DSA3102: Lecture 3 (Section 3.1, 3.2)

Domain: length n vector

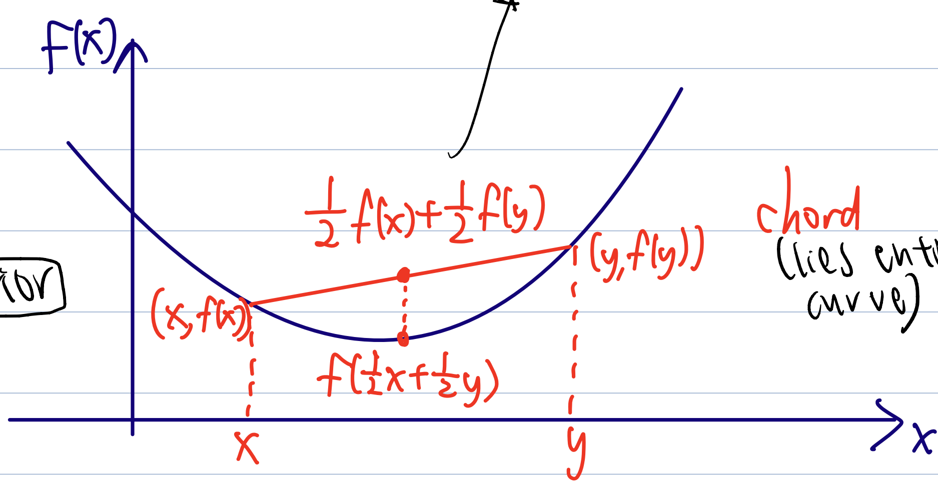
Def: A function $f: \mathbb{R}^n \to \mathbb{R}$ is convex if (i) dom f is a convex set and

Range: returns a number

ii) $\forall x, y \in \text{dom} f$ & $\forall \theta \in [0,1]$ chord lies above function

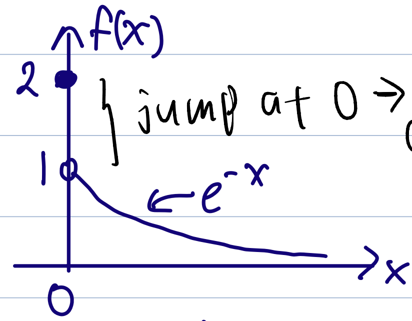
$$f(\theta x + (1-\theta) y) \leq \theta f(x) + (1-\theta) f(y). \quad \text{———} \quad (*)$$

convex function continuous $\boxed{\text{on its interior}}$



chord (lies entirely above curve)

Consider the f^n $f: [0,\infty) \to \mathbb{R}$
not continuous



jump at 0 → not continuous

Def: A function $f: \mathbb{R}^n \to \mathbb{R}$ is <u>strictly convex</u> if

$$f(\theta x + (1-\theta) y) < \theta f(x) + (1-\theta) f(y).$$

for all $\theta \in (0,1)$ & $x \neq y$.

convex but not strictly convex
↳ affine function

$\forall x, y \in \operatorname{dom} f$

Def: f is (strictly) concave if $-f$ is (strictly) convex.

First-order condition

$f : (a,b) \to \mathbb{R}$

f is <u>diffble at $x \in (a,b)$</u> if $\lim_{\varepsilon \to 0} \dfrac{f(x+\varepsilon) - f(x)}{\varepsilon}$ exist.

& we will then denote the derivative at x as

$$f'(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

f is <u>diff^{ble} on (a,b)</u> if f is diff^{ble} $\forall x \in (a,b)$.

differentiable component wise

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is diff^{ble} (on dom f), i.e., ∇f exists at every point in dom f, $\nabla f = \left(\dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_n} \right)^T$ gradient vector

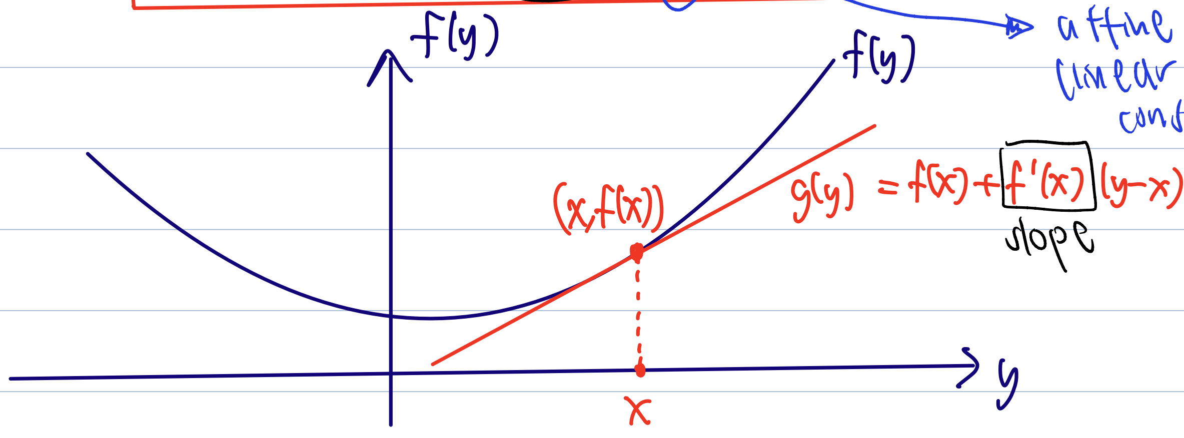
one time differentiable

$f'(x)$

then f is convex $\iff$ dom f is convex &

$$\boxed{f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \forall x, y \in \operatorname{dom} f}$$

affine in y (linear and constant everywhere else)

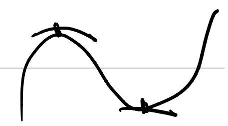


The affine f^n $g(y) = f(x) + \nabla f(x)^T (y - x)$ is a first-order always below

Taylor approx of f at x that underestimates $f(y)$.

