Spring 2020 SIT22004

ICT Problem Solving

Algorithm Design and Analysis

March 4, 2020

Analyzing algorithms

How to compare algorithms ?

Given algorithms to solve a problem, how do we compare algorithms ?

The **rate of growth** of the time or space requirements to solve a **large instance** of the problem.

The size of the problem

Input size: the amount of data given as the input

Examples

sorting : the length of the list containing the data

graph coloring : $|V| + |E|$ for $G = (V, E)$

matrix multiplication : the number of elements in the matrices

Running time

# of primitive operations (steps) executed

Complexity

Time Complexity

The number of time steps to solve a problem

Expressed as a function of problem size

Space Complexity

The amount of memory space to solve a problem

Expressed as a function of problem size

Worst case complexity

The worst possible complexity over all inputs of a given problem size

Average case complexity

The average complexity over all inputs of a given problem size

Optimal algorithm

(Tight) Lower bound $\mathbf{L}(\boldsymbol{n})$

The minimum time complexity over **all possible** algorithms

(Tight) Upper bound $\mathbf{U}(\boldsymbol{n})$

The minimum time complexity over **all known** algorithms

Optimal algorithm

An algorithm whose time complexity is the same as the lower bound of the problem, $\mathrm{L}(n)$

Introduction

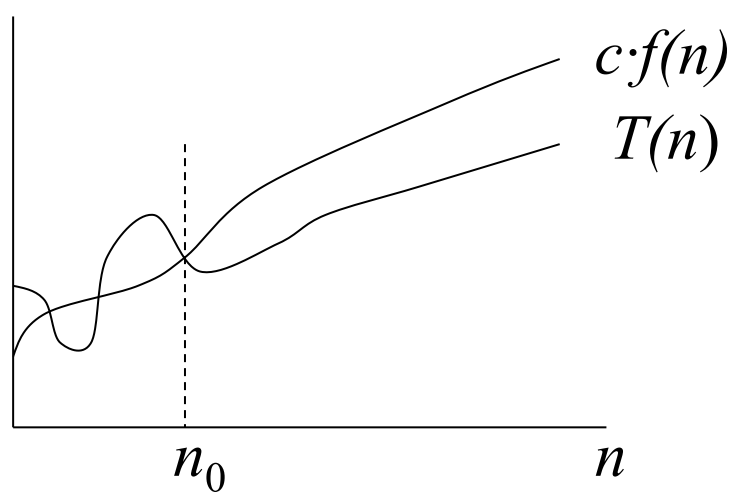
Measuring efficiency of algorithms

$\boxed{\text{time complexity}}$: T(n)

space complexity : S(n)

"Big-Oh" notation: e.g., $T(n) = O(f(n))$

$T(n)$ is $O(f(n))$ $\Leftrightarrow$ There exist positive constants c and n_0 such that $T(n) \le c \cdot f(n)$ for all $n \ge n_0$



Example

Let $T(n) = n^2/2 + 3n + 10$

Then $T(n) = O(n^2)$! Why?

$T(n) \leq c \cdot n^2$ for all $n \geq 1$ and $c = 27/2$!!!

Is $T(n)$ also $O(n^3)$?

Is $T(n)$ also $O(n)$?

The Maximum Subsequence Sum Problem

Given a sequence of n numbers, $X=(x_1, x_2, \ldots, x_n)$, find a subsequence $X^* \subseteq X$ such that

(i) the numbers in X^* is contiguous in X

(ii) the sum of the numbers in X^* is the maximum over all contiguous subsequences of X

X =

(31,	-41,	59,	26,	-53,	58,	97,	-93,	-23,	84)
x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}

187

3 7

X*=X[3..7]

Observations

$$X = (x_1, x_2, x_3, \dots , x_n)$$

What if $x_i > 0$ for all $1 \leq i \leq n$?

What if $x_i < 0$ for all $1 \leq i \leq n$?

Brute Force Algorithm

How many subsequences?

