Week 5. Xiaochen. Focus on Q3.

Q1. Show that the following functions $f: R^n \to R$ are convex.

(a). $f(x) = \max_{i=1,2,\cdots,k} \|A^{(i)}x - b^{(i)}\|$

(b). $f(x) = \sum_{i=1}^{r} |x|_{[i]}$. where $|x| = (|x_1|, |x_2|, \cdots, |x_n|)$

$x_{[i]}$ = i-th largest component of x.

(a). $g_i(x) = \|A^{(i)}x - b^{(i)}\|$ is convex . composition of norm + affine.

$f(x) = \max_{i=1,2,\cdots,k} g_i(x)$. $\therefore$ is convex. (pointwise max)

(b) $f(x) = \max_{1\le i_1 < i_2 < \cdots < i_r \le n} \underbrace{|x_{i_1}| + |x_{i_2}| + \cdots + |x_{i_n}|}$ convex

the max of sum of r components = sum of r largest components.

Q2. Let f be a convex function. Define $g(x) = \inf\limits_{\alpha>0} \frac{f(\alpha x)}{\alpha}$.

(a). Show that for all $t \geq 0$, $g(tx) = t g(x)$ (homogeneous)

(b). Show that if h is also homogeneous and $h(x) \leq f(x)$. $\forall x$. then we have $h(x) \leq g(x)$ $\forall$ x. (g is the largest homogeneous underestimator of f)

(c). Prove that g is convex.

(a). $t>0$: $g(tx) = \inf\limits_{\alpha>0} \frac{f(\alpha tx)}{\alpha} = t \inf\limits_{\alpha>0} \frac{f(\alpha t x)}{\alpha t} = t \inf\limits_{\beta>0} \frac{f(\beta x)}{\beta}$

$= t\, g(x).$

$t=0$: $g(0) = 0$.

(b). $h(x) = \frac{h(tx)}{t} \leq \frac{f(tx)}{t}$ $\forall t>0.$

$h(x) \leq \inf\limits_{t>0} \frac{f(tx)}{t} = g(x)$

(c) * perspective function of f : $h(x,t) = t f(x/t)$ is convex.

$g(x) = \inf\limits_{\alpha>0} \frac{f(\alpha x)}{\alpha} = \inf\limits_{\alpha>0} \alpha f(x/\alpha) = \inf\limits_{\alpha>0} h(x,\alpha)$. is convex.

Q3. Find conjugate of following functions.

(a) Max function. $f(x) = \max_{i=1,2,\cdots,n} x_i$ on R^n

(b) Sum of largest elements. $f(x) = \sum_{i=1}^{r} x_{[i]}$ on R^n

(c). Piece-wise linear function on R. $f(x) = \max_{i=1,2,\cdots,m} (a_i x + b_i)$ on R. $a_1 \le a_2 \le \cdots \le a_m$. $\forall k\ \exists x$ s.t. $f(x) = a_k x + b_k$

(d) Power function. $f(x) = x^p$ on R_{++} for $p>1$.

(e). Neg. Geometric mean. $f(x) = -(\prod x_i)^{1/n}$ on R^n_{++}.

(f). Neg generalized log for Second order cone. $f(x,t) = -\log(t^2 - x^T x)$ on $\{(x,t) \in R^n \times R \mid \|x\|_2 < t\}$.

i.e. the func cannot go to ∞

Conjugate function. $f^*(y) = \sup_{x \in \operatorname{dom} f} \underline{(y^T x - f(x))}$ $f^*(y)$ is defined iff sup exists.

(a). $f(x) = \max_{1\le i\le n} x_i$. if $y_k < 0$. take $x = t \cdot e_k$. $t<0$

$y^T x = t y_k$. $f(x) = 0$. $\Rightarrow$ $y^T x - f(x) = t y_k$. $t \to -\infty$.

$y^T x - f(x) \to +\infty$.

$\therefore$ if $y \prec 0$. then $f^*(y) = +\infty$. Now we consider $y \succeq 0$

$\sup\ y_1 x_1 + y_2 x_2 + \cdots + y_n x_n - \max_{1\le i\le n} x_i$

idea: all $x_i \to +\infty$. $\underline{(y_1 + \cdots + y_n)} \cdot \infty - \infty$.

$-\max x_i$

(we attempt to upper bound): $\le y_1 \cdot \max x_i + y_2 \max x_i + \cdots + y_n \max x_i$

$= (\Sigma y_i - 1) \max x_i$

if $\Sigma y_i - 1 > 0$. then this upper bound is $\underline{(\text{some positive value}) \cdot (\max x_i)}$

which can $\to \infty$ when $x_i \to \infty$. In fact, since the inequality can take equality when $x_1 = x_2 = \cdots = x_n$, the obj can go to ∞ when $x_i \to \infty$. $\therefore$ $\sup = \infty$

if $\Sigma y_i - 1 < 0$. then the upper bound can go to ∞ when $x_i \to -\infty$. Similarly, since the inequality is tight, this means the obj $\to \infty$ when $x_i \to -\infty$.

