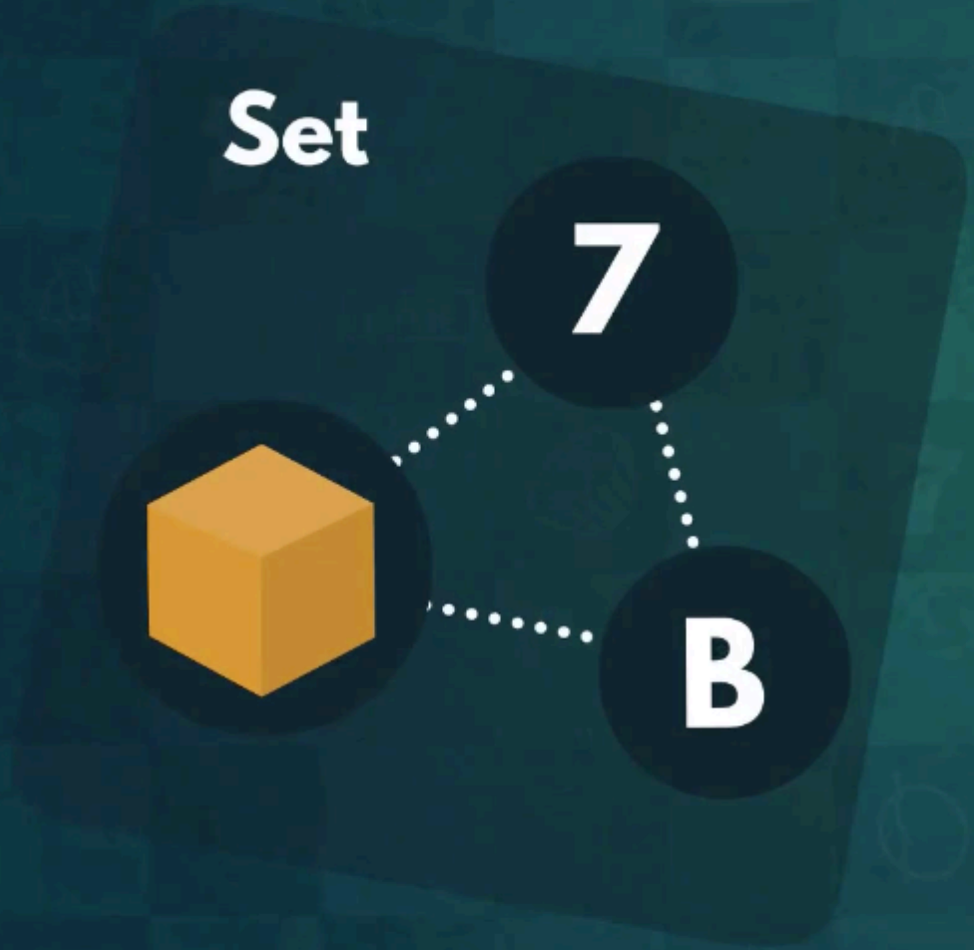
Combinatorics



Combinatorics deal with
combinations of objects from a set

Combinatorics

- Permutations
- Variations
- Combinations

Number of favourable outcomes
OR
Number of all elements in a sample space

Permutations

Permutations represent the number of different possible ways we can **arrange** a number of elements.

$$P(n) = n \times (n-1) \times (n-2) \times \cdots \times 1$$

Permutations

Options for who we put first

Options for who we put second

Options for who we put last

Characteristics of Permutations:

- Arranging **all** elements within the sample space.
- No repetition.
- $P(n) = n \times (n-1) \times (n-2) \times \cdots \times 1 = n!$ (Called "n factorial")

Example:

- If we need to arrange 5 people, we would have P(5) = 120 ways of doing so.

Intuition

n - many elements

already chosen winner

n	n - 1	?

n - 1 many possibilities for the second slot

Factorials

Factorials express the **product** of all integers from **1** to **n** and we denote them with the "!" symbol.

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 1$$

Key Values:

- 0! = 1.
- If n<0, n! does not exist.

Rules for factorial multiplication. (For n>0 and n>k)

- $(n+k)! = n! \times (n+1) \times \cdots \times (n+k)$
- $(n-k)! = \frac{n!}{(n-k+1)\times\cdots\times(n-k+k)} = \frac{n!}{(n-k+1)\times\cdots\times n}$
- $\frac{n!}{k!} = \frac{k!\times(k+1)\times\cdots\times n}{k!} = (k+1)\times\cdots\times n$

Examples: n = 7, k = 4

- $(7+4)! = 11! = 7! \times 8 \times 9 \times 10 \times 11$
- $(7-4)! = 3! = \frac{7!}{4\times5\times6\times\times7}$
- $\frac{7!}{4!} = 5 \times 6 \times 7$

