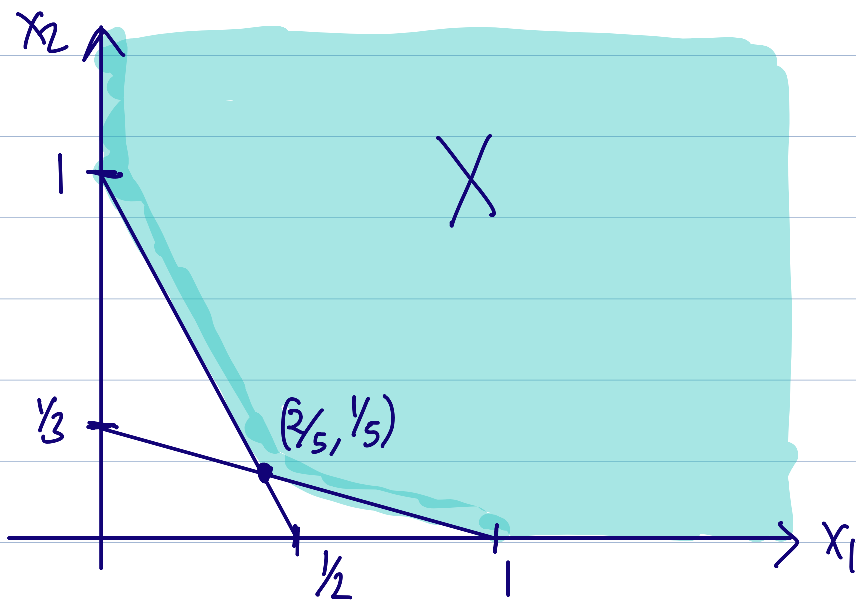
DSA3102 Tutorial 6

$$\min_{x_1, x_2} f_0(x_1, x_2)$$

$2x_1 + x_2 \geq 1 \Rightarrow x_2 \geq 1 - 2x_1$

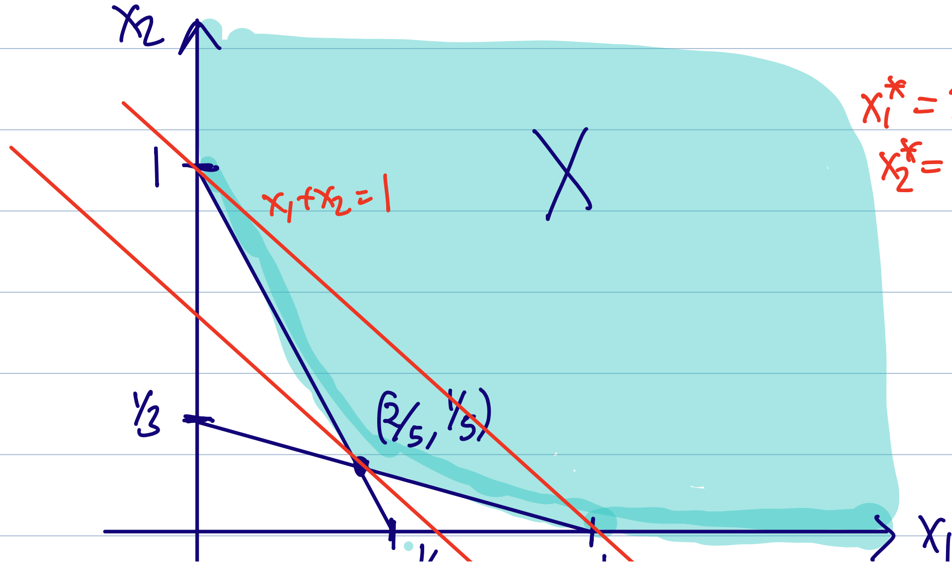
$x_1 + 3x_2 \geq 1 \Rightarrow x_2 \geq \frac{1}{3}(1 - x_1)$

