① every local min is a global min
② First order optimality

DSA3102 Lecture 5 (Secs 4.1 – 4.4.1).

Optimization Problems.

objective function

$$\min_x \boxed{f_0(x)}$$

standard form ⇒ RHS = 0

x: decision variable

subject to

s.t. $f_i(x) \le 0 \quad \forall i = 1, \dots, m$ (m inequality constrains)

$h_i(x) = 0 \quad \forall i = 1, \dots, p.$

f_0: objective function

$x \in \text{dom} f \iff f(x) < \infty.$

$f_i(x) \le 0$: i^{th} inequality contraint

$h_i(x) = 0$: i^{th} equality constraint

must be valid ⇒ cannot extend to ∞

domain of the opt. problem

$$\mathcal{D} = \left(\bigcap_{i=0}^{m} \text{dom} f_i\right) \cap \bigcap_{i=1}^{p} \text{dom}\, h_i$$

(make the function the largest/smallest)

Optimal value

$i \in [m]$ $\quad$ $i \in [p]$.

$$p^* = \inf\{f_0(x) : f_i(x) \le 0 \ \forall i = 1, \dots, m, \ h_i(x) = 0 \ \forall i = 1, \dots, p\}$$

primal

smallest value of the objective function subject to the constrains being met

If the problem is infeasible ($\nexists x$ s.t. $f_i(x) \le 0 \ \forall i$ &

Set is empty

does not exist an i that satisfies all the constrains simultaneously

$h_i(x)=0 \ \forall i$), $p^* = +\infty$.
when taking inf of an empty set

sequence

If $\exists (x_k)_{k=1}^{\infty}$ s.t. x_k is feasible & $f_0(x_k) \to -\infty$ as $k \to \infty$, then the problem is unbounded below, $p^* = -\infty$.

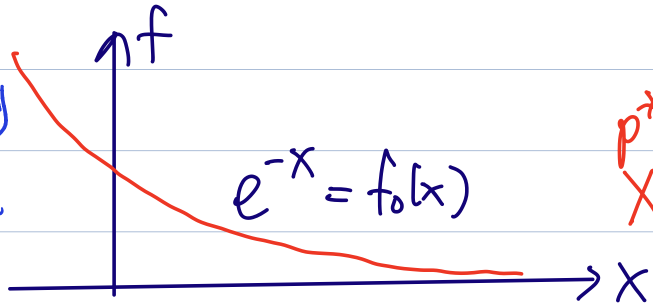
Set of optimal points

$\mathcal{D} \subseteq \mathbb{R}^n$.

feasible inequality constrains

$$X_{opt} = \{x \in \mathcal{D} : f_i(x) \le 0 \ \forall i, \ h_i(x)=0 \ \forall i, \ f_0(x) = p^*\}$$

f is continuous but set is not compact



$p^* = 0$

$X_{opt} = \phi$ (which are the Xs such that when evaluated on $f(x)$ gives $0 \Rightarrow$ NONE!)

If $X_{opt} \neq \phi$, the optimal value is attained or achieved.

Sufficient condition: min a continuous function f_0 over a compact set ($\{x : f_i(x) \le 0 \ \forall i, \ h_i(x)=0 \ \forall i\}$), then $X_{opt} \neq \phi$.
constraint set

A feasible x ($f_i(x) \le 0 \ \forall i$, $h_i(x)=0 \ \forall i$) is $\boxed{\varepsilon\text{-optimal}}$ if $f_0(x) \le p^* + \varepsilon$. $\to$ coming close to the optimum value

x belong to the constrain set (satisfy all the inequality / equality constrains)

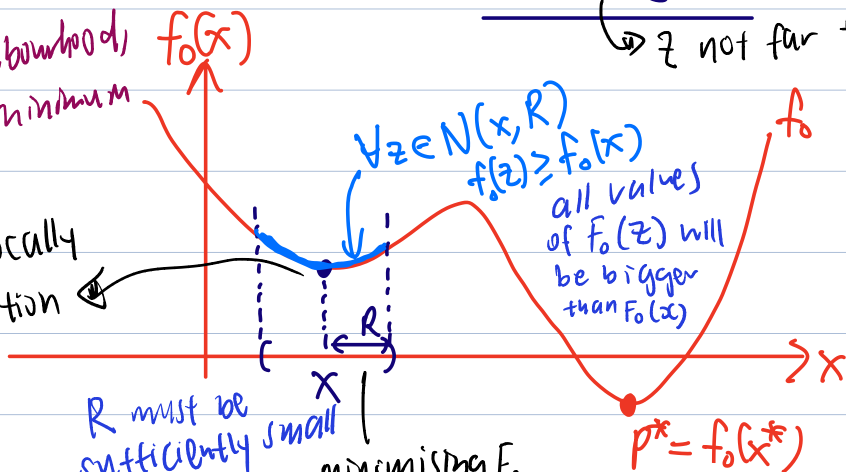
Locally optimal solution

x is locally optimal if $\exists R > 0$ s.t.