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Important Properties

- $n! = (n - 1)! \times n$
- $(n + 1)! = n! \times (n + 1)$

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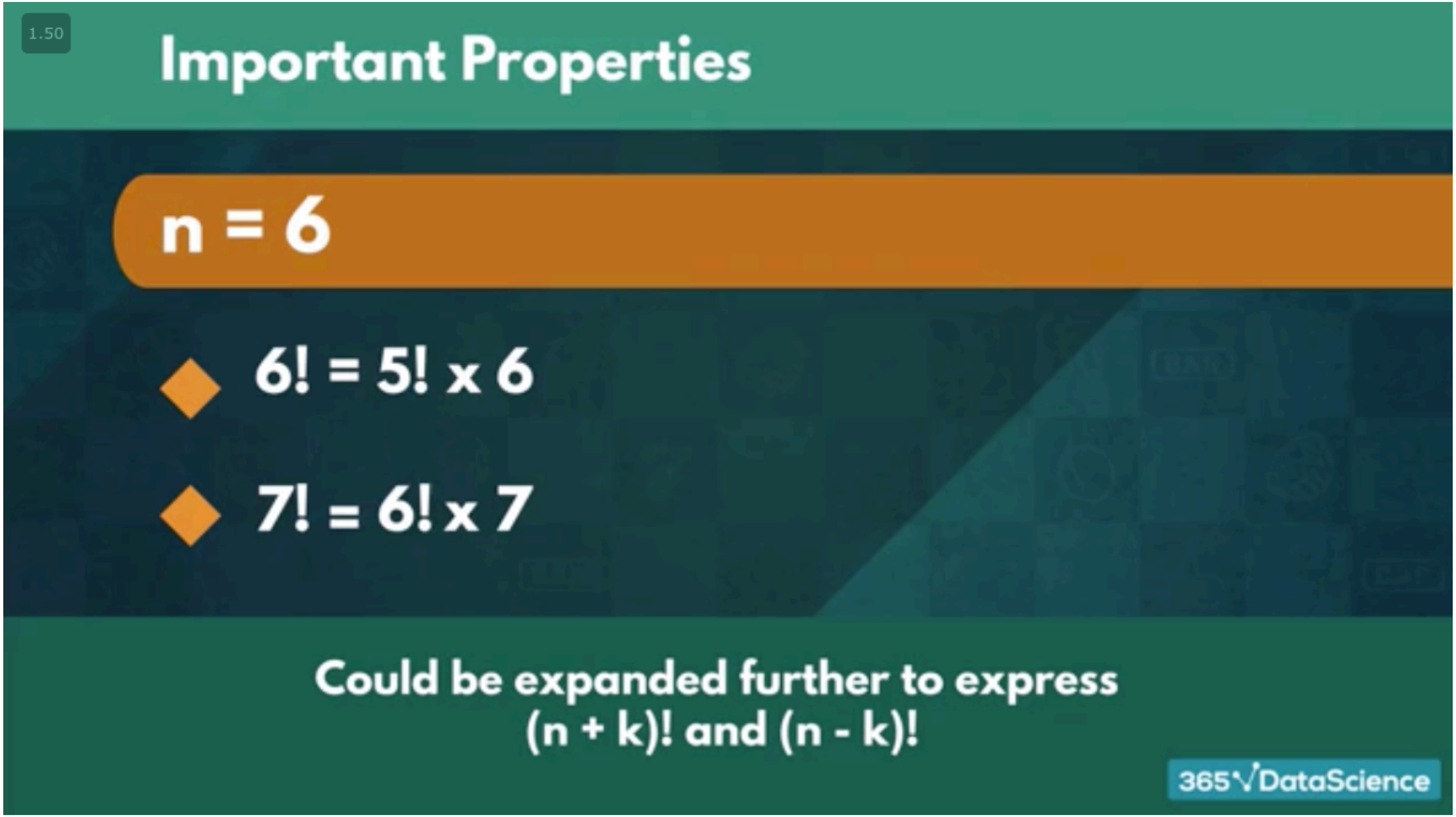
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Important Properties
n = 6
6! = 5! x 6
7! = 6! x 7
Could be expanded further to express
(n + k)! and (n - k)!
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Variations

Variations represent the number of different possible ways we can **pick** and **arrange** a number of elements.

Variations **with** repetition

$$\bar{V}(n,p) = n^p$$

Number of different elements available

Number of elements we are arranging

Variations without repetition

$$V(n,p) = \frac{n!}{(n-p)!}$$

Number of different elements available

Number of elements we are arranging

Intuition behind the formula. (With Repetition)

- We have n-many options for the first element.
- We **still have n-many options** for the second element because repetition is allowed.
- We have n-many options for each of the p-many elements.
- $n \times n \times n \dots n = n^p$

Intuition behind the formula. (Without Repetition)

- We have n-many options for the first element.
- We **only have (n-1)-many options** for the second element because we cannot repeat the value for we chose to start with.
- We have less options left for each additional element.
- $n \times (n-1) \times (n-2) \dots (n-p+1) = \frac{n!}{(n-p)!}$

