Chapter 4

Forward contracts

4.1 Predicting the future price

The asset price now, at $t = 0$, (called the spot price) S_0 is known with certainty, as are all the previous (past) values $t < 0$. We *assert* that, at some time $t = T$ in the future, the corresponding asset price, S_T, can have one of the (discrete) values:

$$S_T \in \{s_1, s_2, \ldots, s_m\}, \tag{4.1}$$

but we don't know which one. This is not due to a lack of information, but it is due to random (human) effects that make the value unpredictable. So S_T is a *discrete* random variable, since there are a finite number of discrete values it can have, according to the model proposed. This simplifies the real situation, but it provides a problem that can be solved much more easily until we are ready to tackle the more difficult problems.

For this random variable, we define a *probability function*:

$$0 \leq P(S_T = s_i) \leq 1 \quad , \quad i = 1, \ldots, m \qquad . \tag{4.2}$$

If we know this function then we are indeed fortunate since this completely describes the random nature of the variable, and we cannot be better informed than this.

For such a discrete random variable, we term the probability function as a *probability mass function*, $f_{S_T}(s)$:

$$P(S_T = s_i) = f_{S_T}(s_i) \qquad . \tag{4.3}$$

The use of the word 'mass' reflects the notion of a 'point mass' in mechanics. The probability is zero everywhere except at certain precise values. This is consistent with financial markets where the price is restricted to a set of *tick values*.

Note that probabilities are always between the limits of 0 and 1, see (4.2), and the total probability must add to one:

$$\sum_{i=1}^{m} f_{S_T}(s_i) = 1 \qquad . \tag{4.4}$$

If we know this function, then we can calculate various properties of f_{S_T}, such as the average value (mean or *expectation value*) of S_T. We use the following notation for this:

$$\mu = \mathbb{E}\left(S_T\right) \equiv \sum_{i=1}^{m} s_i f_{S_T}(s_i) \qquad . \tag{4.5}$$

The expectation value will be a rough guess for S_T, it is *not* the value we expect S_T to be. We can also calculate the expectation value of any *function* of S_T in this way:

$$\mathbb{E}\left(g(S_T)\right) \equiv \sum_{i=1}^{m} g(s_i) f_{S_T}(s_i) \qquad . \tag{4.6}$$

A second important property that can be determined is the *variance* of S_T, defined as follows:

$$\text{var}\left(S_T\right) \equiv \mathbb{E}\left(S_T^2\right) - \left(\mathbb{E}\left(S_T\right)\right)^2 = \mathbb{E}\left((S_T - \mu)^2\right) \qquad . \tag{4.7}$$

The expectation value is a useful quantity for S_T: it is the *best predictor* for S_T in the following sense. Suppose we seek a number a which is the 'best guess' for S_T. We define the error in this guess, by the (random) variable:

$$\varepsilon = a - S_T \qquad . \tag{4.8}$$

So we want to choose a to minimise this error. But since S_T has a range of values, with different probabilities, it is not immediately clear what quantity to minimise. One option is to choose a such that the expectation value of the square of the error is minimised. That is, a is such that:

$$g(a) = \mathbb{E}\left((a - S_T)^2\right) \tag{4.9}$$

is minimum. Thus, we seek the value of a such that the function:

$$g(a) = \sum_{i=1}^{m} (a - s_i)^2 f_{S_T}(s_i) \tag{4.10}$$

is minimum. Every term in (4.10) is positive, thus $g(a) > 0$ for all a. The minimum value of g is then the solution of the equation:

$$\frac{d}{da} g(a) = 0 \qquad . \tag{4.11}$$

That is:

$$\frac{d}{da}\left[\sum_{i=1}^{m} (a - s_i)^2 f_{S_T}(s_i)\right] = 2\left[\sum_{i=1}^{m} (s_i - a) f_{S_T}(s_i)\right] = 0 \qquad . \tag{4.12}$$

This has the solution, using (4.4)

$$a = \mathbb{E}\left(S_T\right) \qquad . \tag{4.13}$$

So, in the sense of minimum expected squared-error, the expected value is the best guess for S_T.

So, if the probability mass function is (publicly) known, then buyer and seller will know the best guess at the future value of the asset, although it is unpredictable, and can choose their strategy accordingly.

4.2 Pricing a forward contract by expectation

Suppose a *forward contract* was established between Alice (long) and Bob (short) at $t = 0$, with expiry $t = T$, at a strike price, X. Remember that this means that Alice is obliged to pay the amount X at time T to Bob for the asset, no matter what its value is. Equally, Bob is obliged to sell the asset to Alice at the price X at time T.