$$f_0(x) = \inf\{f_0(z) : f_i(z) \le 0 \ \forall i,\ h_i(z) = 0 \ \forall i,\ \underbrace{\|z - x\|_2 \le R}_{z \text{ not far from } x}\}$$

when restricted to this (small) neighbourhood, x is a global minimum therein

perceieved locally optimal solution



R must be sufficiently small

minimising f_0 subject to me being sufficiently close to X

x solves the optimization prob.

$$\min_z f_0(z) \quad \text{s.t.} \quad f_i(z) \le 0 \ \forall i, \quad h_i(z) = 0 \ \forall i$$
$$\|x - z\|_2 \le R.$$

By definition, all equality constraints are active, or else infeasible

If x is feasible & $f_i(x) = 0$, the i^{th} inequality constraint is <u>active</u>; $f_i(x) < 0$, the i^{th} ineq. constraint is <u>inactive</u>
strict inequality

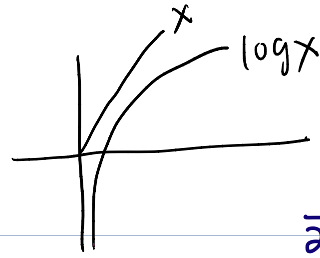
Ex: $\text{dom } f_0 = \mathbb{R}_{++}$ & unconstrained $m = p = 0$.
no constrains

a) $f_0(x) = 1/x$. $p^* = 0$ (inf) but $X_{opt} = \emptyset$ not attained

b) $f_0(x) = -\log x$, $p^* = -\infty \Rightarrow$ prob. is unbounded below
(when x goes to $+\infty$, $\log x$ will go to $+\infty$ and whole thing will be unbounded below $\rightarrow$ go to $-\infty$)

c) $f_0(x) = x \log x$

local min is the global min



2nd derivative

$\frac{1}{x} > 0$: Hessian +ve so strict convexity hence unique global optimum

$f_0(x^*) = p^*$

$= e^{-1} \log e^{-1}$

$= -e^{-1} = p^*$

$X_{opt} = \{ e^{-1} \}.$

$\frac{\partial}{\partial x}(x \log x)$

$= \log x + x \cdot \frac{1}{x} = 0.$

$\log x = -1$

$x^* = e^{-1}$

---

find X that belongs to the constraint set

Feasibility problems.

Find an x that satisfies $f_i(x) \leq 0 \ \forall i, \ h_i(x) = 0 \ \forall i.$

$$\text{find } x \text{ s.t. } f_i(x) \leq 0 \ \forall i, \ h_i(x) = 0 \ \forall i.$$

This can be posed as an optimization problem.

$$\min_x \ \underset{f_0(x)}{\underset{\parallel}{0}} \quad \text{s.t. } f_i(x) \leq 0 \ \forall i, \ h_i(x) = 0 \ \forall i.$$

not making anything small, just finding x that fulfils conditions?

Expressing opt. problem in standard form

Standard form $\Leftrightarrow$ RHS of inequality & equality constraints $= 0$.

(not in standard form)

$$\min_{x \in \mathbb{R}^n} f_0(x) \quad \text{s.t.} \quad l_i \leq x_i \leq u_i \ , \ i = 1, \ldots, k.$$

Rewrite the k constraints as

$x_i \leq u_i \quad \forall i = 1, \ldots, k \qquad -x_i + l_i \leq 0 \quad \forall i = 1, \ldots, k.$

$x_i - u_i \leq 0$

Define $\begin{cases} f_i(x) = -x_i + l_i, & i = 1, \dots, k \\ f_i(x) = x_{i-k} - u_{i-k}, & i = k+1 \dots 2k = m \end{cases}$ inequality constraints

$f_i(x) \le 0 \quad \forall i = 1, \dots, k$ $\qquad$ $f_i(x) \le 0 \quad \forall i = k+1 \dots 2k$

$m = 2k$.

