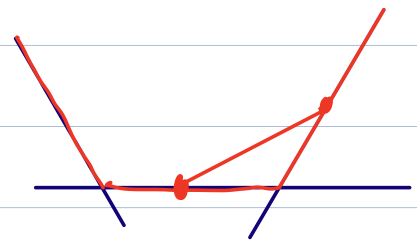
DSA3102 Tutorial 5.

BV 3.21

1.a) $f(x) = \max_{i=1,\dots,k} \|A^{(i)}x - y^{(i)}\| \quad x \in \mathbb{R}^n$

$A^{(i)} \in \mathbb{R}^{m\times n}, \; b^{(i)} \in \mathbb{R}^m$

$f_i(x) = \|A^{(i)}x - y^{(i)}\|$: convex $\because$ composition of affine function with norm.



$f(x) = \max_i f_i(x)$

$\hookrightarrow$ convex

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b) $f(x) = \sum_{i=1}^{r} |x|_{[i]}$

$x \longrightarrow |x| = (|x_1|, |x_2|, \dots, |x_n|)$

$|x_{[1]}| \geq |x_{[2]}| \geq \cdots |x_{[n]}|$.

Rewrite: $f(x) = \max_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} |x_{i_1}| + \cdots + |x_{i_n}|$

Each $|x_{i_1}| + \cdots + |x_{i_r}|$ for fixed $i_1, i_2, \dots, i_r$ is a convex function of x.

$f(x)$: maximum over a bunch of $\binom{n}{r}$ cvx f^{n}.

$\Rightarrow f$ is convex

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2. BV 3.31

a) f : convex $\qquad g(x) = \inf_{\alpha > 0} \dfrac{f(\alpha x)}{\alpha}$

$t > 0$

$$g(tx) = \inf_{\alpha > 0} \frac{f(\alpha t x)}{\alpha} = t \inf_{\alpha > 0} \frac{f(\alpha t x)}{t\alpha}$$

$$= t \inf_{\beta > 0} \frac{f(\beta x)}{\beta} = t\, g(x) \quad //.$$

b) Let h be a homogeneous underestimator of f

$$h(x) = \frac{h(\alpha x)}{\alpha} \le \frac{f(\alpha x)}{\alpha} \qquad \alpha > 0.$$

homogeneous $\quad$ underestimator

$$h(x) \le \frac{f(\alpha x)}{\alpha} \quad \forall \alpha > 0$$

$$h(x) \le \inf_{\alpha > 0} \frac{f(\alpha x)}{\alpha} = g(x)$$

$\Rightarrow g(x)$ is the largest homo. underestimator.

c) $$g(x) = \inf_{\alpha > 0} \frac{f(\alpha x)}{\alpha} = \inf_{t > 0} t f\left(\frac{x}{t}\right).$$

$\alpha = 1/t$

$$h(x,t) = t f\left(\frac{x}{t}\right).$$

If $f(x)$ is convex, then $h(x,t)$ is convex in (x,t).
g is the partial minimization of $h(x,t)$ over t belonging to the convex set $(0,\infty)$.
$\Rightarrow$ g is convex.

BV 3.36.

a) $f(x) = \max_i x_i \qquad x \in \mathbb{R}^n$

$$f^*(y) = \sup_{x\in\mathbb{R}^n} \{ \underbrace{x^T y - \max_i x_i}_{g(x)} \}$$

i) Say y has a negative component say $y_k < 0$.
$x_k = -t \quad (t>0) \qquad x_j = 0 \quad \forall j \neq k.$
$g(x) = -t y_k \to \infty$

ii) Say $\sum_{i=1}^n y_i \neq 1 \quad (\mathbb{1}^T y \neq 1)$ but $y \geq 0$.

Choose $x = t\mathbb{1}_n$. Then $g(x) = t\sum_{i=1}^n y_i - t = t(\sum_{i=1}^n y_i - 1)$
a) $\sum y_i > 1$, $t \to +\infty \Rightarrow g(x) \to \infty$.
$\sum y_i < 1$, $t \to -\infty \Rightarrow g(x) \to \infty$

$\text{dom } f^* = \{ y \mid y \geq 0, \sum_{i=1}^n y_i = 1 \}$. prob. simplex.

