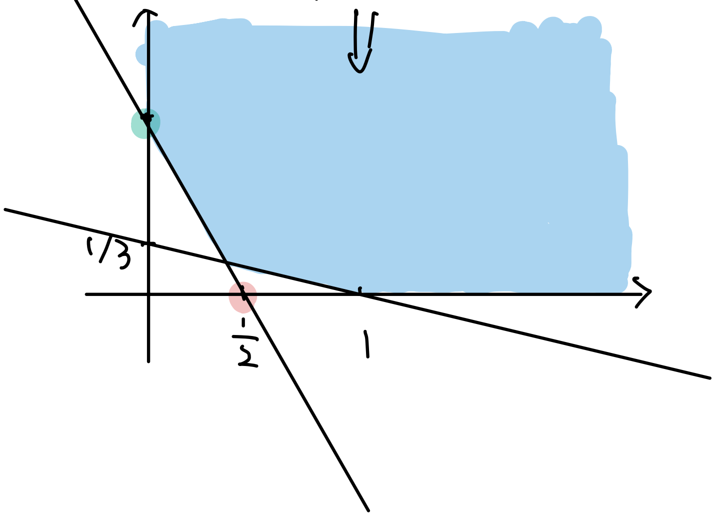
Week 6. Tutorial. Xiaochen. Q1. Q3. Q4. Q5

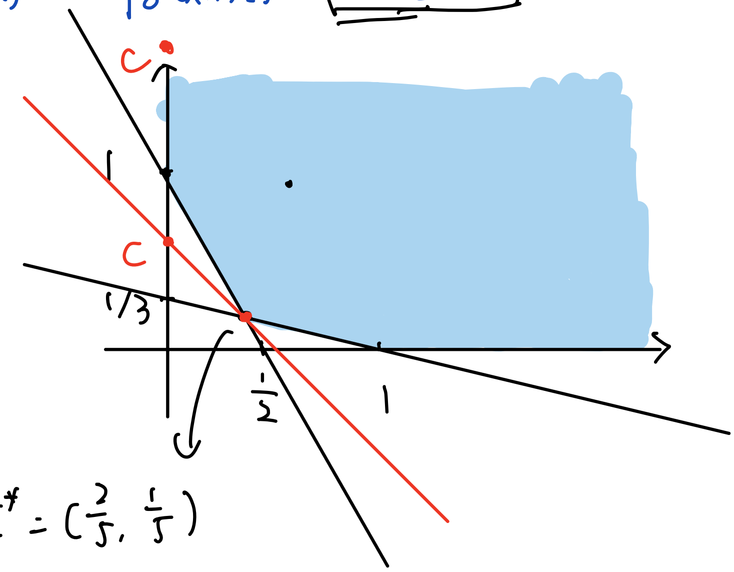
Q1. minimize $f_0(x_1, x_2)$ subject to

$$\begin{cases} 2x_1 + x_2 \geq 1 & \Rightarrow \left(\tfrac{1}{2}, 0\right). \quad (0, 1) \\ x_1 + 3x_2 \geq 1 & \Rightarrow (1, 0). \quad \left(0, \tfrac{1}{3}\right). \\ x_1 \geq 0. \quad x_2 \geq 0 \end{cases}$$

feasible space.



(a) $f_0(x_1, x_2) = \boxed{x_1 + x_2 = c}$



$x^* = \left(\frac{2}{5}, \frac{1}{5}\right)$

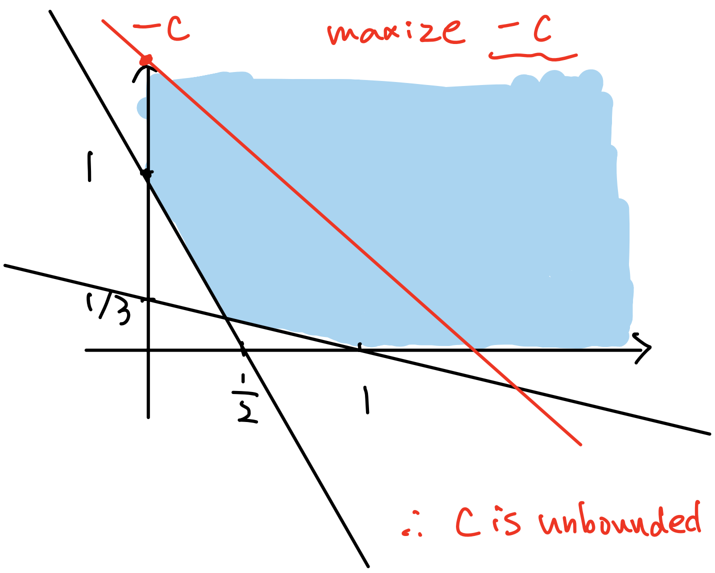
$-x_1 - x_2 = c$

$-x_1 + c = x_2$

(b) $f_0(x_1, x_2) = \underline{-(x_1 + x_2)} = c$

minize c.

