COMP6212: Computational Finance

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This module is in two parts:

- Part I Financial data analysis, taught by Prof. M. Niranjan
- Part II Crypto-currencies and blockchain technology, taught by Dr Jie Zhang

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Financial Equilibrium
Caution: A peculiar and rather personal view


jamesnichollsillustration.blogspot.co.uk

Financial Markets


www.investors411.com

- Generate products and services
- In need of
  - stability against fluctuations (e.g. demand, exchange rate)
  - capital investment (e.g. to modernise, grow)

- Pocess wealth & capital
- Driven by gambling instinct and greed

The Setting

- Finance gets bad publicity; bankers and fund managers are sometimes disliked
- The system can fail badly
- When the system fails, large amounts of tax-payer money is used to bail them out. I don't like this!
- Yet the system is useful
  - Investors interested in future returns
    - Greed?
    - Pay for retirement
  - Firms / Governments looking to raise capital for investment
  - Companies looking for stability; *e.g.* insure against exchange rate fluctuation
- What are the sources of computational problems?
  - Time - present value of money.
  - Uncertainty - of the future.

Overview of the Module

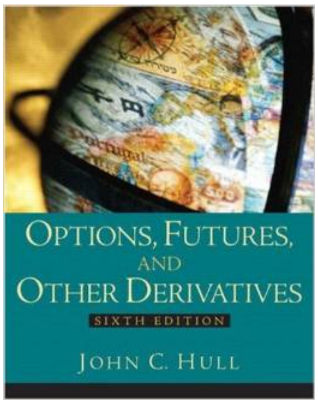
Topics in Part I: Financial Data Analysis

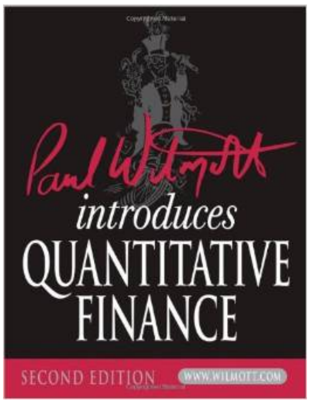
- Portfolio Optimization
- Derivatives Pricing

Keywords:

Mean-Variance optimization, Linear and quadratic programming, Multivariate Gaussian distribution, Constrained optimization, Value at risk and Conditional value at risk, Sharpe ratio, Present value, Stochastic differential equations, Ito's Lemma, Black-Scholes model, Options pricing, Stochastic Simulations and Monte Carlo methods.

Resources





- plus several academic papers.

Financial Instruments (broad classes)

- Bonds
  - Debt instrument to raise capital; delivers periodic payment (*coupon*); has a *face value* on *maturity*. No ownership associated.
- Stocks
  - Own a small *share* of a company; the ownership may be traded in the market; owning the share might earn *dividends*.
- Derivatives
  - Contracts written on the basis of a future value of a a stock, currency etc. Usually there is a time of *maturity* and a promised *payoff* in the contract. Variations in style of *exercising* the contract.

Time: Present Value

- Wealth W_0 deposit in bank and get W_1 after one year
- $W_1 = (1 + r)\, W_0, \quad r$ interest rate
- Compound interest over n years: $W_n = (1 + r)^n\, W_0$
- Define interest rate as r per year; allow compounding at m intervals within the year

$$W_1 = \left(1 + \frac{r}{m}\right)^m W_0$$

- Continuous compounding $m \to \infty$

$$W_1 = \exp(r)\, W_0$$

- Present value of your promise to give me cash C in time t is

$$\exp(-rt)\, C$$

Various Topics We Will Learn

Part I (Topic I): Portfolio Optimization

Portfolios:

- Notion of expected return and risk in investing - balancing it out
- Investing in a portfolio of assets, than in a single asset - "not all eggs in one basket"
- Optimization techniques we will learn and use
  - Linear programming
  - Quadratic programming
  - (Second order cone programming)
  - Inducing sparsity – l_1 or *lasso* regularization
  - Convex optimization using `CVX` toolbox

Various Topics We Will Learn (cont'd)

Part I (Topic II): Derivatives Pricing

Derivatives Pricing (contract in the future, in an uncertain world):

- Brownian motion, Geometric Brownian motion
- Stochastic differential equations

$$\frac{dS}{S} = \mu\, dt + \sigma\, dZ$$

$$dZ = \phi \sqrt{dt}, \quad \phi \sim (0, 1)$$

- Ito's Lemma: Function of a Geometric Brownian Motion

$$dG = \left(\mu S \frac{\partial G}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 G}{\partial S^2} + \frac{\partial G}{\partial t} \right) dt + \sigma S \frac{\partial G}{\partial S}\, dZ$$

- Black-Scholes: options pricing under specific assumptions
- Monte Carlo / Stochastic simulations: general cases

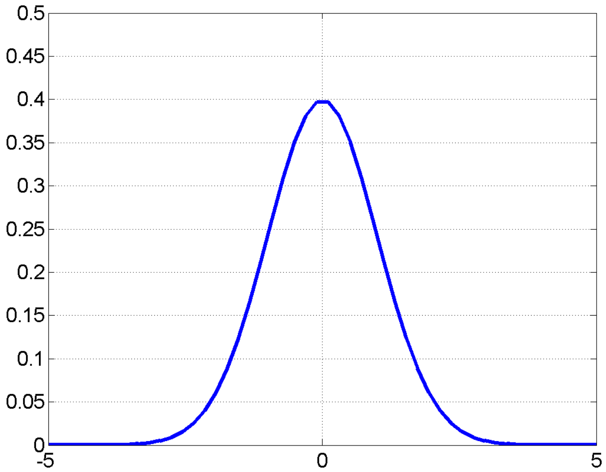
Part I: Portfolio Optimization

- $r_i(t)$ Return on asset i at time t; *i.e.* invest at time $t-1$, what have you earned at time t?
- We think of this as a random variable, say Gaussian distributed with mean μ_i and variance σ_i^2
- The mean is what we expect (on average) to gain by investing
- We think of variance in return as *risk*
- When we look at more than one asset, we can think of how returns on them are correlated: σ_{ij}^2
- A portfolio (investment in N assets) with relative weights π_i
- Return on the portfolio: $r_p = \sum_{i=1}^{N} \pi_i\, r_i = \boldsymbol{\pi}^t \boldsymbol{r}$
- Returns on the portfolio has a multivariate Gaussian distribution

$$\boldsymbol{r} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{pmatrix} \quad \boldsymbol{\mu} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{pmatrix} \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12}^2 & \dots & \sigma_{1N}^2 \\ \sigma_{12}^2 & \sigma_2^2 & \dots & \sigma_{2N}^2 \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{1N}^2 & \sigma_{2N}^2 & \dots & \sigma_N^2 \end{pmatrix}$$

Gaussian Density

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-m)^2}{2\sigma^2}\right\}$$

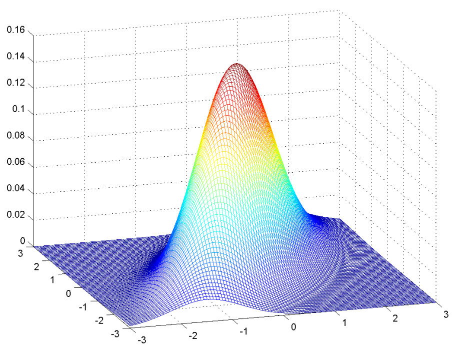


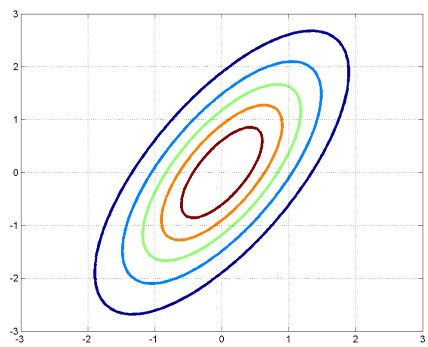

```
x = linspace(-5,5,50);
m = 0;
s = 1;
y = normpdf(x,m,s);
figure(1), clf
plot(x,y,'LineWidth',3);

grid on
```


Multivariate Gaussian Distribution

$$p(\boldsymbol{x}) = \frac{1}{(2\pi)^{d/2}\,|\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\}$$

- Parameters: mean vector $\boldsymbol{\mu}$ covariance matrix $\boldsymbol{\Sigma}$





- How do these shapes change with $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$?

Portfolio Return

- Linear transform of multivariate Gaussian

$$\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{\mu}, \; \boldsymbol{\Sigma}); \;\; \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} \quad \Longrightarrow \quad \boldsymbol{y} \sim \mathcal{N}\left(\boldsymbol{A}\boldsymbol{\mu}, \; \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}^t\right)$$

- Return on our portfolio is a linear transform of the vector of returns

$$r_P = \boldsymbol{\pi}^T \boldsymbol{r}$$

- We can immediately write down the distribution of the return on the portfolio

$$r_P \sim \mathcal{N}\left(\boldsymbol{\pi}^T \boldsymbol{\mu}, \; \boldsymbol{\pi}^T \boldsymbol{\Sigma} \boldsymbol{\pi}\right)$$

- Mean return $M = \boldsymbol{\pi}^T \boldsymbol{\mu}$ and the variance on it $V = \boldsymbol{\pi}^T \boldsymbol{\Sigma} \boldsymbol{\pi}$
- When π changes, M and V change — how?

Efficient Portfolio

- As we change π (*i.e.* invest in different proportions), M and V change
- Not all M and V are realizable.
- We can formulate constrained optimization problems
- For a given risk we tolerate, what is the highest return we can expect

$$\max_{\boldsymbol{\pi}} \; \boldsymbol{\pi}^T \boldsymbol{\mu} \quad \text{subject to } \boldsymbol{\pi}^T \boldsymbol{\Sigma} \boldsymbol{\pi} = \sigma_0$$

- If we hope for (expect) a given return, at what minimum risk can we achieve it?

$$\min_{\boldsymbol{\pi}} \; \boldsymbol{\pi}^T \boldsymbol{\Sigma} \boldsymbol{\pi} \quad \text{subject to } \; \boldsymbol{\pi}^T \boldsymbol{\mu} = r_0$$

- Other constraints possible: $\sum_{i=1}^{N} \pi_i = 1, \; \pi_i \geq 0, \; \alpha \leq \pi_i \leq \beta$

Estimation

- We estimate $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ from historic data and apply optimization to allocate assets.
- We hope the past might be a good reflection of future!
- Estimation:

$$\begin{aligned} \widehat{\boldsymbol{\mu}} &= \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{r}(t) \\ \widehat{\boldsymbol{\Sigma}} &= \frac{1}{T}\sum_{t=1}^{T} \left(\boldsymbol{r}(t) - \widehat{\boldsymbol{\mu}}\right)\left(\boldsymbol{r}(t) - \widehat{\boldsymbol{\mu}}\right)^T \end{aligned}$$

Solving quadratic program in MATLAB

$$\min_{\boldsymbol{x}} \ \boldsymbol{x}^T \boldsymbol{H}\, \boldsymbol{x} + \boldsymbol{f}^T \boldsymbol{x} \text{ such that } \begin{cases} \boldsymbol{A}\, \boldsymbol{x} \leq 0 \\ \boldsymbol{A}_{eq}\, \boldsymbol{x} = b_{eq} \\ lb \leq \boldsymbol{x} \leq ub. \end{cases}$$

```
x = quadprog(H ,f, A, b, Aeq, beq ,lb, ub, x0)
```

Do `>doc quadprog` in `MATLAB` and read more.

