By 2^X we denote the family of all subsets of a finite set X. The elements of 2^X are partially ordered by inclusion. Namely, for subsets $A \neq B \subset X$ we write $A \prec B$ if $A \subset B$. Thereby, 2^X is a partially ordered set. A family $\mathcal{F} \subset 2^X$ is called an *antichain* if any element of $\mathcal{F}$ is not a subset of other elements of $\mathcal{F}$. A family $\mathcal{F} \subset 2^X$ is called a *chain* if for any two elements of $\mathcal{F}$ one of them is subset of the other.

1. (a) Find the number of largest chains ($\varnothing = S_0 \subset S_1 \subset \ldots \subset S_{|X|} = X$).
   (b) Find the number of largest chains that contain the given subset $Y \subset X$.
   (c) Let $\{A_1, A_2, \ldots, A_m\} \subset 2^X$ be an antichain. Prove that $\sum\limits_{i=1}^{m} \frac{1}{\binom{|X|}{|A_i|}} \leq 1$.
   (d) (`Sperner`) Prove that the size of a largest antichain is equal to $\binom{|X|}{\lfloor |X|/2 \rfloor}$.
2. There are five clubs. In a company of eleven students every student attends some clubs. Prove that there are two students A and B such that the student A attends every club attended by B.
3. A detective is investigating a crime. Among n suspects there is a criminal and a witness (the detective doesn't know who is who). Every day, the detective can invite one or more suspects to the police station, and if among those invited there is a witness, but no criminal, then the witness will tell who the criminal is. Find the maximum n for which the detective can solve the case in four days (the witness must tell the detective who the criminal is).
4. (`Erdös`) Let $a_1, a_2, \ldots, a_n \in (-\infty, -1/2] \cup [1/2, +\infty)$. Prove that for any open unit interval I there are at most $\binom{n}{\lfloor n/2 \rfloor}$ vectors $(\varepsilon_i)_{i=1}^{n} \in \{-1, 1\}^n$ such that $\sum \varepsilon_i a_i \in I$.
5. A family $\mathcal{F} \subset 2^X$ doesn't include a chain of size k. Find the maximum possible value of $|\mathcal{F}|$.

Home work

6. 30 students from the same class have decided to visit each other. A student can make several visits in an evening, and on the evening when someone should come to him, he himself doesn't go anywhere. Prove that the minimum number of days during which everyone can visit everyone is 7.
7. Let $\{A_1, A_2, \ldots, A_m\} \subset X$ be an antichain. Using Hall's theorem, show that we can take an antichain $\{A'_1, A'_2, \ldots, A'_m\} \subset X$ such that $|A'_i| = \lfloor |X|/2 \rfloor$.
8. There were 8 problems at a contest. It turned out that any two participants solved different sets of problems, and there was a problem solved by the first of them and not solved by the second one. Find the largest possible number of all correct solutions obtained at the contest.
9. There are n subjects in a school. Each student has received `A` or `F` in each subject. Different students have different tuples of n marks. Moreover, there is no student who studies better than two other students, and there is also no student who studies worse than two other students. Prove that there are at most $2\binom{n-1}{\lfloor n/2 \rfloor}$ students in the school.