The present value of the contract is zero - no money changes hands. However the pay-off for a *forward contract* for the party long (Alice) in the contract is:

$$F_T = S_T - X \qquad , \tag{4.14}$$

that is, the difference between the value of the asset at time T, S_T, and what she paid for it: X.

The key question then is:

$$\boxed{\text{What would be a fair price for X ?}}.$$

Now, both parties are taking a risk entering into the forward contract, since S_T is unpredictable. However, in spite of this unpredictability, there is a well-determined *fair price* for X. By fair price we mean, in which neither party has a *guaranteed* profit/loss.

We assume that both Alice and Bob have a very good knowledge of mathematics and both have (somehow) calculated the probability mass function for the asset price at the expiry time: f_{S_T}.

Thus both Alice and Bob agree that the pay-off is unpredictable, and that the best guess for S_T would be $\mathbb{E}(S_T)$. Since the forward costs nothing at $t = 0$, then, one may think that X is chosen such that the value of the forward at pay-off should also be nothing (to the best estimate).

This implies that the best value for X, according to finding a zero (or minimum) pay-off (4.14) is such that:

$$\mathbb{E}\left(F_T^2\right) \equiv \mathbb{E}\left((S_T - X)^2\right) \quad , \tag{4.15}$$

is minimised, and this means that:

$$X = \mathbb{E}(S_T) \quad . \tag{4.16}$$

However, while mathematically consistent, this is completely wrong. The reason for the error is that one needs to take into account other possible investment opportunities over this time, and risk-free investments in particular.

4.3 Pricing by Arbitrage

We will first state the correct strike price for the forward and then prove the result. The *fair strike price*, F of a *forward contract* expiring at a time T in the future, for an asset with current (spot) value S_0 is:

$$\boxed{F = S_0 e^{rT}} \tag{4.17}$$

where r is the (risk-free) interest rate.

We will derive this through *proof by contradiction*. This is often referred to by the Latin expression *reductio ad absurdum*. To do this we assume that expression (4.17) is *incorrect* and show that this assumption leads to an incorrect outcome. In finance, an incorrect outcome normally means an arbitrage opportunity: an opportunity to make a riskless profit.

Proof:

If we assume that (4.17) is incorrect, that would mean either (a) $F > S_0 e^{rT}$, or (b) $F < S_0 e^{rT}$.

If (a) is true then an arbitrageur could make a profit as follows.

- The price $F > S_0 e^{rT}$ is agreed at $t = 0$. So Alice (long) agrees to pay Bob (short), a price F for the asset at $t = T$.
- Bob goes to the bank, borrows cash to the value S_0, and buys the asset now at a price S_0. Thus Bob already has the asset in his possession that he will need to sell to Alice. However, he has borrowed money in order to do this, and that loan must be repaid.
- At $t = T$ (maturity/expiry) the contract must be settled. Bob has the asset which he delivers to Alice, for which she pays him F in cash. Bob then uses this money to repay his loan.

The cost of this loan has gone up and is now $S_0 e^{rT}$ since it has been gaining interest.

However Bob still makes a profit:

$$\mathsf{Bob's\ \ profit} = F - S_0 e^{rT} > 0 \qquad . \tag{4.18}$$

Bob has made a guaranteed profit, without any risk: he has used a *arbitrage* strategy. On the other hand, it is Alice alone who has been taking all the risk. She has been speculating that in the future, $S_T > F$. She may well be right, but she has taken more risk than necessary.

If (b) is true then Alice can adopt an arbitrage strategy.

- The price $F < S_0 e^{rT}$ is agreed at $t = 0$. So Alice (long) agrees to pay Bob (short), a price F for the asset at $t = T$.
- Alice shorts the asset now (at $t = 0$). She borrows the asset from Dave, sells this asset to Carol and receives S_0 in cash from her. Alice immediately puts the cash into bonds/saving with interest r.
  So Alice has a risk-free investment, but she is obliged to return the asset to Dave, and obliged to buy this from Bob at $t = T$.
- At $t = T$ (maturity/expiry) the contract must be settled. Bob has the asset which he delivers to Alice, for which she pays him F in cash, which Alice can easily cover since her cash has gained interest and is now worth $S_0 e^{rT}$.
  Alice takes the asset and then passes this to Dave as promised.

