5.3 *Problems with one inequality constraint.* Express the dual problem of

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & f(x) \le 0, \end{array}$$

$\leftarrow$ (λ).

with $c \neq 0$, in terms of the conjugate f^*. Explain why the problem you give is convex. We do not assume f is convex.

Q1 Q2 Q5.

$$f^*(y) = \sup_x (y^T x - f(x))$$

Lagrangian $L(x, \lambda) = c^T x + \lambda f(x)$.

dual func $g(\lambda) = \inf_x L(x, \lambda) = \inf_x (c^T x + \lambda f(x))$

$$= - \sup_x (-c^T x - \lambda f(x))$$

$$= -\lambda \sup_x \left(-\frac{c^T}{\lambda} x - f(x)\right)$$

$$= -\lambda f^*(-c/\lambda).$$

dual prob is $\max_\lambda \; -\lambda f^*(-c/\lambda)$ s.t. $\lambda \geq 0$.

$\Leftrightarrow$ $\min_\lambda \; \lambda f^*(-c/\lambda)$ s.t. $\lambda \geq 0$

$f^*(y)$ is a conjugate func. $\therefore$ convex in y.

$\triangleright$ the perspective function of $f^*(y)$ is $h(y, t) = t f^*(y/t)$.

h is convex in y and t

fix $y = -c$. so $\lambda f^*(-c/\lambda) = h(\lambda)$ is convex in λ.

$\therefore$ the dual prob is convex.

5.5 *Dual of general LP.* Find the dual function of the LP

$G \in \mathbb{R}^{m\times n}$

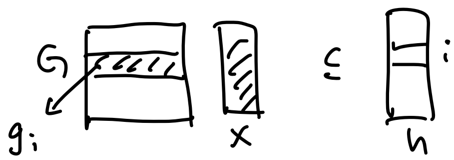
$A \in \mathbb{R}^{p\times n}$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b. \end{array}$$

$g_i^T x \le h_i$ for $i = 1, 2, \cdots, m$.

$\lambda \in \mathbb{R}^m$

$\nu \in \mathbb{R}^p$



Give the dual problem, and make the implicit equality constraints explicit.

Lagrangian: $L(x, \lambda, \nu) = c^T x + \lambda^T(Gx - h) + \nu^T(Ax - b)$

dual func: $g(\lambda, \nu) = \inf_{x\in\mathbb{R}^n} \{ c^T x + \lambda^T(Gx-h) + \nu^T(Ax-b) \}$.

$= -\lambda^T h - \nu^T b + \inf_{x\in\mathbb{R}^n} \{ c^T x + \lambda^T G x + \nu^T A x \}$.

$= -\lambda^T h - \nu^T b$

vector constant u.

$+ \inf_{x\in\mathbb{R}^n} \{ (c^T + \lambda^T G + \nu^T A) x \}$

$$\inf_x u^T x = \begin{cases} 0 & \text{if } u = \vec{0} \\ -\infty & \text{if } u \neq \vec{0} \end{cases} \quad (x = -tu, \quad u^T x = -t\|u\|_2^2)$$

$$g(\lambda, \nu) = \begin{cases} -\lambda^T h - \nu^T b & \text{if } c + G^T\lambda + A^T\nu = 0 \\ -\infty & \text{if } c + G^T\lambda + A^T\nu \neq 0 \end{cases}$$

dual problem: $\max_{\lambda\in\mathbb{R}^m, \nu\in\mathbb{R}^p} g(\lambda, \nu)$. s.t. $\lambda \succeq 0$.

since when $c + G^T\lambda + A^T\nu \neq 0$, g is $-\infty$, can't be max.

$\therefore$ the dual problem is equivalent to.

$$\max_{\lambda\in\mathbb{R}^m, \nu\in\mathbb{R}^p} -\lambda^T h - \nu^T b \quad \text{s.t. } \lambda \succeq 0. \quad c + G^T\lambda + A^T\nu = 0.$$

$$\Leftrightarrow \min_{\lambda\in\mathbb{R}^m, \nu\in\mathbb{R}^p} \lambda^T h + \nu^T b \quad \text{s.t. } \lambda \succeq 0, \quad c + G^T\lambda + A^T\nu = 0.$$

Exercise: write the dual prob of the dual.