$x_1, x_2 \geq 0$



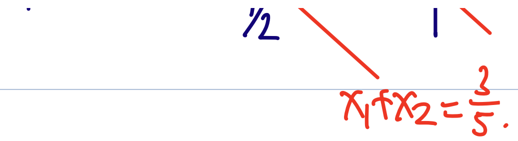
a) $f_0(x_1, x_2) = x_1 + x_2.$

$x_1 + x_2 = c.$

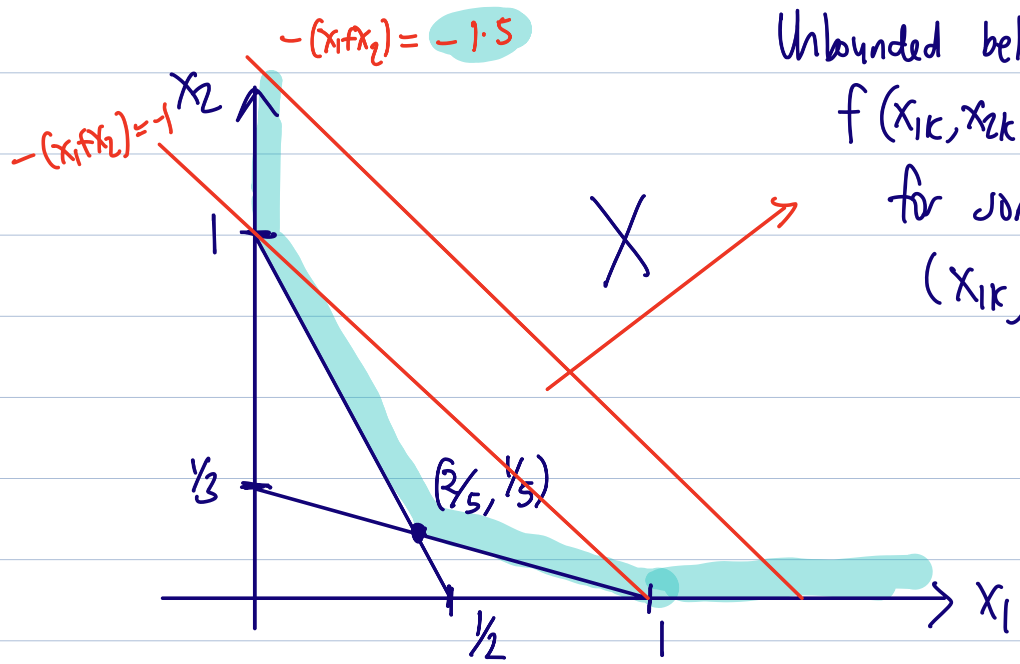


$x_1^* = \frac{2}{5}$

$x_2^* = \frac{1}{5}.$

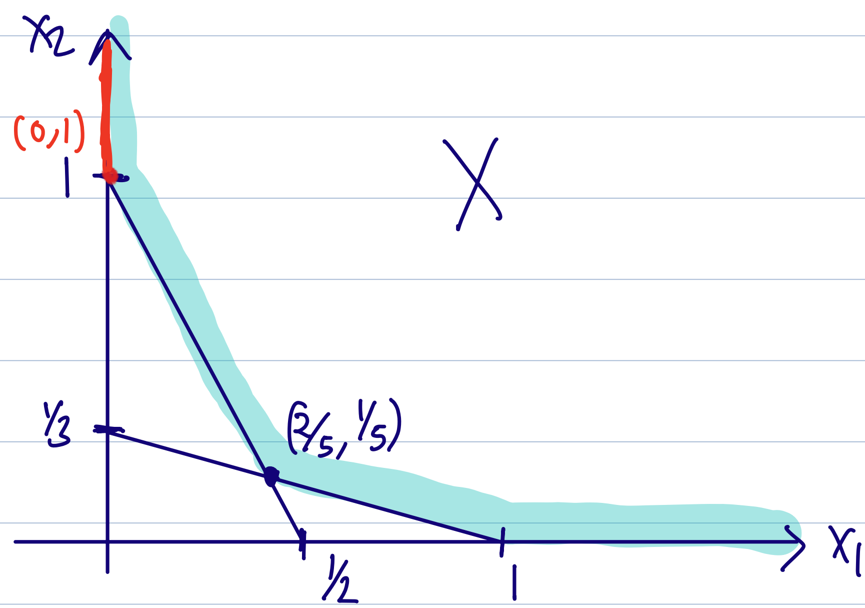


b) $f_0(x_1,x_2) = -(x_1+x_2)$



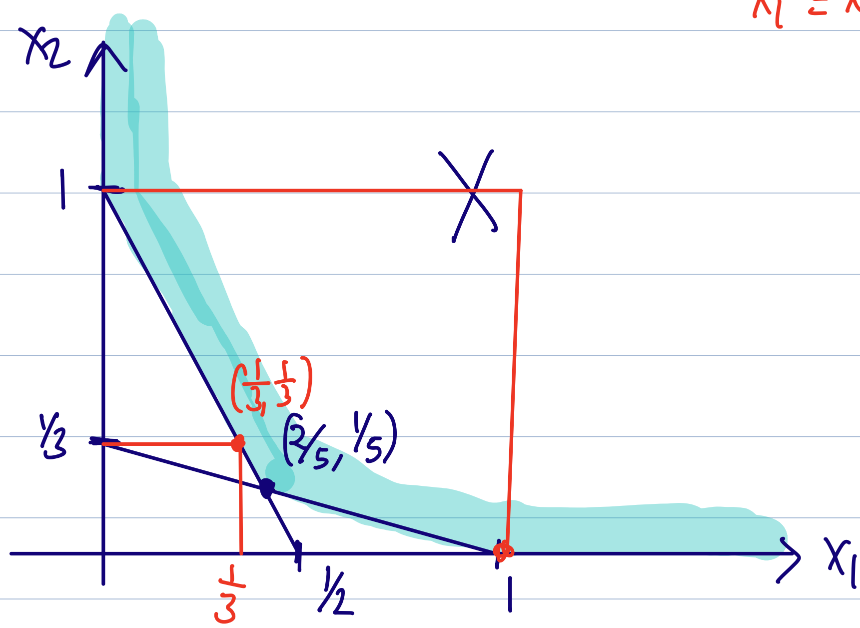
Unbounded below
$f(x_{1k}, x_{2k}) = -\infty.$
for some seq
(x_{1k}, x_{2k})

c) $f_0(x_1,x_2) = x_1$ $\qquad X_{opt} = \{(x_1,x_2)\in\mathbb{R}^2 ;\ x_1 = 0, x_2 \geq 1\}$



