COMP6212 Computational Finance

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Part I: Portfolio Optimization

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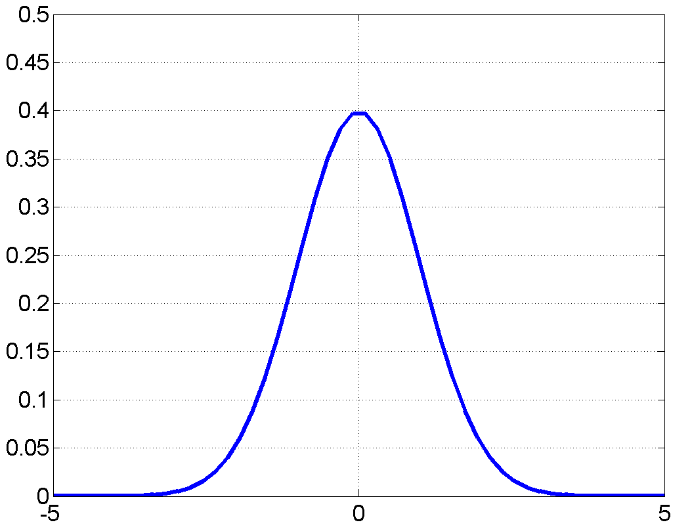
Part I: Portfolio Optimization

- $r_i(t)$ Return on asset i at time t; *i.e.* invest at time $t-1$, what have you earned at time t?
- We think of this as a random variable, say Gaussian distributed with mean μ_i and variance σ_i^2
- The mean is what we expect (on average) to gain by investing
- We think of variance in return as *risk*
- When we look at more than one asset, we can think of how returns on them are correlated: σ_{ij}^2
- A portfolio (investment in N assets) with relative weights π_i
- Return on the portfolio: $r_p = \sum_{i=1}^{N} \pi_i r_i = \boldsymbol{\pi}^t \boldsymbol{r}$
- Returns on the portfolio has a multivariate Gaussian distribution

$$\boldsymbol{r} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{pmatrix} \quad \boldsymbol{\mu} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{pmatrix} \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12}^2 & \cdots & \sigma_{1N}^2 \\ \sigma_{12}^2 & \sigma_2^2 & \cdots & \sigma_{2N}^2 \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{1N}^2 & \sigma_{2N}^2 & \cdots & \sigma_N^2 \end{pmatrix}$$

Gaussian Density

$$p(x) \;=\; \frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{(x-m)^2}{2\sigma^2}\right\}$$

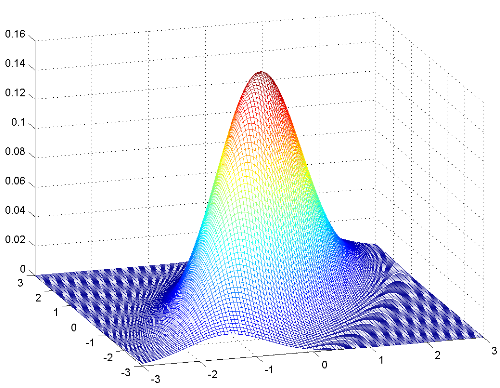


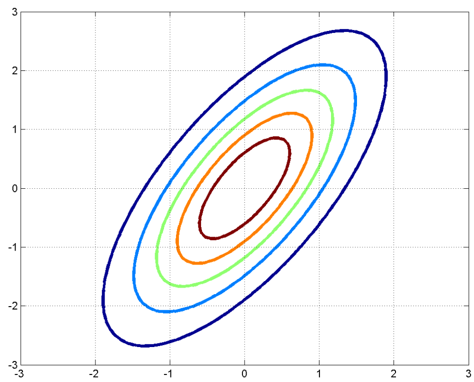

```
x = linspace(-5,5,50);
m = 0;
s = 1;
y = normpdf(x,m,s);
figure(1), clf
plot(x,y,'LineWidth',3);
grid on
```


Multivariate Gaussian Distribution

$$p\left(\boldsymbol{x}\right) \;=\; \frac{1}{(2\pi)^{d/2}\,|\boldsymbol{\Sigma}|^{1/2}}\;\exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\}$$

- Parameters: mean vector $\boldsymbol{\mu}$ covariance matrix $\boldsymbol{\Sigma}$





- How do these shapes change with $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$?

Portfolio Return

- Linear transform of multivariate Gaussian

$$\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{\mu}, \ \Sigma); \ \ \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} \ \ \Longrightarrow \ \ \boldsymbol{y} \sim \mathcal{N}\left(\boldsymbol{A}\boldsymbol{\mu}, \ \boldsymbol{A}\Sigma\boldsymbol{A}^t\right)$$

- Return on our portfolio is a linear transform of the vector of returns

$$r_P = \boldsymbol{\pi}^T \boldsymbol{r}$$

- We can immediately write down the distribution of the return on the portfolio

$$r_P \sim \mathcal{N}\left(\boldsymbol{\pi}^T \boldsymbol{\mu}, \ \boldsymbol{\pi}^T \boldsymbol{\Sigma} \boldsymbol{\pi}\right)$$

- Mean return $M = \boldsymbol{\pi}^T \boldsymbol{\mu}$ and the variance on it $V = \boldsymbol{\pi}^T \boldsymbol{\Sigma} \boldsymbol{\pi}$
- When π changes, M and V change — how?

