5.12 *Analytic centering.* Derive a dual problem for

$$\text{minimize} \quad -\sum_{i=1}^m \log(b_i - a_i^T x)$$

with domain $\{x \mid a_i^T x < b_i,\ i = 1, \dots, m\}$. First introduce new variables y_i and equality constraints $y_i = b_i - a_i^T x$.

$$\min_{x \in \mathbb{R}^n,\ y \in \mathbb{R}_+^m} \ -\sum_{i=1}^m \log y_i \quad \text{s.t.} \quad y = b - Ax. \quad a_i^T \text{ is the } i\text{-th row of } A.$$

Lagrangian: $L(x, y, \overset{\in \mathbb{R}^m}{v}) = -\sum_{i=1}^m \log y_i + v^T(y - b + Ax)$

Dual : $g(v) = \inf_{x \in \mathbb{R}^n,\ y \in \mathbb{R}_+^m} \left(-\sum_{i=1}^m \log y_i + v^T(y - b + Ax)\right)$

First, $g(v) = -\infty$ if $Av^T \neq 0$.

if $Av^T = 0$. $g(v) = \inf_{x, y \in D} \left(\sum_{i=1}^m (v_i y_i - \log y_i) - v^T b \right)$

1) if $\exists\ i$. s.t. $v_i < 0$, then $y_i \to +\infty$. $v_i y_i - \log y_i = -\infty$. $\therefore g = -\infty$.

2) $v \geq 0$. $(v_i y_i - \log y_i)' = v_i - \frac{1}{y_i}$

$y_i^* = 1/v_i$.

$$\therefore \inf \left(\sum_{i=1}^m (v_i y_i - \log y_i) - v^T b \right) = \sum_{i=1}^m (1 + \log v_i) - v^T b$$

$$= m + \sum_{i=1}^m \log v_i - v^T b.$$

$$\therefore g(v) = \begin{cases} m + \sum_{i=1}^m \log v_i - v^T b, & \text{if } v > 0.\ Av^T = 0. \\ -\infty & \text{else where} \end{cases}$$

$\therefore$ Dual problem is $\max_{v} \ m + \sum_{i=1}^m \log v_i - v^T b$. s.t. $v > 0$. $Av^T = 0$.

5.13 *Lagrangian relaxation of Boolean LP.* A *Boolean linear program* is an optimization problem of the form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & x_i \in \{0,1\}, \quad i = 1, \dots, n, \end{array}$$

and is, in general, very difficult to solve. In exercise 4.15 we studied the LP relaxation of this problem,

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & 0 \le x_i \le 1, \quad i = 1, \dots, n, \end{array} \tag{5.107}$$

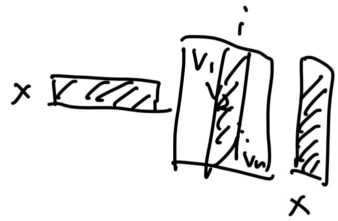
which is far easier to solve, and gives a lower bound on the optimal value of the Boolean LP. In this problem we derive another lower bound for the Boolean LP, and work out the relation between the two lower bounds.

(a) *Lagrangian relaxation.* The Boolean LP can be reformulated as the problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & x_i(1 - x_i) = 0, \quad i = 1, \dots, n, \end{array}$$

which has quadratic equality constraints. Find the Lagrange dual of this problem. The optimal value of the dual problem (which is convex) gives a lower bound on the optimal value of the Boolean LP. This method of finding a lower bound on the optimal value is called *Lagrangian relaxation.*

(b) Show that the lower bound obtained via Lagrangian relaxation, and via the LP relaxation (5.107), are the same. *Hint.* Derive the dual of the LP relaxation (5.107).



(a). multiplier $\mu \in R^m$. $v \in R^n$

$$L(x, \mu, v) = c^T x + \mu^T (Ax - b) - \underbrace{\sum_{i=1}^n v_i x_i (1 - x_i)}$$

$$= c^T x + \mu^T (Ax - b) - v^T x + \sum_{i=1}^n v_i x_i^2$$

$$= c^T x + \mu^T Ax - \mu^T b - v^T x + x^T \operatorname{diag}(v) x$$

$$= x^T \operatorname{diag}(v) x + (A^T \mu + c - v)^T x - \mu^T b$$

$$= \left(\sum_{i=1}^n v_i x_i^2 + (a_i^T \mu + c_i - v_i) x_i \right) - \mu^T b.$$

$$g(\mu, v) = \inf_{x \in D} L(x, \mu, v)$$

if $\exists\, i$. s.t. $v_i < 0$. then $x_i \to +\infty$. $L \to -\infty$.

