

COMP6212: Computational Finance

Mahesan Niranjan

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This module is in two parts:

- Part I** Financial data analysis, taught by Prof. M. Niranjan
- Part II** Crypto-currencies and blockchain technology, taught by Dr Jie Zhang

Spring Semester 2017/2018

Financial Equilibrium

Caution: A peculiar and rather personal view



jamesnichollsillustration.blogspot.co.uk

Financial Markets



www.investors411.com

- Generate products and services
- In need of
 - stability against fluctuations (e.g. demand, exchange rate)
 - capital investment (e.g. to modernise, grow)
- Process wealth & capital
- Driven by gambling instinct and greed

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- What are the sources of computational problems?
 - Time - present value of money.
 - Uncertainty - of the future.

Topics in Part I: Financial Data Analysis

- Portfolio Optimization
- Derivatives Pricing

Overview of the Module

Topics in Part I: Financial Data Analysis

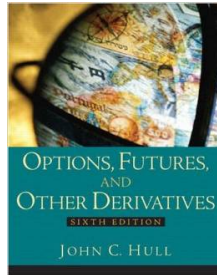
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Keywords:

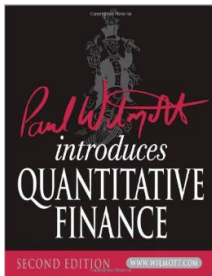
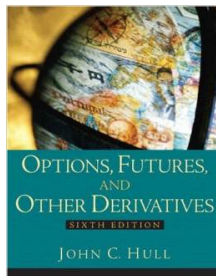
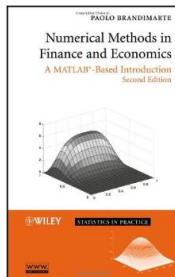
Mean-Variance optimization, Linear and quadratic programming, Multivariate Gaussian distribution, Constrained optimization, Value at risk and Conditional value at risk, Sharpe ratio, Present value, Stochastic differential equations, Ito's Lemma, Black-Scholes model, Options pricing, Stochastic Simulations and Monte Carlo methods.



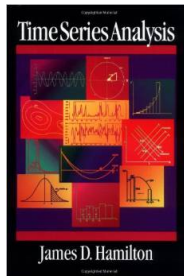
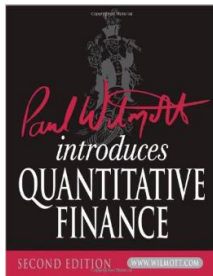
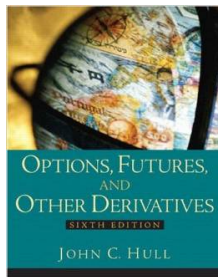
Resources



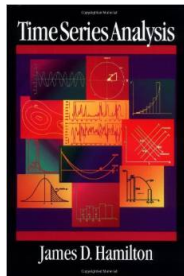
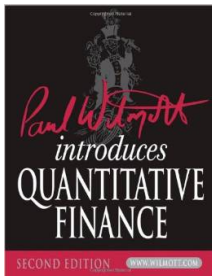
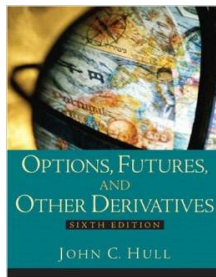
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- plus several academic papers.

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 - Own a small *share* of a company; the ownership may be traded in the market; owning the share might earn *dividends*.
- Derivatives
 - Contracts written on the basis of a future value of a a stock, currency etc. Usually there is a time of *maturity* and a promised *payoff* in the contract. Variations in style of *exercising* the contract.

Time: Present Value

- Wealth W_0 deposit in bank and get W_1 after one year
- $W_1 = (1 + r) W_0$, r interest rate
- Compound interest over n years: $W_n = (1 + r)^n W_0$

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- Present value of your promise to give me cash C in time t is

$$\exp(-rt) C$$

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 - Convex optimization using CVX toolbox

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- Brownian motion, Geometric Brownian motion

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- Black-Scholes: options pricing under specific assumptions
- Monte Carlo / Stochastic simulations: general cases

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- Returns on the portfolio has a multivariate Gaussian distribution

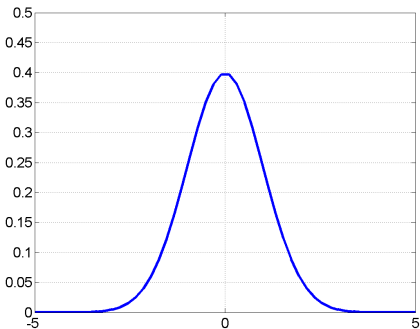
$$\mathbf{r} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{pmatrix} \quad \boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{pmatrix} \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12}^2 & \dots & \sigma_{1N}^2 \\ \sigma_{12}^2 & \sigma_2^2 & \dots & \sigma_{2N}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1N}^2 & \sigma_{2N}^2 & \dots & \sigma_N^2 \end{pmatrix}$$

Gaussian Density

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-m)^2}{2\sigma^2}\right\}$$

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```
x = linspace(-5,5,50);  
m = 0;  
s = 1;  
y = normpdf(x,m,s);  
figure(1), clf  
plot(x,y,'LineWidth',3);  
  
