NICE PROOFS

SCALLOP YE

Proposition 1. *Let f be a continuous function with period T, and let x_0 be a point. If $f(x_0) \neq 0$ and $\int_0^T f(x)dx = 0$, then f has at least two zeros in the interval $I = (x_0, x_0 + T)$.*

Proof. The function f is periodic with period T, so

$$f(x_0 + T) = f(x_0) \neq 0,$$
$$\int_{x_0}^{x_0+T} f(x)dx = \int_0^T f(x)dx = 0.$$

Without loss of generality we may assume that $f(x_0 + T) = f(x_0) > 0$. By the mean value theorem for definite integrals, there exists $c \in I$ such that

$$f(c) = \frac{1}{T} \int_{x_0}^{x_0+T} f(x)dx = 0.$$

Hence f has at least one zero in I. Suppose for sake of contradiction that f has only one zero in I. Again, suppose for sake of contradiction that there exists $m \in I$ such that $f(m) < 0$. So by the intermediate value theorem f has an extra zero in (x_0, m) or $(m, x_0 + T)$, a contradiction. Thus we have

$$f(x) \geq 0 \quad \forall x \in I,$$

and as f has only one zero in I

$$f(x) > 0 \quad \forall x \in I - \{c\}.$$

By the mean value theorem for definite integrals, there exists $x_1 \in (x_0, c)$ and $x_2 \in (c, x_0 + T)$ such that

$$0 < f(x_1) = \frac{1}{c - x_0} \int_{x_0}^{c} f(x)dx,$$
$$0 < f(x_2) = \frac{1}{x_0 + T - c} \int_{c}^{x_0+T} f(x)dx.$$

Then we have

$$\int_{x_0}^{x_0+T} f(x)dx = \int_{x_0}^{c} f(x)dx + \int_{c}^{x_0+T} f(x)dx > 0,$$

contradicting the fact that $\int_{x_0}^{x_0+T} f(x)dx = 0$. This contradiction gives the proof. $\square$