Chapter 6

Bernoulli model

In this chapter, we will construct a simple mathematical model that follows the dynamics of assets and their associated derivatives. We make a very simple assumption about the randomness of the price variation as time progresses. We know the value of the asset (the spot price) at the present time, $t = 0$, and this will be denoted as: S_0. We assume this asset can be bought and sold in unlimited quantities, and that the selling price and buying price is the same, so that there is no *bid/offer spread*.

6.1 Asset values as a coin toss

The simplest possible random process is the *Bernoulli trial*: a random process with only two possible outcomes. For example the toss of a coin with the outcomes being HEADS or TAILS. In one case, the price goes up to $uS_0 > S_0$. In the other the price goes down to $dS_0 < S_0$.

We also introduce a *probability mass function*. The probability that the price increases is given by p,

$$P\left[S_{\delta t} = uS_0\right] = p \quad , \qquad 0 \leq p \leq 1, \tag{6.1}$$

and the probability that the price decreases to dS_0 is

$$P\left[S_{\delta t} = dS_0\right] = q = 1 - p \tag{6.2}$$

and we suppose that we know p. This is indicated schematically in figure 6.1.

Two useful statistical parameters would be the *expected value* of $S_{\delta t}$, the best least-square-error estimate of the future price, and its standard deviation, a measure of the error in this estimate. The *expected value* of the asset in the future is:

$$\boxed{\mathbb{E}\left(S_{\delta t}\right) = puS_0 + qdS_0 = (pu + qd)S_0} \qquad . \tag{6.3}$$

The standard deviation σ, is the square root of the variance

$$\sigma = \sqrt{\mathrm{var}(S_{\delta t})}. \tag{6.4}$$

Since the variance is given by

$$\mathrm{var}(S_{\delta t}) = \mathbb{E}\left(S_{\delta t}^2\right) - \left[\mathbb{E}\left(S_{\delta t}\right)\right]^2 \qquad , \tag{6.5}$$

we find that

$$\mathrm{var}(S_{\delta t}) = p(uS_0)^2 + q(dS_0)^2 - (pu + qd)^2 S_0^2 \qquad , \tag{6.6}$$

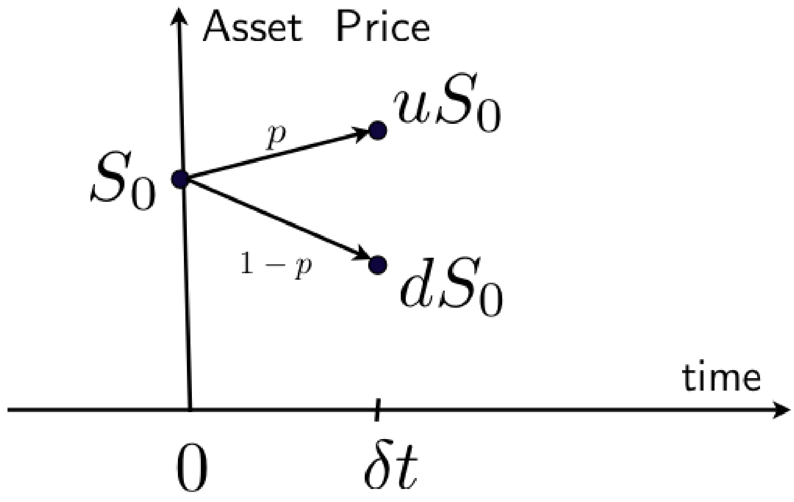


Figure 6.1: The Bernoulli branching process. An asset is priced at S_0 now. A short time in the future δt (measured in years), this price may go up to a value uS_0, with some probability $0 \leq p \leq 1$, or down to dS_0 with the probability $1-p$.

or, by using $p+q=1$,

$$\boxed{\sigma = \sqrt{pq}(u-d)S_0} \quad . \tag{6.7}$$

The uncertainty in the estimate for $S_{\delta t}$ is proportional to the difference in price, $u-d$. Moreover this uncertainty is maximised when:

$$pq = p(1-p) \quad , \tag{6.8}$$

takes the largest value, and that means: $p = \frac{1}{2}$.

6.2 Arbitrage and Risk-Neutrality

For investments, one always has to bear in mind that a risk-free investment of an amount S_0 in, say, a bond with a rate r (annual rate continuously compounded) will grow over a time δt years, to a value of $S_0 e^{r\delta t}$.

