1. There are 512 students, some of whom are boys and some of whom are girls. Each student has 10 different colored caps. Prove that students can put on their caps such that no boy is wearing the same colored cap as a girl.

An event A is called *independent* of events B_1, B_2, …, B_n if for any subset $S \subset \{1, 2, \dots, n\}$ the equality $P\left(A \mid \bigcap_{i \in S} B_i\right) = P(A)$ holds.

2. Prove that an event A is independent of events B_1, B_2, …, B_n iff for any disjoint subsets S_1, $S_2 \subset \{1, 2, \dots, n\}$ the following equality holds:

$$P\Big(A \mid \bigcap_{i_1 \in S_1} B_{i_1} \cap \bigcap_{i_2 \in S_2} \overline{B_{i_2}}\Big) = P(A).$$

Lovász Local Lemma. *Given a set of events A_1, A_2, …, A_n and a directed graph $G = (V, E)$ with vertices 1, 2, …, n. It is known that every event A_i is independent of other events A_j such that $(i, j) \notin E$. If for some numbers x_1, x_2, …, $x_n \in [0, 1)$ and every i the inequality $P(A_i) \le x_i \prod\limits_{(i,j) \in E} (1 - x_j)$ holds, then*

$$P\Big(\bigcap_{1 \le i \le n} \overline{A_i}\Big) \ge \prod_{1 \le i \le n} (1 - x_i) > 0.$$

3. Under the assumptions of **LLL**, prove that for an arbitrary index i and a subset $S \subset V \setminus \{i\}$ of vertices the inequality $P(A_i \mid \bigcap_{j \in S} \overline{A_j}) \le x_i$ holds.

4. Prove the **Lovász Local Lemma**.

5. Prove the symmetric version of **Lovász Local Lemma**:
Let each of the events A_1, A_2, …, A_n be independent of all the others except for at most d events and $P(A_i) \le \frac{1}{e(d+1)}$. Then $P(\bigcap\limits_{1 \le i \le n} \overline{A_i}) > 0$.

6. The vertices of a regular $11n$-gon are colored in n colors (11 vertices are colored in each color). Prove that there are n vertices of different colors, no two of which are adjacent.

HOMEWORK

7. In a round-robin chess tournament, each participant won 10 games and lost 10 games. Prove that all participants can be divided into two teams so that each player beats some representatives of both teams.