Variations

The total number of ways we can **pick and arrange** some elements of a given set

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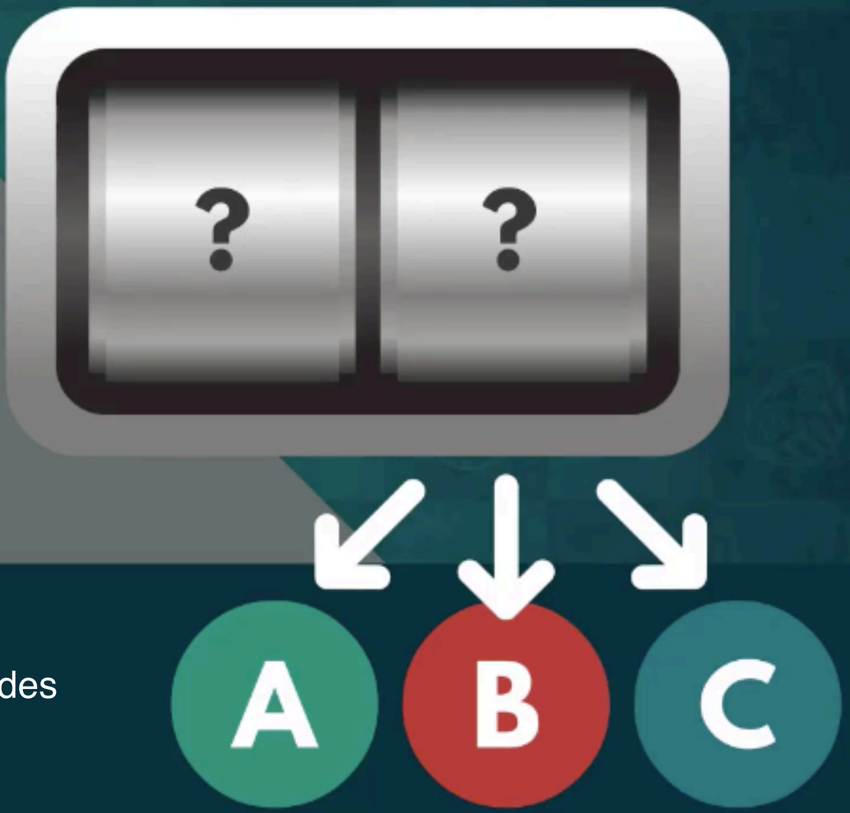
Combination Lock

Variation with repetition

Regardless of which one of the 3 letters we decide to start with, we are going to have 3 different options for the second letter



The total number of variation is:
3x3 = 9

There are 9 different variations of 2-letter passcodes consisting or A, B or C only.