maxize $\underline{-c}$

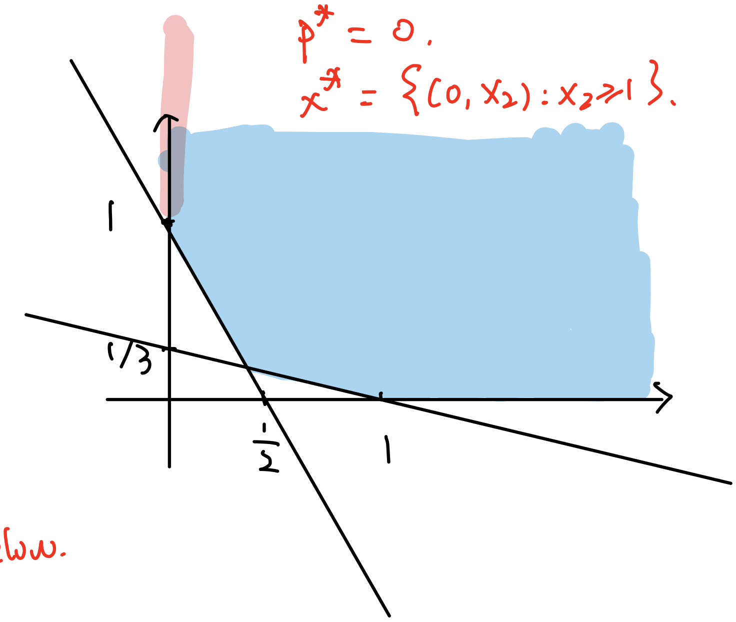


$\therefore$ c is unbounded below.

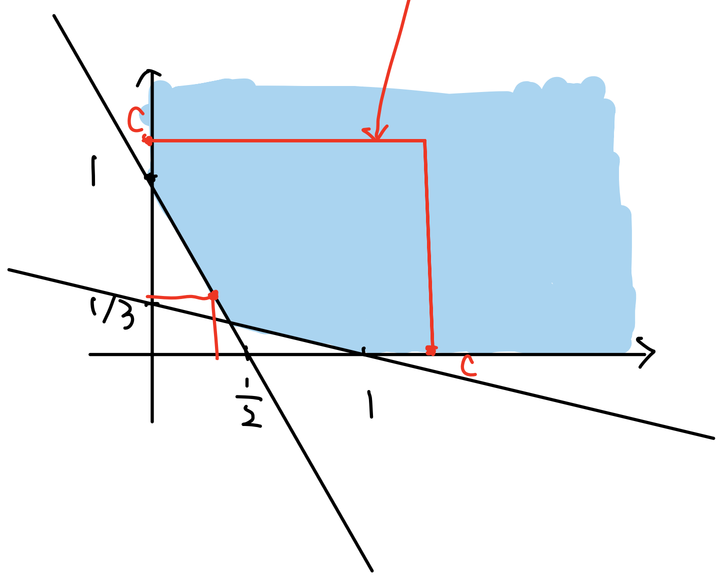
(c) $f_0(x_1, x_2) = \underline{x_1}$

$p^* = 0.$

$x^* = \{(0, x_2) : x_2 \geq 1\}.$



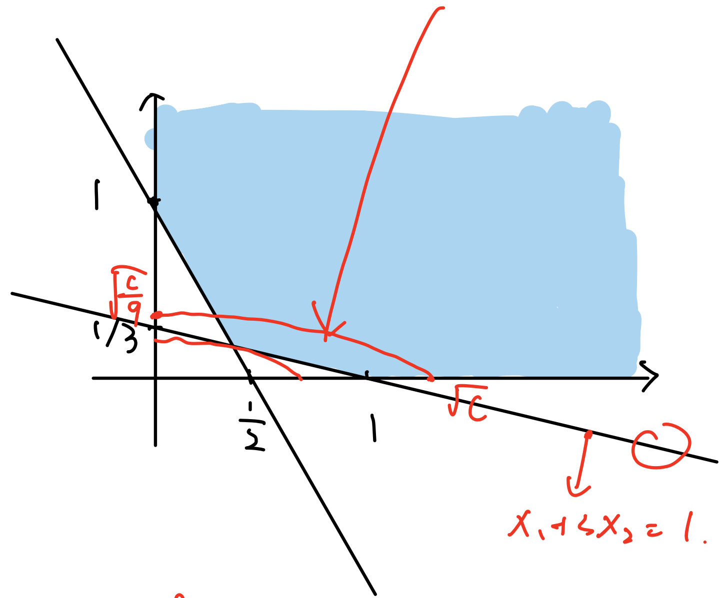
(d). $f_0(x_1,x_2) = \boxed{\max(x_1,x_2) = c}$



