THE PROBLEM OF THE RANDOMLY WALKED DOG

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ABSTRACT. In this article, we present a novel problem in the spirit of the well-known problem of the random walk [2], along with a curious conjecture on the property of its solutions.

1. Introduction

A man walks the dog every day. The dog will prepare $n \geq 3$ cards beforehand, each containing a direction and a positive distance. They start from home and repeat the following process n times: (1) choose a random card; (2) walk the given distance in the given direction; (3) throw the card away.

After these n rounds, they shall be exactly back home. The dog wins a treat if they visited home exactly twice and any other place at most once. Now, help the dog to find n cards that maximize the probability of winning the treat in a walk.

Remark. This maximum probability is later conjectured to equal $2/(n-1)$, despite the lack of a general solution to the problem.

2. Definitions

We now proceed to formally define the problem. The cards the dog prepares should represent a sequence of nonzero vectors that sum to zero.

Definition 2.1. A *step vector* is a nonzero vector in $\mathbb{R}^2$. An *n-step sequence* S, or generally a *step sequence*, is a finite sequence $(s_1, s_2, \ldots, s_n)$ of step vectors; we say that S is *zero-sum* iff $\sum_{i=1}^{n} s_i = (0,0)$.

To study the points the pair visits in a walk, we shall introduce polygonal paths.

Definition 2.2. A *polygonal path* is a finite sequence $(p_0, p_1, \ldots, p_n)$ of points called its *vertices*; the line segments $\overline{p_0p_1}, \overline{p_1p_2}, \ldots, \overline{p_{n-1}p_n}$ are called its *edges*.

The winning conditions for the dog naturally translate to whether the polygonal path formed in a walk does not "intersect itself", or, more precisely, whether the path is simple.

Definition 2.3. A polygonal path $(p_0, p_1, \ldots, p_n)$ is *simple* iff for all $0 < i \leq j < n$,

$$\overline{p_{i-1}p_i} \cap \overline{p_jp_{j+1}} = \begin{cases} \{p_i\} & \text{if } i = j, \\ \{p_0\} \cap \{p_n\} & \text{if } n \geq 3 \text{ and } (i,j) = (1, n-1), \\ \varnothing & \text{otherwise.} \end{cases}$$

Geometrically, a simple path is one in which only consecutive edges intersect and only at their endpoints; it may also be *closed* as specially dealt with in the second case above. Next, we define a way to create a path from a step sequence.

Definition 2.4. Let $S := (s_1, s_2, \ldots, s_n)$ be a step sequence. The *walk* of S is the polygonal path $(p_0, p_1, \ldots, p_n)$ where $p_0 = (0,0)$ and $p_i = p_{i-1} + s_i$ for $1 \le i \le n$.

Finally, we define the value we are maximizing and then the problem.

Definition 2.5. The *simplicity* of a step sequence S is the number of permutations S' of S such that the walk of S' is simple.

Problem 2.6 (The Problem of the Randomly Walked Dog)
Instance: A natural number $n \ge 3$.
Task: Find a zero-sum n-step sequence of maximum simplicity.

This could be viewed as a combinatorial optimization problem, because w.l.o.g. we might restrict step vectors to those in $\mathbb{Z}^2$ of magnitude less than some $f(n)$, hence a finite set of feasible solutions. However, this problem is quite peculiar in that (a) each instance is only a single natural number, (b) the set of feasible solutions is not readily restricted to be finite, and (c) the value of a feasible solution cannot even be computed in reasonable time for n moderately large.

Despite all these peculiarities, the solutions to the problem seem to have the nice property that their values can be expressed in a simple closed form (see Conjecture 3.2). I thus claim that there is a polynomial-time algorithm for the problem, which I invite the reader to prove or disprove.

3. Preliminary Results

Let us fix $n \ge 3$. We first show that it is always possible for the dog to win.

Proposition 3.1. *Let S be a zero-sum n-step sequence. If the step vectors in S are not all collinear, then there exists a permutation S' of S such that the walk of S' is simple.*

Proof. Let $S' := (s_1, s_2, \ldots, s_n)$ be a permutation of S such that, with θ_i being the argument of s_i satisfying $-\pi < \theta_i \le \pi$, the sequence $(\theta_1, \theta_2, \ldots, \theta_n)$ is increasing. Since S is zero-sum, S' is also zero-sum. Let $s_{n+1} := s_1$ and denote by α_i the angle between s_{i+1} and s_i satisfying $0 \le \alpha_i \le \pi$. If $\theta_n - \theta_1 < \pi$, then S' would not be zero-sum. Thus we have $\alpha_n = 2\pi - (\theta_n - \theta_1)$. If there were some $1 \le i < n$ such that $\theta_{i+1} - \theta_i > \pi$, then S' would again not be zero-sum. So we have $\alpha_i = \theta_{i+1} - \theta_i$ for all $1 \le i < n$ and that $\sum_{i=1}^{n} \alpha_i = 2\pi$. Let P be the walk of S', a polygon of total absolute curvature 2π. By Fenchel's theorem (generalized to any closed curve by Milnor [1, Theorem 3.4]), we know that P is convex. Since P does not lie on a line, it follows from [3, Theorem 2.2] that P is simple. $\square$

By computer simulation on Problem 2.6, we observe that the maximum simplicities on record follow a very simple formula, hence the following conjecture.

Conjecture 3.2. *The maximum simplicity of a zero-sum n-step sequence equals*

$$2 \cdot n!/(n-1).$$

References

[1] J. W. Milnor, *On the total curvature of knots*, Ann. of Math. **52** (1950), no. 2, 248–257.
[2] K. Pearson, *The problem of the random walk*, Nature **72** (1905), no. 1865, 294.
[3] S. Ye, *A convex closed curve is simple iff it does not lie on a line*, preprint.