A CONVEX CLOSED CURVE IS SIMPLE IFF IT DOES NOT LIE ON A LINE

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Abstract. A convex curve is a plane curve whose intersection with a line is either a set of at most two points or a line segment. With analytical definitions of convexity and simpleness, we show that a convex closed curve is simple if and only if it does not lie on a line. We note that the same result still holds when topological definitions are used instead.

1. Definitions

We first follow Milnor [2] to define the concepts in a rigorous manner.

Definition 1.1 (Closed curves)**.** A *closed curve* in $\mathbb{R}^n$ is a nonconstant continuous periodic function $\gamma : \mathbb{R} \to \mathbb{R}^n$. We say that a closed curve is *planar* iff its image lies in a plane or on a line.

Here we do not restrict a closed curve to be nonconstant on any interval, considering that every closed curve may be reparameterized to be so, as we will see in Section 4.

One can show that a closed curve has a least positive period. For convenience we redefine the notion of period for closed curves as follows.

Definition 1.2 (Periods)**.** Let γ be a closed curve of least positive period p, and let $x \in \mathbb{R}$. We call p *the period* of γ, and the interval $P := [x, x + p)$ *a period* of γ. We say that a closed curve γ' is *of period* p' iff the period of γ' is p'.

In fact, every closed curve γ of period p can be factored into a unique pair of functions $(\mathring{\gamma} : \mathbb{S}^1 \to \mathbb{R}^n, \tilde{\gamma} : \mathbb{R} \to \mathbb{S}^1)$ such that $\gamma = \mathring{\gamma} \circ \tilde{\gamma}$, where $\mathring{\gamma}$ is a nonconstant continuous function and $\tilde{\gamma}$ is another closed curve $\tilde{\gamma}(t) := (\cos 2\pi t/p, \sin 2\pi t/p)$ whose image is the unit circle $\mathbb{S}^1$. We may call $\mathring{\gamma}$ the *equivalent circular map* of γ, call $\tilde{\gamma}$ the *underlying circle* of γ, and fix these notations.

Note that the restriction of $\tilde{\gamma}$ to every period of γ is continuous and bijective. This fact allows us to translate between the properties of a closed curve γ (on individual periods) and of its equivalent circular map $\mathring{\gamma}$ (on the unit circle).

Definition 1.3 (Simpleness)**.** Let γ be a closed curve of period p. We say that γ is *simple* iff for all $t_1, t_2 \in \mathbb{R}$, $\gamma(t_1) = \gamma(t_2)$ only when $(t_1 - t_2)/p$ is an integer.

Proposition 1.4. *Let γ be a closed curve. Then the following are equivalent:*

1. *γ is simple.*
2. *$\mathring{\gamma}$ is a homeomorphism.*
3. *For every period P of γ, the restriction of γ to P is injective.*
4. *There exists a period P of γ such that the restriction of γ to P is injective.*

Proof. The claim follows immediately from the definitions above. □

Proposition 1.5. *Let $f : \mathbb{R} \to \mathbb{R}^n$ be a continuous periodic function, and let $p > 0$ be a period of f. Suppose that the restriction of f to some interval $[x_0, x_0 + p)$ is injective. Then f is a simple closed curve of period p.*

Proof. The restriction being injective implies that f is nonconstant and that p is the least positive period of f. Thus f is a closed curve of period p. Then by Proposition 1.4 we know that f is simple. $\square$

We now present a unified definition of convex closed curves in all dimensions. Later we will show that the property of convexity implies planarity and is preserved under a bijective affine map (thus consistent across all dimensions).

Definition 1.6 (Convexity)**.** Let γ be a closed curve in $\mathbb{R}^n$. We say that γ is *convex* iff for every hyperplane $H \subset \mathbb{R}^n$, the inverse image $\mathring{\gamma}^{-1}(H)$ either has cardinality ≤ 2, is a nondegenerate arc, or is the unit circle $\mathbb{S}^1$.

