Chapter 16

Volatility

There are many dangers inherent in the use of mathematics for financial purposes. It may look like options could provide insurance to mitigate risk. However, insurance can have the side effect of taking on more risk than appropriate. Even if mathematics can provide means to mitigate risk, it is still important to bear in mind that insurance is only effective if the insurer is actually able to pay out.

The second danger is that we can put too much faith in mathematics. Any model of a real situation will contain approximations. These may be very small approximations, but they can also be quite significant. Within this module, we have assumed that asset prices move through geometric Brownian motion. However, is this actually the case?

In this section we apply statistical methods to financial engineering problems and present a few case studies. This section is merely an illustration of very simple statistical techniques. The combination of advanced statistical analysis and efficient numerical analysis is necessary to make money on capital financial markets.

In our discussion of pricing European options we have emphasised that, when the asset has a geometric Brownian motion, the option price depends on a set of known parameters: r, S, X, T. The interest rate (to borrow money on the market) is publicly known as is the spot price of the asset at that time. Furthermore the option will have an agreed strike price and the expiry date is established up front.

We have already deduced that, as far as the asset dynamics is concerned, there is no dependence on the drift rate μ, the only dependence is on the variability of the price represented by the asset volatility (σ).

So we can consider σ as the missing ingredient needed to calculate the asset price. It is therefore worthwhile to consider volatility in more detail.

16.1 Historic Volatility

The origin of σ is as a parameter of the stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \qquad . \tag{16.1}$$

We already have shown that the solution of this equation is:

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \qquad . \tag{16.2}$$

The evolution of the asset value is defined by two *parameters* and a single random variable. So we could use market data from historical records for the asset price to fit σ. The value of σ obtained in this way, by a statistical analysis is called the *historic volatility*. It is one possible way to evaluate the statistical parameter for our model and we can use standard *regression methods* to extract the values. More importantly, from the historical records, we can use statistical criteria (*goodness of fit*) to quantify whether the model (16.1) is realistic, and this is discussed in section 16.2.

Suppose we have historical data at various discrete times, $t_1, t_2, \ldots, t_N$ (not necessarily equally spaced) where $t_1 < t_2 < \cdots < t_N$. The data are the asset price values at these times, $S_1, S_2, \ldots, S_N$. We want to fit the data to the stochastic differential equation (16.1) which can be *approximated* as the stochastic *difference* equation:

$$\frac{\delta S}{S} = \mu \delta t + \sigma W_{\delta t} \tag{16.3}$$

where $\delta S/S$ would be the fractional price increase over a time δt. That is,

$$\boxed{\frac{S_{i+1} - S_i}{S_i} = \mu(t_{i+1} - t_i) + \sigma W_{t_{i+1} - t_i}} \quad . \tag{16.4}$$

So consider the variable

$$X_i = \frac{S_{i+1} - S_i}{S_i \sqrt{t_{i+1} - t_i}} \qquad , \quad i = 1, 2, \ldots, N-1 \quad . \tag{16.5}$$

Then, given that

$$W_{t_{i+1} - t_i} = \sqrt{t_{i+1} - t_i}\ z \quad , \tag{16.6}$$

where z is a standard normal random variable, the discrete equation (16.4) can be written as:

$$\boxed{X_i = \mu \sqrt{t_{i+1} - t_i} + \sigma z_i} \qquad . \tag{16.7}$$

Now, in mathematical terms, but *not* for a limited sample,

$$\mathbb{E}\left(z_i\right) = 0 \quad . \tag{16.8}$$

However, suppose we have loads of data, then we can invoke the *law of large numbers*. That is we can assume that the *sample average* approaches the *true average*. That is, in mathematical terms,

$$\lim_{N \to \infty} \frac{1}{N-1} \sum_{i=1}^{N-1} z_i \approx 0 \quad . \tag{16.9}$$

Then, applying this idea to equation (16.7) we say that:

$$\lim_{N \to \infty} \frac{1}{N-1} \sum_{i=1}^{N-1} X_i = \mu \frac{1}{N-1} \sum_{i=1}^{N-1} \sqrt{t_{i+1} - t_i} \qquad . \tag{16.10}$$

