30. Aug. Tutorial 0. Xiaochen. xczhu@nus.edu.sg. Q1. 2. 3. 6.

Q1. Is the set $A=\{a\in\mathbb{R}^k : p(0)=1,\ |p(t)|\le 1.\ \forall t\in[\alpha,\beta]\}$

where $p(t)=\underline{a_1}+a_2t+\cdots+a_kt^{k-1}$ convex?

By definition. Assume $a.\ b\in A$. $l\in[0,1]$. $c=la+(1-l)b\in A$?

$P_a(0)=1.\quad P_a(t)=a_1+\underline{a_2t+\cdots+a_kt^{k-1}} \Rightarrow a_1=1$

$P_b(0)=1.\quad P_b(t)=b_1+\underline{b_2t+\cdots+b_kt^{k-1}}. \Rightarrow b_1=1$

$\forall t\in[\alpha.\beta].\ |P_a(t)|\le 1.\ |P_b(t)|\le 1.$

1°. $P_c(0)=1.\quad P_c(0)=c_1=la_1+(1-l)b_1=l+1-l=1$

2°. $|P_c(t)|\le 1$ for all $t\in[\alpha.\beta]$. $c=la+(1-l)b.$

$$P_c(t)=c_1+c_2t+c_3t^2+\cdots+c_kt^{k-1}$$
$$=(la_1+(1-l)b_1)+(la_2+(1-l)b_2)t+\cdots+(la_k+(1-l)b_k)t^{k-1}$$
$$=la_1+\underline{l}a_2t+\underline{l}a_3t^2+\cdots+\underline{l}a_kt^{k-1}$$
$$+(1-l)b_1+(1-l)b_2t+(1-l)b_3t^2+\cdots+(1-l)b_kt^{k-1}$$
$$=l\cdot P_a(t)+(1-l)\cdot P_b(t)$$

$$|P_c(t)|=|l\cdot P_a(t)+(1-l)\cdot P_b(t)|$$
$$\le|l\cdot P_a(t)|+|(1-l)P_b(t)|=l\,\underbrace{|P_a(t)|}_{\le 1}+(1-l)\underbrace{|P_b(t)|}_{\le 1}$$
$$\le l+(1-l)=1.\quad \text{for } t\in[\alpha.\beta].$$

$\therefore\ c\in A. \Rightarrow A$ is convex

Q2. Prove that $A = \{(x_1, x_2, x_3) \in R^3 : \begin{bmatrix} x_1 + x_2 & x_1 - 2x_3 \\ x_1 - 2x_3 & x_2 + 3x_3 \end{bmatrix} \succeq 0\}$.

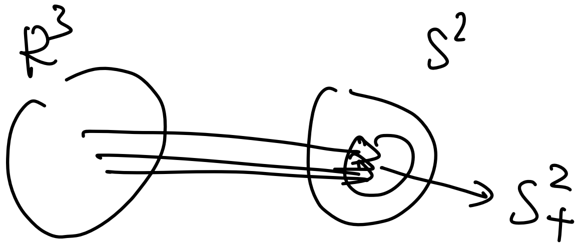
positive semi definite

Lemma: $f: V \to W$. a linear map of

$\forall x, y \in V, \quad c \in R. \quad \begin{cases} f(x+y) = f(x) + f(y) \\ \cancel{c} f(cx) = c f(x). \end{cases}$

if $S \subseteq W$. S is convex. $f^{-1}(S)$ is convex.

$f: R^3 \to S^2 : (x_1, x_2, x_3) \mapsto \begin{bmatrix} x_1 + x_2 & x_1 - 2x_3 \\ x_1 - 2x_3 & x_2 + 3x_3 \end{bmatrix}$ linear map



$A = f^{-1}(\underline{S^2_+})$

convex.

A is convex

Q3.

2.12 Which of the following sets are convex?

(a) A *slab*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha \le a^T x \le \beta\}$.

(b) A *rectangle*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha_i \le x_i \le \beta_i,\ i = 1, \ldots, n\}$. A rectangle is sometimes called a *hyperrectangle* when $n > 2$.

(c) A *wedge*, *i.e.*, $\{x \in \mathbf{R}^n \mid a_1^T x \le b_1,\ a_2^T x \le b_2\}$. $= \{x : a_1^T x \le b_1\} \cap \{x : a_2^T x \le b_2\}$.

convx

(d) The set of points closer to a given point than a given set, *i.e.*,

$$\{x \mid \|x - x_0\|_2 \le \|x - y\|_2 \text{ for all } y \in S\}$$

where $S \subseteq \mathbf{R}^n$.

(e) The set of points closer to one set than another, *i.e.*,

$$\{x \mid \mathbf{dist}(x, S) \le \mathbf{dist}(x, T)\},$$

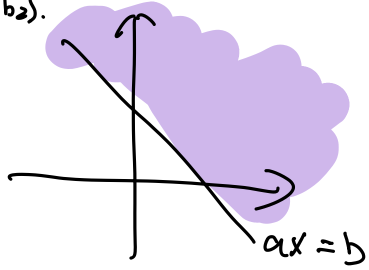
where $S, T \subseteq \mathbf{R}^n$, and

$$\mathbf{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}.$$

(f) [HUL93, volume 1, page 93] The set $\{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbf{R}^n$ with S_1 convex.