In terms of speculation in buying/selling the asset, the Bernoulli model of random price variation has three possible scenarios.

- The future value of the asset is always below the risk-free investment:

  $$S_0 e^{r\delta t} > uS_0 > dS_0 \quad .$$

  In this case it makes no sense to put money in a risky asset rather than invest in bonds. However, there is an arbitrage strategy. One can short the asset, and invest the cash from this sale in a risk-free investment. The bond will grow to the value $S_0 e^{r\delta t}$, which more than covers the cost of buying the asset at time δt and then returning the asset.

  The size of the profit would depend on the cost of the asset at δt in order to return it. The profit is either

  $$S_0 e^{r\delta t} - uS_0 > 0 \quad , \text{ or } \quad S_0 e^{r\delta t} - dS_0 > 0 \quad . \tag{6.9}$$

  depending on whether the price went up or down, as illustrated in figure 6.2(a).

- The asset price is always greater than the risk-free investment:

$$uS_0 > dS_0 > S_0 e^{r\delta t} \qquad .$$

  Now in this case, it makes no sense to invest in bonds.The asset investment, no matter how risky, is guarenteed to outperform the bond.

  In this case, one can also arbitrage. One would borrow S_0 from a bank, and use this money to buy (long) the asset at the price S_0 at $t = 0$. St $t = \delta t$, the asset would be sold for either uS_0 or dS_0. This will enable one to repay the loan, and pocket the difference as profit.

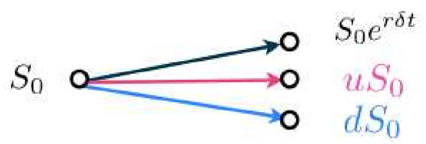
  The size of the profit would be

$$uS_0 - S_0 e^{r\delta t} > 0 \quad , \quad dS_0 - S_0 e^{r\delta t} > 0 \qquad . \tag{6.10}$$

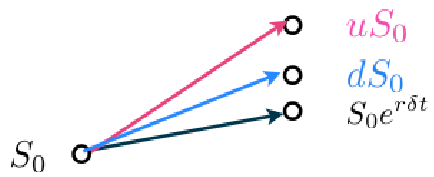
- The only *difficult* situation for a speculator is the risk that going long in the asset could mean doing better or worse than the risk-free bond investment.

$$uS_0 \geq S_0 e^{r\delta t} \geq dS_0$$

  as illustrated in figure 6.2. This is the case we will focus on.



(a) The Bernoulli model for an asset price in which the future price $S_{\delta t}$, is always *below* the values given by an equivalent risk-free investment, $S_0 e^{r\delta t}$. In this case one always takes a short position in the asset.



(b) The Bernoulli model for an asset price in which the prices are always *above* the risk-free investment value, $S_0 e^{r\delta t}$. There is always an advantage in investing in the asset, one would always take a long position.

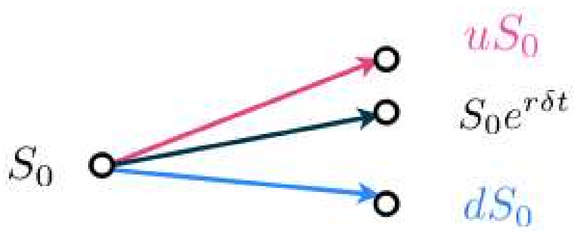


Figure 6.2: The Bernoulli mode in which taking a long position in the asset can out-perform or under-perform an investment in a risk-free investment.

6.2.1 Option as insurance

A speculator can use an option as an insurance against loss. For example, if I hold an asset (a risky investment) then I would be happy if its value increases. However, if the price falls, I risk losing money. One method of covering these losses is to use a *hedging strategy*, in which I simultaneously hold an asset with an option as an insurance, section 2.6.

The hedging scenario we will consider is an investor who simultaneously is *long* in an asset, and *short* in a call option. This portfolio will be balanced in that, at pay-off, if the asset price is high the gain in value from being long in the asset is offset by the losses from being short in the call. Conversely, if the asset value falls then the losses from holding the asset will be offset from the gains in being short the call.