For the portfolio optimization problem, we might have:

```
pi = quadprog(Sigma, [], [], [], mu', rMax, 0, 1, [])
```

Map the problem variables to the function in the tool.

Efficient Frontier

- Given $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$
- What portfolio has the highest return, unconstrained by risk?

$$\max \boldsymbol{\pi}^T \boldsymbol{\mu} \quad \text{subject to} \quad \sum_{i=1}^{N} \pi_i = 1, \quad \text{and} \quad \pi_i \geq 0$$

Linear Programming:

$$\min_{\boldsymbol{x}} \boldsymbol{f}^T \boldsymbol{x} \quad \text{such that} \quad \begin{cases} \boldsymbol{A}\,\boldsymbol{x} \leq \boldsymbol{b} \\ \boldsymbol{A}_{eq}\,\boldsymbol{x} = \boldsymbol{b}_{eq} \\ lb \leq \boldsymbol{x} \leq ub \end{cases}$$

```
x = linprog( f, A, b, Aeq, beq, lb, ub )
w1 = linprog(-mu, [], [], ones(1,N), 1, 0, 0);
r1 = w1 * mu;
```


Efficient Frontier (cont'd)

- What portfolio has lowest variance (unconstrained by expectation)?

$$\min \boldsymbol{\pi}^T \boldsymbol{\Sigma} \boldsymbol{\pi} \quad \text{subject to} \quad \sum_{i=1}^{N} \pi_i = 1$$

```
w2=quadprog(mu,zeros(N,1),[],[],ones(1,N),1,zeros(N,1),[],[]);
r2 = w2' * mu;
```

- Portfolios on the efficient frontier will have returns in range `r1` to `r2`
- Now, select a number of points (returns we need), and find the minimum variance portfolios that will offer these returns!

```
M = linspace(r1, r2, p)
for j=1:p
    ret = M(j);
    w = quadprog(...);
    V(j) = w' * Sigma * w;
end
```

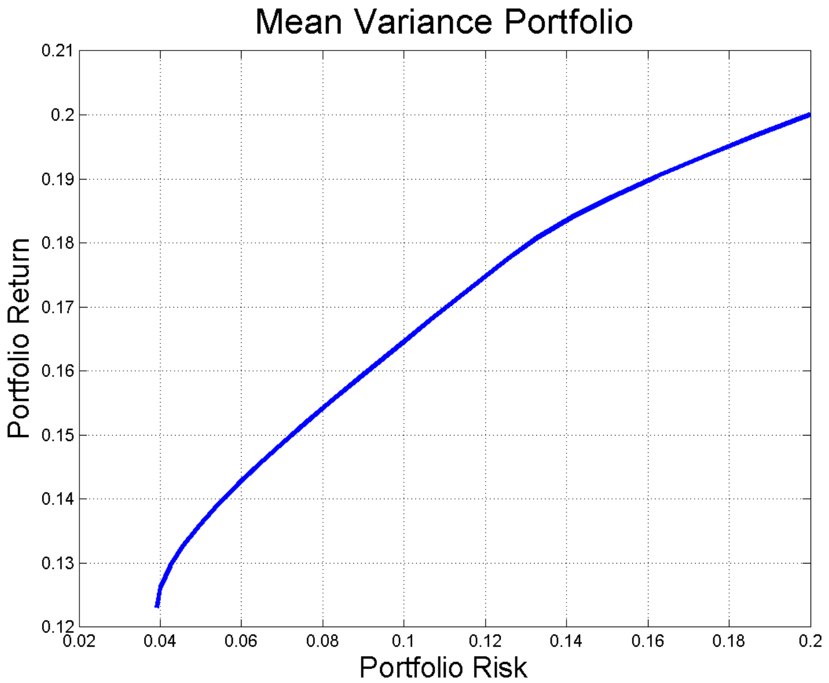
Complete Function to Compute the Efficient Frontier

```
function [PRisk, PRoR, PWts] = NaiveMV(ERet, ECov, NPts)
ERet = ERet(:);      % makes sure it is a column vector
NAssets = length(ERet);  % get number of assets
% vector of lower bounds on weights
V0 = zeros(NAssets, 1);
% row vector of ones
V1 = ones(1, NAssets);
% set medium scale option
options = optimset('LargeScale', 'off');
% Find the maximum expected return
MaxReturnWeights = linprog(-ERet, [], [], V1, 1, V0);
MaxReturn = MaxReturnWeights' * ERet;
% Find the minimum variance return
MinVarWeights = quadprog(ECov,V0,[],[],V1,1,V0,[],[],options);
MinVarReturn = MinVarWeights' * ERet;
MinVarStd = sqrt(MinVarWeights' * ECov * MinVarWeights);
% check if there is only one efficient portfolio
if MaxReturn > MinVarReturn
   RTarget = linspace(MinVarReturn, MaxReturn, NPts);
   NumFrontPoints = NPts;
else
      RTarget = MaxReturn;
      NumFrontPoints = 1;
end
```


Complete Function (cont'd)

```
...
% Store first portfolio
PRoR = zeros(NumFrontPoints, 1);
PRisk = zeros(NumFrontPoints, 1);
PWts = zeros(NumFrontPoints, NAssets);
PRoR(1) = MinVarReturn;
PRisk(1) = MinVarStd;
PWts(1,:) = MinVarWeights(:)';
% trace frontier by changing target return
VConstr = ERet';
A = [V1 ; VConstr ];
B = [1 ; 0];
for point = 2:NumFrontPoints
B(2) = RTarget(point);
Weights = quadprog(ECov,V0,[],[],A,B,V0,[],[],options);
PRoR(point) = dot(Weights, ERet);
PRisk(point) = sqrt(Weights'*ECov*Weights);
PWts(point, :) = Weights(:)';
end
```

Summary: what have we achieved?



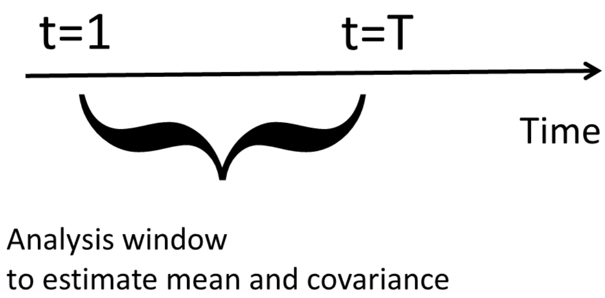
Homework

- Three assets with the following properties:

$$m = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.15 \end{bmatrix}; \quad C = 100 * \begin{bmatrix} 0.005 & -0.010 & 0.004 \\ -0.010 & 0.040 & -0.002 \\ 0.004 & -0.002 & 0.023 \end{bmatrix};$$

- Study the code of function `NaiveMV` and draw the efficient frontier.
- Use the function `frontcon` in `MATLAB` and draw the efficient frontier.

Estimation of Parameters



- Estimate parameters μ and $\boldsymbol{C}$ from data within a window
- Optimize portfolio, invest and wait
- Periodically re-balance the portfolio (re-estimate parameters)
- Trade-off:
  - Need long window for accurate estimation
  - But relationships may not be stationary over long durations
- Shrinkage in covariance estimates

Advances on the Mean-Variance Portfolio

- Do such portfolios make money?

  DeMiguel, J. *et al.* (2009) Optimal versus Naive Diversification: How Inefficient Is the 1/N Portfolio Strategy?, *The Review of Financial Studies* **22**(5): 1915.

- Including transaction costs into the optimization

  Lobo, M.S. *et al.* (2007) Portfolio Optimization with Linear and Fixed Transaction Costs, *Annals of Operations Research* **152**: 341.

- Forcing the portfolio to be sparse and stable

  Brodie, J. *et al.* (2009) Sparse and stable Markowitz portfolios, *PNAS* **106**(30): 12267
  Takeda, A. *et al.* (2013) Simultaneous pursuit of out-of-sample performance and sparsity in index tracking portfolios, *Comput Manag Sci* **10**: 21

- Optimizing the execution of trade

  TBC

Portfolio Performance

- Sharpe Ratio: mean to standard deviation of portfolio return

$$S = \frac{m - r}{\sigma}$$

  r "risk free" interest rate

- Value at Risk (VAR): Value such that probability of loss exceeding this is 0.01.

$$V \text{ such that } P[-G > V] = 0.01$$

> *"if a portfolio of stocks has a one-day 5% VaR of 1 million, there is a 0.05 probability that the portfolio will fall in value by more than 1 million over a one day period"*

```
ValueAtRisk=portvrisk(PortReturn,PortRisk,RiskThreshold,PortValue )
```

- cVAR: Conditional Value at Risk (later)

Empirical Evaluation

DeMiguel, J. *et al.* (2009),

- Comparison of a number of portfolio optimization methods
  - $\frac{1}{N}$ with re-balancing
  - Sample based mean-variance
  - Bayesian methods (of shrinking estimates)
  - Constraints
  - Combination of portfolios (model averaging / mixing)
- No method *consistently* beats the naive strategy!