X[L..U]

L	U	
1	1 2 ⋮ n	n
2	2 3 ⋮ n	n-1
3 ⋮ ⋮	3 4 ⋮ n	n-2
⋮	⋮	⋮
n	n	1
		$\frac{n(n+1)}{2}$

For each subsequence, we need at most n-1 additions

why?

$\therefore$ O(n)

Then, the total number of operations is at most

$$\frac{n(n+1)}{2} \times (n-1) = \frac{1}{2}(n^3 - n)$$

$\therefore$ O(n^3)

Procedure Max-sub

```
begin
        MaxSoFar ← 0
        for L ← 1 to n do
                for U ← L to n do
                        Sum ← 0
                        for I ← L to U do
                                Sum ← Sum+x[I]
                        end-do
                        MaxSoFar ← max { MaxSoFar, Sum}
                end-do
        end-do
end
```

Do we need this?

$O(n^3)$

$O(n^2)$ Algorithm

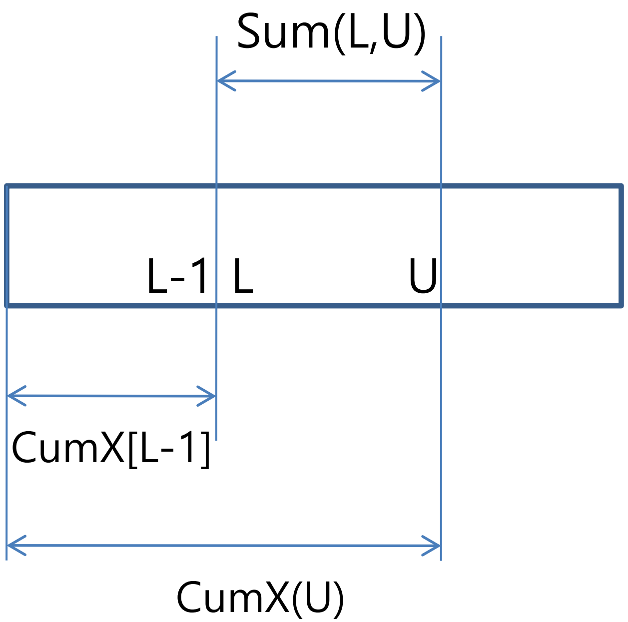
Sum of X[L..U] = Sum(L,U), $1 \le L \le n$, $L \le U \le n$

$$\mathrm{Sum(L,U)} = \begin{cases} X[U] & \text{if } L = U \\ \mathrm{Sum}[L.U-1] + x[U] & \text{otherwise} \end{cases}$$

```
Procedure Max-sub
    begin
        MaxSoFar ← 0
        for L ← 1 to n do
            Sum ← 0
            for U← L to n do
                Sum ← Sum+X[U]
                MaxSoFar ← max { MaxSoFar, Sum}
            end-do
        end-do
    end
```

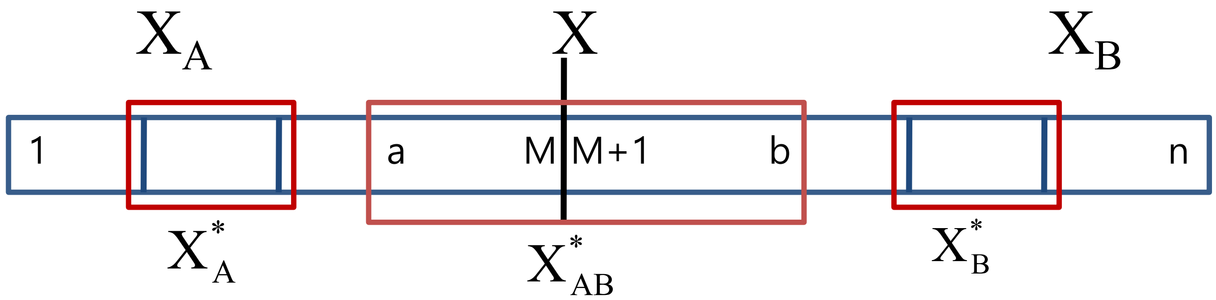
$O(n^2)$

A Yet Another $O(n^2)$ Algorithm




```
Procedure Max-sub
    begin
        CumX[1] ← X[1]
        for I ← 2 to n do
            CumX[ I ] ← CumX[I-1]+X[ I ]      Preprocessing
        end-do
        MaxSoFar ← 0
        for L ← 1 to n do
            for U ← L to n do
                Sum ← CumX[U]-CumX[L-1]
                MaxSoFar ← max { MaxSoFar, Sum}
            end-do
        end-do
    end
```

