DSA3102: Solutions to Tutorial Set 6
Assigned: 15/09/23

1. BV Problem 4.1
   Solutions: The feasible set is the convex hull of $(0, \infty)$, $(0, 1)$, $(2/5, 1/5)$, $(1, 0)$ and $(-\infty, 0)$.
   (a) $x^* = (2/5, 1/5)$
   (b) Unbounded below
   (c) $X_{\text{opt}} = \{(0, x_2) : x_2 \geq 1\}$
   (d) $x^* = (1/3, 1/3)$
   (e) $x^* = (1/2, 1/6)$. This is optimal because it satisfies $2x_1 + x_2 = 7/6 > 1$, $x_1 + 3x_2 = 1$, and
   $$\nabla f_0(x^*) = (1, 3)$$
   is perpendicular to the line $x_1 + 3x_2 = 1$.

2. BV Problem 4.7
   Solutions:
   (a) The domain of the objective is convex because $f_0(x)$ is convex. The sublevel sets are convex because $f_0(x)/(c^T x + d) \leq \alpha$ iff $c^T x + d > 0$ and $f_0(x) \leq \alpha(c^T x + d)$.
   (b) Suppose x is feasible in the original problem. Define $t = 1/(c^T x + d)$ (a positive number), $y = x/(c^T x + d)$. Then $t > 0$ and it is easily verified that t, y are feasible in the transformed problem, with the objective value $g_0(y, t) = f_0(x)/(c^T x + d)$.
   Conversely, suppose y, t are feasible for the transformed problem. We must have $t > 0$, by definition of the domain of the perspective function. Define $x = y/t$. We have $x \in \mathbf{dom}\, f_i$ for $i = 0, 1, \ldots, m$ (again, by definition of perspective). x is feasible in the original problem, because
   $$f_i(x) = g_i(y, t)/t \leq 0, \quad i = 1, \ldots, m, \qquad Ax = A(y/t) = b.$$
   From the last equality, $c^T x + d = (c^T y + dt)/t = 1/t$, and hence,
   $$t = \frac{1}{c^T x + d}, \qquad \frac{f_0(x)}{c^T x + d} = t f_0(x) = g_0(y, t).$$
   Therefore x is feasible in the original problem, with the objective value $g_0(y, t)$. In conclusion, from any feasible point of one problem we can derive a feasible point of the other problem, with the same objective value.
   (c) We first verify that the problem is quasiconvex. The domain of the objective function is convex, and its sublevel sets are convex because for $\alpha \geq 0$, $f_0(x)/h(x) \leq \alpha$ iff $f_0(x) - \alpha h(x) \leq 0$, which is a convex inequality. For $\alpha < 0$, the sublevel sets are empty.

The convex formulation is

$$\begin{array}{ll}\min\limits_{y,t} & g_0(y,t) \\ \text{s.t} & g_i(y,t) \le 0, \quad i=1,\ldots,m \\ & Ay = bt, \tilde{h}(y,t) \le -1\end{array}$$

where g_i is the perspective of f_i and $\tilde{h}$ is the perspective of $-h$. To verify the equivalence, assume first that x is feasible in the original problem. Define $t = 1/h(x)$ and $y = x/h(x)$. Then $t > 0$ and

$$g_i(y,t) = tf_i(y/t) = tf_i(x) \le 0 \qquad Ay = Ax/h(x) = bt.$$

Moreover $\tilde{h}(y,t) = th(y/t) = h(x)/h(x) = 1$ and

$$g_0(y,t) = tf_0(y/t) = f_0(x)/h(x)$$

We see that for every feasible point in the original problem we can find a feasible point in the transformed problem, with the same objective value.
Conversely, assume y, t are feasible in the transformed problem. By definition of perspective, $t > 0$. Define $x = y/t$. We have

$$f_i(x) = f_i(y/t) = g_i(y,t)/t \le 0 \qquad Ax = A(y/t) = b.$$

From the last inequality, we have

$$\tilde{h}(y,t) = -th(y/t) = -th(x) \le -1.$$

This implies that $h(x) > 0$ and $th(x) \ge 1$. And finally, the objective is

$$f_0(x)/h(x) = g_0(t,t)/(th(x)) \le g_0(y,t)$$

We conclude that with every feasible point in the transformed problem there is a corresponding feasible point in the original problem with the same or lower objective value.
Putting the two parts together, we can conclude that the two problems have the same optimal value, and that optimal solutions for one problem are optimal for the other (if both are solvable).
For the example, we have the equivalent problem

