DSA3102 HW3

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1. (Convexity and concavity of optimal value of an LP)
Consider the linear programming problem

$$p^* = \min_{x \in \mathbb{R}^n} c^T x \quad \text{subject to} \quad Ax \le b,$$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^M$. Prove the following statements, or provide a counterexample.

(a) The optimal value of the problem p^*, as a function of c, is concave in c.

(b) The optimal value of the problem p^*, as a function of b, is convex in b (you may assume that the problem is feasible).

(c) The optimal value of the problem p^*, as a function of A, is concave in A.

(a) This is true

$c^T x$ is a linear function of c indexed by x

$p^* = \min_{x \in \mathbb{R}^n} c^T x$ which is a point wise minimum of this linear, hence affine function.

Pointwise infimums are always concave hence p^* is concave (statement is true)

(b) statement is true

Assuming that the problem is feasible
$\Rightarrow$ strong duality holds

$L(\lambda) = c^T x + \lambda^T (Ax - b)$

since strong duality holds,

$$p^* = \inf_{x \in D} \sup_{\lambda \ge 0} c^T x + \lambda^T (Ax - b)$$

$$= \sup_{\lambda \ge 0} \inf_{x} \{ c^T x + \lambda^T (Ax - b) \} \quad (d^*)$$

consider $g(\lambda) = \inf\{(c^T + \lambda^T A)x - \lambda^T b\}$

$$g(\lambda) = \inf\{(A^T\lambda + c)x - b^T\lambda\}$$

$$= \begin{cases} -b^T\lambda & \text{when } A^T\lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

$$p^* = \max_{\lambda \geq 0} (-b^T\lambda) \quad \text{s.t } A^T\lambda + c = 0$$

$$= \max(-b^T\lambda) \quad \text{s.t } A^T\lambda + c = 0 \text{ and } \lambda \geq 0$$

p^* is the point-wise max of linear functions of b $(-b^T u)$ indexed by u

$\Rightarrow$ pointwise supremums are always convex hence p^* is convex in b

statement is true.

(c) statement is false.

⇒ Find counterexample

consider $\min_x x$ s.t $bx \geq 1$

case 1: $b = 0$,

problem infeasible, $p^* \to +\infty$

case 2: $b < 0$,

$p^* \to -\infty$

case 3: $b > 0$

$p^* = \frac{1}{b}$

Hence $p^* = \begin{cases} 1/b & \text{if } a > 0 \\ +\infty & a = 0 \\ -\infty & \text{otherwise} \end{cases}$

This function is clearly not concave

hence statement is false.

2. (Quadratic Programming Duality)

Consider the following quadratic program with H being a positive definite symmetric matrix and x being the decision variable.

$$\min_x \frac{1}{2}x^T Hx \quad \text{s.t.} \quad Ax \geq b.$$

$b - Ax \leq 0$

Show that the dual problem is also a quadratic optimization problem that looks like

$$\max_{\lambda,z} \lambda^T f(b,A,H) - \frac{1}{2}z^T g(b,A,H)z \quad \text{s.t.} \quad \lambda \geq 0, \quad h(b,A,H)^T\lambda = z.$$

Find the functions $f(b,A,H)$, $g(b,A,H)$ and $h(b,A,H)$.

$$L(x,\lambda) = \frac{1}{2}x^T Hx + \lambda^T(b - Ax)$$

$$= \frac{1}{2}x^T Hx - (A^T\lambda)^T x + \lambda^T b$$

$$g(\lambda) = \inf_{x\in D} L(x,\lambda)$$

$$= \inf_{x\in D} \left\{\frac{1}{2}x^T Hx - (A^T\lambda)^T x + \lambda^T b\right\}$$

consider $\nabla_x \left[\frac{1}{2}x^T Hx - (A^T\lambda)^T x + \lambda^T b\right]$

$$= Hx - A^T\lambda = 0$$

This gives $x^* = H^{-1}A^T\lambda$

$$g(\lambda) = \frac{1}{2}(H^{-1}A^T\lambda)^T HH^{-1}A^T\lambda - (A^T\lambda)^T H^{-1}A^T\lambda + \lambda^T b$$

