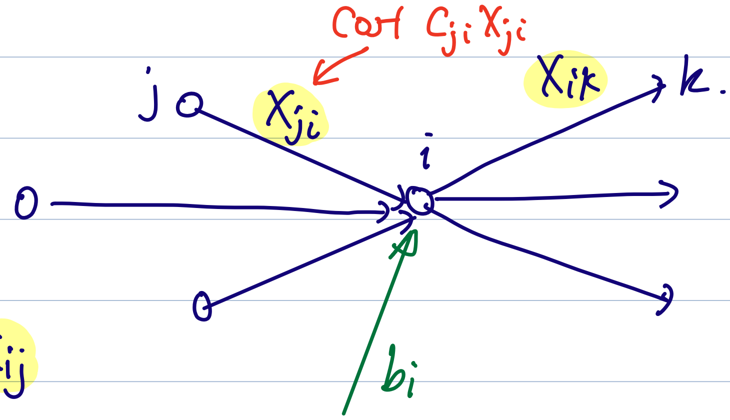
Tutorial 7.

BV 4.12



$$\min_{X = \{x_{ij}\}} C = \sum_{i,j} c_{ij} x_{ij}$$

$\wedge$

$$\underline{l_{ij} \le x_{ij} \le u_{ij}} \quad \forall i,j$$

Conservation of flow

$\forall i \in [n]$, in = out

$$b_i + \sum_{j=1}^{n} x_{ji} = \sum_{k=1}^{n} x_{ik} \quad \text{(affine / linear)}$$

BV 4.23

l_4-norm approximation

$$\min_x \|Ax - b\|_4 = \left(\sum_{i=1}^{m} ((Ax)_i - b_i)^4 \right)^{1/4}$$

$x \in \mathbb{R}^n$

$A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

$$= \left(\sum_{i=1}^{m} (a_i^T x - b_i)^4 \right)^{1/4}$$

Formulate this as a <u>QCQP</u>

$P_i \in S_+^n$, $i=0,1,\dots,m$.

$$\min_{x} \quad \frac{1}{2}x^T P_0 x + q_0^T x + r_0$$

$$\text{s.t.} \quad \frac{1}{2}x^T P_i x + q_i^T x + r_i \le 0 \quad \forall i=1,\dots,m.$$

$$Ax = b.$$

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{m} (a_i^T x - b_i)^4$$

Let $y_i = a_i^T x - b_i$ $\qquad y_i^2 \le z_i$

$i=1,\dots,m \qquad i=1,\dots,m.$

$$\min_{\substack{x \in \mathbb{R}^n,\\ y \in \mathbb{R}^m,\ z \in \mathbb{R}^m}} \sum_{i=1}^{m} z_i^2 = z^T z$$

$y_i = a_i^T x - b_i \qquad y_i^2 \le z_i$

$i=1,\dots,m \qquad i=1,\dots,m.$

$$w = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^{n+2m}$$

Objective: $\displaystyle \min_{w \in \mathbb{R}^{n+2m}} \frac{1}{2} \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \overbrace{\begin{bmatrix} 0_{n\times n} & \underset{}{\overset{n\times m}{0}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \underset{m\times m}{I} \end{bmatrix}}^{P_0} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Equality Constraint:

$$y_i = a_i^T x - b_i \quad \Rightarrow \quad Aw = b.$$

$\Leftarrow$ $i = 1, \dots, m$ $\qquad\qquad \Downarrow$

$$-a_i^T x + y_i = -b_i$$

$$\begin{bmatrix} -A & I & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -b \end{bmatrix}$$

$$\boxed{\begin{array}{l} y_i^2 \le z_i \\ \quad i = 1, \dots, m. \end{array}}$$

Inequality constraint:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}^T P_i \begin{bmatrix} x \\ y \\ z \end{bmatrix} + q_i^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \cancel{r_i} \le 0.$$

$$\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}^T}_{W} \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{W} - \begin{bmatrix} 0_n \\ 0_m \\ 1_m \end{bmatrix}^T \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{W} \le 0.$$

BV 4.28

$$\min_x \ \frac{1}{2} x^T P x + q^T x + r \quad \text{s.t. } Ax \le b$$

$$\min_x \ \sup_{P \in \mathcal{E}} \left\{ \frac{1}{2} x^T P x + q^T x + r \right\} \quad \text{s.t. } Ax \le b$$

a) $\mathcal{E} = \{P_1, \dots, P_K\}$, $P_i \in S_+^n$

$$\min_{x \in \mathbb{R}^n} \max_{i=1,\dots,K} \left\{ \frac{1}{2} x^T P_i x + q^T x + r \right\} \quad \text{s.t. } Ax \le b$$