Sparse Portfolios
Brodie *et al.* (2007) PNAS

- N assets; $\boldsymbol{r}_t$, return vector at time t
- Expected return and covariance:

$$\boldsymbol{r}_t = \begin{pmatrix} r_{1,t} \\ r_{2,t} \\ \vdots \\ r_{N,t} \end{pmatrix} \qquad \boldsymbol{E}\left[\boldsymbol{r}_t\right] = \boldsymbol{\mu} \quad \boldsymbol{E}\left[\left(\boldsymbol{r}_t - \boldsymbol{\mu}\right)\left(\boldsymbol{r}_t - \boldsymbol{\mu}\right)^T\right] = \boldsymbol{C}$$

- Markowitz portfolio

$$\begin{cases} \min_{\boldsymbol{w}} \ \boldsymbol{w}^T \boldsymbol{C} \, \boldsymbol{w} \\ \text{subject to } \boldsymbol{w}^T \boldsymbol{\mu} = \rho \text{ and } \mathbf{1}_N^T \boldsymbol{w} = 1 \end{cases}$$

- Short selling allowed; *i.e.* w_j need not be positive
- Covariance: $\boldsymbol{C} = \boldsymbol{E}\left[\boldsymbol{r}_t \boldsymbol{r}_t^T - \boldsymbol{\mu}\boldsymbol{\mu}^t\right]$

Sparse Portfolios, Brodie *et al.* (2007) *PNAS* (cont'd)

- Mean and covariance estimated from data (expectations as sample averages):
- $\widehat{\boldsymbol{\mu}} = \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{r}_t$
- $\boldsymbol{R}$ $T \times N$ matrix with rows as $\boldsymbol{r}_t^T$
- Optimization problem rewritten as

$$\begin{cases} \widehat{\boldsymbol{w}} = \min_{\boldsymbol{w}} \frac{1}{T} ||\rho \mathbf{1}_T - \boldsymbol{R}\,\boldsymbol{w}||_2^2 \\ \text{subject to } \boldsymbol{w}^T \widehat{\boldsymbol{\mu}} = \rho, \ \boldsymbol{w}^T \mathbf{1}_N = 1 \end{cases}$$

- Often there is strong correlation between returns
  - Assets in the same sector respond in similar ways
- Strong correlations make R ill-conditioned $\Longrightarrow$ numerically unstable optimization
- Solution: regularization
- Brodie *et al.* suggest l_1 regularizer

$$\begin{cases} \widehat{\boldsymbol{w}} = \min_{\boldsymbol{w}} \left[||\,\rho \mathbf{1}_T - \boldsymbol{R}\,\boldsymbol{w}\,||_2^2 + \tau\,||\,\boldsymbol{w}\,||_1\right] \\ \text{subject to } \boldsymbol{w}^T \widehat{\boldsymbol{\mu}} = \rho, \ \boldsymbol{w}^T \mathbf{1}_N = 1 \end{cases}$$

Index Tracking
Brodie *et al.* (2007) (cont'd)

- Passive investor, wishing to get the same return as stock index (*e.g.* FTSE100)
- Invest in all 100 stocks of the FTSE?
- Transaction costs very high
- Can we find a small subset of the 100 stocks (say 10), that will approximate the performance of the index?
- Subset selection / cardinality constrained optimization

$$\begin{cases} \min_{\boldsymbol{w}} \left[||\boldsymbol{y} - \boldsymbol{R}\,\boldsymbol{w}||_2^2 \right] \\ \text{subject to } ||\,\boldsymbol{w}\,||_0 = w_0 \end{cases}$$

- 0^{th} norm $\rightarrow$ number of nonzero elements of $\boldsymbol{w}$ $\rightarrow$ subset of assets
- The above is of combinatorial complexity
- Suboptimal algorithm: greedy search

Brodie *et al.* (2007) (cont'd)

- A convenient proxy to achieve sparsity is *lasso* (l_1 constrained regression)

$$\min_{\boldsymbol{w}} \left[||\boldsymbol{y} - \boldsymbol{R}\,\boldsymbol{w}||_2^2 + \tau\,||\boldsymbol{w}||_1 \right]$$

- Several elements of $\boldsymbol{w}$ will be zero
- Tune τ to achieve different levels of sparsity
- Can incorporate transaction costs into the optimization

$$\min_{\boldsymbol{w}} \left[||\boldsymbol{y} - \boldsymbol{R}\,\boldsymbol{w}||_2^2 + \tau \sum_{i=1}^{N} s_i\,|\,w_i\,| \right]$$

- Transaction costs:
  - Usually have fixed (overhead) part and transaction-dependent part
  - Institutional investors fixed part negligible
  - Small investors can assume fixed cost only

Brodie *et al.* (2007) (cont'd)

Portfolio adjustment (re-balancing)

- We are holding a portfolio $\boldsymbol{w}$
- We want to make an adjustment $\boldsymbol{\Delta_w}$, new portfolio $\boldsymbol{w} + \boldsymbol{\Delta_w}$
- Transaction costs only on the adjustments

$$\begin{cases} \boldsymbol{\Delta_w} \;=\; \min_{\boldsymbol{\Delta_w}} \; \left[||\rho \, \mathbf{1}_T \;-\; \boldsymbol{R} \, (\boldsymbol{w} + \boldsymbol{\Delta_w}) \, ||_2^2 \;+\; \tau \, ||\boldsymbol{\Delta_w}||_1 \right] \\ \text{subject to } \boldsymbol{\Delta_w}^T \, \widehat{\boldsymbol{\mu}} \;=\; 0 \text{ and } \boldsymbol{\Delta_w}^T \, \mathbf{1}_N \;=\; 1 \end{cases}$$

Brodie *et al.* (2007) (cont'd)

Homework:

Coursework 1 will involve confirming some claims in Brodie *et al.*'s paper. Please download the paper and start reading.

Convex Optimization: CVX

- We will use the CVX toolbox within MATLAB to implement optimization
- http://cvxr.com/
- Download, uncompress, set MATLAB to the cvx directory and do cvx_setup
- Take MATLAB back into your working directory

Example of using CVX

```
T = 150; N =  50;
R = randn(T, N);
rho = 0.02;
tau = 1;
mu = rand(N,1);

cvx_begin quiet
variable w(N)
   minimize( norm(rho*ones(T,1)-R*w) + tau*norm(w,1) )
   subject to
         w'*ones(N,1) == 1;
         w'*mu == rho;
         w > 0;
cvx_end

figure(1), clf, bar(w); grid on
```

Note: Data random - probably won't work all the time

Portfolio Optimization with Transaction Costs

Lobo *et al.* (2007) *Ann Oper Res* **152**:341

- Portfolio weights: $\boldsymbol{w} = [w_1 \; w_2 \; \ldots \; w_n]^T$
- Returns: $\boldsymbol{a}$; $E[\boldsymbol{a}] = \bar{\boldsymbol{a}}$; $E\left[(\boldsymbol{a} - \bar{\boldsymbol{a}})(\boldsymbol{a} - \bar{\boldsymbol{a}})^T\right] = \boldsymbol{\Sigma}$
- We consider an adjustment to the portfolio of value $\boldsymbol{x}$
- New portfolio: $\boldsymbol{w} + \boldsymbol{x}$; Wealth: $\boldsymbol{a}^T(\boldsymbol{w} + \boldsymbol{x})$
- Portfolio return and variance:

$$\begin{aligned} E[W] &= \bar{\boldsymbol{a}}^T(\boldsymbol{w} + \boldsymbol{x}) \\ E\left[(W - E[W])^2\right] &= (\boldsymbol{w} + \boldsymbol{x})^T \boldsymbol{\Sigma} (\boldsymbol{w} + \boldsymbol{x}) \end{aligned}$$

- Transaction cost: $\phi(\boldsymbol{x})$
- Budget Constraint:

$$\mathbf{1}^T \boldsymbol{x} + \phi(\boldsymbol{x}) \leq 0$$

Lobo *et al.* (2007) (cont'd)

Possible Optimizations:

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{maximize}} \quad & \bar{\boldsymbol{a}}^T(\boldsymbol{w} + \boldsymbol{x}) \\ \text{subject to} \quad & \mathbf{1}^T \boldsymbol{x} + \phi(\boldsymbol{x}) \leq 0 \\ & \boldsymbol{w} + \boldsymbol{x} \in \mathcal{S} \end{aligned}$$

$\mathcal{S}$ some feasible set (with other constraints)

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \phi(\boldsymbol{x}) \\ \text{subject to} \quad & \bar{\boldsymbol{a}}^T(\boldsymbol{w} + \boldsymbol{x}) \geq r_{\min} \\ & \boldsymbol{w} + \boldsymbol{x} \in \mathcal{S} \end{aligned}$$

Lobo *et al.* (2007) (cont'd)

Modeling Transaction Costs

Costs are separable (usual assumption):

$$\phi(\boldsymbol{x}) \;=\; \sum_{i=1}^{n} \phi_i\,(x_i)$$

$$\phi_i(x_i) \;=\; \begin{cases} \alpha_i^+\,x_i, & x_i \geq 0 \\ -\,\alpha_i^-\,x_i, & x_i \leq 0 \end{cases}$$

- α_i^+ and α_i^- cost rates for buying and selling asset i
- Convex cost function
- $\phi_i \;=\; \alpha_i^+ x_i^+ \;+\; \alpha_i^- x_i^-$ with $x_i^+ \geq 0$ and $x_i^- \geq 0$

Fixed plus linear transaction costs:

$$\phi_i(x_i) \;=\; \begin{cases} 0, & x_i = 0 \\ \beta_i^+ \;+\; \alpha_i^+\,x_i, & x_i \geq 0 \\ \beta_i^- \;-\; \alpha_i^-\,x_i, & x_i \leq 0 \end{cases}$$

- This is non-convex

Diversification Constraints

- Limit the amount of investment in any asset

$$w_i \;+\; x_i \;\leq\; p_i, \quad i = 1, 2, \ldots, n$$

- Limit the fraction of total wealth held in each asset

$$w_i \;+\; x_i \;\leq\; \mathbf{1}^T\;(\,\boldsymbol{w} \;+\; \boldsymbol{x}\,)$$

- Limit exposure in any small group of assets (say in a sector)

$$\sum_{i=1}^{r}\;(w_i \;+\; x_i)_{[i]} \;\leq\; \mathbf{1}^T\;(\,\boldsymbol{w} \;+\; \boldsymbol{x}\,)$$

(Tricky, but can show this is convex – see Eqn (11) in paper)

Lobo *et al.* (2007) (cont'd)

- Constraints on short-selling
  ... on individual asset

$$w_i \; + \; x_i \; \geq \; -s_i, \quad i = 1, ..., n$$

...or as bound on the total short position

$$\sum_{i=1}^{n} (w_i + x_i)_- \; \leq \; S$$

- Collateralization:

$$\sum_{i=1}^{n} (w_i + x_i)_- \; \leq \; \gamma \sum_{i=1}^{n} (w_i + x_i)_+$$

What I have borrowed to sell is smaller than a fraction of what I own

Lobo *et al.* (2007) (cont'd)

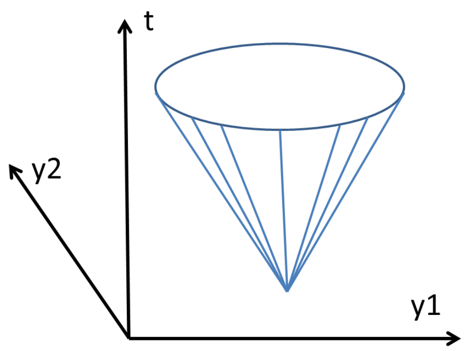
Variance:

$$(\, \boldsymbol{w} \; + \; \boldsymbol{x} \,)^T \; \boldsymbol{\Sigma} \; (\, \boldsymbol{w} \; + \; \boldsymbol{x} \,) \; \leq \; \sigma_{\max}$$

Can also be written as:

$$||\, \boldsymbol{\Sigma}^{1/2} \; (\, \boldsymbol{w} \; + \; \boldsymbol{x} \,) \; || \; \leq \; \sigma_{\max}$$

- This is *Second Order Cone* constraint.