$$= \frac{1}{2}\lambda^T AH^{-1}A^T\lambda - (A^T\lambda)^T H^{-1}A^T\lambda + \lambda^T b$$

$$= \lambda^T b - \frac{1}{2}(A^T\lambda)^T H^{-1}A^T\lambda$$

Dual problem:

$$\max_{\lambda, z} \left\{ \lambda^T f(b, A, H) - \frac{1}{2} z^T g(b, A, H) z \right\}$$

$$\text{s.t } \lambda \geq 0, \quad h(b, A, H)^T \lambda = z$$

$$\Leftrightarrow \max_{\lambda} \left\{ \lambda^T b - \frac{1}{2} (A^T \lambda)^T H^{-1} A^T \lambda \right\}$$

$$z = (A^T \lambda)$$

$$= h(b, A, H)^T \lambda$$

This gives $f(b, A, H) = b$, $\quad g(b, A, H) = H^{-1}$

and $h(b, A, H) = A$

3. (Duality Gap Problem)

Consider the two-dimensional problem

$$\min \quad e^{x_2} \quad \text{subject to} \quad \|x\|_2 - x_1 \le 0$$

where the domain of the optimization variable $x = (x_1, x_2)$ is $\mathbb{R}^2$.

(a) Is this a convex program?

(b) Calculate the primal optimal value p^*

(c) Calculate the dual optimal value d^*. What is the duality gap?

(d) Can you find a Slater vector? Explain intuitively why there is or isn't a duality gap.

(a) Exponential function is always convex

$\Rightarrow e^{x_2}$ is convex

By triangle inequality and homogeneity of the norm, for any $x_1, x_2 \in \mathbb{R}^n$ and any $\theta \in (0,1)$

$$\|\theta x_1 + (1-\theta)x_2\| \le \|\theta x_1\| + \|(1-\theta)x_2\|$$
$$= \theta\|x\| + (1-\theta)\|y\|$$

$\Rightarrow$ Any valid norm is convex hence $\|x\|_2$ the l_2 norm is convex

$-x_1$ is also convex as it is linear to $\|x\|_2 + (-x_1)$ is convex.

The objective function and its constraints are both convex hence problem is a convex opt prob

(b) $p^* = \inf\{ e^{x_2} \mid \|x\|_2 - x_1 \le 0 \}$

consider $\sqrt{x_1^2 + x_2^2} - x_1 \le 0$

$x_1^2 + x_2^2 \le x_1^2 \quad \Rightarrow \quad x_2^2 \le 0$

$\Rightarrow x_2 = 0$

$p^* = e^0 = 1$

(c) $g(\lambda) = \inf_{x_1, x_2} \left\{ \underbrace{e^{x_2}}_{\substack{\geq 0 \\ < \frac{\varepsilon}{2}}} + \lambda \underbrace{\left(\sqrt{x_1^2 + x_2^2} - x_1 \right)}_{\substack{\geq 0 \\ < \frac{\varepsilon}{2}}} \right\}$

consider $x_2 = -t$ and $x_1 = t^3$

$\Rightarrow e^{x_2} = e^{-t}$

$$\sqrt{x_1^2 + t^2} - x_1 = \left(x_1^2 \left(1 + \frac{t^2}{x_1^2} \right) \right)^{1/2} - x_1$$

$$= x_1 \left(1 + \frac{t^2}{x_1^2} \right)^{1/2} - x_1$$

$$= x_1 \left(1 + \frac{t^2}{2x_1^2} + \dots \right) - x_1$$

$$= \frac{t^2}{2x_1} \to 0$$

Hence objective function $< \varepsilon$
≈ 0

$g(\lambda) = 0 \quad \forall \lambda \geq 0$

hence $d^* = \max_{\lambda \geq 0} g(\lambda)$

$= \max(0)$

$= 0$

The duality gap is

$$p^* - d^* = 1 - 0$$
$$= 1$$

(d) Since $\|x\|_2$ is a 2 norm,

we cannot find a slater vector as there does not exist $\|x\|_2 - x_1 < 0$

hence strong duality does not hold and there is a duality gap.

