WHEN IS A CONVEX CLOSED CURVE NOT SIMPLE?

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Abstract. A convex curve is, informally, a plane curve whose intersection with a line is either a set of at most two points or a line segment. With analytical definitions we show that a convex closed curve is simple if and only if it does not lie entirely on a line.

1. Definitions

We first follow Milnor [1] to define the concepts in a more rigorous manner.

Definition 1.1 (Closed curves)**.** A *closed curve* in Euclidean n-space $\mathbb{E}^n$ is a continuous, periodic function $\gamma : \mathbb{R} \to \mathbb{R}^n$ which is not constant in any interval.

One can show that a closed curve has a least positive period. For convenience we redefine the notion of period as follows.

Definition 1.2 (Periods)**.** Let γ be a closed curve of least positive period p, and let $x \in \mathbb{R}$. We call p *the period* of γ, and the interval $P = [x, x+p)$ *a period* of γ. We say that a closed curve γ' is *of period* p' iff the period of γ' is p'.

Definition 1.3 (Simpleness)**.** Let γ be a closed curve of period p. We say that γ is *simple* iff for all $t_1, t_2 \in \mathbb{R}$, $\gamma(t_1) = \gamma(t_2)$ only when $(t_1 - t_2)/p$ is an integer.

Definition 1.4 (Convexity)**.** Let γ be a closed curve in $\mathbb{E}^2$. We say that γ is *convex* iff for every line $L \subset \mathbb{R}^2$, there exists a period P of γ such that the set $\{t \in P : \gamma(t) \in L\}$ either has cardinality ≤ 2 or is a nontrivial interval.

Proposition 1.5. *A closed curve γ in $\mathbb{E}^2$ is convex iff for every $u \in \mathbb{S}^1$ and $y \in \mathbb{R}$, there exists a period P of γ such that the set $\{t \in P : u \cdot \gamma(t) = y\}$ either has cardinality ≤ 2 or is a nontrivial interval.*

Proof. It is clear that every line $L \subset \mathbb{R}^2$ can be specified by a normal vector $u \in \mathbb{S}^1$ and a distance $y \in \mathbb{R}$ from the origin such that $L = \{x \in \mathbb{R}^2 : u \cdot x = y\}$. $\square$

Definition 1.6. Let γ be a closed curve in $\mathbb{E}^n$, let $u \in \mathbb{S}^{n-1}$, and let P be any period of γ. Define $\mu(\gamma, u)$ to be the cardinality of the set $\{t \in P :$ the function $u \cdot \gamma$ attains a local maximum at $t\}$ if it is finite, or ∞ otherwise.

The $\mu(\gamma, u)$ defined above is clearly unique as with many other properties of a periodic function over a period. We then prove a generalization of the necessity part of [1, Lemma 3.3] in the hope that our definitions are equivalent to Milnor's.

Proposition 1.7. *Let γ be a closed curve in $\mathbb{E}^2$. If γ is convex, then for every $u \in \mathbb{S}^1$ either $\mu(\gamma, u) = 1$ or $\mu(\gamma, u) = \infty$.*

Proof. Suppose that $2 \leq \mu(\gamma, u) < \infty$ for some $u \in \mathbb{S}^1$. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $u \cdot \gamma$, let $P = [x, x + p)$ be any period of γ, and let $t_1 < t_2 \in P$ such that f attains local maxima at t_1 and t_2. Suppose that $f(t_1) \geq f(t_2)$. Since f attains a finite number of local maxima in P, there exists a, b, y with $t_1 < a < t_2 < b < x+p$ such that $f(a), f(b) < y < f(t_2) \leq f(t_1)$. By the intermediate value theorem there exists $c_1 \in (t_1, a)$, $c_2 \in (a, t_2)$, and $c_3 \in (t_2, b)$ such that $f(c_i) = y$ for $i = 1, 2, 3$. Since $f(t_2) \neq y$, we know that the set $\{t \in P : u \cdot \gamma(t) = y\}$ is not an interval, and if it is finite, has cardinality > 2; this is a contradiction to convexity. Now suppose that $f(t_1) < f(t_2)$. If $t_1 \neq x$ then the proof is identical, so we assume that $t_1 = x$. It is always possible to find a $\delta > 0$ such that $x - \delta < t_1 < t_2 < x + p - \delta$. Similarly, there exists $c_1 \in (x - \delta, t_1)$ and $c_2 < c_3 \in (t_1, t_2)$ such that $f(c_i) = y$ for $i = 1, 2, 3$. Let $c_4 = c_1 + p \in (t_2, x + p)$ and obtain $f(c_4) = y$. Since $f(t_2) \neq y$ we again have a contradiction. □