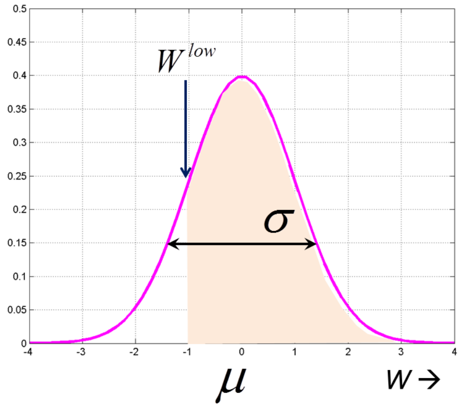


Lobo *et al.* (2007) (cont'd)

Shortfall Risk Constraint:

$$P\left(W \geq W^{\text{low}}\right) \geq \eta$$

$$W = \mathbf{a}^T \left(\mathbf{w} + \mathbf{x}\right) \sim \mathcal{N}\left(\mu, \sigma^2\right)$$



$$P\left(\frac{W - \mu}{\sigma} \leq \frac{W^{\text{low}} - \mu}{\sigma}\right) \leq 1 - \eta$$

Lobo *et al.* (2007) (cont'd)

- But $(W - \mu)/\sigma \sim \mathcal{N}(0, 1)$
- Hence

$$P\left(\frac{W - \mu}{\sigma} \leq \frac{W^{\text{low}} - \mu}{\sigma}\right) = \Phi\left(\left(W^{\text{low}} - \mu\right) / \sigma\right)$$

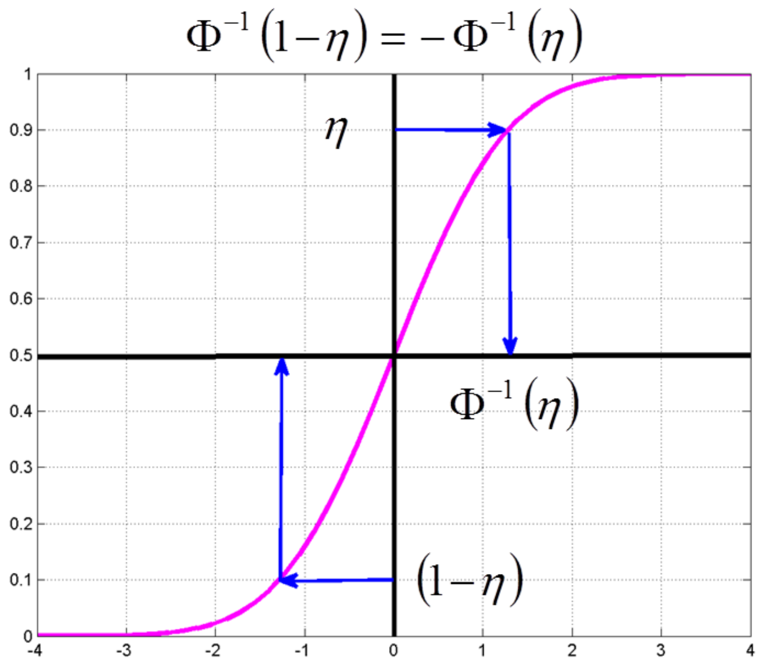
$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left\{-\frac{t^2}{2}\right\} dt$$

- $\frac{W^{\text{low}} - \mu}{\sigma} \leq \Phi^{-1}(1 - \eta)$
- $\Phi^{-1}(1 - \eta) = -\Phi^{-1}(\eta)$

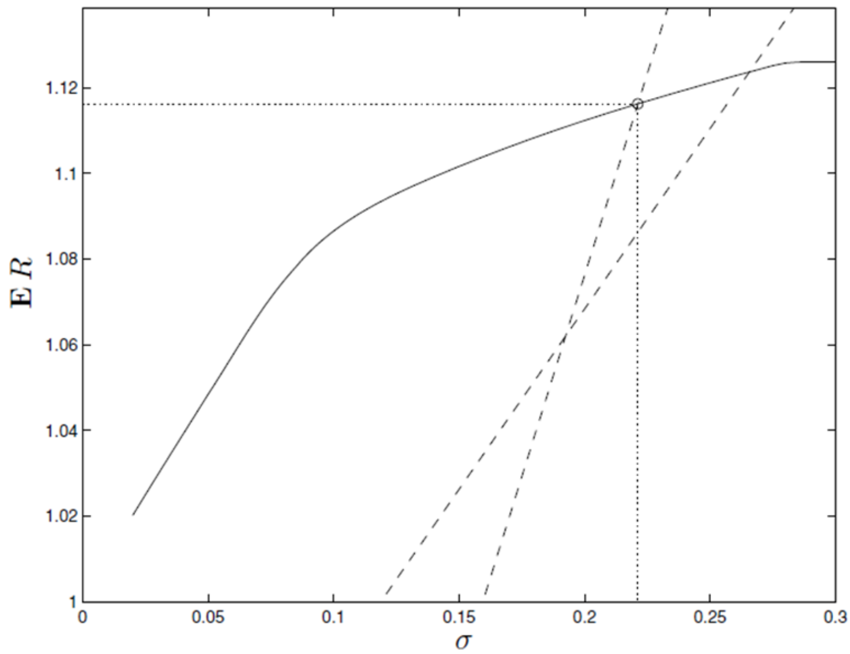
$$\mu - W^{\text{low}} \geq \Phi^{-1}(\eta)\, \sigma$$

- Using $\mu = \bar{\mathbf{a}}^T(\mathbf{w} + \mathbf{x})$ and $\sigma^2 = (\mathbf{w} + \mathbf{x})^T \boldsymbol{\Sigma} (\mathbf{w} + \mathbf{x})$

$$\Phi^{-1}(\eta)\, ||\boldsymbol{\Sigma}^{1/2}(\mathbf{w} + \mathbf{x})|| \leq \bar{\mathbf{a}}^T(\mathbf{w} + \mathbf{x}) - W^{\text{low}}$$



Lobo *et al.* (2007) (cont'd)
Shortfall Risk on M-V Space



Lobo *et al.* (2007) (cont'd)

Question in Assignment 1

$$\text{maximize} \quad \bar{\boldsymbol{a}}^T \left(\boldsymbol{w} + \boldsymbol{x}^+ - \boldsymbol{x}^-\right)$$

$$\text{subject to} \quad \mathbf{1}^T \left(\boldsymbol{x}^+ - \boldsymbol{x}^-\right) + \sum_{i=1}^{n} \left(\alpha_i^+ x_i^+ + \alpha_i^- x_i^-\right) \leq 0$$

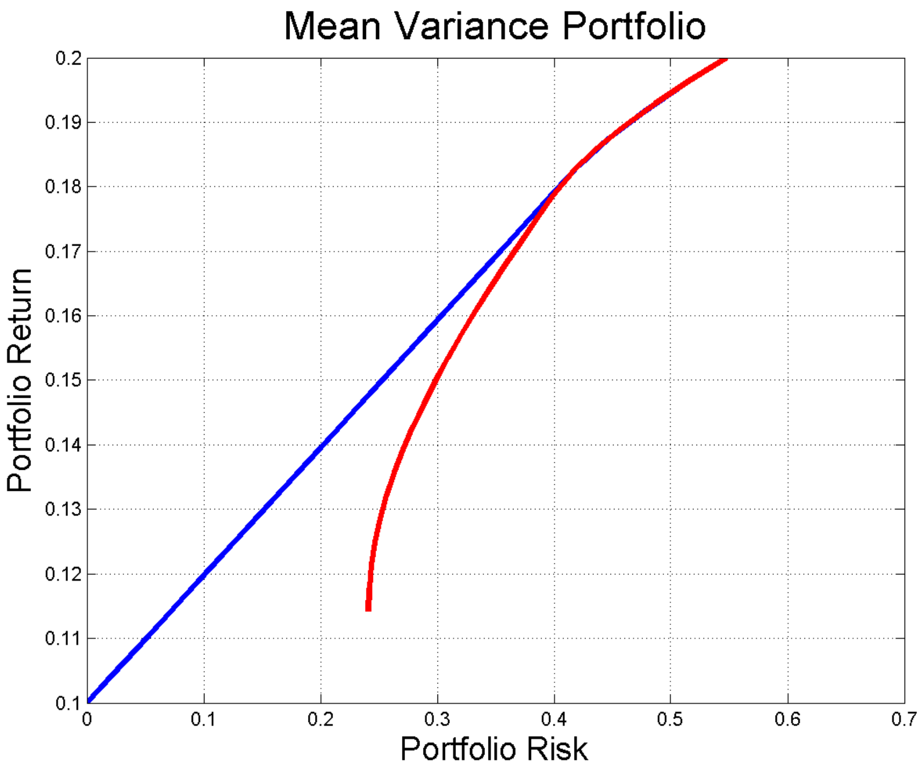
$$x_i^+ \geq 0, \; x_i^- \geq 0, \; i = 1, 2, ..., n$$

$$w_i + x_i^+ - x_i^- \geq s_i, \; i = 1, 2, ..., n$$

$$\Phi^-1(\eta_j) \, ||\boldsymbol{\Sigma}^{1/2} \left(\boldsymbol{w} + \boldsymbol{x}^+ - \boldsymbol{x}^-\right)|| \leq \bar{\boldsymbol{a}}^T \left(\boldsymbol{w} + \boldsymbol{x}^+ - \boldsymbol{x}^-\right) - W_j^{\text{low}}, j = 1, 2$$

Puzzle

```
m1 = [0.15 0.2 0.08 0.1]';
C1 = [ 0.2    0.05    -0.01    0.0
       0.05   0.30     0.015   0.0
      -0.01   0.015    0.10    0.0
       0.0    0.0      0.0     0.0
       ];

m2 = [0.15 0.2 0.08]';
C2 = [ 0.2    0.05    -0.01
       0.05   0.30     0.015
      -0.01   0.015    0.10
       ];

[V1, M1, PWts1] = NaiveMV(m1, C1, 25);
[V2, M2, PWts2] = NaiveMV(m2, C2, 25);

figure(2), clf,
plot(V1, M1, 'b', V2, M2, 'r', 'LineWidth', 3),
title('Mean Variance Portfolio', 'FontSize', 22)
xlabel('Portfolio Risk', 'FontSize',18)
ylabel('Portfolio Return', 'FontSize', 18);
```

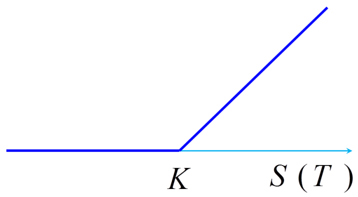


Derivatives Pricing

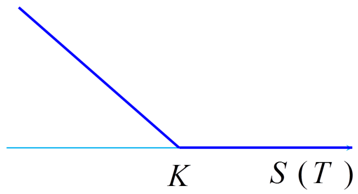
- Efficiency, no-arbitrage and fair price
- Example:
  - Price today $S(0)$
  - A and B enter into a *future* contract to sell/buy at price F at time T
  - A borrows $S(0)$ from the bank, buys the asset and waits till T
  - At time T, A owes the bank $S(0)\exp(rT)$ and has the asset to sell to B.
  - $F = S(0)\exp(rT)$, else arbitrage opportunity