However, she has made a profit, without any risk:

$$\mathsf{Alice's\ \ profit} = S_0 e^{rT} - F > 0 \qquad . \tag{4.19}$$

Now it is Bob alone who has been taking all the risk: speculating that in the future, $S_T < F$, while Alice has no interest at all in the value of S_T since she already has her profit.

So, in case (a) Bob has an opportunity to make a risk-free profit, whereas in case (b) Alice can do so. The principle of fair price means that, at a fair price, there are no arbitrage (free lunch) opportunities. Thus the existence of either (a) or (b) means that either condition could not reflect the fair-price case.

The conclusion is: $F \not> S_0 e^{rT}$ and $F \not< S_0 e^{rT}$. Thus, we must conclude that $F = S_0 e^{rT}$.

4.3.1 Elimination of arbitrage

We have mentioned that arbitrage opportunities should de eliminated quickly in an efficient market. To explain this, let us assume that $F > S_0 e^{rT}$. This was the case where Bob (short) could make a guaranteed profit by buying assets at $t = 0$, so he would be assured of fulfilling his side of the bargain at expiry. However, only a limited amount of asset would be for sale at that price. Hence Bob's purchases would increase the spot price of the asset. As S_0 increases, Bob's profit margin $F - S_0 e^{rT}$ decreases. By taking advantage of the arbitrage opportunity, Bob will eliminate his advantage, and bring the spot price in line with the forward price. This is how the market operates: arbitrage opportunities exist but are usually spotted quickly, taken advantage of by an arbitrageur and thereby eliminated.

4.4 Forward for a non-dividend-paying asset

We will again derive the *fair price* of a forward, but now use arbitrage arguments.

The following scenario exists. The time is $t = 0$, and there is an asset with spot price S_0 and a forward contract with a delivery date T and delivery price X. The aim of the following discussion is to determine what a fair delivery price X should be.

Suppose we have two different investment portfolios: P1 and P2. We will consider the value/price of investment in each portfolio at $t = 0$ and their pay-offs at a later time $t = T$.

P1: consists of two instruments.

- At $t = 0$ (the present) 1 unit *long* in the *forward* on an asset with strike price X and expiry $t = T$.
- Risk-free (bond) investment of value Xe^{-rT},

The contract costs nothing, but the investment in bonds costs Ze^{-rT}, so that the total initial cost of the portfolio (and its value at $t = 0$) is, Xe^{-rT}.

P2: consists of one instrument.

- At $t = 0$, 1 unit long in the asset which has spot price S_0.

The initial value/cost of P2 is S_0.

The values (pay-offs) of the portfolio P1 at the future time T can be calculated. The value of the forward contract will be:

$$S_T - X \quad ,$$

since one buys the asset at the price X, but its true value is S_T. This is the risky part of the portfolio since this may be positive or negative. On the other hand, the bond investment has grown, with interest to:

$$(Xe^{-rT})e^{rT} = X \qquad .$$

Thus the total value of the portfolio P1 at $t = T$ is the sum of the value of both instruments:

$$S_T - X + X = S_T \tag{4.20}$$

The value of P2 is simply the future value of the asset, S_T.

Thus, the values of P1 and P2 are identical, even though they are both unknown and unpredictable. Now let us state the *principle of arbitrage*:

$$\boxed{\text{Two portfolios with the same final value must have the same initial value}}$$

If this is not the case, then an arbitrage opportunity for risk-free profit exists. For example, if the initial investment in P1 cost less than P2 then we would sell P2 (go short in this portfolio) and buy P1 (go long in portfolio 1). That is we would short one unit of the asset (short in P2) and long the forward and invest money in bonds (long P1). If the converse would be true and P1 cost more that P2, then we would sell P1 (short P1) and buy (long) P2.

Thus the arbitrage principle asserts that the initial costs for portfolio P1 and P2 must be the same. That is P1=P2 at $t = 0$, and we obtain:

$$Xe^{-rT} = S_0 \quad , \tag{4.21}$$

or,

$$X = S_0 e^{rT} \qquad . \tag{4.22}$$

This is the expression (4.17) we obtained previously through proof by contradiction.

4.5 Forward pricing for a dividend-paying asset

There are, however, further considerations for the pricing of a forward. Assets may need to be stored, and this may involve storage costs. Holding a share entitles the holder to dividends, a share of the profits of the company, whereas holding a forward for shares does not. In the determination of a fair price for a forward, these costs and benefits need to be accounted for.

