Chapter 15

Greeks

We have seen that the option price is a function of the 5 variables X, T, r, σ and S. The Greeks are the derivatives of the option, with respect to these parameters. Since a derivative measures the rate of change of the variable with respect to the parameter, the Greeks measure the sensitivity (or stress) of the option as the parameters change. This is of great interest to option traders since price difference is often as of much interest as the price itself. Recall a trader is often motivated not by the true value of an option but on what the market perception is of the value.

To start off the chapter, we have to revisit some basic mathematical theorems.

15.1 Mean-value theorem

Let us recall the mean-value theorem, in its integral form. For any function $f(x)$, continuous in the interval $a \leq x \leq b$:

$$\boxed{\int_a^b f(x)dx = (b-a)f(\xi)} \quad , \tag{15.1}$$

where $a < \xi < b$.

15.2 Fundamental theorem of calculus

It follows that, for any continuous function f we have

$$\boxed{\frac{d}{dx}\int_a^x f(u)du = f(x)} \quad . \tag{15.2}$$

Proof:

By definition:

$$\frac{d}{dx}\int_a^x f(u)du = \lim_{h\to 0}\frac{1}{h}\left(\int_a^{x+h} f(u)du - \int_a^x f(u)du\right) = \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(u)du \quad . \tag{15.3}$$

Then according to the mean-value theorem (15.1):

$$\frac{d}{dx}\int_a^x f(u)du = \lim_{h\to 0}\frac{1}{h}\times hf(\xi) \quad , \quad x \leq \xi \leq x+h \quad . \tag{15.4}$$

Then (15.2) follows immediately in this limit as $h \to 0$.

A generalisation of the theorem will be used in this chapter. If $g(x,u)$ is a *continuous* function of the variable x on the interval $a \leq x \leq b$:

$$\boxed{\frac{d}{dx}\int_{a(x)}^{b(x)} g(x,u)du = \int_{a(x)}^{b(x)} \frac{\partial g}{\partial x}(x,u)du + \frac{db}{dx}g(x,b) - \frac{da}{dx}g(x,a)} \quad . \tag{15.5}$$

Proof: By definition:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} g(x,u)du = \lim_{h\to 0}\left(\frac{1}{h}\right)\left[\int_{a(x+h)}^{b(x+h)} g(x+h,u)du - \int_{a(x)}^{b(x)} g(x,u)du\right] \quad . \tag{15.6}$$

The expression on the right can be expanded as:

$$\lim_{h\to 0}\left(\frac{1}{h}\right)\left[\int_{b(x)}^{b(x+h)} g(x+h,u)du + \int_{a(x+h)}^{a(x)} g(x+h,u)du + \int_{a(x)}^{b(x)} \{g(x+h,u) - g(x,u)\}\, du\right] \quad . \tag{15.7}$$

Then, using (15.1) again, the first term is simply:

$$\lim_{h\to 0}\left(\frac{1}{h}\right)\int_{b(x)}^{b(x+h)} g(x+h,u)du = \lim_{h\to 0}\frac{b(x+h)-b(x)}{h}g(x+h,\xi) \quad , \quad b(x) \leq \xi \leq b(x+h) \quad . \tag{15.8}$$

That is, the first term is

$$b'(x)g(x,b(x)) \quad , \tag{15.9}$$

similar the second term is:

$$-a'(x)g(x,a(x)) \quad , \tag{15.10}$$

and the third term is clearly

$$\lim_{h\to 0}\int_{a(x)}^{b(x)}\left\{\frac{g(x+h,u)-g(x,u)}{h}\right\}du = \int_{a(x)}^{b(x)}\frac{\partial g(x,u)}{\partial x}du \quad . \tag{15.11}$$

15.3 Calculating the Greeks

15.3.1 DELTA: Δ

We have previously already encountered the Δ of the European call option. It is given by

$$\boxed{\Delta_c = \frac{\partial c}{\partial S}} \quad . \tag{15.12}$$

Of course, this derivative can be calculated from the solution of the Black-Scholes equation (12.84), but the parameters d_1 and d_2 depend on S, and this makes the direct evaluation cumbersome. It is is much easier to evaluate Δ using the integral definition (risk-neutral) definition of the call.

