30. Aug. Tutorial 0. Xiaochen. xczhu@nus.edu.sg. Q1. 2. 3. 6. 7.

Q1. Is the set $A = \{a \in \mathbb{R}^k : p(0) = 1, |p(t)| \le 1.\ \forall t \in [\alpha, \beta]\}$

where $p(t) = a_1 + a_2 t + \cdots + a_k t^{k-1}$ convex?

$p(0) = 1$. $|p(t)| \le 1$. $\forall t \in [\alpha, \beta]$.

By definition. Assume $a, b \in A$. $\Rightarrow$ $la + (1-l)b \in A$. for $l \in [0, 1]$.

$a_1 = 1$ $\quad$ $b_1 = 1$

$P_a(0) = 1$. $P_b(0) = 1$. $\forall t \in [\alpha, \beta]$. $|P_b(t)| \le 1$. $|P_a(t)| \le 1$

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$c = la + (1-l)b$

$P_c(0) = c_1 = l \cdot a_1 + (1-l) b_1 = l + 1 - l = 1$

$|P_c(t)| \le 1$ for all $t \in [\alpha, \beta]$.

$$
\begin{aligned}
P_c(t) &= c_1 + c_2 \cdot t + \cdots + c_k \cdot t^{k-1} \\
&= (la_1 + (1-l) b_1) + (la_2 + (1-l) b_2) t + \cdots + (la_k + (1-l) b_k) t^{k-1} \\
&= la_1 + la_2 t + la_3 t^2 + \cdots + l a_k t^{k-1} \\
&\quad + (1-l) b_1 + (1-l) b_2 t + \cdots + (1-l) b_k t^{k-1} \\
&= l \cdot P_a(t) + (1-l) \cdot P_b(t).
\end{aligned}
$$

$$
\begin{aligned}
|P_c(t)| = |l \cdot P_a(t) + (1-l) \cdot P_b(t)| &\le |l \cdot P_a(t)| + |(1-l) P_b(t)| \\
&= l \cdot \underbrace{|P_a(t)|}_{\le 1} + (1-l) \cdot \underbrace{|P_b(t)|}_{\le 1} \\
&\le l + (1-l) = 1 \quad \forall t \in [\alpha, \beta].
\end{aligned}
$$

$\therefore c \in A$. $\Rightarrow$ A is convex set.

Q2. Prove that $A = \{(x_1, x_2, x_3) \in R^3 : \begin{bmatrix} x_1 + x_2 & x_1 - 2x_3 \\ x_1 - 2x_3 & x_2 + 3x_3 \end{bmatrix} \succeq 0\}$.

positive semi definite

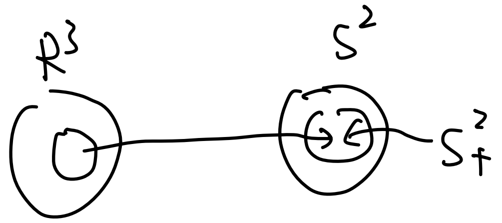
Lemma: $f : V \to W$. f is a linear map if $\begin{cases} \forall x, y. & f(x+y) = f(x) + f(y) \\ \forall c \in R & f(cx) = cf(x) \end{cases}$

$f : R^m \to R^n$. (affine functions linear functions during the [illegible])

$S \subseteq W$. S is convex $f^{-1}(S)$ is convex.

~~A~~ $f : R^3 \to S^2$ $A = f^{-1}(S_+^2)$ A is convex

convex.



Q3.

2.12 Which of the following sets are convex?

(a) A *slab*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha \le a^T x \le \beta\}$.

(b) A *rectangle*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha_i \le x_i \le \beta_i,\ i = 1, \ldots, n\}$. A rectangle is sometimes called a *hyperrectangle* when $n > 2$.

(c) A *wedge*, *i.e.*, $\{x \in \mathbf{R}^n \mid a_1^T x \le b_1,\ a_2^T x \le b_2\}$. $= \{x : a_1^T x \le b_1\} \cap \{x : a_2^T x \le b_2\}$.

(d) The set of points closer to a given point than a given set, *i.e.*,

$$\{x \mid \|x - x_0\|_2 \le \|x - y\|_2 \text{ for all } y \in S\}$$

where $S \subseteq \mathbf{R}^n$.

(e) The set of points closer to one set than another, *i.e.*,

$$\{x \mid \mathbf{dist}(x, S) \le \mathbf{dist}(x, T)\},$$

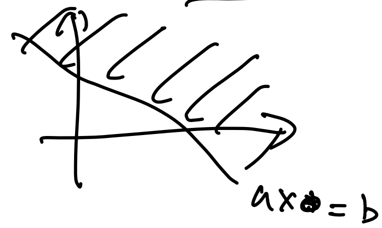
where $S, T \subseteq \mathbf{R}^n$, and

$$\mathbf{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}.$$

(f) [HUL93, volume 1, page 93] The set $\{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbf{R}^n$ with S_1 convex.

