Chapter 8

Probability for continuous variables

Although, in this course, we make a formal distinction between discrete and continuous random variables, this is not necessary. It simply makes the ideas clearer when using a Bernoulli (coin toss) model to understand the concepts.

In mathematical terms, the discrete and continuous variables are equivalent and one does not need to make a formal distinction between them. One merely speaks of a 'measure', which can either be discrete or continuous. Thus all the algebra used to define discrete variable calculations can be extended to the continuous variable, with integrals replacing sums and so on.

8.1 Continuous Random Variable

Consider a random variable which has continuous values. Let X be such a variable limited between the values $a \leq X \leq b$. We define a *probability density function*, $f_X(x)$, as follows:

$$P\left(x \leq X \leq x + dx\right) = f_X(x)dx \qquad . \tag{8.1}$$

Furthermore, by definition: $f_X(x) \geq 0$, and all events must occur within this interval:

$$P\left(a \leq X \leq b\right) = 1 \tag{8.2}$$

In terms of the probability density, this means:

$$\int_a^b f_X(x)dx = 1 \tag{8.3}$$

We can furthermore define the (cumulative) probability distribution:

$$P\left(a \leq X \leq x\right) = F_X(x) = \int_a^x f_X(x')dx' \qquad , \tag{8.4}$$

so that, by the fundamental theorem of calculus, the probability density function is the derivative of the probability distribution:

$$\frac{d}{dx}F_X(x) = f_X(x) \qquad . \tag{8.5}$$

We can then define an expectation operator as,

$$\mathbb{E}\left(X\right) \equiv \int_a^b x f_X(x)dx = \mu_X \tag{8.6}$$

or in general,

$$\mathbb{E}(g(X)) = \int_a^b g(x) f_X(x) dx \qquad . \tag{8.7}$$

In the same manner, the variance is defined by

$$\mathrm{var}(X) \equiv \mathbb{E}\left(X^2\right) - (\mathbb{E}(X))^2 = \sigma_X^2 \tag{8.8}$$

where σ_X is the standard deviation.

8.2 Normal distribution

The standard normal distribution (also called a Gaussian distribution) has the (cumulative) probability distribution

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}u^2} du \qquad -\infty < z < +\infty \qquad . \tag{8.9}$$

This distribution is also, more commonly, written as, $\Phi(z)$. This has the corresponding probability density:

$$n(z) = \frac{dN(z)}{dz} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \qquad -\infty < z < +\infty \tag{8.10}$$

This function is sketched in figure 8.1, and $N(z)$ is shown in figure 8.2. We note that:

$$N(+\infty) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}u^2} du = 1 \qquad . \tag{8.11}$$

With a change of variable: $u = \sqrt{\alpha} y$, we furthermore find

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\alpha y^2} dy = \sqrt{\frac{2\pi}{\alpha}} \qquad . \tag{8.12}$$

We can use this relation to calculate other integrals of interest. Consider taking the partial derivative with respect to α of both sides then:

$$\left(-\frac{\partial}{\partial \alpha}\right) \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\alpha y^2} dy = \left(-\frac{\partial}{\partial \alpha}\right) \sqrt{\frac{2\pi}{\alpha}} \qquad . \tag{8.13}$$

This leads to the result:

$$\frac{1}{2} \int_{-\infty}^{+\infty} y^2 e^{-\frac{1}{2}\alpha y^2} dy = \frac{1}{2} \frac{\sqrt{2\pi}}{\alpha^{3/2}} \qquad . \tag{8.14}$$

This enables us to determine the mean (expected value) and variance of the standard normal distribution.

$$\mu_X = \mathbb{E}(X) = \int_{-\infty}^{+\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 0 \qquad , \tag{8.15}$$

since the integrand is an odd function. The variance is given by

$$\mathrm{var}(X) = \mathbb{E}\left(X^2\right) - (\mathbb{E}(X))^2 = \int_{-\infty}^{+\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx - 0 = 1 \qquad , \tag{8.16}$$

where we have used (8.14), taking $\alpha = 1$. Using this result, the standard normal distribution has mean 0 and variance 1. This is why it is called the standard normal distribution.

