Homework1 A0239929J

1. Consider the set defined by the following inequalities

$$S = \{x \in \mathbb{R}^2 : \overset{y}{x_1} \geq \overset{x}{x_2} - 1, \overset{x}{x_2} \geq 0\} \bigcup \{x \in \mathbb{R}^2 : \overset{y}{x_1} \leq \overset{x}{x_2} - 1, \overset{x}{x_2} \leq 0\}$$

(a) Draw the set S. Is it convex?

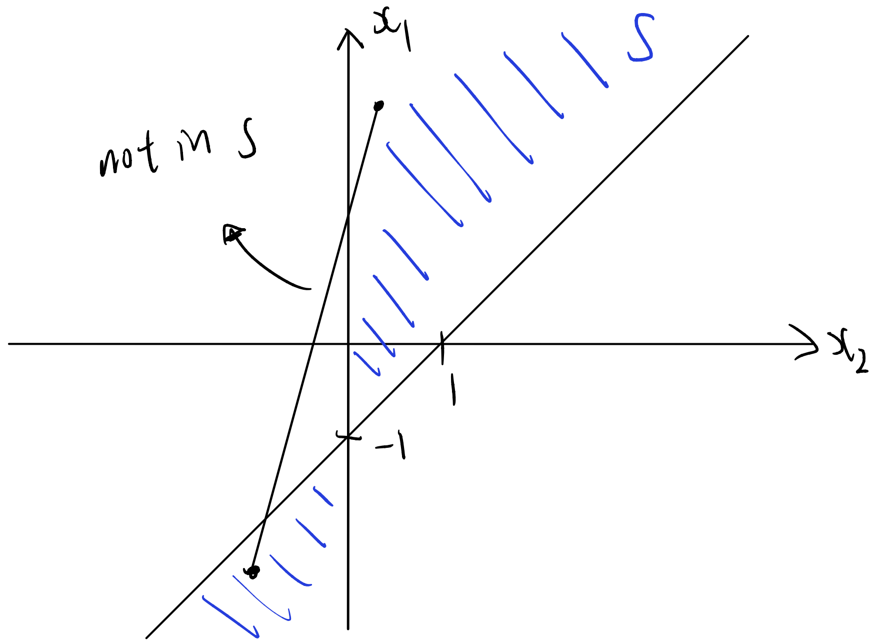
(b) Show that the set S can be described as a single quadratic inequality of the form

$$q(x) = x^T Ax + 2b^T x + c \leq 0$$

for a matrix $A \in \mathbf{S}^2$ (2 by 2 symmetric matrix), $b \in \mathbb{R}^2$ and $c \in \mathbb{R}$ which you should determine. *Hint: Note that q is continuous, $q \leq 0$ on S and $q > 0$ on S^c, hence q must be 0 on the boundary of S.*

(c) What is the convex hull of the set S?

(a)



set S is drawn as the blue shaded region

set S is not convex as the definition of convex sets state that $S \subseteq \mathbb{R}^n$ is convex if the line segment between any 2 points in S lie in S

(not the case as seen in the diagram)

(b)

consider the following 2 inequalities

$x_1 \geq x_2 - 1, \quad x_2 \geq 0$
$x_1 \leq x_2 - 1, \quad x_2 \leq 0$ } symmetric about x_2 axis

S rewritten as $\left\{ x \in \mathbb{R}^2 : \left(1 - \frac{x_1+1}{x_2}\right)\left(1 + \frac{x_1+1}{x_2}\right) \leq 0 \right\}$

$$\left(1 - \frac{x_1+1}{x_2}\right)\left(1 + \frac{x_1+1}{x_2}\right) \leq 0$$

$$1 - \frac{x_1^2 + 2x_1 + 1}{x_2^2} \leq 0$$

$$x_2^2 \leq x_1^2 + 2x_1 + 1$$

$$x_2^2 - x_1^2 - 2x_1 - 1 \leq 0$$

$-2 = 2(f \;\; 0)$
$\Rightarrow f = -1$

$x_2^2 - x_1^2 = (x_1 \;\; x_2)\begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}$
$= ax_1^2 + bx_1x_2 + cx_1x_2 + dx_2^2$

$b, c = 0$
$a = -1, \; d = 1$

written as

$$(x_1 \;\; x_2)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 2\begin{pmatrix} -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - 1 \leq 0$$

(in the form stated)

where $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $c = -1$
symmetric

(c)

Convex hull : smallest convex set that contains the original set

$(0,0)$ belongs to both subsets $\Rightarrow$ part of convex hull

For $x_1 \geq 0$, convex hull includes $(t, 0)$ where $t \geq 0$

For $x_1 \leq 0$, convex hull includes $(t, 0)$ where $t \leq 0$

Hence convex hull is the entire $\mathbb{R}^2$ plane as set S is non-convex so it has to include all points in x-axis and extending in all directions

2. The following is a very important result in linear programming duality. Let $A \in \mathbb{R}^{m\times n}$ and $y \in \mathbb{R}^m$. Show that one and only one of the following two conditions is satisfied.

