WHEN IS A CONVEX CLOSED CURVE NOT SIMPLE?

SCALLOP YE

Abstract. A convex curve is a plane curve whose intersection with a line is either a set of at most two points or a line segment. With analytical definitions of convexity and simpleness, we show that a convex closed curve is simple if and only if it does not lie on a line. We conjecture that the same result still holds when geometrical definitions are used instead.

1. Definitions

We first follow Milnor [1] to define the concepts in a rigorous manner.

Definition 1.1 (Closed curves)**.** A *closed curve* in Euclidean n-space $\mathbb{E}^n$ is a nonconstant continuous periodic function $\gamma : \mathbb{R} \to \mathbb{R}^n$. We say that a closed curve is *planar* iff its image lies on a line or in a plane.

Here we do not restrict a closed curve to be light[1], considering that every closed curve may be reparameterized to be light, as we will see in Section 3.

One can show that a closed curve has a least positive period. For convenience we redefine the notion of period for closed curves as follows.

Definition 1.2 (Periods)**.** Let γ be a closed curve of least positive period p, and let $x \in \mathbb{R}$. We call p *the period* of γ, and the interval $P = [x, x+p)$ *a period* of γ. We say that a closed curve γ' is *of period* p' iff the period of γ' is p'.

In fact, every closed curve γ of period p can be factored into a unique pair of functions $(\mathring{\gamma} : \mathbb{S}^1 \to \mathbb{R}^n, \tilde{\gamma} : \mathbb{R} \to \mathbb{S}^1)$ such that $\gamma = \mathring{\gamma} \circ \tilde{\gamma}$, where $\mathring{\gamma}$ is a nonconstant continuous function and $\tilde{\gamma}(t) := (\cos 2\pi t/p, \sin 2\pi t/p)$. We may call $\mathring{\gamma}$ the *underlying map* of γ, call $\tilde{\gamma}$ the *intermediate map* of γ, and fix these notations.

Note that the restriction of $\tilde{\gamma}$ to every period of γ is bijective. This fact allows us to translate between the properties of a closed curve γ (on individual periods) and of its underlying map $\mathring{\gamma}$ (on the unit circle).

Definition 1.3 (Simpleness)**.** Let γ be a closed curve of period p. We say that γ is *simple* iff for all $t_1, t_2 \in \mathbb{R}$, $\gamma(t_1) = \gamma(t_2)$ only when $(t_1 - t_2)/p$ is an integer.

Proposition 1.4. *Let γ be a closed curve. Then the following are equivalent:*

(1) *γ is simple.*
(2) *$\mathring{\gamma}$ is a homeomorphism.*
(3) *For every period P of γ, the restriction of γ to P is injective.*
(4) *There exists a period P of γ such that the restriction of γ to P is injective.*

Proof. The claim follows immediately from the definitions above. □

[1] A continuous function $f : S_1 \to S_2$ is called *light* iff $f^{-1}(y)$ is totally disconnected for each $y \in S_2$ (see [2, Definition 13.1]). A closed curve is light iff it is not constant on any interval.

Proposition 1.5. *Let $f : \mathbb{R} \to \mathbb{R}^n$ be a continuous periodic function, and let $p > 0$ be a period of f. Suppose that the restriction of f to some interval $[x_0, x_0 + p)$ is injective. Then f is a simple closed curve of period p.*

Proof. The restriction being injective implies that f is nonconstant and that p is the least positive period of f. Thus f is a closed curve of period p. Then by Proposition 1.4 we know that f is simple. $\square$

We now present a unified definition of convex closed curves in all dimensions. Later we will show that the property of convexity implies planarity and is preserved under a linear isomorphism (thus consistent across all dimensions).

Definition 1.6 (Convexity)**.** Let γ be a closed curve in $\mathbb{E}^n$. We say that γ is *convex* iff for every hyperplane $H \subset \mathbb{R}^n$, the inverse image $\mathring{\gamma}^{-1}(H)$ either has cardinality ≤ 2, is a nondegenerate arc, or is the entire unit circle.