Rmk: If f is convex & diff^{ble} & $x_0 \in \text{dom} f$ satisfies

gradient $\nabla f(x_0) = 0$

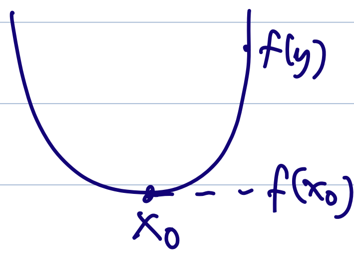


then the first-order optimality condition yields

$$f(y) \geq f(x_0) \qquad \forall y \in \text{dom} f$$

stationary point found by differentiating and setting to 0

gradient vanishes (set to 0)



$\Rightarrow$ x_0 is a global min of f

this is not convex

For convex f^n every local min is a global min.

Rmk: f is strictly convex iff (i) dom f is convex &

$$f(y) > f(x) + \nabla f(x)'(y-x) \qquad \forall x, y \in \text{dom} f, \; x \neq y$$

Rmk: $x_0 \in \text{dom} f$ s.t. $\nabla f(x_0) = 0$ & f is strictly convex,

then x_0 is <u>the</u> global minimum.

Rmk: $f: \mathbb{R}^n \to \mathbb{R}$ is convex iff $\forall x \in \text{dom} f$ and all directions

v, $g(t) = f(x+tv)$ is convex in $t \in \{t \in \mathbb{R} : x+tv \in \text{dom} f\}$

function of t

↳ all possible shifts from a fixed x (within domain of f)

proof 2 direction

Proof of first-order optimality condition:

(for one dimension)

NTS: If $f: \mathbb{R}^1 \to \mathbb{R}$ with dom f convex & f diff^{ble} on dom f,

then f is convex iff

$$f(y) \geq f(x) + f'(x)(y-x) \quad \forall\, x, y \in \text{dom} f$$

($\Rightarrow$) Assume f is convex. Take any $x, y \in \text{dom} f$ ($x \neq y$) and any $t \in (0,1]$. Then $(1-t)x + ty \in \text{dom} f$ and by the convexity of f

convex combo parameter

can put in function

$$f((1-t)x + ty) \leq (1-t) f(x) + t f(y)$$

$f\{(1-t)x + ty\}$
$\leq f(x) - tf(x) + tf(y)$

$$f(y) \geq f(x) + \frac{1}{t}\left[f((1-t)x + ty) - f(x)\right]$$

$$= f(x) + \frac{y-x}{t(y-x)}\left[f(x + t(y-x)) - f(x)\right]$$

$$= f(x) + (y-x) \cdot \underbrace{\frac{1}{t(y-x)}\left[f(x + t(y-x)) - f(x)\right]}_{f'(x)}$$

Now take $t \downarrow 0$.

definition of derivative

$$f'(x_0) = \lim_{s \downarrow 0} \frac{f(x_0 + s) - f(x_0)}{s}$$

$$f(y) \geq f(x) + (y-x) f'(x) \qquad ///.$$

($\Leftarrow$) Assume that f satisfies the 1st order opt condition.

WTS $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$

Pick $x, y \in \text{dom} f$, $x \neq y$ & $\theta \in [0,1]$. Let $z = \theta x + (1-\theta) y \in \text{dom}(f)$

everything that is placed inside f must be inside $\mathrm{dom}(f)$

$$\left.\begin{aligned} \theta f(x) &\geq \theta f(z) + \boxed{\theta f'(z)(x-z)} \\ (1-\theta) f(y) &\geq (1-\theta) f(z) + \boxed{(1-\theta) f'(z)(y-z)} \end{aligned}\right\} +$$

$$\Downarrow$$

$$\theta f(x) + (1-\theta) f(y) \geq f(\theta x + (1-\theta) y) \qquad ///$$

$$\theta(x-z) + (1-\theta)(y-z) = \underline{\theta x + (1-\theta) y} - \underline{z} = 0.$$

Second-order condition $\quad f: \mathbb{R}^n \to \mathbb{R}$.

Assume f is twice diff'ble, i.e., its Hessian exists at each point in dom f.

matrix of 2nd order derivatives

$$\underset{n \times n}{\nabla^2 f(x)} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ & \ddots & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \quad i, j \in [n].$$

e.g. $f(x) = 4x^2 - 6x + 7$

$f''(x) = 8 > 0$ (positive semi-definite)

f is convex iff (i) dom f is convex

(ii) $\nabla^2 f(x)$ is positive semidefinite $\forall x \in \mathrm{dom} f$

, i.e., $\forall v \in \mathbb{R}^n$, $\underbrace{v^T \nabla^2 f(x) v}_{\text{number}} \geq 0$.

Proof of second-order opt. condition for $n=1$.