To simplify the calculation of the fair price for the forward, let us consider the value of the dividend. Suppose we are considering a forward contract with maturity at $t = T$, and a share/asset that has two dividend payments in this time-frame. The dividend payment d_1 is paid at time t_1 and the dividend (of value) d_2 is paid at time t_2, where: $0 \leq t_1 \leq t_2 \leq T$, as indicated in figure 4.1. Suppose these payments and dates of payments are known in advance. Typically the dividend dates are known, but the value of the payments are not, in general, so the forward price will be an approximation.

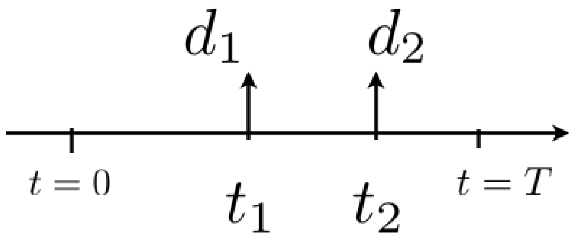

Figure 4.1: Dividend payments of values d_1 and d_2 payable at times t_1 and t_2, respectively, on an asset.

The present day value of the dividends d_1 and d_2, their worth at $t = 0$, is easily calculated. We simply discount each payment:

$$D_0 = d_1 e^{-rt_1} + d_2 e^{-rt_2} \tag{4.23}$$

Equally, we can calculate the future value of these dividends, say at $t = T$:

$$D_T = D_0 e^{rT} = d_1 e^{r(T-t_1)} + d_2 e^{r(T-t_2)} \tag{4.24}$$

We can understand this expression as follows. Once we receive the first payment d_1 at t_1 we immediately invest the cash in bonds, and this will grow in value to, $d_1 e^{r(T-t_1)}$ by a time T. The same applies for the second payment.

In general for n dividend payments $\{d_1, d_2, \ldots, d_n\}$ at times $\{t_1, t_2, \ldots, t_n\}$, the present-day value is:

$$\boxed{D_0 = \sum_{i=1}^{n} d_i e^{-rt_i}} \quad . \tag{4.25}$$

To obtain the fair price for the forward contract, we consider the two portfolios, P1 and P2, again. Now we will take into account the benefits gained by the asset holder.

The pay-off for P2 is now, the asset value and the cash accrued from the dividends, invested in bonds, (4.24):

$$S_T + D_T$$

In order for P1 to match this pay-off value, a further investment of D_0 in bonds needs to be made at the start.

By the principle of arbitrage, the initial value of the two portfolios should now be the same. That is:

$$X e^{-rT} + D_0 = S_0 \tag{4.26}$$

This gives us the correct (fair) strike price, X, for a forward on a dividend-paying asset:

$$\boxed{X = (S_0 - D_0)e^{rT}} \quad , \tag{4.27}$$

where, T is the time to expiry (in years), S_0 is the current (spot) price of the asset, D_0 is the present-day value of the dividends, and r is the continuously-compounded annual (risk-free) interest rate (in units per year). Thus the fair price of the forward has gone down.

4.6 Cost of carry

On the other hand, the holder of an asset might be liable to costs. For example, if the underlying asset was a commodity (gold, oil, wheat, coffee) then the person holding the commodity would be required to store this somewhere, or pay transport costs, or insurance costs. We term these costs arising from holding the asset, *costs to carry.*

These costs are effectively negative dividends. Let us assume that two charges are due, u_1 at time t_1 and u_2 at time t_2 (see figure 4.2). We then follow the same procedure as above. Their present day cost, their worth at $t = 0$, is given by:

$$U_0 = u_1 e^{-rt_1} + u_2 e^{-rt_2} > 0, \tag{4.28}$$

and the future value of these costs at $t = T$ by:

$$U_T = U_0 e^{rT} = u_1 e^{r(T-t_1)} + u_2 e^{r(T-t_2)} > 0. \tag{4.29}$$

Portfolio P1 is unaffected by costs to carry, but we need to re-evaluate P2. The pay-off for P2 will now be:

$$S_T - U_T$$

which is less than before. P1 should have the same pay-off at $t = T$, and our initial investment in bonds can be therefore be reduced to $Xe^{-rT} - U_0$. We now equate the new initial investments, and find:

$$Xe^{-rT} - U_0 = S_0. \tag{4.30}$$

This leads to:

$$\boxed{X = (S_0 + U_0)e^{rT}}, \tag{4.31}$$

and we find that the fair price of the forward has increased due to costs to carry.