grid on
```

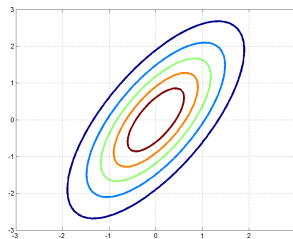
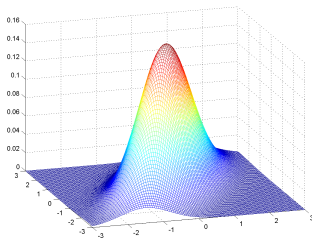
Multivariate Gaussian Distribution

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} \exp \left\{ -\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\}$$

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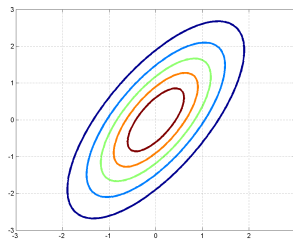
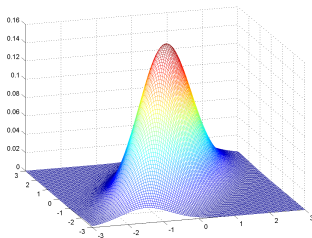
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- How do these shapes change with $\boldsymbol{\mu}$ and Σ ?

- Linear transform of multivariate Gaussian

$$\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}); \mathbf{y} = \mathbf{A} \mathbf{x} \implies \mathbf{y} \sim \mathcal{N}(\mathbf{A} \boldsymbol{\mu}, \mathbf{A} \boldsymbol{\Sigma} \mathbf{A}^t)$$

Portfolio Return

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- When $\boldsymbol{\pi}$ changes, M and V change — how?

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- Other constraints

possible: $\sum_{i=1}^N \pi_i = 1, \pi_i \geq 0, \alpha \leq \pi_i \leq \beta$

Estimation

- We estimate μ and Σ from historic data and apply optimization to allocate assets.
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- Estimation:

$$\begin{aligned}\hat{\mu} &= \frac{1}{T} \sum_{t=1}^T \mathbf{r}(t) \\ \hat{\Sigma} &= \frac{1}{T} \sum_{t=1}^T (\mathbf{r}(t) - \hat{\mu}) (\mathbf{r}(t) - \hat{\mu})^T\end{aligned}$$

Solving quadratic program in MATLAB

$$\min_{\mathbf{x}} \mathbf{x}^T \mathbf{H} \mathbf{x} + \mathbf{f}^T \mathbf{x} \text{ such that } \begin{cases} \mathbf{A} \mathbf{x} \leq \mathbf{b} \\ \mathbf{A}_{eq} \mathbf{x} = \mathbf{b}_{eq} \\ \mathbf{lb} \leq \mathbf{x} \leq \mathbf{ub}. \end{cases}$$

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pi = quadprog(Sigma, [], [], [], mu', rMax, 0, 1, [])
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Map the problem variables to the function in the tool.

Efficient Frontier

- Given μ and Σ
- What portfolio has the highest return, unconstrained by risk?

$$\max \pi^T \mu \quad \text{subject to} \quad \sum_{i=1}^N \pi_i = 1, \quad \text{and} \quad \pi_i \geq 0$$

- Given μ and Σ
- What portfolio has the highest return, unconstrained by risk?

$$\max \pi^T \mu \quad \text{subject to} \quad \sum_{i=1}^N \pi_i = 1, \quad \text{and} \quad \pi_i \geq 0$$

Linear Programming:

$$\min_x \mathbf{f}^T \mathbf{x} \quad \text{such that} \quad \begin{cases} \mathbf{A} \mathbf{x} \leq \mathbf{b} \\ \mathbf{A}_{eq} \mathbf{x} = \mathbf{b}_{eq} \\ lb \leq \mathbf{x} \leq ub \end{cases}$$

```
x = linprog( f, A, b, Aeq, beq, lb, ub )  
w1 = linprog(-mu, [], [], ones(1,N), 1, 0, 0);  
r1 = w1 * mu;
```


Efficient Frontier (cont'd)

- What portfolio has lowest variance (unconstrained by expectation)?

$$\min \pi^T \Sigma \pi \quad \text{subject to} \quad \sum_{i=1}^N \pi_i = 1$$

Efficient Frontier (cont'd)

- What portfolio has lowest variance (unconstrained by expectation)?

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```
w2=quadprog(mu,zeros(N,1),[],[],ones(1,N),1,zeros(N,1),[],[]);  
r2 = w2' * mu;
```

- Portfolios on the efficient frontier will have returns in range r1 to r2

Efficient Frontier (cont'd)

- What portfolio has lowest variance (unconstrained by expectation)?

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- Now, select a number of points (returns we need), and find the minimum variance portfolios that will offer these returns!

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```
M = linspace(r1, r2, p)  
for j=1:p  
    ret = M(j);  
    w = quadprog(...);  
    V(j) = w' * Sigma * w;  
end
```

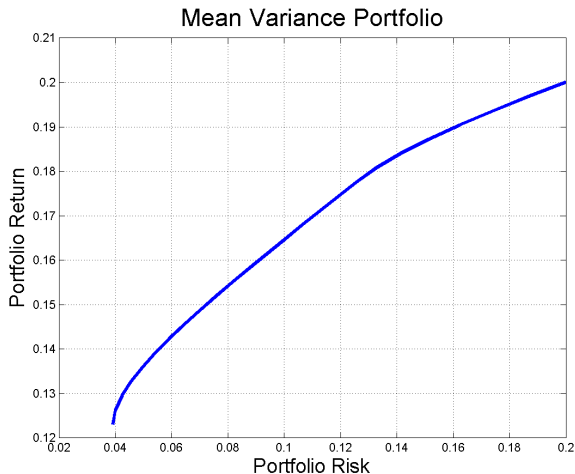
Complete Function to Compute the Efficient Frontier

```
function [PRisk, PRoR, PWts] = NaiveMV(ERet, ECov, NPts)
ERet = ERet(:);          % makes sure it is a column vector
NAssets = length(ERet); % get number of assets
% vector of lower bounds on weights
V0 = zeros(NAssets, 1);
% row vector of ones
V1 = ones(1, NAssets);
% set medium scale option
options = optimset('LargeScale', 'off');
% Find the maximum expected return
MaxReturnWeights = linprog(-ERet, [], [], V1, 1, V0);
MaxReturn = MaxReturnWeights' * ERet;
% Find the minimum variance return
MinVarWeights = quadprog(ECov, V0, [], [], V1, 1, V0, [], [], options);
MinVarReturn = MinVarWeights' * ERet;
MinVarStd = sqrt(MinVarWeights' * ECov * MinVarWeights);
% check if there is only one efficient portfolio
if MaxReturn > MinVarReturn
    RTarget = linspace(MinVarReturn, MaxReturn, NPts);
    NumFrontPoints = NPts;
else
    RTarget = MaxReturn;
    NumFrontPoints = 1;
end
```

Complete Function (cont'd)