Proposition 1.7. *A closed curve γ in $\mathbb{E}^n$ is convex iff for every $u \in \mathbb{S}^{n-1}$ and $y \in \mathbb{R}$, there exists a period P of γ such that the set $\{t \in P : u \cdot \gamma(t) = y\}$ either has cardinality ≤ 2 or is a nondegenerate interval.*

Sketch of proof. A normal vector $u \in \mathbb{S}^{n-1}$ and an offset $y \in \mathbb{R}$ from the origin specify a hyperplane $H := \{x \in \mathbb{R}^n : u \cdot x = y\}$, and every hyperplane in $\mathbb{R}^n$ can be specified in this manner. It only remains to notice that an interval within *some* period of the closed curve γ specifies an arc on the unit circle or the entire unit circle (the domain of the underlying map $\mathring{\gamma}$), and every arc on the unit circle and the entire unit circle can be specified in this manner. $\square$

Proposition 1.8. *A convex closed curve is planar.*

Proof. Let γ be a closed curve in $\mathbb{E}^n$ ($n \geq 3$) which is not planar, and let P be any period of γ. Then there exists $c_1 < c_2 < c_3 < c_4 \in P$ such that the points $p_1 := \gamma(c_1), p_2 := \gamma(c_2), p_3 := \gamma(c_3), p_4 := \gamma(c_4)$ are noncoplanar. Let $H \subset \mathbb{R}^n$ be a hyperplane such that $p_1, p_3, p_4 \in H$ but $p_2 \notin H$, and let $u \in \mathbb{S}^{n-1}$ be a vector normal to H. Since $u \cdot \gamma(c_1) = u \cdot \gamma(c_3) = u \cdot \gamma(c_4) \neq u \cdot \gamma(c_2)$, the set $\{t \in P : u \cdot \gamma(t) = y\}$ is not an interval, and if it is finite, has cardinality > 2. Hence γ is not convex by Proposition 1.7. $\square$

Proposition 1.9. *Let γ be a closed curve in $\mathbb{E}^2$ of image C, let γ' be a planar closed curve in $\mathbb{E}^n$ of image C' where $n \geq 2$, and let $P \subset \mathbb{R}^n$ be a plane such that $C' \subset P$. Suppose that there exists a linear isomorphism $T : \mathbb{R}^2 \to P$ such that $\gamma' = T \circ \gamma$. Then γ is convex iff γ' is convex.*

Sketch of proof. The intersection of the plane P and a hyperplane $H \subset \mathbb{R}^n$ is either the empty set, a line, or the entire plane P (when $n > 2$); conversely, every line in P and the entire plane P (when $n > 2$) can be written as such an intersection. It only remains to notice that both T and T^{-1} transforms lines into lines and the entire plane into the other plane, in a one-to-one manner. $\square$

We then prove a generalization of the necessity part of [1, Lemma 3.3] in the hope that our definitions are equivalent to Milnor's.

Definition 1.10. Let γ be a closed curve in $\mathbb{E}^n$, let $u \in \mathbb{S}^{n-1}$, and let P be any period of γ. Define $\mu(\gamma, u)$ to be the cardinality of the set $\{t \in P : \text{the function } u \cdot \gamma \text{ attains a local maximum at } t\}$ if it is finite, or ∞ otherwise.

Proposition 1.11. *Let γ be a closed curve in $\mathbb{E}^n$. If γ is convex, then for every $u \in \mathbb{S}^{n-1}$ either $\mu(\gamma, u) = 1$ or $\mu(\gamma, u) = \infty$.*

Proof. Suppose that $2 \leq \mu(\gamma, u) < \infty$ for some $u \in \mathbb{S}^{n-1}$. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $u \cdot \gamma$, let $P := [x, x+p)$ be any period of γ, and let $t_1 < t_2 \in P$ such that f attains local maxima at t_1 and t_2. Suppose that $f(t_1) \geq f(t_2)$. Since f attains a finite number of local maxima in P, there exists a, b, y with $t_1 < a < t_2 < b < x+p$ such that $f(a), f(b) < y < f(t_2) \leq f(t_1)$. By the intermediate value theorem there exists $c_1 \in (t_1, a)$, $c_2 \in (a, t_2)$, and $c_3 \in (t_2, b)$ such that $f(c_i) = y$ for $i = 1, 2, 3$. Since $f(t_2) \neq y$, we know that γ is not convex, a contradiction. Now suppose that $f(t_1) < f(t_2)$. If $t_1 \neq x$ then the proof is identical, so we assume that $t_1 = x$. Take $\delta := (x + p - t_2)/2$. Similarly, there exists c_1, c_2, c_3, y with $x - \delta < c_1 < t_1 = x < c_2 < c_3 < t_2 < x + p - \delta$ and $y < f(t_1) < f(t_2)$ such that $f(c_i) = y$ for $i = 1, 2, 3$. Let $c_4 := c_1 + p$ and obtain $f(c_4) = y$ with $c_4 \in (t_2, x+p)$. Since $f(t_2) \neq y$, we again have a contradiction. □