$x_1 = x_2$ and $x_1 + 3x_2 = 1$

$\Rightarrow x_1^* = \frac{1}{3}. \quad x_2^* = \frac{1}{3}.$

(e). $f_0(x_1,x_2) = x_1^2 + 9x_2^2 = c$



$\nabla f_0(x_1,x_2) = (1,3)$

$\Rightarrow x_1 = \frac{1}{2}. \quad x_2 = \frac{1}{6} \in$ feasible.

Q3. (a). minimize $c^T x$ subject to $Ax=b$.

(b) minimize $c^T x$ subject to $a^T x \le b$.

(c). minimize $c^T x$ subject to $l \le x \le u$.

$n \in null(A)$
$t \cdot n \in null(A)$.

(a). Null space of A is $\{v : Av = 0\}$.

1° if x_0 is a solution to $Ax=b$. then all solutions will be given as $x_0 + n$ where $n \in nullspace(A)$

$c^T x = c^T x_0 + c^T n$ find $\min\{c^T x_0 + c^T n : n \in null\,space(A)\}$.

(1). if $\exists\ n_0 \in nul(A)$. s.t. $c^T n_0 \neq 0$.

WLOG. let $c^T n_0 > 0$. $n = t n_0$. then $c^T n = t c^T n_0$

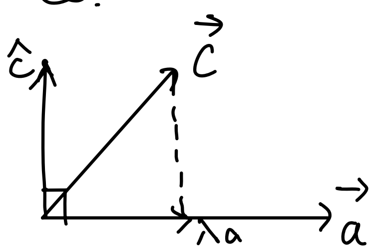
$c^T n \to -\infty$ when $t \to -\infty$. $\therefore$ unbounded below.

(2). $\forall\ n \in nul(A)$. $c^T n = 0$. c is orthogonal to null space.

$\Leftrightarrow$ c is in the row space of A $\Leftrightarrow$ $c = A^T \lambda$ for some λ.

then $c^T x = \lambda^T A x = \lambda^T b$.

2°. feasible space is ϕ. $\Rightarrow$ ∞.



(b). min $c^T x$. $a^T x \le b$.

pal to a

$c = \underline{\lambda a} + \underline{\hat{c}}$. $\lambda \in \mathbb{R}$. $a^T \hat{c} = \hat{c}^T a = 0$.

min $c^T x = \lambda a^T x + \hat{c}^T x$.

1°. if $\lambda > 0$. want: $a^T x \to -\infty$ while $\hat{c}^T x = 0$.

$x = -t a$ then $a^T x = -t \|a\|_2^2 \le b$ when $t \to \infty$

$c^T x = \lambda a^T x + 0 = -t \lambda a^T a = -t \lambda \|a\|_2^2 \to -\infty$

$\therefore$ unbounded. below

2°. if $\lambda \le 0$.

(1). $\hat{c} = 0$. $c \parallel a$. then $c^T x = \lambda a^T x \geq \underline{\lambda b}$. $x^* = \{x : a^T x = b\}$.

(2). $\hat{c} \neq 0$. want: $\underline{\hat{c}^T x} \to -\infty$. while $x \in$ feasible space.

$x = -t \hat{c} + \frac{a}{\|a\|_2^2} \cdot b$.

$a^T x = 0 + \frac{\|a\|_2^2}{\|a\|_2^2} b = b$ $\therefore$ $x \in$ feasible space.

Then $c^T x = (\lambda a + \hat{c})^T x$

$$= \lambda \underline{a^T x} + \hat{c}^T x$$

$$= \lambda b + (-t) \|\hat{c}\|^2 + \frac{\overset{0}{\overset{\parallel}{a \cdot \hat{c}^T}}}{\heartsuit} \cdot b$$

$$= \lambda b - t \|\hat{c}\|^2 \to -\infty. \text{ when } t \to \infty.$$

$\therefore$ unbounded below.

$$\therefore \quad p^* = \begin{cases} \lambda b & \text{if } c = a \cdot \lambda \quad \lambda \le 0. \\ -\infty. & \text{otherwise.} \end{cases}$$

(c). minimize $c^T x$ subject to $l \le x \le u$.

min: $c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$.

for every $i = 1, 2, \cdots, n$:

$$x_i^* = \begin{cases} l_i & \text{if } c_i > 0. \\ u_i & \text{if } c_i < 0. \end{cases}$$

$$p^* = \sum_{i=1}^{n} \cdot \left(\max(c_i, 0) \cdot l_i + \min(c_i, 0) \cdot u_i \right) = l^T c^+ + u^T c^-$$

where $c_i^+ = \max(c_i, 0)$ $\quad c_i^- = \min(c_i, 0)$

Q4. BV 4.9. Consider LP: min $c^T x$ subject to $Ax \le b$.

