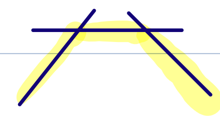
Tutorial 8

BV 5.3

$$\min_{x \in \mathbb{R}^n} c^T x \quad \text{s.t. } f(x) \le 0.$$

$f: \mathbb{R}^n \to \mathbb{R}$ not necessarily convex.



Lagrangian: $L(\overset{\in \mathbb{R}^n}{x}, \overset{\in \mathbb{R}}{\lambda}) = c^T x + \lambda f(x)$

Lagrange dual function: $g(\lambda) = \inf_{x \in \mathbb{R}^n} \{ c^T x + \lambda f(x) \}$

Dual problem: $\max_{\lambda \ge 0} g(\lambda)$

$$f^*(y) = \sup_x \{ x^T y - f(x) \}$$

$$\begin{aligned} g(\lambda) &= \inf_{x \in \mathbb{R}^n} \{ c^T x + \lambda f(x) \} \\ &= \lambda \inf_{x \in \mathbb{R}^n} \{ (\tfrac{c}{\lambda})^T x + f(x) \} \\ &= -\lambda \sup_{x \in \mathbb{R}^n} \{ (-\tfrac{c}{\lambda})^T x - f(x) \} \\ &= -\lambda f^*(-\tfrac{c}{\lambda}) \end{aligned}$$

Dual problem: $\min_{\lambda \ge 0} \underbrace{\lambda f^*(-\tfrac{c}{\lambda})}_{h(\lambda)}$

$h(\lambda)$ is convex due to perspective function

$f(x)$ is convex, then $\tilde{h}(x,t) = t f(\frac{x}{t})$ is convex (x,t).

$h(t) \triangleq \tilde{h}(x,t)$ is convex.

BV 5.5

$$\min_{x\in\mathbb{R}^n} c^T x \quad \text{s.t.} \quad \begin{array}{lll} Gx \preceq h & G\in\mathbb{R}^{m\times n} & m \text{ ineq.} \\ Ax=b. & A\in\mathbb{R}^{p\times n} & p \text{ eq.} \end{array}$$

$g_i^T x \le h_i \quad \forall i=1,\dots,m.$

Lagrangian: $L(x,\lambda,v) = c^T x + \lambda^T(Gx-h) + v^T(Ax-b)$

$(\lambda_1,\dots,\lambda_m)$

$$L(x,\lambda,v) = c^T x + \sum_{i=1}^{m} \lambda_i (g_i^T x - h_i) + \dots$$

$$= c^T x + \lambda^T (Gx-h)$$

Lagrange dual function:

$$g(\lambda,v) = \inf_{x\in\mathbb{R}^n} \left\{ c^T x + \lambda^T(Gx-h) + v^T(Ax-b) \right\}$$

$$= \inf_{x\in\mathbb{R}^n} \left\{ (c^T + \lambda^T G + v^T A)x - \lambda^T h - v^T b \right\}.$$

$$= -\lambda^T h - v^T b + \inf_{x\in\mathbb{R}^n} \left\{ (c^T + \lambda^T G + v^T A)x \right\}$$

If $c^T + \lambda^T G + v^T A = 0$, the inf is 0.

If $c^T + \lambda^T G + v^T A \neq 0$, the inf is $-\infty$.

Dual Problem: $\max_{\lambda\in\mathbb{R}^m_+,\, v\in\mathbb{R}^p} g(\lambda,v)$

$$g(\lambda, \nu) = \begin{cases} -\lambda^T h - \nu^T b & \text{if } c + G^T \lambda + A^T \nu = 0. \\ -\infty & \text{else} \end{cases}$$

$$\min_{\lambda \in \mathbb{R}^m_+, \nu \in \mathbb{R}^p} \lambda^T h + \nu^T b \quad \text{s.t.} \quad c + G^T \lambda + A^T \nu = 0.$$

<u>Dual Problem</u>

$$\min_{\lambda \in \mathbb{R}^m, \nu \in \mathbb{R}^p} \lambda^T h + \nu^T b \quad \text{s.t.} \quad c + G^T \lambda + A^T \nu = 0. \quad \lambda \geq 0.$$

Exercise: Take the dual of the dual.

<u>BV 5.7 (a, b)</u>

$$\min_{x \in \mathbb{R}^n} \max_{i=1,\dots,m} \{a_i^T x + b_i\}.$$

a) $$\min_{x \in \mathbb{R}^n, y \in \mathbb{R}^m} \left(\max_{i=1,\dots,m} y_i\right) \quad \text{s.t.} \quad y_i = a_i^T x + b_i, \quad i = 1, \dots, m.$$

primal

Lagrangian: $$L(x, y, \nu) = \max_{i=1,\dots,m} y_i + \sum_{i=1}^m \nu_i (a_i^T x + b_i - y_i)$$

$\nu = (\nu_1, \dots, \nu_m)$ dual variables.

