For a prime number p and an integer $x \neq 0$ by $v_p(x)$ we denote the exponent of p in the prime factorization of x, i.e. if $\alpha = v_p(x)$, then $p^\alpha \mid x$ and $p^{\alpha+1} \nmid x$. By definition, we set $v_p(0) = \infty$.

1. Let x, $y \in \mathbb{Z}$ and let p be a prime number. Prove that

$$v_p(xy) = v_p(x) + v_p(y) \quad \text{and} \quad v_p(x+y) \geq \min\bigl(v_p(x), v_p(y)\bigr).$$

Prove the following assertions:

2. Let x, $y \in \mathbb{Z}$ and $n \in \mathbb{N}$. If p is a prime number such that $p \mid x - y$, $p \nmid x$, $p \nmid y$, and $p \nmid n$, then $v_p(x^n - y^n) = v_p(x - y)$.

3. Let x, $y \in \mathbb{Z}$. If p is an odd prime number such that $p \mid x - y$, $p \nmid x$, and $p \nmid y$, then $v_p(x^p - y^p) = v_p(x - y) + 1$.

4. (a) Let x, $y \in \mathbb{Z}$ and $n \in \mathbb{N}$. If p is an odd prime number such that $p \mid x - y$, $p \nmid x$, and $p \nmid y$, then $v_p(x^n - y^n) = v_p(x - y) + v_p(n)$.
   (b) Let x, $y \in \mathbb{Z}$ and n is an odd positive integer. If p is an odd prime number such that $p \mid x - y$, $p \nmid x$, and $p \nmid y$, then $v_p(x^n + y^n) = v_p(x + y) + v_p(n)$.

5. For odd integers x and y, an even integer n the following equality holds

$$v_2(x^n - y^n) = v_2(x - y) + v_2(x + y) + v_2(n) - 1.$$

The latter three assertions (**4a**, **4b**, and **5**) constitute the **LTE** lemma. Let's use this lemma to solve the following problems.

6. Suppose a and b are positive real numbers such that $a - b$, $a^2 - b^2$, $a^3 - b^3$, ... are all positive integers. Show that a and b must be positive integers.

7. Let $k > 1$ be an integer. Show that there exists infinitely many positive integers n such that $n \mid 1^n + 2^n + \ldots + k^n$.

HOMEWORK

8. Let a, $b \in \mathbb{N}$. Prove that there is only a finite number of $n \in \mathbb{N}$ such that $\left(a + \frac{1}{2}\right)^n + \left(b + \frac{1}{2}\right)^n$ is an integer.

9. (a) Prove that $3^n \mid 2^{3^n} + 1$ for any $n \in \mathbb{N}$.
   (b) Does there exist a positive integer n which is divisible by exactly 2000 different prime numbers and such that $2^n + 1$ is divisible by n?