DSA3102: Lagrangian and Related Functions

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Consider an optimization problem

$$\min_{x\in\mathbb{R}^n} f_0(x) \quad \text{s.t.} \quad f_i(x) \le 0,\ i \in [m] \quad h_i(x) = 0,\ i \in [p]. \tag{1}$$

The **(primal) feasible set** is

$$X = \{x \in \mathbb{R}^n : f_i(x) \le 0,\ i \in [m] \quad h_i(x) = 0,\ i \in [p]\}. \tag{2}$$

The **primal optimal value** is

$$p^* = \inf\{f_0(x) : x \in X\}. \tag{3}$$

The **Lagrangian** $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ is

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x). \tag{4}$$

The **Lagrange dual function** $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ is

$$g(\lambda, \nu) = \inf_{x\in\mathbb{R}^n} L(x, \lambda, \nu) = \inf_{x\in\mathbb{R}^n} \left\{ f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x) \right\}. \tag{5}$$

The **Lagrange dual problem** is

$$\max_{\lambda\in\mathbb{R}^m, \nu\in\mathbb{R}^p} g(\lambda, \nu) \quad \text{s.t.} \quad \lambda \ge 0. \tag{6}$$

The **(dual) feasible set** is

$$\{(\lambda, \nu) \in \mathbb{R}^m \times \mathbb{R}^p : \lambda \ge 0\}. \tag{7}$$

The **dual optimal value** is

$$d^* = \max_{\lambda\ge 0, \nu\in\mathbb{R}^p} g(\lambda, \nu). \tag{8}$$

Consider the special case where there are no equality constraints, i.e., $p = 0$ in (1). It also holds that

$$p^* = \inf_{x\in\mathbb{R}^n} \sup_{\lambda\ge 0} L(x, \lambda) \tag{9}$$

because

$$\sup_{\lambda\ge 0} L(x, \lambda) = \begin{cases} f_0(x) & f_i(x) \le 0,\ i = 1, \ldots, m \\ +\infty & \text{else} \end{cases}.$$

By the definition of the dual function (combining (5) and (8)),

$$d^* = \sup_{\lambda\ge 0} \inf_{x\in\mathbb{R}^n} L(x, \lambda). \tag{10}$$

From here, **weak duality**

$$d^* \le p^* \tag{11}$$

is obvious to see.