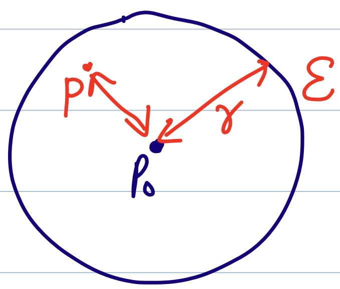
$$\equiv \min_{x \in \mathbb{R}^n, t \in \mathbb{R}} \; t \quad \text{s.t. } \frac{1}{2} x^T P_i x + q^T x + r \le t \quad \forall i = 1,\dots,K$$
$$Ax \le b.$$

QCQP $\begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{R}^{n+1}$

b) $\mathcal{E} = \{ P \in S_+^n : -\gamma I \preceq P - P_0 \preceq \gamma I \}$

$\Updownarrow$

$\lambda_{max}(P - P_0) \le \gamma$



$$\min_x \sup_{P \in \mathcal{E}} \left\{ \frac{1}{2} x^T P x + q^T x + r \right\} \quad \text{s.t. } Ax \le b$$

$$\sup_{P \in \mathcal{E}} \left\{ \frac{1}{2} x^T P x \right\} = \sup_{-\gamma I \preceq Q \preceq \gamma I} \left\{ \frac{1}{2} x^T (P_0 + Q) x \right\}$$

$$= \frac{1}{2} \left(x^T P_0 x + \sup_{-\gamma I \preceq Q \preceq \underline{\gamma I}} x^T Q x \right)$$

$$\boxed{\sup_{x \ne 0} \frac{x^T Q x}{x^T x} = \lambda_1(Q)}$$

$$= \frac{1}{2} \left(x^T P_0 x + x^T \gamma I x \right).$$

$$= \frac{1}{2} x^T P_0 x + \gamma x^T x.$$

$$\min_x \; \frac{1}{2} x^T (P_0 + \gamma I) x + q^T x + r \quad \text{s.t. } Ax \le b.$$

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c) $\mathcal{E} = \left\{ P_0 + \sum_{i=1}^{K} P_i u_i \; : \; \|u\|_2 \le 1 \right\}$.

Objective writes

$$\min_x \sup_{P \in \mathcal{E}} \frac{1}{2} x^T P x$$

$$\equiv \min_x \sup_{u : \|u\|_2 \le 1} \frac{1}{2} x^T \left(P_0 + \sum_{i=1}^{K} P_i u_i \right) x$$

$$\equiv \min_x \frac{1}{2} \left(x^T P_0 x + \sup_{u : \|u\|_2 \le 1} \sum_{i=1}^{K} \underbrace{(x^T P_i x)}_{w_i} u_i \right).$$

$$\sup_{u : \|u\|_2 \le 1} \sum_{i=1}^{K} w_i u_i = \sup_{u : \|u\| \le 1} u^T w = \|w\|_2$$

$$\equiv \min_x \frac{1}{2} \left(x^T P_0 x + \underbrace{\left(\sum_{i=1}^{K} (x^T P_i x)^2 \right)^{1/2}}_{\text{Convex}} \right) + q^T x + r$$

We know that the function $g_i(x) \triangleq x^T P_i x$ is convex in x, since $P_i \succeq 0$.

The second term $\left(\sum_i \overbrace{(x^T P_i x)}^{\text{convex in } x}{}^2 \right)^{1/2}$ is

$$h(g_1(x), g_2(x), \dots, g_K(x))$$

$h(y) = \underbrace{\|y\|_2}_{}$. nonnegative

$|y|$

h is convex & when the arguments are nonnegative, h is non-decreasing in its arguments. Hence, the 2nd term is convex.

$$\equiv \min_{x,y,u,t} \overbrace{\tfrac{1}{2}x^TP_0x}^{u} + \overbrace{\|y\|_2}^{t} + q^Tx + r$$

$$\tfrac{1}{2}x^TP_ix \le y_i \qquad i=1,\dots,K$$

$$Ax \le b$$

$\tfrac{1}{2}x^TP_0x \le u$ $\qquad$ $\|y\| \le t$ SOCP

$\Downarrow$ SOCP

$\|Ax+b\| \le c^Tx + d$

$$\left\| \begin{bmatrix} P_0^{1/2}x \\ 2u - 1/4 \end{bmatrix} \right\| \le 2u + \tfrac{1}{4}$$

$$\left\| \begin{bmatrix} P_i^{1/2}x \\ 2y_i - \tfrac{1}{4} \end{bmatrix} \right\| \le 2y_i + \tfrac{1}{4}$$

$$\|P_0^{1/2}x\|^2 + (2u - \tfrac{1}{4})^2 \le (2u + \tfrac{1}{4})^2$$

$$x^TP_0x + -u \le u$$

$$\tfrac{1}{2}x^TP_0x \le u.$$