Now in (15.5), if $g(x,a) = g(x,b) = 0$, the integrand vanishes at the limits, then we have the special case:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} g(x,u)du = \int_{a(x)}^{b(x)} \frac{\partial g}{\partial x}(x,u)du \quad . \tag{15.13}$$

The price/value of the call option is given in terms of an integral by:

$$c = \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-d_2}^{+\infty}[Se^{(r-\frac{1}{2}\sigma^2)T+z\sigma\sqrt{T}} - X]e^{-\frac{1}{2}z^2}dz \quad , \tag{15.14}$$

so that the Δ_c is given by

$$\Delta_c = \frac{\partial}{\partial S}\left\{\frac{e^{-rT}}{\sqrt{2\pi}}\int_{-d_2}^{+\infty}[Se^{(r-\frac{1}{2}\sigma^2)T+z\sigma\sqrt{T}} - X]e^{-\frac{1}{2}z^2}dz\right\} \quad . \tag{15.15}$$

By definition $-d_2$ is the value at which the integrand is zero. Furthermore as $z \to +\infty$, the integrand is zero at the other limit. Thus we can use (15.13) and find:

$$\Delta_c = \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-d_2}^{+\infty}\frac{\partial}{\partial S}\{Se^{(r-\frac{1}{2}\sigma^2)T+z\sigma\sqrt{T}} - X\}e^{-\frac{1}{2}z^2}dz \quad . \tag{15.16}$$

Application of the partial derivative gives us a single integral to evaluate

$$\Delta_c = \frac{e^{-rT}e^{rT}}{\sqrt{2\pi}}\int_{-d_2}^{+\infty}e^{-\frac{1}{2}\sigma^2T+\sigma\sqrt{T}z-\frac{1}{2}z^2}dz \quad . \tag{15.17}$$

For this integral, we define $v = z - \sigma\sqrt{T}$ to simplify, and this gives us the final result

$$\Delta_c = \frac{1}{\sqrt{2\pi}}\int_{-d_2-\sigma\sqrt{T}}^{+\infty}e^{-\frac{1}{2}v^2}dv = 1 - N(-d_1) = N(d_1) \quad . \tag{15.18}$$

as found previously. (We could also note that, in comparison with the evaluation of the Black-Scholes equation, the partial derivative with respect to S removes the term proportional to X entirely, and removes the S from the integral term proportional to S, giving the same result.)

15.3.2 THETA: θ

The θ of an option is the rate of change of a derivative with respect to time.

So for the call option:

$$\boxed{\theta_c = \frac{\partial c}{\partial t} = -\frac{\partial c}{\partial T}} \quad . \tag{15.19}$$

This can be calculated as shown below. Starting with the integral (15.14), we have:

$$\theta_c = -\frac{\partial}{\partial T}\left\{\frac{1}{\sqrt{2\pi}}\int_{-d_2}^{+\infty}[Se^{-\frac{1}{2}\sigma^2T+z\sigma\sqrt{T}} - Xe^{-rT}]e^{-\frac{1}{2}z^2}dz\right\} \quad . \tag{15.20}$$

This gives

$$\theta_c = \frac{1}{\sqrt{2\pi}}\int_{-d_2}^{+\infty}\left[\left(\frac{1}{2}\sigma^2 - \frac{z\sigma}{2\sqrt{T}}\right)Se^{-\frac{1}{2}(z-\sigma\sqrt{T})^2} - rXe^{-rT}e^{-\frac{1}{2}z^2}\right]dz \quad . \tag{15.21}$$

$$\theta_c = \frac{\sigma S}{2\sqrt{T}\sqrt{2\pi}}\int_{-d_2}^{+\infty}(\sigma\sqrt{T} - z)e^{-\frac{1}{2}(z-\sigma\sqrt{T})^2}dz - \frac{rXe^{-rT}}{\sqrt{2\pi}}\int_{-d_2}^{+\infty}e^{-\frac{1}{2}z^2}dz \quad . \tag{15.22}$$