$$\min \frac{1}{m}\mathbf{tr}(tF_0 + y_1F_1 + \ldots, +y_nF_n) \quad \text{s.t.} \quad \det(tF_0 + y_1F_1 + \ldots + y_nF_n)^{1/m} \ge 1$$

with domain

$$\{(y,t) : t > 0, tF_0 + y_1F_1 + \ldots + y_nF_n \succ 0\}$$

3. BV Problem 4.8 (a)–(c)
To help you, the optimal value for (a) is

$$p^* = \begin{cases} +\infty & b \notin \mathcal{R}(A) \\ \lambda^T b & c = A^T\lambda \textit{ for some } \lambda \\ -\infty & \lambda \textit{ otherwise}\end{cases}.$$

For part (b),

$$p^* = \begin{cases} \lambda b & c = a\lambda \textit{ for some } \lambda \le 0 \\ -\infty & \lambda \textit{ otherwise}\end{cases}$$

For part (c),

$$p^* = l^Tc^+ + u^Tc^-$$

where $c_i^+ = \max\{c_i, 0\}$ *and similarly for* c_i^-*.*
Solutions:

(a) There are three possibilities

- The problem is infeasible $b \notin \mathcal{R}(A)$. The optimal value is ∞.
- The problem is feasible, and c is orthogonal to the nullspace of A. We can decompose c as

$$c = A^T\lambda + \hat{c}, \quad A\hat{c} = 0$$

  Here $\hat{c}$ is the component in the nullspace of A and $A^T\lambda$ is the component orthogonal to the nullspace. If $\hat{c} = 0$, then on the feasible set the objective function reduces to a constant:

$$c^Tx = \lambda^T Ax + \hat{c}^Tx = \lambda^T b.$$

  The optimal value is $\lambda^T b$. All feasible solutions are optimal.
- The problem is feasible, and c is not in the range of A^T and $\hat{c} \neq 0$. The problem is unbounded $p^* = -\infty$. To verify this, note that $x = x_0 - t\hat{c}$ feasible for all t; as t goes to infinity, the objective value decreases unboundedly.

So we obtain the first part.

(b) This problem is always feasible. The vector c can be decomposed into a component parallel to a and a component orthogonal to a:

$$c = a\lambda + \hat{c}.$$

with $a^T\hat{c} = 0$.

- If $\lambda > 0$, the problem is unbounded below. Choose $x = -ta$ and let t go to infinity:

$$c^Tx = -tc^Ta = -t\lambda\|a\|_2^2 \to -\infty.$$

  and

$$a^Tx - b = -ta^Ta - b \leq 0$$

  for large t, so x is feasible for large t. Intuitively, by going very far in the direction $-a$, we find feasible points with arbitrarily negative objective values.
- If $\hat{c} \neq 0$, the problem is unbounded below. Choose $x = ba - t\hat{c}$ and let $t \to \infty$.
- If $c = a\lambda$ for some $\lambda \leq 0$, the optimal value is $c^Tab = \lambda b$.

So we obtain the second part.

(c) The objective and the constraints are separable: The objective is a sum of terms c_ix_i, each dependent on one variable only; each constraint depends on only one variable. We can therefore solve the problem by minimizing over each component of x independently. The optimal x_i^* minimizes c_ix_i subject to the constraint $l_i \leq x_i \leq u_i$. If $c_i > 0$, then $x_i^* = l_i$; if $c_i < 0$, then $x_i^* = u_i$; if $c_i = 0$, then any x_i in the interval $[l_i, u_i]$ is optimal. Therefore, the optimal value of the problem is

$$p^* = l^Tc^+ + u^Tc^-$$

where $c_i^+ = \max\{c_i, 0\}$ and $c_i^- = \max\{-c_i, 0\}$.