Change of variables

ϕ^{-1} exists (eg. x^2 is not 1-1)

If $\exists\ \phi : \mathbb{R}^n \to \mathbb{R}^n$ and is one-to-one.

$\tilde{f}_i(z) = f_i(\phi(z)) \quad i = 0, 1, \dots, m \qquad x = \phi(z)$.

$\tilde{h}_i(z) = h_i(\phi(z))$

Consider new opt. prob. $\min_z \tilde{f}_0(z)$

s.t. $\tilde{f}_i(z) \le 0\ \forall i$, $\tilde{h}_i(z) = 0\ \forall i$

If x solves the original opt. prob., z solves the transformed opt. problem. Get x by passing through ϕ

Transformation of objective and constraints.

$\psi_0 : \mathbb{R} \to \mathbb{R}$ monotonically $\uparrow$

$\psi_i : \mathbb{R} \to \mathbb{R}$ $\quad \psi_i(u) \le 0$ iff $u \le 0$.

$\psi_{m+1}, \dots, \psi_{m+p} : \mathbb{R} \to \mathbb{R}$ s.t. $\psi_i(u) = 0$ when $u = 0$.

$\left.\begin{array}{l} \tilde{f}_i(x) = \psi_i(f_i(x)), \quad i = 0, 1, \dots, m. \\ \tilde{h}_i(x) = \psi_{m+i}(h_i(x)) \quad \forall i = 1, \dots, p \end{array}\right\}$ create new objective and constraint function

new equality constraint function

equivalently minimize $f_0(x)$ since φ is monotonically increasing

Transformed problem $\quad \min_x \tilde{f}_0(x)$

s.t. $\quad \tilde{f}_i(x) \le 0 \;\; \forall i, \quad \tilde{h}_i(x) = 0 \;\; \forall i.$

constraints are preserved

non-negative

Ex: $\quad \min_x \|Ax-b\|_2$. : least squares problem.

Take $\varphi_0(u) = u^2$ on non-negative real, u^2 is monotonically increasing

$$\min_x \|Ax-b\|_2^2 = x^T A^T A x - 2b^T A x + b^T b.$$

which is diff'ble in x!

We can use the previous theory because $\varphi_0|_{\mathbb{R}_+}$ is monotonically $\uparrow$.

Slack variables $\quad f_i(x) \le 0 \iff \exists s_i \ge 0$ s.t. $f_i(x) + s_i = 0$.

$i = 1, \dots, m \qquad$ slack

Transformed problem.

m inequality constraints

$$\min_{x \in \mathbb{R}^n,\, s \in \mathbb{R}^m} f_0(x) \quad \text{s.t.} \quad \begin{aligned} & s_i \ge 0 \quad i = 1, \dots, m \\ & f_i(x) + s_i = 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p. \end{aligned}$$

(x, s): decision variable.

x is opt. for original problem

$\iff (x, s)$ is optimal for transformed problem & $s_i = -f_i(x)$

the pair is the decision variable

Epigraph problem form

trying to minimise this $\Rightarrow$ push $f_0(x)$ using a value t and making t as small as possible

$$\min_x \; f_0(x) \quad \text{s.t.} \quad f_i(x) \le 0 \;\; \forall i, \quad h_i(x) = 0 \;\; \forall i$$

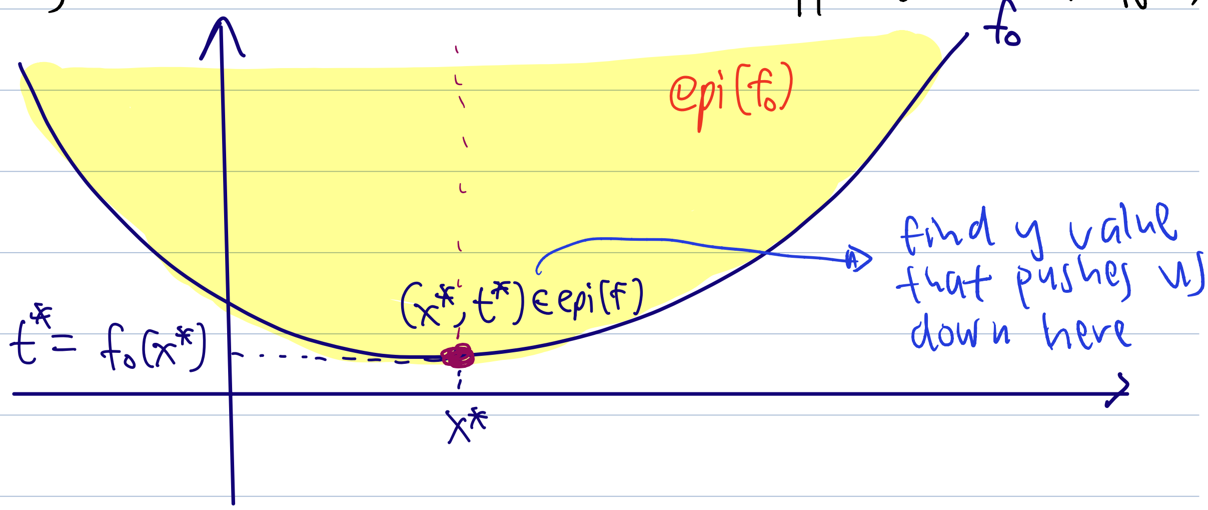
$$\min_{x,t \in \mathbb{R}} \; t \quad \text{s.t.} \quad t \ge f_0(x), \quad f_i(x) \le 0 \;\; \forall i, \quad h_i(x) = 0 \;\; \forall i$$

