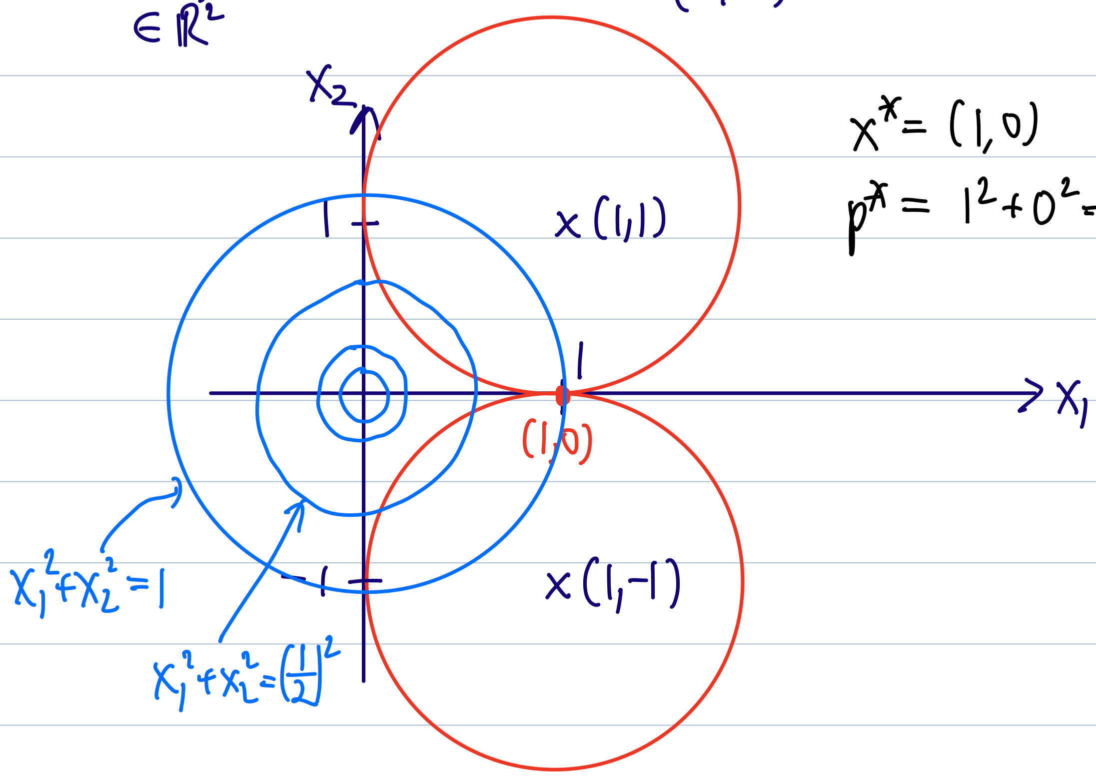
DSA3102 Tutorial 10

1. BV 5.26

$$\min_{x=(x_1,x_2)\in\mathbb{R}^2} x_1^2+x_2^2 \quad \text{s.t.} \quad \begin{aligned}(x_1-1)^2+(x_2-1)^2 &\le 1\\ (x_1-1)^2+(x_2+1)^2 &\le 1.\end{aligned}$$

(1,0)



$x^* = (1,0)$

$p^* = 1^2+0^2 = 1.$

b) Stationarity:

$$L(x,\lambda_1,\lambda_2) = x_1^2+x_2^2+\lambda_1\left[(x_1-1)^2+(x_2-1)^2-1\right] + \lambda_2\left[(x_1-1)^2+(x_2+1)^2-1\right]$$

$$\frac{\partial L}{\partial x_1}=0 \Rightarrow \quad 2x_1+2\lambda_1(x_1-1)+2\lambda_2(x_1-1)=0$$

$$\frac{\partial L}{\partial x_2}=0 \Rightarrow \quad 2x_2+2\lambda_1(x_2-1)+2\lambda_2(x_2+1)=0$$

Primal Feasibility: $(x_1-1)^2+(x_2-1)^2 \le 1$

$(x_1-1)^2+(x_2+1)^2 \le 1$.

Dual Feasibility: $\lambda_1 \ge 0,\ \lambda_2 \ge 0$.

Comp. Slackness: $\lambda_1 [\underbrace{(x_1-1)^2+(x_2-1)^2-1}_{=0}]=0$

$\lambda_2 [\underbrace{(x_1-1)^2+(x_2+1)^2-1}_{=0}]=0$

Only primal feasible $\underline{(x_1,x_2)=(1,0)}$.

$\underline{\lambda_1 \ge 0,\ \lambda_2 \ge 0}$, $2-2\lambda_2=0$, $-2\lambda_1+2\lambda_2=0$

$\Downarrow$ $\underline{\lambda_1=\lambda_2}$

$\underline{\lambda_1>0,\ \lambda_2=\lambda_1>0}$

$\lambda_1=\lambda_2=0$ $2=0$.