Efficient Portfolio

- As we change π (*i.e.* invest in different proportions), M and V change
- Not all M and V are realizable.
- We can formulate constrained optimization problems
- For a given risk we tolerate, what is the highest return we can expect

$$\max_{\boldsymbol{\pi}} \ \boldsymbol{\pi}^T \boldsymbol{\mu} \quad \text{subject to } \boldsymbol{\pi}^T \boldsymbol{\Sigma} \boldsymbol{\pi} = \sigma_0$$

- If we hope for (expect) a given return, at what minimum risk can we achieve it?

$$\min_{\boldsymbol{\pi}} \ \boldsymbol{\pi}^T \boldsymbol{\Sigma} \boldsymbol{\pi} \quad \text{subject to } \ \boldsymbol{\pi}^T \boldsymbol{\mu} = r_0$$

- Other constraints possible: $\sum_{i=1}^{N} \pi_i = 1, \ \pi_i \geq 0, \ \alpha \leq \pi_i \leq \beta$

Estimation

- We estimate μ and Σ from historic data and apply optimization to allocate assets.
- We hope the past might be a good reflection of future!
- Estimation:

$$
\begin{aligned}
\widehat{\mu} &= \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{r}(t) \\
\widehat{\Sigma} &= \frac{1}{T}\sum_{t=1}^{T} \left(\boldsymbol{r}(t) - \widehat{\mu}\right)\left(\boldsymbol{r}(t) - \widehat{\mu}\right)^{T}
\end{aligned}
$$

Solving quadratic program in MATLAB

$$
\min_{\boldsymbol{x}} \; \boldsymbol{x}^{T}\boldsymbol{H}\boldsymbol{x} + \boldsymbol{f}^{T}\boldsymbol{x} \text{ such that } \begin{cases} \boldsymbol{A}\boldsymbol{x} \leq 0 \\ \boldsymbol{A}_{eq}\boldsymbol{x} = b_{eq} \\ lb \leq \boldsymbol{x} \leq ub. \end{cases}
$$

```
x = quadprog(H ,f, A, b, Aeq, beq ,lb, ub, x0)
```

Do `>doc quadprog` in MATLAB and read more.

For the portfolio optimization problem, we might have:

```
pi = quadprog(Sigma, [], [], [], mu', rMax, 0, 1, [])
```

Map the problem variables to the function in the tool.

Efficient Frontier

- Given $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$
- What portfolio has the highest return, unconstrained by risk?

$$\max \boldsymbol{\pi}^T \boldsymbol{\mu} \quad \text{subject to} \ \sum_{i=1}^{N} \pi_i = 1, \ \text{and} \quad \pi_i \geq 0$$

Linear Programming:

$$\min_{\boldsymbol{x}} \boldsymbol{f}^T \boldsymbol{x} \ \text{such that} \ \begin{cases} \boldsymbol{A}\,\boldsymbol{x} \leq \boldsymbol{b} \\ \boldsymbol{A}_{eq}\,\boldsymbol{x} = \boldsymbol{b}_{eq} \\ lb \leq \boldsymbol{x} \leq ub \end{cases}$$

```
x = linprog( f, A, b, Aeq, beq, lb, ub )
w1 = linprog(-mu, [], [], ones(1,N), 1, 0, 0);
r1 = w1 * mu;
```


Efficient Frontier (cont'd)

- What portfolio has lowest variance (unconstrained by expectation)?

$$\min \boldsymbol{\pi}^T \boldsymbol{\Sigma} \boldsymbol{\pi} \quad \text{subject to} \ \sum_{i=1}^{N} \pi_i = 1$$

```
w2=quadprog(mu,zeros(N,1),[],[],ones(1,N),1,zeros(N,1),[],[]);
r2 = w2' * mu;
```

- Portfolios on the efficient frontier will have returns in range `r1` to `r2`
- Now, select a number of points (returns we need), and find the minimum variance portfolios that will offer these returns!

```
M = linspace(r1, r2, p)
for j=1:p
    ret = M(j);
    w = quadprog(...);
    V(j) = w' * Sigma * w;
end
```