Options

- Call: right to buy at price K at time T



- Put: right to sell at price K at time T



- Exercise of contract
  - European style: only at time T
  - American style: any time in $0 \rightarrow T$

- Example: Put-Call Parity
  - Portfolio P_1: European Call + cash $K \exp(-rT)$
  - Portfolio P_2: European Put + one share of underlying stock
  - Values at time $t = 0$

$$\begin{array}{ll} P_1 & C + K \exp -rT \\ P_2 & P + S(0) \end{array}$$

  - Value of portfolios at time $t = T$

$$\begin{array}{lll} S(T) > K & P_1 & [S(T) - K] + K = S(T) \\ & P_2 & 0 + S(T) = S(T) \\ S(T) < K & P_1 & 0 + K = K \\ & P_2 & [K - S(T)] + S(T) = K \end{array}$$

  - Both portfolios having the same value at time $t = T$ should also have the same value at $t = 0$.

$$C + K \exp(-rT) = P + S(0)$$

- Geometric Brownian motion for stock price

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S(t)dW(t) \\ \frac{dS(t)}{S(t)} &= \mu dt + \sigma dW(t) \end{aligned}$$

- Stochastic differential equation for the log of the process

$$F(S,t) = \log S(t)$$

- Ito's lemma tells us about increments dF
- Terms needed to apply Ito's lemma

$$\boxed{\begin{aligned} \frac{\partial F}{\partial t} &= 0 \\ \frac{\partial F}{\partial S} &= \frac{1}{S} \\ \frac{\partial^2 F}{\partial S^2} &= -\frac{1}{S^2} \end{aligned}}$$

$$\begin{aligned} dF &= \left(\frac{\partial F}{\partial t} + \mu S\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2}\right) dt + \sigma S \frac{\partial F}{\partial S} dW \\ &= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW \end{aligned}$$

$$\log S(t) = \log S(0) + \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma dW(t)$$

- $dW(t) = \epsilon\sqrt{t}$ where $\epsilon \sim \mathcal{N}(0,1)$

$$\log S(t) \sim \mathcal{N}\left[\log S(0) + \left(\mu - \frac{\sigma^2}{2}\right) t, \;\; \sigma^2 t\right]$$

- Log of asset price has a normal distribution
- Also

$$S(t) = S(0)\exp\left((\mu - \sigma^2/2)t + \sigma\sqrt{t}\epsilon\right)$$

Black-Scholes Model

- Model

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

- Change in option price

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}dt$$

- At maturity

$$f(S(T), T) = \max\{S(T) - K, 0\}$$

- Consider a portfolio
  - Own Δ stocks (long)
  - One call option sold

$$\Pi = \Delta\, S - f(S, t)$$

$$\begin{aligned} d\Pi &= \Delta dS - df \\ &= \left(\Delta - \frac{\partial f}{\partial S}\right) dS - \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right) dt \end{aligned}$$

- Term in dS (stochastic) can be eliminated by choosing Δ

$$\Delta = \frac{\partial f}{\partial S}$$

- With this choice of Δ (balance between short and long), the portfolio is riskless.
- $d\Pi = r\,\Pi\,dt$
- Eliminating $d\Pi$

$$\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right) dt = r\left(f - S\frac{\partial f}{\partial S}\right) dt$$

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = 0$$

- Partial differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = 0$$

- Boundary condition
  - European Call: $f(S, T) = \max\{S - K, 0\}$
  - European Put: $f(S, T) = \max\{K - S, 0\}$
- Black-Scholes

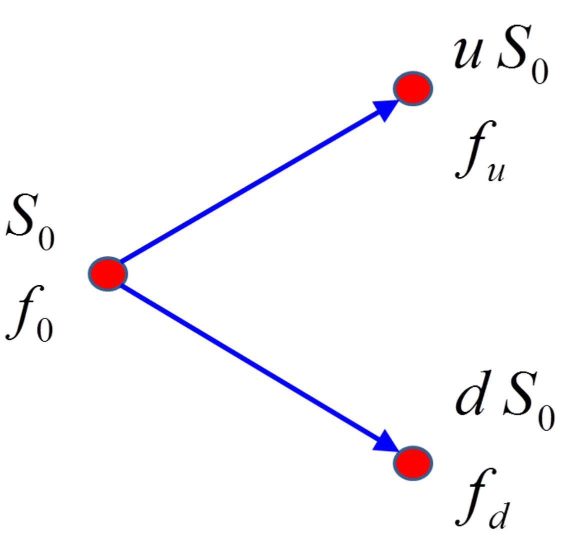
$$C = S_0 \mathcal{N}(d_1) - K \exp(-rT)\mathcal{N}(d_2)$$

$$\begin{aligned} d_1 &= \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\ d_2 &= \frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \\ &= d_1 - \sigma\sqrt{T} \\ \mathcal{N}(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp(-y^2/2)dy \end{aligned}$$

- Put-Call parity

$$P = K\exp(-rT)\mathcal{N}(-d_2) - S_0\mathcal{N}(-d_1)$$

Binomial Lattice



Options Pricing on a Binomial Model

- Construct a portfolio:
  - A riskless bond, initial price $B_0 = 1$ and future value $B_1 = \exp(r\delta t)$
  - Underlying asset, initial value S_0
  - Number of stocks Δ, number of bonds Ψ
- Initial value of this portfolio

$$\Pi_0 = \Delta S_0 + \Psi$$

- Future value depends on price movement up or down

$$\begin{cases} \Pi_u = \Delta S_0 u + \Psi \exp(r\delta t) \\ \Pi_d = \Delta S_0 d + \Psi \exp(r\delta t) \end{cases}$$

- We can solve for a portfolio that will replicate option payoff

$$\begin{aligned} \Delta S_0 u + \Psi \exp(r\delta t) &= f_u \\ \Delta S_0 d + \Psi \exp(r\delta t) &= f_d \end{aligned}$$

and solve for Δ and Ψ

- ... solving

$$\begin{aligned} \Delta &= \frac{f_u - f_d}{S_0(u-d)} \\ \Psi &= \exp(-r\delta t)\frac{uf_d - df_u}{u-d} \end{aligned}$$

- No arbitrage $\Longrightarrow$ initial value of this portfolio should be f_0

$$\begin{aligned} f_0 &= \Delta S_0 + \Psi \\ &= \frac{f_u - f_d}{(u-d)} + \exp(-r\delta t)\frac{uf_d - df_u}{u-d} \\ &= \exp(-r\delta t)\left\{\frac{--}{u-d} f_u + \frac{--}{u-d} f_d /\right\} \end{aligned}$$

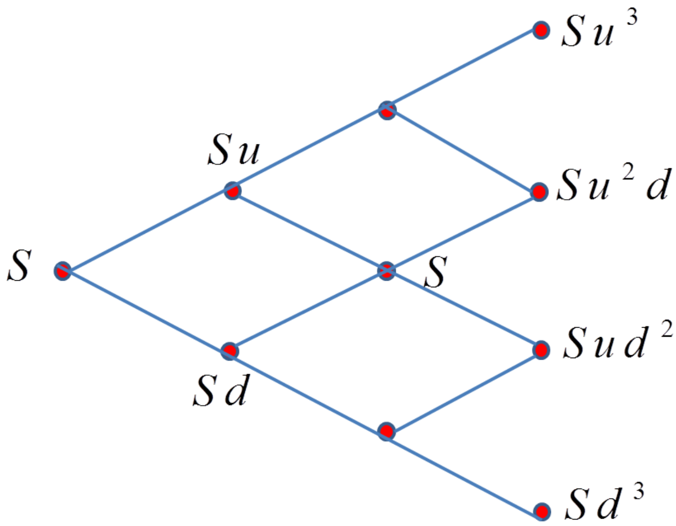
- Defining probabilities

$$\pi_u = \frac{\exp(r\delta t) - d}{u-d} \text{ and } \pi_d = \frac{u - \exp(r\delta t)}{u-d}$$

option price interpreted as discounted expected value

$$f_0 = \exp(-r\delta t)\left(\pi_u f_u + \pi_d f_d\right)$$

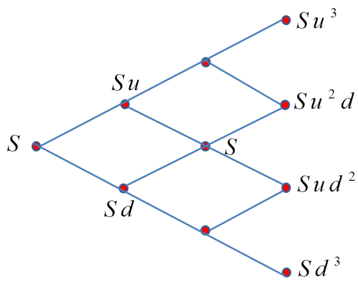
Binomial Lattice



Calibrating a Binomial Lattice

- When are these equivalent?

$$dS = r\, S\, dt + \sigma\, S\, dW$$



- Log normal distribution

$$\log\left(S_{t+\delta t}\right) \sim \mathcal{N}\left(\left(r - \sigma^2/2\right),\ \sigma^2\,\delta t\right)$$

- Mean and variance of log normal distribution

  (log of the variable is normal, what is mean and variance of the variable?)

$$\begin{aligned} E\left[\, S_{t+\delta t}\,\right] &= \exp(r\,\delta t) \\ \mathrm{Var}\left[\, S_{t+\delta t}\,\right] &= \exp(2r\delta t)\left(\exp(\sigma^2\delta t) - 1\right) \end{aligned}$$

Calibrating binomial lattice (cont'd)

- Mean for the lattice

$$E\left[S_{t+\delta t}\right] \;=\; p\,u\,S_t \;+\; (1-p)\,d\,S_t$$

- Equating the means...

$$puS_t \;+\; (1-p)dS_t \;=\; \exp(r\,\delta t)\,S_t$$

$$p \;=\; \frac{\exp(r\,\delta t) \;-\; d}{u \;-\; d}$$

- Variance on the lattice

$$\begin{aligned}\mathrm{Var}\left[S_{t+\delta t}\right] &= E\left[S_{t+\delta t}^2\right] \;-\; E^2\left[S_{t+\delta t}\right] \\ &= S_t^2\left(pu^2 + (1-p)d^2\right) - S_t^2\exp(2r\delta t)\end{aligned}$$

... which from the dynamical model is...

$$\mathrm{Var}\left[S_{t+\delta t}\right] \;=\; S_t^2\exp(2r\delta t)\left(\exp(\sigma^2\delta t) - 1\right)$$

(cont'd)

- Equating the two variances

$$S_t^2\exp(2r\delta t)\left(\exp(\sigma^2\delta t) - 1\right) \;=\; S_t^2\left(pu^2 + (1-p)d^2\right) - S_t^2\exp(2r\delta t)$$

Which reduces to

$$\exp(2r\delta t + \sigma^2\delta t) \;=\; pu^2 \;+\; (1-p)d^2$$

Substitute for p and simplify

$$\exp(2r\delta t + \sigma^2\delta t) \;=\; (u+d)\exp(r\delta t) \;-\; 1$$

...and because $u = 1/d$,

$$u^2\exp(r\delta t) \;-\; u\;\left(1 + \exp(2r\delta t + \sigma^2\delta t)\right) \;+\; \exp(r\delta t) = 0$$

... a quadratic equation in u.

