Row Reduction Echelon Form

AND ITS APPLICATIONS IN BALANCING CHEMICAL EQUATIONS

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Echelon Form

Echelon form means that the matrix is in one of two states:

- Row echelon form.
- Reduced row echelon form.

This means that the matrix meets the following three requirements:

1. The first number in the row (called a leading coefficient) is 1. Note: some authors don't require that the leading coefficient is a 1; it could be any number. You may want to check with your instructor to see which version of this rule they are adhering to).
2. Every leading 1 is to the right of the one above it.
3. Any non-zero rows are always above rows with all zeros.

The following examples are of matrices in echelon form:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & .2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Row Echelon Form

A matrix is in row echelon form if it meets the following requirements:

- The first non-zero number from the left (the "Leading Coefficient ") is always to the right of the first non-zero number in the row above.
- Rows consisting of all zeros are at the bottom of the matrix.

$$\begin{bmatrix} 1 & a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & 2 & a_4 & a_5 \\ 0 & 0 & 0 & 1 & a_6 \end{bmatrix}$$ Row echelon form. "a" can represent any number.

Technically, the **leading coefficient** can be any number. However, the majority of Linear Algebra textbooks do state that the leading coefficient must be the number 1. To add to the confusion, some definitions of row echelon form state that there must be zeros both above *and* below the leading coefficient.

It's therefore best to follow the definition given in the textbook you're following (or the one given to you by your professor). If you're unsure (i.e. it's Sunday, your homework is due and you can't get hold of your professor), it safest to use 1 as the leading coefficient in each row.

If the leading coefficient in each row is the **only** non-zero number in that column, the matrix is said to be in reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & a_1 & 0 & b_1 \\ 0 & 1 & a_2 & 0 & b_2 \\ 0 & 0 & 0 & 1 & b_3 \end{bmatrix}$$
A 3x5 matrix in reduced row echelon form.

Row echelon forms are commonly encountered in linear algebra, when you'll sometimes be asked to convert a matrix into this form. The row echelon form can help you to see what a matrix represents and is also an important step to solving systems of linear equations.

Reduced Row Echelon Form

Reduced row echelon form is a type of matrix used to solve systems of linear equations. Reduced row echelon form has four requirements:

- The first non-zero number in the first row (**the leading entry**) is the number 1.
- The second row also starts with the number 1, which is further to the right than the leading entry in the first row. For every subsequent row, the number 1 must be further to the right.
- The leading entry in each row must be the only non-zero number in its column.
- Any non-zero rows are placed at the bottom of the matrix.

$$\begin{bmatrix} 1 & 0 & a_1 & 0 & b_1 \\ 0 & 1 & a_2 & 0 & b_2 \\ 0 & 0 & 0 & 1 & b_3 \end{bmatrix}$$
A 3x5 matrix in reduced row echelon form.

Algorithm of Reduced Row Echelon Form:

Step 1. Begin with an $m \times n$ matrix A. If $A = 0$, go to Step 7.

Step 2. Determine the leftmost non-zero column.

Step 3. Use elementary row operations to put a 1 in the topmost position (we call this position pivot position) of this column.

Step 4. Use elementary row operations to put zeros (strictly) below the pivot position.

Step 5. If there are no more non-zero rows (strictly) below the pivot position, then go to Step 7.

Step 6. Apply Step 2-5 to the submatrix consisting of the rows that lie (strictly below) the pivot position.

Step 7. The resulting matrix is in row-echelon form.

Further proceed as follows, we can reduce a Row Echelon Form to the Reduced Row Echelon Form

Step 8. Determine all the leading ones in row-echelon form obtained in Step 7.

Step 9. Determine the right most column containing a leading one (we call this column pivot column).