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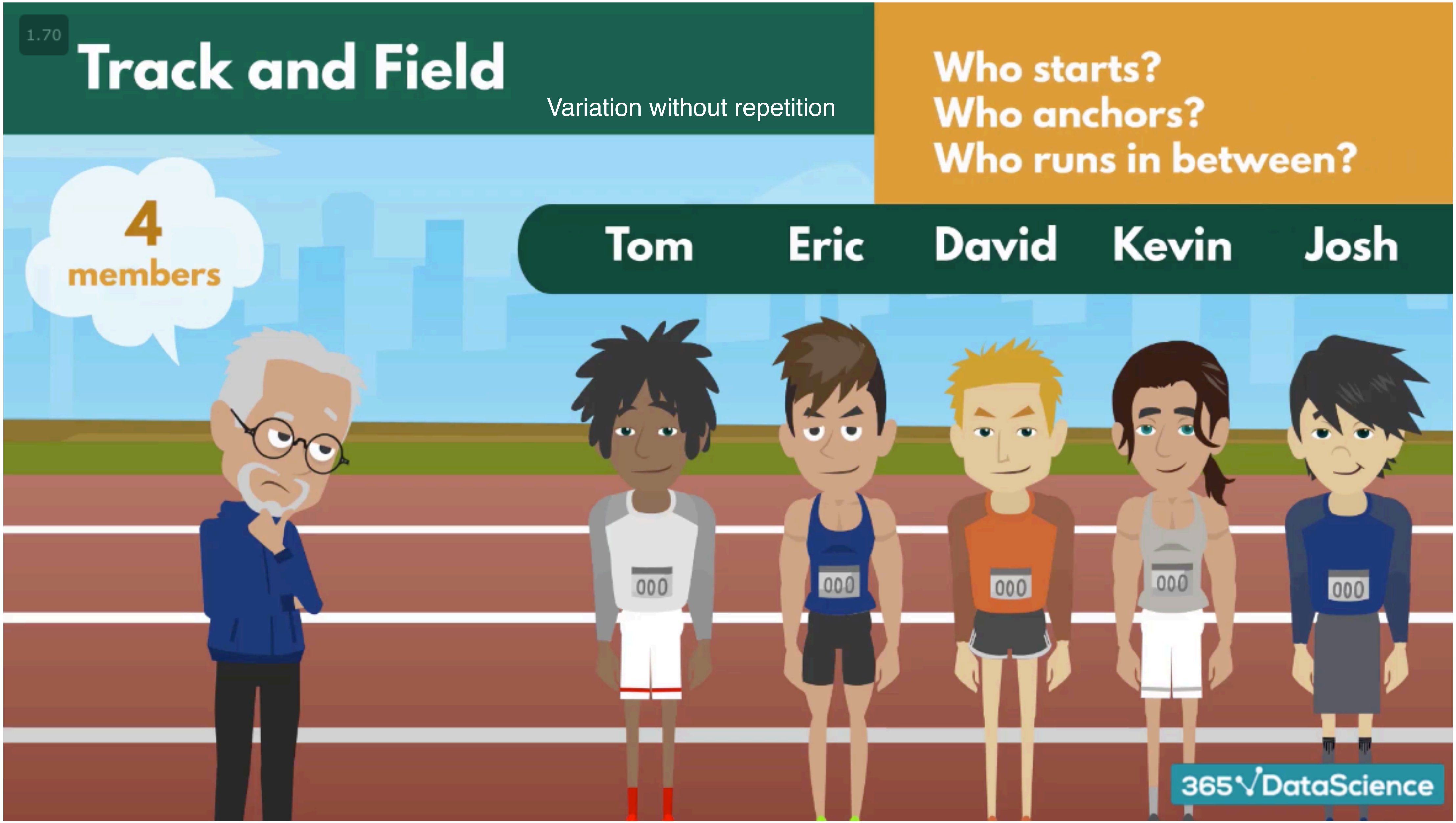
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Track and Field
Variation without repetition
Who starts?
Who anchors?
Who runs in between?
4
members
Tom
Eric
David
Kevin
Josh
000
000
000
000
000