The KKT conditions do not give a solution for $x_1, x_2, \lambda_1, \lambda_2$.

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c) $g(\lambda_1,\lambda_2) = \inf_{x_1,x_2} L(x_1,x_2,\lambda_1,\lambda_2)$

$= \inf_{x_1,x_2} \{ x_1^2+x_2^2+\lambda_1[(x_1-1)^2+(x_2-1)^2-1]$

$+\lambda_2[(x_1-1)^2+(x_2+1)^2-1] \}$

$$= \inf_{x_1, x_2} (1+\lambda_1+\lambda_2)x_1^2 + (1+\lambda_1+\lambda_2)x_2^2 - 2(\lambda_1+\lambda_2)x_1 - 2(\lambda_1-\lambda_2)x_2 + \lambda_1 + \lambda_2$$

L reaches its minimum when

$$x_1 = \frac{\lambda_1+\lambda_2}{1+\lambda_1+\lambda_2}, \quad x_2 = \frac{\lambda_1-\lambda_2}{1+\lambda_1+\lambda_2}.$$

$$g(\lambda_1,\lambda_2) = \begin{cases} -\frac{(\lambda_1+\lambda_2)^2 - (\lambda_1-\lambda_2)^2}{1+\lambda_1+\lambda_2} + \lambda_1 + \lambda_2 & 1+\lambda_1+\lambda_2 \geq 0 \\ -\infty & \text{else.} \end{cases}$$

Lagrange dual problem:

$$\max_{\lambda_1,\lambda_2 \geq 0} -\frac{(\lambda_1+\lambda_2)^2 - (\lambda_1-\lambda_2)^2}{1+\lambda_1+\lambda_2} + \lambda_1 + \lambda_2 \quad \text{s.t. } 1+\lambda_1+\lambda_2 \geq 0$$

$$\max_{\lambda_1,\lambda_2 \geq 0} \frac{\lambda_1+\lambda_2 - (\lambda_1-\lambda_2)^2}{1+\lambda_1+\lambda_2}$$

Since $g(\lambda_1,\lambda_2)$ is symmetric ($g(\lambda_1,\lambda_2) = g(\lambda_2,\lambda_1)$) the optimum, if it exists, must be $\lambda_1 = \lambda_2$.

$$\max_{\lambda_1 \geq 0} g(\lambda_1,\lambda_2) = \frac{2\lambda_1}{1+2\lambda_1}$$

$d^* = 1$ achieved when $\lambda_1 \to +\infty$.

Strong duality holds! $p^* = d^* = 1$.

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3. $\min_{x \in \mathbb{R}^2} f(x)$ s.t. $\underline{5x_1 + x_2 \geq 4}$

explicit constraint

$$f(x) = \begin{cases} 10x_1 + 3x_2, & \underline{(x_1, x_2) \in \{0,1\}^2} \\ +\infty & \text{else.} \end{cases}$$

implicit constraint

b) $L(x, \lambda) = 10x_1 + 3x_2 + \lambda(4 - 5x_1 - x_2)$

Dual function:

$$g(\lambda) = \inf_{(x_1, x_2) \in \{0,1\}^2} \{10x_1 + 3x_2 + \lambda(4 - 5x_1 - x_2)\}$$

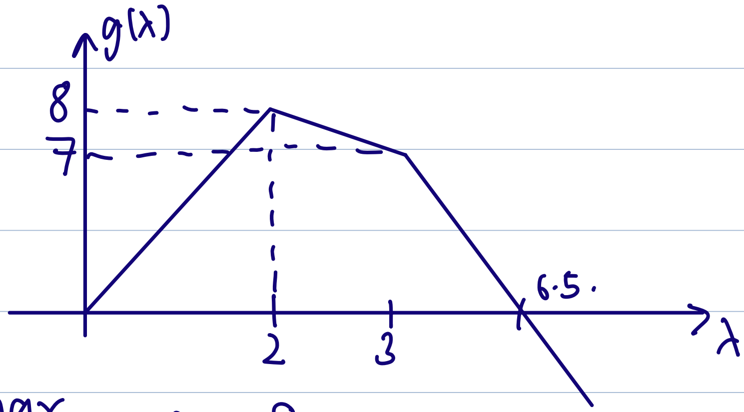
$$= \inf_{(x_1, x_2) \in \{0,1\}^2} \{4\lambda + (10 - 5\lambda)x_1 + (3 - \lambda)x_2\}$$

implicit constraint

$$g(\lambda) = \begin{cases} 4\lambda & \lambda \in [0,2] \\ 10 - \lambda & \lambda \in [2,3] \\ 13 - 2\lambda & \lambda \in [3, \infty) \end{cases}$$

c) Primal: $10x_1 + 3x_2 \quad (x_1, x_2) \in \{0,1\}^2$

$(1,0) \quad p^* = 10, \quad 5x_1 + x_2 \geq 4$



$$d^* = \max_{\lambda \geq 0} g(\lambda) = 8$$

Duality gap $= p^* - d^* = 10 - 8 = 2.$

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2. BV 5.27

$$\min_x \quad \|Ax-b\|_2^2 \quad \text{s.t.} \quad Gx = h. \quad G \in \mathbb{R}^{p \times n}$$