The sufficiency part of the lemma seems way more tricky to prove analytically, especially the case when $\mu(\gamma, u) = \infty$. Milnor's proof, on the other hand, made too much use of geometrical methods despite his analytical definitions. It is not entirely clear, for example, why it is always possible to rotate a line about one of its points of intersection with a polygon so that the number of intersections is not decreased. Also, we have this particular case when a polygon in the shape of 凸 (lit. convex; however the shape itself is concave) is intersected by a horizontal line: the set of points of intersection either has cardinality 2 or is infinite, but when infinite the set may still be disconnected. This case is not seen to be handled in Milnor's proof and does not seem trivial otherwise. I think, however, that the lemma is true but still needs an analytical proof for the sake of completeness.

2. Analytical Proof

Now that we have checked the equivalence of definitions, we shall first present a lemma and then our main theorem. Note that it suffices to give proofs for the two-dimensional case as a convex closed curve is planar.

Lemma 2.1. *Let $f : (a, b) \cup (b, c) \to \mathbb{R}^2$ be a nonconstant function, and let $p \in \mathbb{R}^2$. Suppose that there exists $x_1, x_2 \in (a, b) \cup (b, c)$ such that the points $p, f(x_1), f(x_2)$ are noncollinear. Then there exists $x_1' \in (a, b)$ and $x_2' \in (b, c)$ such that the points $p, f(x_1'), f(x_2')$ are noncollinear.*

Proof. Suppose that for all $x_1' \in (a, b)$ and $x_2' \in (b, c)$, the points $p, f(x_1'), f(x_2')$ are collinear. Then we have $x_1, x_2 \in A$ where A is either (a, b) or (b, c). Let B be the other interval. Since f is nonconstant, there exists $x_0 \in B$ such that $f(x_0) \neq p$. But $f(x_0)$ cannot be collinear with $p, f(x_1)$ and with $p, f(x_2)$ at the same time. □

Theorem 2.2. *A convex closed curve is simple iff it does not lie on a line.*

Proof. The image of a simple closed curve is homeomorphic to $\mathbb{S}^1$ and thus cannot be one-dimensional. Let γ be a convex closed curve in $\mathbb{E}^2$ which is not simple and does not lie on a line. Let $X := [x_1, x_1 + p)$ be a period of γ such that $\gamma(x_1) = \gamma(x_1')$ for some $x_1' \in (x_1, x_1 + p)$. Then there exists $x_2, x_3 \in (x_1, x_1') \cup (x_1', x_1 + p)$ such that the points $\gamma(x_1), \gamma(x_2), \gamma(x_3)$ are noncollinear. By Lemma 2.1 we may assume that $x_2 \in (x_1, x_1')$ and $x_3 \in (x_1', x_1 + p)$. Now let P be any period of γ. Then there exists $c_1, c_2, c_1', c_3 \in P$ such that $\gamma(c_1) = \gamma(c_1')$, that $c_1 < c_2 < c_1'$, and that the points $p_1 := \gamma(c_1), p_2 := \gamma(c_2), p_3 := \gamma(c_3)$ are noncollinear. Let $u \in \mathbb{S}^1$ be a vector perpendicular to $\overline{p_1 p_3}$. Then we have $u \cdot \gamma(c_1) = u \cdot \gamma(c_1') = u \cdot \gamma(c_3) \neq u \cdot \gamma(c_2)$, a contradiction to convexity. □

3. Geometrical Consideration

Definition 3.1 (Parameterizations)**.** Let γ be a closed curve in $\mathbb{E}^n$, and let $C \subset \mathbb{R}^n$. We say that γ is a *parameterization* of C iff the image of γ is C. Two closed curves are said to be *reparameterizations* of each other iff their images equal.

Proposition 3.2. *Every closed curve has a light reparameterization.*

Proof. The claim follows from [2, Corollary 13.4] by viewing maps $f : \mathbb{S}^1 \to \mathbb{R}^n$ as maps $f : [0,1] \to \mathbb{R}^n$ with $f(0) = f(1)$. □

Definition 3.3 (Simple sets)**.** Let $C \subset \mathbb{R}^n$. We say that C is *simple* iff C is homeomorphic to the unit circle $\mathbb{S}^1$.

