Chapter 3

Arbitrage and the value of money

3.1 Arbitrage

The term *arbitrage* has been mentioned previously. However it is one of the most important concepts in capital markets, and therefore merits extensive discussion.

By definition *speculation* involves uncertainty and therefore the risk of loss. A good speculator would argue that they are taking a *calculated risk*. Ideally, however, a trader seeks to make profit with no risk: a guaranteed profit.

Arbitrage is any method of *risk-free* trading. It relies on finding a market inefficiency, in which an asset is being sold for different prices. Arbitrage is simply buying this asset at the lower price and simultaneously selling it at the higher price. The simultaneously buying and selling ensures that no risk is taken since the possible change in value over time of the asset, which leads to uncertainty, is eliminated.

More generally, *arbitrage* is the (self-financing) simultaneous buying and selling of securities, currency, or commodities in different markets or in derivative forms in order to take advantage of differing prices for the same asset.

An example of arbitrage is the trade in insurance. An *insurance broker* (Bob) can sell you (Alice) an insurance policy for you and your car (for £500), but avoid taking any risk by (simultaneously) buying the same insurance *cheaper*, say at £400, from an *underwriter* (Carol), as shown in figure 3.1. Thus Bob has bought and sold a risky asset, for a profit or £100, but avoided any risk. Bob is the arbitrageur in this scenario, while Alice has her insurance policy (she is the policy holder) underwritten by Carol (the policy writer).

The key to arbitrage is finding a different price for exactly the same asset; this is termed *mispricing* or a *market inefficiency*.

This system of *passing the buck* doesn't stop there, the underwriter can then pass on (sell) this risk to a *reinsurer*. So now the insurance company is the *arbitrageur* in which the risk is off-loaded at a profit. The reinsurer in turn could find a way of repackaging and reselling their insurance risks. This creates an entire industry based upon packaging up risky products and trading them. Each layer of repackaging and reselling makes the pricing more complicated and this is where mathematics is an indispensable tool to estimating the value of these products. So often, it is the company that holds the risk (the reinsurer in this case) that has needs the best mathematical advice.

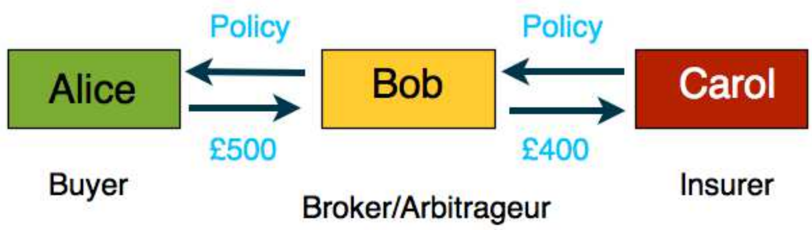

Figure 3.1: Arbitrage in insurance: Alice is the policy holder, Bob the insurance broker, Carol the policy writer. Bob is the *arbitrageur* in this scheme. He takes on the risk by selling Alice an insurance policy, but then immediately off-loads this risk onto Carol, the underwriter. He does this at a profit of £100 and although he is trading in risk, the final position is that he is able to get rid of all risk to himself, and his profit is completely risk free.

3.1.1 Example from FX

We can illustrate the basic principle of arbitrage in a financial market by considering an example from foreign exchange (FX). The foreign exchange market is particularly well arbitraged, as it is active 24 hours a day on exchanges around the globe, and it has great fluidity. We take the unrealistic assumption that the bid/offer spread is zero: that is we can buy and sell an asset at the same price. In this example, the trader is buying and selling currencies.

1	GBP	buys	1.580	USD
1	USD	buys	0.860	EUR
1	GBP	buys	1.200	EUR

Table 3.1: Schematic version of an FX exchange table converting between: USD (\$), GBP (£), EUR (€). We (falsely) assume the bid and offer prices are the same.

Suppose we have the following conversion table between three currencies (Table 3.1). The first line means that:

$$\pounds 1 \text{ buys } \$1.580$$

and conversely that:

$$\$1.580 \text{ buys } \pounds 1$$

So perhaps a currency speculator (Dave) might take the view that the € will (in the near future) go up in value versus the £. Dave has a large quantity of £. He will then take a long position in € (buying € with his £) hoping that the price will go up later so he can convert these € back to pounds (sell) and make a profit.

A currency arbitrageur (Eric) notices table 3.1, and spots a risk-free method of making a profit.

Eric calculates that there is a misplacing in this table, as can be easily seen from the following check. According to this table:

$$1\text{GBP buys } 1.580 \text{ USD}$$

and this

$$1.580 \text{ USD buys } 1.580 \times 0.860 = 1.359 \text{ EUR}$$

This method of buying € seems to be much better value than what the market is offering for a direct purchase, line 3 of Table 3.1 : £1 buys €1.200. So Eric can take advantage of the difference in prices.