The sufficiency part of the lemma seems way more tricky to prove analytically, especially the case when $\mu(\gamma, u) = \infty$. Milnor's proof, on the other hand, made too much use of geometrical methods despite his analytical definitions. It is not entirely clear, for example, why it is always possible to rotate a line about one of its points of intersection with a polygon so that the number of intersections is not decreased. Also, we have this particular case when a polygon in the shape of 凸 (lit. convex; however the shape itself is concave) is intersected by a horizontal line: the set of points of intersection either has cardinality 2 or is infinite, but when it is infinite the set may still be unconnected. This case is not seen to be handled in Milnor's proof and does not seem trivial otherwise. I think, however, that the lemma is true but still needs an analytical proof for the sake of completeness.

2. Proof

Now that we have checked the equivalency of definitions, we shall first present a lemma and then our main theorem.

Lemma 2.1. *Let $f : (a, b) \cup (b, c) \to \mathbb{R}^2$ be a continuous function which is not constant, and let $p \in \mathbb{R}^2$. Suppose that there exists $x_1, x_2 \in (a, b) \cup (b, c)$ such that the points $p, f(x_1), f(x_2)$ are noncollinear. Then there exists $x_1' \in (a, b), x_2' \in (b, c)$ such that the points $p, f(x_1'), f(x_2')$ are noncollinear.*

Proof. Suppose that for all $x_1' \in (a, b)$ and $x_2' \in (b, c)$, the points $p, f(x_1'), f(x_2')$ are collinear. Then we have $x_1, x_2 \in A$ where A is either (a, b) or (b, c). Let B be the other interval. Since f is not constant, there exists $x_0 \in B$ such that $f(x_0) \neq p$. But $f(x_0)$ cannot be collinear with $p, f(x_1)$ and with $p, f(x_2)$ at the same time. □

Theorem 2.2. *A convex closed curve is simple iff it does not lie entirely on a line.*

Proof. Suppose that γ is a simple closed curve in $\mathbb{E}^2$ which lies entirely on a line. Then there exists a simple closed curve $\bar{\gamma}$ in $\mathbb{E}^1$. Let $P = [x, x+p)$ be any period of $\bar{\gamma}$. Since $\bar{\gamma}$ is not constant, there exists $t \in (x, x + p)$ such that $\bar{\gamma}(t) \neq \bar{\gamma}(x) = \bar{\gamma}(x + p)$. By the intermediate value theorem there exists $c_1 \in (x, t), c_2 \in (t, x + p)$ such that $\bar{\gamma}(c_1) = \bar{\gamma}(c_2) = y$ for some y between $\bar{\gamma}(t)$ and $\bar{\gamma}(x)$. But then $0 < (c_2 - c_1)/p < 1$, a contradiction to simpleness.

Now suppose that γ is a convex closed curve in $\mathbb{E}^2$ which is not simple and does not lie entirely on a line. Let $X = [x_1, x_1 + p)$ be a period of γ such that $\gamma(x_1) = \gamma(x_1')$ for some $x_1' \in (x_1, x_1 + p)$. Then there exists $x_2, x_3 \in (x_1, x_1') \cup (x_1', x_1 + p)$

such that the points $\gamma(x_1), \gamma(x_2), \gamma(x_3)$ are noncollinear. By Lemma 2.1 we may assume that $x_2 \in (x_1, x_1')$ and $x_3 \in (x_1', x_1 + p)$. Now let P be any period of γ. Then there exists $c_1, c_1', c_2, c_3 \in P$ such that $\gamma(c_1) = \gamma(c_1')$, that $c_1 < c_3 < c_1'$, and that the points $p_1 = \gamma(c_1), p_2 = \gamma(c_2), p_3 = \gamma(c_3)$ are noncollinear. Let $u \in \mathbb{S}^1$ be a vector perpendicular to $\overline{p_1 p_2}$. Then we have $u \cdot \gamma(c_1) = u \cdot \gamma(c_1') = u \cdot \gamma(c_2) \neq u \cdot \gamma(c_3)$, a contradiction to convexity. $\square$

References

[1] J. W. Milnor, *On the Total Curvature of Knots*, Ann. of Math. **52** (1950), no. 2, 248-257.