NICE PROOFS

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Proposition 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with period T, and let $x_0 \in \mathbb{R}$. If $f(x_0) \neq 0$ and $\int_0^T f(x)dx = 0$, then f has at least two zeros in the interval $I = (x_0, x_0 + T)$.*

Proof. The function f is periodic with period T, so

$$f(x_0 + T) = f(x_0) \neq 0,$$
$$\int_{x_0}^{x_0+T} f(x)dx = \int_0^T f(x)dx = 0.$$

Without loss of generality we may assume that $f(x_0 + T) = f(x_0) > 0$. By the mean value theorem for definite integrals, there exists a $c \in I$ such that

$$f(c) = \frac{1}{T}\int_{x_0}^{x_0+T} f(x)dx = 0.$$

Hence f has at least one zero in I. Suppose for sake of contradiction that f has only one zero in I. Again, suppose that $f(x) < 0$ for some $x \in I$. But then by the intermediate value theorem f has an extra zero in (x_0, x) or $(x, x_0 + T)$, a contradiction. Thus we have $f(x) \geq 0$ for all $x \in I$. Since f is continuous, there exists an $a \in I$ such that

$$f(x) \geq \frac{f(x_0)}{2} \quad \forall x \in (x_0, a).$$

By the properties of the Riemann integral

$$\begin{aligned} \int_{x_0}^{a} f(x)dx \geq \int_{x_0}^{a} \frac{f(x_0)}{2}dx = \frac{a - x_0}{2}f(x_0) > 0, \\ \int_{a}^{x_0+T} f(x)dx \geq 0, \\ \int_{x_0}^{x_0+T} f(x)dx = \int_{x_0}^{a} f(x)dx + \int_{a}^{x_0+T} f(x)dx > 0. \end{aligned}$$

This contradicts the fact that $\int_{x_0}^{x_0+T} f(x)dx = 0$. Thus f has at least two zeros in I, as desired. □

Proposition 2. *Let $f : (a, b) \to \mathbb{R}$ be a continuous function. Suppose that there exists a unique $x_0 \in (a, b)$ such that f attains a local maximum at x_0, and that f does not attain a local minimum anywhere. Then f also attains a global maximum at x_0.*

Proof. Suppose for sake of contradiction that $f(x_1) > f(x_0)$ for some $x_1 \in (a, b)$. Without loss of generality we may assume that $x_1 > x_0$. Since f attains a local maximum at x_0, there exists a $c \in (x_0, x_1)$ such that $f(x) \leq f(x_0)$ for all $x \in (x_0, c)$.

Suppose that $f(x) = f(x_0)$ for some $x \in (x_0, c)$. But then f attains a local maximum at x, a contradiction. Hence $f(x) < f(x_0)$ for all $x \in (x_0, c)$.

Now let $c' \in (x_0, c)$ and obtain $f(c') < f(x_0)$. Since f is continuous, there exists a $c'' \in [x_0, x_1]$ such that $f(x) \geq f(c'')$ for all $x \in [x_0, x_1]$, by the maximum principle. Thus we have $f(c'') \leq f(c') < f(x_0) < f(x_1)$ and that $c'' \in (x_0, x_1)$. But then f attains a local minimum at c'', a contradiction. Thus $f(x) \leq f(x_0)$ for all $x \in (a, b)$, as desired. $\square$