- the system of linear equations $Ax = y$ admits a non-negative solution $x \geq 0$; — 1
- there exists $z \in \mathbb{R}^m$ such that $z^T A \geq 0$ and $z^T y < 0$. — 2

You should use the *strict separating hyperplane* theorem.

Firstly, we show that we cannot have both 1 and 2 (cannot hold at the same time)

Take $z^T Ax$,

case 1: $z^T A x = z^T (Ax)$

$= z^T (y) < 0$

since by 1: $Ax = y$ and by 2: $z^T y < 0$

case 2: $z^T Ax = (z^T A) x$ (take transpose)

≥ 0

since by 1: $x \geq 0$ and by 2: $z^T A \geq 0$

Next, WTS: If 1 does not hold then 2 must hold

Let $A = \begin{pmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{pmatrix}$ where $a_1, a_2 \ldots a_n$ are the columns of matrix A

we define $Q = \text{cone}(a_1, a_2 \ldots a_n)$

$= \{ s \in \mathbb{R}^m : s = \sum_{i=1}^{n} \lambda_i a_i , \lambda_i \geq 0 \ \forall i \}$

which is non-empty, closed and convex

Then $Ax = \sum_{i=1}^{n} x_i a_i$, $\exists x$ such that $Ax = y$ and $x \geq 0$
iff $y \in Q$

Assume ① does not hold,

then $\underline{y \notin Q}$

Strict Version

Recall seperating hyperplane Thm,

$\Leftrightarrow \exists \alpha, \beta \in \mathbb{R}^m$, $\alpha \neq 0$ such that

$\alpha^T y > \beta$ and $\alpha^T s < \beta$, $\forall s \in Q$
(and $y \notin Q$)

$0 \in Q$, $\beta > 0$, $\lambda a_i \in C$ $\forall \lambda > 0$

$\Rightarrow$ since $\alpha^T s < \beta$, $\forall s \in Q$

we have $\alpha^T(\lambda a_i) \in Q$, $\forall \lambda > 0$

$\Leftrightarrow \alpha^T(\lambda a_i) < \beta \Rightarrow \alpha^T a_i < \frac{\beta}{\lambda}$, $\forall \lambda > 0$

since $\beta > 0$ as $\lambda \to \infty \Rightarrow \alpha^T a_i \leq 0$

Set $z = -\alpha$, we get $z^T y < 0$ and $z^T a_i \geq 0$ $\forall i$

$\Rightarrow z^T A \geq 0$ (since a_i are columns of A)

Hence showing ② holds

3. The *polar* of an arbitrary set $C \subset \mathbb{R}^n$ is defined as the set

$$C^\circ := \{y \in \mathbb{R}^n : y^T x \le 1 \text{ for all } x \in C\}$$

(a) Let $C \subset \mathbb{R}^n$ be any set, not necessarily convex. Is C° convex? Justify your answer carefully.

(b) Recall that a *cone* K is such that if $x \in K$ and $\theta \ge 0$, then $\theta x \in K$. Recall that the *dual cone* is defined as

$$K^* := \{y \in \mathbb{R}^n : y^T x \ge 0 \text{ for all } x \in K\}$$

It is known that the polar of the cone K, denoted as K°, and the dual cone, denoted as K^*, are related as follows:

$$K^\circ = -cK^*$$

for some *positive* number c. Find c.

(c) Show carefully that the polar of the unit ball $\mathcal{B}(0,1) := \{x \in \mathbb{R}^n : \|x\|_2 \le 1\}$ is the unit ball. You may find the Cauchy–Schwarz inequality (i.e., $x^T y \le \|x\|_2 \|y\|_2$) useful.

(a) C° convex $\Rightarrow$ $\forall y_1, y_2 \in C^\circ$ and $\theta \in [0,1]$ then $\theta y_1 + (1-\theta) y_2 \in C^\circ$

since $y_1, y_2 \in C^\circ$,

$y_1^T x \le 1$ and $y_2^T x \le 1$

we take linear combination of y_1 and y_2 with x

$\Rightarrow (1-\theta) y_1^T x + \theta y_2^T x \le (1-\theta)(1) + (\theta)(1)$

$= 1 - \theta + \theta$

$= 1$

This shows that linear combi of y_1 and y_2 satisfies conditions for C°

hence $\forall x \in C$, $(1-\theta) y_1 + \theta y_2 \in C^\circ$

(b) K is a cone

$\Rightarrow$ polar of cone $K^{0} = \{y \in \mathbb{R}^n : y^T x \leq 0, \forall x \in \}$

since x is either all non-positive or all non-negative (but not both),

we are able to derive for $\forall c$:

$$y^T x \leq 0 \iff -(y^T x) \geq -(0)$$
$$-y^T x \geq 0$$

Hence proving $K^{0} = -K^{*}$ where $c = -1$

(c) Polar of unit ball denoted as

$B^\circ(0,1) = \{ y \in \mathbb{R}^n : y^T x \leq 1 \text{ for } \forall x \in B(0,1) \}$