decision variables

minimise an upper bound on the objective function

$\text{epi } f = \{(x,t) : f(x) \le t\}$

trying to find smallest possible upper bound of $f_0(x)$



Equivalent to the original prob.
(x,t) is opt. for transformed problem $\Leftrightarrow$ x is opt for the original prob. & $t = f_0(x)$.

Convex Optimization Problem.

eg. If f_1 is convex: $\{x : f_1(x) \le 0\}$ convex sets

$$\min_x \; f_0(x) \quad \text{s.t.} \quad f_i(x) \le 0 \;\; \forall i = 1, \dots, m, \quad a_i^T x - b_i = 0 \;\; \forall i = 1, \dots, p.$$

intersection of convex sets $\Downarrow$ convex

hyperplane $\Rightarrow$ convex

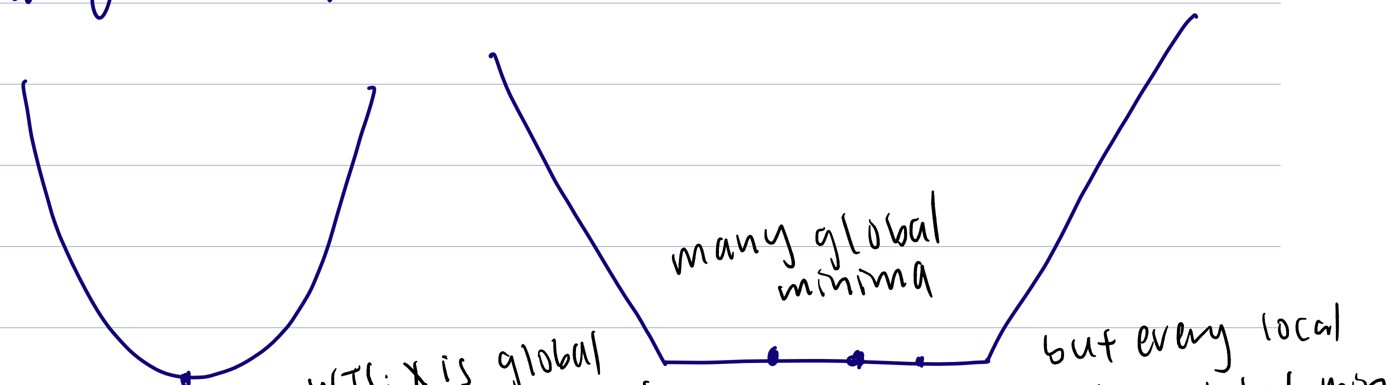
$f_0, f_1, \dots, f_m$ are convex function.

$\because \{f_i\}_{i=1}^m$ are convex f^ns & $a_i^T x = b_i$ are hyperplanes, the feasible region is convex.

hence objective function $f_0(x)$ is convex

Rmk: If f_0 is <u>quasiconvex</u> & the constraint set is convex, we say that problem is a <u>quasiconvex opt. problem.</u>

Prop: For any convex opt. problem, any local minimum is a global minimum.



WTS: x is global min too

Pf: Let x be a local minimum, i.e., $\exists R > 0$

$$f_0(x) = \inf\{f_0(z) : z \text{ feasible}, \|z - x\|_2 \le R\}.$$

$\Downarrow$

$f_i(z) \le 0 \ \forall i, \ h_i(z) = 0 \ \forall i$

Suppose, to the contrary, that x is not a global min.