```
...
% Store first portfolio
PRoR = zeros(NumFrontPoints, 1);
PRisk = zeros(NumFrontPoints, 1);
PWts = zeros(NumFrontPoints, NAssets);
PRoR(1) = MinVarReturn;
PRisk(1) = MinVarStd;
PWts(1,:) = MinVarWeights(:)';
% trace frontier by changing target return
VConstr = ERet';
A = [V1 ; VConstr ];
B = [1 ; 0];
for point = 2:NumFrontPoints
    B(2) = RTarget(point);
    Weights = quadprog(ECov,V0,[],[],A,B,V0,[],[],options);
    PRoR(point) = dot(Weights, ERet);
    PRisk(point) = sqrt(Weights'*ECov*Weights);
    PWts(point, :) = Weights(:)';
end
```

Summary: what have we achieved?



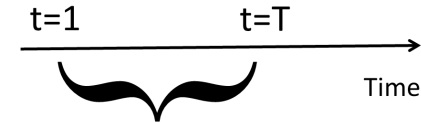
Homework

- Three assets with the following properties:

$$m = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.15 \end{bmatrix}; \quad C = 100 * \begin{bmatrix} 0.005 & -0.010 & 0.004 \\ -0.010 & 0.040 & -0.002 \\ 0.004 & -0.002 & 0.023 \end{bmatrix};$$

- Study the code of function `NaiveMV` and draw the efficient frontier.
- Use the function `frontcon` in MATLAB and draw the efficient frontier.

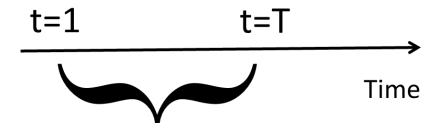
Estimation of Parameters



Analysis window
to estimate mean and covariance

- Estimate parameters μ and C from data within a window

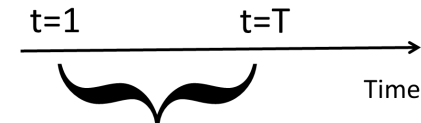
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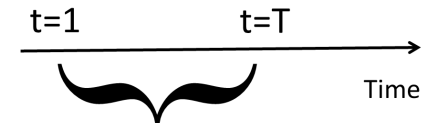
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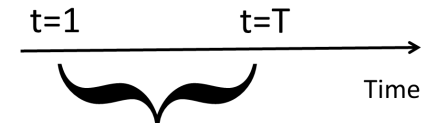
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- Shrinkage in covariance estimates

Advances on the Mean-Variance Portfolio

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- Optimizing the execution of trade

TBC

Portfolio Performance

- Sharpe Ratio: mean to standard deviation of portfolio return

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r “risk free” interest rate

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- cVAR: Conditional Value at Risk (later)

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 - Combination of portfolios (model averaging / mixing)
- No method *consistently* beats the naive strategy!

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- Brodie *et al.* suggest l_1 regularizer

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Brodie *et al.* (2007) (cont'd)

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- The above is of combinatorial complexity
- Suboptimal algorithm: greedy search

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- Transaction costs:
 - Usually have fixed (overhead) part and transaction-dependent part
 - Institutional investors fixed part negligible
 - Small investors can assume fixed cost only

Brodie *et al.* (2007) (cont'd)

Portfolio adjustment (re-balancing)

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- We want to make an adjustment $\Delta_{\mathbf{w}}$, new portfolio $\mathbf{w} + \Delta_{\mathbf{w}}$
- Transaction costs only on the adjustments

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Homework:

Coursework 1 will involve confirming some claims in Brodie *et al.*'s paper. Please download the paper and start reading.

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- Download, uncompress, set MATLAB to the cvx directory and do `cvx_setup`
- Take MATLAB back into your working directory

Example of using CVX

```
T = 150; N = 50;
R = randn(T, N);
rho = 0.02;
tau = 1;
mu = rand(N,1);

cvx_begin quiet
variable w(N)
    minimize( norm(rho*ones(T,1)-R*w) + tau*norm(w,1) )
    subject to
        w'*ones(N,1) == 1;
        w'*mu == rho;
        w > 0;
cvx_end

figure(1), clf, bar(w); grid on
```

Note: Data random - probably won't work all the time

Portfolio Optimization with Transaction Costs

Lobo *et al.* (2007) *Ann Oper Res* **152**:341

- Portfolio weights: $\mathbf{w} = [w_1 \ w_2 \ \dots \ w_n]^T$
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Lobo *et al.* (2007) (cont'd)

Modeling Transaction Costs

Costs are separable (usual assumption):

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- Limit exposure in any small group of assets (say in a sector)

$$\sum_{i=1}^r (w_i + x_i)_{[i]} \leq \mathbf{1}^T (\mathbf{w} + \mathbf{x})$$

(Tricky, but can show this is convex – see Eqn (11) in paper)

- Constraints on short-selling
... on individual asset

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- Collateralization:

$$\sum_{i=1}^n (w_i + x_i)_- \leq \gamma \sum_{i=1}^n (w_i + x_i)_+$$

What I have borrowed to sell is smaller than a fraction of what I own

Variance:

$$(\mathbf{w} + \mathbf{x})^T \Sigma (\mathbf{w} + \mathbf{x}) \leq \sigma_{\max}$$

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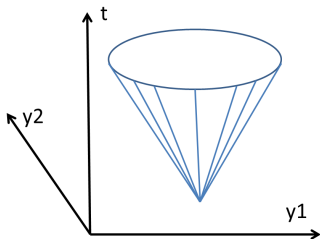
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- This is *Second Order Cone* constraint.



Shortfall Risk Constraint:

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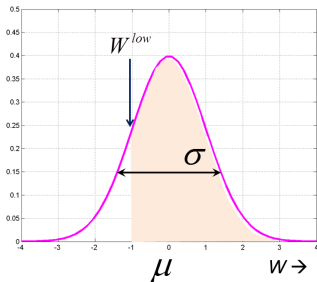
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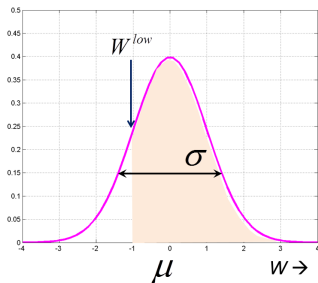
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$$P\left(\frac{W - \mu}{\sigma} \leq \frac{W^{\text{low}} - \mu}{\sigma}\right) \leq 1 - \eta$$

- But $(W - \mu)/\sigma \sim \mathcal{N}(0, 1)$

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$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z \exp\left\{-\frac{t^2}{2}\right\} dt$$

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- $\Phi^{-1}(1 - \eta) = -\Phi^{-1}(\eta)$

$$\mu - W^{\text{low}} \geq \Phi^{-1}(\eta) \sigma$$

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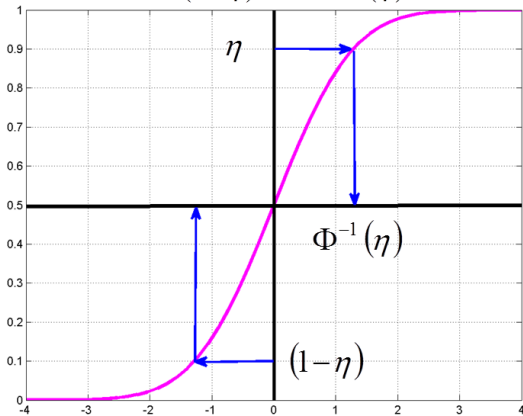
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- Using $\mu = \bar{\mathbf{a}}^T(\mathbf{w} + \mathbf{x})$ and $\sigma^2 = (\mathbf{w} + \mathbf{x})^T \Sigma (\mathbf{w} + \mathbf{x})$

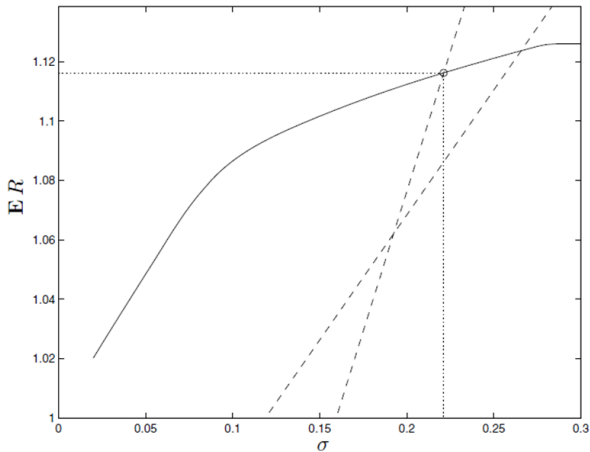
$$\Phi^{-1}(\eta) \|\Sigma^{1/2}(\mathbf{w} + \mathbf{x})\| \leq \bar{\mathbf{a}}^T(\mathbf{w} + \mathbf{x}) - W^{\text{low}}$$

$$\Phi^{-1}(1-\eta) = -\Phi^{-1}(\eta)$$



Lobo *et al.* (2007) (cont'd)

Shortfall Risk on M-V Space



Lobo *et al.* (2007) (cont'd)

Question in Assignment 1

$$\text{maximize} \quad \bar{\mathbf{a}}^T (\mathbf{w} + \mathbf{x}^+ - \mathbf{x}^-)$$

$$\text{subject to} \quad \mathbf{1}^T (\mathbf{x}^+ - \mathbf{x}^-) + \sum_{i=1}^n (\alpha_i^+ x_i^+ + \alpha_i^- x_i^-) \leq 0$$

$$x_i^+ \geq 0, x_i^- \geq 0, i = 1, 2, \dots, n$$

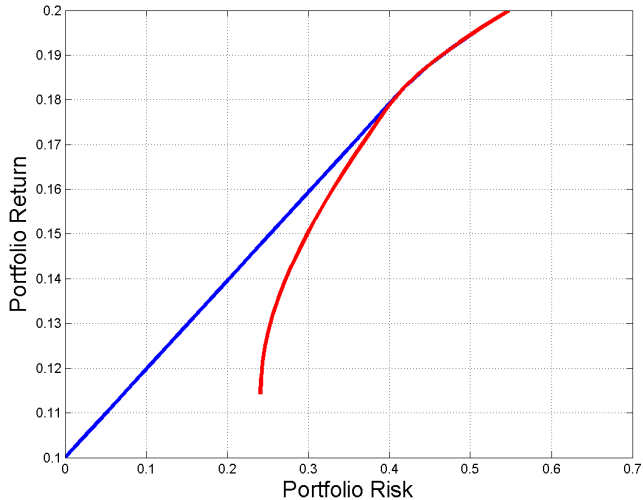
$$w_i + x_i^+ - x_i^- \geq s_i, i = 1, 2, \dots, n$$

$$\Phi^{-1}(1(\eta_j)) \|\Sigma^{1/2} (\mathbf{w} + \mathbf{x}^+ - \mathbf{x}^-)\| \leq \bar{\mathbf{a}}^T (\mathbf{w} + \mathbf{x}^+ - \mathbf{x}^-) - W_j^{\text{low}}, j = 1, 2$$

Puzzle

