1. Proof that a polynomial with $k \in \mathbb{N}$ nonzero coefficients has no more than $2k-1$ different real roots. Show that the estimate is sharp, i.e. for every k construct a polynomial with k nonzero coefficients that has exactly $2k-1$ real roots.

Let a_1, a_2, ... be a sequence of nonzero numbers. We say that two consequent terms a_i and a_{i+1} make *a sign change* if $a_i \cdot a_{i+1} < 0$. The number of sign changes in an arbitrary sequence is defined as the number of sign changes in its subsequence consisting of all nonzero terms.

2. Prove that the number of sign changes in the sequence $a_0, a_0+a_1, a_1+a_2, \ldots, a_{n-1}+a_n, a_n$ is not greater than the number of sign changes in a sequence $a_0, a_1, \ldots, a_n$.
3. (`Descartes' rule of signs`) Let $p \in \mathbb{R}[x]$. By $C(p)$ we denote number of sign changes in the sequence of coefficients of $p(x)$. Prove that
   (a) the number of positive roots of $p(x)$ and $C(p)$ have the same parity;
   (b) the number of positive roots of $p(x)$ is less than or equal to $C(p)$.
4. Let $p(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0$ be a nonzero polynomial with integer coefficients such that $p(r) = p(s) = 0$ for some positive integers $r < s$. Prove that $a_k \leq -s$ for some k.

Let $p(x) \in \mathbb{R}[x]$ be a polynomial without multiply roots. The sequence of polynomials $p_0(x), p_1(x), \ldots, p_n(x)$ is called *Sturm sequence* of $p(x)$ if it possesses the following properties:

a) $\deg p_0(x) > \deg p_1(x) > \ldots > \deg p_n(x)$;
b) $p_0(x) = p(x)$ and $p_1(x) = p'(x)$;
c) if $p_i(x_0) = 0$ for some $x_0 \in \mathbb{R}$ then $p_{i-1}(x_0)p_{i+1}(x) < 0$;
d) the last polynomial $p_n(x)$ doesn't have real roots.

For given x_0 by $S(x_0)$ we denote the number of sign changes in the sequence $p_0(x_0), p_1(x_0), \ldots, p_n(x_0)$.

5. Prove that the number of roots of $p(x)$ on a interval (a, b) (we do not exclude the cases $a = -\infty$ or $b = +\infty$) is equal to $S(a) - S(b)$.

Now let us construct the Sturm sequence for a polynomial $p(x)$. We already know that $p_0(x) = p(x)$ and $p_1(x) = p'(x)$. To construct $p_{k+1}(x)$ from $p_k(x)$ and $p_{k-1}(x)$, divide the latter by the former and take the reminder with the opposite sign: $p_{k-1}(x) = q(x)p_k(x) - p_{k+1}(x)$.

6. Prove the constructed sequence $p_0(x), p_1(x), \ldots, p_n(x)$ is a Sturm sequence for the polynomial $p(x)$.

HOMEWORK

7. Prove that a polynomial of the form $x^n + a_{n-3}x^{n-3} + a_{n-4}x^{n-4} + \ldots + a_1x + a_0$, with at least one $a_k \neq 0$, cannot have all its roots real.