DSA3102: Homework 3
Assigned: 05/10/23, Due: 26/10/2023

Instructions:

- Please do all the five problems below. A (non-empty) subset of them will be graded.
- Write (legibly) on pieces of paper or Latex your answers. Convert your submission to PDF format.
- Submit to the Assignment 3 folder under Assignments in Canvas before 23:59 on 26/10/2023.

1. (Convexity and concavity of optimal value of an LP)

   Consider the linear programming problem

   $$p^* = \min_{x \in \mathbb{R}^n} c^T x \quad \text{subject to} \quad Ax \leq b,$$

   where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^M$. Prove the following statements, or provide a counterexample.

   (a) The optimal value of the problem p^*, as a function of c, is concave in c.

   (b) The optimal value of the problem p^*, as a function of b, is convex in b (you may assume that the problem is feasible).

   (c) The optimal value of the problem p^*, as a function of A, is concave in A.

2. (Quadratic Programming Duality)

   Consider the following quadratic program with H being a positive definite symmetric matrix and x being the decision variable.

   $$\min_x \ \frac{1}{2} x^T H x \quad \text{s.t.} \quad Ax \geq b.$$

   Show that the dual problem is also a quadratic optimization problem that looks like

   $$\max_{\lambda, z} \lambda^T f(b, A, H) - \frac{1}{2} z^T g(b, A, H) z \quad \text{s.t.} \quad \lambda \geq 0, \quad h(b, A, H)^T \lambda = z.$$

   Find the functions $f(b, A, H)$, $g(b, A, H)$ and $h(b, A, H)$.

3. (Duality Gap Problem)

   Consider the two-dimensional problem

   $$\min \quad e^{x_2} \quad \text{subject to} \quad \|x\|_2 - x_1 \leq 0$$

   where the domain of the optimization variable $x = (x_1, x_2)$ is $\mathbb{R}^2$.

   (a) Is this a convex program?

   (b) Calculate the primal optimal value p^*

(c) Calculate the dual optimal value d^*. What is the duality gap?

(d) Can you find a Slater vector? Explain intuitively why there is or isn't a duality gap.

4. (Duality Gap?)

Consider the two-dimensional problem

$$\min_{x\in\mathbb{R}^2} f(x) \quad \text{subject to} \quad x_1 \leq 0$$

where

$$f(x) = e^{-\sqrt{x_1 x_2}} \quad \text{for all} \quad x \in \mathbf{dom}\, f = \mathbb{R}^2_+$$

(a) Is this a convex optimization problem?

(b) Evaluate the primal optimal value p^*.

(c) Form the Lagrangian $L(x, \lambda)$, dual function $g(\lambda)$, and find the dual optimal value $d^* = \sup_{\lambda \geq 0} g(\lambda)$.

(d) Is there a duality gap and why?

5. (Simple Convex Optimization)

Consider the problem

$$\min_{x\in\mathbb{R}} x^2 + 1 \quad \text{subject to} \quad (x-2)(x-4) \leq 0.$$

The decision variable is $x \in \mathbb{R}$.

(a) State the feasible set, the optimal value p^*, and the optimal solution.

(b) Plot the objective $x^2 + 1$ versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ versus x for a few positive values of λ. Verify the lower bound property $p^* \geq \inf_{x\in\mathbb{R}} L(x, \lambda)$ for $\lambda \geq 0$. Derive and sketch the Lagrange dual function $g(\lambda)$.

(c) State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution d^*. Does strong duality hold?