DSA3102: Solutions to Homework 2
Assigned: 11/09/23, Due: 05/10/2023

Instructions:

- Please do all the five problems below. A (non-empty) subset of them will be graded.
- Write (legibly) on pieces of paper or Latex your answers. Convert your submission to PDF format.
- Submit to the Assignment 2 folder under Assignments in Canvas before 23:59 on 05/10/2023.

1. For which sets of $\alpha \in \mathbb{R}$ are the following functions convex?

   (a) (2 points) $f(x) = \sin x + \alpha x$ with $\mathbf{dom}\, f = \mathbb{R}$
   Solution: $f''(x) = -\sin x$ which is not positive definite. So no α ensures that f is convex.

   (b) (3 points) $f(x) = \sin x + \alpha x^2$ with $\mathbf{dom}\, f = \mathbb{R}$
   Solution: $f''(x) = -\sin x + 2\alpha$ which is positive definite if $\alpha \geq \frac{1}{2}$.

   (c) (5 points) $f(x_1, x_2) = (5-\alpha)x_1^2 + 10x_1x_2 + x_2^2 + 4\alpha x_1$ with $\mathbf{dom}\, f = \mathbb{R}^2$
   Solution: The Hessian of f is

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 2(5-\alpha) & 10 \\ 10 & 2 \end{bmatrix}$$

   The northwest entry is non-negative if $\alpha \leq 5$. The southeast entry is definitely positive. The determinant is non-negative if $\alpha \leq -20$. So $\nabla^2 f(x_1, x_2)$ is psd, i.e., f is convex, if $\alpha \leq -20$.

2. (Ky Fan's inequality)
   Let $x_i \in (0, 1/2]$ and let $\gamma_i \in (0, 1)$ for $i = 1, \ldots, n$ be real numbers satisfying $\sum_{i=1}^n \gamma_i = 1$. It is known that either

$$\text{(A)} \qquad \frac{\prod_{i=1}^n x_i^{\gamma_i}}{\prod_{i=1}^n (1-x_i)^{\gamma_i}} \leq \frac{\sum_{i=1}^n \gamma_i x_i}{\sum_{i=1}^n \gamma_i (1-x_i)}$$

   or

$$\text{(B)} \qquad \frac{\prod_{i=1}^n x_i^{\gamma_i}}{\prod_{i=1}^n (1-x_i)^{\gamma_i}} \geq \frac{\sum_{i=1}^n \gamma_i x_i}{\sum_{i=1}^n \gamma_i (1-x_i)}$$

   is true. Which is true and why?

   Hint: Consider the convexity/concavity properties of the function $f(x) = \ln x - \ln(1-x)$ *with* $\mathbf{dom}\, f = [0, 1/2]$ *and then apply Jensen's inequality.*

   When does equality hold?

Solution: The hint asks us to consider the properties of $f(x) = \ln x - \ln(1-x)$. Upon differentiation, we obtain

$$f''(x) = -\frac{1}{x^2} + \frac{1}{(1-x)^2} = \frac{2x-1}{x^2(1-x)^2} \leq 0$$

so the function f is concave. By Jensen's inequality applied to the convex combination $\sum_i \gamma_i x_i$, we obtain

$$f\left(\sum_i \gamma_i x_i\right) \geq \sum_i \gamma_i f(x_i)$$

In other words,

$$\ln\left(\frac{\sum_i \gamma_i x_i}{1 - \sum_i \gamma_i x_i}\right) \geq \sum_i \gamma_i \ln\left(\frac{\sum_i x_i}{1 - \sum_i x_i}\right)$$

Upon rearrangement (taking exp on both sides), we obtain

$$\frac{\sum_i \gamma_i x_i}{\sum_i \gamma_i (1-x_i)} \geq \prod_i \frac{x_i^{\gamma_i}}{(1-x_i)^{\gamma_i}}$$

so inequality (A) is true.

Equality holds if and only if all the $x_i, i = 1, \ldots, n$ are equal to some common $x \in (0, 1/2]$.

3. (Conjugate of the conjugate of a quadratic on bounded interval)

   Consider the convex function $f : [0,1] \to \mathbb{R}$ defined as

   $$f(x) = \frac{1}{2}x^2.$$

   The domain of f is $\mathbf{dom}\, f = [0,1]$.

   (a) Find the conjugate function $f^*(y)$ specifying various ranges of y carefully.

   (b) Now find the conjugate of f^*. Call it f^{**}.

   (c) Is $f^{**}(x)$ the same as $f(x)$?

   Solutions:

   (a) Consider

   $$f^*(y) = \sup_{x \in [0,1]} \left\{ xy - \frac{x^2}{2} \right\}.$$

   Note that $f^*(y)$ is finite for all y. When $y < 0$ both xy and $-\frac{x^2}{2}$ are monotonically decreasing and so the sup is attained at 0 and $f^*(y) = 0$. When $y \in [0,1]$, direct differentiation yields that $x^* = y$ and $f^*(y) = \frac{y^2}{2}$. Finally, when $y > 1$, the function $xy - \frac{x^2}{2}$ is monotonically increasing on the interval $[0,1]$ and the sup is attained at $x = 1$. This yields $f^*(y) = y - \frac{1}{2}$. In summary

   $$f^*(y) = \begin{cases} 0 & y < 0 \\ \frac{y^2}{2} & y \in [0,1] \\ y - \frac{1}{2} & y > 1 \end{cases}.$$

   Note that the domain of f^* is the whole real line $\mathbb{R}$ and $f^*(y)$ is increasing on $[0, \infty)$.