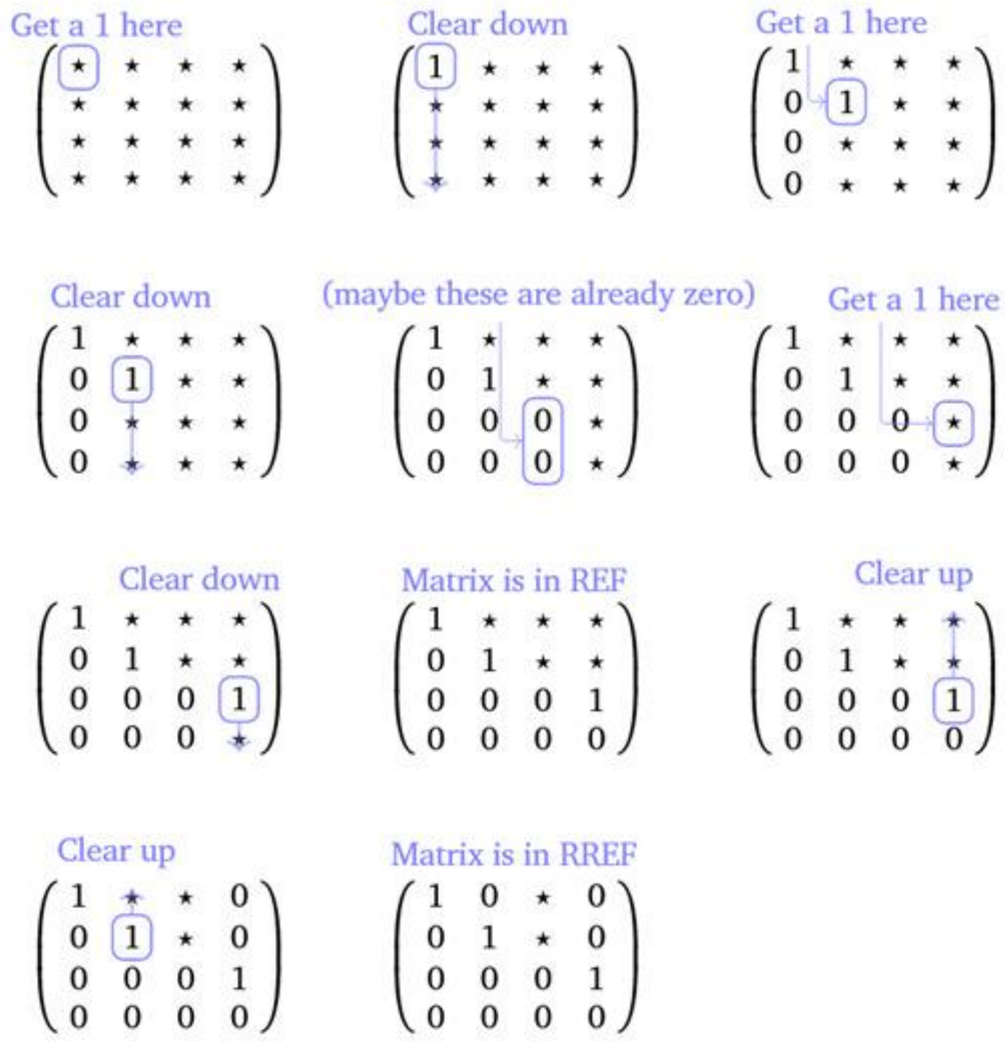
Step 10. Use type III elementary row operations to erase all the non-zero entries above the leading one in the pivot column.

Step 11. If there are no more columns containing leading ones to the left of the pivot column, then go to Step 13.

Step 12. Apply Step 9-11 to the submatrix consisting of the columns that lie to the left of the pivot column.

13. The resulting matrix is in reduced row-echelon form.

Working of Reduced Row Echelon Form Algorithm:



A *pivot position* of a matrix is an entry that is a pivot of a row echelon form of that matrix.

A *pivot column* of a matrix is a column that contains a pivot position.

Applications for Reduced Row Echelon Form:

Row reduction is very useful for balancing chemical equations

Sodium hydroxide (NaOH) reacts with sulphuric acid (H_2SO_4) to yield sodium sulphate (Na_2SO_4) and water,

$$\mathrm{NaOH} + \mathrm{H_2SO_4} \to \mathrm{Na_2SO_4} + \mathrm{H_2O}.$$

Balance the equation.

In balancing the equation, let p, q, r and s be the unknown variables such that

$$p\mathrm{NaOH} + q\mathrm{H_2SO_4} \to r\mathrm{Na_2SO_4} + s\mathrm{H_2O}.$$

We compare the number of Sodium (Na), Oxygen (O), Hydrogen (H) and Sulphur (S) atoms of the reactants with the number of atoms of the products. We obtain the following set of equations:

$$\mathrm{Na}: p = 2r$$

$$\mathrm{O}: p + 4q = 4r + s$$

$$\mathrm{H}: p + 2q = 2s$$

$$\mathrm{S}: q = r.$$

Re-writing these equations in standard form, we have a homogeneous system $\boldsymbol{Ax} = \boldsymbol{0}$ of linear equations with p, q, r and s

$$p - 2r = 0$$

$$p + 4q - 4r - s = 0$$

$$p + 2q - 2s = 0$$

$$q - r = 0,$$

or

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 4 & -4 & -1 \\ 1 & 2 & 0 & -2 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \boldsymbol{0}, \quad \text{where} \quad \boldsymbol{A} = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 4 & -4 & -1 \\ 1 & 2 & 0 & -2 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

The augmented system becomes

$$[A \quad \mathbf{0}] = \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 1 & 4 & -4 & -1 & 0 \\ 1 & 2 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right].$$

Since the right hand side is the zero vector, we work with the matrix A because any row operation will not change the zeros.