(g) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, *i.e.*, the set $\{x \mid \|x - a\|_2 \le \theta\|x - b\|_2\}$. You can assume $a \ne b$ and $0 \le \theta \le 1$.

half space: $\{x \in \mathbb{R}^n : a^T x \ge b\}$.



$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \ge b$.

convex

intersection

(a). $\{x \in \mathbb{R}^n \mid \alpha \le a^T x \le \beta\} = \{x : a^T x \ge \alpha\} \cap \{x : a^T x \le \beta\}$.

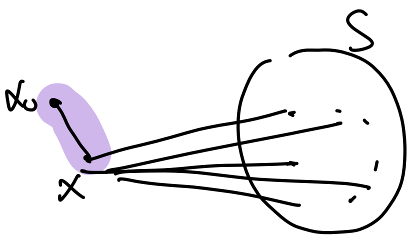
(b). $\{x \in \mathbb{R}^n \mid \alpha_i \le x_i \le \beta_i,\ i = 1, 2, \cdots, n\}$

$$= \bigcap_{i=1,\cdots,n} \{x \in \mathbb{R}^n : x_i \in [\alpha_i, \beta_i]\}$$

$$\underbrace{\alpha_i \le \overset{a^T x}{\overset{\shortparallel}{x_i}} \le \beta_i}_{\text{convex by (a).}} \iff 0 \cdot x_1 + 0 \cdot x_2 + \cdots + 1 \cdot x_i + 0 x_{i+1} + \cdots + 0 \cdot x_n$$

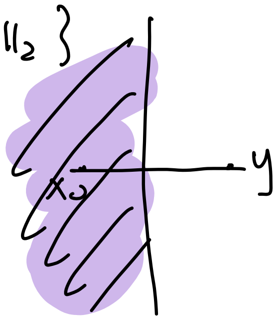
$\therefore A$ is convex.

(d).



$\{x : \|x - x_0\|_2 \le \|x - y\|_2 \text{ for all } y \in S\}$.

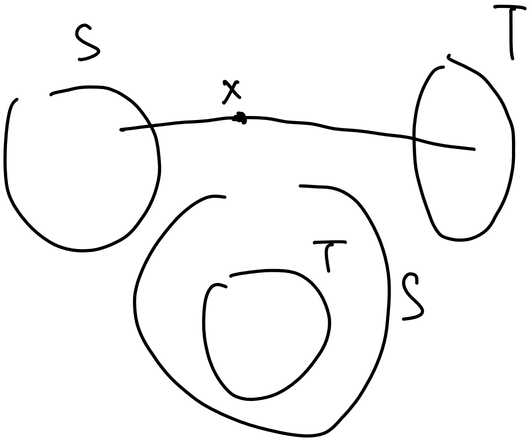
$= \bigcap_{y \in S} \{x : \|x - x_0\|_2 \le \|x - y\|_2\}$



$\|x - x_0\|_2 \le \|x - y\|_2 \iff \|x - x_0\|_2^2 \le \|x - y\|_2^2$

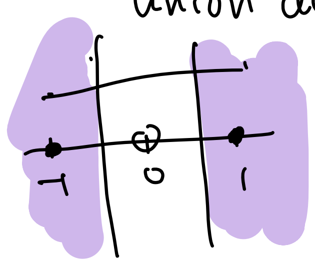
$\iff (x - x_0)^T (x - x_0) \le (x - y)^T (x - y)$

(e).



$\{x : \exists s \in S.\ \forall t \in T.\ \|x - s\|_2 \le \|x - t\|_2\}$

union does not preserve convexity.



$T = \{0\}$ $S = \{-1, 1\}$

(f) The set $A = \{x : x + S_2 \subseteq S_1\}$. where. $S_1, S_2 \subseteq \mathbb{R}^n$. S_1 convex

$$x + S_2 = \{x + y : y \in S_2\}$$

$$A = \{x : x + y \in S_1 \text{ for all } y \in S_2\} = \bigcap_{y \in S_2} \underline{\{x : x + y \in S_1\}}$$

$\{x : x+y \in S_1\}$ $\quad f : x \mapsto x+y$. linear. S_1 convex

$= f^{-1}(S_1)$ convex. $\quad \therefore$ convex.

(g) $A = \{x : \|x-a\|_2 \le \theta \|x-b\|_2\}$.

$= \{x : \underline{\|x-a\|_2^2 \le \theta^2 \|x-b\|_2^2}\}$

$(x-a)^T(x-a) \le \theta^2 (x-b)^T(x-b)$.

