Chapter 13

Risk-neutral pricing

This chapter derives the Black-Scholes formula through *risk-neutral expectation*, or Martingale representation. This way we obtain an equation for the option in terms of an integral rather than a PDE.

13.1 Binomial model revisited

Recall the simple Bernoulli process (binomial model) for asset price variation. The current price of the asset is S_0, and over a short time δt, the probability that the price goes up to uS_0 is p, and the probability that it decreases to dS_0, is $q = 1 - p$.

Then, as already seen, the *expected value* of $S_{\delta t}$ is given by

$$\mathbb{E}\left(S_{\delta t}\right) = puS_0 + (1-p)dS_0 \qquad . \tag{13.1}$$

For the call option with expiry δt, we found that its fair price is

$$c_0 = e^{-r\delta_t}\{p^* c_u + (1-p^*)c_d\} \qquad , \tag{13.2}$$

where

$$c_u = \max(uS_0 - X, 0), \quad c_d = \max(dS_0 - X, 0) \qquad , \tag{13.3}$$

are the *known* values of the call option at expiry, and the *risk-neutral measure* is:

$$p^* = \frac{e^{r\delta t} - d}{u - d} \qquad . \tag{13.4}$$

Given the analogy of their formula with the *expected value* formula for $S_{\delta t}$, we could write this as

$$\boxed{c_0 = e^{-r\delta t}\mathbb{E}^*\left(c_{\delta t}\right)} \qquad . \tag{13.5}$$

where we defined $\mathbb{E}^*\left(\right)$ as the *risk-neutral probability* or ('measure').

Previously it was shown that under this measure:

$$\boxed{e^{-r\delta t}\mathbb{E}^*\left(S_{\delta t}\right) = S_0} \qquad . \tag{13.6}$$

That is under the *discounted* ($e^{-r\delta t}$) *risk-neutral* ($\mathbb{E}^*$) expectation, the asset price is the same in the future as it is now. Hence, $e^{-r\delta t}S_{\delta t}$ is a *martingale*.

13.2 Geometric Brownian motion

Consider an asset with geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \qquad . \tag{13.7}$$

where μ is the drift rate, σ is the volatility, and W_t is the Wiener process.

It was found that this could be integrated using Itô's lemma to give:

$$S_{\delta t} = S_0 e^{(\mu - \frac{1}{2}\sigma^2)\delta t + \sigma W_{\delta t}} \qquad . \tag{13.8}$$

Since $W_{\delta t} = z\sqrt{\delta t}$, and z is a standard normal variable, this price is random.

It was demonstrated previously, by integrating over the probability density for z, that the expected value of the price

$$\mathbb{E}\left(S_{\delta t}\right) = S_0 e^{\mu \delta t} \qquad . \tag{13.9}$$

So one *recipe* to ensure that the asset price has the desired martingale property (13.6):

$$\boxed{e^{-r\delta t}\mathbb{E}^*\left(S_{\delta t}\right) = S_0} \qquad . \tag{13.10}$$

would be to simply replace μ by r. Then we could use the usual *expectation* (integration over the standard normal distribution) but changing μ to r.

This *ad hoc* prescription will certainly ensure that μ disappears from the final option price formula as found in the Black-Scholes formula.

The question is - does this prescription lead to the correct formula and is this how we can define a *risk-neutral measure* for the geometric Brownian motion?

13.3 Forward Pricing By Risk-Neutral Expectation

Consider the *long* position in a forward contract in which the strike price, F, is agreed now ($t = 0$) with expiry $t = T$. The asset price now (S_0) is known, however at $t = T$, its value, S_T, is not known. We will assume that the asset has a geometric Brownian motion so that:

$$S_T = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T} \qquad . \tag{13.11}$$

The pay-off for the 'long' position at $t = T$ is

$$f_T = S_T - F \qquad . \tag{13.12}$$

we could say that f_T is the *value* of the forward to the holder at $t = T$.

But the *fair value* for the forward at $t = 0$ should be zero. In signing a forward contract there is no exchange of money, and both parties would agree that the contract is worthless (if correctly priced!).

According to our risk-neutral recipe, if we want to calculate the fair price for the forward now, then we should calculate the discounted risk-neutral expectation. That is:

$$f_0 = \mathbb{E}^*\left(e^{-rT} f_T\right) \qquad . \tag{13.13}$$

Our conjecture as to the meaning of $\mathbb{E}^*\left(\right)$ is that we should replace μ by r and just integrate over z. We know that this works for the case (13.10).

Since $\mathbb{E}^*()$ is a linear operator, and F is a constant:

$$f_0 = \mathbb{E}^*\left(e^{-rT}(S_T - F)\right) = \mathbb{E}^*\left(e^{-rT}S_T\right) - Fe^{-rT} \qquad . \tag{13.14}$$

By our definition of $\mathbb{E}^*()$, see the martingale equation (13.10), we have:

$$f_0 = S_0 - e^{-rT}F \qquad . \tag{13.15}$$

Now we specified that the contract should have no cost. Therefore $f_0 = 0$, and this gives

$$\boxed{F = S_0 e^{rT}} \qquad . \tag{13.16}$$

as the 'fair' strike price for such a contract. This agrees with the earlier derivation based on *arbitrage* and gives us some confidence that we are doing the right thing.

In fact, the simple idea "replace μ by r" was the Nobel prize winning idea of Black, Scholes and Merton.