8.3 Dirac delta function

A special case of a random event is a certain event. We can define a probability density for such a case by borrowing a function from Physics.

The Dirac δ-function [1] is a generalized function (measure) with the properties:

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ +\infty & x = 0 \end{cases} \qquad \text{and} \qquad \int_{-\infty}^{+\infty} \delta(x)dx = 1 \qquad . \tag{8.17}$$

It then follows that, for any function $g(x)$, and any constant a:

$$\int_{-\infty}^{+\infty} g(x)\delta(x-a)dx = \int_{-\infty}^{+\infty} g(a)\delta(x-a)dx = g(a)\int_{-\infty}^{+\infty} \delta(x-a)dx = g(a) \qquad .$$

So, a continuous random variable X, which has the value a with certainty, is given by a probability density function:

$$f_X(x) = \delta(x-a) \qquad . \tag{8.18}$$

Clearly the mean (expected value) of the variable is just

$$\mathbb{E}\,(X) = a \tag{8.19}$$

and it has zero variance.

The δ-function is a very useful mathematical concept. However, it requires care to define it properly in a mathematical sense. The best way to define it is as a limiting function. For example:

$$\delta(x-a) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x-a)^2}{2\sigma^2}\right) \qquad , \tag{8.20}$$

or

$$\delta(x-a) = \lim_{\sigma\to 0} \frac{1}{2\sigma}\left(\theta(a-\sigma) - \theta(a+\sigma)\right) \qquad , \tag{8.21}$$

where θ is the step function. In fact, the *step function* is the (cumulative) probability distribution function for the delta function. With

$$\theta(x) = F_X(x) = \begin{cases} 0 & . & x < 0 \\ \frac{1}{2} & , & x = 0 \\ 1 & , & x > 0 \end{cases} \quad , \tag{8.22}$$

we find:

$$\delta(x) = F'_X(x) \qquad . \tag{8.23}$$

The distribution is well defined and finite, and to be strictly mathematical, whenever one sees a delta function it is usually just a shorthand for the derivative of the step function.

8.3.1 Change of variable

Suppose $Y = h(X)$ is a monotonic function of X, for all $x_2 > x_1$, $y(x_2) > y(x_1)$, for which an inverse exists, $h^{-1}(Y) = X$. Then we can calculate the probability density of Y knowing the probability density of X. Let $g_Y(y)$ be the required density, that is

$$P\left(y \leq Y \leq y + dy\right) = g_Y(y)dy \tag{8.24}$$

[1]Paul Dirac was a Nobel prize winner in Physics (he formulated relativistic quantum mechanics).

We start with the statement, which follows from the monotonicity,

$$P(Y \leq y(x)) = P(X \leq x) \tag{8.25}$$

Then, differentiation of both sides gives:

$$\frac{d}{dx} P(Y \leq y(x)) = \frac{d}{dx} P(X \leq x) \tag{8.26}$$

Using the chain rule, this becomes:

$$\frac{dy}{dx} \frac{d}{dy} P(Y \leq y(x)) = f_X(x) \tag{8.27}$$

This can be arranged in the form:

$$\frac{dy}{dx} g_Y(y) = f_X(x) \tag{8.28}$$

$$\boxed{g_Y(y) = \frac{1}{(dy/dx)} f_X(x)} \quad . \tag{8.29}$$

This equation describes the change of variable which gives a change of measure.

8.4 Joint, Marginal and Conditional probability

Consider a pair of continuous random variables X and Y, which may or may not be independent. One can then define a joint probability density, $f_{XY}(x, y)$ as follows:

$$f_{XY}(x, y) dx dy = P(x \leq X \leq x + dx, y \leq Y \leq y + dy) \quad . \tag{8.30}$$

The *marginal density*, for X is then:

$$f_X(x) = \int f_{XY}(x, y) dy \quad , \tag{8.31}$$

and for Y,

$$f_Y(y) = \int f_{XY}(x, y) dx \quad . \tag{8.32}$$

Continuing the analogy, the expected values for the random variables are then defined as the integrals:

$$\mathbb{E}(X) = \int x f_X(x) dx = \int \int x f_{XY}(x, y) dy dx \quad , \tag{8.33}$$

and

$$\mathbb{E}(Y) = \int y f_Y(y) dy = \int \int y f_{XY}(x, y) dy dx \quad . \tag{8.34}$$