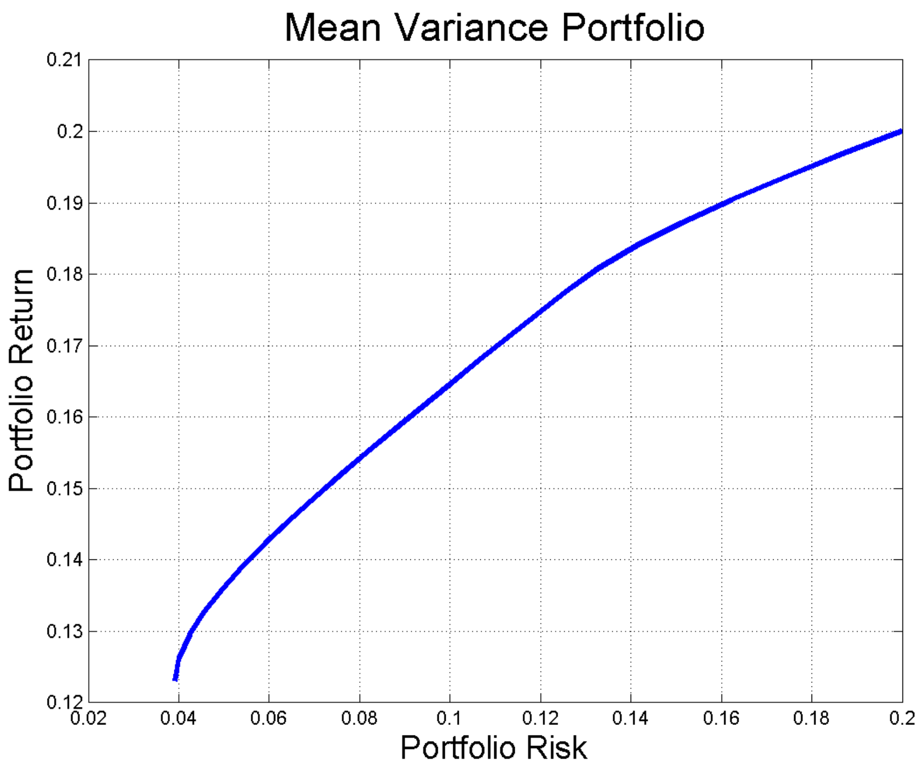
Complete Function to Compute the Efficient Frontier

```
function [PRisk, PRoR, PWts] = NaiveMV(ERet, ECov, NPts)
ERet = ERet(:);       % makes sure it is a column vector
NAssets = length(ERet);  % get number of assets
% vector of lower bounds on weights
V0 = zeros(NAssets, 1);
% row vector of ones
V1 = ones(1, NAssets);
% set medium scale option
options = optimset('LargeScale', 'off');
% Find the maximum expected return
MaxReturnWeights = linprog(-ERet, [], [], V1, 1, V0);
MaxReturn = MaxReturnWeights' * ERet;
% Find the minimum variance return
MinVarWeights = quadprog(ECov,V0,[],[],V1,1,V0,[],[],options);
MinVarReturn = MinVarWeights' * ERet;
MinVarStd = sqrt(MinVarWeights' * ECov * MinVarWeights);
% check if there is only one efficient portfolio
if MaxReturn > MinVarReturn
   RTarget = linspace(MinVarReturn, MaxReturn, NPts);
   NumFrontPoints = NPts;
else
      RTarget = MaxReturn;
      NumFrontPoints = 1;
end
```


Complete Function (cont'd)

```
...
% Store first portfolio
PRoR = zeros(NumFrontPoints, 1);
PRisk = zeros(NumFrontPoints, 1);
PWts = zeros(NumFrontPoints, NAssets);
PRoR(1) = MinVarReturn;
PRisk(1) = MinVarStd;
PWts(1,:) = MinVarWeights(:)';
% trace frontier by changing target return
VConstr = ERet';
A = [V1 ; VConstr ];
B = [1 ; 0];
for point = 2:NumFrontPoints
B(2) = RTarget(point);
Weights = quadprog(ECov,V0,[],[],A,B,V0,[],[],options);
PRoR(point) = dot(Weights, ERet);
PRisk(point) = sqrt(Weights'*ECov*Weights);
PWts(point, :) = Weights(:)';
end
```

Summary: what have we achieved?



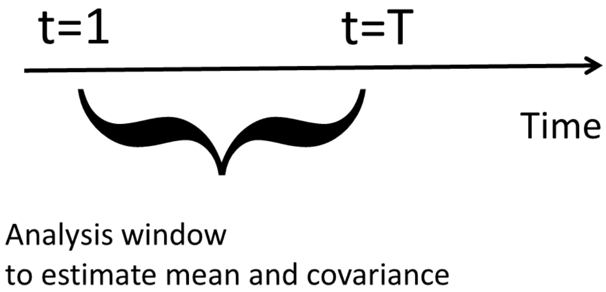
Homework

- Three assets with the following properties:

$$m = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.15 \end{bmatrix}; \quad C = 100 * \begin{bmatrix} 0.005 & -0.010 & 0.004 \\ -0.010 & 0.040 & -0.002 \\ 0.004 & -0.002 & 0.023 \end{bmatrix};$$

- Study the code of function `NaiveMV` and draw the efficient frontier.
- Use the function `frontcon` in `MATLAB` and draw the efficient frontier.