"Divide and Conquer" Algorithm



$M = \lfloor (n+1)/2 \rfloor$

MaxA MaxAB MaxB

(MaxCA+MaxCB)

$$\text{MaxX} = \max\{\ \underset{X^*_A}{\text{MaxA}},\ \underset{X^*_B}{\text{MaxB}},\ \underset{X^*_{AB}}{\text{MaxAB}}\ \}$$

$$T(n) = 2T(n/2) + M(n)$$

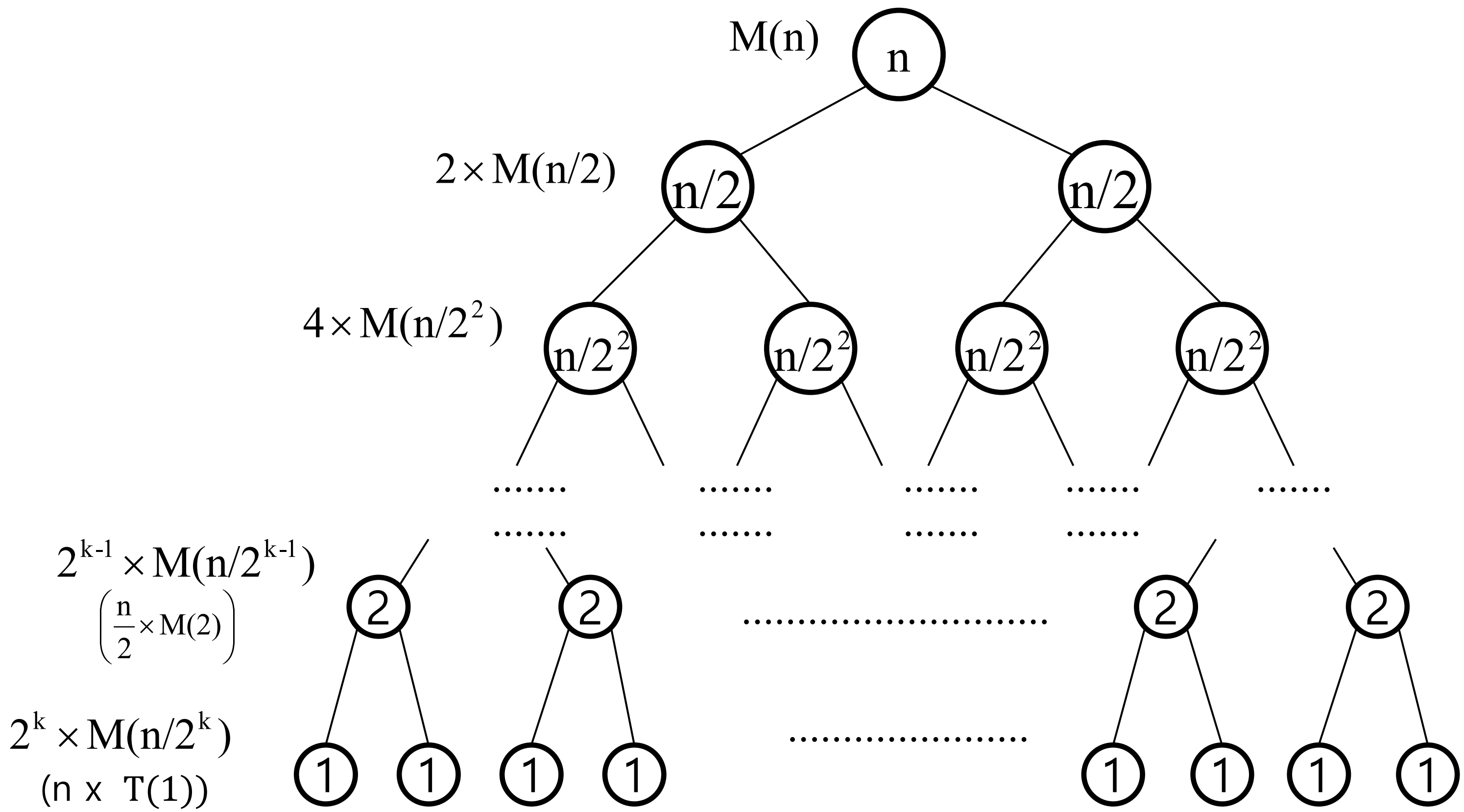
Compute MaxA: $T(n/2)$

Compute MaxB: $T(n/2)$

$\boxed{\text{Divide and Merge}}$: $M(n)$

Need to compute MaxAB

$$T(n) = 2T(n/2) + M(n)$$



$$\boxed{n = 2^k}$$

How many levels? $\log_2 n$

Why?

$n/2^k = 1 \quad \therefore\ k = \log_2 n$

What is T(1)?

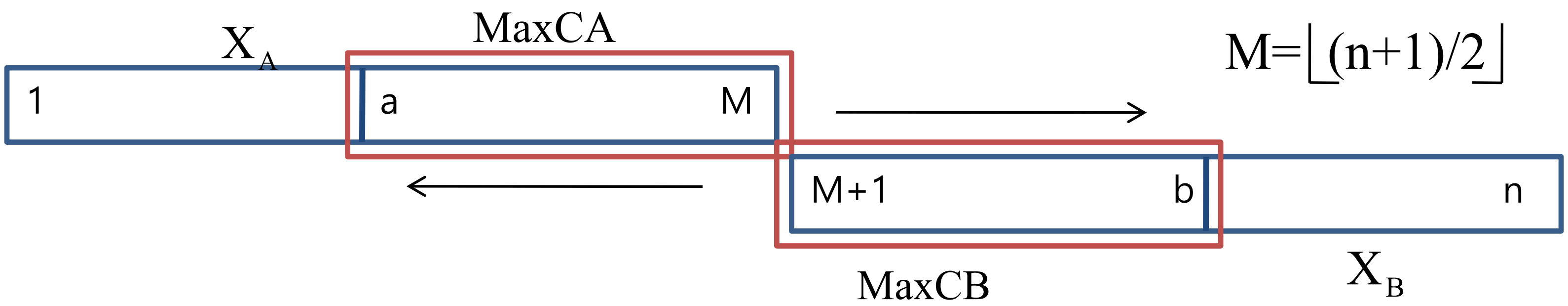
No further division!

$$\therefore \quad \mathrm{MaxX}(X[I..I]) = \begin{cases} X[I] & \text{if } X[I] > 0 \quad (X^* = X[I..I]) \\ 0 & \text{if } X[I] \le 0 \quad (X^* = \phi) \end{cases}$$

What is M(n) for n > 1?

It depends on how to compute MaxAB!

How to Compute MaxAB



$M = \lfloor (n+1)/2 \rfloor$

MaxAB=MaxCA+MaxCB

where

MaxCA = max{ Sum(1, M), Sum(2, M), ……………………, Sum(M-1, M), Sum(M, M) }

MaxCB = max{ Sum(M+1, M+1), Sum(M+1, M+2), ……, Sum(M+1, n-1), Sum(M+1, n) }

Note

$$\mathrm{Sum}(I, M) = \begin{cases} X[I] & \text{if } I = M \\ \mathrm{Sum}(I+1, M) + X[I] & \text{otherwise} \end{cases}$$