```
m1 = [0.15 0.2 0.08 0.1]';  
C1 = [ 0.2    0.05   -0.01    0.0  
      0.05    0.30    0.015   0.0  
     -0.01    0.015   0.10    0.0  
      0.0     0.0     0.0     0.0  
      ];  
  
m2 = [0.15 0.2 0.08]';  
C2 = [ 0.2    0.05   -0.01  
      0.05    0.30    0.015  
     -0.01    0.015   0.10  
      ];  
  
[V1, M1, PWts1] = NaiveMV(m1, C1, 25);  
[V2, M2, PWts2] = NaiveMV(m2, C2, 25);  
  
figure(2), clf,  
plot(V1, M1, 'b', V2, M2, 'r', 'LineWidth', 3),  
title('Mean Variance Portfolio', 'FontSize', 22)  
xlabel('Portfolio Risk', 'FontSize', 18)  
ylabel('Portfolio Return', 'FontSize', 18);
```


Mean Variance Portfolio



- Efficiency, no-arbitrage and fair price

Derivatives Pricing

- Efficiency, no-arbitrage and fair price
- Example:
 - Price today $S(0)$
 - A and B enter into a *future* contract to sell/buy at price F at time T

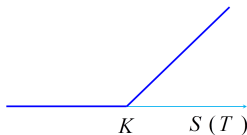
- Efficiency, no-arbitrage and fair price
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 - At time T , A owes the bank $S(0)\exp(rT)$ and has the asset to sell to B .

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 - $F = S(0)\exp(rT)$, else arbitrage opportunity

Options

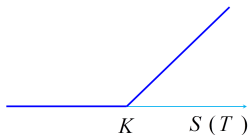
Options

- Call: right to buy at price K at time T

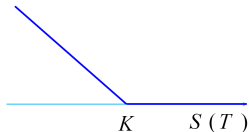


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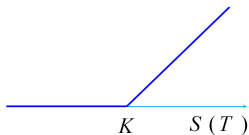


- Put: right to sell at price K at time T

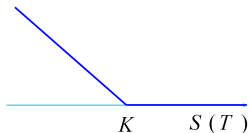


Options

- Call: right to buy at price K at time T



- Put: right to sell at price K at time T



- Exercise of contract
 - European style: only at time T
 - American style: any time in $0 \rightarrow T$

- Example: Put-Call Parity
 - Portfolio P_1 : European Call + cash $K \exp(-rT)$

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$$P_1 = C + K \exp(-rT)$$

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- Both portfolios having the same value at time $t = T$ should also have the same value at $t = 0$.

$$C + K \exp(-rT) = P + S(0)$$

- Geometric Brownian motion for stock price

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t)$$

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$$F(S, t) = \log S(t)$$

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- Stochastic differential equation for the log of the process

$$F(S, t) = \log S(t)$$

- Ito's lemma tells us about increments dF
- Terms needed to apply Ito's lemma

$$\begin{aligned}\frac{\partial F}{\partial t} &= 0 \\ \frac{\partial F}{\partial S} &= \frac{1}{S} \\ \frac{\partial^2 F}{\partial S^2} &= -\frac{1}{S^2}\end{aligned}$$

$$dF = \left(\frac{\partial F}{\partial t} + \mu S \frac{\partial F}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} \right) dt + \sigma S \frac{\partial F}{\partial S} dW$$

$$\begin{aligned}
 dF &= \left(\frac{\partial F}{\partial t} + \mu S \frac{\partial F}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} \right) dt + \sigma S \frac{\partial F}{\partial S} dW \\
 &= \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dW
 \end{aligned}$$

$$\log S(t) = \log S(0) + \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma dW(t)$$

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- $dW(t) = \epsilon \sqrt{t}$ where $\epsilon \sim \mathcal{N}(0, 1)$

$$\log S(t) \sim \mathcal{N} \left[\log S(0) + \left(\mu - \frac{\sigma^2}{2} \right) t, \sigma^2 t \right]$$

- Log of asset price has a normal distribution

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- Log of asset price has a normal distribution
- Also

$$S(t) = S(0) \exp \left((\mu - \sigma^2/2)t + \sigma \sqrt{t} \epsilon \right)$$

Black-Scholes Model

- Model

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

Black-Scholes Model

- Model

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

- Change in option price

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}dt$$

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- At maturity

$$f(S(T), T) = \max\{S(T) - K, 0\}$$