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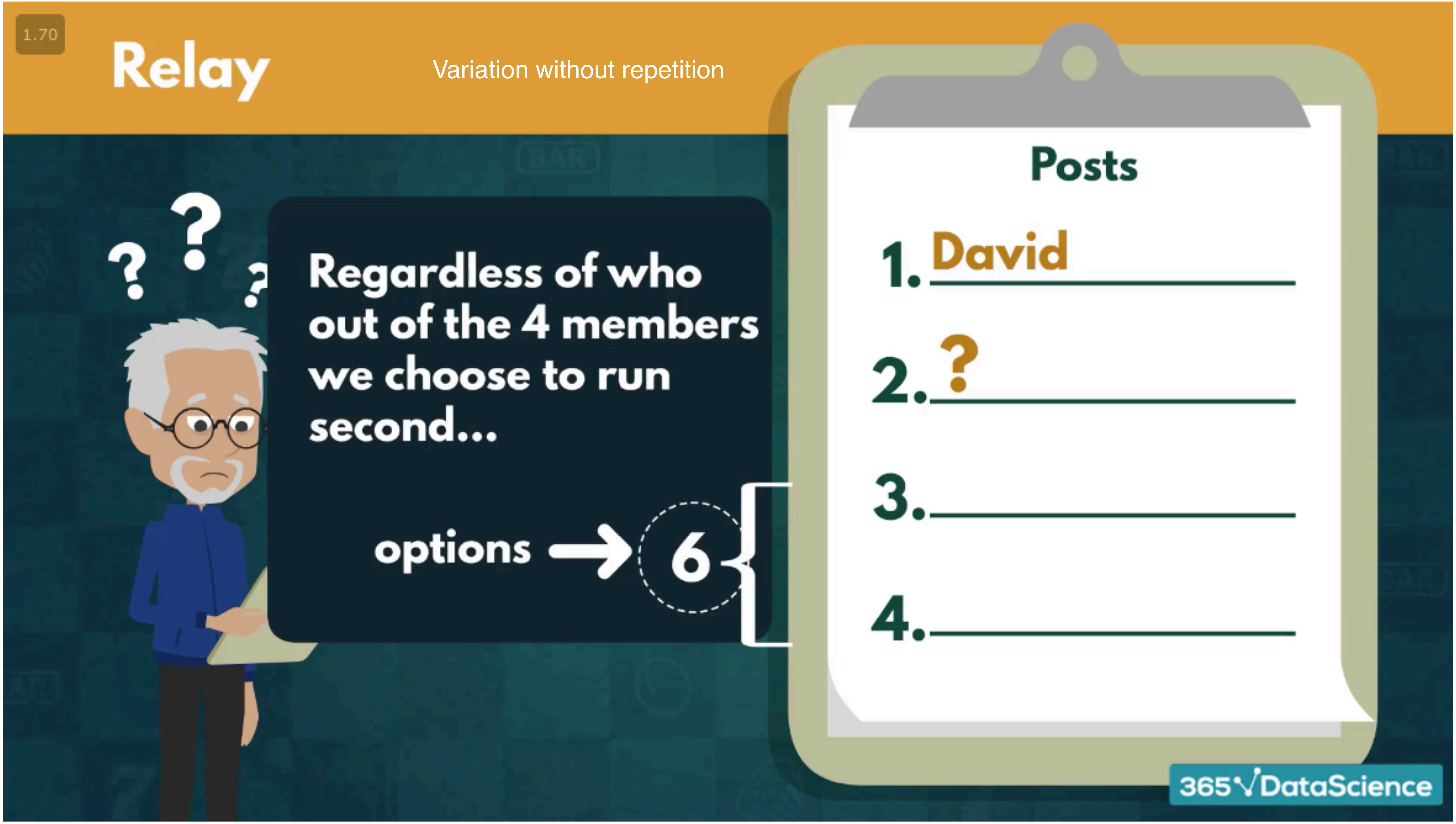
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Relay
Variation without repetition
Regardless of who out of the 4 members we choose to run second...
options → 6
Posts
1. David
2. ?
3.
4.
