

# THE PROBLEM OF THE RANDOMLY WALKED DOG

SCALLOP YE

ABSTRACT. In 1905, Karl Pearson [1] proposed on *Nature* the famous problem of the random walk. In this article, we present a new combinatorial optimization problem in the spirit of Pearson's problem.

## 1. INTRODUCTION

A man walks the dog every day. The dog will prepare  $n$  cards beforehand. Each card contains a positive distance and a direction that is neither identical nor opposite to that on another card. Once the cards are ready, the walk begins.

They start from a point  $O$  and repeat the following process  $n$  times:

- (1) choose a card randomly;
- (2) turn to the direction given on the card;
- (3) walk the distance given on the card in a straight line;
- (4) throw the card away.

After these  $n$  rounds, the walk ends and the dog wins a treat if they visited any point *twice or more* in the walk, except if the *only* point they visited twice or more is  $O$  and it was *only* visited once at the beginning and once at the end of the walk.

Now, the dog asks for your help in maximizing the probability of winning the treat in a walk. The man is known to sometimes restrict the value of  $n$  and sometimes not. Note that too much calculation exhausts the dog.

*Remark.* As one might have seen, this problem differs from Pearson's one in that it is not inherently probabilistic. We will be using permutations instead to formally define the problem in the next section.

## 2. DEFINITIONS

Let us take a look at the cards first. The cards the dog prepares should represent a sequence of noncollinear vectors in  $\mathbb{R}^2$ , or in  $\mathbb{Z}^2$  without loss of generality. We may call these vectors *steps* and define

**Definition 2.1.** An  $n$ -step sequence  $S$ , or generally a step sequence, is a finite sequence  $(s_1, s_2, \dots, s_n)$  of noncollinear vectors in  $\mathbb{Z}^2$ .

Then in order to study the points the pair visits in a walk, we shall introduce polygonal paths:

**Definition 2.2.** A *polygonal path*  $P$  is a finite sequence  $(p_1, p_2, \dots, p_n)$  of points called its *vertices*; the line segments  $\overline{p_1 p_2}, \overline{p_2 p_3}, \dots, \overline{p_{n-1} p_n}$  are called its *edges*.

The winning conditions for the dog naturally translate to whether the polygonal path formed in a walk does not “intersect itself”, or in other words whether the path is simple.

**Definition 2.3.** A polygonal path  $(p_1, p_2, \dots, p_n)$  is *simple* iff for all  $1 < i \leq j < n$ ,

$$\overline{p_{i-1}p_i} \cap \overline{p_jp_{j+1}} = \begin{cases} \{p_i\} & \text{if } i = j, \\ \{p_1\} \cap \{p_n\} & \text{if } n > 3 \text{ and } (i, j) = (2, n-1), \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that a simple polygonal path can be *closed*, as specially dealt with in the second case above. We also want the same to be the case for our problem, because if it were not the dog would be so bored as all it has to do is close the path!

Then, we would like to form a polygonal path by *walking* the steps:

**Definition 2.4.** Let  $S = (s_1, s_2, \dots, s_n)$  be a step sequence. The *walk* of  $S$ , denoted  $\text{walk}(S)$ , is the polygonal path  $(p_0, p_1, \dots, p_n)$  where  $p_0 = (0, 0)$  and  $p_i = p_{i-1} + s_i$  for all  $1 \leq i \leq n$ .

Finally, we define the value we are optimizing and then the problem:

**Definition 2.5.** Let  $S$  be a step sequence. The *simplicity* of  $S$ , denoted  $\text{simp}(S)$ , is the number of permutations  $S'$  of  $S$  such that  $\text{walk}(S')$  is simple.

**Problem 2.6** (THE PROBLEM OF THE RANDOMLY WALKED DOG)

*Instance:* A natural number  $n \geq 2$ .

*Task:* Find an  $n$ -step sequence of minimum simplicity.

## REFERENCES

- [1] Pearson, K. The Problem of the Random Walk. *Nature* **72**, 294 (1905).