MTH 420/520 Homework 6. **Yesh Godse**
Date: May 31, 2019
Prof. M. Peszynska

Problem A-1. Find $u_j(t)$ so that $u(x,t) = \sum_{j=1}^{\text{inf}} u_j(t)sin(jx)$ solves the following:

$$c\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = f, x \in (0,\pi), u(0,t) = u(\pi,t) = 0$$
$$\text{supplemented by the initial condition} u(x,0) = u_{init}(x)$$

See Problem III.7 in class notes for full problem statement

Solution 1. separating each term of the equation gives us the following.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\sum_{j=1}^{\text{inf}} u_j(t)sin(jx) \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\sum_{j=1}^{\text{inf}} u_j(t)cos(jx)$$
$$= \sum_{j=1}^{\text{inf}} u_j'(t)sin(jx) \qquad \frac{\partial^2 u}{\partial x^2} = -u_j(t)j^2 sin(jx)$$

We recombine and divide out the $\sum sin(jx)$ to get

$$cu_j'(t) + j^2 k u_j(t) = f_j \tag{1}$$

To solve this, first we analyze the general solution by assuming $f_j = 0$ and $u_j(0) \neq 0$. For readability, we let $u_j(t) = y$

$$cy' + kj^2 y = 0$$
$$cy'\frac{1}{y} = -kj^2$$
$$c\int y'\frac{1}{y}dy = -kj^2\int dt$$
$$cln(y) = -kj^2 t + \alpha_1$$
$$u_j(t) = y = e^{\frac{\alpha_1}{c}} e^{\frac{-kj^2}{c}} \tag{2}$$

We apply the initial condition $u(x,0) = u_i nit(x)$ to get $\alpha_1 = cln(u_{init}(x))$. Thus, we can rewrite (2) as follows:

$$u_j(t) = y = e^{ln(u_{init}(x))} e^{\frac{-kj^2}{c}} \tag{3}$$

Similarly, we evaluate (1) with $f_j \neq 0$ and $u_j(0) = 0$ to find the particular solution.

$$y' + \frac{kj^2}{c}y = \frac{f_j}{c}$$
$$\text{let } \mu = e^{\frac{kj^2 t}{c}} \text{ (integrating factor)}$$
$$\mu(y' + \frac{kj^2}{c}y) = \mu(\frac{f_j}{c})$$

Solving for y gives us

$$u_j(t) = y = \frac{f_j}{c} + \alpha_2 e^{\frac{-kj^2 t}{c}} \tag{4}$$

If $u_j(0) = 0$, $cu_j'(j) = f_j$:

$$\alpha_2 = -\frac{f_j}{kj^2}$$
$$\implies u_j(t) = \frac{f_j}{kj^2} - \frac{f_j}{uj^2}e^{\frac{-kj^2 t}{c}} \tag{5}$$

Combining (5) and (3) gives us the final solution for $u_j(t)$:

$$u_j(t) = e^{\frac{-kj^2}{c}t}\left(e^{ln(u_{init}(x))} - \frac{f_j}{kj^2}\right) + \frac{f_j}{kj^2} \tag{6}$$

Problem A-2. Find $u_j(t)$ so that $u(x,t) = \sum_{j=1}^{\inf} u_j(t)sin(jx)$ solves the following:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0, x \in (0,\pi), u(0,t) = u(\pi,t) = 0$$
supplemented by the initial condition$u(x,0) = u_{init}(x)$

This is called the wave equation.
See Problem III.8 in class notes for full problem statement

Solution 2. separating each term of the equation gives us the following.

$$\frac{\partial^2 u}{\partial t^2} = \sum_{j=1}^{\inf} u_j''(t)sin(jx) \qquad \frac{\partial^2 u}{\partial x^2} = -\sum_{j=1}^{\inf} u_j(t)sin(jx)j^2$$

We recombine and divide out the $\sum sin(jx)$ to get

$$u_j''(t) - c^2 j^2 u_j(t) = f_j \tag{7}$$

(7) is a 2nd order homogeneous ODE of the form $Ay'' + By' + Cy = 0$. It can be solved by finding an r which satisfies $Ar^2 + Br + C = 0$, where the solution $y(t) = e^{rt}$ From this procedure, we get $r = \pm jc$ and $u_j(t) = e^{jct}$ or e^{-jct}. By the principal of superposition, the general solution to the wave equation is:

$$u_j(t) = \alpha_1 e^{jct} + \alpha_2 e^{-jct} \tag{8}$$

We apply the initial conditions $u(x, 0) = u_{init}(x)$ and $\frac{\partial u}{\partial t}(x, 0) = v_{init}(x)$ to get:

$$\alpha_1 + \alpha_2 = u_{init}(x) \qquad\qquad jc\alpha_1 - jc\alpha_2 = v_{init}(x)$$