4. BV Problem 4.9
Solution: Make a change of variables $y = Ax$. The problem is equivalent to

$$\min_y \; c^TA^{-1}y \quad \text{s.t.} \quad y \preceq b$$

If $A^{-T}c \preceq 0$, the optimal solution is $y = b$, with $p^* = c^TA^{-1}b$. Otherwise, the LP is unbounded below.

5. BV Problem 4.11
Solutions:

(a) Equivalent to LP

$$\min_{t,x} \; t \quad \text{s.t.} \quad Ax - b \preceq t\mathbf{1}, Ax - b \succeq -t\mathbf{1}$$

(b) Equivalent to LP

$$\min_{s,x} \; \mathbf{1}^T s \quad \text{s.t.} \quad Ax - b \preceq s, Ax - b \succeq -s$$

Assume x is fixed in this problem, and we optimize only over s. The constraints say that

$$-s_k \leq a_k^T x - b_k \leq s_k.$$

The objective function of the LP is separable, so we achieve the optimum over s by choosing

$$s_k = |a_k^T x - b_k|$$

and obtain the optimal value $p^*(x) = \|Ax - b\|_1$. Therefore optimizing over t and s simultaneously is equivalent to the original problem.

(c) Equivalent to the LP

$$\min_{x,y} \; \mathbf{1}^T y \quad \text{s.t.} \quad -y \preceq Ax - b \preceq y, -\mathbf{1} \preceq x \preceq \mathbf{1}$$

(d) Equivalent to the LP

$$\min_{x,y} \; \mathbf{1}^T y \quad \text{s.t.} \quad -y \preceq x \preceq y, -\mathbf{1} \preceq Ax - b \preceq \mathbf{1}$$

Another good solution is to write $x = x^+ - x^-$ and to express the problem as

$$\min_{x^+, x^-} \; \mathbf{1}^T x^+ + \mathbf{1}^T x^- \quad \text{s.t.} \quad -\mathbf{1} \preceq A(x^+ - x^-) - b \preceq \mathbf{1}, x^+, x^- \succeq 0.$$

(e) Equivalent to the LP

$$\min_{x,y,t} \; \mathbf{1}^T y + t \quad \text{s.t.} \quad -y \preceq Ax - b \preceq y, -\mathbf{1}t \preceq x \preceq t\mathbf{1}$$

6. BV Problem 4.12

Solution: This can be formulated as the LP

$$\min \; C = \sum_{i,j=1}^{n} c_{ij} x_{ij}$$

subject to

$$b + \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = 0, \qquad , i = 1, \ldots, n$$
$$l_{ij} \leq x_{ij} \leq u_{ij}.$$

7. BV Problem 4.21(a)

Solutions: If $A \succ 0$, the solution is

$$x^* = -\frac{1}{\sqrt{c^T A^{-1} c}} A^{-1} c, \qquad p^* = -\|A^{-1/2} c\|_2$$

This can be shown as follows. We make a change of variables $y = A^{1/2} x$, and write $\tilde{c} = A^{-1/2} c$. With this new variable the optimization problem becomes

$$\min_{y} \; \tilde{c}^T y, \quad \text{s.t.} \quad y^T y \leq 1$$

The answer is $y^* = -\tilde{c}/\|\tilde{c}\|_2$.

In the general case, we can make a change of variables based on the eigenvalue decomposition

$$A = Q\mathrm{diag}(\lambda)Q^T = \sum_{i=1}^n \lambda_i q_i q_i^T.$$

We define $y = Qx$, $b = Qc$, and express the problem as

$$\min \sum_{i=1}^n b_i y_i, \quad \text{s.t.} \quad \sum_{i=1}^n \lambda_i y_i^2 \leq 1$$

If $\lambda_i > 0$ for all i, the problem reduces to the case we already discussed. Otherwise, we can distinguish several cases.

- $\lambda_n < 0$: The problem is unbounded below. By letting $y_n \to \pm\infty$, we can make any point feasible.
- $\lambda_n = 0$: If for some i, $b_i \neq 0$ and $\lambda_i = 0$, the problem is unbounded below.
- $\lambda_n = 0$, and $b_i = 0$ for all i with $\lambda_i = 0$. In this case we can reduce the problem to a smaller one with all $\lambda_i > 0$.