Knowledge Representation Techniques

Logical Equivalence

- Two prepositions **α and β** are equivalent if and only if,
  - for all interpretations that **α** is true **β** is also true and
  - for all interpretations that **α** is false **β** is also false.
- Logical equivalence is represented as **α** ≡ **β.**
- It can be verified using truth table method.
- If **α** ≡ **β,** then we can substitute **α** for **β** and vice versa in any compound preposition.

Logical Equivalence

Equivalence	Name of Identity
p$\wedge T \equiv p$ p$\vee F \equiv p$	Identity Laws
p$\wedge F \equiv F$ p$\vee T \equiv T$	Domination Laws
p$\wedge p \equiv p$ p$\vee p \equiv p$	Idempotent Laws
$\neg(\neg p) \equiv p$	Double Negation Law
p$\wedge q \equiv q \wedge p$ p$\vee q \equiv q \vee p$	Commutative Laws
(p$\wedge q) \wedge r \equiv p \wedge (q \wedge r)$ (p$\vee q) \vee r \equiv p \vee (q \vee r)$	Associative Laws
p$\wedge(q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ p$\vee(q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Ditributive Laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's Laws
p$\wedge(p \vee q) \equiv p$ p$\vee(p \wedge q) \equiv p$	Absorption Laws
p$\wedge \neg p \equiv F$ p$\vee \neg p \equiv T$	Negation Laws

Logical Equivalence

$$\mathrm{p} \rightarrow q \equiv \neg p \vee q$$
$$\mathrm{p} \rightarrow q \equiv \neg q \rightarrow \neg p$$
$$\mathrm{p} \wedge q \equiv \neg(q \rightarrow \neg p)$$
$$(\mathrm{p} \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$$
$$(\mathrm{p} \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$$
$$(\mathrm{p} \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$$
$$(\mathrm{p} \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$$

$$\mathrm{p} \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$
$$\mathrm{p} \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$
$$\mathrm{p} \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$
$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Prove
$$\begin{aligned}
\neg(p \rightarrow q) &\equiv \neg(\neg p \vee q) && \textit{Using first equivalence of Conditionals}\\
&\equiv \neg(\neg p) \wedge \neg q && \textit{Using De Morgan's law}\\
&\equiv p \wedge \neg q && \textit{Using Double negation law}
\end{aligned}$$

Prove
$$\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \textit{Using De Morgan's law}\\
&\equiv \neg p \wedge (\neg(\neg p) \vee \neg q) && \textit{Using De Morgan's law}\\
&\equiv \neg p \wedge (p \vee \neg q) && \textit{Using Double negation law}\\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \textit{Using Distributive law}\\
&\equiv F \vee (\neg p \wedge \neg q) && \textit{Using Negation Law}\\
&\equiv \neg p \wedge \neg q && \textit{Using Identity Law}
\end{aligned}$$

Logical Entailment

- If $\alpha \vDash \beta$ (α entails β), then in any world where α is true, β is true, too.
- For example, if
  - **α: "It is a Tuesday in January"** and
  - **β: "It is a Tuesday,"** then we know that **$\alpha \vDash \beta$**.
  - If it is true that it it a Tuesday in January, we also know that it is a Tuesday.
- Entailment is different from implication.
  - Implication is a logical connective between two propositions.
  - Entailment, on the other hand, is a relation that means that if all the information in α is true, then all the information in β is true.

Inference from KB

- Inference is a process by which new sentences/facts are derived from existing sentences in the KB
  - the inference process is implemented on a computer
- Assume an inference procedure *i* that derives (or proves) a sentence α from the KB: KB $\vdash_i \alpha$

Soundness: An inference procedure is **sound**

If $KB \vdash_i \alpha$ then it is true that $KB \models \alpha$

Completeness: An inference procedure is **complete**

If $KB \models \alpha$ then it is true that $KB \vdash_i \alpha$

Inference from KB

- |= means "logically follows"
- $|\!-_i$ means "can be derived from"
- If your inference algorithm derives only things that follow logically from the KB, the inference is sound
- If everything that follows logically from the KB can be derived using your inference algorithm, the inference is complete

Inference from KB

Truth-table approach

Problem: $KB \models \alpha$?

