Probability Theory

Need of Probability

- So far in course, everything was deterministic
- If I walk with my umbrella, I **will not** get wet.
- **But:** there is some chance my umbrella will break!
- Intelligent systems must take possibility of failure into account…
  - May want to have backup umbrella in city that is often windy and rainy
- … but should not be excessively conservative
  - Two umbrellas not worthwhile for city that is usually not windy
- Need quantitative notion of uncertainty

Events and Sample Space

- Event
  - any collection of results or outcomes of a procedure
- Simple Event
  - an outcome or an event that cannot be further broken down into simpler components
- Sample Space:
  - for a procedure Sample Space consists of all possible simple events; that is, the sample space consists of all outcomes that cannot be broken down any further

Events and Sample Space

- Sample space Ω - set of all possible outcomes of a random experiment
  - Dice roll: {1, 2, 3, 4, 5, 6}
  - Coin toss: {Tails, Heads}
- Event space $\mathcal{F}$ - subsets of elements in a sample space
  - Dice roll: {1, 2, 3} or {2, 4, 6}
  - Coin toss: {Tails}

Events and Sample Space

- A pair of dice are rolled. The sample space has 36 simple events:

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

  where the pairs represent the numbers rolled on each dice.

- Which elements of the sample space correspond to the event that the sum of each dice is 4?

Probability

- The word 'Probability' means the chance of occurring of a particular event.
- It is generally possible to predict the future of an event quantitatively with a certain probability of being correct.
- The probability is used in such cases where the outcome of the trial is uncertain.

- P - denotes a probability.
- A, B, and C - denote specific events.
- P(A) - denotes the probability of event A occurring.

$$P(A) = \frac{\text{number of cases favourable to A}}{\text{number of possible outcomes}}$$

Probability

- Probability of an Event Defined over $(\Omega, \mathcal{F})$ s.t.
  - $0 < P(a) < 1$ for all a in $\mathcal{F}$
  - $P(\Omega) = 1$

- Probability of an event which is certain to occur is one.
- Probability of an event which is impossible to zero.
- If the probability of happening of an event P(A) and that of not happening is P(A), then $P(A) + P(A') = 1$, where, $0 \leq P(A) \leq 1$, $0 \leq P(A') \leq 1$.

Event Relations

- **Equally Likely Events:** Events are said to be equally likely if one of them cannot be expected to occur in preference to others. In other words, it means each outcome is as likely to occur as any other outcome.
  - *Example:* When a die is thrown, all the six faces, i.e., 1, 2, 3, 4, 5 and 6 are equally likely to occur.

- **Mutually Exclusive or Disjoint Events:** Events are called mutually exclusive if they cannot occur simultaneously.
  - *Example:* Suppose a card is drawn from a pack of cards, then the events getting a jack and getting a king are mutually exclusive because they cannot occur simultaneously.

Event Relations

- **Exhaustive Events:** The total number of all possible outcomes of an experiment is called exhaustive events.
  - Example: In the tossing of a coin, either head or tail may turn up. Therefore, there are two possible outcomes. Hence, there are two exhaustive events in tossing a coin.

- **Dependent Event:** Events are said to be dependent if occurrence of one affect the occurrence of other events.

- **Independent Events:** Events A and B are said to be independent if the occurrence of any one event does not affect the occurrence of any other event.

$$P\,(A \cap B) = P\,(A)\,P\,(B).$$

Event Relations

Example: A coin is tossed thrice, and all 8 outcomes are equally likely

A: "The first throw results in heads."

B: "The last throw results in Tails."

Prove that event A and B are independent.

Solution:

Sample Space: [HHH, HHT, HTH, THH, TTT, TTH, THT, HTT]

A: [HHH, HHT, HTH, HTT]

B: [HHT, TTT, THT, HTT]

A∩B: [HHT, HTT]

$P(A) = \frac{4}{8} = \frac{1}{2}$ $\qquad$ $P(B) = \frac{4}{8} = \frac{1}{2}$

$P(A \cap B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Event Relations

Theorem1: If A and B are two mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

Proof: Let the n=total number of exhaustive cases

n_1= number of cases favorable to A.

n_2= number of cases favorable to B.

Now, we have A and B two mutually exclusive events. Therefore, n_1+n_2 is the number of cases favorable to A or B.

$$P(A \cup B) = \frac{\text{favorable cases}}{\text{Total number of exhaustive cases}} = \frac{n_1 + n_2}{n} = \frac{n_1}{n} + \frac{n_2}{n}$$

But we have, $P(A) = \frac{n_1}{n}$ and $P(B) = \frac{n_2}{n}$

Hence, $P(A \cup B) = P(A) + P(B)$.

Event Relations

Theorem2: If A and B are two events that are not mutually exclusive, then

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Proof: Let n = total number of exhaustive cases

n_1 = number of cases favorable to A

n_2 = number of cases favorable to B

n_3 = number of cases favorable to both A and B

But A and B are not mutually exclusive. Therefore, A and B can occur simultaneously. So, $n_1 + n_2 - n_3$ is the number of cases favorable to A or B.

Therefore, $P(A \cup B) = \frac{n_1 + n_2 - n_3}{n} = \frac{n_1}{n} + \frac{n_2}{n} - \frac{n_3}{n}$

But we have, $P(A) = \frac{n_1}{n}$, $P(B) = \frac{n_2}{n}$ and $P(A \cap B) = \frac{n_3}{n}$

Hence, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Event Relations

- If a, b are disjoint/mutually exclusive, then
  - $p(a \cup b) = p(a) + p(b)$
- If a, b are non-disjoint, then
  - $p(a \cup b) = p(a) + p(b) - p(a \cap b)$