Proposition 1.7. *A closed curve γ in $\mathbb{R}^n$ is convex iff for every $u \in \mathbb{S}^{n-1}$ and $y \in \mathbb{R}$, there exists a period P of γ such that the set $\{t \in P : u \cdot \gamma(t) = y\}$ either has cardinality ≤ 2 or is a nondegenerate interval.*

Proof. Suppose that there exists a hyperplane $H \subset \mathbb{R}^n$ such that $\mathring{\gamma}^{-1}(H)$ is neither $\mathbb{S}^1$ nor an arc, and if it is finite, has cardinality > 2. Then we can find a normal vector $u \in \mathbb{S}^{n-1}$ and an offset $y \in \mathbb{R}$ such that $H = \{x \in \mathbb{R}^n : u \cdot x = y\}$. We claim that there does not exist a period P of γ such that the set $T := \{t \in P : u \cdot \gamma(t) = y\}$ either has cardinality ≤ 2 or is a nondegenerate interval, because otherwise

$$\mathring{\gamma}^{-1}(H) = \tilde{\gamma}(\gamma^{-1}(H)) = \tilde{\gamma}(\gamma^{-1}(H) \cap P) = \tilde{\gamma}(T)$$

would either have cardinality ≤ 2 (when T has cardinality ≤ 2), be a nondegenerate arc (when T is a nondegenerate proper subinterval of P), or be $\mathbb{S}^1$ (when $T = P$).

Now suppose that there exists $u \in \mathbb{S}^{n-1}$ and $y \in \mathbb{R}$ such that for every period P of γ, the set $T := \{t \in P : u \cdot \gamma(t) = y\}$ is not an interval, and if it is finite, has cardinality > 2. Let H be the hyperplane $H := \{x \in \mathbb{R}^n : u \cdot x = y\}$. Then $\mathring{\gamma}^{-1}(H)$ is neither $\mathbb{S}^1$ nor an arc, and if it is finite, has cardinality > 2, since otherwise

$$T = \gamma^{-1}(H) \cap P = \tilde{\gamma}^{-1}(\mathring{\gamma}^{-1}(H)) \cap P$$

would either have cardinality ≤ 2 for any period P of γ (when $\mathring{\gamma}^{-1}(H)$ has cardinality ≤ 2), be a nondegenerate proper subinterval of P for some period $P := [x_0, x_0 + p)$ of γ such that $\tilde{\gamma}(x_0) \notin \mathring{\gamma}^{-1}(H)$ (when $\mathring{\gamma}^{-1}(H)$ is a nondegenerate arc), or be P for any period P of γ (when $\mathring{\gamma}^{-1}(H) = \mathbb{S}^1$). $\square$

Proposition 1.8. *A convex closed curve is planar.*

Proof. Let γ be a closed curve in $\mathbb{R}^n$ ($n \geq 3$) which is not planar, and let P be any period of γ. Then there exists $c_1 < c_2 < c_3 < c_4 \in P$ such that the points $p_1 := \gamma(c_1), p_2 := \gamma(c_2), p_3 := \gamma(c_3), p_4 := \gamma(c_4)$ are noncoplanar. Observe that every plane in $\mathbb{R}^n$ can be written as an intersection of hyperplanes, and so we can find a hyperplane $H \subset \mathbb{R}^n$ such that $p_1, p_3, p_4 \in H$ but $p_2 \notin H$. Now let $u \in \mathbb{S}^{n-1}$ be a vector normal to H. Since $u \cdot \gamma(c_1) = u \cdot \gamma(c_3) = u \cdot \gamma(c_4) \neq u \cdot \gamma(c_2)$, the set $\{t \in P : u \cdot \gamma(t) = y\}$ is not an interval, and if it is finite, has cardinality > 2. Hence γ is not convex by Proposition 1.7. $\square$

Proposition 1.9. *Let $1 \leq n \leq m$ be integers, let γ be a closed curve in $\mathbb{R}^n$ of image C, and let γ' be a closed curve in $\mathbb{R}^m$ of image C' such that $C' \subset F$ for some*

n-flat $F \subset \mathbb{R}^m$. Suppose that there exists a bijective affine map $f : \mathbb{R}^n \to F$ such that $\gamma' = f \circ \gamma$. Then γ is convex iff γ' is convex.

