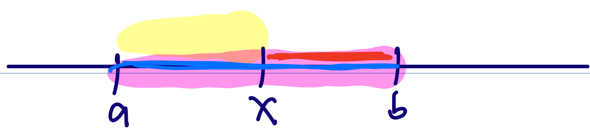
Tutorial 4 BV Problems 3.1, 3.5, 3.13, 3.16 (convexity only) and Q7.

3.1 $f: \mathbb{R} \to \mathbb{R}$ convex, $a, b \in \operatorname{dom} f$, $a < b$



$$f(\theta x + (1-\theta) y) \leq \theta f(x) + (1-\theta) f(y) \quad x, y \in \operatorname{dom} f \quad \theta \in [0,1].$$

$x = a$, $y = b$, $\theta = \frac{b-x}{b-a} \in [0,1]$. $1 - \theta = 1 - \frac{b-x}{b-a}$

$$= \frac{x-a}{b-a}$$

$$f\left(\frac{b-x}{b-a} \cdot a + \frac{x-a}{b-a} \cdot b\right)$$

$$\leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$$

$$\Downarrow$$

$$f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) \quad -(*) \qquad ///.$$

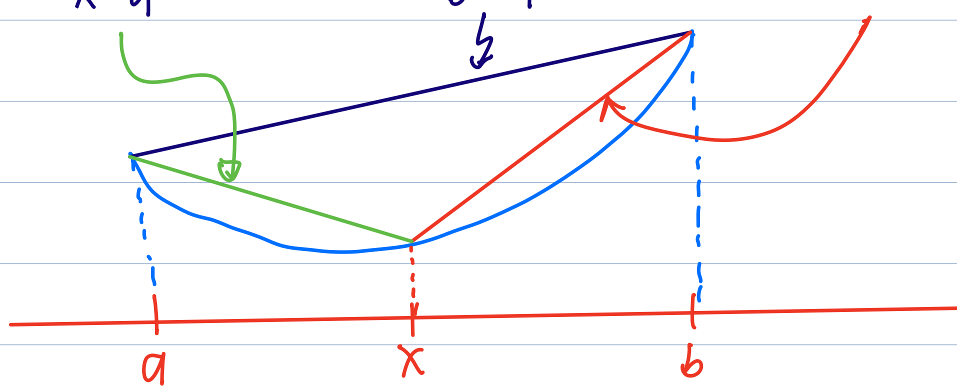
b) Subtract $f(a)$ on both sides of $(*)$

$$f(x) - f(a) \leq \frac{b-x}{b-a} f(a) - f(a) + \frac{x-a}{b-a} f(b).$$

$$= -\frac{x-a}{b-a} f(a) + \frac{x-a}{b-a} f(b)$$

$$= \frac{x-a}{b-a}\left(f(b)-f(a)\right)$$

$$\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(x)}{b-x}.$$



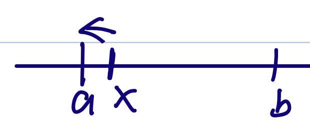
Slope is non-decreasing for convex f^n.

c) $$\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(x)}{b-x}.$$

WTS: $$f'(a) \leq \frac{f(b)-f(a)}{b-a} \leq f'(b)$$

Recalling: $$f'(x_0) = \lim_{\varepsilon \downarrow 0.} \frac{f(x_0+\varepsilon)-f(x_0)}{\varepsilon}$$

$x_0 \in (a,b)$ $$= \lim_{y \to x_0} \frac{f(y)-f(x_0)}{y-x_0}$$

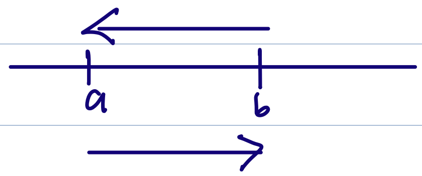


Since $\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a}$ holds $\forall x \in (a,b)$,

LHS as $x \to a^+$, $\frac{f(x)-f(a)}{x-a} \to f'(a)$

$$f'(a) \leq \frac{f(b)-f(a)}{b-a} \leq f'(b). \quad -(**)$$

d) From $(**)$ & $b>a$, $\dfrac{f'(b)-f'(a)}{b-a} \geq 0. \quad -(***)$



Let $b \to a^+$, $(***)$ yields
$f''(a) \geq 0$.

Let $a \to b^-$, $f''(b) \geq 0$. //.