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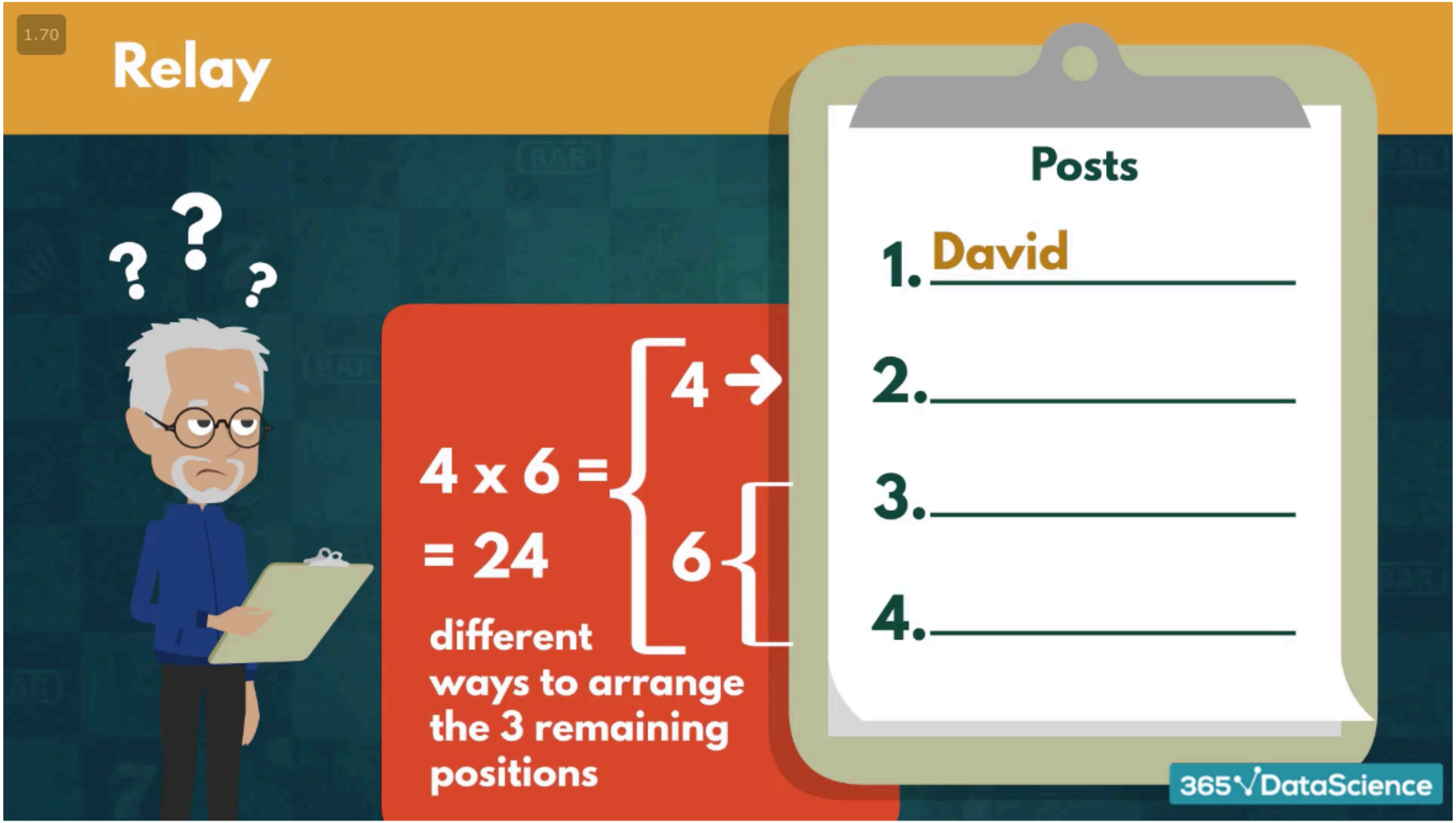
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Relay
Posts
1. David
2.
3.
4.
4 x 6 =
= 24
different ways to arrange the 3 remaining positions
4
6
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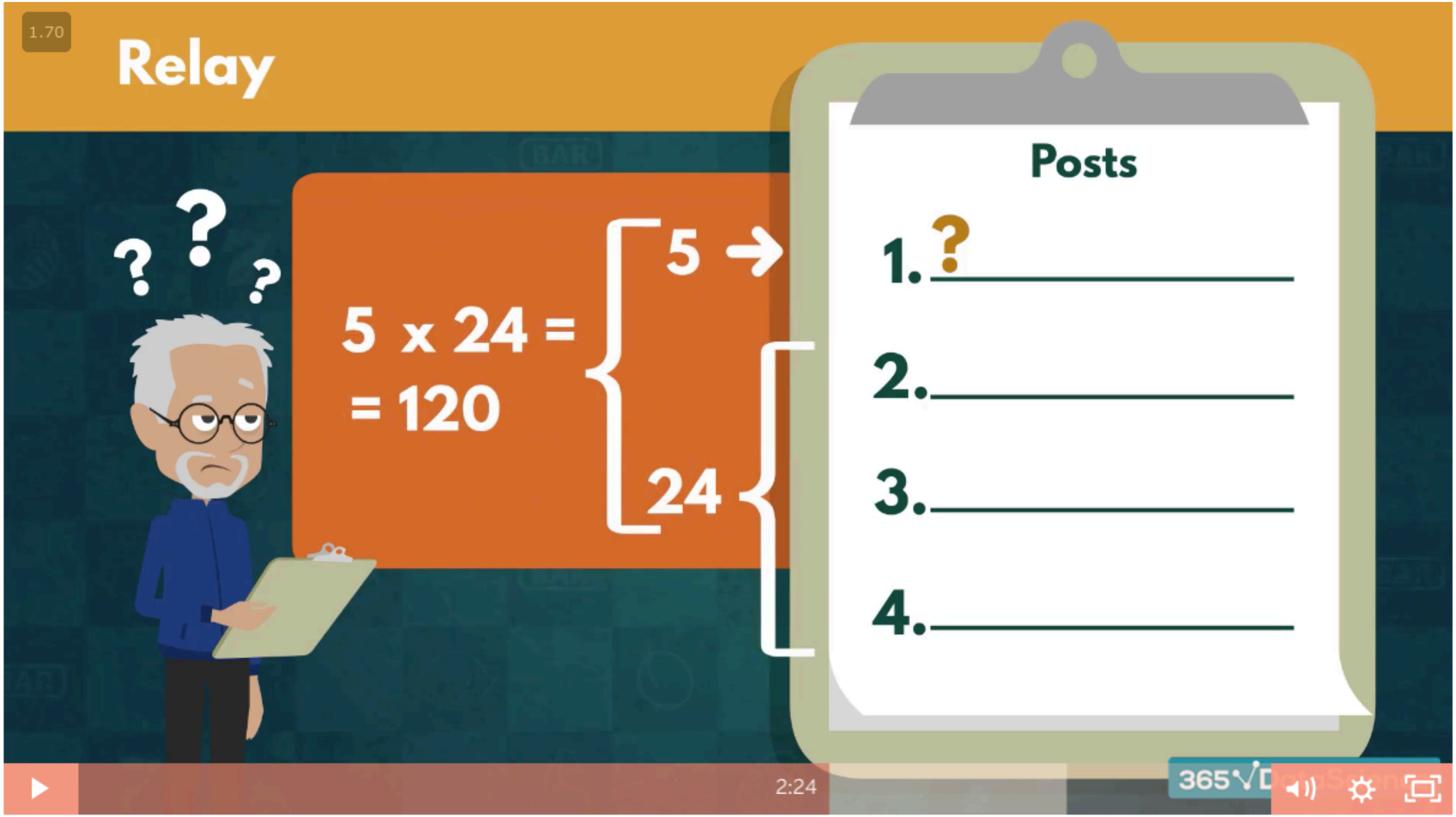