Estimation of Parameters



- Estimate parameters $\boldsymbol{\mu}$ and $\boldsymbol{C}$ from data within a window
- Optimize portfolio, invest and wait
- Periodically re-balance the portfolio (re-estimate parameters)
- Trade-off:
  - Need long window for accurate estimation
  - But relationships may not be stationary over long durations
- Shrinkage in covariance estimates

Advances on the Mean-Variance Portfolio

- Do such portfolios make money?

DeMiguel, J. *et al.* (2009) Optimal versus Naive Diversification: How Inefficient Is the 1/N Portfolio Strategy?, *The Review of Financial Studies* **22**(5): 1915.

- Including transaction costs into the optimization

Lobo, M.S. *et al.* (2007) Portfolio Optimization with Linear and Fixed Transaction Costs, *Annals of Operations Research* **152**: 341.

- Forcing the portfolio to be sparse and stable

Brodie, J. *et al.* (2009) Sparse and stable Markowitz portfolios, *PNAS* **106**(30): 12267
Takeda, A. *et al.* (2013) Simultaneous pursuit of out-of-sample performance and sparsity in index tracking portfolios, *Comput Manag Sci* **10**: 21

- Optimizing the execution of trade

TBC

Portfolio Performance

- Sharpe Ratio: mean to standard deviation of portfolio return

$$S = \frac{m - r}{\sigma}$$

  r "risk free" interest rate
- Value at Risk (VAR): Value such that probability of loss exceeding this is 0.01.

$$V \text{ such that } P[-G > V] = 0.01$$

> *"if a portfolio of stocks has a one-day 5% VaR of 1 million, there is a 0.05 probability that the portfolio will fall in value by more than 1 million over a one day period"*

```
ValueAtRisk=portvrisk(PortReturn,PortRisk,RiskThreshold,PortValue )
```

- cVAR: Conditional Value at Risk (later)

Empirical Evaluation

DeMiguel, J. *et al.* (2009),

- Comparison of a number of portfolio optimization methods
  - $\frac{1}{N}$ with re-balancing
  - Sample based mean-variance
  - Bayesian methods (of shrinking estimates)
  - Constraints
  - Combination of portfolios (model averaging / mixing)
- No method *consistently* beats the naive strategy!

Sparse Portfolios
Brodie *et al.* (2007) PNAS

- N assets; $\boldsymbol{r}_t$, return vector at time t
- Expected return and covariance:

$$\boldsymbol{r}_t = \begin{pmatrix} r_{1,t} \\ r_{2,t} \\ \vdots \\ r_{N,t} \end{pmatrix} \qquad \boldsymbol{E}[\boldsymbol{r}_t] = \boldsymbol{\mu} \qquad \boldsymbol{E}\left[(\boldsymbol{r}_t - \boldsymbol{\mu})(\boldsymbol{r}_t - \boldsymbol{\mu})^T\right] = \boldsymbol{C}$$

- Markowitz portfolio

$$\begin{cases} \min_{\boldsymbol{w}} \ \boldsymbol{w}^T \boldsymbol{C} \boldsymbol{w} \\ \text{subject to } \boldsymbol{w}^T \boldsymbol{\mu} = \rho \text{ and } \boldsymbol{1}_N^T \boldsymbol{w} = 1 \end{cases}$$

- Short selling allowed; *i.e.* w_j need not be positive
- Covariance: $\boldsymbol{C} = \boldsymbol{E}\left[\boldsymbol{r}_t \boldsymbol{r}_t^T - \boldsymbol{\mu}\boldsymbol{\mu}^t\right]$

Sparse Portfolios, Brodie *et al.* (2007) *PNAS* (cont'd)

- Mean and covariance estimated from data (expectations as sample averages):
- $\widehat{\boldsymbol{\mu}} = \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{r}_t$
- $\boldsymbol{R}$ $T \times N$ matrix with rows as $\boldsymbol{r}_t^T$
- Optimization problem rewritten as

$$\begin{cases} \widehat{\boldsymbol{w}} = \min_{\boldsymbol{w}} \frac{1}{T} ||\rho \boldsymbol{1}_T - \boldsymbol{R}\boldsymbol{w}||_2^2 \\ \text{subject to } \boldsymbol{w}^T \widehat{\boldsymbol{\mu}} = \rho, \ \boldsymbol{w}^T \boldsymbol{1}_N = 1 \end{cases}$$

- Often there is strong correlation between returns
  - Assets in the same sector respond in similar ways
- Strong correlations make R ill-conditioned $\Longrightarrow$ numerically unstable optimization
- Solution: regularization
- Brodie *et al.* suggest l_1 regularizer

$$\begin{cases} \widehat{\boldsymbol{w}} = \min_{\boldsymbol{w}} \left[|| \rho \boldsymbol{1}_T - \boldsymbol{R}\boldsymbol{w} ||_2^2 + \tau || \boldsymbol{w} ||_1 \right] \\ \text{subject to } \boldsymbol{w}^T \widehat{\boldsymbol{\mu}} = \rho, \ \boldsymbol{w}^T \boldsymbol{1}_N = 1 \end{cases}$$

Index Tracking

Brodie *et al.* (2007) (cont'd)

