Chapter 14

Martingales

The purpose of this chapter is to make the connection between the Black-Scholes equation and measure theory. These approaches are equivalent mathematically, so one ends up with the exactly the same result at the end of the calculation. However, they take entirely different approaches: one through a differential equation, and the other through an integral equation.Many of the more advanced literature of financial mathematics uses martingales extensively, and so it is important to be aware of its existence. Although you could explore Financial Mathematics without thinking about Martingales, you would miss out on lots of elegant mathematics.

14.1 Feynman-Kac theorem

The formal connection between the Black-Scholes partial-differential equation and the risk-neutral measure is given by the Feynman-Kac (1949) relation. The ideas developed by Richard Feynman and Mark Kac (pronounced 'katz') were in fact developed as a general application in Physics that relates random (stochastic) processes to the solutions of parabolic partial-differential equations. It just so happens that this is exactly the problem we encounter in option pricing.

Theorem:

We will state this relation (and outline the proof) in a very simplified version as it applies to the Black-Scholes problem. Consider a one-dimensional convection-diffusion process in space and time. The process has a drift rate $\mu(x,t)$ and diffusion coefficient, $\frac{1}{2}\sigma^2$, described by the equation:

$$\boxed{\frac{\partial}{\partial t}u(x,t) + \mu(x,t)\frac{\partial}{\partial x}u(x,t) + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2}{\partial x^2}u(x,t) = 0} \quad , \tag{14.1}$$

where the space variable has the domain, $-\infty < x < +\infty$. Suppose there is a boundary condition (equivalent to the pay-off function) at time $t = T$, given by:

$$u(x,T) = h(x) \quad . \tag{14.2}$$

Then the solution of (14.1) subject to (14.2) is given by the *conditional expectation*:

$$\boxed{u(x,t) = \mathbb{E}\left(h(X_T)|X_t = x\right)} \quad 0 \leq t \leq T \quad , \tag{14.3}$$

where X_t follows the stochastic differential equation:

$$\boxed{dX_t = \mu(X_t,t)dt + \sigma(X_t,t)dW_t} \quad , \tag{14.4}$$

and W_t is a Wiener process.

Comment: The Feynman-Kac formula might appear to be a retrograde step: we have a nice linear parabolic PDE (14.1) which we convert back into its SDE (14.4) and expectation integral (14.3). However, the Feynman-Kac theorem is a mathematical bridge between the stochastic differential equation and integral equation methods where the independent variable behaves as (14.4).

Proof:

Consider the stochastic process, $u(X_t, t)$, where we are replacing the continuous (independent) variable x by the stochastic variable X_t which follows (14.4). Then, according to Ito's lemma,

$$du(X_t,t) = \left(\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2(X_t,t)\frac{\partial^2 u}{\partial X_t^2}\right)dt + \frac{\partial u}{\partial X_t}dX_t \quad . \tag{14.5}$$

Now equation (14.4) can be used for the last term on the RHS to give

$$du(X_t,t) = \left(\frac{\partial u}{\partial t} + \mu(X_t,t)\frac{\partial u}{\partial X_t} + \frac{1}{2}\sigma^2(X_t,t)\frac{\partial^2 u}{\partial X_t^2}\right)dt + \sigma(X_t,t)\frac{\partial u}{\partial X_t}dW_t \quad . \tag{14.6}$$

Then, formally, integrating from t (the present) to T (the future):

$$u(X_T,T) - u(X_t,t) = \int_t^T \left(\frac{\partial u}{\partial t} + \mu(X_t,t)\frac{\partial u}{\partial X_t} + \frac{1}{2}\sigma^2(X_t,t)\frac{\partial^2 u}{\partial X_t^2}\right)dt + \int_t^T \sigma(X_t,t)\frac{\partial u}{\partial X_t}dW_t \quad . \tag{14.7}$$

The LHS contains the difference of two random terms. Now we apply the boundary conditions. We calculate the (conditional) expectation of both sides, that is fixing the random variable X_t at x (the present) and then integrating over the future;

$$\begin{aligned}\mathbb{E}\left(u(X_T,T) - u(X_t,t)|X_t = x\right) &= \mathbb{E}\left(\int_t^T \left(\frac{\partial u}{\partial t} + \mu(X_t,t)\frac{\partial u}{\partial X_t} + \frac{1}{2}\sigma^2(X_t,t)\frac{\partial^2 u}{\partial X_t^2}\right)dt|X_t = x\right) \\ &+\mathbb{E}\left(\int_t^T \sigma(X_t,t)\frac{\partial u}{\partial X_t}dW_t|X_t = x\right) \quad .\end{aligned}$$

This simplifies to:

$$\begin{aligned}\mathbb{E}\left(u(X_T,T) - u(x,t)|X_t = x\right) &= \int_t^T \left(\frac{\partial u}{\partial t} + \mu(x,t)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 u}{\partial x^2}\right)dt \\ &+\mathbb{E}\left(\int_t^T \sigma(x,t)\frac{\partial u}{\partial x}dW_t\right) \quad .\end{aligned} \tag{14.8}$$

