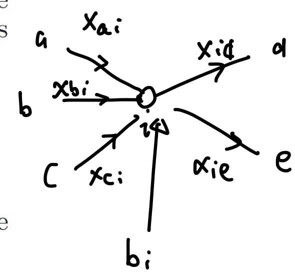
Tutorial week 7. Xiaochen. Q1.Q2.Q3.Q4.

Q1. **4.12** *Network flow problem.* Consider a network of n nodes, with directed links connecting each pair of nodes. The variables in the problem are the flows on each link: x_{ij} will denote the flow from node i to node j. The cost of the flow along the link from node i to node j is given by $c_{ij}x_{ij}$, where c_{ij} are given constants. The total cost across the network is



$$C = \sum_{i,j=1}^{n} c_{ij}x_{ij}.$$

Each link flow x_{ij} is also subject to a given lower bound l_{ij} (usually assumed to be nonnegative) and an upper bound u_{ij}.

The external supply at node i is given by b_i, where $b_i > 0$ means an external flow enters the network at node i, and $b_i < 0$ means that at node i, an amount $|b_i|$ flows out of the network. We assume that $\mathbf{1}^T b = 0$, *i.e.*, the total external supply equals total external demand. At each node we have conservation of flow: the total flow into node i along links and the external supply, minus the total flow out along the links, equals zero.

The problem is to minimize the total cost of flow through the network, subject to the constraints described above. Formulate this problem as an LP.

$$\min_{x=\{x_{ij}\}} \quad \underbrace{C = \sum_{i,j\in[n]^2} c_{ij}x_{ij}}_{\text{linear}} \qquad \text{s.t.} \quad l_{ij} \le x_{ij} \le u_{ij} \ \text{ for all } i,j \quad \text{linear}$$

$$\underbrace{b_i + \sum_{j=1}^{n} x_{ji} = \sum_{k=1}^{n} x_{ik}}_{\text{linear.}} \qquad \text{for all } i.$$

Q2. **4.23** *ℓ_4-norm approximation via QCQP.* Formulate the ℓ_4-norm approximation problem

$$\text{minimize} \quad \|Ax-b\|_4 = \left(\sum_{i=1}^m (a_i^T x - b_i)^4\right)^{1/4}$$

as a QCQP. The matrix $A \in \mathbf{R}^{m\times n}$ (with rows a_i^T) and the vector $b \in \mathbf{R}^m$ are given.

QCQP:

$$\min_x \ \frac{1}{2}x^T P_0 x + q_0^T x + r_0.$$

$$\text{s.t.} \begin{cases} \frac{1}{2}x^T P_i x + q_i^T x + r_i \le 0. & i=1,\cdots,m. \\ Ax = b. \end{cases}$$

$$\min_x \left(\sum_{i=1}^m (a_i^T x - b_i)^4\right)^{1/4}$$

$\Leftrightarrow \quad \min_x \sum_{i=1}^m (a_i^T x - b_i)^4$ this is because $x^{1/4}$ is monotonic.

$\Leftrightarrow \quad \min_{x,t\in R^m} \sum_{i=1}^m t_i$ s.t. $(a_i^T x - b_i)^4 \le t_i$ epigraph form, but not QCQP

$\Leftrightarrow \quad \min_{x,z\in R^m} \sum_{i=1}^m z_i^2$ s.t. $(a_i^T y - b_i)^4 \le z_i^2$ change of variable: $t_i = z_i^2$

$\Leftrightarrow \quad \min_{x,z\in R^m} \sum_{i=1}^m z_i^2$ s.t. $(a_i^T x - b_i)^2 \le z_i$

This is a QCQP in $\begin{pmatrix} x \\ z \end{pmatrix} \in R^{n+m}$:

$$\sum_{i=1}^m z_i^2 = \begin{pmatrix} x \\ z \end{pmatrix}^T \begin{pmatrix} 0_{n\times n} & 0_{n\times m} \\ 0_{n\times m} & I_{m\times m} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}$$