if $v \succeq 0$. $\quad x_i^* = \dfrac{-a_i^T \mu - c_i + v_i}{2 v_i}$

$$\therefore\ g(\mu, v) = L(x^*, \mu, v) = -\mu^T b - \frac{1}{4} \sum_{i=1}^n (c_i + a_i^T \mu - v_i)^2 / v_i$$

$$\therefore\ g(\mu, v) = \begin{cases} -\mu^T b - \frac{1}{4} \sum_{i=1}^n (c_i + a_i^T \mu - v_i)^2 / v_i & \text{if } v \succ 0 \\ -\infty & \text{elsewhere} \end{cases}$$

$\therefore$ The dual problem is $\max_{\substack{v>0 \\ \mu\in\mathbb{R}^m}} -\mu^T b - \frac{1}{4}\sum_{i=1}^{n} (c_i + a_i^T\mu - v_i)^2/v_i$

s.t. $\mu \geq 0$.

Still too complicated, we try to find, given μ, what is the optimal value for v.

$$\Longleftrightarrow \max_{\mu\in\mathbb{R}^m} -\mu^T b + \sup_{v>0}\left[-\frac{1}{4}\sum_{i=1}^{n}(c_i+a_i^T\mu-v_i)^2/v_i\right], \quad \mu\geq 0.$$

we try to find $\sup_{v>0} -\sum_{i=1}^{n} (c_i+a_i^T\mu-v_i)^2/v_i$ (μ fixed).

$$\Leftrightarrow \sup_{v>0} -(\underbrace{c_i+a_i^T\mu}_{=k}-v_i)^2/v_i \Leftrightarrow \sup_{v>0} \underbrace{-\frac{k^2}{v_i} - v_i} + 2k$$

$(-\frac{k^2}{v_i} - v_i)' = \frac{k^2}{v_i^2} - 1$ $\therefore$ increasing when $k^2 \geq v_i^2$

if $k\geq 0$. increasing when $v_i \leq k \Rightarrow$ max when $v_i = k \Rightarrow \sup = 0$

if $k<0$. increasing when $v_i \leq -k \Rightarrow$ max when $v_i = -k \Rightarrow \sup = 4k$

$$\therefore \sup_{v} (c_i+a_i^T\mu-v_i)^2/v_i = \begin{cases} 0 & \text{if } c_i+a_i^T\mu\geq 0 \\ 4(c_i+a_i^T\mu) & \text{if } c_i+a_i^T\mu<0 \end{cases} = \min(0, 4(c_i+a_i^T\mu)).$$

$$\therefore \max_{\mu} -\mu^T b + \sum_{i=1}^{n} \min(0, c_i+a_i^T\mu). \quad \text{s.t. } \mu\geq 0.$$

(b) Show that the lower bound obtained via Lagrangian relaxation, and via the LP relaxation (5.107), are the same. *Hint.* Derive the dual of the LP relaxation (5.107).

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & 0 \leq x_i \leq 1, \quad i=1,\ldots,n, \end{array}$$

LP relaxation $\Leftrightarrow$ $\min c^T x$. s.t. $Ax \leq b$. and $-x(1-x) \leq 0$.

$$\overset{\text{gap 0.}}{\underset{\text{dual}}{\Longleftrightarrow}} \max_{u,v} \inf_{x}\left[c^T x + u^T(Ax-b) - \sum_{i=1}^{n} x_i(1-x_i)v_i\right], \quad u. v. \geq 0$$

This is the same as Lagragian relaxation except $v\geq 0$.

but we show in (a) that LR also requires $v\geq 0$.

$\therefore$ They are the same.

4(a). (primal) $\min_{x\in\mathbb{R}^2} x_1+2x_2$ s.t. $2x_1+x_2\geq 2$. $x\geq 0$. Find d^*. p^*.

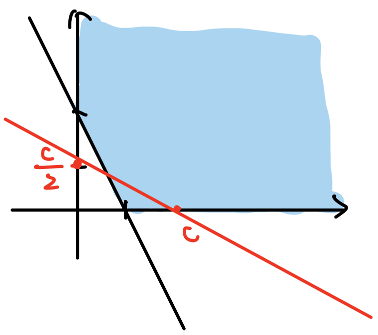
(b). (primal). $\min_{x\in\mathbb{R}} x^2$. s.t. $x\leq a$. $a>0$ or $a\leq 0$. Find d^*. p^*.

(c). (primal). $\min_{x\in\mathbb{R}} x^3$ s.t. $x\geq 0$. Find d^*. p^*.

(d). (primal). $\min_{x\in\mathbb{R}} f_0(x)$. s.t. $x\leq 0$. where $f_0(x)=\begin{cases} -\sqrt{x}. & x>0 \\ 1. & x=0. \\ +\infty. & x<0.\end{cases}$

Verify the convexity. Find d^*. p^*.

(a).



$x_1+2x_2=c$ $\quad p^*=1$.