Solving this system of equations gives

$$\alpha_1 = \alpha_2 = \frac{u_{init}(x) - v_{init}(x)}{2}$$

Substituting into (8) gives us the final solution given our initial conditions:

$$u_j(t) = \Big(\frac{u_{init}(x) - v_{init}(x)}{2}\Big)\Big(e^{jct} + e^{-jct}\Big) \tag{9}$$

Problem B-1. See Problem III.9 in class notes

Solution 3. The following code shows the original vector of frequencies for the C Major scale alongside a more accurate vector of frequencies for that scale with the ratios approximated by observation. The frequency that sounded the most 'off' was 466. Online research on the ratios I came up with shows that they are the same ratios used by the Ancient Greeks, but are not necessarily the most accurate way to compute the frequencies of the C Major scale given the frequency of middle C.

```
t=linspace(0,2*pi,2000);s=sin(220*t);

% bad version
for f=[262 294 330 349 392 440 466 524]
    s=sin(f*t);
    plot(t,s);
    pause;
    sound(s);
end

% C major triad is C, G, E: [262, 327.5, 393.0]

figure();
ratios = [1 9/8 5/4 4/3 3/2 5/3 15/8 2];

% good version computed with ratios
for f = 262 * ratios
```

```
    s=sin(f*t);
    plot(t,s);
    pause;
    sound(s);
end
```

Problem B-2. See Problem III.10 in class notes

Solution 4. The following text show the piano notes (letters) for the memorable start to Kanye West's "Runaway". The notes are from the sixth and fifth octaves, and are spaced out relative to their spacing in the song.

```
6|e-------e-------e---------|
5|------------------------e-|

6|------D-------D-------D---|

6|------------C-------C-----|
5|----D---------------------|

6|--C-----------------------|
6|----------C-------a-------|

6|----------------e---------|
5|a-------G-----------------|
```

The following MATLAB code plays the notes an octave down and plots the frequencies.

```
t=linspace(0,2*pi,2000);s=sin(220*t);
for f=[1318.5 1318.5 1318.5 659.3 1174.7...
        1174.7 1174.7 587.3 1046.5 1046.5...
        1046.5 523.3 440 440]/2
    s=sin(f*t);
    plot(t,s);
    pause(1);
    sound(s);
end
```

Problem D-2. See Problem III.12 in class notes

Solution 5. In Part A of the Fourier Transform Activity, we use `fft()` to find the Discrete Fourier Transform (DFT) of $x = 7sin(2\pi * 20t - 0.3);$. In this example, the amplitude $R = 7$, the frequency $f = 20$, and a phase shift of 0.3. x is composed of a single sin wave, so its frequency plot is relatively simple:

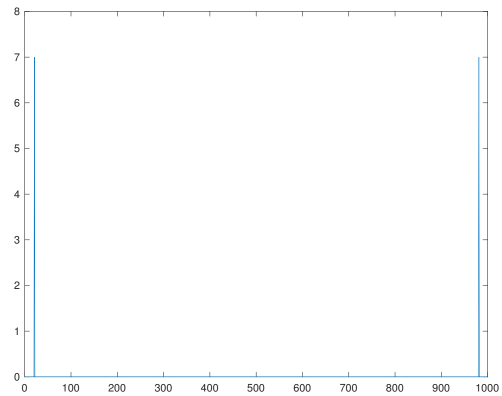
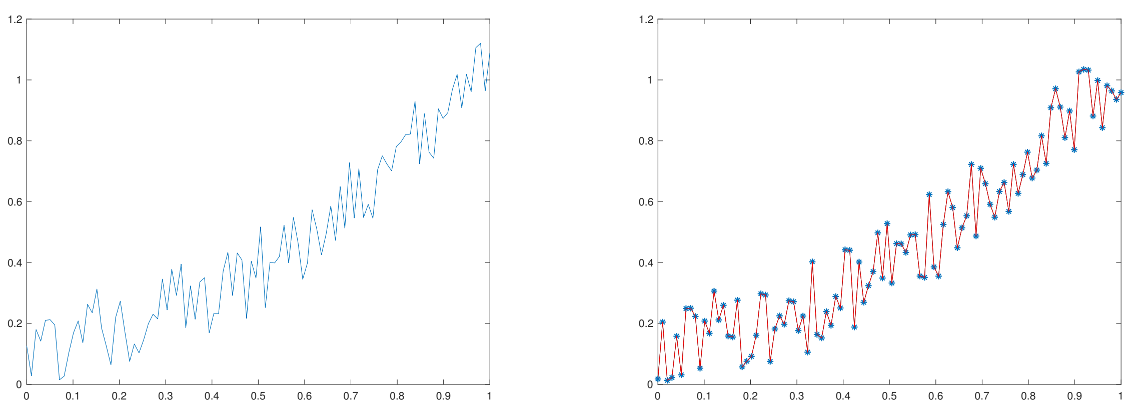


FIGURE 1. Frequency plot of `fft(7*sin(2*pi*20*t-0.3))`

In Part B of the activity, we use the fourier transform and the inverse fourier transform to recreate a chaotic signal.