The extension of the conditional probability from discrete to continuous variables is as follows. One defines the conditional density:

$$f_{X|Y}(x|y) \equiv \frac{f_{XY}(x, y)}{f_Y(y)} \quad , \quad f_Y(y) \neq 0 \quad . \tag{8.35}$$

and similarly for $f_{Y|X}(y|x)$. Traditionally statistical models are classified into a fixed time snapshot (cross sectional) or a time-dependent movie (longtitudinal).

In this course we come across conditional probabilities mostly in terms of *transition probabilities.* That is, we are considering a longitudinal time-dependent stochastic process such as: *given* the price of an asset today, what is the probability of it having a certain value at a future time.

Then the *conditional expectation* is defined as:

$$\mathbb{E}\left(Y|X\right) = \int y f_{Y|X}(y|x) dy \quad , \tag{8.36}$$

and the *conditional expectation theorem* takes the form:

$$\mathbb{E}\left(Y\right) = \mathbb{E}\left(\mathbb{E}\left(Y|X\right)\right) = \int\int y f_{Y|X}(y|x) f_X(x) dy dx \quad . \tag{8.37}$$

So in our studies of finance, we might have a stochastic (random) variable S_t as the asset price at a time t, and we wish to calculate its future expected value:

$$\mathbb{E}\left(S_T|S_t\right) \quad , \quad T \geq t \quad .$$

This *conditional expectation* is used in the definition of a *martingale*, namely that X_t is a martingale if:

$$\mathbb{E}\left(X_T|X_t\right) = X_t \quad , \quad \text{for all} \;\; T \geq t \quad . \tag{8.38}$$

8.4.1 The central-limit theorem

The second 'big' convergence theorem of probability theory states that a large sample converges towards a normal distribution. This is formally expressed as the *central-limit theorem*:

$$\boxed{\lim_{n\to\infty} P\left(\frac{S_n/n - \mu_X}{\sigma_X/\sqrt{n}} \leq z\right) = N(z)} \quad . \tag{8.39}$$

where,

$$S_n \equiv X_1 + X_2 + \cdots + X_n \quad ,$$

is the sum of n independent, identically-distributed random variables (discrete or continuous), each with a mean, μ_X, and standard deviation, σ_X, and $N(z)$ is the standard normal distribution function given by:

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}u^2} du \quad .$$

A sketch of this function is given in figure (8.2).

8.5 Binomial to normal

Here is an example of how this might apply to a problem in finance. A share price changes each day at random. It either increases by δs, or decreases by δs, with probabilities p and $1-p$, respectively, as shown in figure 8.3. So we have our usual Bernoulli process.

If X_i is the price change over a time interval $\delta\ t$.

$$P\left(X_i = +\delta s\right) = p \quad , \quad 0 \leq p \leq 1 \quad . \tag{8.40}$$