Lagrangian $L(x,\lambda_1.\lambda_2.\lambda_3)$

$$= x_1+2x_2+\lambda_1(2-x_1-x_2)+\lambda_2(-x_1)+\lambda_3(-x_2)$$

$$= 2\lambda_1+(1-2\lambda_1-\lambda_2)x_1+(2-\lambda_1-\lambda_3)x_2$$

$$\therefore\ g(\lambda_1.\lambda_2.\lambda_3)=\inf_{x_1.x_2.} L(x.\lambda_1.\lambda_2.\lambda_3)$$

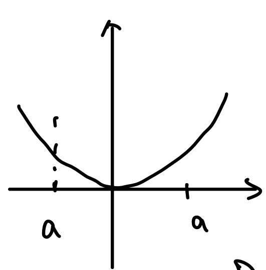
$$=\begin{cases} 2\lambda_1 & \text{if } 1=2\lambda_1+\lambda_2 \text{ and } \lambda_1+\lambda_3=2. \\ -\infty & \text{elsewhere}\end{cases}$$

Dual prob: $\max_{\lambda_1.\lambda_2.\lambda_3} 2\lambda_1$ s.t. $\underbrace{2\lambda_1+\lambda_2=1}_{\Downarrow\ 2\lambda_1\leq 1}$. $\underbrace{\lambda_1+\lambda_3=2}_{\Downarrow\ \lambda_1\leq 2}$. $\lambda_1.\lambda_2.\lambda_3\geq 0$.

$d^*=1$. $d^*=p^*$.

(b)

$\min_{x\in\mathbb{R}} x^2$. s.t. $x\leq a$. $a>0$ or $a\leq 0$. Find d^*. p^*.



if $a>0$. $p^*=0$; if $a<0$ $p^*=a^2$.

$L(x,\lambda)= x^2+\lambda(x-a)= x^2+\lambda x-a\lambda$

$g(\lambda)= L(-\frac{\lambda}{2},\lambda)= -\lambda^2/4-a\lambda$

Dual prob: $\max_{\lambda} \underbrace{-\frac{\lambda^2}{4}-a\lambda}_{c(\lambda)}$. $\lambda\geq 0$.

$c'(\lambda)=-\frac{\lambda}{2}-a$ increasing when $-2a\geq\lambda$.

if $a>0$: alway decreasing, $\max=c(0)=0$; if $a<0$: $\lambda=-2a$, $d^*=a^2$

(c). $\min_{x\in\mathbb{R},} \quad x^3. \quad \text{s.t. } x \geq 0. \qquad p^* = 0.$

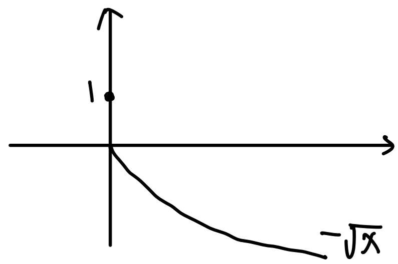
$L(x,\lambda) = x^3 - \lambda x \qquad g(\lambda) = \inf_x (x^3 - \lambda x) = -\infty. \qquad \therefore d^* = \max g = -\infty.$

The duality gap $p^* - d^* = +\infty.$

(d) (primal). $\min_{x\in\mathbb{R}} f_0(x). \quad \text{s.t. } x \leq 0.$ where $f_0(x) = \begin{cases} -\sqrt{x}. & x > 0 \\ 1. & x = 0. \\ +\infty. & x < 0. \end{cases}$

Verify the convexity. Find $d^*. p^*.$

$p^* = 1$ when $x^* = 0.$



Lagrange: $L(x,\lambda) = f_0(x) + \lambda x$

$g(\lambda) = \inf_x \left(f_0(x) + \lambda x\right)$

$= \begin{cases} -\infty & \text{if } \lambda \leq 0 \\ \underline{\inf_{x\geq 0} \left(f_0(x) + \lambda x\right)} & \end{cases}$ $= \min\left(1 + \lambda \cdot 0, \ \min_{x>0} -\sqrt{x} + \lambda x\right)$

direvative: $-\frac{1}{2\sqrt{x}} + \lambda \geq 0$

increase when $x \geq \frac{1}{4\lambda^2}$

$\therefore \min_{x>0} -\sqrt{x} + \lambda x = -\frac{1}{2\lambda} + \frac{1}{4\lambda} = -\frac{1}{4\lambda}$

$\therefore g(\lambda) = \begin{cases} -\infty. & \text{if } \lambda \leq 0 \\ -\frac{1}{4\lambda}. & \text{if } \lambda > 0. \end{cases}$

$\therefore \max_\lambda -\frac{1}{4\lambda}. \quad \lambda \geq 0 \Rightarrow d^* = 0$

$p^* - d^* = 1 - 0 = 1$