$(a_i^T x - b_i)^2 \le z_i$: $\quad x^T a_i a_i^T x + b_i^2 - 2b_i a_i^T x - z_i \le 0.$

$$\therefore \begin{pmatrix} x \\ z \end{pmatrix}^T \begin{pmatrix} a_i a_i^T & 0_{n\times m} \\ 0_{n\times m} & 0_{m\times m} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} - \begin{pmatrix} \overset{n\times 1}{2b_i a_i^T} \\ \underset{m\times 1}{e_i} \end{pmatrix}^T \begin{pmatrix} x \\ z \end{pmatrix} \le 0.$$

Q3.

4.28 *Robust quadratic programming.* In §4.4.2 we discussed robust linear programming as an application of second-order cone programming. In this problem we consider a similar robust variation of the (convex) *quadratic* program

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Ax \preceq b. \end{array}$$

For simplicity we assume that only the matrix P is subject to errors, and the other parameters (q, r, A, b) are exactly known. The robust quadratic program is defined as

$$\begin{array}{ll} \text{minimize} & \sup_{P\in\mathcal{E}}((1/2)x^T P x + q^T x + r) \\ \text{subject to} & Ax \preceq b \end{array}$$

where $\mathcal{E}$ is the set of possible matrices P.

For each of the following sets $\mathcal{E}$, express the robust QP as a convex problem. Be as specific as you can. If the problem can be expressed in a standard form (*e.g.*, QP, QCQP, SOCP, SDP), say so.

(a) A finite set of matrices: $\mathcal{E} = \{P_1, \ldots, P_K\}$, where $P_i \in \mathbf{S}^n_+$, $i = 1, \ldots, K$.

(b) A set specified by a nominal value $P_0 \in \mathbf{S}^n_+$ plus a bound on the eigenvalues of the deviation $P - P_0$:

$$\mathcal{E} = \{P \in \mathbf{S}^n \mid -\gamma I \preceq P - P_0 \preceq \gamma I\}$$

where $\gamma \in \mathbf{R}$ and $P_0 \in \mathbf{S}^n_+$,

(c) An ellipsoid of matrices:

$$\mathcal{E} = \left\{ P_0 + \sum_{i=1}^K P_i u_i \;\middle|\; \|u\|_2 \le 1 \right\}.$$

You can assume $P_i \in \mathbf{S}^n_+$, $i = 0, \ldots, K$.

(a). $\min\limits_{x} \sup\limits_{P\in\mathcal{E}} \left(\frac{1}{2}x^T P x + q^T x + r\right)$ s.t. $Ax \preceq b$

$\Leftrightarrow$ $\min\limits_{x} \max\limits_{i=1,2,\cdots,k} \left(\frac{1}{2}x^T P_i x + q^T x + r\right)$ s.t. $Ax \preceq b$.

$\Leftrightarrow$ $\min\limits_{x,t\in R} t$ s.t. $\max\limits_{i=1,2,\cdots,k} \left(\frac{1}{2}x^T P_i x + q^T x + r\right) \le t$ and $Ax \preceq b$

$\Leftrightarrow$ $\min\limits_{x,t\in R} t$ s.t. $\frac{1}{2}x^T P_i x + q^T x + r \le t$ for $i=1,\cdots,k$ and $Ax \preceq b$

This is a QCQP. in $\binom{x}{t} \in R^{n+1}$. You can try write this as standard form w.r.t. $\binom{x}{t} \in R^{n+1}$

(b). First, $P - P_0 \le \gamma I$ means $P - P_0 - \gamma I$ is NSD

$\Leftrightarrow$ $\lambda_{max}(P - P_0 - \gamma I) = \lambda_{max}(P - P_0) - \gamma \le 0$.

$P - P_0 \ge -\gamma I$ means $P - P_0 + \gamma I$ is PSD.