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- At maturity

$$f(S(T), T) = \max\{S(T) - K, 0\}$$

- Consider a portfolio
 - Own Δ stocks (long)
 - One call option sold

$$\Pi = \Delta S - f(S, t)$$

$$\begin{aligned}d\Pi &= \Delta dS - df \\&= \left(\Delta - \frac{\partial f}{\partial S} \right) dS - \left(\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt\end{aligned}$$

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- Term in dS (stochastic) can be eliminated by choosing Δ

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- $d\Pi = r \Pi dt$

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- $d\Pi = r \Pi dt$
- Eliminating $d\Pi$

$$\left(\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt = r \left(f - S \frac{\partial f}{\partial S} \right) dt$$

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = 0$$

- Partial differential equation

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = 0$$

- Boundary condition

- European Call: $f(S, T) = \max \{S - K, 0\}$
- European Put: $f(S, T) = \max \{K - S, 0\}$

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- European Put: $f(S, T) = \max \{K - S, 0\}$

- Black-Scholes

$$C = S_0 \mathcal{N}(d_1) - K \exp(-rT) \mathcal{N}(d_2)$$

$$d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$= d_1 - \sigma\sqrt{T}$$

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp(-y^2/2) dy$$

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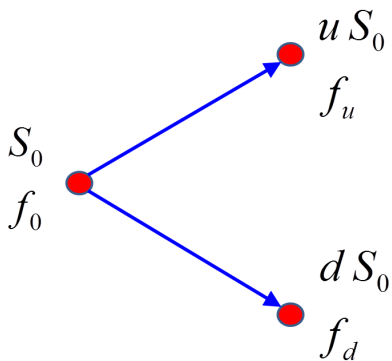
$$= d_1 - \sigma\sqrt{T}$$

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp(-y^2/2) dy$$

- Put-Call parity

$$P = K \exp(-rT) \mathcal{N}(-d_2) - S_0 \mathcal{N}(-d_1)$$

Binomial Lattice



Options Pricing on a Binomial Model

- Construct a portfolio:
 - A riskless bond, initial price $B_0 = 1$ and future value $B_1 = \exp(r\delta t)$
 - Underlying asset, initial value S_0
 - Number of stocks Δ , number of bonds Ψ

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- Future value depends on price movement up or down

$$\begin{cases} \Pi_u = \Delta S_0 u + \Psi \exp(r\delta t) \\ \Pi_d = \Delta S_0 d + \Psi \exp(r\delta t) \end{cases}$$

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$$\begin{cases} \Pi_u = \Delta S_0 u + \Psi \exp(r\delta t) \\ \Pi_d = \Delta S_0 d + \Psi \exp(r\delta t) \end{cases}$$

- We can solve for a portfolio that will replicate option payoff

$$\Delta S_0 u + \Psi \exp(r\delta t) = f_u$$

$$\Delta S_0 d + \Psi \exp(r\delta t) = f_d$$

and solve for Δ and Ψ

- ... solving

$$\Delta = \frac{f_u - f_d}{S_0(u - d)}$$

$$\psi = \exp(-r\delta t) \frac{uf_d - df_u}{u - d}$$

- ... solving

$$\Delta = \frac{f_u - f_d}{S_0(u - d)}$$

$$\Psi = \exp(-r\delta t) \frac{uf_d - df_u}{u - d}$$

- No arbitrage \implies initial value of this portfolio should be f_0

$$\begin{aligned} f_0 &= \Delta S_0 + \Psi \\ &= \frac{f_u - f_d}{(u - d)} + \exp(-r\delta t) \frac{uf_d - df_u}{u - d} \\ &= \exp(-r\delta t) \left\{ \frac{f_u}{u - d} + \frac{f_d}{u - d} \right\} \end{aligned}$$

- ... solving

$$\Delta = \frac{f_u - f_d}{S_0(u - d)}$$

$$\Psi = \exp(-r\delta t) \frac{uf_d - df_u}{u - d}$$

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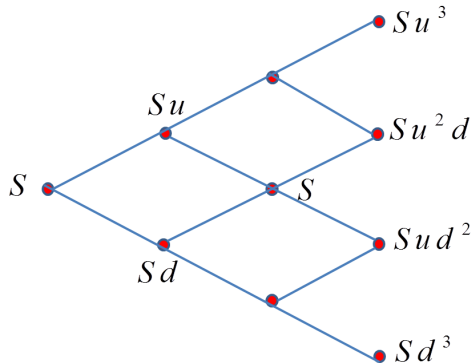
- Defining probabilities

$$\pi_u = \frac{\exp(r\delta t) - d}{u - d} \text{ and } \pi_d = \frac{u - \exp(r\delta t)}{u - d}$$

option price interpreted as discounted expected value

$$f_0 = \exp(-r\delta t) (\pi_u f_u + \pi_d f_d)$$

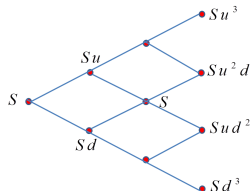
Binomial Lattice



Calibrating a Binomial Lattice

- When are these equivalent?

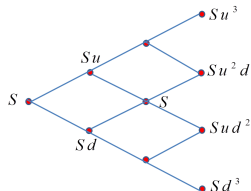
$$dS = r S dt + \sigma S dW$$



Calibrating a Binomial Lattice

- When are these equivalent?

$$dS = r S dt + \sigma S dW$$



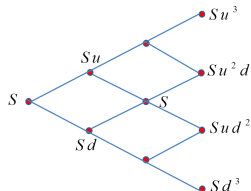
- Log normal distribution

$$\log(S_{t+\delta t}) \sim \mathcal{N}((r - \sigma^2/2), \sigma^2 \delta t)$$

Calibrating a Binomial Lattice

- When are these equivalent?

$$dS = r S dt + \sigma S dW$$



- Log normal distribution