$$u = \frac{(1+\exp(2r\delta t + \sigma^2\delta t)) \; + \; \sqrt{(1+\exp(2r\delta t+\sigma^2\delta t)^2 - 4\exp(2r\delta t)}}{2\exp(r\delta t)}$$

Taylor series expansion of $\exp(x)$

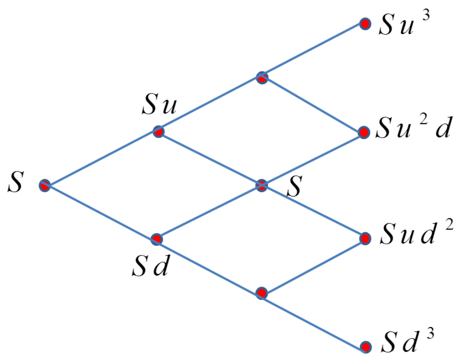
$$\left(1+\exp(2r\delta t + \sigma^2\delta t)\right)^2 - 4\exp(2r\delta t) \;\approx\; (2+(2r+\sigma^2)\delta t)^2 - 4(1+2r\delta t) \;\approx\; 4\sigma^2\delta t$$

$$\begin{aligned}
u &\approx \frac{2+(2r+\sigma^2)\delta t \; + \; 2\sigma\sqrt{\delta t}}{2\exp(r\delta t)} \\
&\approx \left(1 + r\delta t + \frac{\sigma^2}{2}\delta t + \sigma\sqrt{\delta t}\right)(1 - r\delta t) \\
&\approx 1 + r\delta t + \frac{\sigma^2}{2}\delta t + \sigma\sqrt{\delta t} - r\delta t \\
&= 1 + \sigma\sqrt{\delta t} + \frac{\sigma^2}{2}\delta t
\end{aligned}$$

Calibrating the Binomial Lattice (cont'd)

$$\begin{aligned}
u &= \exp(\sigma\sqrt{\delta t}) \\
d &= \exp(-\sigma\sqrt{\delta t}) \\
p &= \frac{\exp(r\delta t) - d}{u - d}
\end{aligned}$$

$$dS = r\,S\,dt + \sigma\,S\,dW$$



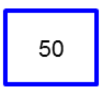
Example

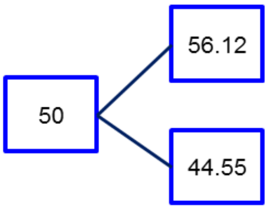
- European call option; $S_0 = K = 50$; $r = 0.1$; $\sigma = 0.4$; maturity in five months.

```
>> call = blsprice(50, 50, 0.1, 5/12, 0.4)
call =
      6.1165
```

We can now build the lattice

δt	$1/12$	0.0833
u	$\exp(\sigma\sqrt{t})$	1.1224
d	$1\,/\,u$	0.8909
p	$(exp(r\delta t) - d)\,/\,(u - d)$	0.5073

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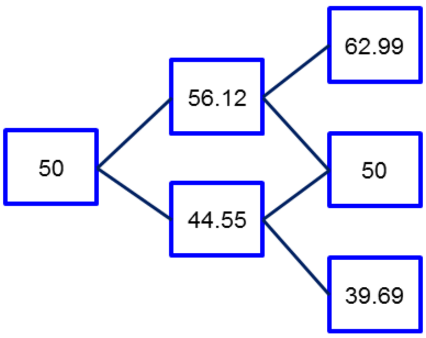
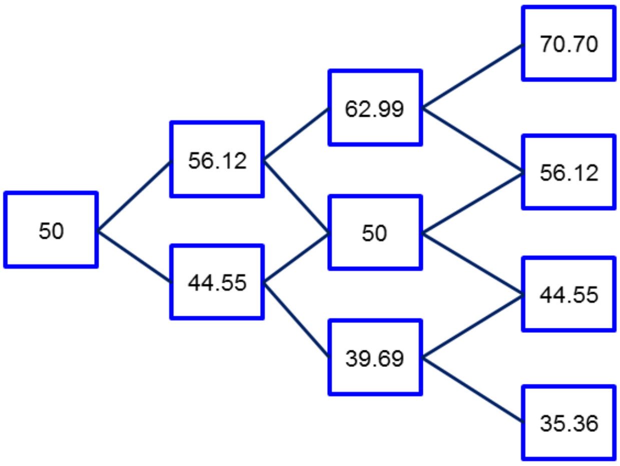
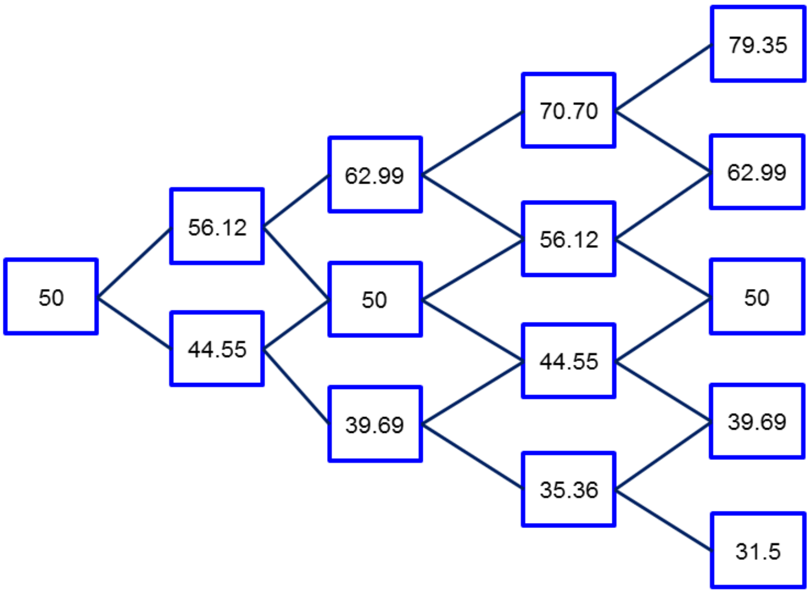
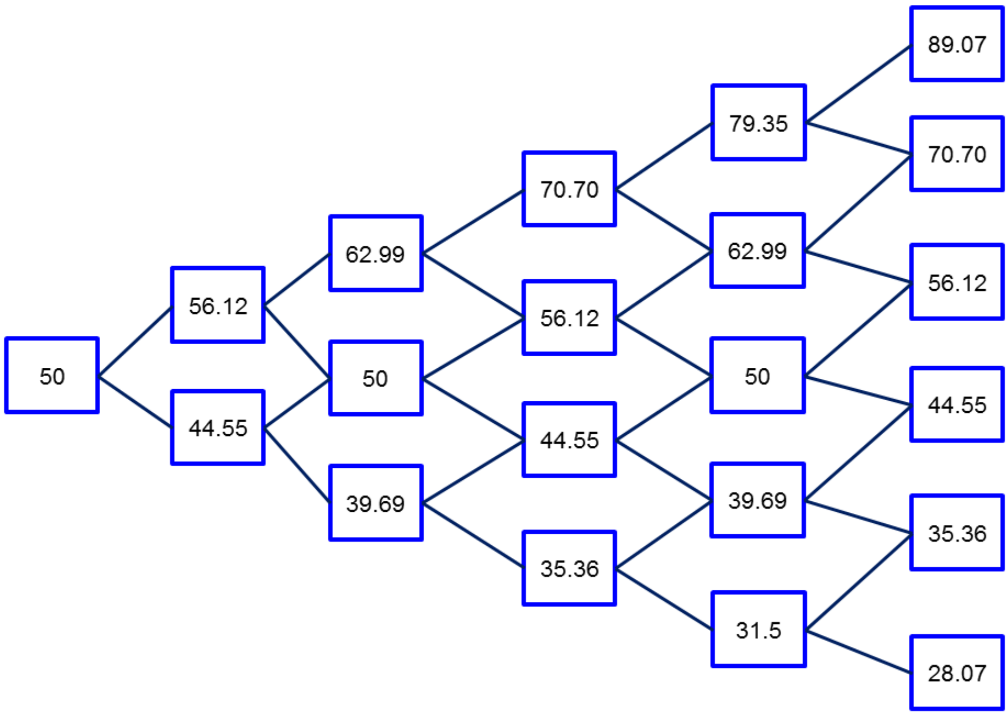
50
44.55