5. The relative entropy between two vectors $x, y \in \mathbb{R}^n_{++}$ is defined as

$$\sum_{k=1}^{n} x_k \log\left(\frac{x_k}{y_k}\right).$$

This is a convex function, jointly in x and y. In the following problem we calculate the vector x that minimizes the relative entropy with a given vector y, subject to equality constraints on x:

$$\min_{x \in \mathbb{R}^n} \sum_{k=1}^{n} x_k \log\left(\frac{x_k}{y_k}\right), \quad \text{s.t. } \overset{z \in \mathbb{R}^m}{Ax = b}, \underline{\mathbf{1}^T x = 1} \quad v \in \mathbb{R}.$$

The domain is $\mathbb{R}^n_{++}$ and the given parameters are $y \in \mathbb{R}^n_{++}$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Note that $\mathbf{1}^T x = 1$ means that x is a probability vector.

$$(A^T z)^T x$$
$$= \sum (A^T z)_k x_k.$$
$$= \sum (a_k^T z) \cdot x_k$$

Derive the Lagrange dual of this problem and simplify it to get

$$\max_{z \in \mathbb{R}^m} b^T z - \log \sum_{k=1}^{n} y_k e^{a_k^T z}$$

where a_k is the k-th column of A. Note that $z \in \mathbb{R}^m$ is the Lagrange multiplier associated with the constraint $Ax = b$.

$$v(x_1 + \cdots + x_n).$$

Lagrangian : $$L(x, v, z) = \sum_{k=1}^{n} (x_k \log x_k - x_k \log y_k) - v(\mathbf{1}^T x - 1) - z^T(Ax - b)$$

$$= \sum_{k=1}^{n} \left(x_k \log x_k - x_k \log y_k - v x_k - (A^T z)_k \cdot x_k \right) + v + z^T b.$$

dual func $$g(v, z) = \inf_x L(x, v, z).$$

find $\inf_x$ $$x_k \log x_k - x_k \log x_k - v x_k - \underbrace{(A^T z)_k}_{a_k^T \cdot z} \cdot x_k.$$

derivative : $$\log x_k + 1 - \log y_k - v - a_k^T z = 0.$$

min will be taken when derivative is zero.

$$\log x_k = \log y_k + v - 1 + a_k^T z$$

$$x_k^* = y_k \exp(a_k^T z + v - 1)$$

$$g(v, z) = L(x^*, v, z)$$

$$= \sum_{k=1}^{n} \Big[y_k \exp(a_k^T z) \exp(v-1) (a_k^T z + v - 1)$$
$$- v y_k \exp(a_k^T z + v - 1)$$
$$- a_k^T z \cdot y_k \exp(a_k^T z + v - 1) \Big] + v + z^T b.$$

$$= -\sum_{k=1}^{n} \left[y_k \exp(a_k^T z) \exp(v-1) \right] + v + z^T b.$$

$$= -\exp(v-1) \sum_{k=1}^{n} \left[y_k \exp(a_k^T z) \right] + v + z^T b$$

dual prob: $\max\limits_{v\in\mathbb{R},\, z\in\mathbb{R}^m} \; -\exp(v-1)\sum_{k=1}^{n}\left[y_k \exp(a_k^T z)\right] + v + z^T b$

fix z. find optimal v^*. let $= T$. (constant when fixing z)

derivative of $-\exp(v-1)T + v + z^T b$. w.r.t. v.

$$-T\cdot \exp(v-1) + 1 = 0.$$

$$\iff \exp(v-1) = \frac{-1}{-T} = \frac{1}{T}$$

$$\iff v^* = -\log T + 1.$$

dual prob: $\max\limits_{z\in\mathbb{R}^m} \; \boxed{-\exp(v^*-1)} \sum_{k=1}^{n} y_k \exp(a_k^T z) + v^* + z^T b.$

$v^* = -\log T + 1.$

$$\begin{cases} -\exp(-\log T) = -\frac{1}{T} \\ v^* = -\log T + 1 \end{cases}$$

dual prob: $\max\limits_{z\in\mathbb{R}^m} \; -\frac{1}{\sum_{k=1}^{n} y_k \exp(a_k^T z)} \cdot \sum_{k=1}^{n} y_k \exp(a_k^T z) - \log\left(\sum_{k=1}^{n} y_k \exp(a_k^T z)\right) + 1 + z^T b$

$$\iff \max_{z\in\mathbb{R}^m} \; \cancel{-1} - \log\left(\sum_{k=1}^{n} y_k \exp(a_k^T z)\right) + z^T b \cancel{+1}$$