So, the investor has a portfolio which can contain the following instruments:

1. a risk-free bond (with interest rate, r)
2. the risky asset, spot price S_0
3. a (European) call option, premium at time $t = 0$, c with a certain *strike price*, with expiry date δt.

Now, the value of the option c is not known. So, how can we estimate it?

6.2.2 Pricing a European call option

We assume that the asset has a behaviour similar to that shown in figure 6.2. Consider a portfolio with two investments at $t = 0$:

1. *long* in $\Delta > 0$ units, of the asset taken at spot price S_0
2. *short* in 1 call option on the asset, strike price X, expiring at δt

In order to price the option, we want to balance the portfolio such that it is risk-neutral: there is no dependence in the pay-off whether the asset price has gone up or down. In that case, the portfolio should replicate the risk-free bond. Otherwise, there is an arbitrage opportunity.

We know the value of the option at expiry (pay-off) for the holder (*long position*):

$$\boxed{c_T = \max(S_T - X, 0)} \quad , \tag{6.11}$$

although, of course we don't know S_T.

If the asset price goes up, $S_0 \to uS_0$, the pay-off for the call option is:

$$c_u = \max(uS_0 - X, 0) \quad , \tag{6.12}$$

So for the investor with the portfolio above, the value of the portfolio at δt is:

$$\Delta uS_0 - c_u \quad . \tag{6.13}$$

On the other hand, if the price goes down, $S_0 \to dS_0$, the value/price of the call option decreases,

$$c_d = \max(dS_0 - X, 0) \quad , \tag{6.14}$$

and the value of the portfolio is:

$$\Delta dS_0 - c_d \quad . \tag{6.15}$$

For the portfolio to have the same value in either case, its pay-off at δt must be the same. That is:

$$\Delta uS_0 - c_u = \Delta dS_0 - c_d \quad . \tag{6.16}$$

This can be arranged if:

$$\boxed{\Delta = \frac{(c_u - c_d)}{(u-d)S_0}} \quad . \tag{6.17}$$

The quantity Δ is called the *delta* of the portfolio, and this strategy of balancing an asset and an option is called *delta hedging*. We will see later that in the continuous limit, when we consider $S_{\delta t}$ to be a continuous rather than discrete variable,

$$\Delta = \frac{\text{diff. in call value}}{\text{diff. in the asset value}} = \frac{\partial c}{\partial S} \quad . \tag{6.18}$$

The portfolio value at δt when we use delta hedging is:

$$A = \Delta u S_0 - c_u = \frac{(c_u - c_d)uS_0}{(u-d)S_0} - c_u \quad , \tag{6.19}$$

that is,

$$A = \frac{(dc_u - uc_d)}{(u-d)}. \tag{6.20}$$

A will be the same regardless of whether the asset goes up or down in value. It is designed that way. Note the symmetry in the result for A when u and d are exchanged. Hence, there is no uncertainty (risk) in this value.

Now we need to consider the 3rd investment – the bond. We could have chosen to put all our money in risk-free bonds. If we had invested an amount, $Ae^{-r\delta t}$ at $t = 0$, this would be worth A at δt, giving the same value as the delta-hedged portfolio.

The arbitrage principle states that given two identical end values for two risk-free portfolios, the starting values at $t = 0$ of these two portfolios must be the same. Thus:

$$\Delta S_0 - c = Ae^{-r\delta t} \quad . \tag{6.21}$$

where c is the premium (price) of the call. By simple rearrangement, we find that c is given by

$$c = \Delta S_0 - Ae^{-r\delta t} = \frac{(c_u - c_d)}{(u-d)} - \frac{(dc_u - uc_d)}{(u-d)}e^{-r\delta t} \quad . \tag{6.22}$$

That is:

$$c = \left[\frac{(e^{r\delta t} - d)}{(u-d)}c_u + \frac{(u - e^{r\delta t})}{(u-d)}c_d\right] e^{-r\delta t} \quad . \tag{6.23}$$