$\exists$ feasible $y \neq x$ s.t. $f_0(y) < f_0(x)$

feasible.

if $\|y - x\|_2 \le R$ then it would belong to the set, which cannot happen

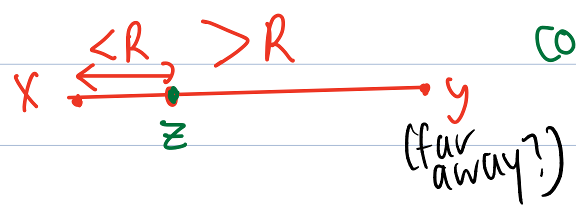
We know that $\|y - x\|_2 > R$. Consider the point

try to make y pretty close to x

$$z = (1-\theta)x + \theta y, \qquad \theta = \frac{R}{2\|x - y\|}.$$

constraint set is convex

z: feasible.



$$\|z-x\| = \|(1-\theta)x+\theta y - x\|$$
$$= \|\theta(y-x)\| = \theta\|y-x\| = \frac{R}{2\cancel{\|x-y\|}}\cancel{\|y-x\|}$$
$$= R/2.$$

By the convexity of f_0

$$f_0(z) = f_0((1-\theta)x+\theta y) \le (1-\theta)f_0(x)+\theta \overset{< f_0(x)}{f_0(y)}$$
$$< (1-\theta)f_0(x)+\theta f_0(x) = f_0(x).$$

$\|z-x\| \le R/2 < R$

z inside pink set but all the z's have higher f_0 values

This contradicts the local optimality of x.

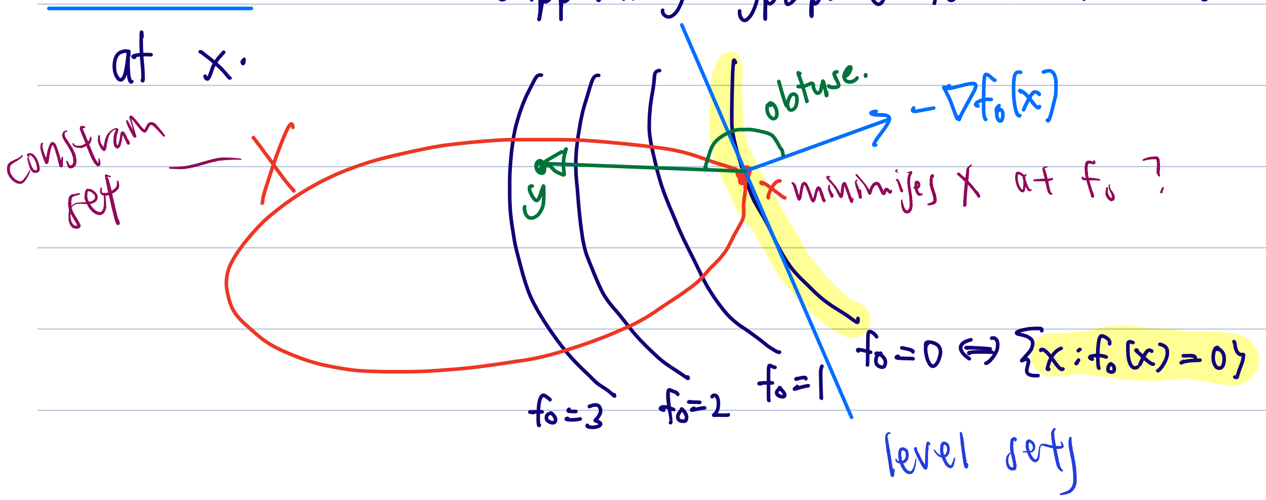
Optimality condition for diff'ble f_0

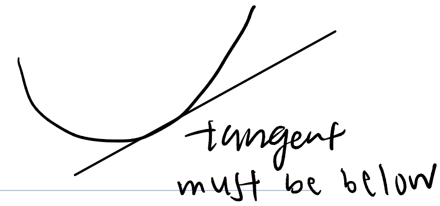
f_i : convex
$i=0,1,\dots,m.$

Thm: Let $X=\{x : f_i(x)\le 0\ \forall i,\ a_i^Tx-b_i=0\ \forall i\}$
affine constrains
be the feasible set.

x is optimal iff $x\in X$ &
$$\nabla f_0(x)^T(y-x)\ge 0 \quad \forall y\in X. \quad \text{—}(*)$$
↳ defines an acute angle

$-\nabla f_0(x)$ defines a supporting hyperplane to the feasible set X at x.