Using the first-order opt conditions for $x, y \in \mathrm{dom} f$, $x < y$ (prevent division by 0)

$$f(y) \geq f(x) + f'(x)(y-x)$$

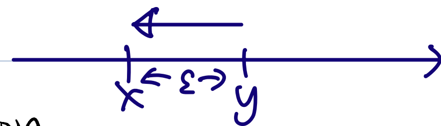
$$f(x) \geq f(y) + f'(y)(x-y)$$

$f'(x)(y-x) \leq f(y) - f(x) \leq f'(y)(y-x)$

Divide both sides by $(y-x)^2$

$$\frac{f'(x)}{y-x} \leq \frac{f(y)-f(x)}{(y-x)^2} \leq \frac{f'(y)}{y-x}$$

$$\Rightarrow \quad \frac{f'(y)-f'(x)}{y-x} \geq 0$$ (use endpoints)



Take $y \to x^+$ treat f' as its own function

$\forall x \in \text{dom} f \quad f''(x) \geq 0$ ///

anything that is twice differentiable is 1 time differentiable

($\Leftarrow$) Assume $f''(x) \geq 0 \ \forall x \in \text{dom} f$.

satisfies

WTS: f is convex, and in particular, 1st order opt cond.

⇕ equivalent

f is convex

By the mean-value theorem (eg. Taylor's Thm)

$$f(y) = f(x) + f'(x)(y-x) + \underbrace{\tfrac{1}{2} \underbrace{f''(z)}_{\geq 0} \underbrace{(y-x)^2}_{\geq 0}}_{\geq 0}$$

for some z between x and y.

$$\Rightarrow \quad f(y) \geq f(x) + f'(x)(y-x)$$ ///.

Ex: Quadratic Functions. $f: \mathbb{R}^n \to \mathbb{R}$. symmetric matrix

$f(x) = \frac{1}{2} x^T P x + q^T x + r \qquad P \in S^n, \ q \in \mathbb{R}^n, \ r \in \mathbb{R}$.

$\nabla f(x) = Px + q$ (Matrix Cookbook)

$\nabla^2 f(x) = P$

matrix of 2nd order derivative

$\Rightarrow$ f is convex if P is PSD.

f is concave if P is negative semidefinite.

f is strictly convex if P is $\underline{PD}$ ($P \succ 0$)
+ve definite

positive definite: $\forall v \in \mathbb{R}^n \setminus \{0\}$, $\quad v^T P v > 0$.

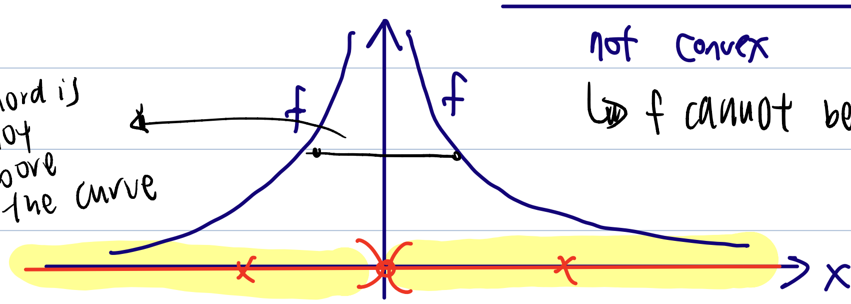
Example: $\quad f(x) = \frac{1}{x^2}$ for $\underline{\text{dom} f = \mathbb{R} \setminus \{0\}}$.

not convex

$\hookrightarrow$ f cannot be convex

on the real line, convex sets are definite intervals

chord is not above the curve



two seperate (disjoint) intervals cannot be convex

$f'(x) = -2/x^3$, $\quad f''(x) = 6/x^4 > 0$. $\forall x \neq 0$, i.e., $\forall x \in \text{dom} f$.

2nd order optimality condition holds $\Rightarrow$ but still not convex as dom(f) is not convex

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set $\quad \text{dom} f = \mathbb{R} \setminus \{0\}$

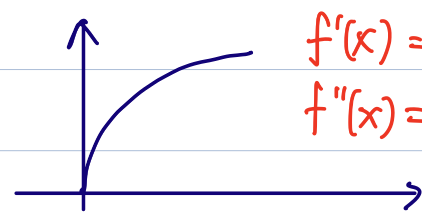
counter example $\Rightarrow$ $x, y = 1, -1 \qquad \theta = \frac{1}{2}$, $\quad z = \theta x + (1-\theta) y = 0 \overset{??}{\in} \text{dom} f \Rightarrow$ NO

Examples:

Powers: $f(x) = x^a$ is convex on $\mathbb{R}_{++}$ if $a \geq 1$, $a \leq 0$
positive numbers

concave if $a \in [0, 1]$

$a = \frac{1}{2}$, $\quad f(x) = \sqrt{x}$



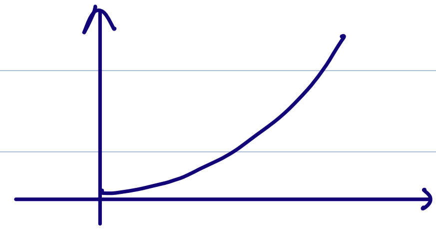
$f'(x) = \frac{1}{2} x^{-1/2}$

$f''(x) = -\frac{1}{4} x^{-3/2} < 0$

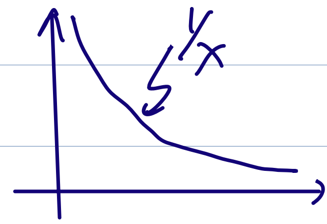
as x is positive

differentiate twice to prove convex/concave

$a = 2 \quad f(x) = x^2$

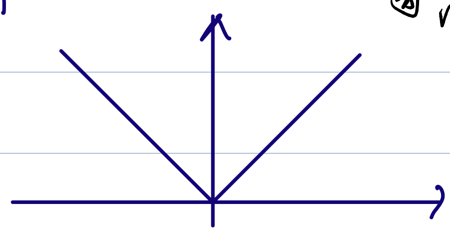


$a = -1, \ f(x) = 1/x$ on $\mathbb{R}_{++}$



Powers of absolute values : $f(x) = |x|^p$ on $\mathbb{R}$, $p \geq 1$.

$p = 1 \quad f(x) = |x| \rightarrow$ not differentiable at 0



check by observation

Logarithm: $f(x) = \log x$, $x \in \mathbb{R}_{++}$ is concave on $\mathbb{R}_{++}$

$f'(x) = \frac{1}{x}, \ f''(x) = -\frac{1}{x^2} < 0.$

Negative Entropy : $f(x) = \underbrace{\sum_{i=1}^{n} x_i \log x_i}_{x_1 \log x_1 + x_2 \log x_2 + \cdots}$, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $x \in \mathbb{R}^n_{++}$ (not 0)

Claim: f is convex.

$[\nabla f(x)]_1 = \frac{\partial f}{\partial x_1} = x_1 \frac{1}{x_1} + \log x_1 = 1 + \log x_1$

$[\nabla f(x)]_i = 1 + \log x_i$

differentiating $1+\log x_i$ wrt i

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \begin{cases} 1/x_i & i=j \\ 0 & i\neq j \end{cases}$$

Hessian $\nabla^2 f(x) = \begin{bmatrix} 1/x_1 & & 0 \\ & \ddots & \\ 0 & & 1/x_n \end{bmatrix} \succ 0.$

diagonal matrix is positive definite $\Leftrightarrow$ all the components on diagonal are positive

Strictly convex. (PD)

Max function: $f(x) = \max_{1\le i\le n} x_i$, $x \in \mathbb{R}^n$ is convex.

have things that are non-differentiable so cannot use first/second order conditions

fix $x, y \in \mathbb{R}^n$, $\theta \in [0,1]$

$$\begin{aligned} f(\theta x + (1-\theta) y) &= \max_{1\le i\le n} \{\theta x_i + (1-\theta) y_i\} \\ &\le \max_i \theta x_i + \max_i (1-\theta) y_i \\ &= \theta \max_i x_i + (1-\theta) \max_i y_i \\ &= \theta f(x) + (1-\theta) f(y) \end{aligned}$$

////

scalars does not affect max

Note: $\max_x \{f_1(x) + f_2(x)\} \le \max f_1(x) + \max f_2(x)$

Quadratic over linear

$f(x,y) = x^2/y$ $\quad \text{dom} f = \{(x,y) \in \mathbb{R}^2 : y > 0\}$

$\nabla f(x,y) = \begin{bmatrix} 2x/y \\ -x^2/y^2 \end{bmatrix}$, $\quad \nabla^2 f(x,y) = \begin{bmatrix} 2/y & -2x/y^2 \\ -2x/y^2 & +2x^2/y^3 \end{bmatrix}$

gradient vector

f_{xx} $\quad$ f_{yx}

f_{xy} $\quad$ f_{yy}

check this one

check if its true

$2/y > 0$ $\quad \det(\nabla^2 f(x,y)) = (2/y)(+2x^2/y^3) - \left(-\frac{2x}{y^2}\right)^2$

$= 0$

Hessian has null space at dim 1 $\Rightarrow$ cannot be positive definite

$\nabla^2 f(x,y) \succeq 0 \quad \forall (x,y) \in \text{dom} f$

Hence f is convex. (not strictly convex).

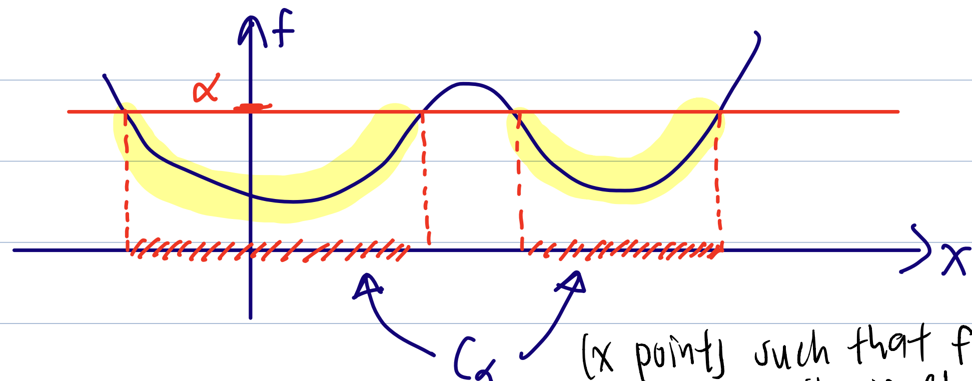
$f(x) = \underline{a}^T \underline{x} + b \qquad f: \mathbb{R}^n \to \mathbb{R}.$ Affine functions are both convex and concave.