- We need to check all possible interpretations for which the KB is true (models of KB) whether α is true for each of them

Truth table:

- enumerates truth values of sentences for all possible interpretations (assignments of True/False to propositional symbols)

Example:

		KB		α	
P	Q	$P \vee Q$	$P \Leftrightarrow Q$	$(P \vee \neg Q) \wedge Q$	
True	*True*	*True*	*True*	*True*	✓
True	*False*	*True*	*False*	*False*	
False	*True*	*True*	*False*	*False*	
False	*False*	*False*	*True*	*False*	

Inference from KB

Truth-table approach

$KB = (A \vee C) \wedge (B \vee \neg C) \qquad \alpha = (A \vee B)$

A	B	C	$A \vee C$	$(B \vee \neg C)$	KB	α
True	True	True	True	True	True	True
True	True	False	True	True	True	True
True	False	True	True	False	False	True
True	False	False	True	True	True	True
False	True	True	True	True	True	True
False	True	False	False	True	False	True
False	False	True	True	False	False	False
False	False	False	False	True	False	False

KB entails α

- The **truth-table approach** is **sound and complete** for the propositional logic!!

Inference from KB

Limitations of the truth table approach.

- What is the computational complexity of the truth table approach?
- Exponential in the number of the propositional symbols
  - 2^n Rows in the table have to be filled
  - the truth table is exponential in the number of propositional symbols (we checked all assignments)

- More efficient approach: **Inference rules**

Inference rules for logic

- **Modus ponens**

$$\frac{A \Rightarrow B, \quad A}{B}$$

← premise

← conclusion

- If both sentences in the premise are true then conclusion is true.
- The modus ponens inference rule is **sound.**
  - We can prove this through the truth table.

A	B	$A \Rightarrow B$
False	*False*	*True*
False	*True*	*True*
True	*False*	*False*
True	$\boxed{\textit{True}}$	*True*

Inference Rules

Modus Ponens: if we know an implication and its antecedent to be true, then the consequent is true as well.

If it is raining, then Harry is inside.

It is raining.

Harry is inside.

Inference Rules

And Elimination

- If an And proposition is true, then any one atomic proposition within it is true as well.
- For example, if we know that Harry is friends with Ron and Hermione, we can conclude that Harry is friends with Hermione.

Double Negation Elimination

- A proposition that is negated twice is true.
- For example, consider the proposition “It is not true that Harry did not pass the test”. We can parse it the following way: “It is not true that (Harry did not pass the test)”, or “¬(Harry did not pass the test)”, and finally “¬(¬(Harry passed the test)).”
- The two negations cancel each other, marking the proposition “Harry passed the test” as true.

Inference rules for logic

- **And-elimination**

$$\frac{A_1 \wedge A_2 \wedge \quad A_n}{A_i}$$

- **And-introduction**

$$\frac{A_1 , A_2 , \quad A_n}{A_1 \wedge A_2 \wedge \quad A_n}$$

- **Or-introduction**

$$\frac{A_i}{A_1 \vee A_2 \vee \ldots A_i \vee \quad A_n}$$

Inference rules for logic

- **Elimination of double negation** $$\frac{\neg\neg A}{A}$$

- **Unit resolution** $$\frac{A \vee B, \quad \neg A}{B}$$

A special case of

- **Resolution** $$\frac{A \vee B, \quad \neg B \vee C}{A \vee C}$$

- All of the above inference rules **are sound.** We can prove this through the truth table, similarly to the **modus ponens** case.

Inference Rules

Rule of Inference	Tautology	Name
p $p \rightarrow q$ $\therefore q$	$(p \wedge (p \rightarrow q)) \rightarrow q$	Modus Ponens
$\neg q$ $p \rightarrow q$ $\therefore \neg p$	$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$	Modus Tollens
$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$\neg p$ $p \vee q$ $\therefore q$	$(\neg p \wedge (p \vee q)) \rightarrow q$	Disjunctive Syllogism
p $\therefore (p \vee q)$	$p \rightarrow (p \vee q)$	Addition
$(p \wedge q) \rightarrow r$ $\therefore p \rightarrow (q \rightarrow r)$	$((p \wedge q) \rightarrow r) \rightarrow (p \rightarrow (q \rightarrow r))$	Exportation
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \rightarrow q \vee r$	Resolution

