1. Proof that a polynomial with $k \in \mathbb{N}$ nonzero coefficients has no more than $2k - 1$ different real roots. Show that the estimate is sharp, i.e. for every k construct a polynomial with k nonzero coefficients that has exactly $2k - 1$ real roots.

Let a_1, a_2, ... be a sequence of nonzero numbers. We say that two consequent terms a_i and a_{i+1} make *a sign change* if $a_i \cdot a_{i+1} < 0$. The number of sign changes in an arbitrary sequence is defined as the number of sign changes in its subsequence consisting of all nonzero terms.

2. Prove that the number of sign changes in a sequence a_0, a_1, ..., a_n is even (odd) if $a_0 \cdot a_n > 0$ (< 0).
3. (`Descartes' rule of signs`) Let $p \in \mathbb{R}[x]$. By $C(p)$ we denote the number of sign changes in the sequence of coefficients of $p(x)$. Prove that
   (a) the number of positive roots of $p(x)$ and $C(p)$ have the same parity;
   (b) the number of positive roots of $p(x)$ is less than or equal to $C(p)$.
4. Let $a_0 \neq 0$, $a_n \neq 0$ and $2m$ subsequent coefficients of a polynomial $p(x) = a_0 + a_1x + \ldots + a_nx^n$ be zero. Prove that $p(x)$ has at least $2m$ imaginary roots.

Let $p(x) \in \mathbb{R}[x]$ be a polynomial without multiply real roots. The sequence of polynomials $p_0(x)$, $p_1(x)$, ..., $p_n(x)$ is called *Sturm sequence* of $p(x)$ if it possesses the following properties:

a) $\deg p_0(x) > \deg p_1(x) > \ldots > \deg p_n(x)$;
b) $p_0(x) = p(x)$ and $p_1(x) = p'(x)$;
c) if $p_i(x_0) = 0$ for some $x_0 \in \mathbb{R}$ then $p_{i-1}(x_0)p_{i+1}(x) < 0$;
d) the last polynomial $p_n(x)$ doesn't have real roots.

For given x_0 by $S(x_0)$ we denote the number of sign changes in the sequence $p_0(x_0)$, $p_1(x_0)$, ..., $p_n(x_0)$.

5. Prove that the number of roots of $p(x)$ on a interval (a, b) (we do not exclude the cases $a = -\infty$ or $b = +\infty$) is equal to $S(a) - S(b)$.

Now let us construct the Sturm sequence for a polynomial $p(x)$. We already know that $p_0(x) = p(x)$ and $p_1(x) = p'(x)$. To construct $p_{k+1}(x)$ from $p_k(x)$ and $p_{k-1}(x)$, divide the latter by the former and take the reminder with the opposite sign: $p_{k-1}(x) = q(x)p_k(x) - p_{k+1}(x)$.

6. Prove the constructed sequence $p_0(x)$, $p_1(x)$, ..., $p_n(x)$ is a Sturm sequence for the polynomial $p(x)$.

HOMEWORK

7. Let $\lambda_1 < \lambda_2 < \ldots < \lambda_n \in \mathbb{R}$. Prove that the number of sign changes in a sequence a_1, a_2, ..., a_n is greater than or equal to the number of roots of the quasi-polynomial $a_1e^{\lambda_1 x} + a_2e^{\lambda_2 x} + \ldots + a_ne^{\lambda_n x}$, and both these numbers have the same parity.