Suppose $y \in \text{dom } f^*$

$$g(x) = x^T y - \max_i x_i \le 0.$$

$$x^T y \le \max_i x_i$$

$$\sum x_i y_i \le \left(\max_i x_i\right) \sum y_i = \max_i x_i$$

$g(x) \le 0 \quad \forall x \in \mathbb{R}^n$ & equality is attained when $x = 0$.

$$f^*(y) = \begin{cases} 0 & y \in \{y : y \ge 0, \sum_{i=1}^n y_i = 1\} \\ \infty & \text{else} \end{cases}$$

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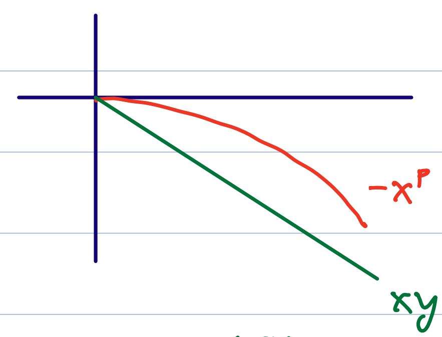
d) $f(x) = x^p$ on $\mathbb{R}_{++}$ $\quad \underline{p > 1.}$ $\quad x \mapsto x^p$ is strictly convex on $\mathbb{R}_{++}$

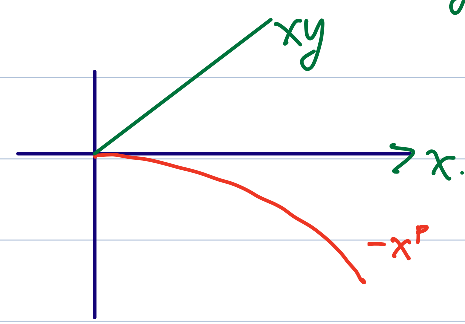
q satisfy $\frac{1}{p} + \frac{1}{q} = 1.$

$$f^*(y) = \sup_{x>0} \{\underbrace{xy - x^p}_{g(x)}\}$$



i) $y < 0$: $\quad \sup_{x>0} g(x) = 0$

$\quad\quad = f^*(y)$



ii) $y > 0$:

$$g'(x) = y - p x^{p-1} = 0$$

$$x^* = \left(\frac{y}{p}\right)^{1/(p-1)}$$

$$f^*(y) = g(x^*) = \left(\frac{y}{p}\right)^{\frac{1}{p-1}} \cdot y - \left(\left(\frac{y}{p}\right)^{\frac{1}{p-1}}\right)^p$$

$$= (p-1)\left(\frac{y}{p}\right)^q$$

iii) $y=0$ $\quad f^*(y)=0.$

$$f^*(y)=\begin{cases} 0 & y\leq 0 \\ (p-1)\left(\frac{y}{p}\right)^q & y>0 \end{cases}$$

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4. (BV 3.37) $\quad f(X)=\mathrm{tr}(X^{-1}), \quad X\in S_{++}^n$

$$f^*(Y)=\sup_{X\in S_{++}^n}\{\underbrace{\mathrm{tr}(XY)-\mathrm{tr}(X^{-1})}_{x^Ty-f(x)}\}$$

$$Y=\sum_{i=1}^n \lambda_i v_iv_i^T, \quad \lambda_i: \text{real} \qquad v_i: \text{orthonormal}$$

Suppose Y is not negative semidefinite. Say $\lambda_1>0$.

$$X=\sum_{i=1}^n \mu_i v_iv_i^T, \qquad \mu_1=t, \quad \mu_2=\cdots=\mu_n=1$$

$$f^*(Y)=\sup_{X\succ 0} \overbrace{\mathrm{tr}(XY)-\mathrm{tr}(X^{-1})}^{g(X)}$$

$$\geq \mathrm{tr}\left(\left(\sum_{i=1}^n \mu_i v_iv_i^T\right)\left(\sum_{i=1}^n \lambda_i v_iv_i^T\right)\right)-\mathrm{tr}(X^{-1})$$

$$=\mu_1\lambda_1+\mu_2\lambda_2+\cdots+\mu_n\lambda_n-\left(\frac{1}{t}+(n-1)\right)$$

$$= t\lambda_1+ - - - - .$$

If we let $t\to+\infty$, $g(X)\to+\infty$

$\mathrm{dom}\, f^* = -S^n_+$; set of negative semidefinite matrices.

$Y \in \mathrm{dom}\, f^*$

$Y \prec 0 \qquad g(X) = \mathrm{tr}(XY) - \mathrm{tr}(X^{-1})$

$\nabla g(X) = Y + X^{-2} = 0 \Rightarrow X^* = (-Y)^{-1/2}$

$f^*(Y) = \mathrm{tr}((-Y)^{-1/2}\, Y) - \mathrm{tr}((-Y)^{1/2})$

$= -2\,\mathrm{tr}((-Y)^{1/2})$