Now let us define the symbol

$$\boxed{p^* = \frac{(e^{r\delta t} - d)}{(u-d)}} \quad . \tag{6.24}$$

Then, since (6.2):

$$d \leq e^{r\delta t} \leq u \quad , \tag{6.25}$$

it follows that:

$$0 \leq p^* \leq 1 \quad . \tag{6.26}$$

We note that

$$1 - p^* = \frac{(u - e^{r\delta t})}{(u-d)} \quad . \tag{6.27}$$

Then we can write the formula for a European call option premium at $t = 0$:

$$\boxed{c = [p^* c_u + (1-p^*)c_d]\, e^{-r\delta t}} \quad . \tag{6.28}$$

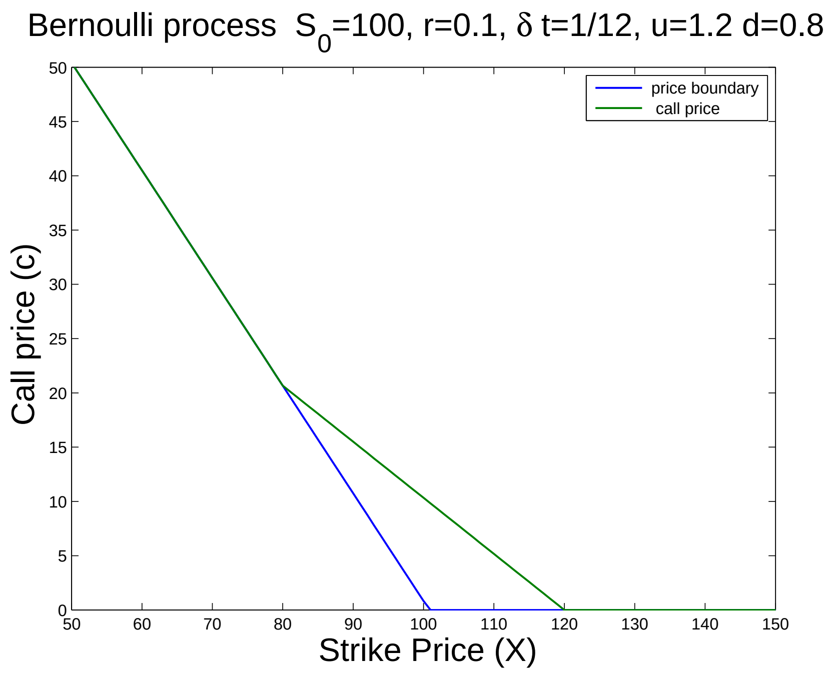


Figure 6.3: The fair price of a call option, c, as a function of strike price, X, for an asset that follows a Bernoulli process.The parameters for the asset are $S_0 = 100$, $u = 1.2$ and $d = 0.8$, with $\delta t = 1/12$ and $r = 0.10$. The green line shows the option price calculated using the formula (6.28), the blue line is the option price boundary $S_0 - Xe^{-r\delta t}$.

6.2.3 The call value decreases with increasing strike price

Then, for a range of strike prices, X, one can calculate the option price. An illustration of the variation of c with respect to X is shown in figure 6.3. From this figure it is clear that as the strike price increases the option value decreases in value (or stays the same when its price is zero). This makes sense in financial terms using the usual arbitrage argument.

Suppose the converse were true, that is given $X_2 > X_1$, suppose that the market price had $c_2 > c_1$. Then one could make a risk free profit by going short in the option for the strike price X_2 and long in the call at X_1, Take Bob to be the arbitrageur long in the option at X_1 (to Alice) and short at X_2 (to Carol).

Then, at the expiry date, if the asset is worth $S_{\delta t} > X_2$, then both call options will be exercised. Bob buys the asset for X_1 and then sells it to Carol for X_1 making a net profit of: $c_2 - c_1 + X_2 - X_1 > 0$.

If the asset ends up slightly lower in the range $X_2 > S_{\delta t} > X_1$, The Bob will exercise his call, buy the asset worth $S_{\delta t}$ for X_1 and make a net profit $c_2 - c_1 + S_{\delta t} - X_1 > 0$.

The final possibility is that the asset falls below X_1, then no one will exercise their call option. Bob still makes a profit of $c_2 - c_1$, since neither call option would be exercised.

As shown in the figure, and in keeping with common sense, the call option will be worthless when $X > uS_0$ since it is impossible for the asset ever to exceed X and thus the call option is worthless. Also shown in the figure is the option price for $X < dS_0$. The price follows a line along $S_0 - Xe^{-r\delta t}$. This result follows from arbitrage arguments and will be explained in more detail in the following chapter.