62.99
56.12
50
50
44.55
39.69


50
56.12
44.55
62.99
50
39.69
70.70
56.12
44.55
35.36


50
56.12
44.55
62.99
50
39.69
70.70
56.12
44.55
35.36
79.35
62.99
50
39.69
31.5


50
56.12
44.55
62.99
50
39.69
70.70
56.12
44.55
35.36
79.35
62.99
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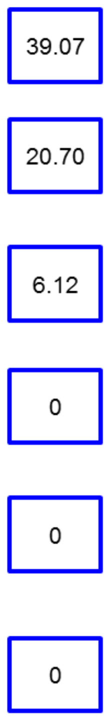
44.55
35.36
28.07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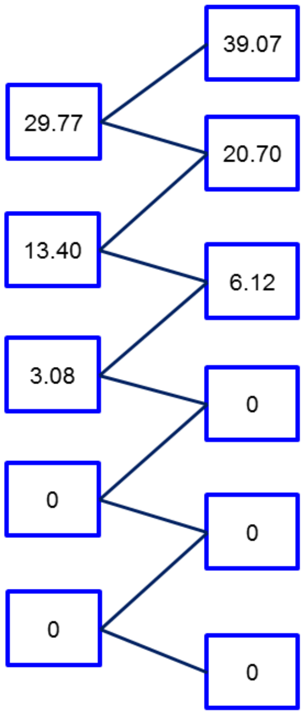
20.70
6.12
0
0
0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6.12
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0
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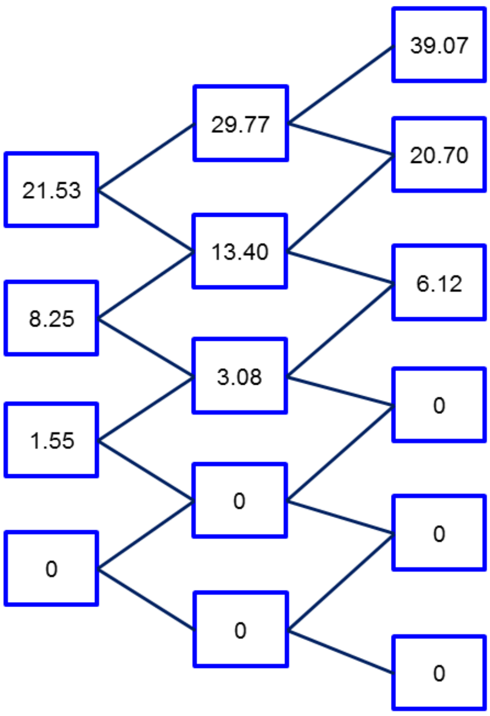
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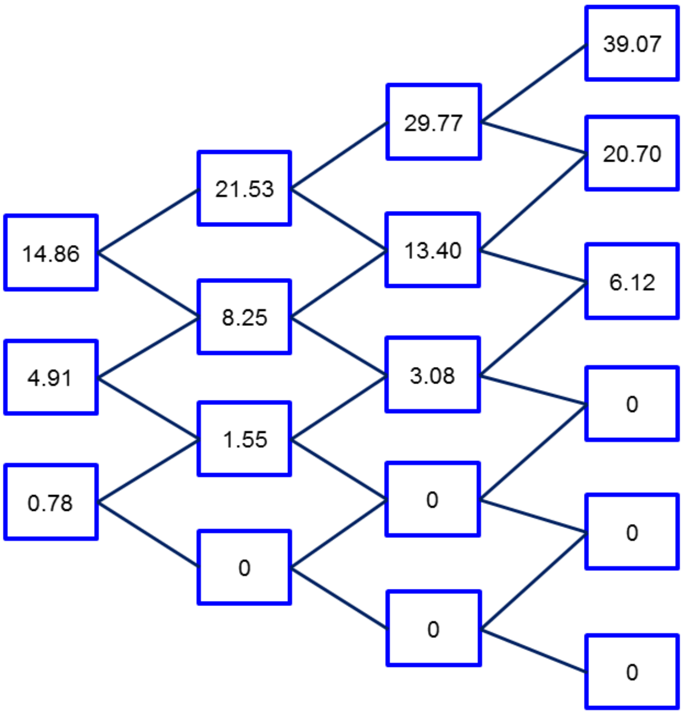
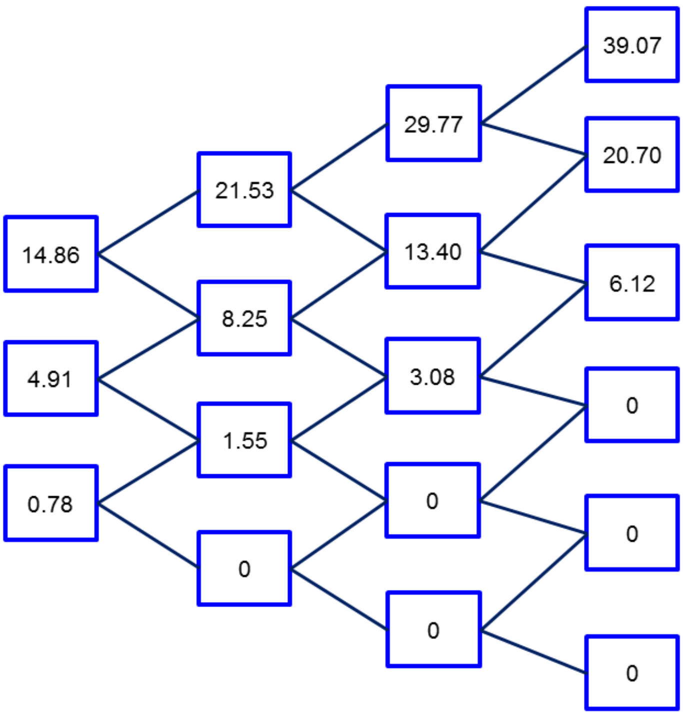
6.12
3.08
1.55
0
0
0
0
0
0





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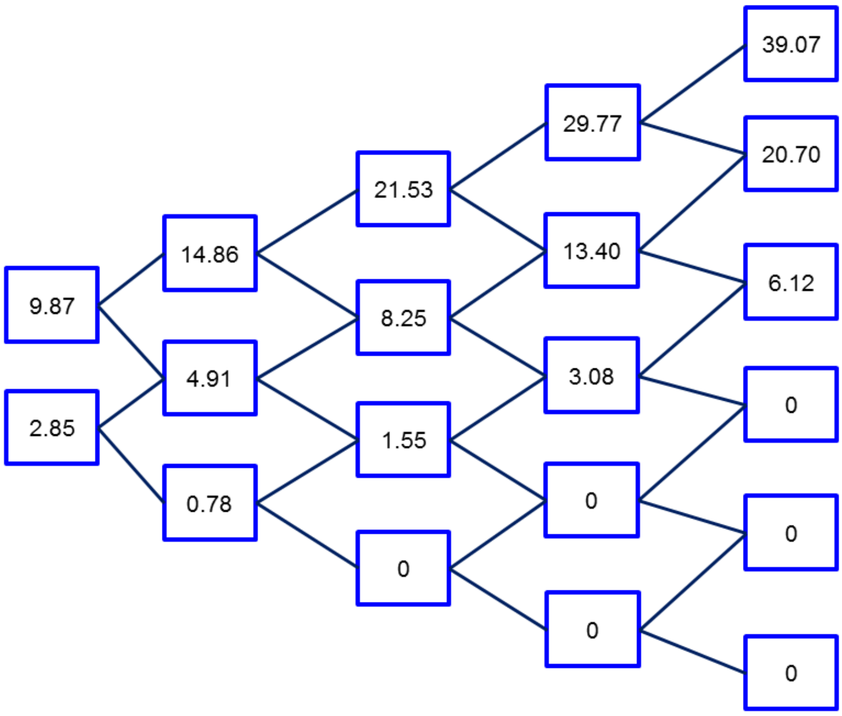
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29.77
20.70
21.53
13.40
14.86
6.12
9.87
8.25
3.08
4.91
0
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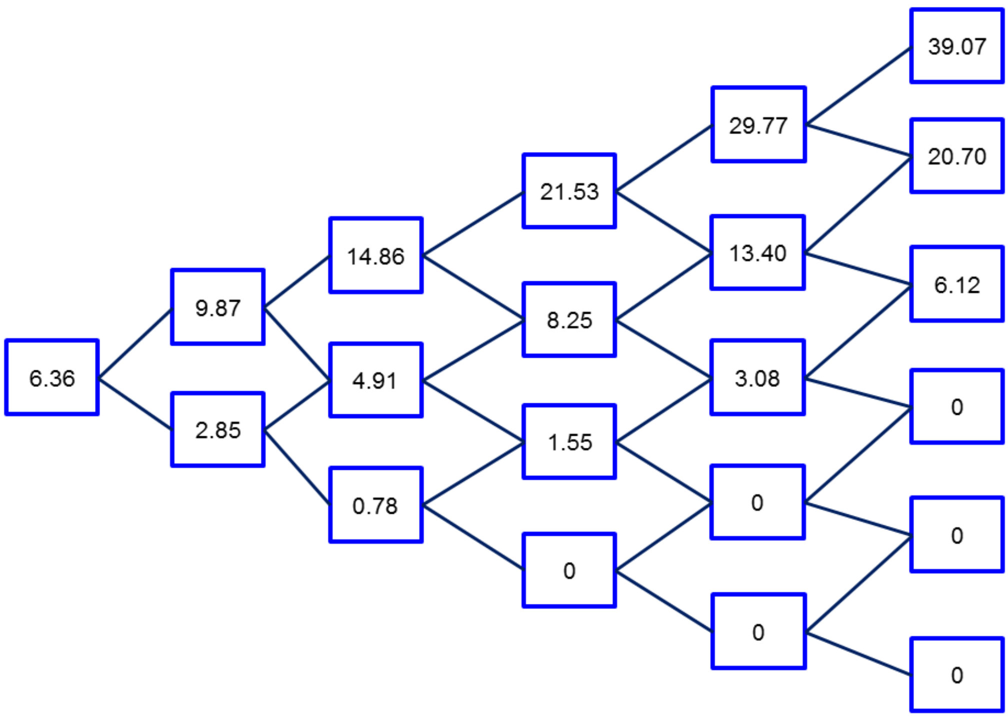
0
0
0
0
0


39.07
29.77
20.70
21.53
13.40
14.86
6.12
9.87
8.25
6.36
3.08
4.91
0
2.85
1.55
0.78
0
0
0
0
0

Pricing European Call Option by Binomial Lattice

```
function [price, lattice] = LatticeEurCall(S0,K,r,T,sigma,N)

deltaT = T/N;
u=exp(sigma * sqrt(deltaT));
d=1/u;
p=(exp(r*deltaT) - d)/(u-d);
lattice = zeros(N+1,N+1);

for i=0:N
   lattice(i+1,N+1)=max(0 , S0*(u^i)*(d^(N-i)) - K);
end

for j=N-1:-1:0
   for i=0:j
      lattice(i+1,j+1) = exp(-r*deltaT) * ...
         (p * lattice(i+2,j+2) + (1-p) * lattice(i+1,j+2));
   end
end

price = lattice(1,1);
```


Pricing American Style Put Option

```
function price = AmPutLattice(S0,K,r,T,sigma,N)
deltaT = T/N;
u=exp(sigma * sqrt(deltaT));
d=1/u;
p=(exp(r*deltaT) - d)/(u-d);
discount = exp(-r*deltaT);
p_u = discount*p;
p_d = discount*(1-p);
SVals = zeros(2*N+1,1);
SVals(N+1) = S0;

[...]
```

Pricing American Style Put Option (cont'd)

```
function price = AmPutLattice(S0,K,r,T,sigma,N)



[...]

for i=1:N
    SVals(N+1+i) = u*SVals(N+i);
    SVals(N+1-i) = d*SVals(N+2-i);
end
PVals = zeros(2*N+1,1);
for i=1:2:2*N+1
    PVals(i) = max(K-SVals(i),0);
end

[...]
```


Pricing American Style Put Option (cont'd

```
function price = AmPutLattice(S0,K,r,T,sigma,N)



[...]
for tau=1:N
    for i= (tau+1):2:(2*N+1-tau)
        hold = p_u*PVals(i+1) + p_d*PVals(i-1);
        PVals(i) = max(hold, K-SVals(i));
    end
end
price = PVals(N+1);
```

- Decisions at every point during backtracking

$$f_{i,j} \;=\; \max\left\{K - S_{i,j},\;\; \exp(-r\delta t)\left(p\, f_{i+1,j+1} \;+\; (1-p)\, f_{i,j+1}\right)\right\}$$

- We will look at inference as expectations...

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x)\, dx$$

- Consider the integral

$$I = \int_0^1 g(x)\, dx$$

- Think of this as computing the expected value

(of a function of a uniform random variable):

$$E[g(U)], \quad \text{where} \quad U \sim (0,1)$$

- We approximate the integral by

$$\widehat{I}_m = \frac{1}{m} \sum_{i=1}^{m} g(U_i)$$

- Where will we use this?
- European call option

$$f = \exp(-rT)\, E[f_T]$$

- f_T is payoff at maturity T; fair price is discounted expected payoff
- $f_T = \max\left\{0, \quad S(0)\exp((r - \sigma^2/2)T + \sigma\sqrt{T}\epsilon) - K\right\}$

```
% BlsMC1.m
function Price = BlsMC1(S0,K,r,T,sigma,NRepl)
nuT = (r - 0.5*sigma^2)*T;
siT = sigma * sqrt(T);
DiscPayoff = exp(-r*T)*max(0, S0*exp(nuT+siT*randn(NRepl,1))-K);
Price = mean(DiscPayoff);
```

```
> S0=50; K=60; r=0.05; T=1; sigma=0.2;
> randn('state', 0);
> BlsMC1(S0, K, r, T, sigma, 1000)
ans =
   1.2562
```

Is this a good approach?