- Passive investor, wishing to get the same return as stock index (*e.g.* FTSE100)
- Invest in all 100 stocks of the FTSE?
- Transaction costs very high
- Can we find a small subset of the 100 stocks (say 10), that will approximate the performance of the index?
- Subset selection / cardinality constrained optimization

$$\begin{cases} \min_{\boldsymbol{w}} \; \left[\, ||\boldsymbol{y} \;-\; \boldsymbol{R}\,\boldsymbol{w}||_2^2 \,\right] \\ \text{subject to } ||\, \boldsymbol{w}\, ||_0 \;=\; w_0 \end{cases}$$

- 0^{th} norm $\rightarrow$ number of nonzero elements of $\boldsymbol{w}$ $\rightarrow$ subset of assets
- The above is of combinatorial complexity
- Suboptimal algorithm: greedy search

Brodie *et al.* (2007) (cont'd)

- A convenient proxy to achieve sparsity is *lasso* (l_1 constrained regression)

$$\min_{\boldsymbol{w}} \left[||\boldsymbol{y} \;-\; \boldsymbol{R}\,\boldsymbol{w}||_2^2 \;+\; \tau\,||\boldsymbol{w}||_1 \right]$$

- Several elements of $\boldsymbol{w}$ will be zero
- Tune τ to achieve different levels of sparsity
- Can incorporate transaction costs into the optimization

$$\min_{\boldsymbol{w}} \left[||\boldsymbol{y} \;-\; \boldsymbol{R}\,\boldsymbol{w}||_2^2 \;+\; \tau \sum_{i=1}^{N} s_i \,|\, w_i \,| \right]$$

- Transaction costs:
  - Usually have fixed (overhead) part and transaction-dependent part
  - Institutional investors fixed part negligible
  - Small investors can assume fixed cost only

Brodie *et al.* (2007) (cont'd)

Portfolio adjustment (re-balancing)

- We are holding a portfolio $\boldsymbol{w}$
- We want to make an adjustment $\boldsymbol{\Delta_w}$, new portfolio $\boldsymbol{w} + \boldsymbol{\Delta_w}$
- Transaction costs only on the adjustments

$$\begin{cases} \boldsymbol{\Delta_w} = \min_{\boldsymbol{\Delta_w}} \left[|| \rho \mathbf{1}_T - \boldsymbol{R} \left(\boldsymbol{w} + \boldsymbol{\Delta_w} \right) ||_2^2 + \tau || \boldsymbol{\Delta_w} ||_1 \right] \\ \text{subject to } \boldsymbol{\Delta_w}^T \widehat{\boldsymbol{\mu}} = 0 \text{ and } \boldsymbol{\Delta_w}^T \mathbf{1}_N = 1 \end{cases}$$

Brodie *et al.* (2007) (cont'd)

Homework:

Coursework 1 will involve confirming some claims in Brodie *et al.*'s paper. Please download the paper and start reading.

Convex Optimization: CVX

- We will use the `CVX` toolbox within `MATLAB` to implement optimization
- `http://cvxr.com/`
- Download, uncompress, set `MATLAB` to the `cvx` directory and do `cvx_setup`
- Take `MATLAB` back into your working directory

Example of using CVX

```
T = 150; N =  50;
R = randn(T, N);
rho = 0.02;
tau = 1;
mu = rand(N,1);

cvx_begin quiet
variable w(N)
   minimize( norm(rho*ones(T,1)-R*w) + tau*norm(w,1) )
   subject to
         w'*ones(N,1) == 1;
         w'*mu == rho;
         w > 0;
cvx_end

figure(1), clf, bar(w); grid on
```

Note: Data random - probably won't work all the time

Portfolio Optimization with Transaction Costs

Lobo *et al.* (2007) *Ann Oper Res* **152**:341

- Portfolio weights: $\boldsymbol{w} = [w_1 \; w_2 \; ... \; w_n]^T$
- Returns: $\boldsymbol{a}$; $E[\boldsymbol{a}] = \bar{\boldsymbol{a}}$; $E\left[(\boldsymbol{a} - \bar{\boldsymbol{a}})(\boldsymbol{a} - \bar{\boldsymbol{a}})^T\right] = \Sigma$
- We consider an adjustment to the portfolio of value $\boldsymbol{x}$
- New portfolio: $\boldsymbol{w} + \boldsymbol{x}$; Wealth: $\boldsymbol{a}^T(\boldsymbol{w} + \boldsymbol{x})$
- Portfolio return and variance:

$$\begin{aligned} E[W] &= \bar{\boldsymbol{a}}^T(\boldsymbol{w} + \boldsymbol{x}) \\ E\left[(W - E[W])^2\right] &= (\boldsymbol{w} + \boldsymbol{x})^T \Sigma (\boldsymbol{w} + \boldsymbol{x}) \end{aligned}$$

- Transaction cost: $\phi(\boldsymbol{x})$
- Budget Constraint:

$$\boldsymbol{1}^T\boldsymbol{x} + \phi(\boldsymbol{x}) \leq 0$$

Lobo *et al.* (2007) (cont'd)