The first integral can be done by direct integration, and the second integral can be expressed in terms of the normal distribution:

$$\theta_c = -\frac{\sigma S}{2\sqrt{T}\sqrt{2\pi}}e^{-\frac{1}{2}(-d_2-\sigma\sqrt{T})^2} - rXe^{-rT}N(d_2) \quad . \tag{15.23}$$

$$\theta_c = -\frac{\sigma S}{2\sqrt{T}\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2} - rXe^{-rT}N(d_2) \quad . \tag{15.24}$$

Alternatively one can directly differentiate the expression for the call, although this is a bit more intricate:

$$\theta_c = -\frac{\partial}{\partial T}\{SN(d_1) - Xe^{-rT}N(d_2)\} \quad . \tag{15.25}$$

$$\theta_c = -S\frac{\partial d_1}{\partial T}N'(d_1) - rXe^{-rT}N(d_2) + Xe^{-rT}\,\frac{\partial d_2}{\partial T}N'(d_2) \quad . \tag{15.26}$$

with

$$\frac{\partial d_1}{\partial T} = \frac{-\ln(S/X) + (r+\frac{1}{2}\sigma^2)T}{2\sigma T^{3/2}} \quad , \tag{15.27}$$

and

$$\frac{\partial d_2}{\partial T} = \frac{-\ln(S/X) + (r-\frac{1}{2}\sigma^2)T}{2\sigma T^{3/2}} \quad . \tag{15.28}$$

Furthermore,

$$N'(d_1) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2} \quad , \tag{15.29}$$

and

$$N'(d_2) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_2^2} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(d_1-\sigma\sqrt{T})^2} = \frac{S}{X}e^{rT} \times \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2} \quad . \tag{15.30}$$

Then after a fair bit of algebra we get,

$$\boxed{\theta_c = -rXe^{-rT}N(d_2) - \frac{S\sigma e^{-\frac{1}{2}d_1^2}}{2\sqrt{2\pi T}}} \quad , \tag{15.31}$$

which agrees with (15.24) as it should.

Both terms in this expression are negative, which implies that

$$\boxed{\theta_c \leq 0} \quad . \tag{15.32}$$

This means that as the maturity date (expiry date) approaches the call option always loses value (all other variables being fixed). This property can be seen from figure 15.1 in which we see the value of the call decreasing as the maturity/expiry date approaches.

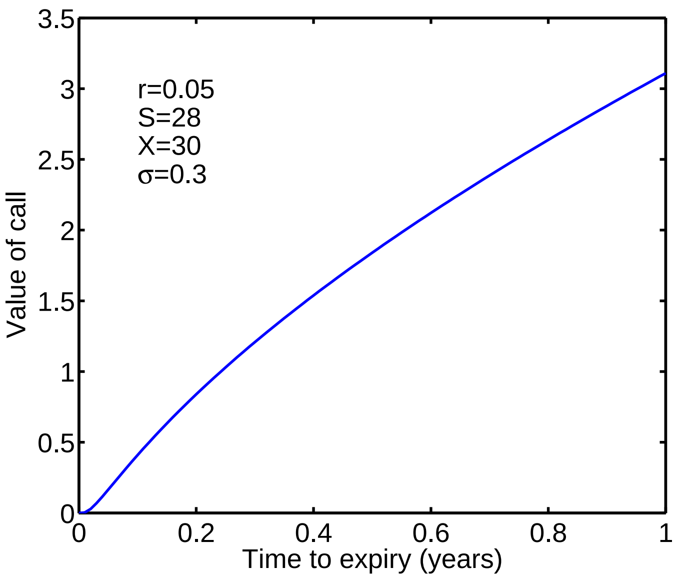

Figure 15.1: Dependence of the value of a call option, according to the Black-Scholes equation, on time to expiry.