Pf: Suppose $x \in X$ and satisfies (*), i.e.,

$$\nabla f_0(x)^T (y - x) \geq 0$$

Since f_0 is convex & diff^{ble} → first order optimality

$$f_0(y) \geq f_0(x) + \nabla f_0(x)^T (y - x) \quad \forall y$$
$$\geq f_0(x)$$

$\forall y \in X \Rightarrow x$ is globally optimal

Now suppose $x \in X$ is optimal but (*) does not hold.

$\exists y \in X$ s.t. $\nabla f_0(x)^T (y - x) < 0$

Consider the line segment convex combi belong to X

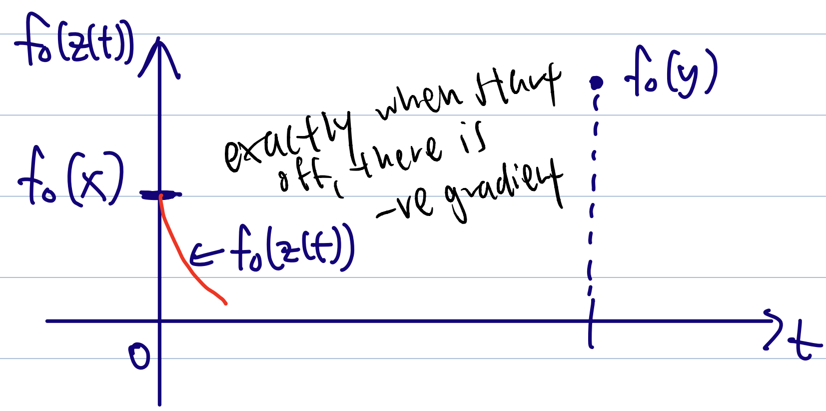
$$z(t) = ty + (1-t)x \quad t \in [0,1]$$

$z(t)$ is feasible $\forall t \in [0,1]$ $\because$ X is convex. make use of set convexity

$$\frac{d}{dt} f_0(z(t)) = \nabla f_0(z(t))^T (y - x)$$

$$\frac{d}{dt} f_0(z(t)) \Big|_{t=0} = \underline{\nabla f_0(x)^T (y - x) < 0.}$$

when $t=0$, $z(t) = x$



For t small enough, $f_0(z(t)) < f_0(x)$. This contradicts

$z(t)$ has a smaller f_0 value than the perceived optimal x

the optimality of x.

Unconstrained Problem

no inequality
↑ and no equality constrains

f_0 convex & problem is unconstrained $(m=p=0)$

$\nabla f_0(x_0)=0 \Rightarrow x_0$ is optimal.

X if there is a constrain set

Show this from the more general characterization.

$\forall x,y \in \mathrm{dom}\, f_0 \qquad \nabla f_0(x)^T(y-x) \geq 0.$

Take $y = x - t\nabla f_0(x)$, then

$t \in \mathbb{R}$.

$$\nabla f_0(x)^T(-t\nabla f_0(x)) \geq 0.$$

$$-t\,\|\nabla f_0(x)\|_2^2 \geq 0$$

$\Rightarrow \nabla f_0(x)=0$. (gradient evaluated at x_0 is 0)

Unconstrained Quadratic Optimization

not necessarily invertible

$P \geq 0$

$$f_0(x) = \frac{1}{2}x^T P x + q^T x + r \qquad P \in S_+^n \text{ +ve semi-definite}$$

Necessary & Suff. for x to be globally opt. is

$$\nabla f_0(x) = 0 \Rightarrow Px + q = 0.$$

range (column space)

i) If $-q \notin \mathcal{R}(P)$, no solution $\Rightarrow f_0$ is unbounded below.

ii) If $P \succ 0$, $x^* = -P^{-1}q$ is the unique opt. sol^{n}.

some 0 eigenvalues

have solutions

iii) If P is singular but $-q \in \mathcal{R}(P)$, the set of opt sol^{n}

$$X_{opt} = -P^{+}q + \mathcal{N}(P)$$

P^{+} is the pseudo-inverse of P.