Possible Optimizations:

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{maximize}} \quad & \bar{\boldsymbol{a}}^T(\boldsymbol{w} + \boldsymbol{x}) \\ \text{subject to} \quad & \boldsymbol{1}^T\boldsymbol{x} + \phi(\boldsymbol{x}) \leq 0 \\ & \boldsymbol{w} + \boldsymbol{x} \in \mathcal{S} \end{aligned}$$

$\mathcal{S}$ some feasible set (with other constraints)

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \phi(\boldsymbol{x}) \\ \text{subject to} \quad & \bar{\boldsymbol{a}}^T(\boldsymbol{w} + \boldsymbol{x}) \geq r_{\min} \\ & \boldsymbol{w} + \boldsymbol{x} \in \mathcal{S} \end{aligned}$$

Lobo *et al.* (2007) (cont'd)

Modeling Transaction Costs

Costs are separable (usual assumption):

$$\phi(\boldsymbol{x}) = \sum_{i=1}^{n} \phi_i(x_i)$$

$$\phi_i(x_i) = \begin{cases} \alpha_i^+ x_i, & x_i \geq 0 \\ -\alpha_i^- x_i, & x_i \leq 0 \end{cases}$$

- α_i^+ and α_i^- cost rates for buying and selling asset i
- Convex cost function
- $\phi_i = \alpha_i^+ x_i^+ + \alpha_i^- x_i^-$ with $x_i^+ \geq 0$ and $x_i^- \geq 0$

Fixed plus linear transaction costs:

$$\phi_i(x_i) = \begin{cases} 0, & x_i = 0 \\ \beta_i^+ + \alpha_i^+ x_i, & x_i \geq 0 \\ \beta_i^- - \alpha_i^- x_i, & x_i \leq 0 \end{cases}$$

Diversification Constraints

- Limit the amount of investment in any asset

$$w_i + x_i \leq p_i, \quad i = 1, 2, ..., n$$

- Limit the fraction of total wealth held in each asset

$$w_i + x_i \leq \mathbf{1}^T (\boldsymbol{w} + \boldsymbol{x})$$

< gama(i)

- Limit exposure in any small group of assets (say in a sector)

gama(i)<1

$$\sum_{i=1}^{r} (w_i + x_i)_{[i]} \leq \mathbf{1}^T (\boldsymbol{w} + \boldsymbol{x})$$

< gama(i)

(Tricky, but can show this is convex – see Eqn (11) in paper)

Lobo *et al.* (2007) (cont'd)

- Constraints on short-selling
  ... on individual asset

$$w_i \, + \, x_i \, \geq \, -s_i, \quad i = 1, ..., n$$

...or as bound on the total short position

$$\sum_{i=1}^{n} (w_i + x_i)_- \, \leq \, S$$

- Collateralization:

$$\sum_{i=1}^{n} (w_i + x_i)_- \, \leq \, \gamma \sum_{i=1}^{n} (w_i + x_i)_+$$

What I have borrowed to sell is smaller than a fraction of what I own

Lobo *et al.* (2007) (cont'd)

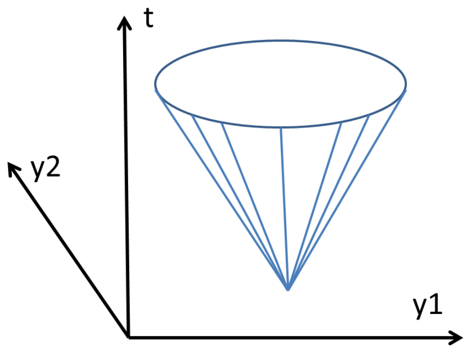
Variance:

$$(\, \boldsymbol{w} \, + \, \boldsymbol{x} \,)^T \, \Sigma \, (\, \boldsymbol{w} \, + \, \boldsymbol{x} \,) \, \leq \, \sigma_{\max}$$

Can also be written as:

$$|| \, \boldsymbol{\Sigma}^{1/2} \, (\, \boldsymbol{w} \, + \, \boldsymbol{x} \,) \, || \, \leq \, \sigma_{\max}$$

- This is *Second Order Cone* constraint.

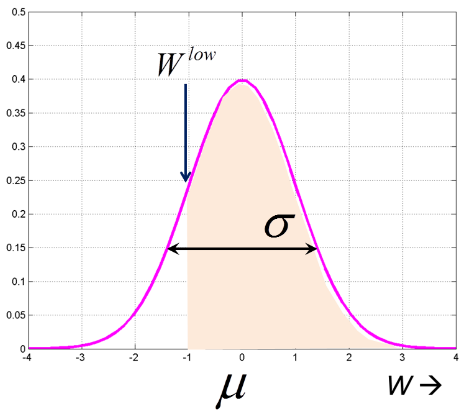


Lobo *et al.* (2007) (cont'd)