This can be explained by noting that the further away the expiry date, the more time the asset has to increase (or decrease) in value. For the holder of the call, larger price increases mean increases in the possible pay-offs at expiry. On the other hand, larger price decreases will not affect the value of the call option. Price decreases below the option strike price will still result in zero pay-off, no matter how large the fall in value is. There is no loss for the holder since the option need not be exercised. Therefore, on the whole the option will be more valuable the further away the date of expiry is, due to the possible price increases in the asset value.

15.3.3 GAMMA (Γ)

The Γ of an option is the second derivative of the function with respect to the spot price S:

$$\boxed{\Gamma_{\text{call}} = \frac{\partial^2 c}{\partial S^2} = \frac{\partial \Delta}{\partial S}} \qquad . \tag{15.33}$$

$$\Gamma_{\text{call}} = \frac{\partial N(d_1)}{\partial S} = \frac{\partial d_1}{\partial S} N'(d_1) \qquad . \tag{15.34}$$

$$\Gamma_{\text{call}} = \frac{\partial}{\partial S} \left\{ \frac{\ln(S/X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right\} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \qquad . \tag{15.35}$$

$$\Gamma_{\text{call}} = \frac{1}{S} \frac{e^{-\frac{1}{2}d_1^2}}{\sigma\sqrt{T}\sqrt{2\pi}} \geq 0 \qquad . \tag{15.36}$$

Geometrically Γ is proportional to the curvature of c. Since $\frac{\partial^2 c}{\partial S^2} > 0$ (always) - this means the call is always concave upwards (convex). This is illustrated in figure 15.2.

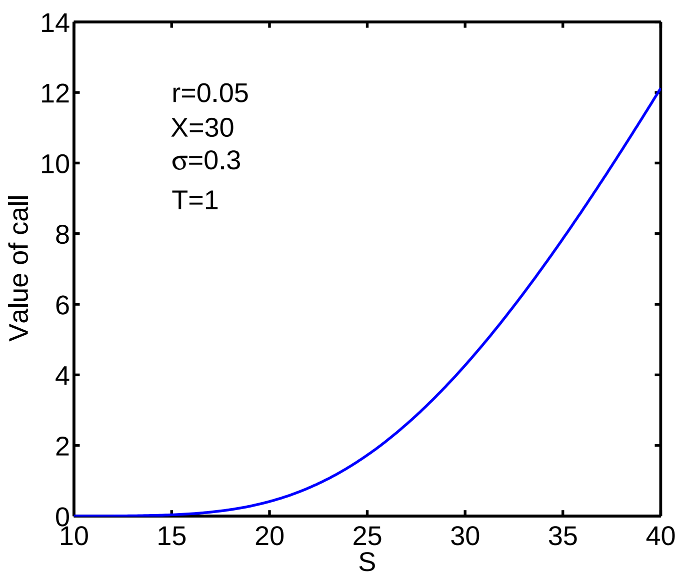

Figure 15.2: Dependence of the value of a call option, according to the Black-Scholes equation, on current asset price.

15.3.4 Sensitivity with respect to X: the dual Delta

Oddly enough there does not appear to be a Greek letter associated with the change in strike price. This parameter is sometimes called the *dual Delta* because of its close relation to the Δ. This follows since:

$$\frac{\partial c}{\partial X} = \frac{\partial}{\partial X}\left[\frac{e^{-rT}}{\sqrt{2\pi}}\int_{-d_2}^{+\infty} e^{-\frac{1}{2}z^2}\left(S\,e^{(r-\frac{1}{2}\sigma^2)T+z\sigma\sqrt{T}}-X\right)dz\right] \quad . \tag{15.37}$$

It follows that,

$$\boxed{\frac{\partial c}{\partial X} = -e^{-rT}N(d_2) \leq 0} \quad . \tag{15.38}$$

Thus, as X increases, c decreases and this is shown in figure 15.3. In financial terms, for a higher strike price (all other factors remaining constant), the pay-off range for the call option decreases, and the pay-offs will be less.