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Relay
5 x 24 =
= 120
5
24
Posts
1. ?
2.
3.
4.
2:24


1.70
Relay
The further down the order we go, the fewer options we are left with
5
4
3
2
Posts
1.
2.
3.
4.
We cannot use the same person/ element twice
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Relay

$$\underline{\ 5\ } \times \underline{\ 4\ } \times \underline{\ 3\ } \times \underline{\ 2\ } =$$

= **120** different options of how to arrange our team for the competition

Combinations

Combinations represent the number of different possible ways we can **pick** a number of elements.

$$C(n,p) = \frac{n!}{(n-p)!\,p!}$$

Combinations

Total number of elements in the sample space

Number of elements we need to select

Characteristics of Combinations:

- Takes into account double-counting. (Selecting Johny, Kate and Marie is the same as selecting Marie, Kate and Johny)
- All the different permutations of a single combination are different variations.
- $C = \frac{V}{P} = \frac{n!/(n-p)!}{p!} = \frac{n!}{(n-p)!p!}$
- Combinations are symmetric, so $C_p^n = C_{n-p}^n$, since selecting p elements is the same as omitting n-p elements.

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Combinations

The number of different ways we can pick certain elements of a set

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Combinations with separate sample spaces

Combinations represent the number of different possible ways we can **pick** a number of elements.

$$C = n_1 \times n_2 \times \cdots \times n_p$$

Combinations

Size of the first sample space.

Size of the second sample space.

Size of the last sample space.

Characteristics of Combinations with separate sample spaces:

- The option we choose for any element does not affect the number of options for the other elements.
- The order in which we pick the individual elements is arbitrary.
- We need to know the size of the sample space for each individual element. $(n_1, n_2 \dots n_p)$

Winning the Lottery

To win the lottery, you need to satisfy two distinct independent events:

- Correctly guess the "Powerball" number. (From 1 to 26)
- Correctly guess the 5 regular numbers. (From 1 to 69)

Total number of Combinations

$$C = \frac{69!}{64!\,5!} \times 26$$

$C_{5\ numbers}$

$C_{Powerball\ number}$

Intuition behind the formula:

- We consider the two distinct events as a combination of two elements with different sample sizes.
  - One event has a sample size of 26, the other has a sample size of C_5^{69}.
- Using the "favoured over all" formula, we find the probability of any single ticket winning equals $1/(\frac{69!}{64!5!} \times 26)$.

Technology Conference example 1

- Pick 3 people to represent your company


Technology Conference example 1
$V^{10}_{3} = 720$
You would be counting every group of 3 people several times
Picking Alex, Sarah and Dave is the same as picking Alex, Dave and Sarah
Variations don't take into account double counting elements
All the different permutations of a single combination are different variations
Any of the 6 possible permutations for Alex, Sarah and Dave is a different variation, but NOT a different combination.
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Technology Conference example 1

We are going to have 6 variations for ANY combination

Six times fewer combinations than variations

$$V_3^{10} = 10 \times 9 \times 8 = 720$$

$$C_3^{10} = \frac{720}{6} = 120$$

ways of choosing who represents the c

Technology Conference example 1

Recall:

- We can say that all the different permutations of a single combination are different variations

$$\left.\begin{array}{l} P_3 = 6 \\ C_3^{10} = 120 \end{array}\right\} V_3^{10} = 720$$

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Formula

What's the number of combinations for choosing p-many elements out of a sample space of n elements?

The number of combinations equals the number of variations, over the number of permutations

$$C_p^n = \frac{V_p^n}{P_p}$$

Formula

What's the number of combinations for choosing p-many elements out of a sample space of n elements?

The number of combinations equals the number of variations, over the number of permutations

$$C_p^n = \frac{n!}{p!\,(n-p)!}$$

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Formula

$n \Rightarrow 10$

$p \Rightarrow 3$

$$C_3^{10} = \frac{10!}{3!\,7!} = \frac{8 \times 9 \times 10}{1 \times 2 \times 3} = \frac{720}{6} = 120$$

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Symmetry of Combinations

Let's see the algebraic proof of the notion that selecting p-many elements out of a set of n is the same as omitting n-p many elements.

For starters, recall the combination formula:

$$\boldsymbol{C(n,p) = \frac{n!}{(n-p)!\,p!}}$$

If we plug in $n - p$ for p, we get the following:

$$\boldsymbol{C(n,n-p) = \frac{n!}{(n-(n-p))!\,(n-p)!} = \frac{n!}{(n-n+p))!\,(n-p)!} = \frac{n!}{p!\,(n-p)!} = \frac{n!}{(n-p)!\,p!} = C(n,p)}$$

Therefore, we can conclude that $\boldsymbol{C(n,p) = C(n,n-p)}$.

Combinations With Repetition

Combinations represent the number of different possible ways we can **pick** a number of elements. In special cases we can have repetition in combinations and for those we use a different formula.

$$\bar{C}(n,p) = \frac{(n+p-1)!}{(n-1)!\,p!}$$

Combinations with repetition

Total number of elements in the sample space

Number of elements we need to select

Now that you know what the formula looks like, we are going to walk you through the process of deriving this formula from the Combinations *without* repetition formula. This way you will be able to fully understand the intuition behind and not have to bother memorizing it.

Applications of Combinations with Repetition

To understand how combinations with repetition work you need to understand the instances where they occur.

We use combinations with repetition when the events we are dealing with, have sufficient quantity of each of the distinct values in the sample space.

One such example is the toppings on a pizza.

We can order extra of any given topping, so **repetition is allowed**. However, we **do not care about the order** in which the toppings are put on top of the pizza, so we cannot use variations.

Similar examples include picking the ice-cream flavours for a Sundae melt or the players for a Fantasy Football Team.

Pizza Example

To get a better grasp of the number of combinations we have, let us explore a specific example.

