WHEN IS A CONVEX CLOSED CURVE NOT SIMPLE?

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Abstract. A convex curve is a plane curve whose intersection with a line is either a set of at most two points or a line segment. With analytical definitions of convexity and simpleness, we show that a convex closed curve is simple if and only if it does not lie entirely on a line. We conjecture that the same result still holds when geometrical definitions are used instead.

1. Definitions

We first follow Milnor [1] to define the concepts in a rigorous manner.

Definition 1.1 (Closed curves)**.** A *closed curve* in Euclidean n-space $\mathbb{E}^n$ is a continuous periodic surjection $\gamma : \mathbb{R} \to C \subset \mathbb{R}^n$ which is not constant on any interval.

One can show that a closed curve has a least positive period. For convenience we redefine the notion of period for closed curves as follows.

Definition 1.2 (Periods)**.** Let γ be a closed curve of least positive period p, and let $x \in \mathbb{R}$. We call p *the period* of γ, and the interval $P = [x, x+p)$ *a period* of γ. We say that a closed curve ϕ is *of period* p' iff the period of ϕ is p'.

In fact, every closed curve γ of period p can be factored into two functions $\gamma^* : \mathbb{S}^1 \to C$ and $f : \mathbb{R} \to \mathbb{S}^1$ such that $\gamma = \gamma^* \circ f$, where γ^* is a continuous surjection which is not constant on any arc and $f(t) := (\cos 2\pi t/p, \sin 2\pi t/p)$. We may call γ^* the *underlying map* of γ and fix this notation.

Note that the restriction of f to every period of γ is bijective. This fact allows us to translate between the properties of the periodic function γ (on individual periods) and of the underlying map γ^* (on the circle).

Definition 1.3 (Simpleness)**.** Let γ be a closed curve of period p. We say that γ is *simple* iff for all $t_1, t_2 \in \mathbb{R}$, $\gamma(t_1) = \gamma(t_2)$ only when $(t_1 - t_2)/p$ is an integer.

Proposition 1.4. *Let γ be a closed curve. Then the following are equivalent:*

(1) *γ is simple.*
(2) *γ^* is a homeomorphism.*
(3) *For every period P of γ, the restriction of γ to P is bijective.*
(4) *There exists a period P of γ such that the restriction of γ to P is bijective.*

Proof. The claim follows immediately from the definitions above. □

Proposition 1.5. *Let $f : \mathbb{R} \to C \subset \mathbb{R}^n$ be a continuous periodic surjection, and let $p > 0$ be a period of f. Suppose that the restriction of f to some interval $[x_0, x_0+p)$ is bijective. Then f is a simple closed curve of period p.*

Proof. The restriction being bijective implies that p is the least positive period of f. Thus f is a closed curve of period p. Then by Proposition 1.4 we have f simple. □

Definition 1.6 (Convexity)**.** Let γ be a closed curve in $\mathbb{E}^2$. We say that γ is *convex* iff for every line $L \subset \mathbb{R}^2$, there exists a period P of γ such that the set $\{t \in P : \gamma(t) \in L\}$ either has cardinality ≤ 2 or is a nondegenerate interval.

Here we see that an arc on the circle corresponds to an interval on *some* period.

Proposition 1.7. *A closed curve γ in $\mathbb{E}^2$ is convex iff for every $u \in \mathbb{S}^1$ and $y \in \mathbb{R}$, there exists a period P of γ such that the set $\{t \in P : u \cdot \gamma(t) = y\}$ either has cardinality ≤ 2 or is a nondegenerate interval.*

Proof. A normal vector $u \in \mathbb{S}^1$ and a distance $y \in \mathbb{R}$ from the origin specifies a line $L = \{x \in \mathbb{R}^2 : u \cdot x = y\}$, and every line in $\mathbb{R}^2$ can be specified in this manner. $\square$

We then prove a generalization of the necessity part of [1, Lemma 3.3] in the hope that our definitions are equivalent to Milnor's.

Definition 1.8. Let γ be a closed curve in $\mathbb{E}^n$, let $u \in \mathbb{S}^{n-1}$, and let P be any period of γ. Define $\mu(\gamma, u)$ to be the cardinality of the set $\{t \in P :$ the function $u \cdot \gamma$ attains a local maximum at $t\}$ if it is finite, or ∞ otherwise.

