Tutorial 0

Q1: Set $T = \{a \in \mathbb{R}^k : p(0) = 1,\ |p(t)| \le 1 \ \forall t \in [\alpha, \beta]\}$ convex

$$p(t) = \sum_{i=1}^{k} a_i t^{i-1} \quad \text{polynomial.}$$

$$T = T_1 \cap T_2$$

$T_1 = \{a \in \mathbb{R}^k : p(0) = 1\} = \{a \in \mathbb{R}^k : a_1 = 0\}$ Convex

$a \in T_1,\ b \in T_1 \quad a_1 = b_1 = 0.$

$\theta \in [0,1]. \quad [\theta a + (1-\theta) b]_1 = 0$

$\Rightarrow \theta a + (1-\theta) b \in T_1$

$T_2 = \{a \in \mathbb{R}^k : |p(t)| \le 1 \ \forall t \in [\alpha, \beta]\}$ Convex

$$= \bigcap_{t \in [\alpha, \beta]} T_2^{(t)}$$

$T_2^{(t)} = \{a \in \mathbb{R}^k : |p(t)| \le 1\}$ convex

$$= \left\{a \in \mathbb{R}^k : -1 \le \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}^T \begin{bmatrix} 1 \\ t \\ \vdots \\ t^{k-1} \end{bmatrix} \le 1\right\}$$

$$= \underbrace{\left\{a \in \mathbb{R}^k : -1 \le \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}^T \begin{bmatrix} 1 \\ t \\ \vdots \\ t^{k-1} \end{bmatrix}\right\}}_{\text{halfspace}} \cap \underbrace{\left\{a : \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}^T \begin{bmatrix} 1 \\ t \\ \vdots \\ t^{k-1} \end{bmatrix} \le 1\right\}}_{\text{halfspace}}$$

2. $$A = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \begin{bmatrix} x_1 + x_2 & x_1 - 2x_3 \\ x_1 - 2x_3 & x_2 + 3x_3 \end{bmatrix} \succeq 0 \right\}$$

If C is convex & f is an affine $f^{\underline{n}}$, then the set $\{x : f(x) \in C\}$ is convex.

↑ pre-image of C under f.

$$f : \mathbb{R}^3 \to S^2$$

$$f(x_1, x_2, x_3) = \begin{bmatrix} x_1 + x_2 & x_1 - 2x_3 \\ x_1 - 2x_3 & x_2 + 3x_3 \end{bmatrix}$$

Claim : f is linear.

i) $f(\alpha x_1, \alpha x_2, \alpha x_3) = \alpha f(x_1, x_2, x_3) \qquad \alpha \in \mathbb{R}.$

ii) $f(x_1 + z_1, x_2 + z_2, x_3 + z_3) = f(x_1, x_2, x_3) + f(z_1, z_2, z_3)$

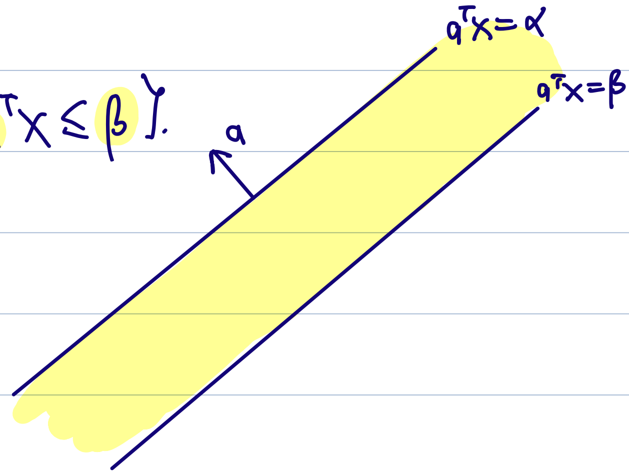
$C = S_+^2$: set of PSD matrices of size 2×2.

$$\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : f(x_1, x_2, x_3) \in S_+^2 \}$$
$$= \left\{ (x_1, x_2, x_3) : \begin{bmatrix} x_1 + x_2 & x_1 - 2x_3 \\ x_1 - 2x_3 & x_2 + 3x_3 \end{bmatrix} \succeq 0 \right\} = A$$

S_+^2 : convex , f : linear $\Rightarrow A$ is convex.

Q3: BV (2.12).

a) Slab: $S = \{x \in \mathbb{R}^n : \alpha \le a^T x \le \beta\}$.



$S = S_1 \cap S_2$

$S_1 = \{x : a^T x \ge \alpha\}$

$S_2 = \{x : a^T x \le \beta\}$ halfspaces convex

$\Rightarrow S$ is convex

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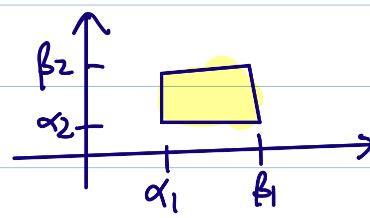
b) $R = \{x : \alpha_i \le x_i \le \beta_i \ \forall i \in [n]\}$.

$$= \bigcap_{i \in [n]} \underbrace{\{x : \alpha \le x^T e_i \le \beta\}}_{\text{slab : convex}}$$



$$e_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ position}$$

$\Downarrow$

R convex

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c) Wedge: $W = \{x : a_1^T x \le b_1, a_2^T x \le b_2\}$ convex