13.4 European call options by risk-neutral measure

Now we want to test this idea for a more complicated case, a European call option. So, according to our conjecture, to price the call option, our approach is as in the binomial case but now modified to continuous variables:

$$c_0 = \mathbb{E}^*\left(e^{-rT}c_T\right) \qquad . \tag{13.17}$$

We know the formula for the future value at expiry:

$$c_T = \max(S_T - X, 0) \qquad . \tag{13.18}$$

Then,

$$c_0 = e^{-rT}\mathbb{E}^*\left(\max(S_T - X, 0)\right) \qquad , \tag{13.19}$$

where $\mathbb{E}^*()$ means, replace μ by r and integrate.

This means that:

$$S_T = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma z\sqrt{T}} \qquad , \tag{13.20}$$

should have r instead of μ, and we rewrite (13.19) as:

$$c_0 = e^{-rT}\mathbb{E}\left(\max[S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma z\sqrt{T}} - X, 0]\right) \qquad . \tag{13.21}$$

We can now drop the * since we have substituted μ by r, and can now calculate the expectation by integrating over z.

$$c_o = \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \max[S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma z\sqrt{T}} - X, 0] e^{-\frac{1}{2}z^2}\, dz \qquad . \tag{13.22}$$

As before, because of the max function in the integrand, the integrand is non-zero only when

$$S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma z\sqrt{T}} \geq X \qquad . \tag{13.23}$$

This corresponds to a z-value,

$$z \geq -\left\{\frac{\ln(S_0/X) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right\} \qquad . \tag{13.24}$$

We recognise the term in the curly bracket as d_2, previously encountered (12.71). So applying this cut-off, we find that:

$$c_0 = \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-d_2}^{+\infty} \left(S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma z\sqrt{T}} - X\right) e^{-\frac{1}{2}z^2} \, dz \quad . \tag{13.25}$$

Separating this integral into two terms we have

$$c_0 = \frac{e^{-rT}}{\sqrt{2\pi}} S_0 e^{rT} \int_{-d_2}^{+\infty} e^{-\frac{1}{2}z^2+\sigma z\sqrt{T}-\frac{1}{2}\sigma^2 T} \, dz + \frac{e^{-rT}(-X)}{\sqrt{2\pi}} \int_{-d_2}^{+\infty} e^{-\frac{1}{2}z^2} \, dz \quad . \tag{13.26}$$

In the first term, we change variable: $z - \sigma\sqrt{T} = v$, in order to complete the square, so that;

$$c_0 = \frac{S_0}{\sqrt{2\pi}} \int_{-d_2-\sigma\sqrt{T}}^{+\infty} e^{-\frac{1}{2}v^2} \, dv - \frac{Xe^{-rT}}{\sqrt{2\pi}} \int_{-d_2}^{+\infty} e^{-\frac{1}{2}z^2} \, dz \quad . \tag{13.27}$$

We can now rewrite $d_2 + \sigma\sqrt{T} = d_1$. As for the integrals, these are now related to the (cumulative) standard normal distribution through symmetry around v or $z = 0$.

This gives the final result:

$$\boxed{c_0 = S_0 N(d_1) - Xe^{-rT} N(d_2)} \quad . \tag{13.28}$$

where

$$d_1 = d_2 + \sigma\sqrt{T} = \frac{\ln(S/X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad . \tag{13.29}$$

This alternative method, using the integral directly and by changing the probability distribution rather than moving through the partial-differential equations and then using the Green function to solve, is simpler. However, while the methods are equivalent in this case, the correspondence for more complex options is not so simple.

It appears that the *recipe* we have used is a valid approach, although we have avoided the detailed proof of the conjecture, as this requires additional mathematical framework. In mathematical terms, there is a more honest way of showing that μ is effectively replaced by r.

13.5 Pricing the put option

The *put option* can now be priced through the put-call parity. Alternatively, we can use the risk-neutral measure:

$$\boxed{p = e^{-rT}\mathbb{E}^* (p_T)} \quad . \tag{13.30}$$

Recalling the expression for p_T, the value of the put at expiry:

$$p = e^{-rT}\mathbb{E}^* (\max (X - S_T, 0)) \quad . \tag{13.31}$$

Then following our interpretation of $\mathbb{E}^* ()$, we write:

$$p = \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}z^2} \max\left(X - S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}z}, 0\right) dz \quad . \tag{13.32}$$

We now evaluate the integral as before. Our first step is to eliminate the max function. Again, this function creates a cut-off in the range of integration, so that the integrand function is only non-zero when:

$$X - S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}z} \geq 0 \quad . \tag{13.33}$$

Thus:

$$\ln(X/S_0) \geq (r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}z \quad , \tag{13.34}$$

giving:

$$z \leq -d_2 \quad , \tag{13.35}$$

where d_2 is given by (12.71). Then we have:

$$p = \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{-d_2} e^{-\frac{1}{2}z^2} \left(X - S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}z} \right) dz \quad . \tag{13.36}$$

This is now in a form that can be reduced to a compact expression. Then, after repeating the same steps as for the put option, which we leave as an exercise, we arrive at the expression:

$$p = Xe^{-rT}N(-d_2) - S_0 N(-d_1) \quad , \tag{13.37}$$

which is the correct result. That is, it agrees with the expression derived from the call option price using put-call parity, which in turn was derived from the solution of the Black-Scholes partial-differential equation.