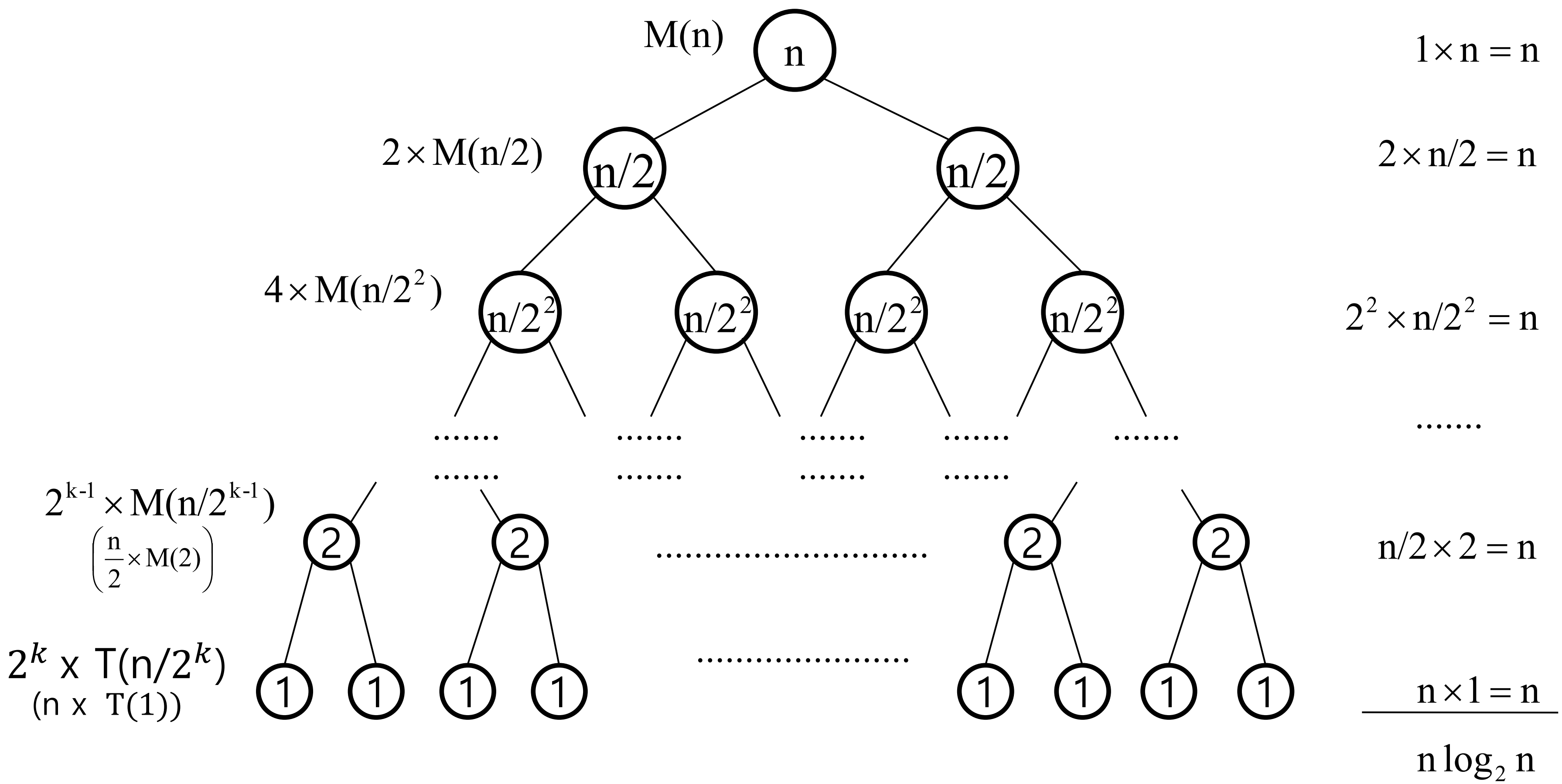
$\therefore$ O(M) time to compute MaxCA

Similarly, MaxCB can also be computed O(M) time

$\therefore$ M(n)=O(M)+O(M)=O(n) if n>1

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T(n/2) + n & \text{if } n > 1 \end{cases}$$

