DSA3102: Definite Matrices

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In this document, we explain what is a positive definite and positive semidefinite matrix and how to check if a symmetric matrix is positive definite or positive semidefinite.

All matrices below are square, contain real entries and are symmetric.

Definition 1. *A matrix M is* positive definite *if $x^T Mx > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.*

Definition 2. *A matrix M is* positive semidefinite *if $x^T Mx \geq 0$ for all $x \in \mathbb{R}^n$.*

We will not spend much time on negative definite and negative semidefinite matrices. It should be understood that M is negative (semi)definite if and only if $-M$ is positive (semi)definite.

Proposition 1. *The following are equivalent.*

1. *M is positive definite;*
2. *All eigenvalues of M are positive;*
3. *All upper left minors of M are positive;*
4. *$M = V^T V$ for some non-singular (full rank) $V \in \mathbb{R}^{n\times n}$ (this is called the Cholesky decomposition).*

What is a *upper left minor*? For a 2×2 matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{1}$$

the upper left minors are the determinants of the $(1, 1)$ entry and the entire matrix, i.e., a and $ad - bc$ respectively. For a 3×3 matrix,

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \tag{2}$$

the upper left minors are the determinants of the $(1, 1)$ entry, the 2×2 submatrix at the top left, and the entire matrix, i.e., a and $ae - bd$ and $a(ei - fh) - d(bi - ch) + g(bf - ce)$ respectively.

For example, the matrix

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{3}$$

is positive definite. Its eigenvalues are 1 and 3. Can you find what V is?

Proposition 2. *The following are equivalent.*

1. *M is positive semidefinite;*
2. *All eigenvalues of M are nonnegative;*
3. *All* **principal** *minors of M are nonnegative;*

4. $M = V^T V$ for **some** *V (may be singular, may even not be square).*

I need to describe what *principal minors* are. For a 2×2 matrix, the principal minors are the determinants of the $(1,1)$ entry, the $(2,2)$ entry and the entire matrix. We need to check one more thing (the determinant of the $(2,2)$ entry) compared to the positive definite case. For example,

$$M = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

is such that all upper left minors are nonnegative (all zeros) but it is not positive semidefinite. There is one principle minor in the bottom right corner (at the $(2,2)$ location) that is negative (-1). In fact, the matrix is negative semidefinite as the eigenvalues are 0 and -1. How many principal minors do you have to check for the 3×3 matrix in (2)? Here they are.

$$a, \quad e, \quad i, \quad \det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right), \quad \det\left(\begin{bmatrix} a & c \\ g & i \end{bmatrix}\right), \quad \det\left(\begin{bmatrix} a & b \\ d & e \end{bmatrix}\right), \quad \det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) \tag{4}$$

The first three are obtained by deleting 2 rows and 2 columns of the same indices. The next 3 are obtained by deleting 1 row and 1 column of the same index. The final determinant is obtained by considering the entire matrix, i.e., not deleting any row and column.