Replace row 2 with row two minus row one *i.e.*, $R_2 \leftrightarrow R_2 - R_1$. Similarly, replace row three with row three minus row one *i.e.*, $R_3 \leftrightarrow R_3 - R_1$. These first set of row operations reduces A to

$$\sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 4 & -2 & -1 \\ 0 & 2 & 2 & -2 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

In the second set of row operations, we replace row three by two times row three minus row two or $R_3 \leftrightarrow 2R_3 - R_2$ and replace row four by four times row four minus row two or $R_4 \leftrightarrow 4R_4 - R_2$ to yield

$$\sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 4 & -2 & -1 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & -2 & 1 \end{bmatrix}.$$

In the third stage of the elimination process, we replace row four with 3 times row four plus row three *i.e.*, $R_4 \leftrightarrow 3R_4 + R_3$ to yield the row echelon matrix or upper triangular U,

$$U = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 4 & -2 & -1 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We now reduce U to row reduced echelon form R as follows: First, we reduce the pivots to unity in rows two and three via $R_2 \leftrightarrow \frac{1}{4}R_2$ and $R_3 \leftrightarrow \frac{1}{6}R_3$ to obtain

$$\sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Replace row one by row one plus two times row three *i.e.*, $R_1 \leftrightarrow R_1 + 2R_3$ and row two by row two plus half row three, that is $R_2 \leftrightarrow R_2 + \frac{1}{2}R_3$. These two operations replaces all nonzeros above the pivots to zero resulting in the row reduced echelon form R

$$R = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (3)$$

The solution to $\boldsymbol{Ax} = \mathbf{0}$ reduces to $R\boldsymbol{x} = 0$ where $\boldsymbol{x}$ is actually the nullspace of $\boldsymbol{A}$ which is equivalent to the nullspace of R. Hence,

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \mathbf{0}.$$

Upon expanding, we have

$$p - s = 0 \quad \text{or} \quad p = s$$

$$q - \frac{1}{2}s = 0 \quad \text{or} \quad q = \frac{1}{2}s$$

$$r - \frac{1}{2}s = 0 \quad \text{or} \quad r = \frac{1}{2}s,$$

the nullspace solution

$$\boldsymbol{x} = \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} s$$

There are three pivot variables p, q, r and one free variable s. We set $s = 2$, so that $p = 2, q = 1$ and $r = 1$. We remark that this is not the only solution since there is

a free variable s, the nullspace solution is infinitely many. Therefore, the chemical equation can be '"balanced" as

$$2\mathrm{NaOH} + \mathrm{H_2SO_4} \rightarrow \mathrm{Na_2SO_4} + 2\mathrm{H_2O}.$$

Implementation of Reduced Row Echelon Form using Python:

```
def ToReducedRowEchelonForm(M):
    if not M:
        return
    lead = 0
    rowCount = len(M) #number of row in M
    columnCount = len(M[0]) #number of column in M
    for r in range(rowCount):
        if lead >= columnCount:
            return
        i = r
        while M[i][lead] == 0:
            i += 1
            if i == rowCount:
                i = r
                lead += 1
                if columnCount == lead:
                    return
        M[i],M[r] = M[r],M[i]
        lv = M[r][lead]
        M[r] = [ mrx/float(lv) for mrx in M[r]]
        for i in range(rowCount):
            if i != r:
                lv = M[i][lead]
                M[i] = [ iv - lv*rv for rv,iv in zip(M[r],M[i])]
        lead += 1


mtx = [
   [ 1, 0, -2, 0],
   [ 1, 4, -4, -1],
   [ 1, 2, 0, -2],
   [ 0, 1, -1, 0]]

ToReducedRowEchelonForm( mtx )

for rw in mtx:
  print (', '.join( (str(rv) for rv in rw) ))
```

Output of the code:

```
1.0, 0.0, 0.0, -1.0
0.0, 1.0, 0.0, -0.5
0.0, 0.0, 1.0, -0.5
0.0, 0.0, 0.0, 0.0
```

Conclusion:

In this project, we have shown how to balance chemical equations using row echelon form. By this project we can balance any chemical equation and it will help in chemical research for better results and more accuracy in production. In actual fact, the echelon form alone could have been used and we still have the same solution but reducing it to rref makes the solution easily deduced.

References:

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2. Noble, B. and Daniel, J.W. (1988) Applied Linear Algebra. 3rd Edition, 90-97, 103, 140-149.

3. Gabriel, C.I. and Onwuka, G.I. (2015) Balancing of Chemical Equations Using Matrix Algebra.Journal of Natural Sciences Research, 3, 29-36.