A is square and nonsingular. A^{-1}

Change of variable. $y = Ax$. $x = A^{-1} y$

min $\underbrace{\boxed{c^T A^{-1}}}_{\text{given vector}} y$ where $y \le b$. (similar to Q3c)

if $(c^T A^{-1})_i > 0$ then $y_i \to -\infty$. then $c^T A^{-1} y \to -\infty$.

$\therefore$ unbounded below.

if $c^T A^{-1} \le 0$ then $y_i^* = b_i$ then $(c^T A^{-1} y)^* = c^T A^{-1} b$.

Q5. BV. 4.11. Formulate as LPs.

$\min_x \ c^Tx + d$ where $Gx \le h, \ Ax = b$

(a). Minimize $\|Ax-b\|_\infty$.

(b). $\min \ \|Ax-b\|_1$

(c). $\min_x \ \|Ax-b\|_1$. where $\|x\|_\infty \le 1$

(d). $\min_x \ \|x\|_1$ where $\|Ax-b\|_\infty \le 1$.

(e). $\min_x \ \|Ax-b\|_1 + \|x\|_\infty$

Epigraph form.

$\min_x f(x)$ s.t. ....

$\Leftrightarrow \min_{(x,t)} t$ s.t. $f(x) \le t$ .....

(a). $\min_{x, t\in\mathbb{R}} t$ s.t. $\|Ax-b\|_\infty \le t$

$\Leftrightarrow \forall i. \ |Ax-b|_i \le t$

$\Leftrightarrow \forall i. \ -t \le (Ax-b)_i \le t$

$\Leftrightarrow \begin{pmatrix} -t \\ -t \\ \vdots \\ -t \end{pmatrix} \le Ax-b \le \begin{pmatrix} t \\ t \\ t \\ \vdots \\ t \end{pmatrix} \Leftrightarrow -t\mathbf{1} \le Ax-b$ and $Ax-b \le t\cdot\mathbf{1}$

affine.

$\therefore$ LP.

(b). $\min_x \ \|Ax-b\|_1$

$\Leftrightarrow \min_{x, t\in\mathbb{R}} t$ s.t. $\|Ax-b\|_1 \le t$

$\Leftrightarrow \sum_{i=1}^n |Ax-b|_i \le t$ cannot be written as an affine constraint.

$\min_{x, S\in\mathbb{R}^n} \sum_{i=1}^n S_i$ s.t. $|Ax-b|_i \le S_i$

$\Leftrightarrow -S_i \le (Ax-b)_i \le S_i \Leftrightarrow Ax-b \ge -S$. $Ax-b \le S$.

$\therefore$ LP: $\min_{x.S\in\mathbb{R}^n} \mathbf{1}^T S$ s.t. $Ax-b \ge -S$ and $Ax-b \le S$.

(c). $\min_x \ \|Ax-b\|_1$ where $\|x\|_\infty \le 1$

$\min_{x.S\in\mathbb{R}^n} \mathbf{1}^T S$ s.t. $Ax-b \ge -S$ and $Ax-b \le S$. and $\|x\|_\infty \le 1$

$\Leftrightarrow |x|_i \le 1$ for all i

$\therefore$ LP. $\Leftrightarrow -1 \le x \le 1$

(d). $\min_x \ \|x\|_1$. where $\|Ax-b\|_\infty \le 1$

$\overset{(b)}{\Leftrightarrow} \min_{x, S\in\mathbb{R}^n} \mathbf{1}^T S$ s.t. $-S \le x \le S$ and $\|Ax-b\|_\infty \le 1$

$\Leftrightarrow 1 \le Ax-b \le 1$

$\therefore$ LP.

(e). $\min_{x} \underbrace{\|Ax-b\|_1}_{(b)} + \underbrace{\|x\|_\infty}_{(a)}$

$$\therefore \min_{\substack{x, s \in \mathbb{R}^n \\ t \in \mathbb{R}}} \mathbf{1}^T s + t \quad \text{s.t.} \quad |Ax-b|_i \le s_i \text{ for all } i$$

$$\text{and} \quad \|x\|_\infty \le t$$

$$\Leftrightarrow \quad -s \le Ax-b \le s$$

$$\text{and } -t\mathbf{1} \le x \le t\mathbf{1}$$