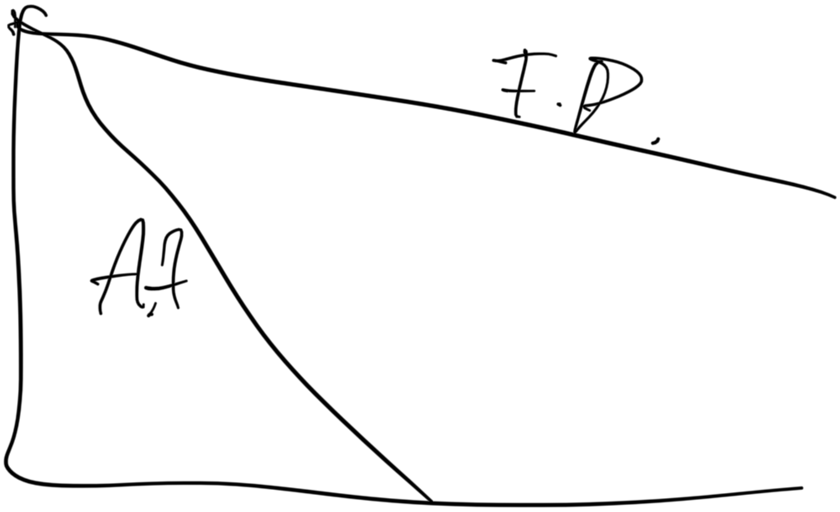
"sketching"
row sampling

$W \sim$ basis for A



i.f.

1 iteration of BiG
+1 iteration of GMRES



$$Ax = f$$

$$\mathrm{cond}(A) \gg 1$$

P left

$PAx = Pf$

$cond(PA) \ll cond(A)$

$P \cdot z$ is cheap

$P = A^{-1}$ — ~~ideal~~

$AP^{-1} y = f \quad cond(AP^{-1}) \ll cond(A)$

$x = P^{-1} y$ is cheap

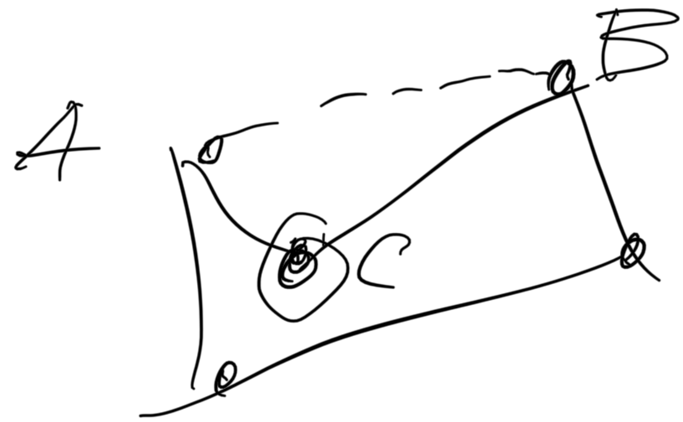
2

o

Sparse matrix

$\Longrightarrow$ graph

Elimination



$u_2 u_2^T \longrightarrow$

Spars

$$(u_2 + R_2)(u_2 + R_2)^T$$

$$\approx u_2 u_2^T + u_2 R_2^T + R_2 u_2^T + \cancel{R_2 R_2^T}$$