4. (Duality Gap?)

Consider the two-dimensional problem

$$\min_{x \in \mathbb{R}^2} f(x) \quad \text{subject to} \quad x_1 \le 0$$

where

$$f(x) = e^{-\sqrt{x_1 x_2}} \quad \text{for all} \quad x \in \mathbf{dom}\, f = \mathbb{R}^2_+$$

(a) Is this a convex optimization problem?

(b) Evaluate the primal optimal value p^*.

(c) Form the Lagrangian $L(x, \lambda)$, dual function $g(\lambda)$, and find the dual optimal value $d^* = \sup_{\lambda \ge 0} g(\lambda)$.

(d) Is there a duality gap and why?

(a) Exponential function is always convex

$\Rightarrow$ $e^{-\sqrt{x_1 x_2}}$ is convex

Linear constraints always convex

The objective function and its constraints are both convex hence problem is a convex opt prob

(b) $p^* = \inf \{ e^{-\sqrt{x_1 x_2}} \mid x_1 \le 0, \ x \in \mathbb{R}^2_+ \}$

since $x \in \mathrm{dom} f = \mathbb{R}^2_+$ and $x_1 \le 0$

This implies $x_1 = 0$

hence $p^* = e^{-0}$

$= 1$

(c) $g(\lambda) = \inf_{\substack{x_1, x_2 \\ \in \mathbb{R}_+}} \{ e^{-\sqrt{x_1 x_2}} + \lambda x_1 \}$

claim: $\forall \lambda \geq 0$, $g(\lambda) = 0 \Rightarrow d^* = \max g(\lambda)$
$= 0$

Take $x_1 = \frac{1}{t}$, $x_2 = t^2$

$\Rightarrow e^{-\sqrt{x_1 x_2}} = e^{-\sqrt{t}} < \frac{\varepsilon}{2}$

$x_1 \leq 0$ is affine, don't have to find $\bar{x}$ s.t. $\bar{x}_1 < 0$

$(\bar{x}_1, \bar{x}_2)$

as such, take $\min_{x_1, x_2 \geq 0} e^{-\sqrt{x_1 x_2}} = 0$

Hence $d^* = \max 0$
$= 0$

(d) Duality gap exist since

$p^* - d^* = 1 - 0$
$= 1$

(cannot find slater vector)

5. (Simple Convex Optimization)

Consider the problem

$$\min_{x \in \mathbb{R}} x^2 + 1 \quad \text{subject to} \quad (x-2)(x-4) \leq 0.$$

The decision variable is $x \in \mathbb{R}$.

(a) State the feasible set, the optimal value p^*, and the optimal solution.

(b) Plot the objective $x^2 + 1$ versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ versus x for a few positive values of λ. Verify the lower bound property $p^* \geq \inf_{x \in \mathbb{R}} L(x, \lambda)$ for $\lambda \geq 0$. Derive and sketch the Lagrange dual function $g(\lambda)$.

(c) State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution d^*. Does strong duality hold?