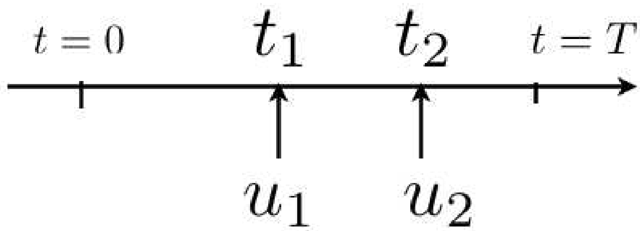

Figure 4.2: Cost-to-carry payments of values u_1 and u_2 payable at times t_1 and t_2, respectively, on holding an asset.

We can therefore consider these costs as negative dividend payments:

$$D_0 \to -U_0 \qquad .$$

Please note that it is easy to make a mistake by getting the sign wrong in these calculations. Be careful.

Example

Calculate the fair forward price of a barrel of oil, for expiry in 6 months time, given the following data.
The spot price is $S_0 = \$60$
3-month storage cost of oil is \$2 per barrel (payable in arrears)
The annual interest rate, continuously compounded, is 5%.

The answer is given by formula (4.31). S_0 is known, and U_0 is given by (4.28). We note that $r = 0.05$ (per year) in decimal and $T = 0.5$ (years). There will be two payments for storage. The first after 3 months, the second after 6 months.

$$U_0 = \$2e^{-0.05\times 0.25} + \$2e^{-0.05\times 0.50} = \$3.9257 \qquad .$$

Then, using formula (4.31):

$$X = (\$60.0000 + \$3.9257)e^{0.05\times 0.50} = \$65.544 \qquad .$$

4.7 Interest-rate parity

If the asset is a foreign currency, then the price of the asset is the *exchange rate*. The trade in foreign currencies is an extremely liquid market which follows, to a good approximation, an entirely unpredictable motion. An illustration of this is given in figure 4.3 where the US dollar versus the UK pound is shown. This is effectively the price of buying a pound in US dollars. So if £1 = \$ 1.5610, this means that the price of a dollar is \$ 1 = £0.6404.

Suppose a UK company needs to buy equipment from an American supplier. This payment will have to be made in \$. If the payment is due in 6 months time, then the cost in pounds in 6 months time is uncertain, because the future exchange rate is uncertain. The company could enter a foreign exchange forward contract agreeing a future exchange rate to buy dollars. The question is, what is the fair price for this forward contract?

An exchange-rate forward contract is not a standard forward contract since the asset, in this case dollars, can be invested in a US dollar account and gain interest. Therefore the asset is similar to a dividend paying asset, in which the holder of the asset (dollars) can use this to their advantage by earning interest. So the price of the forward would be less than the value given by equation (4.17). However, the interest rate earned on the US dollar account will differ from the UK pound interest rate.

In order to calculate the price, we again use the arbitrage principle considering two portfolios. We have two interest rates to think about, let's call these r_{GBP} for bond investments in Sterling and r_{USD} as the interest rate that would be in US treasury notes - their equivalent of UK gilts.

As before P1 consists of two instruments.

- At $t = 0$ (the present) 1 unit *long* in the *forward* on the asset with strike price X and expiry $t = T$. That is, the promise to buy 1 dollar at a future time T for a price X agreed now.
- Risk-free investment of value $Xe^{-r_{\mathsf{GBP}}T}$, in UK bonds with interest rate r_{GBP}.

The total initial cost of the portfolio is, $Xe^{-r_{\mathsf{GBP}}T}$, and the final (pay-off) value will be $S_T = 1$.

P2: consists of one instrument.

- At $t = 0$, buy $e^{-r_{\mathsf{USD}}T}$ units in the asset (US dollars) at the current exchange S_0.

The initial value/cost of P2 is $e^{-r_{\mathsf{USD}}T}S_0$. But immediately putting our $e^{-r_{\mathsf{USD}}T}$ units (dollars) into a USD account will be worth at $t = T$, with interest:

$$e^{-r_{\mathsf{USD}}T} \times e^{r_{\mathsf{USD}}T} = 1$$

dollars. That is, the final pay-off value will be 1 dollar. (Of course we can amend the units to any final amount necessary).

Since the pay-offs are the same, the value of the initial investments would also be the same by *arbitrage*:

$$Xe^{-r_{\mathsf{GBP}}T} = e^{-r_{\mathsf{USD}}T}S_0, \tag{4.32}$$

$$\boxed{X = \left(S_0 e^{-r_{\mathsf{USD}}T}\right) e^{r_{\mathsf{GBP}}T} = S_0 e^{(r_{\mathsf{GBP}} - r_{\mathsf{USD}})T},} \tag{4.33}$$

and we have obtained a fair value for the exchange rate in the forward contract. The modification to the standard forward price $S_0 \to S_0 e^{-r_{\mathsf{USD}}T}$. The initial investment S_0 becomes smaller, similar to the modification for a dividend-paying asset: equation (4.27). This is not too surprising, as the USD is an asset of additional value to the holder because it can gain risk-free USD interest.