(g) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, *i.e.*, the set $\{x \mid \|x - a\|_2 \le \theta \|x - b\|_2\}$. You can assume $a \ne b$ and $0 \le \theta \le 1$.

half space: $\{x \in R^n \mid a^T x \ge b\}$.



$a^T x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$

(a). $\{x \in R^n \mid \alpha \le a^T x \le \beta\} = \{x \in R^n : a^T x \ge \alpha\} \cap \{x \in R^n : a^T x \le \beta\}$. Convex.

Alternatively. $f : x \mapsto a^T x$. $R^n \to R$ $A = f^{-1}([\alpha, \beta])$ convex.

(b). $\{x \in R^n \mid \alpha_i \le x_i \le \beta_i \text{. for } i = 1, 2, \cdots, n\}$.

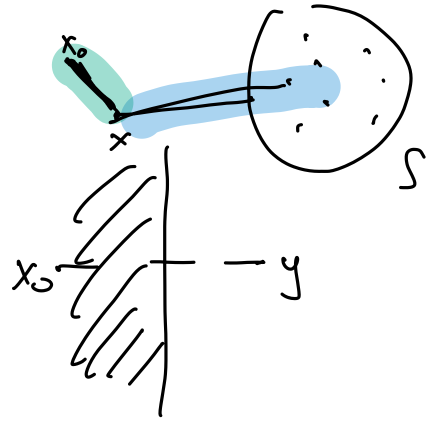
$= \bigcap_{i=1,\cdots,n} \{x \in R^n \mid \alpha_i \le x_i \le \beta_i\}$ $\leftarrow$ $\{x : a^T x \in [\alpha, \beta]\}$

$a_i = 1$. $a_j = 0$. for $j \ne i$.

$$\begin{array}{r} \\ \\ i\text{-th} \\ \\ \\ \end{array}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} x_i \end{bmatrix} = x_i$$

(d). $A = \{x \in R^n \mid \|x - x_0\|_2 \le \|x - y\|_2 \text{ for all } y \in S\}$.

$= \bigcap_{y \in S} \{x : \|x - x_0\|_2 \le \|x - y\|_2\}$.



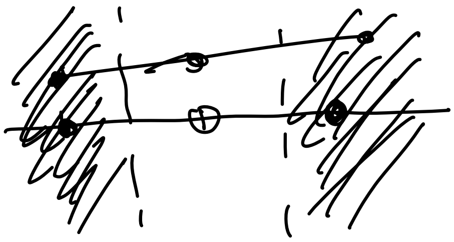
$\|x - x_0\|_2^2 \le \|x - y\|_2^2$

$(x - x_0)^T (x - x_0) \le (x - y)^T (x - y) \Rightarrow$ half space.

A set where x is closer to a given set S than to a point x_0.

$\Leftrightarrow \exists y \in S$. s.t. $\|x - x_0\|_2 \ge \|x - y\|_2$. $\bigcup_{y \in S} \{x : \|x - x_0\|_2 \ge \|x - y\|_2\}$.

(e). the set of x s.t. $\mathrm{dist}(x, S_1) \le \mathrm{dist}(x, S_2)$.



$S_1 = \{-1, +1\}$ $\quad \{x_0\}$.

$S_2 = \{0\}$

$$\bigcup_{y \in S_1} \{x : \|x - x_0\| \ge \|x - y\|\}.$$

$$x \ge |x-1| \ \cup \ x \ge |x+1|$$

f). The set $A = \{x : x + S_2 \subseteq S_1\}$, $S_1, S_2 \subseteq \mathbb{R}^n$. S_1 convex.

$$x + S_2 = \{x + y : y \in S_2\}$$

$$A = \{x : x + y \in S_1 \ \forall y \in S_2\} = \bigcap_{y \in S_2} \boxed{\{x : x + y \in S_1\}}$$

$f : x \mapsto x + y$. $\quad f^{-1}(S_1)$. convex

(g). set $A = \{x : \|x - a\|_2 \le \theta \|x - b\|_2\}$.

$$= \{x : \|x - a\|_2^2 \le \theta^2 \|x - b\|_2^2\}.$$

$$(x-a)^T(x-a) \le \theta^2 (x-b)^T(x-b).$$

$$x^T x - \left[\frac{2a - 2\theta^2 b}{(1+\theta)^2}\right]^T x + \bigcirc \le 0 \qquad x^2 - 2bx + \bigcirc = (x-b)^2 + \bigcirc$$

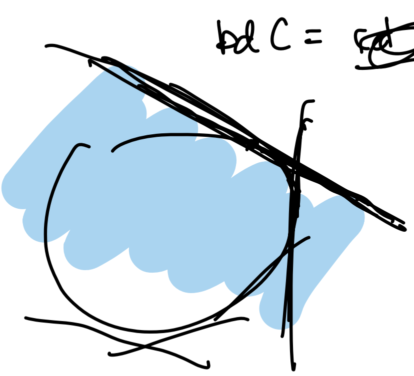
$$\left(x^T - \frac{(a - \theta^2 b)^T}{(1+\theta)^2}\right)\left(x - \frac{a - \theta^2 b}{(1+\theta)^2}\right) \le R^2$$

$$(x^T - x_0^T)(x - x_0) \le R^2. \qquad \{x : \|x - x_0\| \le R\}.$$

Q6. **2.24** *Supporting hyperplanes.*

(a) Express the closed convex set $\{x \in \mathbf{R}^2_+ \mid x_1 x_2 \geq 1\}$ as an intersection of halfspaces.