$\nabla f(x) = \underline{a}, \quad \underset{n\times n}{\nabla^2 f(x)} = \underline{\underline{0}} \left.\begin{matrix} \succeq 0 \\ \preceq 0 \end{matrix}\right\} \begin{matrix} \text{convex} \\ \text{concave} \end{matrix}$

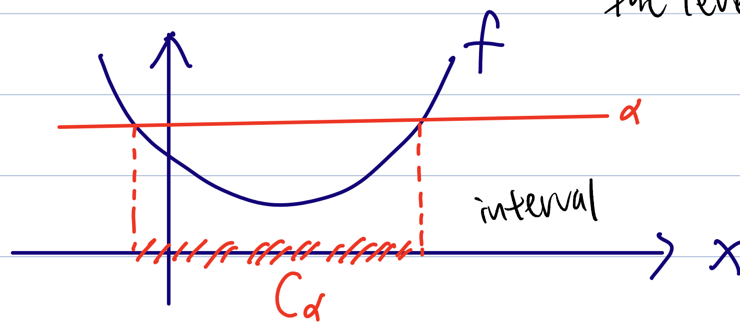
Sublevel Sets.

$f: \mathbb{R}^n \to \mathbb{R}$, its α-sublevel set

$C_\alpha = \{x \in \text{dom} f : f(x) \le \alpha\}$



(x points such that f lies below the level α)



Fact: If f is convex, then $C_\alpha = \{x \in \text{dom} f : f(x) \le \alpha\}$ is convex $\forall \alpha \in \mathbb{R}$.

dom(f) is convex

Pf: $x, y \in C_\alpha$. WTS: $\theta x + (1-\theta) y \in C_\alpha$.

$\Downarrow$

$f(x) \le \alpha,\ f(y) \le \alpha$

$$f(\theta x + (1-\theta) y) \le \theta f(x) + (1-\theta) f(y)$$
$$\le \theta \alpha + (1-\theta)\alpha = \alpha$$

$\Rightarrow \theta x + (1-\theta) y \in C_\alpha$

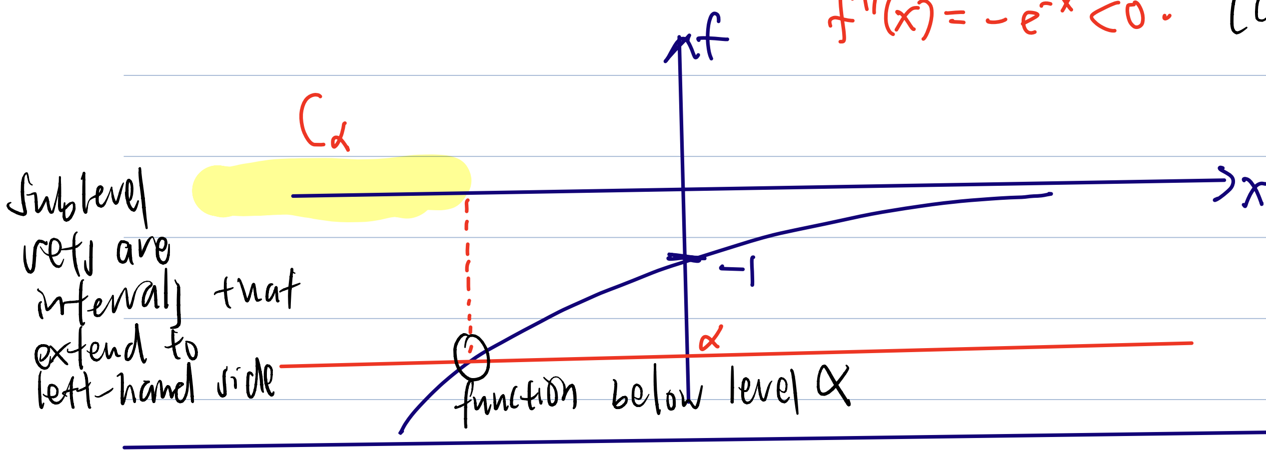
The converse is not true.

$\Downarrow$

If all α-sublevel sets are convex sets, f may not be a convex fn.

$f(x) = -e^{-x}$ $\quad x \in \mathbb{R}$ $\quad f'(x) = e^{-x}$

$f''(x) = -e^{-x} < 0$. (concave)



Epigraph:

The graph of a function $f: \mathbb{R}^n \to \mathbb{R}$ is the set of points

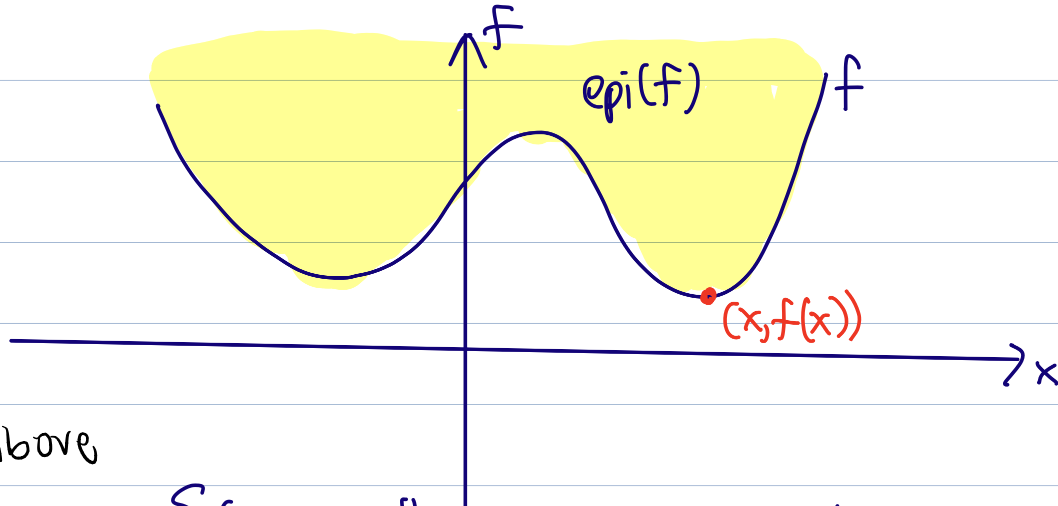
$$\{(x, f(x)) : x \in \operatorname{dom} f\} \subseteq \mathbb{R}^{n+1}$$

$\cap \quad \cap$

$\mathbb{R}^n \quad \mathbb{R}$.

cartesian product of

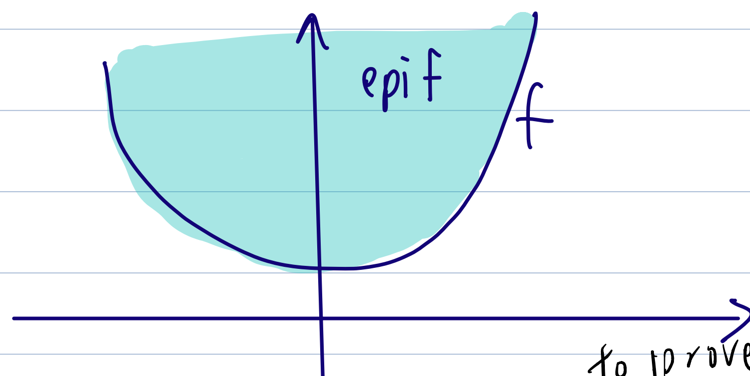
$\mathbb{R}^n$ and $\mathbb{R}$



epi ⇒ above

$epi(f) = \{(x,t) \in \mathbb{R}^{n+1} : x \in dom f, f(x) \le t\}$.

Thm: f is convex function $\Leftrightarrow$ epi(f) is a convex set.



Pf:

to prove convex set: take 2 elements and prove convex combi inside the set

($\Rightarrow$) Pick $(x,t), (x',t') \in epi(f)$.

$f(x) \le t, \quad f(x') \le t'$

NTS: $\forall \theta \in [0,1]$, $\theta(x,t) + (1-\theta)(x',t') \in epi(f)$.

$f(\theta x + (1-\theta)x') \le \theta f(x) + (1-\theta) f(x')$ ⇐ f is convex function

$\le \theta t + (1-\theta)t'$

$\Rightarrow \theta(x,t) + (1-\theta)(x',t') \in epi(f)$ as desired.

($\Leftarrow$) Now suppose epi(f) is a convex set.

i.e., $(x,t) \in \text{epi}(f)$, $(x',t') \in \text{epi}(f)$

$$\theta(x,t) + (1-\theta)(x',t') \qquad \forall \theta \in [0,1]$$

$$= (\theta x + (1-\theta)x', \theta t + (1-\theta)t') \in \text{epi}(f).$$

convex combi also in epi(f)

By the def of epi(f), $f(x) \le t$, $f(x') \le t'$ → comes from the fact that these 2 are in epi(f)

$$f(\theta x + (1-\theta)x') \le \underbrace{\theta t + (1-\theta)t'}_{\text{scalar}} \quad -(*)$$

$t \geq f(x) \Leftrightarrow (x,t) \in \text{epi}(f)$

Now choose $t = f(x)$, $t' = f(x')$, which yields from $(*)$

$$f(\theta x + (1-\theta)x') \le \theta f(x) + (1-\theta) f(x')$$

$\Rightarrow f$ is a convex f^n ////.

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convex intersect convex → convex

Operations that preserve convexity

Conic combination of convex functions

→ is still convex

$f_1, \ldots, f_k$ are convex f^n, then $\sum_{i=1}^{k} w_i f_i$ is convex

if $w_i \ge 0$ $\forall i \in [k]$.

(it scale by negative, it become concave)

weight

Composition with affine map. (or linear)

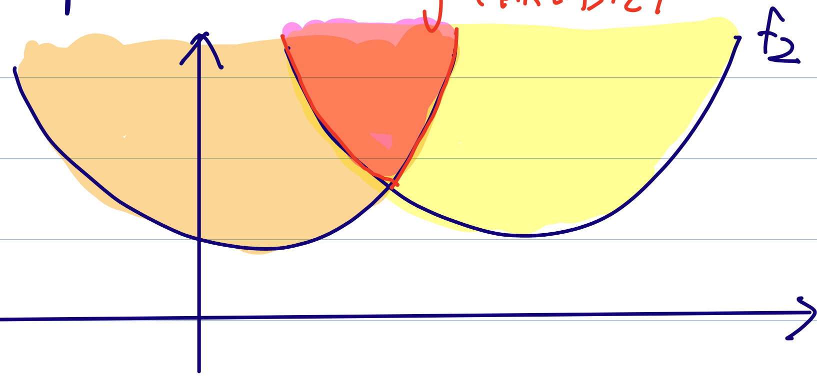
$f: \mathbb{R}^n \to \mathbb{R}$, $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^n$.