$p \geq 1$ ineq.

a) Lagrangian: $L(x,\nu) = \|Ax-b\|_2^2 + \nu^T(Gx-h)$

$$g(\nu) = \inf_{x \in \mathbb{R}^n} \Big\{ \underbrace{\|Ax-b\|_2^2}_{x^TA^TAx - 2x^TA^Tb} + \nu^T(Gx-h) \Big\}$$

$$\nabla_x L(x,\nu) = 2A^TAx - 2A^Tb + G^T\nu = 0.$$

$$x^* = \frac{1}{2}(A^TA)^{-1}(2A^Tb - G^T\nu^*)$$

$$g(\nu) = -\frac{1}{4}(G^T\nu - 2A^Tb)^T(A^TA)^{-1}(G^T\nu - 2A^Tb) - \nu^Th + b^Tb.$$

b) KKT conditions:

Stationarity: $2A^TAx^* - 2A^Tb + G^Tv^* = 0.$

$$\underline{2A^T}(Ax^* - b) + G^Tv^* = 0 \quad \updownarrow n. \qquad -(S)$$

$n \qquad p$

Primal Feasibility: $Gx^* = h. \quad \updownarrow p. \qquad -(PF)$

(S) $$x^* = (A^TA)^{-1}(A^Tb - \tfrac{1}{2}G^Tv^*).$$

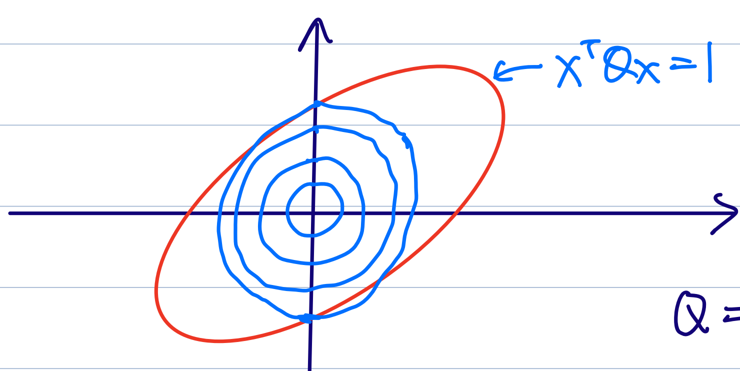
Plugging this in (PF) yields

$$G(A^TA)^{-1}(A^Tb - \tfrac{1}{2}G^Tv^*) = h$$

$$v^* = -2(G(A^TA)^{-1}G^T)^{-1}(h - G(A^TA)^{-1}A^Tb)$$

5(c) $\qquad \min_x f(x) \quad$ s.t. $\quad h(x) = 0 \qquad x^TQx = 1$

$f(x) = \underline{\|x\|^2} \quad$ s.t. $\quad x^TQx - 1 = h(x). \qquad Q \in S^n_{++}$



$$Q = \sum_{i=1}^{n} \lambda_i q_i q_i^T$$

$$L(x,v) = x^Tx - v(x^TQx - 1)$$

$$\nabla_x L(x,v) = 2x - v(2Qx) = 0$$

$$\Leftrightarrow \quad (I - \nu Q)x = 0$$

$$\Leftrightarrow \quad \nu Q x = x. \qquad (5)$$

x is an eigenvector of νQ, with eigenvalue 1.

$$\nu Q = \sum_i (\nu \lambda_i) q_i q_i^T$$

(PF) $\qquad \underline{x^T Q x = 1.} \qquad \qquad \nu Q x = x$

To ensure that x is an e-vector of νQ with e-value 1, we set $\nu > 0$ s.t. $\nu \lambda_i = 1$ for some $i = 1, \ldots, n$.

There are n possible choices for ν $\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}\right)$.

PF $\qquad x^T\left(\frac{1}{\nu} x\right) = 1 \Rightarrow \qquad \nu = \|x\|^2$

Want to minimize $\|x\|^2$. Choose ν to be the smallest possible out of the n choices. $\nu = \frac{1}{\lambda_1}$

Choose x to be the eigenvector corresponding to the largest eigenvalue (λ_1) of Q.