d) $f_0(x_1, x_2) = \max\{x_1, x_2\} = \|x\|_\infty$

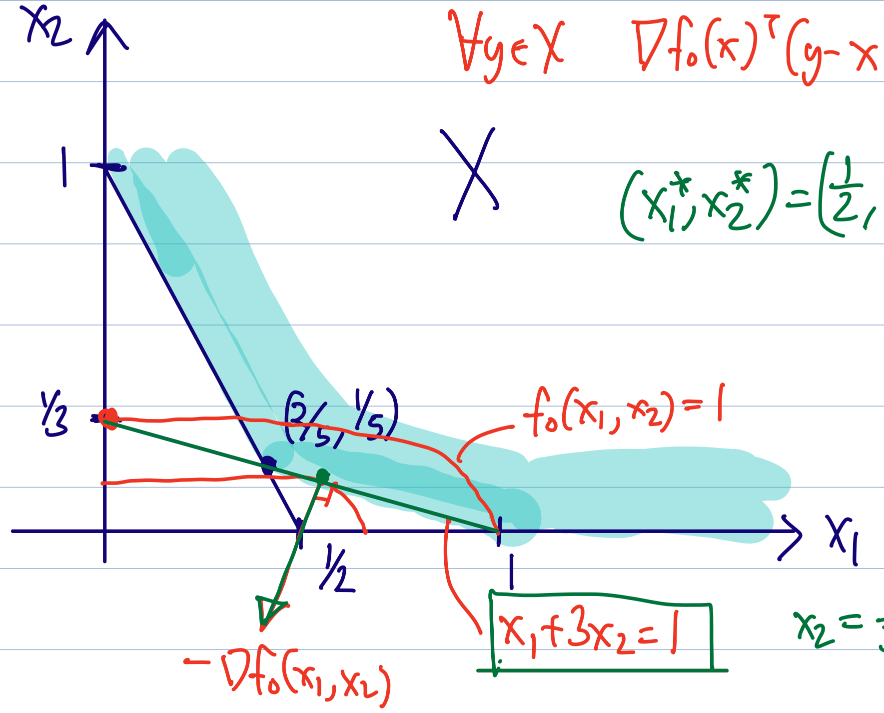
$x_1^* = x_2^* = \frac{1}{3}.$



e) $f_0(x_1, x_2) = x_1^2 + 9x_2^2 = c$

$\forall y \in X \quad \nabla f_0(x)^T (y - x) \geq 0.$

$(x_1^*, x_2^*) = (\frac{1}{2}, \frac{1}{6})$



$x_2 = \frac{1}{3}(1 - x_1)$

$\nabla f_0(x_1, x_2) = (2x_1, 18x_2)^T$

$$\nabla f_0(x_1,x_2)\Big|_{x_1^*, x_2^*} = \underline{(1,3)}$$

$\nabla f_0(x_1,x_2)\Big|_{x_1^*=\frac{1}{?},\, x_2^*=\frac{1}{6}}$ is perpendicular to $x_1+3x_2=1$.