$\Leftrightarrow$ $\lambda_{min}(P - P_0 + \gamma I) = \lambda_{min}(P - P_0) + \gamma \ge 0$

$\therefore$ $\lambda_{max}(P - P_0) \le \gamma$ $\quad \lambda_{min}(P - P_0) \ge -\gamma$

$$\min_x \ \sup_{P\in\mathcal{E}} \left(\tfrac{1}{2}x^TPx + q^Tx + r\right) \quad \text{s.t. } Ax \preceq b$$

$$\Leftrightarrow \min_x \ q^Tx + r + \sup_{P\in\mathcal{E}} \left(\tfrac{1}{2}x^TPx\right) \quad \text{s.t. } Ax \le b.$$

$-\gamma \le \lambda(Q) \le \gamma$

$\Updownarrow$

Now we focus on $\sup_{P\in\mathcal{E}} (\frac{1}{2}x^TPx)$. let $Q = P - P_0$. then $-\gamma I \preceq Q \preceq \gamma I$

We try to find $\sup_{-\gamma I \preceq Q \preceq \gamma I} \frac{1}{2}x^T(Q+P_0)x$

$$= \tfrac{1}{2}x^TP_0x + \sup_{-\gamma I \preceq Q \preceq \gamma I} \tfrac{1}{2}x^TQx.$$

non-negative.

write $Q = V\Lambda V^T$ (eigenvector decomposition, $\Lambda_{ii} = \lambda_i(Q)$)

$$\tfrac{1}{2}x^TV\Lambda V^Tx = \tfrac{1}{2}(V^Tx)^T\Lambda(V^Tx) = \tfrac{1}{2}u^T\Lambda u = \tfrac{1}{2}\sum_{i=1}^n \lambda_i (u_i)^2$$

hence, $\frac{1}{2}x^TQx$ is maximized (over Q) when all eigenvalues of Q is maximized (as γ)

$$\therefore \sup_{-\lambda I \preceq Q \preceq \lambda I} \tfrac{1}{2}x^TQx = \tfrac{1}{2}x^T(\gamma I)x$$

$$\therefore \quad = \tfrac{1}{2}x^TP_0x + \tfrac{1}{2}x^T(\gamma I)x \qquad \text{(QP)}$$

$\therefore$ the opt prob $\Leftrightarrow$ $\min_x \ q^Tx + r + \frac{1}{2}x^TP_0x + \frac{1}{2}x^T(\gamma I)x$ s.t. $Ax \le b$.

(c). $\mathcal{E} = \left\{P_0 + \sum_{i=1}^K P_iu_i \;\middle|\; \|u\|_2 \le 1\right\}$ $\quad P_i \in S_+^n$.

$$\min_x \ \sup_{P\in\mathcal{E}} \left(\tfrac{1}{2}x^TPx + q^Tx + r\right) \quad \text{s.t. } Ax \preceq b$$

$$\Leftrightarrow \min_x \ q^Tx + r + \sup_{P\in\mathcal{E}} \left(\tfrac{1}{2}x^TPx\right) \quad \text{s.t. } Ax \le b.$$

$$\sup_{P\in S} \left(\tfrac{1}{2}x^TPx\right) = \sup_{u:\|u\|_2\le 1} \tfrac{1}{2}x^T\left(P_0 + \sum_{i=1}^K P_iu_i\right)x$$

$$= \tfrac{1}{2}x^TP_0x + \sup_{u:\|u\|_2\le 1} \tfrac{1}{2}\sum_{i=1}^K \overbrace{(x^TP_ix)}^{w_i}u_i$$

$$\sup_{u:\|u\|_2\le 1} \tfrac{1}{2}\sum w_iu_i = \sup_{u:\|u\|_2\le 1} \tfrac{1}{2}w^Tu \qquad w^Tu \le \|w\|_2\,\|u\|_2 \le \underline{\|w\|_2}$$

$$\therefore \sup_u \tfrac{1}{2}\sum_{i=1}^K (x^TP_ix)u_i = \tfrac{1}{2}\left(\sum_{i=1}^K (x^TP_ix)^2\right)^{\frac{1}{2}}$$

$$\therefore \min_x \ \underbrace{q^Tx + r + \tfrac{1}{2}x^TP_0x}_{\text{convex}} + \tfrac{1}{2}\left(\sum_{i=1}^K \underbrace{(x^TP_ix)^2}_{\text{convex}}\right)^{1/2} \quad \text{s.t. } Ax \le b.$$

↳ l_2 norm is non-decreasing convex.