(A) Original chaotic signal

(B) Fourier transform (points) and inverse fourier transform (red line) of signal

FIGURE 2. Using Fourier transform to analyze signal, reconstructing it with Inverse Fourier transform

In Part C of the activity, we extend Fourier transforms to two dimensions by using the Fourier basis functions $sin(n\pi x) * sin(m\pi x)$. The following plot shows these basis functions on the unit square.

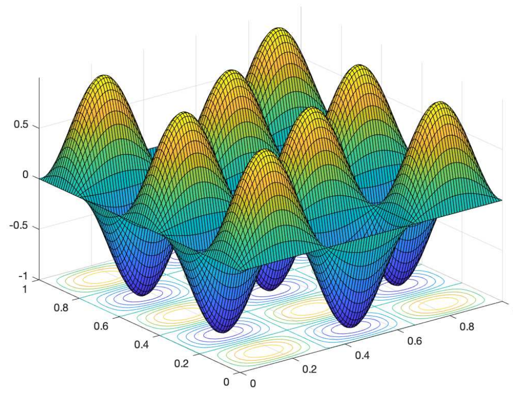
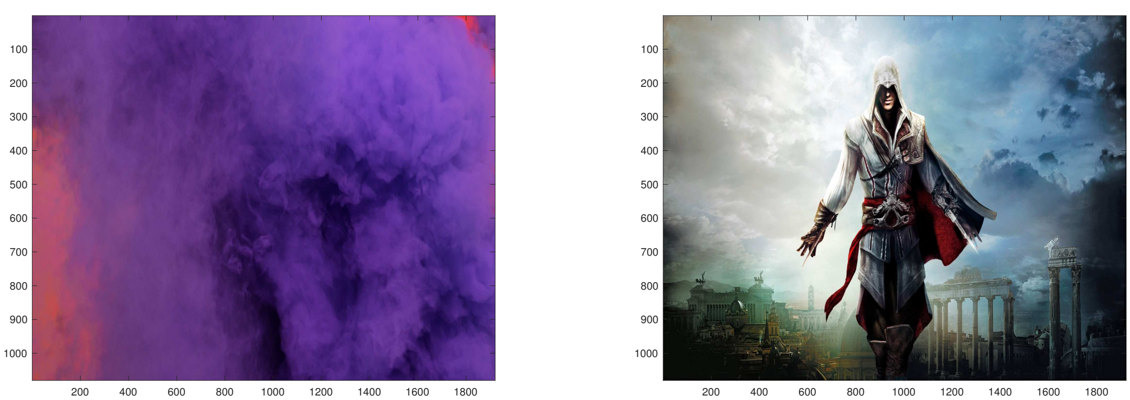


FIGURE 3. Fourier basis functions in 2d

In Part D of the activity we use the 2D extension of the Fourier transform to analyze the frequencies and amplitudes of two images. The plots of these images' reconstruction via the 2D version of Inverse Fourier Transform is shown below

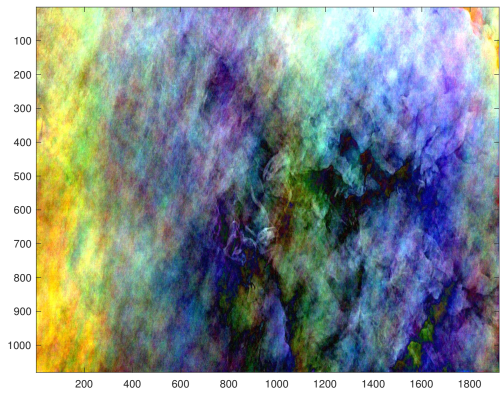


(A) ifft2() of smoke image

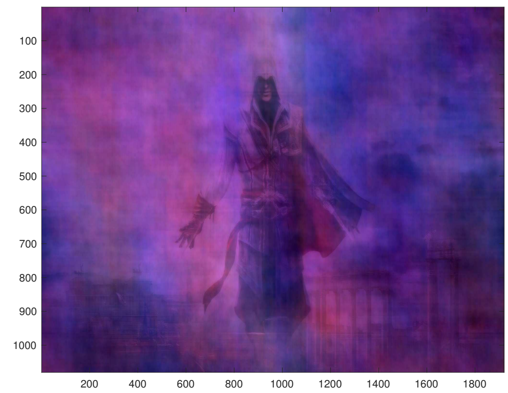
(B) ifft2() of assassin image

FIGURE 4. Using 2D Fourier transform and Inverse Fourier transform to analyze and reconstruct images

We can also use the computed Fourier transforms to mix the frequencies of the "smoke" and "assassin" images, resulting in these two mixed art-pieces:



(A) mixed image 1



(B) mixed image 2

FIGURE 5. Using 2D Fourier transform and Inverse Fourier transform to analyze and reconstruct images