4.7 BV

$$\min_x \frac{f_0(x)}{c^Tx+d} \quad \text{s.t.} \quad \begin{matrix} f_i(x)\le 0 \ \forall i \\ Ax=b. \end{matrix}$$

$f_0, f_1, \ldots, f_m$: convex

NTS: $\frac{f_0(x)}{c^Tx+d} = g_0(x)$ is quasiconvex

$$S_\alpha(g_0) = \left\{ x : \frac{f_0(x)}{c^Tx+d} \le \alpha \right\}.$$

$$= \left\{ x : f_0(x) \le \alpha(c^Tx+d) \right\}$$

$$= \{ x : \underbrace{f_0(x) - \alpha(c^Tx+d)}_{\text{convex.}} \le 0 \}. \ : \text{convex.}$$

b) Introduce $t = \frac{1}{c^Tx+d} > 0$. Define $y = \frac{x}{c^Tx+d} = xt$.

$x = \frac{y}{t}$

$$\min_x \frac{f_0(x)}{c^Tx+d} \quad \text{s.t.} \quad \begin{matrix} f_i(x)\le 0 \ \forall i \\ Ax=b. \end{matrix}$$

① $Ax=b \Rightarrow A\left(\frac{y}{t}\right)=b \Rightarrow Ay=bt \Rightarrow Ay-bt=0$

② $f_i(x) \le 0 \Leftrightarrow f_i(\frac{y}{t}) \le 0 \Leftrightarrow t f_i(\frac{y}{t}) \le 0$

Define $g_i(y,t)$: perspective of f_i

$$g_i(y,t) = t f_i(\tfrac{y}{t})$$

Rmk: If f_i is convex, so is g_i.

$\Leftrightarrow$ $g_i(y,t) \le 0$. $\forall i$

③ Objective: $\dfrac{f_0(x)}{c^T x + d} = t f_0(\frac{y}{t}) = g_0(y,t)$.

becomes a convex f^n

Hence the opt prob. is convex.

4.9: (LP): $\min_x c^T x$ s.t. $Ax \le b$.

A: square, non-singular.

Make the substitution $y = Ax$, i.e, $x = A^{-1} y$

(LP'): $\min_{y \in \mathbb{R}^n} c^T (A^{-1} y)$ s.t. $y \le b$.

$\min_{y \in \mathbb{R}^n} (A^{-T} c)^T y$ s.t. $y \le b$.

Suppose, $A^{-T} c \not\le 0$, i.e, $\exists\, i \in [n]$ s.t. $(A^{-T} c)_i > 0$.

Then take y to be $(0, 0, 0, \cdots, 0, \underset{\uparrow}{L}, 0, \cdots, 0)$

$$(A^{-T}c)^T y = L\,\underbrace{(A^{-T}c)_i}_{>0}$$ | ith component

Now take $L \to -\infty$. Then the objective is unbounded below. If $(A^{-T}c) \not\leq 0$, then the $p^* = -\infty$.

So $A^{-T}c \leq 0$. Under this condition, $\min (A^{-T}c)^T y$ s.t. $y \leq b$. The optimum/min is achieved at $y^* = b$.

$$p^* = \begin{cases} c^T A^{-1} b & , A^{-T}c \leq 0 \\ -\infty & , \text{else} \end{cases}.$$

BV 4.11

a) $\min_x \left(\|Ax - b\|_\infty = \max_{1 \leq i \leq m} |(Ax-b)_i| \right)$

$\min c^T x$ s.t. $Ax = b$. $x \geq 0$ or $Gx \leq f$

Epigraph form: Use $t \geq 0$ to upper bound $\max_{1\leq i \leq m} |(Ax-b)_i|$

$$\min_{t \geq 0, x \in \mathbb{R}^n} t \quad \text{s.t.} \quad \max_{1 \leq i \leq m} |(Ax-b)_i| \leq t.$$

$b \in \mathbb{R}^m$

$$|(Ax-b)_i| \leq t. \quad \forall i \in [m].$$

$$-t \leq (Ax-b)_i \leq t \quad \forall i \in [m].$$

$$Ax - b \leq t \cdot \mathbb{1}$$
$$-Ax + b \leq -t \mathbb{1}$$

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b) $\min_{x \in \mathbb{R}^n} \|Ax - b\|_1 = \sum_{i=1}^{m} |a_i^T x - b_i|$, $b \in \mathbb{R}^m$

Introduce another variable $s \in \mathbb{R}^m$

$$\|y\|_1 = \sum_{i=1}^{m} |y_i|$$

$$\min_{x \in \mathbb{R}^n, s \in \mathbb{R}^m} \sum_{i=1}^{m} s_i$$

$$\boxed{\begin{aligned} Ax - b &\leq s \\ Ax - b &\geq -s. \end{aligned}}$$

Assuming x is fixed, then

The constraints say that $\boxed{-s_i \leq a_i^T x - b_i \leq s_i}$ $i \in [m]$

$$\Updownarrow$$

$$|a_i^T x - b_i| \leq s_i$$