$g: \mathbb{R}^m \to \mathbb{R}$ $\qquad g(x) = f(Ax + b)$.

$$\text{dom } g = \{x \in \mathbb{R}^m : Ax + b \in \text{dom } f\}.$$

If f is convex (concave), so is g.

Pointwise max/supremum



epi(f_1) and epi(f_2) are convex sets

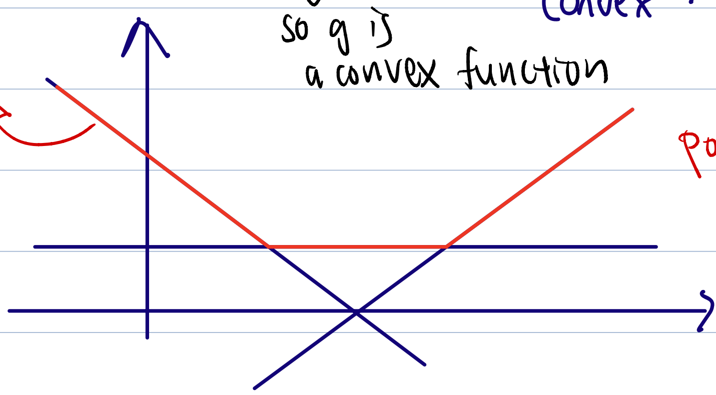
If f_1, f_2 are convex, $g(x) = \max\{f_1(x), f_2(x)\}$ is a convex f^{n} with dom g = dom $f_1 \cap$ dom f_2.

Observe that epi (g) = epi $(f_1) \cap$ epi (f_2) : convex

so g is a convex function

convex

maximum $\downarrow$ also convex



pointwise supremum of convex function is convex

Ex: Sum of the r largest components.

$$f(x) = \sum_{i=1}^{r} x_{[i]} \qquad x \in \mathbb{R}^n, \quad x_{[i]} : i^{th} \text{ largest comp.}$$

$$x_{[1]} \geq x_{[2]} \geq \cdots x_{[n]}.$$

$$f(x) = \max\{x_{i_1} + x_{i_2} + \cdots + x_{i_r} : 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}$$

Pointwise max of $\binom{n}{r}$ linear (convex) functions.

maximum to some extent

Pointwise supremum $\quad f(\cdot, y)$ is a convex f^n. $\quad$ look at $f(x,y)$ as a function of x

freeze y (dot represent argument to g)

If $\forall y \in \mathcal{A}$, $f(x,y)$ is convex in x, then

$$g(x) = \sup_{y \in \mathcal{A}} f(x,y)$$

is convex in x.

each 1 is convex

$$\underbrace{\mathrm{epi}(g)}_{\text{convex}} = \bigcap_{y \in \mathcal{A}} \underbrace{\mathrm{epi}(f(\cdot, y))}_{\text{convex}} \quad \text{convex}$$

Maximum eigenvalue: $\quad X \in S^n$ symmetric matrix $\quad \color{red}{\lambda_{max}(A) = \sup_{y \neq 0} \frac{y^T A y}{y^T y}}$ → DSA4102

$f(X) = \lambda_{max}(X)$: largest eigenvalue → convex

$\color{red}{\text{Rayleigh quotient.}}$

$$f(X) = \sup \{ \overbrace{y^T X y}^{\text{linear function}} : \|y\| = 1 \}$$

This is the pointwise supremum of a family of linear functions in X (indexed by y). Hence f is convex

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Composition $\quad h: \mathbb{R}^k \to \mathbb{R}, \quad g: \mathbb{R}^n \to \mathbb{R}^k$

$f = h \circ g: \mathbb{R}^n \to \mathbb{R}$.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "g"] \arrow[rr, bend right, "f"'] & \mathbb{R}^k \arrow[r, "h"] & \mathbb{R} \end{tikzcd}$$

$f(x) = h(g(x))$ $\quad$ $\text{dom } f = \{x \in \text{dom } g : g(x) \in \text{dom } h\}$

$k=1$ and all f^ns are twice diff^{ble}.

$f(x) = h(g(x))$ $\quad$ use 2nd order optimality condition

chain rule $f'(x) = g'(x)h'(g(x))$

$f''(x) = g'(x)h''(g(x))g'(x) + h'(g(x))g''(x)$

$= \underbrace{h''(g(x))}_{\geq 0}\underbrace{(g'(x))^2}_{\geq 0} + \underbrace{h'(g(x))}_{\geq 0}\underbrace{g''(x)}_{\geq 0} \geq 0.$

Ex: g is convex & h is convex & non-decreasing.

$\Downarrow$ $\qquad$ $\Downarrow$ $\qquad$ $\Downarrow$

$g'' \geq 0$ $\qquad$ $h'' \geq 0$ $\qquad$ $h' \geq 0$.