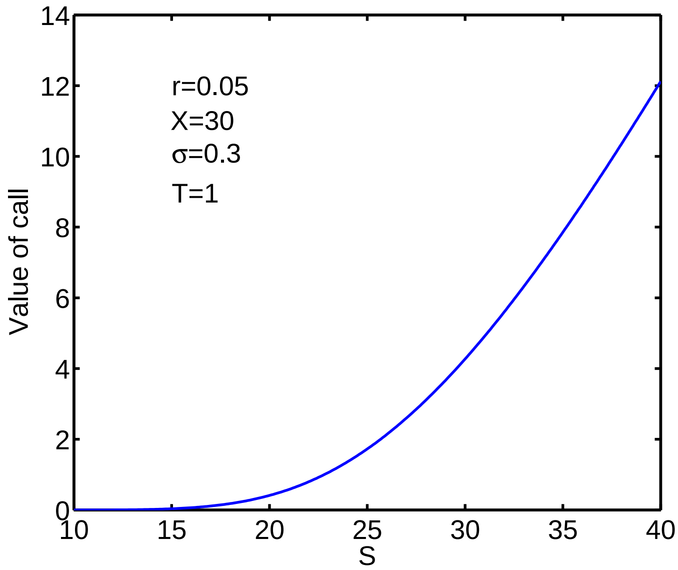

Figure 15.3: Dependence of the value of a call option, according to the Black-Scholes equation, on strike price.

15.3.5 VEGA:

The variation in the option price, as σ changes is called the "Vega" ($\mathcal{V}$. (Note: there is no such letter as *vega* in the Greek alphabet. In fact, the etymology of the name is Arabic not Greek! It is the name of a star: one of the brightest stars in our sky, around 25 light years from our solar system and the brightest star in the constellation Lyra.)

The *vega* of a call option is defined as:

$$\boxed{\mathcal{V}_{\text{call}} = \frac{\partial c}{\partial \sigma}} \quad . \tag{15.39}$$

Now using financial reasoning, we would anticipate that as σ increases the asset has more likelihood of reaching high and low values. The low values ($S_T < X$) contribute nothing to the pay-off of a call option, but the high values ($S_T > X$) add significant value to the eventual pay-off. Thus, one would expect

$$\mathcal{V}_{\text{call}} = \frac{\partial c}{\partial \sigma} \geq 0 \quad . \tag{15.40}$$

We can obtain a formula in the normal way by differentiating the integrand:

$$\mathcal{V}_{\text{call}} = \frac{S_0 e^{-rT}}{\sqrt{2\pi}} \int_{-d_2}^{+\infty} e^{(r-\frac{1}{2}\sigma^2)T+z\sigma\sqrt{T}-\frac{1}{2}z^2} \left(-\sigma T + \sqrt{T}z\right) dz \quad . \tag{15.41}$$

$$\mathcal{V}_{\text{call}} = \frac{S_0\sqrt{T}}{\sqrt{2\pi}} \int_{-d_2}^{+\infty} e^{-\frac{1}{2}(z-\sigma\sqrt{T})^2} \left(z - \sigma\sqrt{T}\right) dz \quad . \tag{15.42}$$

Changing variable, let $z - \sigma\sqrt{T} = v$ we get:

$$\mathcal{V}_{\text{call}} = \frac{S_0\sqrt{T}}{\sqrt{2\pi}} \int_{-d_2-\sigma\sqrt{T}}^{+\infty} e^{-\frac{1}{2}v^2} v \; dv \quad . \tag{15.43}$$

Now we note that the lower limit $-d_2 - \sigma\sqrt{T} = -d_1$, and that the integral can be done directly to give:

$$\boxed{\mathcal{V}_{\text{call}} = \frac{S_0\sqrt{T}}{\sqrt{2\pi}} \; e^{-\frac{1}{2}d_1^2}} \quad . \tag{15.44}$$

This expression is always non-negative:

$$\boxed{\mathcal{V}_{\text{call}} \geq 0} \quad . \tag{15.45}$$

The *vega* of a put is the same as that of the call. This can be seen from the put-call parity, and therefore:

$$\mathcal{V}_{\text{put}} = \mathcal{V}_{\text{call}} \quad . \tag{15.46}$$

15.3.6 RHO

The sensitivity with respect to the interest rate is defined as:

$$\boxed{\rho_{\text{call}} = \frac{\partial c}{\partial r}} \quad . \tag{15.47}$$

$$\rho_{\text{call}} = \frac{\partial}{\partial r}\left\{\frac{e^{-rT}}{\sqrt{2\pi}} \int_{-d_2}^{+\infty} \left(S_0 \; e^{(r-\frac{1}{2}\sigma^2)T+z\sigma\sqrt{T}} - X\right) \; e^{-\frac{1}{2}z^2} dz\right\} \quad . \tag{15.48}$$

