WHEN IS A CONVEX CLOSED CURVE NOT SIMPLE?

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Abstract. In this article, we provide analytical definitions of closed curves and of simpleness and convexity of closed curves. We will show that a convex closed curve is simple if and only if it does not lie entirely on a line.

1. Definitions

We first follow Milnor [1] to define the concepts in a more rigorous manner.

Definition 1.1 (Closed curves)**.** A *closed curve* in Euclidean n-space $\mathbb{E}^n$ is a continuous, periodic function $\gamma : \mathbb{R} \to \mathbb{R}^n$ which is not constant in any interval.

One can show that a closed curve has a least positive period. For convenience we redefine the notion of period as follows.

Definition 1.2 (Periods)**.** Let γ be a closed curve of least positive period p, and let $x \in \mathbb{R}$. We call p *the period* of γ, and the interval $P = [x, x + p)$ *a period* of γ. We say that a closed curve γ' is *of period* p' iff the period of γ' is p'.

Here we restricted periods to be right-open, since every definition in this article that relies on right-open periods can be proved equivalent to a left-open version.

Definition 1.3 (Simpleness)**.** Let γ be a closed curve of period p. We say that γ is *simple* iff for all $t_1, t_2 \in \mathbb{R}$, $\gamma(t_1) = \gamma(t_2)$ only when $(t_1 - t_2)/p$ is an integer.

Definition 1.4 (Convexity)**.** Let γ be a closed curve in $\mathbb{E}^2$. We say that γ is *convex* iff for every line $L \subset \mathbb{R}^2$, there exists a period P of γ such that the set $\{t \in P : \gamma(t) \in L\}$ either has cardinality ≤ 2 or is a nontrivial interval.

Proposition 1.5. *A closed curve γ in $\mathbb{E}^2$ is convex iff for every $u \in \mathbb{S}^1$ and $y \in \mathbb{R}$, there exists a period P of γ such that the set $\{t \in P : u \cdot \gamma(t) = y\}$ either has cardinality ≤ 2 or is a nontrivial interval.*

Proof. It is clear that every line $L \subset \mathbb{R}^2$ can be specified by a normal vector $u \in \mathbb{S}^1$ and a distance $y \in \mathbb{R}$ from the origin such that $L = \{x \in \mathbb{R}^2 : u \cdot x = y\}$. □

Definition 1.6. Let γ be a closed curve in $\mathbb{E}^n$, let $u \in \mathbb{S}^{n-1}$, and let P be any period of γ. Define $\mu(\gamma, u)$ to be the cardinality of the set $\{t \in P : \text{the function } u \cdot \gamma \text{ attains a local maximum at } t\}$ if it is finite, or ∞ otherwise.

The $\mu(\gamma, u)$ defined above is clearly unique as with many other properties of a periodic function over a period. We then prove a generalization of the necessity part of [1, Lemma 3.3] in the hope that our definitions are equivalent to Milnor's.

Proposition 1.7. *Let γ be a closed curve in $\mathbb{E}^2$. If γ is convex, then for every $u \in \mathbb{S}^1$ either $\mu(\gamma, u) = 1$ or $\mu(\gamma, u) = \infty$.*

Proof. Suppose that $2 \leq \mu(\gamma, u) < \infty$ for some $u \in \mathbb{S}^1$. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $u \cdot \gamma$, let $P = [x, x+p)$ be any period of γ, and let $t_1 < t_2 \in P$ such that f attains local maxima at t_1 and t_2. Suppose that $f(t_1) \geq f(t_2)$. Since f attains a finite number of local maxima in P, there exists a, b, y with $t_1 < a < t_2 < b < x+p$ such that $f(a), f(b) < y < f(t_2) \leq f(t_1)$. By the intermediate value theorem there exists $c_1 \in (t_1, a)$, $c_2 \in (a, t_2)$, and $c_3 \in (t_2, b)$ such that $f(c_i) = y$ for $i = 1, 2, 3$. Since $f(t_2) \neq y$, we know that the set $\{t \in P : u \cdot \gamma(t) = y\}$ is not an interval, and if it is finite, has cardinality > 2; this is a contradiction to convexity. Now suppose that $f(t_1) < f(t_2)$. If $t_1 \neq x$ then the proof is identical, so we assume that $t_1 = x$. It is always possible to find a $\delta > 0$ such that $x - \delta < t_1 < t_2 < x + p - \delta$. Similarly, there exists $c_1 \in (x - \delta, t_1)$ and $c_2 < c_3 \in (t_1, t_2)$ such that $f(c_i) = y$ for $i = 1, 2, 3$. Let $c_4 = c_1 + p \in (t_2, x+p)$ and obtain $f(c_4) = y$. Since $f(t_2) \neq y$ we again have a contradiction. $\square$