WTS: $\overset{\circ}{B}(0,1) = B(0,1)$

($\Rightarrow$) prove $B^\circ(0,1) \subseteq B(0,1)$

consider $x \in B^\circ(0,1)$

For any $y \in$ unit ball $B(0,1)$, we have inner product

$y^T x \leq 1$ (definition of polar)

$\leq \|y\|_2 \|x\|_2$ (by cauchy schwarz inequality)

since inequality holds for $\forall y \in B(0,1)$, we can choose y to be x itself

$x^T x \leq \|x\|_2 \|x\|_2$

$\Rightarrow \|x\|_2^2 \leq \|x\|_2^2$ ($\|x\|_2^2$ less than equal to itself $\Rightarrow \|x\|_2^2$ is non-negative)

$\Rightarrow \|x\|_2$ also non-negative

since $\|x\|_2^2 \leq 1$ and $\|y\|_2$ is non-negative,

it follows that $\|x\|_2 \leq 1$ hence

$\forall x \in B^\circ(0,1)$ satisfies $\|x\|_2 \leq 1$ and belongs inside unit ball

($\Leftarrow$) prove $B(0,1) \subseteq B^\circ(0,1)$

consider $y \in B(0,1)$,

WTS: for this given y, $y^T x \leq 1$ holds for $\forall x \in B^\circ(0,1)$

let x be a point in $B^\circ(0,1)$, by the definition of polar: $y^T x \leq 1$

This inequality holds for $\forall y$ in $B(0,1)$ so it will be satisfied for any value of x we choose

Hence $B(0,1) \subseteq B^\circ(0,1)$

Thus this proves equality

4. (a) For x, y both positive scalars, show that

$$ye^{x/y} = \sup_{\alpha>0} \alpha(x+y) - y\alpha \log \alpha.$$

(log here refers to the natural logarithm.) Use the above result to prove that the function $f : \mathbb{R}^n_{++} \to \mathbb{R}$ defined as

$$f(x,y) = ye^{x/y}$$

is convex.

(b) Show that the function $f : \mathbb{R}^2_+ \to \mathbb{R}$

$$f(x) = (\sqrt{x_1} + \sqrt{x_2})^2$$

is concave by examining the definiteness of its Hessian.

Differentiate both sides

$$\frac{\partial}{\partial \alpha} ye^{\frac{x}{y}} = \frac{\partial}{\partial \alpha}\left\{ \sup_{\alpha>0} \alpha(x+y) - y\alpha\log\alpha \right\}$$

$$\Rightarrow (x+y) - y\left\{ \log\alpha + \alpha\left(\frac{1}{\alpha}\right)\right\} = 0$$

$$x + y - y\log\alpha - y = 0$$

$$x - y\log\alpha = 0 \Rightarrow \log\alpha = \frac{x}{y}$$

$$\alpha = e^{\frac{x}{y}}$$ (maximising initial expression)

Hence

$$\sup_{\alpha>0} \alpha(x+y) - y\alpha\log\alpha = e^{\frac{x}{y}}(x+y) - y(e^{\frac{x}{y}})(\log e^{\frac{x}{y}})$$

$$= e^{\frac{x}{y}}\left\{ x+y - y\left(\frac{x}{y}\right)\right\}$$

$$= ye^{\frac{x}{y}}$$

(addition of linear functions)

since linear functions are convex and addition preserve linearity, $f(x,y)$ is convex

(b) $f(x_1, x_2) = (\sqrt{x_1} + \sqrt{x_2})^2$

$$\nabla f(x_1, x_2) = \begin{bmatrix} 1 + 2\sqrt{x_2}\left(\frac{1}{2}\right)\frac{1}{\sqrt{x_1}} \\ 1 + 2\sqrt{x_1}\left(\frac{1}{2}\right)\frac{1}{\sqrt{x_2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \frac{\sqrt{x_2}}{\sqrt{x_1}} \\ 1 + \frac{\sqrt{x_1}}{\sqrt{x_2}} \end{bmatrix}$$

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} -\frac{\sqrt{x_2}}{2(x_1)^{\frac{3}{2}}} & \frac{1}{2\sqrt{x_1}\sqrt{x_2}} \\ \frac{1}{2\sqrt{x_1}\sqrt{x_2}} & -\frac{\sqrt{x_1}}{2(x_2)^{\frac{3}{2}}} \end{bmatrix}$$

→ principal minor is non-positive too

$-\frac{\sqrt{x_2}}{2(x_1)^{\frac{3}{2}}} < 0$ given $\sqrt{x_2} > 0$ and $(x_1)^{\frac{3}{2}} > 0$

$$\det\{\nabla f(x_1, x_2)\} = \frac{-\sqrt{x_2}}{2(x_1)^{\frac{3}{2}}} \cdot \frac{-\sqrt{x_1}}{2(x_2)^{\frac{3}{2}}} - \left(\frac{1}{2\sqrt{x_1}\sqrt{x_2}}\right)^2$$

$$= 0$$

$\nabla^2 f(x_1, x_2) \leq 0$ hence f is concave