$\therefore$ the objectie function is convex. We try to write it as SOCP

$$\Leftrightarrow \min_{x, y \in \mathbb{R}^k} \frac{1}{2} x^T P_0 x + \|y\|_2 + q^T x + r$$

$$\text{s.t.} \quad \frac{1}{2} x^T P_i x \le y_i \quad Ax \le b$$

$$\min_x \quad f^T x$$

$$\text{s.t.} \quad \|A_i x + b\|_2 \le c_i^T x + d_i$$

for $i = 1, \cdots, m$

$$Fx = g.$$

$$\Leftrightarrow \min_{x \in \mathbb{R}^n, y \in \mathbb{R}^k, u \in \mathbb{R}, t \in t} u + t + q^T x + r$$

$$\text{s.t.} \quad \frac{1}{2} x^T P_i x \le y_i, \quad Ax \le b. \quad \frac{1}{2} x^T P_0 x \le u. \quad \|y\|_2 \le t$$

$P_i = (P_i^{\frac{1}{2}})^T P_i^{\frac{1}{2}}$ since P_i is PSD. ($P_i^{1/2}$ is PSD)

$$(P_i^{1/2} x)^T (P_i^{1/2} x) \le 2 y_i$$

L2-norm $\le c^T x + d$ is sum of squares $\le$ square.

$$(P_i^{1/2} x)^T (P_i^{1/2} x) + (y_i + a)^2 \le (y_i + b)^2$$

Want: $a^2 = b^2$, $2b - 2a = 2$. $\Rightarrow a = -\frac{1}{2}$, $b = \frac{1}{2}$

$$\Leftrightarrow (P_i^{1/2} x)^T (P_i^{1/2} x) + (y_i - \tfrac{1}{2})^2 \le (y_i + \tfrac{1}{2})^2$$

$$\Leftrightarrow \left\| \begin{pmatrix} P_i^{1/2} x \\ y_i - \frac{1}{2} \end{pmatrix} \right\|_2 \le y_i + \frac{1}{2}$$

Similarly, $\frac{1}{2} x^T P_0 x \le u \Leftrightarrow \left\| \begin{pmatrix} P_0^{1/2} x \\ u - \frac{1}{2} \end{pmatrix} \right\|_2 \le u + \frac{1}{2}$.

$\therefore$ SOCP.

Q4. **4.33** Express the following problems as convex optimization problems.

(a) Minimize $\max\{p(x), q(x)\}$, where p and q are posynomials.

(b) Minimize $\exp(p(x)) + \exp(q(x))$, where p and q are posynomials.

Posynomials: $f(x) = \sum_{i=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$.

f may not be convex, but consider c.o.v $y_i = \log x_i$ $\quad x_i = e^{y_i}$

then $\quad c x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} = \exp(a_1 y_1 + a_2 y_2 + \cdots + a_n y_n + \log c)$

becomes convex in y.

(a). $\min\limits_{x} \max(p(x), q(x))$

$\Leftrightarrow \min\limits_{x, t \in R} t$, $\quad p(x) \leq t. \quad q(x) \leq t.$ $\qquad$ p and q may not be convex.

do change of variables. $y_i = \log x_i$

(b). $\min\limits_{x} \exp(p(x)) + \exp(q(x))$

$\Leftrightarrow \min\limits_{x, t_1 \in R, t_2 \in R} \exp(t_1) + \exp(t_2)$ $\quad$ s.t. $\quad p(x) \leq t_1. \quad q(x) \leq t_2.$

p and q may not be convex so do change of variables to $y_i = \log x_i$