Proposition 1.9. *Let γ be a closed curve in $\mathbb{E}^2$. If γ is convex, then for every $u \in \mathbb{S}^1$ either $\mu(\gamma, u) = 1$ or $\mu(\gamma, u) = \infty$.*

Proof. Suppose that $2 \leq \mu(\gamma, u) < \infty$ for some $u \in \mathbb{S}^1$. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $u \cdot \gamma$, let $P = [x, x + p)$ be any period of γ, and let $t_1 < t_2 \in P$ such that f attains local maxima at t_1 and t_2. Suppose that $f(t_1) \geq f(t_2)$. Since f attains a finite number of local maxima in P, there exists a, b, y with $t_1 < a < t_2 < b < x+p$ such that $f(a), f(b) < y < f(t_2) \leq f(t_1)$. By the intermediate value theorem there exists $c_1 \in (t_1, a)$, $c_2 \in (a, t_2)$, and $c_3 \in (t_2, b)$ such that $f(c_i) = y$ for $i = 1, 2, 3$. Since $f(t_2) \neq y$, we know that the set $\{t \in P : u \cdot \gamma(t) = y\}$ is not an interval, and if it is finite, has cardinality > 2; this is a contradiction to convexity. Now suppose that $f(t_1) < f(t_2)$. If $t_1 \neq x$ then the proof is identical, so we assume that $t_1 = x$. It is always possible to find a $\delta > 0$ such that $x - \delta < t_1 < t_2 < x + p - \delta$. Similarly, there exists $c_1 \in (x - \delta, t_1)$ and $c_2 < c_3 \in (t_1, t_2)$ such that $f(c_i) = y$ for $i = 1, 2, 3$. Let $c_4 = c_1 + p \in (t_2, x + p)$ and obtain $f(c_4) = y$. Since $f(t_2) \neq y$ we again have a contradiction. $\square$

The sufficiency part of the lemma seems way more tricky to prove analytically, especially the case when $\mu(\gamma, u) = \infty$. Milnor's proof, on the other hand, made too much use of geometrical methods despite his analytical definitions. It is not entirely clear, for example, why it is always possible to rotate a line about one of its points of intersection with a polygon so that the number of intersections is not decreased. Also, we have this particular case when a polygon in the shape of 凸 (lit. convex; however the shape itself is concave) is intersected by a horizontal line: the set of points of intersection either has cardinality 2 or is infinite, but when it is infinite the set may still be disconnected. This case is not seen to be handled in Milnor's proof and does not seem trivial otherwise. I think, however, that the lemma is true but still needs an analytical proof for the sake of completeness.

2. Analytical Proof

Now that we have checked the equivalence of definitions, we shall first present a lemma and then our main theorem.

Lemma 2.1. *Let $f : (a,b) \cup (b,c) \to \mathbb{R}^2$ be a continuous nonconstant function, and let $p \in \mathbb{R}^2$. Suppose that there exists $x_1, x_2 \in (a,b) \cup (b,c)$ such that the points $p, f(x_1), f(x_2)$ are noncollinear. Then there exists $x_1' \in (a,b), x_2' \in (b,c)$ such that the points $p, f(x_1'), f(x_2')$ are noncollinear.*

Proof. Suppose that for all $x_1' \in (a,b)$ and $x_2' \in (b,c)$, the points $p, f(x_1'), f(x_2')$ are collinear. Then we have $x_1, x_2 \in A$ where A is either (a,b) or (b,c). Let B be the other interval. Since f is nonconstant, there exists $x_0 \in B$ such that $f(x_0) \neq p$. But $f(x_0)$ cannot be collinear with $p, f(x_1)$ and with $p, f(x_2)$ at the same time. □

Theorem 2.2. *A convex closed curve is simple iff it does not lie entirely on a line.*

Proof. The image of a simple closed curve is homeomorphic to $\mathbb{S}^1$ and thus cannot be one-dimensional. Let γ be a convex closed curve in $\mathbb{E}^2$ which is not simple and does not lie entirely on a line. Let $X = [x_1, x_1+p)$ be a period of γ such that $\gamma(x_1) = \gamma(x_1')$ for some $x_1' \in (x_1, x_1+p)$. Then there exists $x_2, x_3 \in (x_1, x_1') \cup (x_1', x_1+p)$ such that the points $\gamma(x_1), \gamma(x_2), \gamma(x_3)$ are noncollinear. By Lemma 2.1 we may assume that $x_2 \in (x_1, x_1')$ and $x_3 \in (x_1', x_1+p)$. Now let P be any period of γ. Then there exists $c_1, c_1', c_2, c_3 \in P$ such that $\gamma(c_1) = \gamma(c_1')$, that $c_1 < c_3 < c_1'$, and that the points $p_1 = \gamma(c_1), p_2 = \gamma(c_2), p_3 = \gamma(c_3)$ are noncollinear. Let $u \in \mathbb{S}^1$ be a vector perpendicular to $\overline{p_1 p_2}$. Then we have $u \cdot \gamma(c_1) = u \cdot \gamma(c_1') = u \cdot \gamma(c_2) \neq u \cdot \gamma(c_3)$, a contradiction to convexity. □

3. Geometrical Consideration

Definition 3.1 (Simple sets)**.** Let $C \subset \mathbb{R}^n$. We say that C is *simple* iff C is homeomorphic to the unit circle $\mathbb{S}^1$.

