THE PROBLEM OF THE RANDOMLY WALKED DOG

SCALLOP YE

Abstract. In 1905, Karl Pearson [1] proposed on *Nature* the famous problem of the random walk. In this article, we present a novel combinatorial optimization problem in the spirit of Pearson's one, along with a curious conjecture on the property of its solutions.

1. Introduction

A man walks the dog every day. The dog will prepare n cards beforehand ($n \geq 3$), each containing a direction and a distance. They start from home and repeat the following process n times:

(1) choose a card randomly;
(2) turn to the direction given on the card;
(3) walk the distance given on the card in a straight line;
(4) throw the card away.

After these n rounds, they shall be exactly back home. The dog wins a treat if they visited home exactly twice and any other place at most once.

Now, you are to instruct the dog how to maximize the probability of winning the treat in a walk. Note that too much calculation exhausts the dog.

Remark. As one might have seen, this problem differs from Pearson's one in that it is not inherently probabilistic. We will be using permutations instead to formally define the problem in the next section. It is later conjectured that the maximum probability required is exactly $2/(n-1)$, despite the lack of a general solution.

2. Definitions

Let us take a look at the cards first. The cards the dog prepares should represent a sequence of nonzero vectors that sum to zero. We may then define

Definition 2.1. A *step vector* is a nonzero vector in $\mathbb{R}^2$. An *n-step sequence* S, or generally a *step sequence*, is a finite sequence $(s_1, s_2, \ldots, s_n)$ of step vectors; we say that S is *zero-sum* iff $\sum_{i=1}^{n} s_i = (0,0)$.

Then in order to study the points the pair visits in a walk, we shall introduce polygonal paths:

Definition 2.2. A *polygonal path* P is a finite sequence $(p_0, p_1, \ldots, p_n)$ of points called its *vertices*; the line segments $\overline{p_0p_1}, \overline{p_1p_2}, \ldots, \overline{p_{n-1}p_n}$ are called its *edges*.

The winning conditions for the dog naturally translate to whether the polygonal path formed in a walk does not "intersect itself", or in other words whether the path is simple.

Definition 2.3. A polygonal path $(p_0, p_1, \ldots, p_n)$ is *simple* iff for all $0 < i \leq j < n$,

$$\overline{p_{i-1}p_i} \cap \overline{p_j p_{j+1}} = \begin{cases} \{p_i\} & \text{if } i = j, \\ \{p_0\} \cap \{p_n\} & \text{if } n \geq 3 \text{ and } (i,j) = (1, n-1), \\ \varnothing & \text{otherwise.} \end{cases}$$

Geometrically, a simple path is one in which only consecutive edges intersect and only at their endpoints; it may also be *closed* as specially dealt with in the second case above. Then, we define a way to create a path from a step sequence:

Definition 2.4. Let $S = (s_1, s_2, \ldots, s_n)$ be a step sequence. The *walk* of S is the polygonal path $(p_0, p_1, \ldots, p_n)$ where $p_0 = (0, 0)$ and $p_i = p_{i-1} + s_i$ for $1 \leq i \leq n$.

Finally, we define the value we are maximizing and then the problem:

Definition 2.5. The *simplicity* of a step sequence S is the number of permutations S' of S such that the walk of S' is simple.

Problem 2.6 (The Problem of the Randomly Walked Dog)
Instance: A natural number $n \geq 3$.
Task: Find a zero-sum n-step sequence of maximum simplicity.

This can be viewed as a combinatorial optimization problem, because without loss of generality we may restrict step vectors to those in $\mathbb{Z}^2$ with magnitudes less than some $f(n)$, hence a finite set of feasible solutions. However, this problem is quite peculiar in that (a) each instance is only a single natural number, (b) the set of feasible solutions is not easily restricted to be finite, and (c) the value of a feasible solution is not easily computed even for moderately large n.

Despite all these peculiarities, the solutions to the problem seem to have such a nice property that their values are in a simple closed form (see Conjecture 3.2). One would naturally think that a clever algorithm ought to be found, if only one that runs in linear time.

3. Preliminary Results

Let us fix $n \geq 3$. We first present a proposition which implies that it is always possible for the dog to win.

Proposition 3.1. *Let S be a zero-sum n-step sequence. If the step vectors in S are not all collinear, then there exists a permutation S' of S such that the walk of S' is simple.*

Proof. Let $S' = (s_1, s_2, \ldots, s_n)$ be a permutation of S such that, with θ_i being the argument of s_i satisfying $-\pi < \theta_i \leq \pi$, the sequence $(\theta_1, \theta_2, \ldots, \theta_n)$ is increasing. Since S is zero-sum, S' is also zero-sum. Let $s_{n+1} = s_1$ and denote by α_i the angle between s_{i+1} and s_i satisfying $0 \leq \alpha_i \leq \pi$. If $\theta_n - \theta_1 < \pi$, then S' would not be zero-sum. Thus we have $\alpha_n = 2\pi - (\theta_n - \theta_1)$. If there were some $1 \leq i < n$ such that $\theta_{i+1} - \theta_i > \pi$, then S' would again not be zero-sum. So we have $\alpha_i = \theta_{i+1} - \theta_i$ for all $1 \leq i < n$ and that $\sum_{i=1}^{n} \alpha_i = 2\pi$. Let P be the walk of S', a polygon of total absolute curvature 2π. By Fenchel's theorem (generalized to any closed curve by Milner [2, Theorem 3.4]), we know that P is convex. A convex polygon whose edges are not all collinear is simple (why?). $\square$

By computer simulation on Problem 2.6, we observe that the maximum simplicity on record follows a very simple formula, hence the following conjecture.

Conjecture 3.2. *The maximum simplicity of a zero-sum n-step sequence is exactly* $2 \cdot n!/(n-1)$.

References

[1] K. Pearson, *The Problem of the Random Walk*. Nature **72** (1905), 294.
[2] J. W. Milnor, *On the Total Curvature of Knots*. Ann. of Math. **52** (1950), no. 2, 248-257.