$x^Tx + a^Ta - \underline{a^Tx - x^Ta} \le \theta^2 \cdot (x^Tx - \underline{b^Tx - x^Tb} + b^Tb)$

$x^Tx - \dfrac{(2a - 2\theta^2 b)^T}{(1-\theta)^2} x + \bigcirc \le 0$

$x^2 - 2bx + \bigcirc$

$= (x-b)^2 + \bigcirc$

$x_0 = \dfrac{a - \theta^2 b}{(1-\theta)^2}$

$x^Tx - 2x_0^Tx + \bigcirc \le 0$

$(x^T - x_0^T)(x - x_0) \le R^2$

$\underline{\|x - x_0\|_2} \le R.$

Q6. **2.24** *Supporting hyperplanes.*

(a) Express the closed convex set $\{x \in \mathbf{R}^2_+ \mid x_1 x_2 \geq 1\}$ as an intersection of halfspaces.

(b) Let $C = \{x \in \mathbf{R}^n \mid \|x\|_\infty \leq 1\}$, the ℓ_∞-norm unit ball in $\mathbf{R}^n$, and let $\hat{x}$ be a point in the boundary of C. Identify the supporting hyperplanes of C at $\hat{x}$ explicitly.

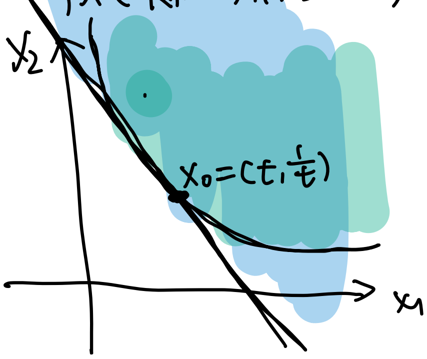
a. s.t. for all $x \in C$. $a^T x \leq a^T x_0$.

$\{x : a^T x = a^T x_0\}$ to be the supporting hyperplane.

$\{x : a^T x \leq a^T x_0\}$ is a supporting ~~hyper space~~ half space that contains C.

A convex set C = intersection of supporting half spaces at all its boundary points

(a). $\{x \in R^2_+ : x_1 x_2 \geq 1\}$



$x_1 x_2 = 1$.

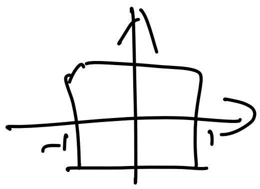
bd point $x_0 = (t, \frac{1}{t})$.

$y = \frac{1}{x}$. $y' = -\frac{1}{x^2}$.

$$\boxed{x_2 = -\frac{1}{t^2} \cdot x_1 + \frac{2}{t}.}$$

$-\frac{1}{t^2} x_1 - x_2 = -\frac{2}{t}$ (supporting ~~half~~ hyperplane)

$$C = \bigcap_{t \in R^+} \{x : \frac{1}{t^2} x_1 + x_2 \geq \frac{2}{t}\}$$



(b). $C = \{x \in R^n : \|x\|_\infty \leq 1\}$. $\hat{x} \in \text{bd } C$

$\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_n|\}$.

$\hat{x} \in \text{bd } C \Leftrightarrow \|\hat{x}\|_\infty = 1 \Leftrightarrow \max\{|\hat{x}_1|, |\hat{x}_2|, \cdots, |\hat{x}_n|\} = 1$

a. s.t. for all $x \in C$. $a^T x \leq a^T \hat{x}$

for $\forall x$. $a_1 x_1 + a_2 x_2 + \cdots + a_i x_i + \cdots + a_n x_n$

$\leq a_1 \hat{x}_1 + a_2 \hat{x}_2 + \cdots + a_i \hat{x}_i + \cdots + a_n \hat{x}_n$

$$\begin{cases} a_i \geq 0 & \text{if } \hat{x}_i = 1 \\ a_i \leq 0 & \text{if } \hat{x}_i = -1 \\ a_i = 0 & \text{if } \hat{x}_i \in (-1, 1) \end{cases}$$

a. s.t. $\{x : a^T x = a^T \hat{x}\}$

Q7. **2.32** Find the dual cone of $\{Ax \mid x \succeq 0\}$, where $A \in \mathbf{R}^{m \times n}$.

$K =$ $\quad x \geqslant 0$

$v \succeq 0 \Rightarrow$ for all $i = 1, 2, \cdots, n$. $v_i \geqslant 0$

K is cone. then the dual cone $K^* = \{y : x^T y \geqslant 0$ for all $x \in K\}$

$k = A \cdot x \quad x \succeq 0$

$K^* = \{y : k^T y \geqslant 0.$ for $k \in K\}$.

$= \{y : (Ax)^T y \geqslant 0$ for all $x \succeq 0\}$.

$= \{y : x^T (A^T y) \geqslant 0$ for all $x \succeq 0\}$

$\tilde{K} = \{y : A^T y \geqslant 0\}$.

1°. $K^* \subseteq \tilde{K}$. $\Rightarrow$ $x^T A^T y \geqslant 0$. $\forall x \succeq 0$.

$\overset{?}{\Rightarrow}$ $A^T y \geqslant 0$ $\Rightarrow y \in \tilde{K}$

assume $(A^T y)_i < 0$. X

$x_i = 1$. $x_j = 0$ for $j \neq i$

$x^T (A^T y) = (A^T y)_i < 0$.

2°. $\tilde{K} \subseteq K^*$: $A^T y \geqslant 0$. $\Rightarrow x^T (A^T y) \geqslant 0$. ✓

$\therefore K^* = \tilde{K}$