Shortfall Risk Constraint:

$$P\left(W \geq W^{\text{low}}\right) \geq \eta$$

$$W = \boldsymbol{a}^T \left(\boldsymbol{w} + \boldsymbol{x}\right) \sim \mathcal{N}\left(\mu, \sigma^2\right)$$



$$P\left(\frac{W - \mu}{\sigma} \leq \frac{W^{\text{low}} - \mu}{\sigma}\right) \leq 1 - \eta$$

Lobo *et al.* (2007) (cont'd)

- But $(W - \mu)/\sigma \sim \mathcal{N}(0, 1)$
- Hence

$$P\left(\frac{W - \mu}{\sigma} \leq \frac{W^{\text{low}} - \mu}{\sigma}\right) = \Phi\left(\left(W^{\text{low}} - \mu\right) / \sigma\right)$$

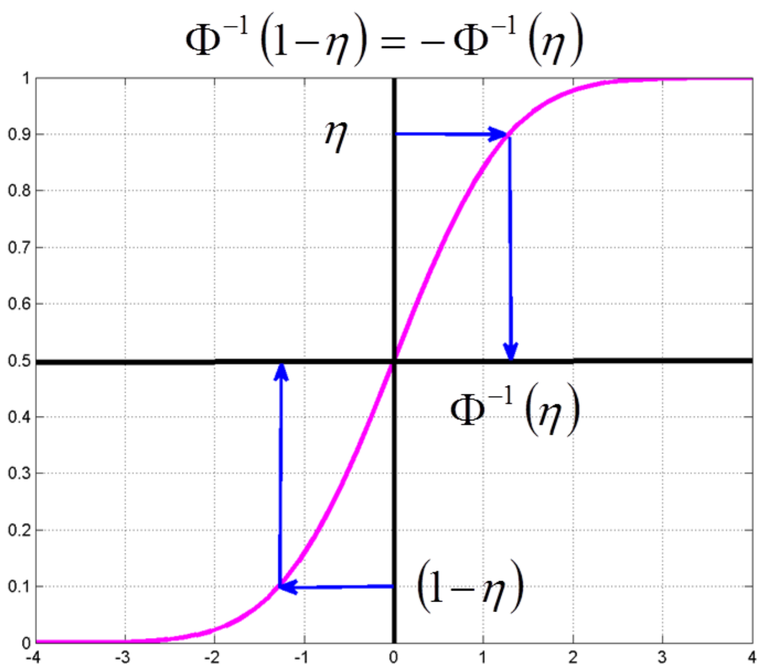
$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left\{-\frac{t^2}{2}\right\} dt$$

- $\frac{W^{\text{low}} - \mu}{\sigma} \leq \Phi^{-1}(1 - \eta)$
- $\Phi^{-1}(1 - \eta) = -\Phi^{-1}(\eta)$

$$\mu - W^{\text{low}} \geq \Phi^{-1}(\eta)\, \sigma$$

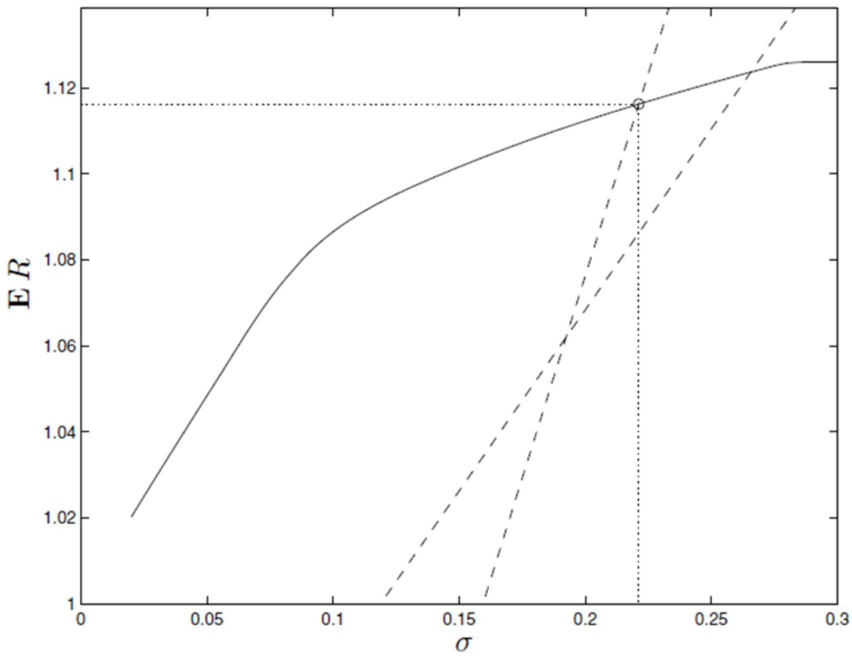
- Using $\mu = \bar{\boldsymbol{a}}^T(\boldsymbol{w} + \boldsymbol{x})$ and $\sigma^2 = (\boldsymbol{w} + \boldsymbol{x})^T \boldsymbol{\Sigma} (\boldsymbol{w} + \boldsymbol{x})$

$$\Phi^{-1}(\eta)\, ||\boldsymbol{\Sigma}^{1/2}(\boldsymbol{w} + \boldsymbol{x})|| \leq \bar{\boldsymbol{a}}^T(\boldsymbol{w} + \boldsymbol{x}) - W^{\text{low}}$$