- Different answers on different runs

```
> S0=50; K=60; r=0.05; T=1; sigma=0.2;
> randn('state', 0);
> BlsMC1(S0, K, r, T, sigma, 1000)
ans =
   1.2562
> BlsMC1(S0, K, r, T, sigma, 1000)
ans =
   1.8783
> BlsMC1(S0, K, r, T, sigma, 1000)
ans =
   1.7864
```

- What if we had large number of samples?

```
> BlsMC1(S0, K, r, T, sigma, 1000000)
ans =
   1.6295
> BlsMC1(S0, K, r, T, sigma, 1000000)
ans =
   1.6164
> BlsMC1(S0, K, r, T, sigma, 1000000)
ans =
   1.6141
```


Sampling: Inverse Transform

- Sample X from $f(x)$; Cumulative distribution $F(x)$

  - Draw $U \sim U(0.1)$
  - Return $X = F^{-1}(U)$

$$
\begin{aligned}
P\{X \le x\} &= P\{F^{-1}(U) \le x\} \\
&= P\{U \le F(x)\} \\
&= F(x)
\end{aligned}
$$

- Example: Exponential distribution $X \sim \exp(\mu)$
- Cumulative

$$F(x) = 1 - \exp(-\mu x)$$

- Inverse

$$x = -\frac{1}{\mu}\log(1 - U)$$

- Distributions of U and $(1 - U)$ are the same
  Hence return: $-\log(U)/\mu$

Sampling: Acceptance-Rejection Method

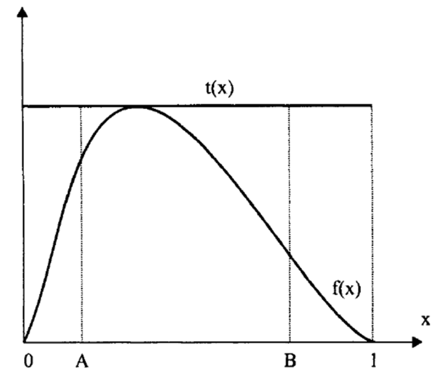
- Probability density function: $f(x)$
- Consider a known function $t(x)$, such that

$$t(x) \geq f(x), \quad \forall x \in \mathcal{I}$$

- $\mathcal{I}$ is the support for f (region in which it is defined)
- $t(x)$ is a probability density of normalized

$$r(x) = t(x)/c \qquad c = \int_{\mathcal{I}} t(x)\, dx$$

1. Generate $Y \sim r$
2. Generate $U \sim U(0,1)$
3. If $U \leq f(Y)/t(Y)$ return $X = Y$
   Else go to 1



Homework
Page 235, Brandimarte

- $f(x) = 30(x^2 - 2x^3 + x^4), \qquad x \in [0,1]$
- Algorithm
  1. Draw U_1 and U_2
  2. If $U2 \leq 16(U_1^2 - 2U_1^3 + U_1^4)$
     accept $X = U_1$
     Else
     go to 1
- Exercise:
  - Draw the graph of $f(x)$
  - Simulate 1000 samples using above algorithm
  - Draw a histogram to the same scale as $f(x)$ – do they match? Is it better with 100000 samples?
  - On average, how many trials were needed through the accept-reject loop for each sample?

Variance Reduction

- Independent samples X_i
- Sample mean (estimates mean $\mu = E[X_i]$ from n samples)

$$\overline{X}(n) = \frac{1}{n}\sum_{i=1}^{n} X_i$$

- Sample variance

$$S^2(n) = \frac{1}{(n-1)}\sum_{i=1}^{n}\left[X_i - \overline{X}(n)\right]^2$$

- Error of the estimator

$$\begin{aligned} E\left[(\overline{X}(n)-\mu)^2\right] &= \mathrm{Var}\left[\overline{X}(n)\right] \\ &= \mathrm{Var}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \\ &= \frac{1}{n^2} \times n \times \sigma^2 = \frac{\sigma^2}{n} \end{aligned}$$

- Two points:
  - More samples n reduces the variance in estimation
  - Variance reduction schemes can control σ^2

Variance reduction: Antithetic Sampling

- Pair of sequences

$$\left\{ \begin{array}{cccc} X_1^{(1)} & X_1^{(2)} & \ldots & X_1^{n} \\ X_2^{(1)} & X_2^{(2)} & \ldots & X_2^{n} \end{array} \right\}$$

- Columns (horizontally) are independent
- $X_1^{(i)}$ and $X_2^{(i)}$ are dependent.
- Sample is a function of each pair: $X^{(i)} = \left(X_1^{(i)} + X_2^{(i)}\right)/2$
- Variance

$$\begin{aligned} \mathrm{Var}\left[\overline{X}(n)\right] &= \frac{1}{n}\mathrm{Var}\left[X^{(i)}\right] \\ &= \frac{1}{4n}\left\{\mathrm{Var}(X_1^{(i)}) + \mathrm{Var}(X_2^{(i)}) + 2\,\mathrm{Cov}(X_1^{(i)}, X_2^{(i)})\right\} \\ &= \frac{1}{2n}\mathrm{Var}(X)\,(1+\rho) \end{aligned}$$

- Uniform random number $\{U_k\}$ and $\{1-U_k\}$ as sequences.

Application

```
function [Price, CI] = BlsMC2(S0,K,r,T,sigma,NRepl)
nuT = (r - 0.5*sigma^2)*T;
siT = sigma * sqrt(T);
DiscPayoff = exp(-r*T)*max(0, S0*exp(nuT+siT*randn(NRepl,1))-K);
[Price, VarPrice, CI] = normfit(DiscPayoff);
```

```
function [Price, CI] = BlsMCAV(S0,K,r,T,sigma,NPairs)
nuT = (r - 0.5*sigma^2)*T;
siT = sigma * sqrt(T);
Veps = randn(NPairs,1);
Payoff1 = max( 0 , S0*exp(nuT+siT*Veps) - K);
Payoff2 = max( 0 , S0*exp(nuT+siT*(-Veps)) - K);
DiscPayoff = exp(-r*T) * 0.5 * (Payoff1+Payoff2);
[Price, VarPrice, CI] = normfit(DiscPayoff);
```


Homework

Test the two functions: BlsMC and BlsMCAV

(Brandimarte, p248)

```
> randn('state', 0)
> [Price, CI] = BlsMC2(50,50,0.05,1,0.4,200000)
Price=
          9.0843
CI =
          9.0154
          9.1532
\pause
> (CI(2)-CI(1))/Price
ans =
        0.0152
\pause
> randn('state', 0)
> [Price, CI] = BlsMCAV(50,50,0.05,1,0.4,200000)
Price=
          9.0553
CI =
           8.9987
           9.1118
\pause
> (CI(2)-CI(1))/Price
ans =
        0.0125
```

Approximating option prices with a neural network

- We have seen three tools for pricing options
  - Closed form Black-Scholes
  - Binomial lattice
  - Monte Carlo

- How well can the relationship between asset price and option price be approximated?

Hutchinson *et al.* (1994) "A nonparametric approach to pricing and hedging derivative securities via learning networks", *Journal of Finance* **49**(3): 851

$$\boldsymbol{x} = [\, S/X \quad (T-t)\,]^T$$

$$c = \sum_{j=1}^{J} \lambda_j \, \phi_j(\boldsymbol{x}) + \boldsymbol{w}^T \boldsymbol{x} + w_0$$

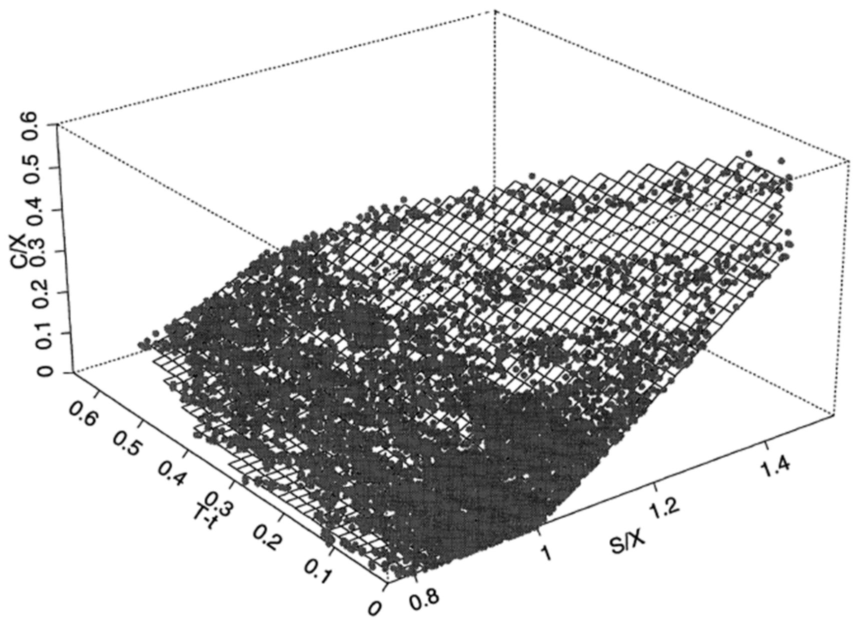


Figure 4. Simulated call option prices normalized by strike price and plotted versus

$$
\begin{aligned}
\widehat{C/X} = {} & -0.06\sqrt{\begin{bmatrix} S/X - 1.35 \\ T - t - 0.45 \end{bmatrix}' \begin{bmatrix} 59.79 & -0.03 \\ -0.03 & 10.24 \end{bmatrix} \begin{bmatrix} S/X - 1.35 \\ T - t - 0.45 \end{bmatrix} + 2.55} \\
& - 0.03\sqrt{\begin{bmatrix} S/X - 1.18 \\ T - t - 0.24 \end{bmatrix}' \begin{bmatrix} 59.79 & -0.03 \\ -0.03 & 10.24 \end{bmatrix} \begin{bmatrix} S/X - 1.18 \\ T - t - 0.24 \end{bmatrix} + 1.97} \\
& + 0.03\sqrt{\begin{bmatrix} S/X - 0.98 \\ T - t + 0.20 \end{bmatrix}' \begin{bmatrix} 59.79 & -0.03 \\ -0.03 & 10.24 \end{bmatrix} \begin{bmatrix} S/X - 0.98 \\ T - t + 0.20 \end{bmatrix} + 0.00} \\
& + 0.10\sqrt{\begin{bmatrix} S/X - 1.05 \\ T - t + 0.10 \end{bmatrix}' \begin{bmatrix} 59.79 & -0.03 \\ -0.03 & 10.24 \end{bmatrix} \begin{bmatrix} S/X - 1.05 \\ T - t + 0.10 \end{bmatrix} + 1.62} \\
& + 0.14S/X - 0.24(T - t) - 0.01. \qquad (9)
\end{aligned}
$$