$$\log(S_{t+\delta t}) \sim \mathcal{N}((r - \sigma^2/2), \sigma^2 \delta t)$$

- Mean and variance of log normal distribution

(log of the variable is normal, what is mean and variance of the variable?)

$$\begin{aligned} E[S_{t+\delta t}] &= \exp(r \delta t) \\ \text{Var}[S_{t+\delta t}] &= \exp(2r \delta t) (\exp(\sigma^2 \delta t) - 1) \end{aligned}$$

Calibrating binomial lattice (cont'd)

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- Mean for the lattice

$$E[S_{t+\delta t}] = p u S_t + (1 - p) d S_t$$

Calibrating binomial lattice (cont'd)

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$$E[S_{t+\delta t}] = p u S_t + (1 - p) d S_t$$

- Equating the means...

$$p u S_t + (1 - p) d S_t = \exp(r \delta t) S_t$$

$$p = \frac{\exp(r \delta t) - d}{u - d}$$

Calibrating binomial lattice (cont'd)

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- Variance on the lattice

$$\begin{aligned}\text{Var}[S_{t+\delta t}] &= E[S_{t+\delta t}^2] - E^2[S_{t+\delta t}] \\ &= S_t^2 (p u^2 + (1 - p) d^2) - S_t^2 \exp(2r \delta t)\end{aligned}$$

Calibrating binomial lattice (cont'd)

- Mean for the lattice

$$E[S_{t+\delta t}] = p u S_t + (1 - p) d S_t$$

- Equating the means...

$$p u S_t + (1 - p) d S_t = \exp(r \delta t) S_t$$

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... which from the dynamical model is...

$$\text{Var}[S_{t+\delta t}] = S_t^2 \exp(2r \delta t) (\exp(\sigma^2 \delta t) - 1)$$

- Equating the two variances

$$S_t^2 \exp(2r\delta t) (\exp(\sigma^2\delta t) - 1) = S_t^2 (pu^2 + (1-p)d^2) - S_t^2 \exp(2r\delta t)$$

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Which reduces to

$$\exp(2r\delta t + \sigma^2\delta t) = pu^2 + (1-p)d^2$$

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Which reduces to

$$\exp(2r\delta t + \sigma^2\delta t) = pu^2 + (1-p)d^2$$

Substitute for p and simplify

$$\exp(2r\delta t + \sigma^2\delta t) = (u + d) \exp(r\delta t) - 1$$

...and because $u = 1/d$,

$$u^2 \exp(r\delta t) - u (1 + \exp(2r\delta t + \sigma^2\delta t)) + \exp(r\delta t) = 0$$

... a quadratic equation in u .

$$u = \frac{(1 + \exp(2r\delta t + \sigma^2\delta t)) + \sqrt{(1 + \exp(2r\delta t + \sigma^2\delta t))^2 - 4\exp(2r\delta t)}}{2\exp(r\delta t)}$$

Taylor series expansion of $\exp(x)$

$$(1 + \exp(2r\delta t + \sigma^2\delta t))^2 - 4\exp(2r\delta t) \approx (2 + (2r + \sigma^2)\delta t)^2 - 4(1 + 2r\delta t) \approx 4\sigma^2\delta t$$

$$\begin{aligned} u &\approx \frac{2 + (2r + \sigma^2)\delta t + 2\sigma\sqrt{\delta t}}{2\exp(r\delta t)} \\ &\approx \left(1 + r\delta t + \frac{\sigma^2}{2}\delta t + \sigma\sqrt{\delta t}\right)(1 - r\delta t) \\ &\approx 1 + r\delta t + \frac{\sigma^2}{2}\delta t + \sigma\sqrt{\delta t} - r\delta t \\ &= 1 + \sigma\sqrt{\delta t} + \frac{\sigma^2}{2}\delta t \end{aligned}$$

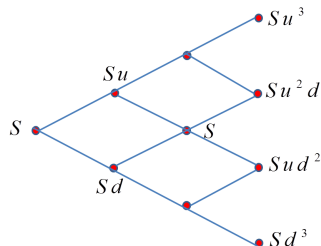
Calibrating the Binomial Lattice (cont'd)

$$u = \exp(\sigma\sqrt{\delta t})$$

$$d = \exp(-\sigma\sqrt{\delta t})$$

$$p = \frac{\exp(r\delta t) - d}{u - d}$$

$$dS = r S dt + \sigma S dW$$



Example

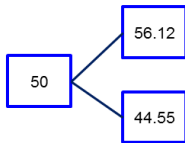
- European call option; $S_0 = K = 50$; $r = 0.1$; $\sigma = 0.4$; maturity in five months.

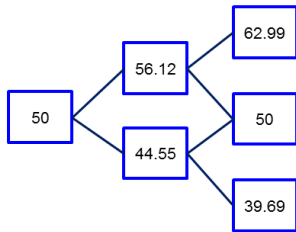
```
>> call = blsprice(50, 50, 0.1, 5/12, 0.4)
call =
    6.1165
```

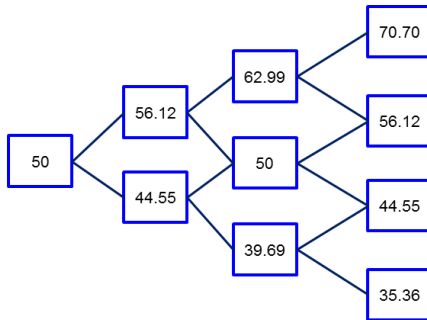
We can now build the lattice

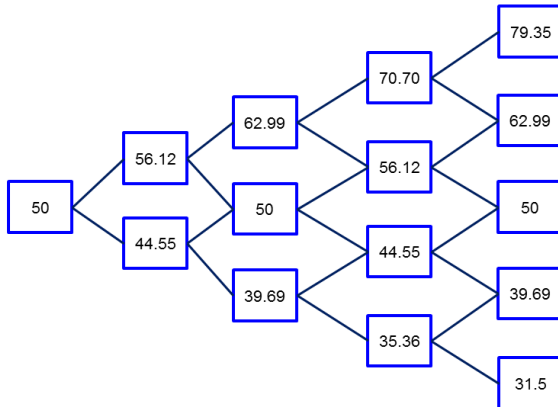
δt	$1/12$	0.0833
u	$\exp(\sigma\sqrt{t})$	1.1224
d	$1/u$	0.8909
p	$(\exp(r\delta t) - d) / (u - d)$	0.5073

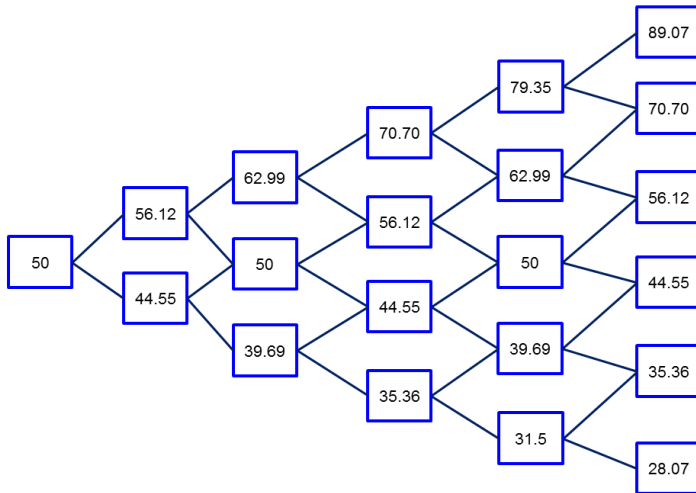
50











39.07

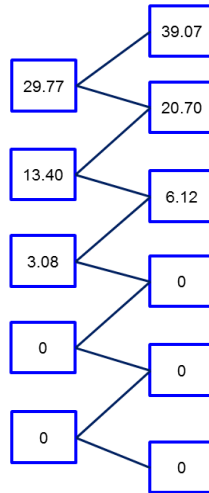
20.70

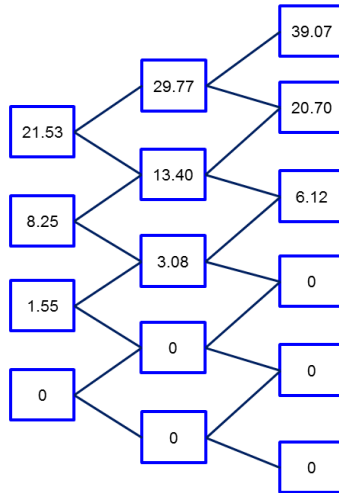
6.12

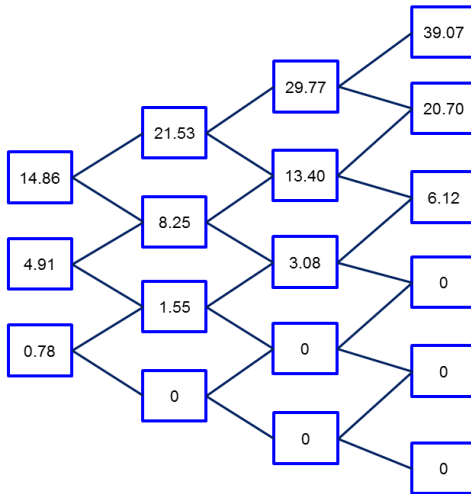
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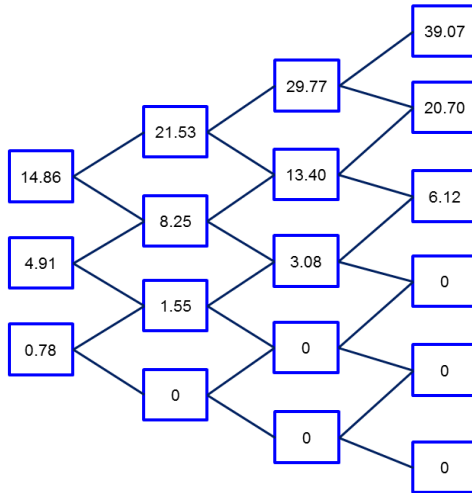
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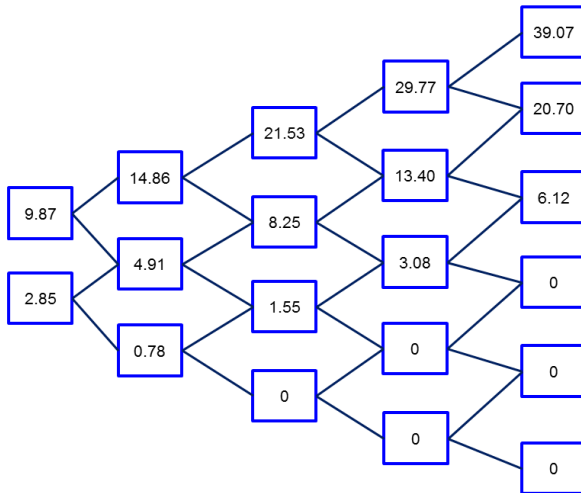
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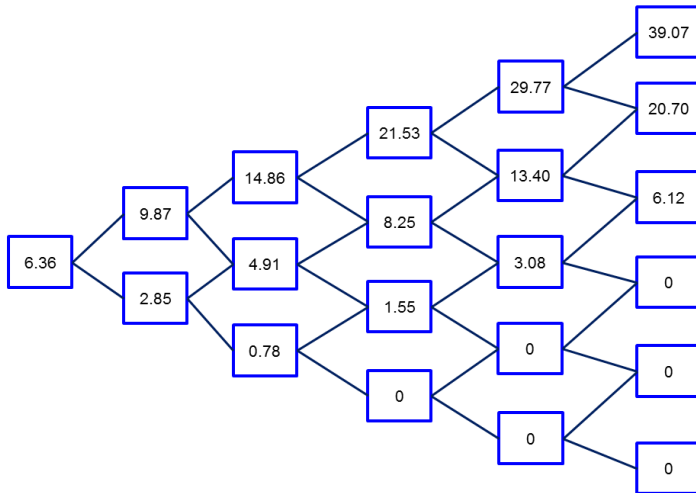












Pricing European Call Option by Binomial Lattice