The argument as to why the price is correct was eloquently explained by Keynes (1923) “If by lending dollars in New York for one month the lender could earn interest at the rate of $5\frac{1}{2}\%$ per annum, whereas by lending sterling in London for one month he could only earn interest at the rate of 4%, then the preference observed above for holding funds in New York rather than in London is wholly explained. That is to say, forward quotations for the purchase of the currency of the dearer money market tend to be cheaper than spot quotations by a percentage per month equal to the excess of the interest which can be earned in a month in the dearer market over what can be earned in the cheaper.”

The foreign exchange is one of the largest trading markets, along with the range of associated derivates: interest rate swaps, and currency options. We compare whether this *theoretical model* agrees with the market prices in table (4.3). In this example, we are comparing forward rates versus the pound for three currencies: the US dollar (USD), the Norwegian Kroner (NOK) and the Euro (EUR). At the time the data for the rates were taken the interest rates were: 0.4% for the GBP, the US rate was 0.1%, the Norwegian rate was higher than the UK rate at 1.5% while the Euro was trading at a *negative* interest rate. So, holding Euros incurs a charge by the bank rather than being offered interest payments.

We have used the formula (4.33) to provide the theoretical estimates. For example, the 3-month forward rate for the pound against the dollar would be the number of dollars the pound can buy. So the (spot price) of the dollar according to the market is, (see figure 4.3):

$$S_0 = \frac{1}{1.5663} = £0.6384 \tag{4.34}$$

So the forward rate at 3-months (0.25 years) would be:

$$F = S_0 e^{(r_{\mathsf{GBP}} - r_{\mathsf{USD}})T} \quad , \tag{4.35}$$

that is:

$$F = \frac{1}{1.5663} \times \exp\left[(0.004 - 0.001)0.25\right] \approx £0.63893 \tag{4.36}$$

That is, a fair price for a 3-month forward is: £1 for $1/0.63893 \approx \$1.5651$. The result is quite close to the market price, figure (4.3), 1.5656, but not as accurate as the Euro rate. Getting the calculation right is very important since foreign exchange is one of the largest traded assets market, valued at around 4 trillion dollars per day in 2012: $\$\ 4 \times 10^{12}$! The futures market is smaller, only about 40 billion dollars per day. For comparison, the Gross Domestic Product of the UK in 2013 was 2.7 trillion dollars.

In fact, the interest rates we have quoted can vary quite dramatically. The interest rates we have quoted are not the bond rates, or Bank of England interest rates which are fairly stable. Rather these rate are the *money market rates*: the cost at which banks will lend to customers. Even more confusingly there are interbank rates, such as the LIBOR (London Interbank Offered Rate) which applies to money loaned between the big financial institutions.

Country currency & Market interest rate	spot rate market	1 month forward market (theory)	3 month forward market (theory)
USD (0.10%)	1.5663	1.5661 (1.5659)	1.5656 (1.5651)
NOK (1.5%)	8.6379	8.6461 (8.6458)	8.6653 (8.6617)
EUR (-0.02%)	1.1707	1.1703 (1.1703)	1.1694 (1.1695)

Figure 4.3: Forward rates for the pound against three currencies: the US dollar (USD) the Norwegian Kroner (NOK) and the Euro (EUR). The market prices were taken on 11/02/13. The theoretical prices are given by the formula (4.33) and given in brackets beside the market rate. At this time the UK interest rate was 0.4%, that is a continuous rate given by $r \approx 0.004$ per year. Since the USD interest rate is lower than the GBP interest rate, the forward rate decreases as the delivery date is later. In contrast, since the NOK rate is higher than the GBP rate, the forward rate increases.

Again, if the forward rate does not agree with the formula above, an arbitrage opportunity exists. For example, if the Euro interest rate were 2% and the Sterling interest rate 1%, and the current spot price for Euros was:

$$S_0 = £0.8300 \quad . \tag{4.37}$$

Then the correct (fair) price for a *forward contract* to buy/sell Euro in 6 months' time is:

$$X_{\mathsf{f}} = S_0 e^{(r_{\mathsf{GBP}} - r_{\mathsf{EUR}})T} = £0.8300 \times e^{-0.01 \times 0.5} \approx £0.8259 \quad . \tag{4.38}$$

Suppose the market price for the forward was somehow different, say:

$$X_{\mathsf{m}} = £0.8400 \tag{4.39}$$

for the 6-month forward for Euro.