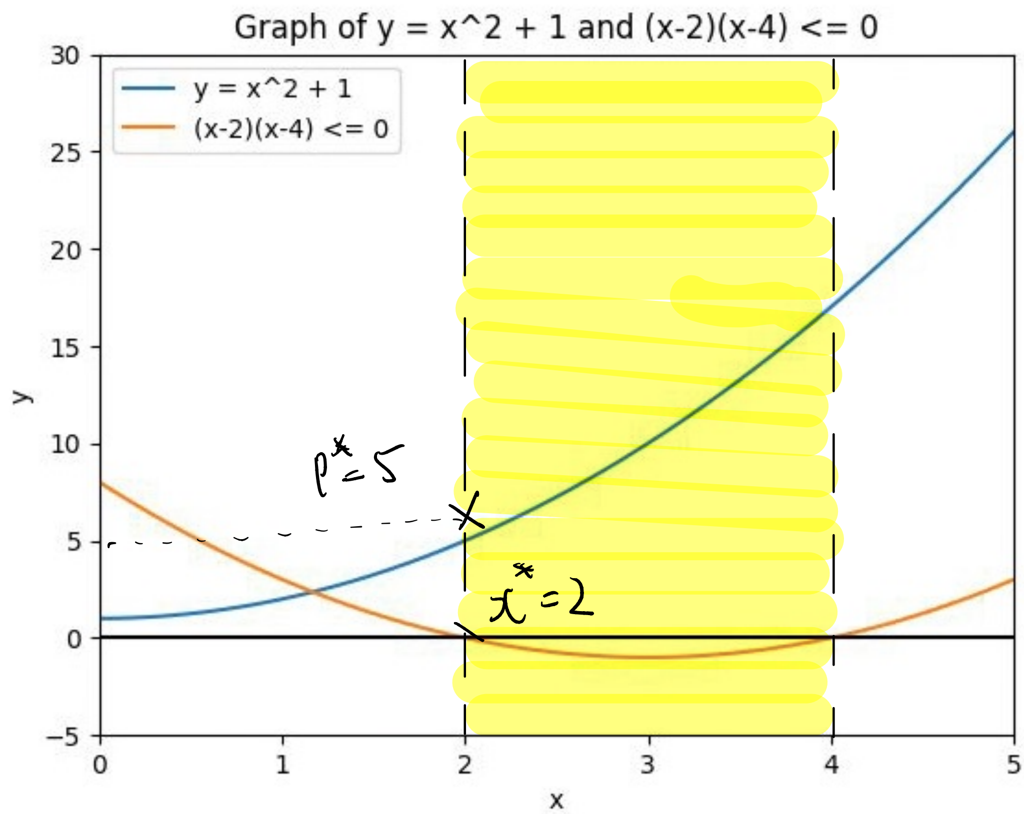
a) The feasible set is $[2, 4]$

optimal solution $x^* = 2$

Hence $p^* = 2^2 + 1$

$= 5$

(b)

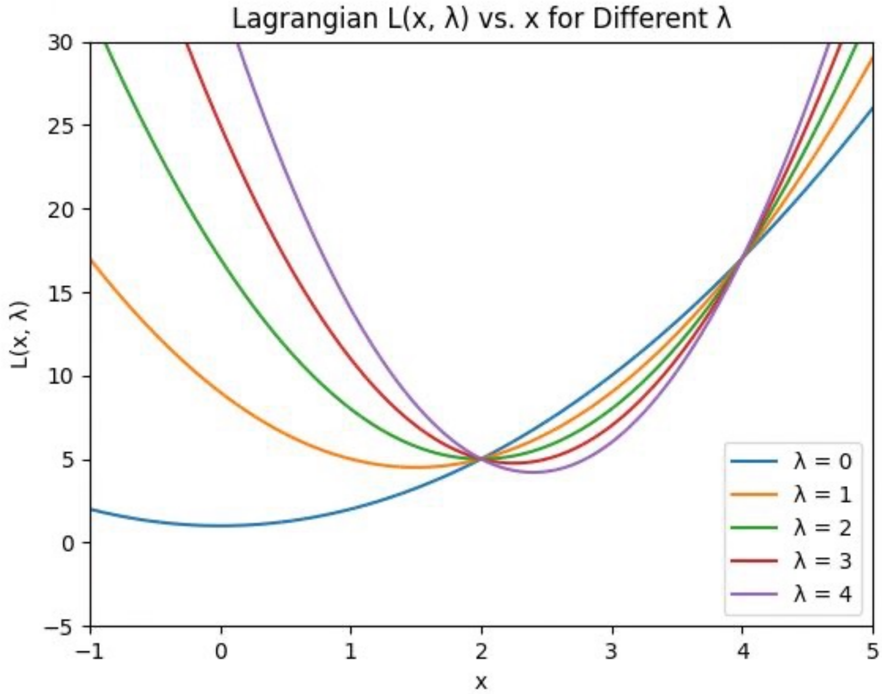


Langrangian $\mathcal{L}(x,\lambda)$

$$= x^2+1+\lambda(x-2)(x-4)$$

$$= x^2+1+\lambda\{x^2-6x+8\}$$

$$= (1+\lambda)x^2-6\lambda x+(1+8\lambda)$$



$$g(\lambda) = \inf_{x\in D}\Big\{\underbrace{(1+\lambda)x^2-6\lambda x+(1+8\lambda)}_{h(x)}\Big\}$$