```
function [price, lattice] = LatticeEurCall(S0,K,r,T,sigma,N)

deltaT = T/N;
u=exp(sigma * sqrt(deltaT));
d=1/u;
p=(exp(r*deltaT) - d)/(u-d);
lattice = zeros(N+1,N+1);

for i=0:N
    lattice(i+1,N+1)=max(0 , S0*(u^i)*(d^(N-i)) - K);
end

for j=N-1:-1:0
    for i=0:j
        lattice(i+1,j+1) = exp(-r*deltaT) * ...
            (p * lattice(i+2,j+2) + (1-p) * lattice(i+1,j+2));
    end
end

price = lattice(1,1);
```

Pricing American Style Put Option

```
function price = AmPutLattice(S0,K,r,T,sigma,N)
deltaT = T/N;
u=exp(sigma * sqrt(deltaT));
d=1/u;
p=(exp(r*deltaT) - d)/(u-d);
discount = exp(-r*deltaT);
p_u = discount*p;
p_d = discount*(1-p);
SVals = zeros(2*N+1,1);
SVals(N+1) = S0;

[...]
```

Pricing American Style Put Option (cont'd)

```
function price = AmPutLattice(S0,K,r,T,sigma,N)
```

```
 [...]
```

```
for i=1:N
```

```
    SVals(N+1+i) = u*SVals(N+i);
```

```
    SVals(N+1-i) = d*SVals(N+2-i);
```

```
end
```

```
PVals = zeros(2*N+1,1);
```

```
for i=1:2:2*N+1
```

```
    PVals(i) = max(K-SVals(i),0);
```

```
end
```

```
 [...]
```

Pricing American Style Put Option (cont'd)

```
function price = AmPutLattice(S0,K,r,T,sigma,N)
```

```
[...]
for tau=1:N
    for i= (tau+1):2:(2*N+1-tau)
        hold = p_u*PVals(i+1) + p_d*PVals(i-1);
        PVals(i) = max(hold, K-SVals(i));
    end
end
price = PVals(N+1);
```

- Decisions at every point during backtracking

$$f_{i,j} = \max \{ K - S_{i,j}, \exp(-r\delta t) (p f_{i+1,j+1} + (1-p) f_{i,j+1}) \}$$

- We will look at inference as expectations...

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x)f_X(x) dx$$

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- Think of this as computing the expected value
(of a function of a uniform random variable):

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- Think of this as computing the expected value
(of a function of a uniform random variable):

$$E[g(U)], \text{ where } U \sim (0,1)$$

- We approximate the integral by

$$\hat{I}_m = \frac{1}{m} \sum_{i=1}^m g(U_i)$$

- Where will we use this?

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- European call option

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- What if we had large number of samples?