If we compare this with the expression for the *discounted* expected value of $S_{\delta t}$, then we have:

$$e^{-r\delta t}\mathbb{E}\left(S_{\delta t}\right) = \left[pS_u + (1-p)S_d\right]e^{-r\delta t} \qquad . \tag{6.29}$$

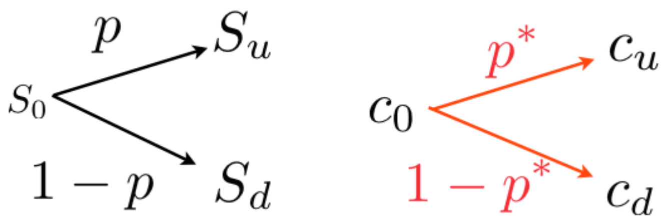


Figure 6.4: The *real-world* (left) and *risk-neutral* world (right). In the real world the asset value has a random future defined by its probability distribution. The probability of the asset value going up is $0 \leq p \leq 1$. However, in the risk-neutral world the price of a derivative based on this underlying asset is not random. We have a pseudo-probability p^*, unrelated to p, that determines the price of any derivative based on the (underlying) asset through the discounted, risk-neutral expectation.

the expected value of the asset, discounted to the present time,$t = 0$.

However, c is not a random variable – it is an unknown parameter (the price/value of a European call). Yet c appears to follow the rules of a random variable with a pseudo-probability (called the risk - neutral measure [1] p^*.

An extremely important point is that here p^* is independent of p. In fact, in the option price formula (6.28) we see no dependence on p, but we do see a dependence on $r, \delta t$, and on S_0, u, d and the strike price X.

6.3 The risk-neutral world

In the rest of this book we will repeated encounter the paradigm of a *real world*: in which the asset prices varies randomly and unpredictably (figure 6.4) according to real probability. The uncertain future evolves from the certain present to the uncertain future.

In parallel with the real world we have a *risk-neutral world* in which the option price is determined by a pseudo-probability and evolves *backwards* in time from the unknown future to the known present.

Recall that previously we discussed the notion of *measure* as simply being a number between 0 and 1. Although our instinct is to associate this concept with a coin toss or a roll of a die, the concept is much more general. We have found that option prices can be related to a *measure*, a number p^* which has the requisite limits: $0 \leq p \leq 1$. This measure is *not* a probability in any sense of that word, but it is a measure nonetheless. The fact that p^* does not depend on p in any way should confirm the statement that p^* is not related to the random process.

The concept of a risk neutral measure is *fundamental* to derivative pricing. In fact, the existence of the risk neutral measure is the basis of the fundamental theorem of asset pricing - see below.

6.4 Δ-hedging for a put option

Alternatively, instead of balancing an asset holding with a *call option*, one could consider a combination of asset and put option. Suppose we hold an asset, then our concern would be the fall in value of our asset. To counterbalance this one could take a long position in a put option, that is have the right to

[1] measure is another word for probability

sell the asset at a fixed price X, even if the market price $S_{\delta t}$ was much less.

Consider a portfolio that long Δ units of asset and short 1 unit of a put option at a strike price X, would be, at $t = 0$:

$$\Delta S_0 - p \qquad . \tag{6.30}$$

Then at the expiry time δt, the value would be either:

$$\Delta u S_0 - p_u \qquad \text{or} \qquad \Delta d S_0 - p_d \qquad , \tag{6.31}$$

depending on whether the asset value goes up or down, respectively. Following the same line of argument, the Δ-hedging that ensures the portfolio has the same value either way:

$$\Delta = \frac{p_u - p_d}{(u-d)S_0} \qquad , \tag{6.32}$$

and, since $p_d \geq p_d$, $u \geq d$, this confirms that $\Delta < 0$. That is, in order to *hedge*, one should be short in the asset and short in the put. Similarly, if one is long in the put, one should be long in the asset to create a risk-free portfolio. This leads to the expression for the value/price of a put option, p_E now $(t = 0)$, which expires at δt, and which has a strike price, X:

$$\boxed{p = \left[p^* p_u + (1-p^*)p_d\right] e^{-r\delta t}} \qquad , \tag{6.33}$$

where, as before:

$$p_d = \max(X - dS_0, 0) \qquad , \qquad p_u = \max(X - uS_0, 0) \qquad , \tag{6.34}$$

and p^* is given by (6.24).