Inference from KB

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$

Inference from KB

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P **From 1 and And-elim**
5. R **From 2,4 and Modus ponens**
6. Q **From 1 and And-elim**
7. $(Q \wedge R)$ **From 5,6 and And-introduction**
8. S **From 7,3 and Modus ponens**

Proved: S

Inference from KB

Prove that given the following information

“It is not sunny this afternoon and it is colder than yesterday”,

“We will go swimming only if it is sunny”,

“If we do not go swimming, then we will take a canoe trip”, and

“If we take a canoe trip, then we will be home by sunset”

We can derive the conclusion

“We will be home by sunset”.

Inference from KB

Define prepositions

p: “It is sunny this afternoon”

q: “It is colder than yesterday”

r: “We will go swimming”

s: “We will take a canoe trip”

t: “We will be home by sunset”

Create KB

- It is not sunny this afternoon and it is colder than yesterday. $\neg p \land q$
- We will go swimming only if it is sunny. $r \rightarrow p$
- If we do not go swimming then we will take a canoe trip. $\neg r \rightarrow s$
- If we take a canoe trip, then we will be home by sunset. $s \rightarrow t$

Inference from KB

Create KB

- It is not sunny this afternoon and it is colder than yesterday. **$\neg p \wedge q$**
- We will go swimming only if it is sunny. **$r \rightarrow p$**
- If we do not go swimming then we will take a canoe trip. **$\neg r \rightarrow s$**
- If we take a canoe trip, then we will be home by sunset. **$s \rightarrow t$**

Inference

1.	$\neg p \wedge q$	**given**
2.	$\neg p$	**And-elimination**
3.	$r \rightarrow p$	**given**
4.	$\neg r$	**Modus Tollens using (2) and (3)**
5.	$\neg r \rightarrow s$	**given**
6.	s	**Modus Ponens using (4) and (5)**
7.	$s \rightarrow t$	**given**
8.	t	**Modus Ponens using (6) and (7)**

Assignment 2

Q1. Check if statements are valid, contingent, or unsatisfiable

$(a)\ (p \Rightarrow q) \vee (q \Rightarrow p)$

$(b)\ p \wedge (p \Rightarrow \neg q) \wedge q$

$(c)\ (p \Rightarrow (q \wedge r)) \Leftrightarrow (p \Rightarrow q) \wedge (p \Rightarrow r)$

$(d)\ (p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \wedge q) \Rightarrow r)$

Q2. Check if statements are Logically Equivalent

$(a)\ (p \Rightarrow q \vee r)$ and $(p \wedge q \Rightarrow r)$

$(b)\ (p \Rightarrow (q \Rightarrow r))$ and $(p \wedge q \Rightarrow r)$

$(c)\ (p \wedge q \Rightarrow r)$ and $(p \wedge r \Rightarrow q)$

$(d)\ ((p \Rightarrow q \vee r) \wedge (p \Rightarrow r))$ and $(q \Rightarrow r)$

Assignment 2

Q3. Use Truth Table method to check the following entailment

$(a)\ \{p \Rightarrow q \vee r\} \vDash (p \Rightarrow r)$

$(b)\ \{p \Rightarrow r\} \vDash (p \Rightarrow q \vee r)$

$(c)\ \{q \Rightarrow r\} \vDash (p \Rightarrow q \vee r)$

$(d)\ \{p \Rightarrow q \vee r, p \Rightarrow r\} \vDash (q \Rightarrow r)$

Q4. Given KB (set of prepositions) check if given statement can be derived or not.

$(a)\ \{p \vee q, p \vee \neg q, \neg p \vee q\}$ and $(\neg p \vee \neg q)$

$(b)\ \{p \Rightarrow r, q \Rightarrow r, p \vee q\}$ and r

$(c)\ \{p \Rightarrow r, q \Rightarrow r, p \vee q\}$ and $\neg r$

$(d)\ \{p \Rightarrow q \vee r, q \Rightarrow r\}$ and $p \wedge q$