The simplest way to approach this integral is by moving the pre-factor e^{-rT} into the integral and then differentiating. The term proportional to S_0 disappears, and the term involving X gets an additional factor $-T$. Hence the integral reduces to:

$$\boxed{\rho_{\text{call}} = XTe^{-rT}N(d_2)} \quad . \tag{15.49}$$

This expression is always ≥ 0. When interest rates increase, the price/value of call options increase.

This makes sense in that the price of the call option can be written as

$$c = \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{+\infty} \left(S_0 \; e^{-\frac{1}{2}\sigma^2 T+z\sigma\sqrt{T}} - Xe^{-rT}\right) \; e^{-\frac{1}{2}z^2} dz \quad . \tag{15.50}$$

Thus, an increase in r will have the same effect as a decrease in X. We have already seen that a decrease in X leads to an increase in the call price, and so we need to find the result we did for consistency:

$$\rho_{\text{call}} = \frac{\partial c}{\partial r} \geq 0 \quad . \tag{15.51}$$

15.4 Black-Scholes computer codes

It is easy to find programs to calculate the Black-Scholes formula, or even write your own.

Here are two programs that you can use: the first uses octave/latlab while the second is written in C.

```
%
%    filename bs_calculator

%    Black-Scholes formula calculator
%    coded in Octave (MATLAB compatible)
%
%    Calculates the prices for European call and put options
%    along with some of the Greeks for a given set of inputs:
%
%   INPUT
%   X - strike price
%   S - spot price
%   r - interest rate
%   sigma - volatility
%   T - expiry
%
%   OUTPUT
%   C - value of the call option
%   DeltaC - value of the Delta for the call
%   VegaC - value of the Vega for the call
%   RhoC - value of the Rho for the call
%   GammaC - value of the Gamma for the call
%   ThetaC - value of the Theta for the call


clear

r=0.05 ;        %  interest rate per annum
X=60   ;        %  strike price
S=50   ;        %  asset spot price
sigma=0.20 ;    % volatility  >=0

T=1.0  ;        % time to expiry in years >=0

if T < 0 || sigma < 0
disp(" Warning sigma and/or T <0")
endif
if S < 0 || X < 0
disp(" Warning S and/or  X <0")
endif

d1=(log(S/X)+(r+0.5*sigma^2)*T)/(sigma*sqrt(T));
d2=(log(S/X)+(r-0.5*sigma^2)*T)/(sigma*sqrt(T));
Nd1=0.5*(1.+erf(d1/sqrt(2))) ; % uses the Octave error function: erf
Nd2=0.5*(1.+erf(d2/sqrt(2))); %
```

```
C=S*Nd1-X*exp(-r*T)*Nd2  % call price
DeltaC= Nd1                  % delta call
RhoC = X*T*exp(-r*T)*Nd2
VegaC=S*sqrt(T)/ (sqrt(2*pi))*exp(-0.5*d1*d1) % vega for call
GammaC = 1./ (S*sigma*sqrt(T)*sqrt(2*pi))*exp(-0.5*d1*d1)
ThetaC= -sigma*S*exp(-0.5*d1*d1)/(2*sqrt(T)*sqrt(2*pi))-r*X*exp(-r*T)*Nd2
%    Put options prices and Greeks using parity relation
P=C-S+X*exp(-r*T)            % put price
DeltaP=DeltaC-1              % delta put
RhoP = RhoC-T*X*exp(-r*T) % rho put
VegaP=VegaC % vega put
GammaP =GammaC % Gamma put
ThetaP= ThetaC+r*X*exp(-r*T) % Theta put
```