(b) Now, we evaluate the conjugate of the function $f^*(y)$ above. Consider,

$$f^{**}(x) = \sup_{y\in\mathbb{R}} \{xy - f^*(y)\}.$$

When $x < 0$, the function $g(y) = xy - f^*(y)$ is unbounded as $y \to -\infty$. When $x > 1$, similarly, $g(y)$ is unbounded as $y \to \infty$. When $x \in [0, 1]$; say $y < 0$, then $g(y) = 0$ so $f^{**}(x) = 0$; say $y > 1$, then $g(y) = xy - y + \frac{1}{2}$ which is maximized when $y \to 1^-$, which gives $f^{**}(x) = x - \frac{1}{2}$; say $y \in [0, 1]$, then $g(y) = xy - \frac{y^2}{2}$ which gives $f^{**}(x) = \frac{1}{2}x^2$. But $\frac{1}{2}x^2 \geq x - \frac{1}{2}$ for all $x \in [0, 1]$, so when $x \in [0, 1]$, $f^{**}(x) = \frac{1}{2}x^2$. In summary,

$$f^{**}(x) = \frac{1}{2}x^2$$

with domain $\mathbf{dom}\, f^{**} = [0, 1]$.

(c) We see that $f = f^{**}$. This is also corroborated by the fact that f is convex, closed and proper.

4. (Reformulating a problem into an SOCP)

Consider the following optimization problem:

$$\begin{array}{ll} \min\limits_{x} \max\limits_{k=1,\ldots,n} & \left|\log(a_k^T x) - \log(b_k)\right| \\ \text{subject to} & x \succeq 0, \end{array}$$

where we assume that $b_i > 0$ and $\log(a_i^T x) = -\infty$ when $a_i^T x \leq 0$. Formulate this problem as a second-order cone program (SOCP).

Solutions: Note that

$$\left|\log(a_k^T x) - \log(b_k)\right| = \log \max\left\{\frac{a_k^T x}{b_k}, \frac{b_k}{a_k^T x}\right\},$$

so we can rewrite the optimization problem as

$$\min_{x} \max_{k=1,\ldots,n} \log \max\left\{\frac{a_k^T x}{b_k}, \frac{b_k}{a_k^T x}\right\}.$$

Since log is monotonically increasing, this is equivalent to

$$\min_{x} \max_{k=1,\ldots,n} \max\left\{\frac{a_k^T x}{b_k}, \frac{b_k}{a_k^T x}\right\}.$$

Now using the epigraph reformulation, we can write the optimization problem as

$$\begin{array}{ll} \min\limits_{t,x} & t \\ \text{subject to} & \dfrac{a_k^T x}{b_k} \leq t,\ \dfrac{a_k^T x}{b_k} \geq \dfrac{1}{t}, \quad k = 1, \ldots, n \\ & x \succeq 0. \end{array}$$

By reformulating the constraint $\frac{a_k^T x}{b_k} \geq \frac{1}{t}$, as

$$\left\| \begin{bmatrix} 2 \\ t - \frac{a_k^T x}{b_k} \end{bmatrix} \right\| \leq t + \frac{a_k^T x}{b_k}, \quad k = 1, \ldots, n,$$

we obtain an SOCP in variables x and t.

5. (Reformulating a non-convex problem into a convex problem)

Consider the following optimization problem

$$\begin{aligned} \min_x \ & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0 \quad i = 1, 2, \ldots, m \\ & x \succeq 0 \end{aligned}$$

where each f_i is quadratic, i.e.,

$$f_i(x) = \frac{1}{2} x^T P_i x + q_i^T x + r_i$$

where $P_i \in \mathbf{S}^n$ ($\mathbf{S}^n$ is the set of $n \times n$ symmetric matrices), $q_i \in \mathbb{R}^n$ and $r_i \in \mathbb{R}$ for all $i = 0, 1, \ldots, m$. We note here that the problem may not be convex since we have not restricted P_i to be positive semidefinite.

Suppose that $q_i \preceq 0$ (all the components of the vector q_i are nonpositive) and P_i has nonpositive off-diagonal elements for all $i = 0, 1, \ldots, m$. Reformulate this problem as a convex problem by considering a change of variables from x_i to $y_i = g(x_i)$ for some suitable function g.

Solutions: Let $g(x) = x^2$ be the square function. Then $y_j = x_j^2$. Because $x_j \geq 0$ we can recover x_j from y_j as $x_j = y_j^{1/2}$. The quadratic functions f_i can then be written as

$$f_i(x) = \frac{1}{2} \sum_{j=1}^{n} (P_i) y_i + \frac{1}{2} \sum_{j \neq k} (P_i)_{jk} \sqrt{y_j y_k} + \sum_{j=1}^{n} (q_i)_j \sqrt{y_j} + r_i$$

Since the $y_i \mapsto \sqrt{y_i}$ and $(y_i, y_k) \mapsto \sqrt{y_j y_k}$ are concave functions and $(q_i)_j, (P_i)_{jk} \leq 0$, the function f_i is convex in y with components $y_i = x_i^2$. Thus the QCQP becomes a convex optimization problem in y.