DSA3102: Solutions to Tutorial Set 7
Assigned: 21/09/23

1. BV Problem 4.12
   Solution: This can be formulated as the LP

$$\min\ C = \sum_{i,j=1}^{n} c_{ij} x_{ij}$$

   subject to

$$b_i + \sum_{j=1}^{n} x_{ji} - \sum_{j=1}^{n} x_{ij} = 0, \qquad , i = 1, \ldots, n$$
$$l_{ij} \le x_{ij} \le u_{ij}.$$

2. BV Problem 4.23
   Solution: We can rewrite the the l_4 norm approximation problem as

$$\min_{x,y,z} \sum_{i=1}^{m} z_i^2$$

   subject to

$$a_i^T x - b_i = y_i, \quad y_i^2 \le z_i, \quad i = 1, \ldots, m.$$

   This is exactly a QCQP.

3. BV Problem 4.28
   Solutions:

   (a) The objective function is a maximum of convex function, hence convex. We can write the problem as

$$\min\ t \quad \text{s.t.} \quad \frac{1}{2} x^T P_i x + q^T x + r \le t, i = 1, \ldots, K, \quad Ax \preceq b$$

   which is a QCQP in the variables x and t.

   (b) For given x, the supremum of $x^T \Delta P x$ over $-\gamma I \preceq \Delta P \preceq \gamma I$ is given by

$$\sup_{-\gamma I \preceq \Delta P \preceq \gamma} x^T \Delta P x = \gamma x^T x$$

   Therefore we can express the robust QP as

$$\min\ \frac{1}{2} x^T (P_0 + \gamma I) x + q^T x + r, \quad \text{s.t.} \quad Ax \preceq b$$

   which is a QP.

(c) For given x, the quadratic objective function is

$$\frac{1}{2}\left(x^TP_0x + \sup_{\|u\|_2\le 1}\sum_{i=1}^K u_i(x^TP_ix)\right) + q^Tx + r = \frac{1}{2}x^TP_0x + \frac{1}{2}\left(\sum_{i=1}^K (x^TP_ix)^2\right)^{1/2} + q^Tx + r.$$

This is a convex function of x: each of the functions x^TP_ix is convex since $P_i \succeq 0$. The second term is a composition $h(g_1(x), \ldots, g_K(x))$ of $h(y) = \|y\|_2$ with $g_i(x) = x^TP_ix$. The functions g_i are convex and nonnegative. The function h is convex and, for $y \in \mathbb{R}^K_+$, nondecreasing in each of its arguments. Therefore the composition is convex.
The resulting problem can be expressed as

$$\min\ \frac{1}{2}x^TP_0x + \|y\|_2 + q^Tx + r$$

subject to

$$\frac{1}{2}x^TP_ix \le y_i, \quad i = 1, \ldots, K, \qquad Ax \preceq b$$

which can be further reduced to an SOCP

$$\min\ u + t + q^Tx$$

subject to

$$\left\|\begin{bmatrix} P_0^{1/2}x \\ 2u - 1/4 \end{bmatrix}\right\|_2 \le 2u + 1/4, \quad \left\|\begin{bmatrix} P_i^{1/2}x \\ 2y_i - 1/4 \end{bmatrix}\right\|_2 \le 2y_i + 1/4, \quad i = 1, \ldots, K, \quad \|y\|_2 \le t, \quad Ax \preceq b.$$

The variables are x, u, t, and $y \in \mathbb{R}^K$.
Note that if we square both sides of the first inequality, we obtain

$$x^TP_0x + (2u - 1/4)^2 \le (2u + 1/4)^2$$

i.e., $x^TP_0x \le 2u$. Similar, the other constraints are equivalent to $\frac{1}{2}x^TP_ix \le y_i$.

4. BV Problem 4.33(a)-(b)
Solutions:

(a) This is equivalent to the GP

$$\begin{aligned} \min_{t,x} \quad & t \\ \text{s.t.} \quad & p(x)/t \le 1, \quad q(x)/t \le 1 \end{aligned}$$

(b) This is equivalent to

$$\begin{aligned} \min_{t_1,t_2,x} \quad & \exp(t_1) + \exp(t_2) \\ \text{s.t.} \quad & p(x) \le t_1, \quad q(x) \le t_2 \end{aligned}$$

Now make the logarithmic change of variables $x_i = e^{y_i}$ (but not to t_i).