Note: minimum value of $\mathcal{L}(x,\lambda)$ over x (ie. $g(\lambda)$) is always less than p^*. when $\lambda=2$, we have $p^*=g(\lambda)$

For $\lambda \le -1$, $g(\lambda) \to -\infty$

when $\lambda > -1$

we find $h'(x) = 2(1+\lambda)x - 6\lambda = 0$

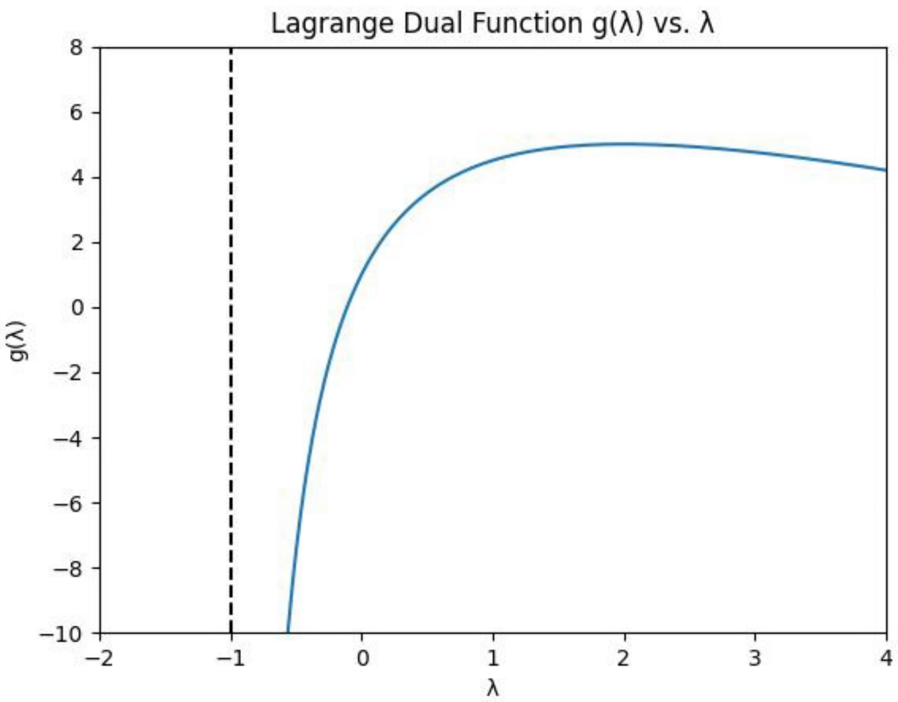
$$\Rightarrow x = \frac{3\lambda}{(1+\lambda)}$$

$$g(\lambda) = (1+\lambda)\left[\frac{3\lambda}{(1+\lambda)}\right]^2 - 6\lambda\left(\frac{3\lambda}{1+\lambda}\right) + (1+8\lambda)$$

$$= \frac{9\lambda^2}{(1+\lambda)} - \frac{18\lambda^2}{(1+\lambda)} + (1+8\lambda)$$

$$= \frac{-9\lambda^2}{(1+\lambda)} + 1 + 8\lambda$$

Hence
$$g(\lambda) = \begin{cases} -\frac{9\lambda^2}{1+\lambda} + (1+8\lambda) & \text{for } \lambda > -1 \\ -\infty & \text{otherwise} \end{cases}$$



(c) $d^* = \max_{\lambda \geq 0} g(\lambda)$ for $\lambda > -1$

$= \max \dfrac{-9\lambda^2}{1+\lambda} + (1+8\lambda)$ s.t $\lambda \geq 0$

Max achieved when $\lambda = 2$,

$d^* = \dfrac{-9(2)^2}{1+2} + (1+8(2))$

$= 5$

$p^* - d^* = 5 - 5$

$= 0$

There is no duality gap hence strong duality holds