$$P\left(X_i = -\delta s\right) = q = 1 - p \tag{8.41}$$

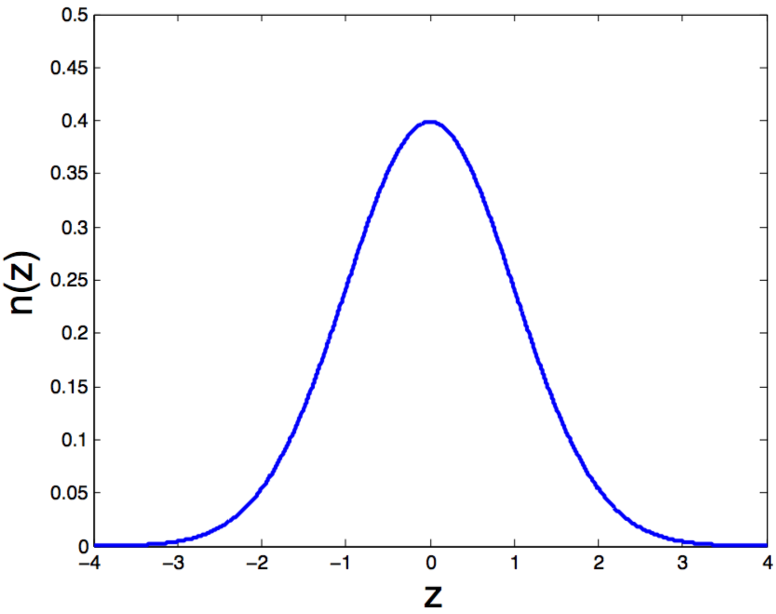

Figure 8.1: The standard normal probability density: $n(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$. We note the function is even: $n(z) = n(-z)$.

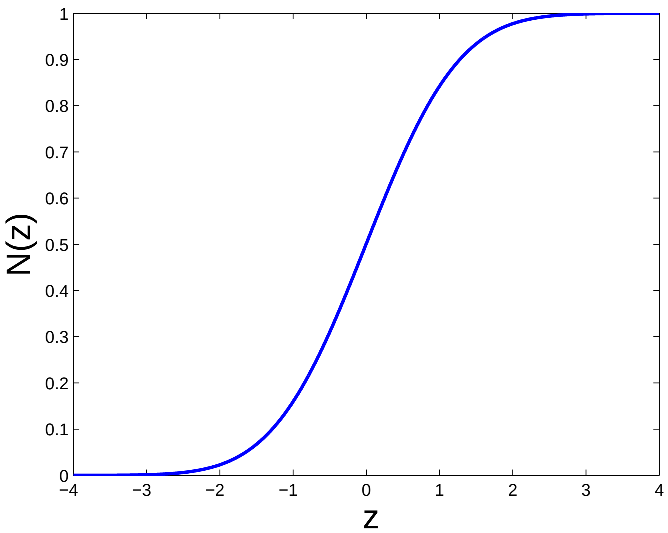

Figure 8.2: The (cumulative) standard normal probability distribution: $N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}u^2} du$. We note the symmetry of the curve about $z = 0$, and this is expressed mathematically through the relation: $N(z) = 1 - N(-z)$. So in particular, $N(0) = 0.5$. A table of values for this function is given at the end of the chapter.

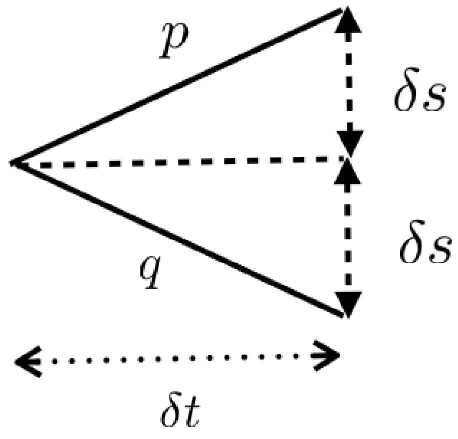

Figure 8.3: An asset price that can go up, by an amount δs with probability p, or down by an amount δs, with probability q, over a time δt.

Then we have:

$$\mu_X = \mathbb{E}(X_i) = (p-q)\delta s \tag{8.42}$$

$$\mathrm{var}(X_i) = \sigma_x^2 = 4pq(\delta s)^2 \quad , \tag{8.43}$$

so that,

$$\sigma_X = 2\sqrt{pq}\ \delta s \quad . \tag{8.44}$$

So after n time steps, the time will be $t = n\delta t$, and the price change will be

$$S_n = X_1 + X_2 \cdots + X_n \quad . \tag{8.45}$$

An example of a typical probability tree is shown in figure (8.4).

We have already seen that a sequence of Bernoulli trials gives a binomial distribution. So after n days the probability that the price will have changed by an amount: $S_n = s_i = (2i-n)\delta s$, will be:

$$P(S_n = s_i) = \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i} \quad . \tag{8.46}$$

The average value (expected value) for the price change is then:

$$\mathbb{E}(S_n) = n(p-q)\delta s \quad , \tag{8.47}$$

and it has a standard deviation:

$$\sigma_n = \sqrt{n}\ \sigma_X = 2\sqrt{pqn}\ \delta s \quad . \tag{8.48}$$

Now, for large n we can use the central limit theorem as an approximation to the binomial distribution.