$\Rightarrow f$ is convex

$e^{g(x)}$

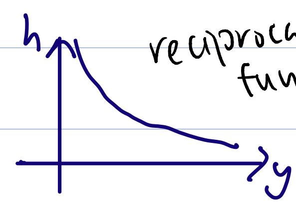
convex function passed through exponential $\Rightarrow$ still convex

Ex: g is convex $\Rightarrow \exp(g(x))$ is convex

Take $h(y) = \exp(y)$: convex & non-decreasing.

Ex: If g is concave & positive, then $1/g(x)$ is convex

$h(y) = 1/y \Rightarrow$ convex $\because y > 0$
decreasing



reciprocal function

$f''(x) = \underbrace{h''(g(x))}_{\geq 0}\underbrace{(g'(x))^2}_{\geq 0} + \underbrace{h'(g(x))}_{\leq 0}\underbrace{g''(x)}_{\leq 0} \geq 0.$

$f(x) = h(g(x))$.

Minimization

also preserves convexity in some sense?

If f is convex in (x,y) & C is a non-empty convex set, then

different conditions from sup case

$$g(x) = \underset{y \in C}{\inf} f(x,y)$$

smallest value

is convex (after minimized) $\text{dom } g = \{x : (x,y) \in \text{dom } f \text{ for some } y \in C\}$.

Pf: Take $x_1, x_2 \in \text{dom } g$. Let $\varepsilon > 0$. Then $\exists\, y_1, y_2 \in C$ s.t.

$$f(x_1,y_1) \le g(x_1) + \varepsilon$$
$$f(x_2,y_2) \le g(x_2) + \varepsilon.$$

inf may not be attained but it is the largest lower bound
$\Rightarrow$ come ε close to the inf

Take $\theta \in [0,1]$

$$g(\theta x_1 + (1-\theta)x_2) = \underset{y \in C}{\inf} f(\theta x_1 + (1-\theta)x_2, y)$$

$y_1, y_2 \in C$

$$\le f(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2)$$

$\in C \because C$ is convex set.

$$\le \theta f(x_1,y_1) + (1-\theta) f(x_2,y_2)$$

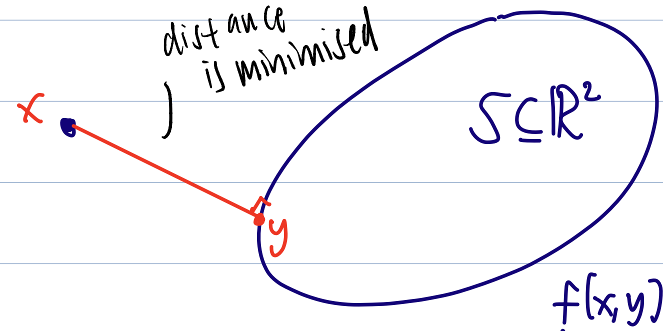
$\because f$ is a convex f^n in (x,y)

$$\le \theta(g(x_1)+\varepsilon) + (1-\theta)(g(x_2)+\varepsilon)$$
$$= \theta g(x_1) + (1-\theta)g(x_2) + \varepsilon$$

Now take $\varepsilon \downarrow 0 \Rightarrow g(\theta x_1 + (1-\theta)x_2) \le \theta g(x_1) + (1-\theta) g(x_2)$ //

g is convex.

Ex: Distance of x to a convex set $S \subseteq \mathbb{R}^n$



convex

$$g(x) = \text{dist}(x,S) = \inf\{\overbrace{\|y - x\|}^{f(x,y)} : y \in S\}$$

convex

Claim: $g(x)$ is convex on $\mathbb{R}^n$. show this is convex

Pf: Note that $\boxed{\|x-y\| = f(x,y)}$ is convex in (x,y).
Fix (x,y) & (x',y') & $\theta \in [0,1]$.

$$f(\overbrace{\theta x + (1-\theta)x'}^{x}, \overbrace{\theta y + (1-\theta)y'}^{y}) \quad \|x-y\|$$
$$= \|(\theta x + (1-\theta)x') - (\theta y + (1-\theta)y')\|$$
$$= \|\theta(x-y) + (1-\theta)(x'-y')\|$$
$$\leq \|\theta(x-y)\| + \|(1-\theta)(x'-y')\|$$
$$= \theta\|x-y\| + (1-\theta)\|x'-y'\|$$
$$= \theta f(x,y) + (1-\theta) f(x',y') \Rightarrow f \text{ is cvx in } (x,y)$$

convex

$$g(x) = \text{dist}(x,S) = \inf\{\|x-y\| : y \in S\}$$

convex f^n in x. convex f^n in (x,y)

f dom f convex $\exists\, \theta \in [0,1]$

$\forall\, x,y \in \text{dom } f$ $\forall\, \theta \in [0,1]$

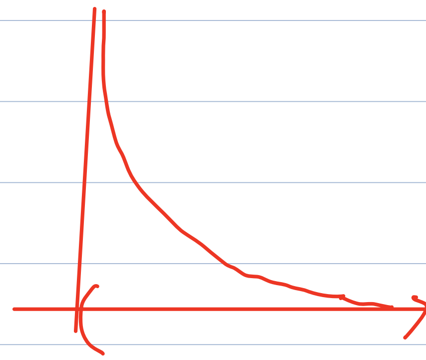
$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

$f(x) = x\log x$ $x > 0$

$f'(x) = x\frac{1}{x} + \log x$

$= 1 + \log x$

$f''(x) = \frac{1}{x} > 0$ $(0,\infty)$



f is convex on $\mathbb{R}_{++}$.