If x_0 is a solution (i.e., $Px_0 = -q$) then $x_0 + x'$ is also a solution for all $x' \in \mathcal{N}(P)$.
(null space $Px' = 0$)

Convex Problem with Equality Constraints

$$\min_{x} f_0(x) \quad \text{s.t.} \quad Ax = b$$ feasible set

$$\Leftrightarrow a_i^T x = b_i \quad \forall i = 1, \dots, m$$

Opt. condition: x is optimal iff

$\forall y$ s.t. $Ay = b$, $\nabla f_0(x)^T (y - x) \geq 0$.

adapt from 1st order optimality condition

Since x is feasible ($Ax = b$). $\Rightarrow y = x + v$ where $v \in \mathcal{N}(A)$ (null space) ($Av = 0$)

x and y are both sols of linear system

$$\Rightarrow y - x \in \mathcal{N}(A)$$

$$\nabla f_0(x)^T (y - x) \geq 0.$$

previously known $\mathcal{N}(A) \perp \mathcal{R}(A^T)$.

gradient orthogonal to nullspace

$$\nabla f_0(x) \perp \mathcal{N}(A)$$

$$\Rightarrow \nabla f_0(x) \in \mathcal{R}(A^T).$$

$\exists u \in \mathbb{R}^p$ s.t. $\nabla f_0(x) + A^T u = 0$.

This is exactly the classical Lagrange multiplier optimality

Condition for problems with equality constraints.

Minimization over the Nonnegative Orthant

set of all vectors that have components that are non-ve

$$\min_x f_0(x) \quad \text{s.t.} \quad x \geq 0 \quad (x_i \geq 0 \quad \forall i \in [n]).$$

Optimality condition

$x \geq 0$ & $\underbrace{\nabla f_0(x)^T(y}_{\text{must be non-negative}}-x) \geq 0 \quad \forall y \geq 0.$ —(*)

y must be feasible

The term $\nabla f_0(x)^T y$ is unbounded below if there exists a component of $\nabla f_0(x)$ being negative

$\Rightarrow \nabla f_0(x) \geq 0$ (in all components)

Since $\nabla f_0(x) \geq 0$ & $y \geq 0$, (*) reduces to

$$-\nabla f_0(x)^T x \geq 0$$

Since $x \geq 0$, $\nabla f_0(x) \geq 0 \Rightarrow \nabla f_0(x)^T x \geq 0$

$$\Rightarrow \boxed{\nabla f_0(x)^T x} = 0.$$

$$\sum_{i=1}^{n} (\nabla f_0(x))_i x_i = 0.$$

inner product

Complementary slackness.

all are non-negative numbers

$$\underbrace{(\nabla f_0(x))_1^{>0} x_1^{=0}}_{=0} + \underbrace{(\nabla f_0(x))_2^{=0} x_2^{>0}}_{=0} = 0$$

how to convert to convex problem?

Quasiconvex optimization

$$\min_x f_0(x) \quad \text{s.t.} \quad \begin{array}{ll} f_i(x) \le 0 \;\; \forall i & \text{convex} \\ Ax = b. & \text{affine.} \end{array}$$

quasiconvex

Represent sublevel sets of a quasiconvex f^n via a family of convex inequalities.

$$\underbrace{f(x) \le t}_{\text{quasiconvex}} \iff \underbrace{\phi_t(x)}_{\text{convex function}} \le 0$$

Eg: $f(x) = \dfrac{p(x)}{q(x)}$ $\quad$ p: convex $\;\; p \ge 0$ $\quad$ $p, q: C \to \mathbb{R}$

$f: C \to \mathbb{R}.$ $\quad$ q: concave $\;\; q > 0$

α: non-negative

f is quasiconvex $\quad S_\alpha(f) = \{x : \overbrace{\frac{p(x)}{q(x)}}^{= f \ge 0} \le \alpha\}$

sub-level sets

$$= \{x : \underbrace{p(x) - \alpha q(x)}_{\text{convex } f^n} \le 0\}$$

$= \phi_\alpha(x)$

convex - concave gives a convex function

$$p^* = \min_x \underbrace{f_0(x)}_{\text{q-convex}} \quad \text{s.t.} \quad \begin{array}{l} f_i(x) \le 0 \\ i = 1, \dots, m. \end{array} \quad Ax = b$$

Conside the convex feasibility problem.

$$\underbrace{\text{find } x}_{\min_x 0} \quad \text{s.t.} \quad \phi_t(x) \le 0.$$

$$f_i(x) \le 0 \qquad Ax = b.$$

$$i = 1, \dots, m$$

primal optimal value

If x is feasible $\Rightarrow$ $\phi_t(x) \le 0$ $\Rightarrow$ $f(x) \le t$ $\Rightarrow$ $p^* \le t$.

If x is infeasible $\Rightarrow$ $p^* \ge t$.