6.4.1 Example

To demonstrate how this principle works, we will calculate the fair price of a call option.

Consider an investment portfolio consisting of

- Δ units of an asset bought at the spot price $S_0 = 30$, at $t = 0$.
- *short* 1 unit of European call options on the asset at strike price X=32, expiring δt =0.3.

The interest rate is: $r = 0.05$ per year.

The future asset price $S_{\delta t}$ has the following *known* probability distribution. The price can increase to a value $uS_0 = 35$ with probability 0.5, or change to a value $dS_0 = 29$ with probability 0.5.

The calculation of the fair option price proceeds in three steps

1. Calculate the value of Δ that ensures a risk-neutral portfolio.
2. Calculate the risk-neutral probability, p^*.
3. Calculate the fair price of the call option at the present time, $t = 0$.

Solution:

1. First, we calculate u and d: $35 = uS_0 = u30$ which gives $u = 35/30$ and $d = 29/30$. Risk neutral means the portfolio value is the same whether the price goes up or down then we equate the two possible outcomes:
$$\Delta 35 - c_u = \Delta 29 - c_d \quad . \tag{6.35}$$
The value of the call option c at expiry time is
$$c_T = \max(S_T - X, 0),$$
so that $c_u = \max(35 - 32, 0) = 3$, and $c_d = \max(29 - 32, 0) = 0$. This gives
$$35\Delta - 3 = 29\Delta, \tag{6.36}$$
from which we find $\Delta = 0.5$. So holding $\Delta = 0.5$ units of asset for every unit short call ensures risk-neutrality.

2. Now p^* can be obtained by (6.28):
$$p^* = \frac{e^{r\delta t} - d}{u - d} = \frac{1.0151 - 0.9667}{1.1667 - 0.9667} = 0.2422 \quad . \tag{6.37}$$

3. The call price can then be determined via
$$c_0 = \left[p^* c_u + (1 - p^*) c_d\right] e^{-r\delta t} \quad , \tag{6.38}$$
or
$$c_0 = [0.2422 \times 3 + 0.7578 \times 0] \times 0.9851 \qquad = 0.7158. \tag{6.39}$$

This is the *fair price* to pay for the option. If the price were anything other than this, one could use *arbitrage*to make a (self-financing) risk-free profit.

6.5 Martingales & the fundamental theorem of asset pricing

We previously defined the discounted expectation under risk-neutral measure to evaluate derivatives at current prices.

$$\mathbb{E}^*\left(g(S)\right) = p^* g(S_u) + (1 - p^*) g(S_d) \quad . \tag{6.40}$$

Thus,
$$f_0 = e^{-r\delta t}\mathbb{E}^*\left(f(S_{\delta t})\right) \quad . \tag{6.41}$$

Consider the value of the asset under this discounted risk-neutral measure,

$$e^{-r\delta t}\mathbb{E}^*\left(S_{\delta t}\right) = e^{-r\delta t}\left[\frac{(e^{r\delta t} - d)}{(u - d)} uS_0 + \frac{(u - e^{r\delta t})}{(u - d)} dS_d\right] \quad . \tag{6.42}$$

Then simplifying the right-hand-side gives us:

$$= \frac{e^{-r\delta t} S_0 (u - d)}{(u - d)} e^{r\delta t} \tag{6.43}$$

that is,
$$\boxed{e^{-r\delta t}\mathbb{E}^*\left(S_{\delta t}\right) = S_0} \quad . \tag{6.44}$$

Thus the discounted risk-neutral expected value of the future asset is the same as its present value. The equation (6.44) is called the *fundamental theorem of asset pricing*[2].

A random variable whose expected value in the future is the same as its present value is called a *martingale*. The expected value of such a portfolio is the same as its present value. This is an example of a *fair game*: one in which the expected profits are zero. We will use this approach later in order to calculate derivative prices for continuous-variable assets.

6.5.1 Option value for low/high strike prices

The price of a European call/put option can also be calculated for the Bernoulli process when either $X > uS_0$ or $X < dS_0$. That is the strike price lies outside the range of the future possible asset prices.