The sufficiency part of the lemma seems way more tricky to prove analytically, especially the case when $\mu(\gamma, u) = \infty$. Milnor's proof, on the other hand, made too much use of geometrical methods despite his analytical definitions. It is not entirely clear, for example, why it is always possible to rotate a line about one of its points of intersection with a polygon so that the number of intersections is not decreased. Also, we have this particular case when a polygon in the shape of 凸 (lit. convex; however the shape itself is concave) is intersected by a horizontal line: the set of points of intersection either has cardinality 2 or is infinite, but when it is infinite the set may still be unconnected. This case is not seen to be handled in Milnor's proof and does not seem trivial otherwise. I think, however, that the lemma is true but it still needs an analytical proof for the sake of completeness.

2. Proof

Now that we have sanity-checked the equivalency of definitions, we shall first present a lemma and then our main theorem.

Lemma 2.1. *Let $f : (a, b) \cup (b, c) \to \mathbb{R}^2$ be a continuous function which is not constant, and let $p \in \mathbb{R}^2$. Suppose that there exists $x_1, x_2 \in (a, b) \cup (b, c)$ such that the points $p, f(x_1), f(x_2)$ are noncollinear. Then there exists $x_1' \in (a, b), x_2' \in (b, c)$ such that the points $p, f(x_1'), f(x_2')$ are noncollinear.*

Proof. Suppose that for all $x_1' \in (a, b)$ and $x_2' \in (b, c)$, the points $p, f(x_1'), f(x_2')$ are collinear. Then we have $x_1, x_2 \in A$ where A is either (a, b) or (b, c). Let B be the other interval. Since f is not constant, there exists $x_0 \in B$ such that $f(x_0) \neq p$. But $f(x_0)$ cannot be collinear with $p, f(x_1)$ and with $p, f(x_2)$ at the same time. $\square$

Theorem 2.2. *A convex closed curve is simple iff it does not lie entirely on a line.*

Proof. Suppose that γ is a simple closed curve in $\mathbb{E}^2$ which lies entirely on a line. Then there exists a simple closed curve $\bar{\gamma}$ in $\mathbb{E}^1$. Let $P = [x, x+p)$ be any period of $\bar{\gamma}$. Since $\bar{\gamma}$ is not constant, there exists $t \in (x, x+p)$ such that $\bar{\gamma}(t) \neq \bar{\gamma}(x) = \bar{\gamma}(x+p)$. By the intermediate value theorem there exists $c_1 \in (x, t), c_2 \in (t, x+p)$ such that $\bar{\gamma}(c_1) = \bar{\gamma}(c_2) = y$ for some y between $\bar{\gamma}(t)$ and $\bar{\gamma}(x)$. But then $0 < (c_2 - c_1)/p < 1$, a contradiction to simpleness.

Now suppose that γ is a convex closed curve in $\mathbb{E}^2$ which is not simple and does not lie entirely on a line. Let $X = [x_1, x_1 + p)$ be a period of γ such that $\gamma(x_1) = \gamma(x_1')$ for some $x_1' \in (x_1, x_1 + p)$. Then there exists $x_2, x_3 \in (x_1, x_1') \cup (x_1', x_1 + p)$

such that the points $\gamma(x_1), \gamma(x_2), \gamma(x_3)$ are noncollinear. By Lemma 2.1 we may assume that $x_2 \in (x_1, x_1')$ and $x_3 \in (x_1', x_1 + p)$. Now let P be any period of γ. Then there exists $c_1, c_1', c_2, c_3 \in P$ such that $c_1 < c_3 < c_1'$, that $\gamma(c_1) = \gamma(c_1')$, and that the points $p_1 = \gamma(c_1), p_2 = \gamma(c_2), p_3 = \gamma(c_3)$ are noncollinear. Let $u \in \mathbb{S}^1$ be a vector perpendicular to $\overline{p_1 p_2}$. Then we have $u \cdot \gamma(c_1) = u \cdot \gamma(c_1') = u \cdot \gamma(c_2) \neq u \cdot \gamma(c_3)$, a contradiction to convexity. $\square$

References

[1] J. W. Milnor, *On the Total Curvature of Knots*, Ann. of Math. **52** (1950), no. 2, 248-257.