BV 3.5: $f: \mathbb{R} \to \mathbb{R}$ convex., $\mathbb{R}_+ \subseteq \operatorname{dom} f$

Running average $F(x) = \frac{1}{x}\int_0^x f(t)\,dt$, $x>0$

NTP: F is convex

Background: $\dfrac{d}{dx}\displaystyle\int_{b(x)}^{a(x)} g(t)\,dt = a'(x)g(a(x)) - b'(x)g(b(x))$

$$F'(x) = \frac{-\int_0^x f(t)dt + \left(\frac{d}{dx}\int_0^x f(t)dt\right)x}{x^2}$$

$$= \frac{-\int_0^x f(t)dt + x f(x)}{x^2}$$

$$= \frac{f(x)}{x} - \frac{1}{x^2}\int_0^x f(t)\,dt.$$

$$F''(x) = \frac{xf'(x) - f(x)}{x^2} - \frac{x^2 f(x) - \int_0^x f(t)\,dt\,(2x)}{x^4}.$$

$$= \frac{f'(x)}{x} - \frac{2f(x)}{x^2} + \frac{2}{x^3}\int_0^x f(t)\,dt$$

$$= \frac{2}{x^3}\int_0^x \underbrace{\left(f(t) - f(x) - f'(x)(t-x)\right)}_{\geq 0} dt \geq 0.$$

$x > 0$

$f(t) \geq f(x) + f'(x)(t-x)$

$\Rightarrow$ F is convex. ////

BV 3.13: $u, v \in \mathbb{R}^n_{++}$

$$D_{kl}(u,v) = \sum_{i=1}^n u_i \log\left(\frac{u_i}{v_i}\right) - u_i + v_i$$

WTS: $D_{kl}(u,v) \geq 0 \quad \forall\ u, v \in \mathbb{R}^n_{++}$.

Pf: The negative entropy $f(v) = \sum_{i=1}^n v_i \log v_i$, $v \in \mathbb{R}^n_{++}$.

We showed in lecture that f is strictly convex.

From the first-order condition,

$$\left(\frac{\partial f}{\partial v_1}, \ldots, \frac{\partial f}{\partial v_n}\right)^T = (\log v_1 + 1, \ldots, \log v_n + 1)^T$$

$$f(u) \geq f(v) + \nabla f(v)^T (u - v) \qquad v, u \in \mathbb{R}^n_{++}$$

$$\sum_i u_i \log u_i \geq \sum_i v_i \log v_i + \sum_i (\log v_i + 1)(u_i - v_i)$$

$$= \sum_i \cancel{v_i \log v_i} + \sum_i u_i \log v_i - \cancel{v_i \log v_i} + (u_i - v_i)$$

$$\underbrace{\sum_i \left(u_i \log\left(\frac{u_i}{v_i}\right) - u_i + v_i\right)}_{D_{kl}(u,v)} \geq 0.$$

BV 3.16

b) $f(x_1, x_2) = x_1 x_2 \quad \operatorname{dom} f = \mathbb{R}^2_{++}$.

$$\nabla f(x_1, x_2) = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A$$

$Ax = \lambda x$

A is neither PSD nor NSD. Hence f is neither convex nor concave.

c) $f(x_1, x_2) = \dfrac{1}{x_1 x_2}, \quad \operatorname{dom} f = \mathbb{R}^2_{++}$

$$\nabla f(x_1, x_2) = \begin{bmatrix} -\dfrac{1}{x_1^2 x_2} \\ -\dfrac{1}{x_2^2 x_1} \end{bmatrix}$$

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} \frac{2}{x_1^3 x_2} & -\frac{1}{x_1^2 x_2^2} \\ -\frac{1}{x_1^2 x_2^2} & \frac{2}{x_2^3 x_1} \end{bmatrix}$$

$$= \frac{1}{x_1 x_2} \begin{bmatrix} 2/x_1^2 & 1/x_1 x_2 \\ 1/x_1 x_2 & 2/x_2^2 \end{bmatrix} \succ 0$$

$2/x_1^2 > 0.$ $\quad \det(\nabla^2 f(x_1, x_2)) = \frac{2}{x_1^2} \cdot \frac{2}{x_2^2} - \left(\frac{1}{x_1 x_2}\right)^2$

$$= \frac{3}{x_1^2 x_2^2} > 0.$$

f is convex.

$$\int_0^x f'(x)(t-x)\,dx$$

$$= f'(x) \int_0^x t - x \, dx$$

$$= f'(x) \left[\frac{(t-x)^2}{-2} \right]_0^x$$

$$= f'(x) \frac{x^2}{2}$$

$$-\frac{2}{x^3} \int_0^x - f'(x)(t-x)\,dt = \frac{2}{x^3} \int_0^x f'(x)(t-x)\,dt$$

$$= \frac{\cancel{2}}{x^{\cancel{3}}} f'(x) \cdot \frac{\cancel{x^2}}{\cancel{2}}$$