$\therefore$ $\sup = \infty$

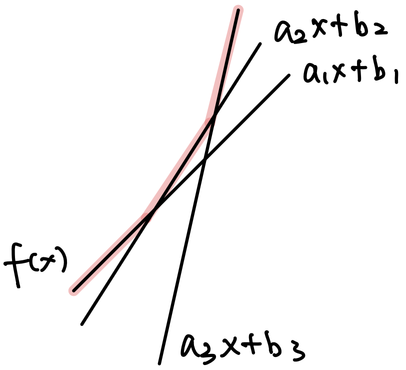
if $\sum y_i - 1 = 0$. upper bound $= 0$. it's tight since when $x_1 = \cdots = x_n = 0$,

obj $= 0$ $\quad \therefore \sup = 0$.

To summarize: $f^*(y) = \begin{cases} 0 & \text{if } 1^T y = 1 \text{ and } y \succeq 0. \\ \infty & \text{elsewhere} \end{cases}$

(b). Very similar to part (a).

(c). $f(x) = \max_i a_i x + b_i$ $\quad a_1 \le a_2 \le \cdots \le a_m$. $\forall k.\ \exists x$ s.t. $f(x) = a_k x + b_k$



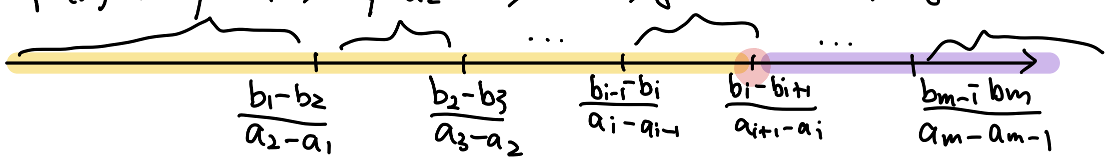
the part with largest a is the last part.

the part with smallest a is the first part.

$$f(x) = \begin{cases} a_1 x + b_1 & \text{for } x \in \left(-\infty, \frac{b_1 - b_2}{a_2 - a_1}\right] \\ a_2 x + b_2 & \text{for } x \in \left[\frac{b_1 - b_2}{a_2 - a_1}, \frac{b_2 - b_3}{a_3 - a_2}\right] \\ \vdots & \\ a_m x + b_m & \text{for } x \in \left[\frac{b_{m-1} - b_m}{a_m - a_{m-1}}, \infty\right) \end{cases}$$

$\sup_x (\underbrace{xy - f(x)}_{g(x)})$ take direvative of :

$g'(x) = y - f'(x) = y - a_1 > y - a_2 > \cdots > y - a_i > \cdots > y - a_m$

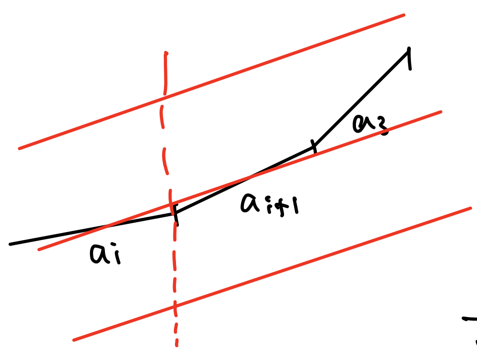


if $y - a_m > 0$. then $g'(x) > 0$ for all x. $\Rightarrow \sup_x (g(x)) = g(\infty) = \infty$

if $y - a_1 < 0$. then $g'(x) < 0$ for all x $\Rightarrow \sup_x (g(x)) = g(-\infty) = \infty$.

if $y \in [a_1, a_m]$. assume $y \in [a_i, a_{i+1}]$ then $y - a_1 > \cdots > y - a_i > 0 > y - a_{i+1} > \cdots > y - a_m$

$\therefore g$ increase on $\left(-\infty, \frac{b_i - b_{i+1}}{a_{i+1} - a_i}\right)$ and decrease on $\left(\frac{b_i - b_{i+1}}{a_{i+1} - a_i}, +\infty\right)$



$\sup_x (xy - f(x))$ is taken when

$x = \frac{b_{i+1} - b_i}{a_{i+1} - a_i}$.

$$f^*(y) = -b_i - (b_{i+1} - b_i) \frac{y - a_i}{a_{i+1} - a_i}$$

(d) $f(x) = x^p \quad x \in R_{++}. \quad p > 1.$

$f^*(y) = \sup_x (xy - x^p) \qquad (xy - x^p)' = y - px^{p-1}$

if $y < 0$ then derivative < 0 for all $x \in R_{++}$. sup is taken when $x \to 0$.

$xy - x^p = 0 \quad . \quad f^*(y) = 0.$

if $y > 0$. then derivative < 0 when $y < px^{p-1}$ i.e. $x > (\frac{y}{p})^{\frac{1}{p-1}}$

> 0 when $x < (y/p)^{\frac{1}{p-1}}$

sup taken at $x = (y/p)^{\frac{1}{p-1}}$.

$f^*(y) = (p-1)\,(y/p)^{p/(p-1)}$

(e) $f(x) = -(\prod x_i)^{1/n}$

$\sup_x \left(x^T y + (\prod x_i)^{1/n} \right)$

if $y_k > 0$. $\Rightarrow y^T x + (\prod x_i)^{1/n} \to \infty$.