5. BV Problem 4.40(a)-(b)
Solution:

(a) The LP can be expressed as

$$\min_x\ c^Tx + d \quad \text{s.t.} \quad \mathbf{diag}(Gx - h) \preceq 0, Ax = b$$

(b) With $P = WW^T$ and $W \in \mathbb{R}^{n\times r}$, the QP can be expressed as

$$\min_{x\in\mathbb{R}^n, t\in\mathbb{R}} \; t + 2q^T x + r \quad \text{s.t.} \quad \begin{bmatrix} I & W^T x \\ x^T W & tI \end{bmatrix} \succeq 0, \mathbf{diag}(Gx - h) \preceq 0, Ax = b$$

(c) With $P_i = W_i W_i^T$ and $W_i \in \mathbb{R}^{n\times r_i}$, the QCQP can be expressed as

$$\min_{x\in\mathbb{R}^n, t_i\in\mathbb{R}, i\in[m]} \; t_0 + 2q_0^T x + r_0$$

subject to

$$t_i + 2q_i^T x + r_i \leq 0, i \in [m], \quad \begin{bmatrix} I & W_i^T x \\ x^T W_i & t_i I \end{bmatrix} \succeq 0, i \in [m], \quad Ax = b.$$

(d) The SOCP can be expressed as

$$\min_x \; c^T x \quad \text{s.t.} \quad \begin{bmatrix} (c_i^T x + d_i)I & A_i x + b_i \\ (Ax_i + b_i)^T & (c_i^T x + d_i)I \end{bmatrix} \succeq 0, i \in [N], Fx = g$$

By the result in the hint, the constraint is equivalent with $\|A_i x + b_i\|_2 < c_i^T x + d_i$ when $c_i^T x + d_i > 0$. We have to check the case $c_i^T x + d_i = 0$ separately. In this case, the LMI constraint means $A_i x + b_i = 0$, so we can conclude that the LMI constraint and the SOC constraint are equivalent.

6. BV Problem 4.43(a)-(c)
Solution:

(a) We use the property that $\lambda_1(x) \leq t$ if and only if $A(x) \preceq tI$. We minimize the maximum eigenvalue by solving the SDP

$$\min_{t,x} \; t \quad \text{s.t.} \quad A(x) \preceq tI$$

(b) $\lambda_1(x) \leq t_1$ if and only if $A(x) \preceq t_1 I$ and $\lambda_m(A(x)) \geq t_2$ if and only if $A(x) \succeq t_2 I$ so we can minimize $\lambda_1 - \lambda_m$ by solving

$$\min_{t_1, t_2, x} \; t_1 - t_2 \quad \text{s.t.} \quad t_2 I \preceq A(x) \preceq t_1 I$$

(c) We first note that the problem is equivalent to

$$\min \; \lambda/\gamma \quad \text{s.t.} \quad \gamma I \preceq A(x) \preceq \lambda I \tag{1}$$

if we take as domain of the objective $\{(\lambda, \gamma) : \gamma > 0\}$. This problem is quasiconvex, and can be solved by bisection: The optimal value is less than or equal to α if and only if the inequalities

$$\lambda \leq \gamma\alpha, \quad \gamma I \preceq A(x) \preceq \lambda I, \quad \gamma > 0$$

(with variables γ, λ, x) are feasible.
Following the hint we can also pose the problem as the SDP

$$\min \; t \quad \text{s.t.} \quad I \preceq sA_0 + y_1 A_1 + \ldots + y_n A_n \preceq tI, s \geq 0 \tag{2}$$

We now verify more carefully that the two problems are equivalent. Let p^* be the optimal value of (1), and p^*_{sdp} is the optimal value of the SDP (2).
Let λ/γ be the objective value of (1), evaluated at a feasible point (γ, λ, x). Define $s = 1/\gamma, y = x/\gamma, t = \lambda/\gamma$. This yields a feasible point in (2), with objective value $t = \lambda/\gamma$. This proves that $p^* \geq p^*_{\text{sdp}}$.

Now suppose that s, y, t are feasible in (2). If $s > 0$, then $\gamma = 1/s, x = y/s, \lambda = t/s$ are feasible in (1) with objective value t. If $s = 0$, we have

$$I \preceq y_1 A_1 + \ldots + y_n A_n \preceq tI$$

Choose $x = \tau y$ with τ sufficiently large so that $A(\tau y) \succeq A_0 + \tau I \succ 0$. We have

$$\lambda_1(\tau y) \leq \lambda_1(0) + \tau t, \quad \lambda_m(\tau y) \geq \lambda_m(0) + \tau.$$

The first inequality is justified as follows:

$$\begin{aligned} A(\tau y) &= A_0 + \tau(y_1 A_1 + \ldots + y_n A_n) \\ &\preceq A_0 + \tau t I \\ &\preceq \lambda_1(0) I + \tau t I \\ &= (\lambda_1(0) + \tau t) I \end{aligned}$$

By using the fact that $\lambda_1(x) \leq t_1$ if and only if $A(x) \leq t_1 I$, we recover the first inequality. Hence, for τ sufficiently large

$$\kappa(x_0 + \tau y) \leq \frac{\lambda_1(0) + \tau t}{\lambda_m(0) + \tau}$$

Letting τ go to infinity, we can construct feasible points in (1), with objective value arbitrarily close to t. We conclude that $t \geq p^*$ if (s, y, t) are feasible in (2). Minimizing over t yields $p^*_{\mathrm{sdp}} \geq p^*$.