Sketch of proof. Suppose that γ is not convex. Then there exists a hyperplane $H \subset \mathbb{R}^n$ such that $\mathring{\gamma}^{-1}(H)$ is neither $\mathbb{S}^1$ nor an arc, and if it is finite, has cardinality > 2. We can find a hyperplane $H' \subset \mathbb{R}^m$ such that $F \cap H' = f(H)$ (why?). It only remains to show that $\mathring{\gamma}'^{-1}(H') = \mathring{\gamma}^{-1}(H)$.

Now suppose that γ' is not convex. Then there exists a hyperplane $H' \subset \mathbb{R}^m$ such that $\mathring{\gamma}'^{-1}(H')$ is neither $\mathbb{S}^1$ nor an arc, and if it is finite, has cardinality > 2. The intersection $F \cap H'$ must be an $(n-1)$-flat in $\mathbb{R}^m$ (why?), so the inverse image $H := f^{-1}(H')$ is a hyperplane in $\mathbb{R}^n$. It only remains to show that $\mathring{\gamma}^{-1}(H) = \mathring{\gamma}'^{-1}(H')$. □

We then prove a generalization of the necessity part of [2, Lemma 3.3] in the hope that our definitions are equivalent to Milnor's.

Definition 1.10. Let γ be a closed curve in $\mathbb{R}^n$, let $u \in \mathbb{S}^{n-1}$, and let P be any period of γ. Define $\mu(\gamma, u)$ to be the cardinality of the set $\{t \in P :$ the function $u \cdot \gamma$ attains a local maximum at $t\}$ if it is finite, or ∞ otherwise.

Proposition 1.11. *Let γ be a closed curve in $\mathbb{R}^n$. If γ is convex, then for every $u \in \mathbb{S}^{n-1}$ either $\mu(\gamma, u) = 1$ or $\mu(\gamma, u) = \infty$.*

Proof. Suppose that $2 \leq \mu(\gamma, u) < \infty$ for some $u \in \mathbb{S}^{n-1}$. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $u \cdot \gamma$, let $P := [x, x+p)$ be any period of γ, and let $t_1 < t_2 \in P$ such that f attains local maxima at t_1 and t_2. Suppose that $f(t_1) \geq f(t_2)$. Since f attains a finite number of local maxima in P, there exists a, b, y with $t_1 < a < t_2 < b < x+p$ such that $f(a), f(b) < y < f(t_2) \leq f(t_1)$. By the intermediate value theorem there exists $c_1 \in (t_1, a)$, $c_2 \in (a, t_2)$, and $c_3 \in (t_2, b)$ such that $f(c_i) = y$ for $i = 1, 2, 3$. Since $f(t_2) \neq y$, we know that γ is not convex, a contradiction.

Now suppose that $f(t_1) < f(t_2)$. If $t_1 \neq x$ then the proof is identical, so we assume that $t_1 = x$. Take $\delta := (x + p - t_2)/2$. Similarly, there exists c_1, c_2, c_3, y with $x - \delta < c_1 < t_1 = x < c_2 < c_3 < t_2 < x+p-\delta$ and $y < f(t_1) < f(t_2)$ such that $f(c_i) = y$ for $i = 1, 2, 3$. Let $c_4 := c_1 + p$ and obtain $f(c_4) = y$ with $c_4 \in (t_2, x+p)$. Since $f(t_2) \neq y$, we again have a contradiction. □

The sufficiency part of the lemma seems way more tricky to prove analytically, especially the case when $\mu(\gamma, u) = \infty$. Milnor's proof, on the other hand, made too much use of geometrical methods despite his analytical definitions. It is not entirely clear, for example, why it is always possible to rotate a line about one of its points of intersection with a polygon so that the number of intersections is not decreased. Also, we have this particular case when a polygon in the shape of 凸 (lit. convex; however the shape itself is concave) is intersected by a horizontal line: the set of points of intersection either has cardinality 2 or is infinite, but when infinite the set may still be disconnected. This case is not seen to be handled in Milnor's proof and does not seem trivial otherwise. I think, however, that the lemma is true but still needs an analytical proof for the sake of completeness.

2. Analytical Result

Now that we have checked the equivalence of definitions, we shall first present a lemma and then our main theorem. Note that it suffices to give proofs for the two-dimensional case as a convex closed curve is planar.