Lobo *et al.* (2007) (cont'd)

Shortfall Risk on M-V Space



Lobo *et al.* (2007) (cont'd)

Question in Assignment 1

$$\text{maximize} \quad \bar{\boldsymbol{a}}^T \left(\boldsymbol{w} + \boldsymbol{x}^+ - \boldsymbol{x}^- \right)$$

$$\text{subject to} \quad \boldsymbol{1}^T \left(\boldsymbol{x}^+ - \boldsymbol{x}^- \right) + \sum_{i=1}^{n} \left(\alpha_i^+ x_i^+ + \alpha_i^- x_i^- \right) \leq 0$$

$$x_i^+ \geq 0, \; x_i^- \geq 0, \; i = 1, 2, ..., n$$

$$w_i + x_i^+ - x_i^- \geq s_i, \; i = 1, 2, ..., n$$

$$\Phi^- 1(\eta_j) \, || \boldsymbol{\Sigma}^{1/2} \left(\boldsymbol{w} + \boldsymbol{x}^+ - \boldsymbol{x}^- \right) || \leq \bar{\boldsymbol{a}}^T \left(\boldsymbol{w} + \boldsymbol{x}^+ - \boldsymbol{x}^- \right) - W_j^{\text{low}}, j = 1, 2$$

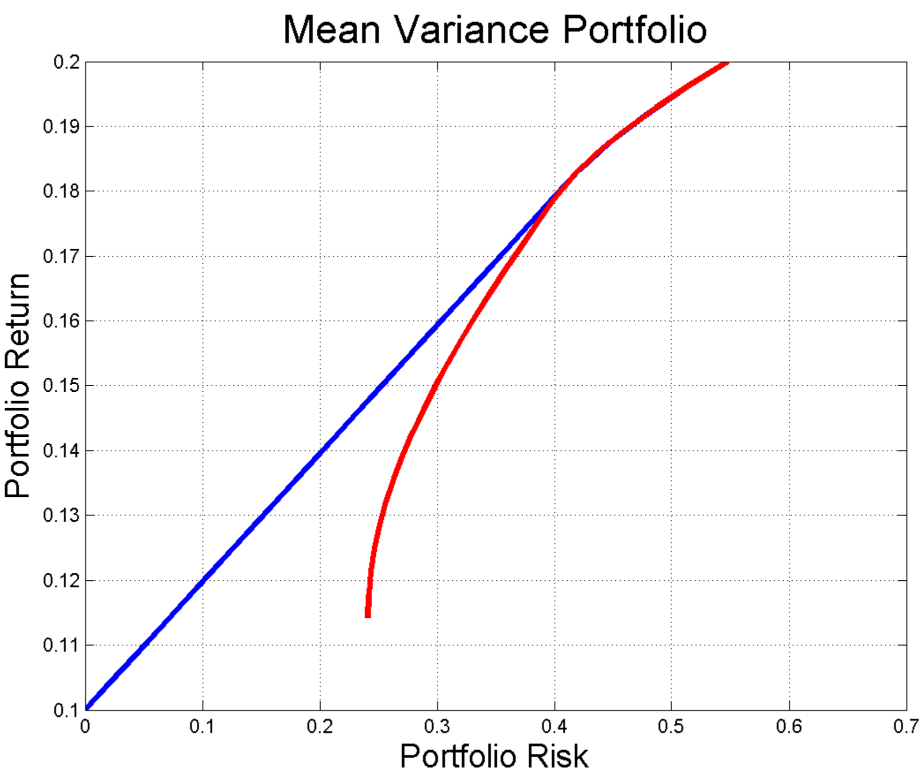
Puzzle

```
m1 = [0.15 0.2 0.08 0.1]';
C1 = [ 0.2    0.05    -0.01    0.0
       0.05   0.30     0.015   0.0
      -0.01   0.015    0.10    0.0
       0.0    0.0      0.0     0.0
       ];

m2 = [0.15 0.2 0.08]';
C2 = [ 0.2    0.05    -0.01
       0.05   0.30     0.015
      -0.01   0.015    0.10
       ];

[V1, M1, PWts1] = NaiveMV(m1, C1, 25);
[V2, M2, PWts2] = NaiveMV(m2, C2, 25);

figure(2), clf,
plot(V1, M1, 'b', V2, M2, 'r', 'LineWidth', 3),
title('Mean Variance Portfolio', 'FontSize', 22)
xlabel('Portfolio Risk', 'FontSize',18)
ylabel('Portfolio Return', 'FontSize', 18);
```


Mean Variance Portfolio
Portfolio Return
Portfolio Risk
0.2
0.19
0.18
0.17
0.16
0.15
0.14
0.13
0.12
0.11
0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7