$\therefore y_k \le 0$. for all k. $y_1 x_1 + \cdots + y_k x_k + (\prod x_i)^{1/n}$.

We try to upper bound the obj

AM-GM: $\dfrac{-y_1 x_1 - y_2 x_2 - \cdots - y_k x_k}{n} \geq \left((-y_1)(-y_2)\cdots(-y_n) \prod x_i \right)^{\frac{1}{n}}$

$= (\prod -y_i)^{\frac{1}{n}} \cdot GM.$

$\therefore x^T y \le -n(\prod -y_i)^{\frac{1}{n}} \cdot GM.$

$\therefore x^T y + (\prod x_i)^{\frac{1}{n}} \le \left(1 - n(\prod -y_i)^{\frac{1}{n}}\right) GM.$

if $(\prod(-y_i))^{\frac{1}{n}} < \frac{1}{n}$ then the upper bound is $+ \bigcirc \cdot GM$. Since AM-GM can take equality when $y_1 x_1 = y_2 x_2 = \cdots = y_n x_n$. This implies that the obj can go infinity when $x_i = \frac{t}{y_i}$ and $t \to +\infty$.

if $(\prod(-y_i))^{\frac{1}{n}} > \frac{1}{n}$. then since $x \in R^n_{++}$. $GM > 0$. $\Rightarrow$ obj ≤ 0.

this is tight, e.g. let $x_i = \frac{t}{y_i}$ and $t \to 0^+$. $\therefore \sup = 0$

if $(\prod(-y_i))^{\frac{1}{n}} = \frac{1}{n}$. then upper bound $= 0$. This is tight, i.e. let $x_i = \frac{t}{y_i}$ and obj $= 0$ (since $\le$ becomes $=$)

$$\therefore f^*(y) = \begin{cases} 0. & \text{if } y \le 0 \text{ and } (\prod(-y_i))^{\frac{1}{n}} \geq \frac{1}{n}. \\ \infty. & \text{elsewhere} \end{cases}$$

(f). $f(x,t) = -\log(t^2 - x^Tx) \quad x \in R^n.\ t \in R.\ \|x\|_2 < t$

$$f^*(y,u) = \sup_{x,t} \underbrace{y^Tx + ut + \log(t^2 - x^Tx)}_{g(\cdot)}$$

first. $u<0$. otherwise $t=\infty$. $y^Tx+ut+\log(t^2-x^Tx) = +\infty$. when $x=\vec{0}$.

then, we attempt to upper bond g.

$x_1y_1 + x_2y_2 + \cdots + x_ny_n \leq \|x\| \ \|y\| = t\|y\|$ Cauthy

$\therefore g \leq t\|y\| + ut + \log(\quad) = t(\|y\|+u) + \log(\quad)$

if $\|y\| \geq -u$. then the upper bound can go infinity when $t \to \infty$. Since the inequality can take equality (check the equal condition for Cauthy inequality), this also means g can go infinite for some (t, x). $\therefore \sup = \infty$

$\therefore$ for $\sup < \infty$, we must have: $u<0$ and $\|y\| < -u$

$$\Rightarrow \quad y^Tx + tu + \log(t^2 - x^Tx) \xRightarrow{\frac{\partial g}{\partial x} = \frac{\partial g}{\partial t} = 0} x = \frac{2y}{u^2 - y^Ty}. \quad \& \quad t = -\frac{2u}{u^2 - y^Ty}$$

You can verify that given the conditions we just solves, this maximizer is in the domain of f.

Q6. Show that $f(x) = -x_1x_2$ $\text{dom} f = R^2_{++}$ is quasiconvex.

using the statement: $f: R^n \to R$ is diff-able f is quasiconvex iff $\text{dom} f$ is convex and $\forall\, x, y \in \text{dom} f$.

$$f(y) < f(x) \Rightarrow \nabla f(x)^T (y-x) \le 0.$$

first order condition:

$$\partial f / \partial x_1 = -x_2 \qquad \frac{\partial f}{\partial x_2} = -x_1 \qquad \nabla f(x) = \begin{bmatrix} -x_2 \\ -x_1 \end{bmatrix}$$

$$\therefore \quad \nabla f(x)^T (y-x) = \begin{bmatrix} -x_2 \\ -x_1 \end{bmatrix}^T \cdot \begin{bmatrix} y_1 - x_1 \\ y_2 - x_2 \end{bmatrix} = -x_2y_1 + x_1x_2 - x_1y_2 + x_1x_2$$

WT: if $-y_1y_2 < -x_1x_2$ then $2x_1x_2 \le x_1y_2 + x_2y_1$

WT: $x_1x_2 < y_1y_2 \Rightarrow 2x_1x_2 \le x_1y_2 + x_2y_1$

$$(x_1y_2 + x_2y_1)^2 \geq 4x_1x_2y_1y_2 \geq 4(x_1x_2)^2$$

$\therefore \quad x_1y_2 + x_2y_1 \geq 2x_1x_2.$ True.