Lemma 2.1. *Let $f : (a, b) \cup (b, c) \to \mathbb{R}^2$ be a nonconstant function, and let $p \in \mathbb{R}^2$. Suppose that there exists $x_1, x_2 \in (a, b) \cup (b, c)$ such that the points $p, f(x_1), f(x_2)$ are noncollinear. Then there exists $x_1' \in (a, b)$ and $x_2' \in (b, c)$ such that the points $p, f(x_1'), f(x_2')$ are noncollinear.*

Proof. Suppose that for all $x_1' \in (a, b)$ and $x_2' \in (b, c)$, the points $p, f(x_1'), f(x_2')$ are collinear. Then we have $x_1, x_2 \in A$ where A is either (a, b) or (b, c). Let B be the other interval. Since f is nonconstant, there exists $x_0 \in B$ such that $f(x_0) \neq p$. But $f(x_0)$ cannot be collinear with $p, f(x_1)$ and with $p, f(x_2)$ at the same time. □

Theorem 2.2. *A convex closed curve is simple iff its image does not lie on a line.*

Proof. A subset of a line cannot be homeomorphic to $\mathbb{S}^1$. Let γ be a convex closed curve in $\mathbb{R}^2$ which is not simple and whose image does not lie on a line. Let $X := [x_1, x_1+p)$ be a period of γ such that $\gamma(x_1) = \gamma(x_1')$ for some $x_1' \in (x_1, x_1+p)$. Then there exists $x_2, x_3 \in (x_1, x_1') \cup (x_1', x_1+p)$ such that the points $\gamma(x_1), \gamma(x_2), \gamma(x_3)$ are noncollinear. By Lemma 2.1 we may assume that $x_2 \in (x_1, x_1')$ and $x_3 \in (x_1', x_1+p)$. Now let P be any period of γ. Then there exists $c_1, c_2, c_1', c_3 \in P$ such that $\gamma(c_1) = \gamma(c_1')$, that $c_1 < c_2 < c_1'$, and that the points $p_1 := \gamma(c_1), p_2 := \gamma(c_2), p_3 := \gamma(c_3)$ are noncollinear. Let $u \in \mathbb{S}^1$ be a vector perpendicular to $\overline{p_1 p_3}$. Then we have $u \cdot \gamma(c_1) = u \cdot \gamma(c_1') = u \cdot \gamma(c_3) \neq u \cdot \gamma(c_2)$, a contradiction to convexity. □

3. Topological Result

The following theorem is the topological analogue to Theorem 2.2.

Theorem 3.1. *The boundary C of a bounded convex set in $\mathbb{R}^2$ is homeomorphic to $\mathbb{S}^1$ iff C does not lie on a line.*

Proof. A subset of a line cannot be homeomorphic to $\mathbb{S}^1$. The remaining claim follows from [1, Proposition 28 and Theorem 32]. □

4. Conceivable Connection Between the Results

It is conceivable that there is a connection between Theorems 2.2 and 3.1, which we now attempt to establish.

Proposition 4.1. *Suppose that $C \subset \mathbb{R}^n$ is a set homeomorphic to $\mathbb{S}^1$. Then C is the image of a simple closed curve.*

Proof. Since C is homeomorphic to $\mathbb{S}^1$, there exists a continuous injection $f : \mathbb{S}^1 \to \mathbb{R}^n$ of image C. Let $g : \mathbb{R} \to \mathbb{S}^1$ be the function $g(t) := (\cos t, \sin t)$. We know that g is continuous on $\mathbb{R}$ and periodic with period 2π, and that the restriction $g|_{[0,2\pi)}$ is injective. Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be the function $f \circ g$. Then γ is continuous on $\mathbb{R}$ and periodic with period 2π, and the restriction $\gamma|_{[0,2\pi)}$ is injective. It follows from Proposition 1.5 that γ is a simple closed curve of image C. □

Definition 4.2 (Reparameterizations)**.** Two closed curves are said to be *reparameterizations* of each other iff their images equal.

Definition 4.3 (Lightness)**.** A continuous function $f : S_1 \to S_2$ is called *light* iff $f^{-1}(y)$ is totally disconnected for each $y \in S_2$ (see [3, Definition 13.1]). A closed curve is light iff it is nonconstant on any interval.