Bisection Method to solve quasiconvex opt.

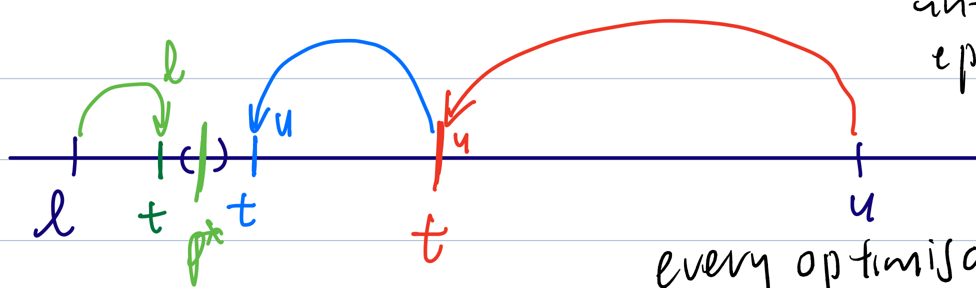
$p^* \in [l, u]$. Fix $\varepsilon > 0$.

repeat (while loop)

1. $t = \frac{l+u}{2}$.
2. Solve the cvx feasibility problem.
3. If feas. $u = t$, else $l = t$.

until $u - l < \varepsilon$.

do bisection until you have eplison approximation



$\phi_t(x) \le 0$

every optimisation problem solved is a convex feasibility problem

Terminate in $\left\lceil \frac{1}{\varepsilon} \log_2 (u - l) \right\rceil$ steps.

Linear Optimization Problem

constraints are also affine

objective is affine in x

$$\min_{x} \quad c^T x + \cancel{d} \quad \text{s.t.} \quad \boxed{Gx \le h}, \ m \text{ inequality}$$
$$Ax = b, \ p \text{ equality}$$

$G \in \mathbb{R}^{m \times n}$, $h \in \mathbb{R}^{m \times 1}$, $A \in \mathbb{R}^{p \times n}$, $b \in \mathbb{R}^{p \times 1}$

convert?

Often we drop d.

Standard form of an LP (Linear Program)

$$\min_{x} \quad c^T x \quad \text{s.t.} \quad Ax = b, \ \boxed{x \ge 0}$$

use a programmed solver

Transform a general LP into standard form LP.

$$\min_{x} \quad c^T x + d \quad \text{s.t.} \quad Gx \le h, \quad Ax = b$$

Introduce slack variables $s \in \mathbb{R}^m$

m inequality constrains that we want to convert to equality constrains

$$\min_{x,s} \quad c^T x + d \quad \text{s.t.} \quad Gx + s = h, \quad Ax = b, \quad s \ge 0.$$

Let $x = x^+ - x^-$, $x^+, x^- \ge 0$

$$x = \begin{pmatrix} 5 \\ 4 \\ -1 \\ -3 \\ 7 \end{pmatrix} \qquad x^+ = \begin{pmatrix} 5 \\ 4 \\ 0 \\ 0 \\ 7 \end{pmatrix} \qquad x^- = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 0 \end{pmatrix}$$

$$c^T x = c^T(x^+ - x^-)$$

$$\min_{(x^+, x^-, s)} \; c^T x^+ - c^T x^- + d \quad \text{s.t.} \quad Gx^+ - Gx^- + s = h$$

new decision variables

$$Ax^+ - Ax^- = b$$

$$s \geq 0, \; x^+ \geq 0, \; x^- \geq 0.$$

This becomes a standard LP with variables $x^+, x^-, s \geq 0$.

the rest is all equality

Ex: Diet Problems.

Healthy diet $\Rightarrow$ m different nutrients of quantities $\geq b_1, b_2, \dots, b_m$

Compose such a diet by choosing nonnegative quantities $x_1, x_2, \dots, x_n$ of n diff foods.

One unit of food $j \in [n]$ contains a_{ij} of nutrient $i \in [m]$ at cost c_j.

Determine cheapest diet that meets dietary requirements pay this amount

$$\min_{x} \; c^T x = \sum_{j=1}^{n} c_j x_j$$

require a certain amount of nutrients

$$\text{s.t.} \quad \sum_{i=1}^{m} a_{ij} x_j \geq b_i \quad \forall i \in [m]$$

nutrients amount

⇒ make sure we have enough nutrients

$Ax = b$

$$x_j \geq 0 \quad \forall j \in [n]$$

⇓ $x \geq 0$.

cannot eat -ve amounts of food