That is: $\mu \approx \hat{\mu}$, where we call $\hat{\mu}$ the *estimator* of μ:

$$\boxed{\hat{\mu} = \frac{\sum_{i=1}^{N-1} X_i}{\sum_{i=1}^{N-1} \sqrt{t_{i+1} - t_i}}} \tag{16.11}$$

If the times are equally spaced, $t_{i+1} - t_i = \Delta t$, for all i, this simplifies to:

$$\boxed{\mu = \frac{1}{(N-1)\Delta t} \sum_{i=1}^{N-1} \frac{S_{i+1} - S_i}{S_i}} \tag{16.12}$$

This gives us a formula to estimate μ. But, as we already know, μ is of no importance in the pricing options: we want σ !

Consider then the variance of equation (16.7), again invoking the law of large numbers:

$$\mathrm{var}(X_i) = \sigma^2 \qquad . \tag{16.13}$$

Replacing the theoretical expectation by the *sample average* leads to the expression, $\sigma^2 \approx \hat{\sigma}^2$,

$$\hat{\sigma}^2 = \frac{1}{(N-1)} \sum_{i=1}^{N-1} \frac{(S_{i+1}-S_i)^2}{S_i^2\ (t_{i+1}-t_i)} - \left[\frac{1}{(N-1)} \sum_{i=1}^{N-1} \frac{(S_{i+1}-S_i)}{S_i\ \sqrt{(t_{i+1}-t_i)}} \right]^2 \qquad . \tag{16.14}$$

$\hat{\sigma}$ is then the estimator of σ. Note that σ^2 has units (year^{-1}), so that σ has units (year$^{-1/2}$). Since this value of σ is based on historical data, it is called $\hat{\sigma}_H$ for *historic volatility.* We can now price the call/put options with the Black-Scholes formula using this historical estimate.

16.2 Statistical tests of geometric Brownian motion

The previous section explained how, given a model (16.1), one could obtain *estimators* for the parameters. In this section we examine the price data for two assets to assess whether they exhibit the characteristics for geometric Brownian motion. The first data set is taken from the New York Stock Exchange. In figure (16.1) the stock (share) price of Wal-Mart Stores Inc. is show as a function of time. The data corresponds to a one-year period (01/02/12-01/12/13) which translates to 252 trading days: the missing days are weekends and holidays. The curve has a prima facie Brownian motion with irregular and unpredictable motion. The data can be analysed in detail.

The drift rate $\hat{\mu}$ for Wal-Mart over this period was negative, using formula (16.11):

$$\hat{\mu} = -0.1145 \quad \text{year}^{-1} \qquad . \tag{16.15}$$

The volatility $\hat{\sigma}$ for Wal-Mart using (16.14) is:

$$\hat{\sigma} = 0.1571 \quad \text{year}^{-1/2} \qquad . \tag{16.16}$$

For the share price in Apple Inc., we have chosen $N = 249$ price values using the opening mid-price each day from April 2012 to April 2013.

The drift rate for the Apple share price, shown in figure (16.2), $\hat{\mu}$ is positive over this time frame, and is given by:

$$\hat{\mu} = +0.4512 \quad \text{year}^{-1} \qquad . \tag{16.17}$$

The volatility $\hat{\sigma}$ for Apple is quite high, and using (16.14) is found to be:

$$\hat{\sigma} = 0.3491 \quad \text{year}^{-1/2} \qquad . \tag{16.18}$$

There are simple *statistical measures* of distributions that allow us to compare the actual distribution with the standard distribution. These measures are the *central moments* of the distribution. Consider the fractional price change:

$$Y_i = \frac{S_{i+1}-S_i}{S_i} \qquad . \tag{16.19}$$

We define the sample mean of this variable:

$$\mu \equiv \frac{1}{N} \sum_{i=1}^{N} Y_i \qquad . \tag{16.20}$$

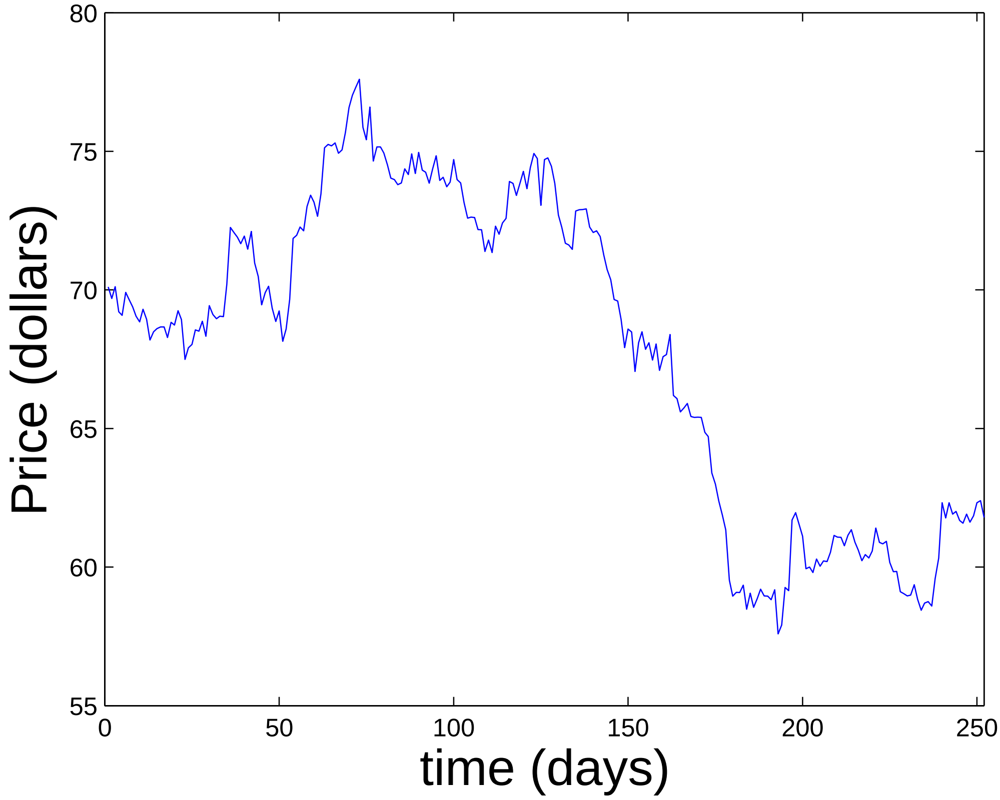

Figure 16.1: The stock price (share price) of Wal-Mart Stores Inc. from the New York Stock Exchange. The y-axis is the opening price each day, given in dollars. The x-axis is time in units of days. The starting date was 1 Feb 2012 and the end date 1 Feb 2013 giving a total of 252 trading days within 1 calendar year.

The n^{th} *sample central moment* is then defined as:

$$m_n \equiv \frac{1}{N}\sum_{i=1}^{N}(Y_i - \mu)^n \quad , \quad n = 0, 1, 2, 3 \ldots \quad . \tag{16.21}$$

Clearly $m_1 = 0$, and the second central moment gives the sample variance through the relation:

$$\mathrm{var}(y) \equiv m_2 \tag{16.22}$$

The *third* central moment is used to define the *skewness* of the distribution:

$$\gamma_1 \equiv \frac{m_3}{m_2^{3/2}} \quad . \tag{16.23}$$

This measures the asymmetry of the distribution, with $\gamma_1 > 0$ meaning a larger number (higher frequency) of large positive values than large negative values.