Consider the call option price when $X < dS_0$, then according to (6.28) we have:

$$c_E = [p^*(uS_0 - X) + (1-p^*)(dS_0 - X)]\, e^{-r\delta t} = [p^*u + (1-p^*)d]\, S_0 e^{-r\delta t} - Xe^{-r\delta t} \quad . \tag{6.45}$$

The first term is the risk-neutral, or martingale, term (6.44) for the asset, and so this can be simplified to its present-day value:

$$c_E = S_0 - Xe^{-r\delta t} \quad . \tag{6.46}$$

This expression can also be derived through *arbitrage* arguments. A portfolio consisting of a long call (strike price X) combined with an investment of $Xe^{-r\delta t}$ in risk-free bonds has a pay-off here of $S_{\delta t}$, since one will always exercise the call option and pay X for an asset worth $S_{\delta t}$. This portfolio would be worth $S_{\delta t}$. Similarly a portfolio consisting just of the asset at $t = 0$, will cost S_0 and be worth $S_{\delta t}$ at δt. Both portfolios have, with certainty, the same value at δt, and they must therefore have the same value at $t = 0$:

$$c + Xe^{-r\delta t} = S_0 \quad . \tag{6.47}$$

On the other hand when $X > uS_0$, the call is worthless at expiry and so:

$$c_E = [p^*0 + (1-p^*)0]\, e^{-r\delta t} = 0 \quad . \tag{6.48}$$

Conversely for the put option, when $X < dS_0$, then according to (6.33) we have:

$$p_E = [p^*0 + (1-p^*)0]\, e^{-r\delta t} = 0 \qquad , \qquad X < S_{\delta t} \qquad . \tag{6.49}$$

While, when $X > uS_0$, we have:

$$p_E = [p^*(X - uS_0) + (1-p^*)(X - dS_0)]\, e^{-r\delta t} = Xe^{-r\delta t} - [p^*u + (1-p^*)d]\, S_0 e^{-r\delta t} \quad . \tag{6.50}$$

Again, we recognise the martingale term and the reduction gives:

$$p_E = Xe^{-r\delta t} - S_0 \quad , \quad X > S_{\delta t} \qquad . \tag{6.51}$$

Again the arbitrage argument supports this result. We can imagine two portfolios, the first of which is long in a put and long in the asset (both one unit). Given we know in advance that we will be in the money with the put (since $X > uS_0$) we will certainly exercise - sell for X and hand over our asset. Thus the portfolio value is X at δt. One would get the same final value by pure bond investment of $Xe^{-r\delta t}$ at $t = 0$. So the argument goes:

$$p + S_0 = Xe^{-r\delta t} \quad ,$$

which, after a simple rearrangement, is confirmation of the result (6.51).

[2]Harrison and Pliska (1981)

6.6 The predictable asset

Suppose the concept of the *risk-neutral world* was tested with the following hypothetical case (*Gedanken-experiment*). If the risk-neutral world really is oblivious to the real world p, then let us choose an extreme situation. Let $p = 1$ and see what happens.

Choosing $p = 1$ means that the asset price is entirely *predictable*. We know for certain that, given the price S_0 at $t = 0$, the price will be uS_0 after a time δt. However, we need to consider the mathematical implication of such an assertion. If the known price is such that $uS_0 > e^{r\delta t}S_0$, then our asset will out-perform any bond. So we would borrow huge amounts from the bank and go long in the asset, since we can pay back our loan and make a self-financing risk-free profit. Arbitrage is possible in the other case, $uS_0 < e^{r\delta t}S_0$, in this case we would always short the asset and put the money received into bonds.

The conclusion from this argument is that any predictable asset must obey the equation: $uS_0 = e^{r\delta t}S_0$. That is, the asset must grow in agreement with any risk-free asset. Hence:

$$u = e^{r\delta t} \qquad . \tag{6.52}$$

If this is the case, then for the risk-neutral measure

$$p^* = \frac{e^{r\delta t} - d}{u - d} = 1 \quad , \quad 1 - p^* = 0 \qquad . \tag{6.53}$$

Thus. according to (6.28):

$$c = \max(uS_0 - X, 0)e^{-r\delta t} = \max(S_0 e^{r\delta t} - X, 0)e^{-r\delta t} \qquad . \tag{6.54}$$

And this result makes perfect sense: If I am offered a call, on a predictable asset with the value $S_0 e^{r\delta t}$ at the expiry time. Then for any X less than this value, the call is worthless, since we will never exercise it.