Proposition 3.4. *A simple subset of* $\mathbb{R}^n$ *has a simple parameterization.*

Proof. Let C be a simple subset of $\mathbb{R}^n$. Then there exists a continuous injection $f : \mathbb{S}^1 \to \mathbb{R}^n$ of image C. Let $g : \mathbb{R} \to \mathbb{S}^1$ be the function $g(t) := (\cos t, \sin t)$. We know that g is continuous on $\mathbb{R}$ and periodic with period 2π, and that the restriction $g|_{[0,2\pi)}$ is injective. Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be the function $f \circ g$. Then γ is continuous on $\mathbb{R}$ and periodic with period 2π, and the restriction $\gamma|_{[0,2\pi)}$ is injective. It follows from Proposition 1.5 that γ is a simple closed curve of image C. □

Definition 3.5 (Weakly convex sets)**.** Let $C \subset \mathbb{R}^2$. We say that C is *weakly convex* iff for every line $L \subset \mathbb{R}^2$, the set $C \cap L$ either has cardinality ≤ 2 or is a nondegenerate line segment.

Proposition 3.6. *A simple closed curve of weakly convex image is convex.*

Proof. Let γ be a simple closed curve of weakly convex image C, and let $L \subset \mathbb{R}^2$ be any line. Since C is weakly convex, the set $C \cap L$ either has cardinality ≤ 2 or is a nondegenerate line segment. Since $\mathring{\gamma}$ is a homeomorphism by Proposition 1.4, the inverse image $\mathring{\gamma}^{-1}(L)$ either has cardinality ≤ 2, is a nondegenerate arc, or is the entire unit circle. Thus γ is convex by definition. □

Definition 3.7 (Irreducibility)**.** Let γ be a closed curve of image C. We say that γ is *irreducible* iff for every period P of γ and every closed interval $[a, b] \subset P$ with $\gamma(a) = \gamma(b)$, we have $\gamma([a, b]) \neq C$.

Being irreducible means that the curve does not contain "redundant loops" which do not contribute to its image and can thus be removed. This sounds like a good property indeed, so let us make some conjectures on it. It is conceivable that 3.8 already has a proof somewhere, and that 3.9 and/or 3.10 have simple proofs.

Conjecture 3.8. *Every closed curve has an irreducible reparameterization.*

Conjecture 3.9. *An irreducible closed curve of simple image is simple.*

Conjecture 3.10. *An irreducible closed curve that lies on a line is convex.*

Conjecture 3.11. *An irreducible closed curve of weakly convex image is convex.*

Proof that $3.9 \wedge 3.10 \wedge 3.13 \implies 3.11$. Let γ be an irreducible closed curve of weakly convex image C. If C is a subset of a line, then γ is convex by Conjecture 3.10. If C is not a subset of a line, then we have C simple by Conjecture 3.13, γ simple by Conjecture 3.9, and thus γ convex by Proposition 3.6. □

We then present two equivalent conjectures. Note that Conjecture 3.13 is the geometrical analog to our main theorem.

Conjecture 3.12. *Let C be the image of a closed curve. If C is weakly convex, then C has a convex parameterization.*

Proof that $3.8 \wedge 3.11 \implies 3.12$. This implication is immediate. □

Proof that $3.13 \implies 3.12$. If C is not simple, then by Conjecture 3.13 we know that C is a subset of a line, which clearly is a line segment and has a convex parameterization. If C is simple, then by Proposition 3.4 there exists a simple parameterization γ of C, which is convex by Proposition 3.6. □

Conjecture 3.13. *Let C be the image of a closed curve. Suppose that C is weakly convex. Then C is simple iff C is not a subset of a line.*

Proof that $3.12 \implies 3.13$. Clearly, a subset of a line cannot be simple. Suppose that C is not a subset of a line. By Conjecture 3.12 there exists a convex parameterization γ of C. Then we have γ simple by Theorem 2.2, and thus C simple by Proposition 1.4. □

References

[1] J. W. Milnor, *On the Total Curvature of Knots*, Ann. of Math. **52** (1950), no. 2, 248-257.
[2] S. B. Nadler, Jr., *Continuum Theory: An Introduction*, CRC Press, Boca Raton, 1992.