Suppose you were a trader based in London. Then arbitrage would proceed as usual - buy low and sell high. Clearly $X_{\mathsf{m}} > X_{\mathsf{f}}$ so one should 'sell' at the forward exchange rate X_{m} and 'buy' at the fair-price exchange rate X_{f}. So we should be able to make a profit:

$$X_{\mathsf{m}} - X_{\mathsf{f}} = £0.0141 \tag{4.40}$$

per unit of asset.

This can be accomplished as follows. Well 'selling' the market forward means taking the *short* position in the forward, let's say for 100,000 units of asset. That is one undertakes the obligation to sell to Alice, 100,000 Euro in six months time at the price $X_{\mathsf{m}} = £0.8400$ per Euro. This may appear risky, but the risk can be eliminated. One needs to have 100,000 Euros in hand in 6 months time to deliver to Alice. So one borrows (in Sterling) in order to buy the correct amount of Euros *now*. One buys: $100,000 \times e^{-r_{EUR}T} = 99,004.98$ Euros now, and this will cost:

$$£0.8300 \times 100,000 \times e^{-0.01} = £82174.136 \qquad ,$$

at today's exchange rate. So we borrow this amount from our UK bank, buy the Euros, move these into our Euro account (which pays the Euro interest rate). So in 6 months time these 99,004.98 Euros will gain interest to be worth 100,000 Euros which we hand over to Alice.

In return, Alice hands over the agreed forward strike price, and gives us: $£84,000$. We use this money to repay the loan for the £82,174.14, and with (UK) interest rate this loan will now cost us:

$$82,174.14 \times e^{0.005} = 82,586.04 \quad ,$$

and we have made a net profit:

$$84,000.00 - 82,586.04 \approx 1,414 \quad .$$

This is the expected profit, given the difference in prices (equation (4.40)), and at no risk.

This forward price between foreign exchange rates is called interest-rate parity.

4.8 Futures

Futures are similar to forward contracts. These are legal obligation to buy/sell at a future date at a future price. However, while forward contracts are direct *over-the-counter* (OTC) arrangements between two parties, futures are traded on an exchange. That is traders can buy and sell futures without having a direct involvement.

One of the most famous exchanges in the Chicago Mercantile Exchange (CME) which opened a Belfast office in 2012 and is the world's largest futures exchange. The contracts that CME trades include futures and options based on agricultural products like wheat, corn and pork bellies, but also includes interest rates, foreign exchange, and even energy and weather.

Traders on the exchange buy and sell each other options and futures contracts. To trade on this market, traders use funds from so-called *margin accounts*. These are a sort of bank account with the exchange and under the oversight of the exchange.

Any forward contract has the danger of default. Although the obligation is legally binding, the other party may fail to uphold the agreement (default on the obligation). The purpose of the exchange is not only to facilitate the trade of these contracts, but also to update their value regularly and transfer money between these margin accounts as time progresses and the values of the futures change. At the start of the contract, the exchange requires traders to deposit money (the *margin*). As the asset changes over time, the (fair) value of the futures changes. The difference between the strike price agreed initially and the (fair) value of the contract, were it to be set up now, is calculated and the price difference is settled for day to day by the exchange transferring money between the two margin accounts. This arrangement by which the values of unpredictable assets are updated, is called *marking to market*. If the price change is such that it goes beyond the funds held in the margin account, then a *margin call* [1] is made to a trader to increase the deposit in their account.

These transfers of funds/payments during the time of the contract are an accounting mechanism. They do not affect the strike price of a *futures contract*.

[1] This term provided the title of a movie released in 2011: Margin Call

4.9 Futures price

Theorem: For a constant interest rate the forward price is the same as the futures price.

Proof: Assume the *delivery date* of a forward and futures is at the same delivery date T (in n days time). Let F_i, G_i be the *fair* forward price and futures price at the end of day i. So $F_0 = X_F$ means the delivery price agreed in the forward at the beginning of day 1. We denote $G_0 = X_G$ as the corresponding price written on the future contract. Our aim is to show that $F_0 = G_0$.

We know the following for certain, at the delivery date, $t = T$: the asset price will be an unknown value $S_n = S_T$, and thus the forward will be worth $F_n = S_T$ to the party long in the forward. Similarly, a party long in a futures will have a pay-off value $G_n = S_T$. And the pay-offs would be $S_T - X_F$ and $S_T - X_G$.