$$T(n) = O(n \log n)$$



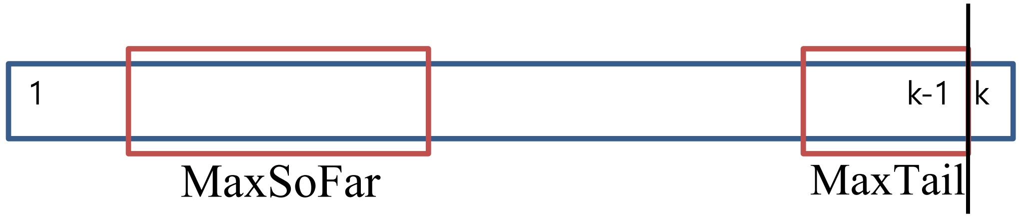
```
function Max-Sub(L, U)
    begin
        if L=U then return(Max{0, X[L]})

        M ← ⌊(L + U)/2⌋
        MaxA = Max-Sub(L, M)
        MaxB = Max-Sub(M+1,U)

        Sum ← 0; MaxCA ← 0
        For I ← M downto L do
                Sum ← Sum+X[I]
                MaxCA ← max{ MaxCA, Sum }
        end-do

        Sum ← 0; MaxCB ← 0
        For I ← M+1 to U do
                Sum ← Sum+X[I]
                MaxCB ← max{ MaxCB, Sum }
        end-do
        MaxAB ← MaxCA + MaxCB
        return( max{ MaxA, MaxB, MaxAB } )
    end
```

O(n) Algorithm



Given MaxSoFar and MaxTail for X[1..k-1], how to update them for X[1..k] ?

Initially,
Maxtail = {0, X[1]}
Maxsofar = Maxtail

In general,
$\text{MaxTail} = \max\{0,\ \text{MaxTail} + X[k]\}$
$\text{MaxSoFar} = \max\{\text{MaxSoFar},\ \text{MaxTail}\}$

```
Procedure Max-sub
    begin
        MaxmSoFar ← 0;   MaxTail ← 0
        for k ← 1 to n
            MaxTail ← max{ 0, MaxTail+X[k] }
            MaxSoFar ← max{ MaxSoFar, MaxTail }
        end-do
    end
```

Example

MaxTail ← max{ 0, MaxTail+X[k] }
MaxSoFar ← max{ MaxSoFar, MaxTail }

$$X[1:10] = (31, -41, \underset{\uparrow\,3}{59}, 26, -53, 58, \underset{\uparrow\,7}{97}, -93, -23, 84)$$

k	MaxTail	MaxSoFar
1	31 (X[1..1]	31 (X[1..1])
2	0 (Ø)	31 (X[1..1])
3	59 (X[3..3])	59 (X[3..3])
4	85 (X[3..4])	85 (X[3..4])
5	32 (X[3..5])	85 (X[3..4])
6	90 (X[3..6])	90 (X[3..6])
7	187 (X[3..7])	187 (X[3..7])
8	94 (X[3..8])	187 (X[3..7])
9	71 (X[3..9])	187 (X[3..7])
10	155 (X[3..10])	187 (X[3..7])

Summary

ALGORITHM		1	2	3	4
Run time in nanoseconds		$1.3 n^3$	$10 n^2$	$47 n \log_2 n$	$48 n$
Time to solve a problem of size	10^3	1.3 secs	10 msecs	.4 msecs	.05 msecs
	10^4				
	10^5				
	10^6				
	10^7				
Max size problem solved in one	sec				
	min				
	hr				
	day				
If n multiplies by 10, time multiplies by					
If time multiplies by 10, n multiplies by					

Extreme Comparison

Algorithm 1 at 533MHz is $0.58n^3$ nanoseconds. Algorithm 4 interpreted at 2.03MHz is $19.5n$ milliseconds, or $19{,}500{,}000n$ nanoseconds.

n	1999 ALPHA 21164A, C, CUBIC ALGORITHM	1980 TRS-80, BASIC, LINEAR ALGORITHM
10	0.6 microsecs	200 millisecs
100	0.6 millisecs	2.0 secs
1000	0.6 secs	20 secs
10,000	10 mins	3.2 mins
100,000	7 days	32 mins
1,000,000	19 yrs	5.4 hrs

Extreme Comparison