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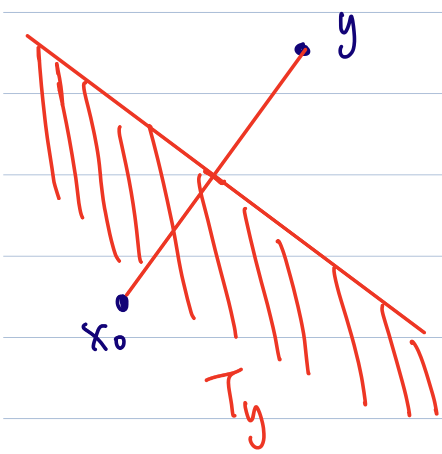
d) $T = \{x : \|x - x_0\| \le \|x - y\|, \forall y \in S\}$

$\uparrow$ not necessarily convex.

$$= \bigcap_{y \in S} \underbrace{\{x : \|x - x_0\|^2 \le \|x - y\|^2\}}_{\text{halfspace: } T_y \text{ convex}}$$

convex

$\cancel{x^Tx} - 2x^Tx_0 + \underline{x_0^Tx_0} \le \cancel{x^Tx} - 2x^Ty + \underline{y^Ty}$

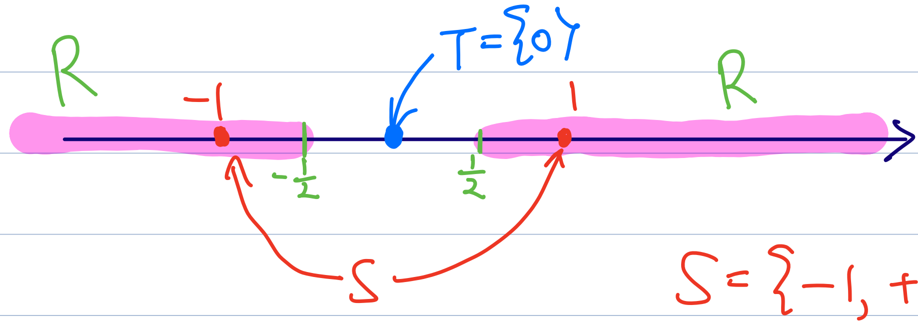


affine term in $x \leq$ constant.

$\{x : x^T a \leq b\}$

e) $R = \{x \mid \text{dist}(x, S) \leq \text{dist}(x, T)\}$. not convex.

$n=1$

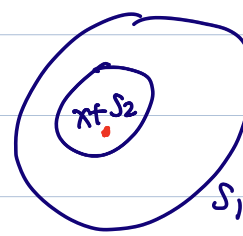


$S = \{-1, +1\}$

f) $A = \{x : x + S_2 \subseteq S_1\}$ $S_1, S_2 \subseteq \mathbb{R}^n$, S_1 : convex.

The condition $x + S_2 \subseteq S_1$.

$\Leftrightarrow$ $x + y \in S_1$ $\forall y \in S_2$



$A = \{x : x+y \in S_1, \forall y \in S_2\}$

$= \bigcap_{y \in S_2} \overbrace{\{x : x+y \in S_1\}}^{A_y}$.

convex fixed convex

////.

$x + S_2 = \{x+y : y \in S_2\}$

$f(x) = x+y$

$A_y = \{x : f(x) \in S_1\}$. convex

g) $\|x-a\|_2^2 \le \theta^2 \|x-b\|_2^2$

$\iff (1-\theta^2)\, x^T x - 2(a-\theta^2 b)^T x + (\|a\|_2^2 - \theta^2 \|b\|_2^2) \le 0.$

$\theta \in [0,1]$

Case A: $\theta = 1 \Rightarrow$ The set is a halfspace.

$0 \le \theta < 1$

Case B: $\theta = \frac{1}{2}$

$\frac{3}{4} x^T x$ + linear part in x + constant ≤ 0.

This is a ball when we compare it to

$\{x : \|x - x_0\|^2 \le R^2\}$.

Consider: $\theta \notin [0,1]$.

Qn 7: BV 2.32

$K = \{\underset{\in \mathbb{R}^m}{Ax} : x \succeq 0\} \qquad A \in \mathbb{R}^{m \times n}$

Take any Ax with $x \succeq 0$ and any $\lambda \in \mathbb{R}_+$.

$\lambda A x = \underbrace{A(\lambda x)}_{\in K} \qquad \lambda x \succeq 0$

$K^* = \{y : y^T z \ge 0 \quad \forall z \in K\}$

WTS: $K^* = \{y : \underbrace{A^T y}_{\in \mathbb{R}^n} \succeq 0\}$

$\tilde{K} = \{y : A^T y \ge 0\}$

WTS: $K^* = \tilde{K}$

$(\Rightarrow)$ $K^* \subseteq \tilde{K}$

Take $y \in K^*$. Then $y^T z \geq 0 \quad \forall z \in K$.

$\Rightarrow \quad y^T (Ax) \geq 0 \quad \forall x \geq 0$

$x^T (A^T y) \geq 0 \quad \forall x \geq 0$

The non-negative orthant is self-dual $\Rightarrow$ $A^T y \geq 0$

$\Rightarrow \quad y \in \tilde{K}$

$(\Leftarrow)$ $\tilde{K} \subseteq K^*$ . . .