```
> BlsMC1(S0, K, r, T, sigma, 1000000)
ans =
    1.6295
> BlsMC1(S0, K, r, T, sigma, 1000000)
ans =
    1.6164
> BlsMC1(S0, K, r, T, sigma, 1000000)
ans =
    1.6141
```

Sampling: Inverse Transform

- Sample X from $f(x)$; Cumulative distribution $F(x)$

- Draw $U \sim U(0,1)$
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- Example: Exponential distribution $X \sim \exp(\mu)$
- Cumulative

$$F(x) = 1 - \exp(-\mu x)$$

- Inverse

$$x = -\frac{1}{\mu} \log(1 - U)$$

- Distributions of U and $(1 - U)$ are the same
Hence return: $-\log(U)/\mu$

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- 3 If $U \leq f(Y)/t(Y)$ return $X = Y$
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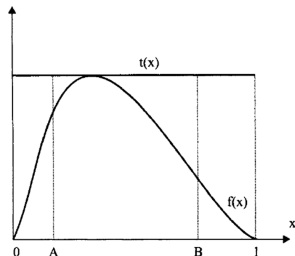
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Homework

Page 235, Brandimarte

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- Exercise:
 - Draw the graph of $f(x)$
 - Simulate 1000 samples using above algorithm
 - Draw a histogram to the same scale as $f(x)$ – do they match? Is it better with 100000 samples?
 - On average, how many trials were needed through the accept-reject loop for each sample?

Variance Reduction

- Independent samples X_i
- Sample mean (estimates mean $\mu = E[X_i]$ from n samples)

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- Two points:
 - More samples n reduces the variance in estimation
 - Variance reduction schemes can control σ^2

Variance reduction: Antithetic Sampling

- Pair of sequences

$$\left\{ \begin{array}{cccc} X_1^{(1)} & X_1^{(2)} & \dots & X_1^n \\ X_2^{(1)} & X_2^{(2)} & \dots & X_2^n \end{array} \right\}$$

- Columns (horizontally) are independent
- $X_1^{(i)}$ and $X_2^{(i)}$ are dependent.

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- Sample is a function of each pair: $X^{(i)} = (X_1^{(i)} + X_2^{(i)}) / 2$
- Variance

$$\begin{aligned} \text{Var} [\bar{X}(n)] &= \frac{1}{n} \text{Var} [X^{(i)}] \\ &= \frac{1}{4n} \left\{ \text{Var}(X_1^{(i)}) + \text{Var}(X_2^{(i)}) + 2 \text{Cov}(X_1^{(i)}, X_2^{(i)}) \right\} \\ &= \frac{1}{2n} \text{Var}(X) (1 + \rho) \end{aligned}$$

- Uniform random number $\{U_k\}$ and $\{1 - U_k\}$ as sequences.

```
function [Price, CI] = BlsMC2(S0,K,r,T,sigma,NRepl)
nuT = (r - 0.5*sigma^2)*T;
siT = sigma * sqrt(T);
DiscPayoff = exp(-r*T)*max(0, S0*exp(nuT+siT*randn(NRepl,1))-K);
[Price, VarPrice, CI] = normfit(DiscPayoff);
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```
function [Price, CI] = BlsMCAV(S0,K,r,T,sigma,NPairs)
nuT = (r - 0.5*sigma^2)*T;
siT = sigma * sqrt(T);
Veps = randn(NPairs,1);
Payoff1 = max( 0 , S0*exp(nuT+siT*Veps) - K);
Payoff2 = max( 0 , S0*exp(nuT+siT*(-Veps)) - K);
DiscPayoff = exp(-r*T) * 0.5 * (Payoff1+Payoff2);
[Price, VarPrice, CI] = normfit(DiscPayoff);
```

Homework

Test the two functions: BlsMC and BlsMCAV

(Brandimarte, p248)

```
> randn('state', 0)
> [Price, CI] = BlsMC2(50,50,0.05,1,0.4,200000)
Price=
    9.0843
CI =
    9.0154
    9.1532
\pause
> (CI(2)-CI(1))/Price
ans =
    0.0152
\pause
> randn('state', 0)
> [Price, CI] = BlsMCAV(50,50,0.05,1,0.4,200000)
Price=
    9.0553
CI =
    8.9987
    9.1118
\pause
> (CI(2)-CI(1))/Price
ans =
    0.0125
```

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$$\mathbf{x} = [S/X \quad (T - t)]^T$$
$$c = \sum_{j=1}^J \lambda_j \phi_j(\mathbf{x}) + \mathbf{w}^T \mathbf{x} + w_0$$

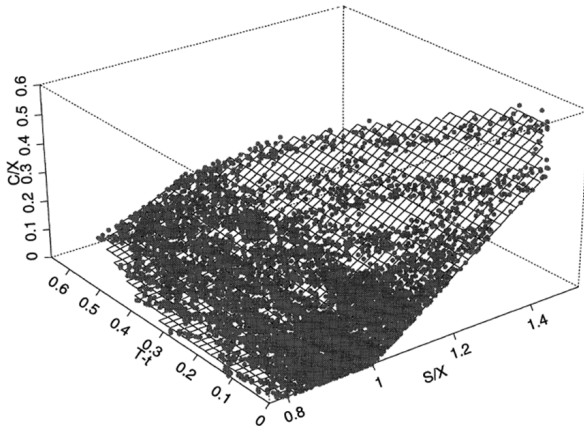


Figure 4. Simulated call option prices normalized by strike price and plotted versus

$$\begin{aligned}
\widehat{C/X} = & -0.06 \sqrt{\begin{bmatrix} S/X - 1.35 \\ T - t - 0.45 \end{bmatrix}, \begin{bmatrix} 59.79 & -0.03 \\ -0.03 & 10.24 \end{bmatrix} \begin{bmatrix} S/X - 1.35 \\ T - t - 0.45 \end{bmatrix} + 2.55} \\
& - 0.03 \sqrt{\begin{bmatrix} S/X - 1.18 \\ T - t - 0.24 \end{bmatrix}, \begin{bmatrix} 59.79 & -0.03 \\ -0.03 & 10.24 \end{bmatrix} \begin{bmatrix} S/X - 1.18 \\ T - t - 0.24 \end{bmatrix} + 1.97} \\
& + 0.03 \sqrt{\begin{bmatrix} S/X - 0.98 \\ T - t + 0.20 \end{bmatrix}, \begin{bmatrix} 59.79 & -0.03 \\ -0.03 & 10.24 \end{bmatrix} \begin{bmatrix} S/X - 0.98 \\ T - t + 0.20 \end{bmatrix} + 0.00} \\
& + 0.10 \sqrt{\begin{bmatrix} S/X - 1.05 \\ T - t + 0.10 \end{bmatrix}, \begin{bmatrix} 59.79 & -0.03 \\ -0.03 & 10.24 \end{bmatrix} \begin{bmatrix} S/X - 1.05 \\ T - t + 0.10 \end{bmatrix} + 1.62} \\
& + 0.14S/X - 0.24(T - t) - 0.01.
\end{aligned} \tag{9}$$