On the right we have the deterministic change (first term) and the expected stochastic change (second term) added together. Given that (14.1) is true, then the first term on the right disappears: this was by design. Next the second term also disappears since the average of the Wiener process is zero:

$$\mathbb{E}\left(dW_t\right) = 0 \quad . \tag{14.9}$$

This means the entire right-hand-side is zero. And we have the result (14.3)

$$u(x,t) = \mathbb{E}\left(u(X_T,T)|X_t = x\right) \tag{14.10}$$

But we know that, according to (14.2),

$$u(X_T,T) = h(X_T) \quad . \tag{14.11}$$

So then, we arrive at the final result:

$$u(x,t) = \mathbb{E}\left(h(X_T)|X_t = x\right) \tag{14.12}$$

which is a stochastic integral.

14.1.1 Application of the Feynman-Kac theorem to forward pricing

We can demonstrate the application of the theorem by considering, for example, forward pricing. The correct result was already found early in the course, and therefore we can verify that it indeed gives the correct answer. If the asset currently has a spot price S_t and the forward matures at a time $T \geq t$ then a fair strike price for the forward contract is:

$$F = S_t e^{r(T-t)} \qquad . \tag{14.13}$$

First we need to derive the partial-differential equation for the value of the forward contract.

To the holder of a forward contract, the party long in the contract, the value of this contract at maturity is:

$$f_T = S_T - F \quad , \tag{14.14}$$

where S_T is the value of the asset at time T. We suppose that S_t follows a geometric Brownian motion with a drift rate and volatility, μ_S and σ_S, then:

$$dS_t = \mu_S S_t dt + \sigma_S S_t dW_t \quad . \tag{14.15}$$

Then consider a balanced (hedged) portfolio *short* one unit of forward and long $\Delta > 0$ units of this asset. At time t the value of this portfolio is:

$$\Pi_t = -f_t + \Delta S_t \quad , \tag{14.16}$$

where the shorthand notation $f_t = f(S_t, t)$ is used. As usual, we make the portfolio risk free and use the arbitrage principle, along with Itô's lemma, to get the usual Black-Scholes equation:

$$\frac{\partial f}{\partial t} + rS_t \frac{\partial f}{\partial S_t} + \frac{1}{2}\sigma_S^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2} - rf = 0 \quad . \tag{14.17}$$

The boundary condition is given by (14.14).

The Black-Scholes equation can be converted to the form (14.1) by the following changes, just set $S_t = x$ and replace:

$$f(S_t, t) = u(x, t) e^{r(t-T)} \qquad . \tag{14.18}$$

The boundary condition (14.14) on u, that we need to be able to solve the equation, becomes:

$$u(x, T) = h(x) = x - F \quad , \tag{14.19}$$

since this expresses the value of the forward contract (to the holder) at the time of expiry, $t = T$.

Then comparing (14.17) with (14.1), we see that:

$$\mu(X_t, t) = rX_t \qquad , \qquad \sigma(X_t, t) = \sigma_S X_t \qquad . \tag{14.20}$$

Then our stochastic differential equation for X_t is (14.4):

$$dX_t = rX_t dt + \sigma_S X_t dW_t \qquad . \tag{14.21}$$

Thus, we have a geometric Brownian motion, but with the *risk-free drift*, r. This explains why we need to use the interest rate r in the stochastic differential equation rather than the drift rate of the asset μ_S. This is the principal reason the theorem is of importance.

Using the results from the previous chapters, we can integrate this stochastic equation to give:

$$X_T = X_t e^{(r - \frac{1}{2}\sigma_S^2)(T-t) + \sigma_S W_{T-t}} \quad . \tag{14.22}$$

Then we can find the solution, using (14.3);

$$\boxed{u(x,t) = \mathbb{E}\left(X_T - F | X_t = x = S_t\right)} \quad . \tag{14.23}$$

This can be evaluated since, using (14.18)

$$f(S_t,t) = u(x,t)e^{r(t-T)} = e^{r(t-T)}\mathbb{E}\left(S_t e^{(r-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S W_{T-t}} - F\right) \tag{14.24}$$

Now:

$$W_{T-t} = \sqrt{T-t}\ z \quad , \tag{14.25}$$

where z has the standard normal distribution, so that:

$$f(S_t,t) = e^{-r(T-t)}\mathbb{E}\left(S_t e^{(r-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\ z} - F\right) \quad . \tag{14.26}$$

The integral is easy enough, and we get:

$$f(S_t,t) = e^{-r(T-t)}\left[S_t e^{r(T-t)} - F\right] \quad . \tag{14.27}$$

Now our usual argument holds that the contract allows the holder to make or lose money, and that the initial value/price should be zero. That is both parties (long and short) agree that the price is fair and that no money should change hands. So:

$$f(S_t,t) = 0 \quad . \tag{14.28}$$

from which it follows that:

$$F = S_t e^{r(T-t)} \quad , \tag{14.29}$$

as found previously (14.13).