Definition 3.2 (Weakly convex sets)**.** Let $C \subset \mathbb{R}^2$. We say that C is *weakly convex*, or simply *convex*, iff for every line $L \subset \mathbb{R}^2$, the set $C \cap L$ either has cardinality ≤ 2 or is a nondegenerate line segment.

Definition 3.3 (Parameterizations)**.** Let γ be a closed curve in $\mathbb{E}^n$, and let $C \subset \mathbb{R}^n$. We say that γ is a *parameterization* of C iff the image of γ is C.

Proposition 3.4. *A simple subset of $\mathbb{R}^n$ has a simple parameterization.*

Proof. Let C be a simple subset of $\mathbb{R}^n$. Then there exists a continuous bijection $f : \mathbb{S}^1 \to C$. Let $g : \mathbb{R} \to \mathbb{S}^1$ be the function $g(t) := (\cos t, \sin t)$. We know that g is continuous on $\mathbb{R}$ and periodic with period 2π, and that the restriction of g to any interval $[x, x+2\pi)$ is bijective. Let $\gamma : \mathbb{R} \to C$ be the function $f \circ g$. Then γ is continuous on $\mathbb{R}$ and periodic with period 2π, and that the restriction of γ to any interval $[x, x+2\pi)$ is bijective. It follows from Proposition 1.5 that γ is a simple closed curve of image C. □

Proposition 3.5. *A simple closed curve of convex image is convex.*

Proof. Let γ be a simple closed curve of convex image C. It follows from Proposition 1.4 that γ^* is a homeomorphism. Let $L \subset \mathbb{R}^2$ be any line. Since C is convex, the set $C \cap L$ either has cardinality ≤ 2 or is a nondegenerate line segment. Thus the inverse image $(\gamma^*)^{-1}(C \cap L)$ either has cardinality ≤ 2 or is a nondegenerate arc. Recall that an arc on the circle corresponds to an interval on some period, and so we have γ convex by definition. □

Definition 3.6 (Irreducibility)**.** Let γ be a closed curve of image C. We say that γ is *irreducible* iff for every period P and every nonempty interval $(a, b) \subset P$ with $\gamma(a) = \gamma(b)$, we have $\gamma(P \backslash (a, b)) \neq C$.

Being irreducible means that the curve does not contain "redundant loops" which do not contribute to its image and can be removed. This sounds like a good property indeed, so let us make some conjectures on it. It is conceivable that 3.7 already has a proof somewhere, and that 3.8 has a simple proof.

Conjecture 3.7. *The image of a closed curve has an irreducible parameterization.*

Conjecture 3.8. *An irreducible closed curve of simple image is simple.*

Conjecture 3.9. *An irreducible closed curve of convex image is convex.*

We then present two equivalent conjectures. Note that Conjecture 3.11 is the geometrical analog to our main theorem.

Conjecture 3.10. *Let C be the image of a closed curve. If C is convex, then C has a convex parameterization.*

Proof that $3.7 \wedge 3.9 \implies 3.10$. This implication is immediate. $\square$

Proof that $3.11 \implies 3.10$. If C is not simple, then by Conjecture 3.11 we know that C is a subset of a line, which clearly is a line segment and has a convex parameterization. If C is simple, then by Proposition 3.4 there exists a simple parameterization γ of C, which is convex by Proposition 3.5. $\square$

Conjecture 3.11. *Let C be the image of a closed curve. Suppose that C is convex. Then C is simple iff C is not a subset of a line.*

Proof that $3.10 \implies 3.11$. Clearly, a subset of a line cannot be simple. Suppose that C is not a subset of a line. By Conjecture 3.10 there exists a convex parameterization γ of C. Then we have γ simple by Theorem 2.2, and thus C simple by Proposition 1.4. $\square$

References

[1] J. W. Milnor, *On the Total Curvature of Knots*, Ann. of Math. **52** (1950), no. 2, 248-257.