According to the central-limit theorem one can make the correspondence:

$$\frac{S_n/n - \mu_X}{\sigma_X/\sqrt{n}} \approx Z \quad , \tag{8.49}$$

where we use Z to indicate a random variable with standard normal distribution.

Now we need to convert this from the discrete variable to the continuous variable: $S_n \to S_t$, $t = n\delta t$. We first rewrite (8.49) as

$$S_n = n\mu_X + \sigma_X\sqrt{n}Z. \tag{8.50}$$

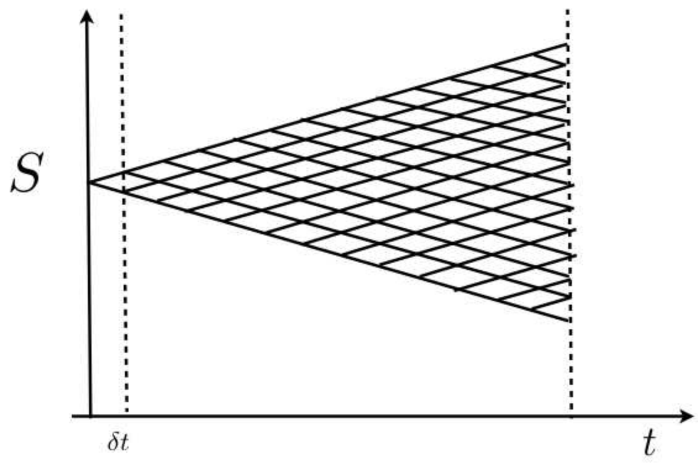

Figure 8.4: A binomial tree constructed from $n = 13$ steps of the elementary process shown in figure 8.3. The asset price has a wide range of values, and the probability of having these values is given by the binomial distribution.

We now want to replace the discrete steps by the continuous time variable. This can be achieved by adding terms of δt.

$$S_t = \frac{n\delta t}{\delta t}\mu_X + \frac{\sigma_X}{\sqrt{\delta t}}\sqrt{n\delta t}Z = t\frac{\mu_X}{\delta t} + \frac{\sigma_X}{\sqrt{\delta t}}\sqrt{t}Z \tag{8.51}$$

Now we define the following variables:

$$\mu_d \equiv \frac{\mu_X}{\delta t} = (p-q)\frac{\delta s}{\delta t} \quad , \tag{8.52}$$

where μ_d is called the *drift rate*, and the *volatility*

$$\sigma_v = \frac{\sigma_X}{\sqrt{\delta t}} \quad . \tag{8.53}$$

Volatility is a measure of the uncertainty in the value of the asset, that is the risk in holding the asset. Then (8.49) can finally be written as:

$$\boxed{S_t = \mu_d t + \sigma_v\sqrt{t}\ Z} \quad , \tag{8.54}$$

where $\mu_d t$ is the *deterministic change* and $\sigma\sqrt{t}Z$ is the *stochastic change*, and Z has the standard normal distribution. S_t therefore has a normal distribution with mean $\mu_d t$ and variance $\sigma_V^2 t$.

Equation (8.54) describes (arithmetic) Brownian motion. In general the term Brownian motion will be used to mean *arithmetic* motion. Later, we will discuss what is known as *exponential* or *geometric* Brownian motion, as this may be more appropriate for the dynamics of asset prices (which tend to be non-negative).

We can visualize Brownian motion. To construct an arithmetic Brownian motion we consider the asset price change starting from $t = 0$, and then look at the value at $t = 1$, which we call S_1 then start again from here to work out the price change over the next day $t = 2$, which we call S_2 and so on. To get the randomness, we need to use the computer to generate the random number Z. Since this is artificial, this is called a *simulation* of the process. In particular, simulations involving random number generation are known as *Monte Carlo* simulations with reference to the famous Casino in Monte Carlo.

