DSA3102: Practice Final

1. Consider the following Boolean linear program:

$$\min_x \; c^T x \quad \text{s.t.} \quad Ax \preceq b, \quad x_i(x_i - 1) = 0, \quad i = 1, \dots, n$$

where $c \in \mathbf{R}^n$, $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$ are given.

(a) (15 points) Show that the dual function of the above problem is

$$g(\lambda, \nu) = \begin{cases} -b^T\mu - \frac{1}{4}\sum_{i=1}^n \frac{(c_i + a_i^T\mu - \nu_i)^2}{\nu_i}, & \nu_i \succeq 0 \\ -\infty & \text{otherwise} \end{cases}$$

where $\mu \in \mathbf{R}^m$ and $\nu \in \mathbf{R}^n$ are the Lagrange multipliers associated to the first and second sets of constraints respectively and a_i is the i-th column of A.

(b) (10 points) Show that a lower bound on the optimal value of the above problem can be obtained as the optimal value of the following problem

$$\max_\mu \; -b^T\mu + \sum_{i=1}^n \min\{0, c_i + a_i^T\mu\}, \quad \text{s.t.} \quad \mu \succeq 0$$

2. (a) (15 points) Solve for the optimal x using the KKT conditions:

$$\max_{x_1,x_2} \; 14x_1 - x_1^2 + 6x_2 - x_2^2 + 7 \quad \text{s.t.} \quad x_1 + x_2 \leq 2 \quad x_1 + 2x_2 \leq 3$$

(b) (10 points) Consider the following problem

$$\max_{x} \; x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \quad \text{s.t.} \quad \sum_{i=1}^{n} x_i = 1 \quad x_i \geq 0, \quad i = 1, \ldots, n$$

where a_i are given positive scalars. Find a global maximum and show that it is unique.

3. (a) (15 points) Formulate the following problem as an equivalent linear program

$$\min_{x} \ \|Ax - b\|_1 + \|x\|_\infty$$

where $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$ are given.

(b) (10 points) Is the following set convex?

$$\mathcal{S} := \{x \in \mathbf{R}^2 : x_1 > 0, x_2 > 0, x_1 \log x_1 + x_2 \log x_2 \leq 2\}.$$

4. (a) (20 points) Consider the gradient descent method with bounded error,

$$x^{(k+1)} = x^{(k)} - s(\nabla f(x^{(k)}) + \epsilon^{(k)})$$

where s is a constant stepsize, $\epsilon^{(k)}$ are error terms satisfying

$$\|\epsilon^{(k)}\| \leq \delta$$

for all k, and f is the positive definite quadratic function

$$f(x) = \frac{1}{2}(x - x^*)^T Q(x - x^*)$$

Let

$$q := \max\{|1 - s\lambda_{\min}(Q)|, |1 - s\lambda_{\max}(Q)|\}$$

and assume that $q < 1$. Show that for all k, we have

$$\|x^{(k)} - x^*\| \leq \frac{s\delta}{1-q} + q^k \|x^{(0)} - x^*\|$$

(b) (5 points) Consider the function f in part (a) of this problem. Suppose we now use Newton's method to find the minimum. Does Newton's method converge? If so how many iterations (Newton steps) are required for convergence to within $\varepsilon = 10^{-10}$ of x^*, i.e., find the minimum k such that $\|x^{(k)} - x^*\| \leq \varepsilon$?