Eric takes £10,000 and buys USD and then uses the USD the bought to buy EUR. After these two trades he has:

$$10,000 \times 1.580 \times 0.860 = 13,588 \text{ EUR}$$

He then *immediately* sells the EUR (hence avoiding any price fluctuations) for GBP and finishes with:

$$\frac{13,588}{1.200} = 11,323 \text{ GBP} \qquad .$$

So, without taking any risk Eric has converted his 10,000 GBP to 11,323 GBP by FX trading.

3.1.2 No arbitrage

We are making a few false assumptions here. Firstly that the bid/off prices are the same - they are not. That stops a great deal of buying and selling activity. The second is that we can buy unlimited quantities of currencies at the same prices. We have mentioned before the effect of buying a limited resource - the price goes up. The effect of selling the price down. Thus Eric continued his arbitrage strategy, say taking his 11,323 and doing the same trick and converting it to 12,821 and so on. Eventually the gap (our profit margin) will be eliminated because of his buying/selling behaviour. He will find it more difficult to by USD for the same price, to buy EUR with USD and to sell EUR for a good price.

In this way, *arbitrageurs* eliminate price differences across exchanges so that some fair price ensues. However, arbitrageurs know that another opportunity will soon arise, because prices keep moving and they aim to take advantage of such opportunities. This is achieved by writing computer programs that monitor all the prices in the market at the same time.

3.2 High-frequency trading and 'stat arb'

So, if just one of the exchange rates moves, all the others need to move to remain consistent. High-frequency arbitrage is when a program is given an *algorithmic trading* instruction to search for these imbalances and trade within a fraction of a millisecond (thousandth of second). This is called *low-latency* trading [1].

In the case of exchange rates the values are strongly correlated. Arbitrage that exploits correlation between assets is called generally called *statistical arbitrage* ('stat. arb.') and is a favourite strategy of *hedge funds*.

3.3 The value of money

In the world of money, timing is very important. This means costs and values are not always what they seem. For example £100 *now*, if invested in a bank, will be worth £110 *a year from now* (if I am lucky enough to find a bank that pays 10% annual equivalent interest). Similarly, a bill for £50 due *in 6 months time*, is worth less than £50 *at today's prices*, since that bill can be covered by putting £48 aside today in this wonderful savings account. The idea, that the value of costs/bills/benefits/payments depends on their due date, is called the *time value of money*.

We say that future costs can be *discounted* (devalued) if we wish to calculate the *present-day value*. This only makes sense because there are (risk-free) interest-bearing investments for our spare cash. We can calculate the advantage of the delayed cost, the *discounting factor*, by considering savings.

[1] Latency is the time is takes for a system to adjust/respond. A low-latency communication channel is one that is extremely fast at responding/switching.

3.4 Simple and Compound Interest

Simple interest on savings means that a single payment is made to the investor, at the end of the savings period. If a sum V_0 (usually called the *principal*) is put into a saving account with annual (simple) interest rate r_s then this will be worth:

$$V_0(1+r_s) \qquad , \tag{3.1}$$

after one year. For example, if the rate of simple interest is 4% per annum, then £200 deposited in such an account will be worth, after one year, $£200(1+0.04) = £208$. Note that interest rates are nearly always quoted in % so we need to convert to decimal form.

In *compound interest*, the interest payments are usually more frequent. These are added to the savings, so that the interest gathers interest. So if we reinvested our principal and simple interest then our new V_0 would be given by equation (3.1). After another year, we would have:

$$\left[V_0(1+r_s)\right](1+r_s) = V_0(1+r_s)^2 \qquad , \tag{3.2}$$

If we put an amount of money V_0 into a compound interest scheme with payments made m times per year, after 1 year, our savings are worth:

$$V_1 = V_0\left(1+\frac{r_c}{m}\right)^m \qquad . \tag{3.3}$$

where we call r_c the *annual compound interest rate*. This is the main difference between simple and compound interest, one is linear (equation (3.1)), whereas the other is a binomial function (equation (3.3)). This r_c is related to what is called the *annual equivalent rate* (AER), R. This is the equivalent *simple interest* for a *compound interest* investment/loan, and, comparing equations (3.3) and (3.1), we can define this via:

$$V_0(1+R) = V_0\left(1+\frac{r_c}{m}\right)^m \tag{3.4}$$

to give

$$\boxed{R = \left(1+\frac{r_c}{m}\right)^m - 1}, \tag{3.5}$$

or:

$$r_c = m\left[(1+R)^{1/m} - 1\right]. \tag{3.6}$$

In the case $r_c/m \ll 1$ (or $R \ll 1$), we can make a binomial expansion in equation (3.5), or equation (3.6), to get the approximation:

$$R \approx r_c \qquad . \tag{3.7}$$

Over t years, our investment (with compound interest) at a rate r_c paid m times per annum, will grow to:

$$V_t = V_0\left[\left(1+\frac{r_c}{m}\right)^m\right]^t = V_0\left(1+\frac{r_c}{m}\right)^{mt} \qquad . \tag{3.8}$$