Dual function: $$g(\nu) = \inf_{x \in \mathbb{R}^n, y \in \mathbb{R}^m} \left[\max_{i=1,\dots,m} y_i + \sum_{i=1}^m \nu_i (a_i^T x + b_i - y_i) \right]$$

If $\sum_{i=1}^{m} v_i a_i = 0$, simplifies

If $\sum_{i=1}^{m} v_i a_i \neq 0$, the infimum $= -\infty$.

Suppose $\sum v_i a_i = 0$,

$$\inf_{y \in \mathbb{R}^m} \left\{ \max_{i=1,\dots,m} y_i - v^T y \right\}$$

$$= - \sup_{y \in \mathbb{R}^n} \left\{ v^T y - \max_{i=1,\dots,m} y_i \right\}$$

$$= \begin{cases} 0 & \text{if} \quad v \geq 0,\ 1^T v = 1 \\ -\infty & \text{else} \end{cases}$$

$$\sup_y v^T y - \max_{i=1,\dots,m} y_i \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \\ t \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$t < 0$.

Suppose $\exists\, k = 1, \dots, m$ s.t. $v_k < 0$. Choose $y = t e_k$.
Then $v^T y - \max_{i=1,\dots,m} y_i = t v_k - 0 \to +\infty \quad t \to -\infty$.
Hence the $\sup = +\infty$

If $1^T v \neq 1$, choose $y = t 1_m$. Then

$$v^T y - \max_{i=1,\dots,m} y_i = v^T(t 1) - t$$
$$= t(1^T v - 1)$$

Suppose $1^T v > 1$, then $t \to \infty$, $\sup = +\infty$
Suppose $1^T v < 1$, then $t \to -\infty$, $\sup = +\infty$

In summary, dual function

$$g(v) = \inf_{\substack{x \in \mathbb{R}^n, \\ y \in \mathbb{R}^m}} \left[\max_{i=1,\dots,m} y_i + \sum_{i=1}^m v_i (a_i^T x + b_i - y_i) \right]$$

$$= \begin{cases} b^T v & \text{if } \underbrace{\sum_{i=1}^m \overset{\in \mathbb{R}^n}{a_i v_i} = 0}_{\text{infimization over } x} \ \& \ \underbrace{\sum_{i=1}^m v_i = 1, \ v_i \geq 0, \forall i}_{\text{inf over } y.} \\ -\infty & \text{else.} \end{cases}$$

$A \in \mathbb{R}^{m \times n}$

$$= \begin{cases} b^T v & A^T v = 0_n, \ 1^T v = 1, \ v \geq 0 \\ -\infty & \text{else.} \end{cases}$$

$$A = \begin{bmatrix} \text{---} \ a_1 \ \text{---} \\ \vdots \\ \text{---} \ a_m \ \text{---} \end{bmatrix}$$

Dual problem: $\max_{v \in \mathbb{R}^m} b^T v$ s.t.

$$A^T v = 0_n, \ 1^T v = 1, \ v \geq 0$$

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b) $$\min_{x \in \mathbb{R}^n} \max_{i=1,\dots,m} \{ a_i^T x + b_i \}.$$

$$\min_{x \in \mathbb{R}^n, t \in \mathbb{R}} \quad t \qquad \text{s.t.} \quad a_i^T x + b_i \le t \quad \forall i = 1, \dots, m.$$

$$a_i^T x + b_i - t \le 0.$$

$\lambda = (\lambda_1, \dots, \lambda_m) \quad \in \mathbb{R}^m$

Lagrangian: $L(t, x, \lambda) = t + \lambda^T (Ax + b - t\mathbb{1})$

$t \in \mathbb{R}$, $x \in \mathbb{R}^n$

$\inf_x \quad \inf_t$

Lagrange dual function: $g(\lambda) = \inf_{x,t} t(1 - \lambda^T \mathbb{1}) + (A^T\lambda)^T x + \lambda^T b.$

To ensure that $g(\lambda) \neq -\infty$, need to ensure $\mathbb{1}^T\lambda = 1$ and $A^T\lambda = 0$

$$g(\lambda) = \begin{cases} \lambda^T b & \text{if} \quad \mathbb{1}^T\lambda = 1 \;\&\; A^T\lambda = 0 \\ -\infty & \text{else} \end{cases}$$

Dual problem: $\max_{\lambda \ge 0} g(\lambda)$ s.t. $\mathbb{1}^T\lambda = 1 \;\&\; A^T\lambda = 0$

$\lambda \in \mathbb{R}^m_+$

$$\max_{\lambda \in \mathbb{R}^m} \lambda^T b \quad \text{s.t.} \quad \mathbb{1}^T\lambda = 1, \quad A^T\lambda = 0, \quad \lambda \ge 0$$

We obtain the same dual opt. problem.