(b) Let $C = \{x \in \mathbf{R}^n \mid \|x\|_\infty \leq 1\}$, the ℓ_∞-norm unit ball in $\mathbf{R}^n$, and let $\hat{x}$ be a point in the boundary of C. Identify the supporting hyperplanes of C at $\hat{x}$ explicitly.



$\text{bd}\, C = \text{cl}\, C \setminus \text{int}\, C$. x_0 on the boundary.

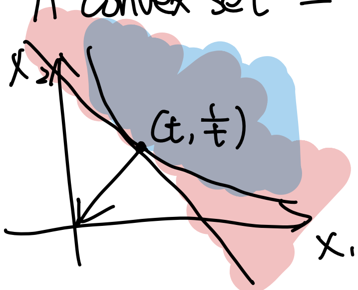
$a^T x_0 \leq a^T x$. for all $x \in C$.

$\{x : a^T x = a^T x_0\}$. is a supporting hyperplane.

$\{x : a^T x \leq a^T x_0\}$ is a supporting half space.

this half space will always contain C.

A convex set = intersection of supporting half spaces at all bundary points.



find sup. half space at boundary point. $(t, \frac{1}{t})$.

$y = \frac{1}{x}$. $-\frac{1}{x^2}$. $\Rightarrow$ $-\frac{1}{t^2} x_1 - y_2 = -\frac{2}{t}$.

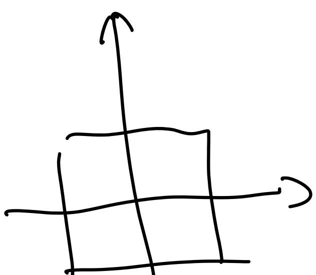
$-\frac{1}{t^2} x_1 - x_2 \leq -\frac{2}{t}$.

$A = \bigcap_{t \in R_+} \{x \in R^2 : -\frac{1}{t^2} x_1 - y_2 \leq -\frac{2}{t}\}$

$a = \begin{pmatrix} -\frac{1}{t^2} \\ -1 \end{pmatrix}$

$a^T x \leq -\frac{2}{t}$.

(b) $C = \{x \in R^n \mid \|x\|_\infty \leq 1\}$. $\hat{x} \in \text{bd}\, C$.



$\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_n|\}$.

$\hat{x} \in \text{bd}\, C \Rightarrow \|\hat{x}\|_\infty = 1$

$\Rightarrow \max\{|\hat{x}_1|, \cdots, |\hat{x}_n|\} = 1$

$a^T x \leq a^T \hat{x}$ $a = (a_1, \cdots, a_n)$.

$a^T x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$ $(\hat{x}_1 + \varepsilon$

$\leq a_1 \hat{x}_1 + a_2 \hat{x}_2 + \cdots + a_n \hat{x}_n$

$a_i \geq 0$. if $\hat{x}_i = 1$ $(\hat{x}_1 - \varepsilon, \hat{x}_2, \cdots, \hat{x}_n)$

$a_i \leq 0$. if $\hat{x}_i = -1$ $\Rightarrow a_1(\hat{x}_1 - \varepsilon) \leq a_1 \cdot \hat{x}_1 \Rightarrow a_1 \geq 0$

$a_i = 0$ if $\hat{x}_i \neq 1, -1$ $a_1 \varepsilon \geq 0$

a^T. $a^T x = a^T \hat{x}$

Q7. **2.32** Find the dual cone of $\overset{K=}{\{Ax \mid x \succeq 0\}}$, where $A \in \mathbf{R}^{m\times n}$.

K is cone then dual cone $K^* = \{y \mid x^T y \geq 0 \text{ for all } x \in K\}$.

$v \succeq 0 \Rightarrow v_i \geq 0$ for all i

$x^T y \geq 0.$ for $x \in K$

$K^* = \{y : (Ax)^T y \geq 0. \text{ for all } x \geq 0\}$ $\quad (Ax)^T$

$= \{y : \boxed{x^T A^T y \geq 0.} \text{ for all } x \geq 0\}.$

$\tilde{K} = \{y : A^T y \geq 0\}.$

Prove $K^* = \tilde{K}$.

1°. $K^* \subseteq \tilde{K}$. $\quad y \in K^*. \Rightarrow x^T A^T y \geq 0 \quad \forall x \geq 0$

want: $y \in \tilde{K}$ $(A^T y \geq 0)$.

$(A^T y)_i < 0$

$x^T = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cdot A^T y =$ (i-th)

$\Rightarrow y \in \tilde{K}. \Rightarrow K^* \subseteq \tilde{K}$

2°. $\tilde{K} \subseteq K^*$