We already know that, for the forward contract:

$$F_0 = S_0 e^{rT} = X_F \quad .$$

Over each of n days the daily interest rate will be:

$$\delta = \frac{rT}{n} \quad . \tag{4.41}$$

So a cash deposit will grow by a factor e^δ each day.

For a *futures* account, the *marking to market* means a cash transfer into/out of the margin account. The amount credited (debited) is the difference in future prices (determined by the market). Let us denote the futures prices at the end of day i by G_i and thus the sequence of values can be written as

$$G_0, G_1, G_2, \ldots, G_n$$

Recall that these futures contracts all have the same maturity date, the end of the nth day.

Consider two investment portfolios:

- Portfolio 1: *long* in bonds, which have a risk-free interest rate r, to the value $F_0 e^{-rT} = S_0$, and *long* in 1 unit of *forward* at delivery price F_0, delivery date T.
- Portfolio 2: *long* in risk-free bonds to the value $G_0 e^{-rT}$, where G_0 has yet to be specified. *long* in $e^{-rT} e^\delta$ units of futures at delivery price G_0, with delivery date T.

  Furthermore, in this portfolio, each day we *increase* the futures holding by a factor e^δ at the market price.

  Recall that we can adjust our holding of futures without any cost, we only need to settle at the maturity date. The futures strike price will vary on a daily basis, and the changes in the strike price are accounted for through the margin account. Thus a futures with strike price G_0 on day 0 will become a futures with a strike price G_1 on day 1, and a transfer of $G_1 - G_0$ will be made to/from the margin account to reflect this change.

For Portfolio 1: the initial investment is $F_0 e^{-rT}$ the final value of the portfolio (at $t = T$) is: S_n.

For Portfolio 2: the initial investment was $G_0 e^{-rT}$, the final value at $t = T$ consists of cash and the futures. The cash payments (positive or negative) should be discounted to the present-day value $t = 0$. The total cash (at present values) then can be written as

$$e^{-n\delta}(G_1 - G_0) + e^{-n\delta}(G_2 - G_1) + \cdots + e^{-n\delta}(G_n - G_{n-1}) = e^{-n\delta}(G_n - G_0) \tag{4.42}$$

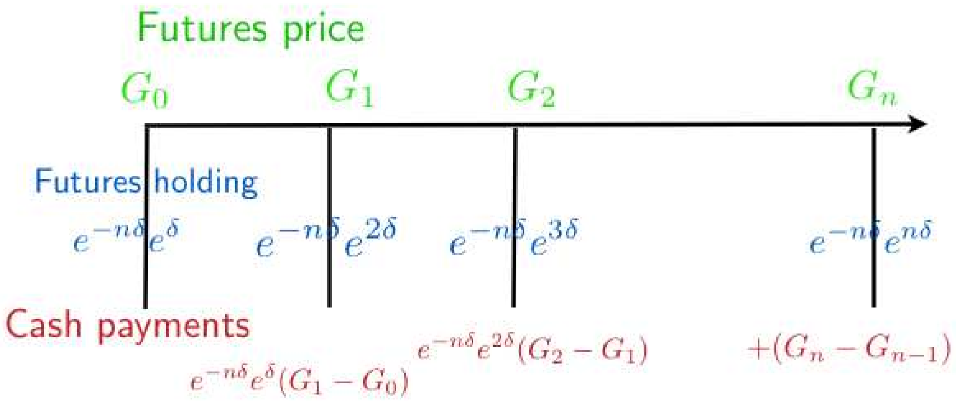

Figure 4.4: Portfolio 2 investment strategy in which futures holdings are increased by a factor e^{δ} each day. The changing value of the futures contract leads to payments into a cash account. These payments can be positive or negative depending how the futures contract prices varies over the time.

The total value of this cash at maturity $t = T$ is then simply $(G_n - G_0) = S_n - G_0$.

Hence the total value of Portfolio 2 at maturity is this cash added to the value of the bond $e^{-rT}G_0e^{rT} = G_0$. Thus the total value is: $S_n - G_0 + G_0 = S_n$. This is the same as portfolio 1. Since both have the same values at maturity, the initial costs should be the same:

$$F_0e^{-rT} = G_0e^{-rT} \qquad . \tag{4.43}$$

Then clearly we have the result we aimed to achieve:

$$F_0 = G_0 \qquad . \tag{4.44}$$