3.5 Continuous interest

Suppose $m \gg 1$, so that there are many frequent interest payments (perhaps on a daily rate). This is, for example, the case when banks lend to each other. Using the Euler formula:

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \qquad , \tag{3.9}$$

we can write

$$V_t \approx V_0 e^{r_c t} \qquad . \tag{3.10}$$

The interest rate r used in the equality:

$$V_t = V_0 e^{rt} \quad , \tag{3.11}$$

is known as the *annual continuously-compounded interest rate.*

This gets a bit confusing with different types of interest. We know that r_c is the rate that applies to each payment, and R and r are derived from this. So to summarise:

$$\boxed{(1+R) = \left(1+\frac{r_c}{m}\right)^m = e^r}. \tag{3.12}$$

R is a good (practical) estimate, while r is the mathematically useful one. We note, in passing, that (see exercises) the following is always true:

$$r < r_c < R \quad ,$$

and for low interest rates, $R \ll 1$, they are about the same, $r_c \approx r \approx R$.

So £100 invested now, with $r = 0.02$ (per annum), after 6 months ($t = 0.5$ years) will be worth: $100\ e^{0.01} \approx 101$ (pounds) approximately.

Monthly compounded annual rate r_c	Annual equivalent Rate R	Continuously compounded annual rate r
0.0010	0.0010	0.0010
0.0198	0.0200	0.0198
0.0489	0.0500	0.0488
0.0957	0.1000	0.0953
0.1408	0.1500	0.1398

Figure 3.2: Comparison of the different expressions of annual interest rate. The numbers given in the table are in units per year (decimal) for the annual equivalent rate R, the annual rate (monthly compounded) r_c corresponding to the AER, and the continuously-compounded annual rate r. For rates less than 0.02 (2%) the values are almost identical.

3.6 Discount factor

We can calculate the (future) cost of a bill in today's prices, by reversing the formula. If the *annual continuously-compounded interest rate* $r = 0.08$ (per year), a bill of £210 due for payment in 6 months time is worth around £202 pounds at *today's prices.* We can deduce this, because £202 can be put into a savings account, so that in 6 months time it can cover the bill of £210 because of the interest gained.

Thus, future costs can be discounted to find their present-day value. So if we have a savings account that pays interest (compounded) m times per year, as in (3.3), then a cost C_1 due in one year's time has a (present day) value (cost):

$$C_0 = \frac{C_1}{\left(1+\frac{r_c}{m}\right)^m} \quad . \tag{3.13}$$

Clearly $C_0 < C_1$ when $r_c > 0$. Moving to a continuous-time model, referring to $t = 0$ as the present time, then a cost C_t arising at time $t > 0$ in the future, has a *present-day value*:

$$\boxed{C_0 = C_t e^{-rt}} \quad . \tag{3.14}$$

The correspondence with (3.10) should be clear. The term e^{-rt} is called the *discount factor*. One must exercise a little care applying this formula, making sure that r and t are all in the same system of units. For example, suppose that we are told that a payment is due on delivery of an order in 8 months time, and the cost will be £2500. Supposing the annual interest rate (continuously-compounded) is 6%, then the *present value* of this *future cost* is:

$$\pounds 2500 \times e^{-0.06\times(8/12)} = \pounds 2401.97 \quad .$$

3.7 Time and approximations

For a description of reality, we make approximations for mathematical convenience. Accountants would hate the way we play around with time and interest. However, such approximations can be the difference between profit and loss, so we need to be aware of these limitations. For example, time (for a trader) is measured in trading time. Once the market closes, at the end of a day and over the weekend, the clock stops, months are not exactly $\frac{1}{12}$ of one year, etc.

In this lecture course, we can get away with the notion of continuous-time trading, so long as we are thinking of investment decisions every few days or so. In practice, the theory needs revision for a real world environment where the interest may be compounded infrequently. Knowing where the approximations are, and how to refine these, is where a *good* mathematician has an advantage and makes the accountants happy (even if they are confused). For the present lecture course, we will be naive about such matters. In reality, there are lots of little equations, calculations and tweaks to make things work properly. When you are working in the real financial markets, you will need these little equations.