Proposition 4.4. *Every closed curve has a light reparameterization.*

Proof. The claim follows from [3, Corollary 13.4] by viewing maps $f : \mathbb{S}^1 \to \mathbb{R}^n$ as maps $f : [0, 1] \to \mathbb{R}^n$ with $f(0) = f(1)$. $\square$

Definition 4.5 (Weakly convex sets)**.** Let $C \subset \mathbb{R}^2$. We say that C is *weakly convex* iff for every line $L \subset \mathbb{R}^2$, the set $C \cap L$ either has cardinality ≤ 2 or is a nondegenerate line segment.

Proposition 4.6. *A simple closed curve of weakly convex image is convex.*

Proof. Let γ be a simple closed curve of weakly convex image C, and let $L \subset \mathbb{R}^2$ be any line. Since C is weakly convex, the set $C \cap L$ either has cardinality ≤ 2 or is a nondegenerate line segment. Since $\mathring{\gamma}$ is a homeomorphism by Proposition 1.4, the inverse image $\mathring{\gamma}^{-1}(L)$ either has cardinality ≤ 2 or is infinite and connected. Thus γ is convex by definition. $\square$

Definition 4.7 (Irreducibility)**.** Let γ be a closed curve of image C. We say that γ is *irreducible* iff for every period P of γ and every closed interval $[a, b] \subset P$ with $\gamma(a) = \gamma(b)$, we have $\gamma([a, b]) \neq C$.

Being irreducible means that the curve does not contain “redundant loops” which do not contribute to its image and can thus be removed. We now show that every closed curve may be reparameterized to be irreducible.

Lemma 4.8. *Every closed curve has an irreducible reparameterization.*

Proof. See `https://math.stackexchange.com/q/4792008` for the moment. $\square$

We make two conjectures that might be of use.

Conjecture 4.9. *An irreducible closed curve whose image lies on a line is convex.*

Conjecture 4.10. *An irreducible closed curve whose image is homeomorphic to* $\mathbb{S}^1$ *is simple.*

It remains to be seen whether these results and conjectures may connect Theorem 2.2 with Theorem 3.1. See also the following section for a failed attempt.

5. Failed Attempt to Find a Connection

We present three conjectures which are *equally false*, with notable counterexamples being space-filling curves.

Conjecture 5.1. *An irreducible closed curve of weakly convex image is convex.*

Proof that $4.9 \wedge 4.10 \wedge 5.3 \implies 5.1$. Let γ be an irreducible closed curve of weakly convex image C. If C lies on a line, then γ is convex by Conjecture 4.9. If C does not lie on a line, then we have C homeomorphic to $\mathbb{S}^1$ by Conjecture 5.3, γ simple by Conjecture 4.10, and thus γ convex by Proposition 4.6. $\square$

Conjecture 5.2. *Let C be the image of a closed curve. If C is weakly convex, then C is the image of a convex closed curve.*

Proof that $5.1 \implies 5.2$. By Lemma 4.8 there exists an irreducible closed curve of image C, which is convex by Conjecture 5.1. $\square$

Proof that $5.3 \implies 5.2$. If C is not homeomorphic to $\mathbb{S}^1$, then by Conjecture 5.3 we know that C lies on a line, which certainly is a line segment and is the image of a convex closed curve. If C is homeomorphic to $\mathbb{S}^1$, then by Proposition 4.1 there exists a simple closed curve of image C, which is convex by Proposition 4.6. $\square$

Conjecture 5.3. *Let C be the image of a closed curve. Suppose that C is weakly convex. Then C is homeomorphic to $\mathbb{S}^1$ iff C does not lie on a line.*

Proof that $5.2 \implies 5.3$. A subset of a line cannot be homeomorphic to $\mathbb{S}^1$. Suppose that C does not lie on a line. By Conjecture 5.2 there exists a convex closed curve γ of image C. Then we have γ simple by Theorem 2.2, and thus C homeomorphic to $\mathbb{S}^1$ by Proposition 1.4. $\square$

References

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[3] S. B. Nadler, Jr., *Continuum Theory: An Introduction*, CRC Press, Boca Raton, 1992.