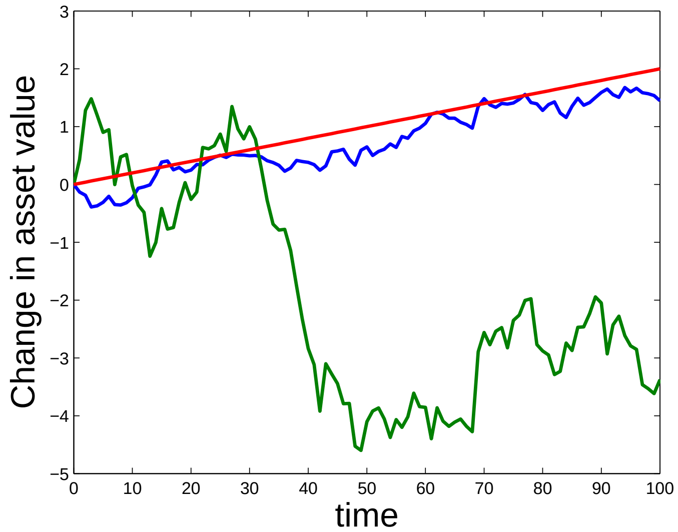

Figure 8.5: Realisations of Brownian motion generated from the equation (8.54) over 100 'days'. The drift rate is $\mu_d = 0.02$ \$ per day. The low volatility process is the blue line with $\sigma = 0.1$ \$ per day$^{\frac{1}{2}}$. The high volatility process $\sigma = 0.4$ is marked in green. The lines are made up of 100 line segments. The price change each day ΔS, in \$, is calculated from: $\mu_d \Delta t + \sigma_v \sqrt{\Delta t} Z$, and for the units used in this case, $\Delta t = 1$. The value of Z is created by a computer random number generator from a standard normal distribution. We see that neither line follows the deterministic straight line (in red) which would end up at $S = 2$ after 100 days.

Some examples of the types of a Monte Carlo simulation of Brownian motion are shown in figure (8.5). These (artificial) lines appear to resemble real asset price movements (although large jumps are more frequent in real movements).

From this relation beteen S_t and the standard normal distribution, we can obtain the probability density of the price change.

$$g(s)ds = P(s \leq S_t \leq s + ds). \tag{8.55}$$

Using the change of variable (8.29), we have:

$$g(s) = \frac{1}{\sigma_V \sqrt{t}} \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \qquad . \tag{8.56}$$

That is:

$$g(s) = \frac{1}{\sigma_V \sqrt{t}} \times \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(s-\mu_d t)^2}{2\sigma_V^2 t}\right] \qquad . \tag{8.57}$$

Since the movement depends explicitly on time, this expression is better written including t in the variables:

$$\boxed{g(s,t) = \frac{1}{\sigma_V \sqrt{2\pi t}} \exp\left[-\frac{(s-\mu_d t)^2}{2\sigma_V^2 t}\right]} \qquad . \tag{8.58}$$

This is the probability density for the price change for an asset which follows the process shown in figure 8.4. S_t has a normal distribution with mean $\mu_d t$ and variance $\sigma_V^2 t$. Thus the expected value and the variance change in time. Note: this function might also be familiar if you have studied the *diffusion equation.*

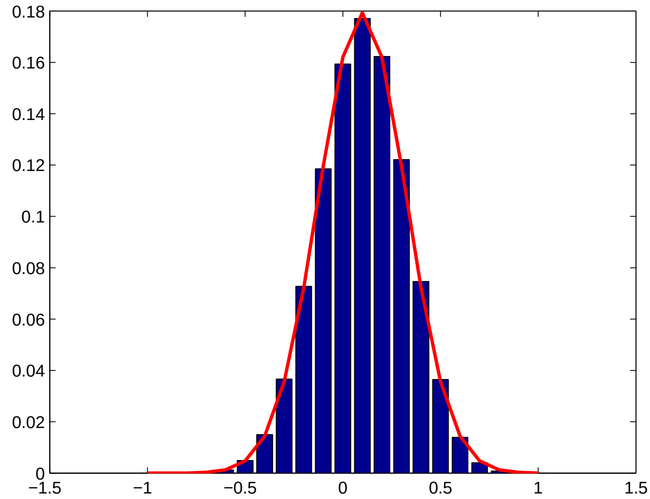

Figure 8.6: The figure shows the probabilities for the asset price change after $n = 20$ days. The horizontal axis is the price change, the vertical axis is the probability. The bar chart show the discrete probabilities given by the binomial distribution. In this case the mode (most likely value) is that the price increases by £0.05. The mean (expected value) is 0.100. These values are compared with the equivalent normal distribution given by the central limit theorem (red line). Clearly the approximate values, given by the central-limit theorem (8.58), are in good agreement with the exact values given by the binomial formula (8.46).