You are ordering a 3-topping pizza from your local pizza place but they only have 6 topping left because its late.

The toppings are as follows:
cheddar **c**heese, **o**nions, **g**reen peppers, **m**ushrooms, **p**epperoni and **b**acon.

Your pizza can have 3 different toppings, or you can repeat a topping up to 3 times.

You can either order a pizza with 3 different toppings, a pizza with 3 identical toppings or a pizza with 2 different toppings but having a double dose of one of them.

Methodology

The methodology we use for such combinations is rather abstract. We like to represent each type of pizza with a special sequence of 0s and 1s. To do so, we first need to select a specific order for the available ingredients.

We can reuse the order we wrote down earlier:
cheddar **c**heese, **o**nions, **g**reen peppers, **m**ushrooms, **p**epperoni and **b**acon.

For convenience we can refer to each ingredient by the associated letter we have highlighted (e.g "c" means cheese, and "o" means onions).

To construct the sequence for each unique type of pizza we follow 2 rules as we go through the ingredients in the order we wrote down earlier.

1. If we want no more from a certain topping, we write a **0 and move to the next topping**.
2. If we want to include a certain topping, we write a **1 and stay on the same topping**.
    - Not going to the next topping allows us to indicate if we want extra by adding another 1, before we move forward. Say, if we want our pizza to have extra cheese, the sequence would begin with "1, 1".
    - Also, we always apply rule 1 before moving on to another topping, so the sequence will actually start with "1, 1, 0".

Pizzas and Sequences

If we need to write a “0” after each topping, then every sequence would consist of 6 zeroes and 3 ones.

Let’s look at some pizzas and the sequences they are expressed with.

A pizza with **cheese** and **extra peperoni** is represented by the sequence 1,0,0,0,0,1,1,0,0.

A vegan variety pizza with **onions, green peppers and mushrooms** would be represented by the sequence 0,1,0,1,0,1,0,0,0.

Now, what kind of pizza would the sequence **0,0,1,0,0,0,1,1,0** represent?

We can put the sequence into the table and see that it represents a cheese pizza with extra bacon.

C	O	G	M	P	B
1,0	0	0	0	1,1,0	0
0	1,0	1,0	1,0	0	0
0	0	1,0	0	0	1,1,0

Always Ending in 0

Notice how all the sequences we have examined end on a 0:

- 1,0,0,0,0,1,1,0,**0**
- 0,1,0,1,0,1,0,0,**0**
- 0,0,1,0,0,0,1,1,**0**

This is no coincidence, since according to rule 1 of our algorithm, we need to add a "0" at the end of the sequence, regardless of whether we wanted bacon or not.

That means that only the first 8 elements of the sequence can take different values.

Each pizza is characterized by the positions of the 3 "1"s in the sequence. Since only 8 of the positions in the sequence can take a value of "1", then the number of different pizzas would be the combination of any 3 of the 8 positions.

C	O	G	M	P	B
1,0	0	0	0	1,1,0	0
0	1,0	1,0	1,0	0	0
0	0	1,0	0	0	1,1,0

Positions

As stated before, we have 3 “1s” and 8 different positions. Therefore, the number of pizzas we can get would be the number of combinations of picking 3 elements out of a set of 8. This means we can transform combinations **with** repetition to combinations **without** repetition.

$$\bar{C}(6,3) = \mathrm{C}(8,3)$$

Let’s observe the values 3 and 8 for a moment and try to generalize the formula. “3” represents the amount of toppings we need to pick, so it is still equivalent to p.

“8” represents the number of positions we have available for the ones. We had 3 + 6, or 9 positions in total, but we knew the last one could not contain a “1”. Thus, we had “n + p - 1” many positions that could contain a 1.

$$\bar{C}(n,p) = \mathrm{C}(\mathrm{n}+\mathrm{p}-1,\mathrm{p})$$

The Final Step

Now that we know the relationship between the number of combinations **with** and **without** repetition, we can plug in "n+p-1" into the combinations without repetitio formula to get:

$$\bar{C}(n,p) = \mathrm{C}(\mathrm{n}+\mathrm{p}-1,\mathrm{p}) = \frac{(\mathrm{n}+\mathrm{p}-1)!}{((\mathrm{n}+\mathrm{p}-1)-\mathrm{p})!\,\mathrm{p}!} = \frac{(\mathrm{n}+\mathrm{p}-1)!}{(\mathrm{n}-1)!\,\mathrm{p}!}$$

This is the exact same formula we showed you at the beginning.
Before we you continue to the next lecture, let's make a quick recap of the algorithm and the formula.

1. We started by ordering the possible values and expressing every combinations as a sequence.
2. We examined that only certain elements of the sequence may differ.
3. We concluded that every unique sequence can be expressed as a combination of the positions of the "1" values.
4. We discovered a relationship between the formulas for combinations with and without repetition.
5. We used said relationship to create a general formula for combinations with repetition.