8.5.1 Example

Suppose we have a binomial process, like the one shown in figure 8.4. Consider an asset that, during each day, either goes up by an amount £0.05, with probability $p = 0.55$ or down by an amount £0.05, with probability $q = 0.45$. Calculate the probability that the asset price will, after 20 days, increase by £.15 or more.

Solution. The increase V of the asset price after n steps depends on the number of times i that the price has increased as $V = (2i - n)\delta s$, where δs is the daily change in asset price. We now need to sum over all probabilities $i \geq v$, where v is the minimum number of daily increases needed to increase the price by £0.15. This minimum number is given by

$$0.15 = (2v - n)\delta s = (2v - 20)0.05,$$

which gives $v = 11.5$. We thus need a minimum number of 12 daily price increases. The probability of i daily price increases is given by

$$P(i \text{ daily increases}) = \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i} \quad . \tag{8.59}$$

We now need to sum these probabilities over all possibilities for i, ranging from 12 to n, or

$$\sum_{i=12}^{20} \frac{20!}{i!(20-i)!} (0.55)^i (0.45)^{20-i} = 0.4143 \quad . \tag{8.60}$$

This is the probability that the asset price increases by £0.15 or more over at the end of a 20-day period.

We can also obtain an approximation to the answer by using the central-limit theorem (8.39). In this case, X_i is the price increase on day i, and so on up to $n = 20$ days. So S_n would then be the total price increase after 20 days, adding up all the daily price increases:

$$S_{20} = X_1 + X_2 + \cdots + X_{20} \quad .$$

We need to know μ_X and σ_X to use the formula (8.39), and for this we need the parameters:

$$\mu_x = \mathbb{E}\left(X\right) = 0.55 \times (+0.05) + 0.45 \times (-0.05) = 0.005 \quad .$$

and

$$\mathbb{E}\left(X^2\right) = 0.55 \times (+0.05)^2 + 0.45 \times (-0.05)^2 = 0.0025$$

Therefore:

$$\sigma_X = \sqrt{0.0025 - 0.000025} = 0.04975$$

So:

$$P(S_n \geq 0.15) = 1 - P(S_n \leq 0.15)$$

$$P(S_n \leq 0.15) = P(S_n/n \leq 0.15/20) = P\left(\frac{S_n/n - \mu_X}{\sigma_X/\sqrt{n}} \leq \frac{0.15/20 - 0.005}{0.04975/\sqrt{20}}\right)$$

$$P(Z \leq 0.2247) \approx N(0.2247) \qquad .$$

The value of $N(0.2247)$ can be looked up in the table or calculated by computer with the value: 0.5889. This gives the final result:

$$P(S_n \geq 0.15) = 1 - 0.5889 = 0.4111 \qquad .$$

Thus, there is roughly a 41% chance of this happening. Of course the approximation we have made, replacing a binomial distribution by a normal distribution means the result is not exact, but it is a very good approximation to the exact result (8.60).

To see why this works so well, we can plot the exact binomial distribution for the price values after 20 days. The probabilities for each price increase are shown as the bar chart (figure 8.6). And on the same graph we plot the normal distribution as a red line, the approximation given by the central limit theorem.

8.5.2 Values of the standard normal probability distribution

Recall the definition:

$$N(z) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}u^2} du \qquad .$$

and that, $N(-z) = 1 - N(z)$.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.00	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.10	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.20	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.30	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.40	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.50	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.60	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.70	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.80	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.90	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.00	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.10	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.20	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.30	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.40	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.50	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.60	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.70	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.80	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.90	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.00	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.10	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.20	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.30	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.40	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.50	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.60	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.70	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.80	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.90	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.00	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.10	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.20	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.30	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.40	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998