The *kurtosis* is a measure of the fatness or narrowness of a distribution generally with respect to the standard normal distribution. It is defined as:

$$\beta_2 \equiv \frac{m_4}{m_2^2} \quad . \tag{16.24}$$

A note of caution. Different definitions of skewness and kurtosis are in use so you should always check your references. The standard normal has zero skewness and kurtosis value 3. A higher value of

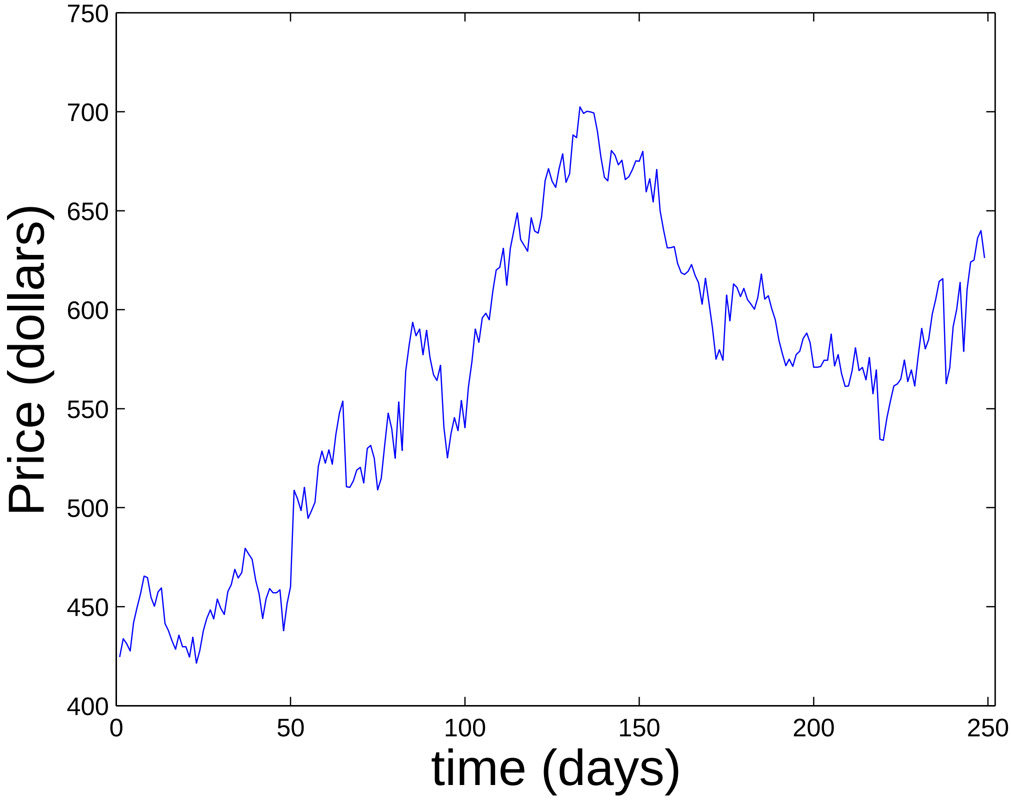

Figure 16.2: The stock price (share price) of Apple Inc. from the New York Stock Exchange. The y-axis is the opening price each day, given in dollars. The x-axis is time in units of days. The starting date was 9 April 2012 and the end date 8 April 2013 giving a total of 249 trading days within 1 calendar year.

kurtosis is associated with a sharper (than standard normal) central peak and a broader than normal distribution for values far from the centre. This is sometimes called a 'fat tailed' distribution or *leptokurtic* to the technically minded. A leptokurtic distribution for price jumps means that big price jumps occur more frequently than one would expect from a normal distribution. The sample data shown in figure (16.4) is rather a small sample, nonetheless it is leptokurtic.

When a process is leptokurtic, one can argue that the normal distribution is not a good fit. A commonly used distribution which is symmetric, and thus has zero skewness, but has fat tails is the Student distribution. In figure (16.5) we show the Student probability density, described by the equation,

$$f(\nu, x) = \frac{\Gamma\left[\frac{1}{2}(1+\nu)\right]}{\sqrt{\nu\pi}\;\Gamma\left[\frac{1}{2}\nu\right]} \times \left(1 + \frac{x^2}{\nu}\right)^{-\frac{1}{2}(1+\nu)} \quad . \tag{16.25}$$

where Γ is the gamma function defined as:

$$\Gamma(a) \equiv \int_0^{+\infty} e^{-u} u^{a-1}\, du \quad , \quad a > 0 \quad . \tag{16.26}$$

The cumulative distribution is given by:

$$P_S(z) = \int_{-\infty}^{z} f(\nu, x)\, dx \quad . \tag{16.27}$$

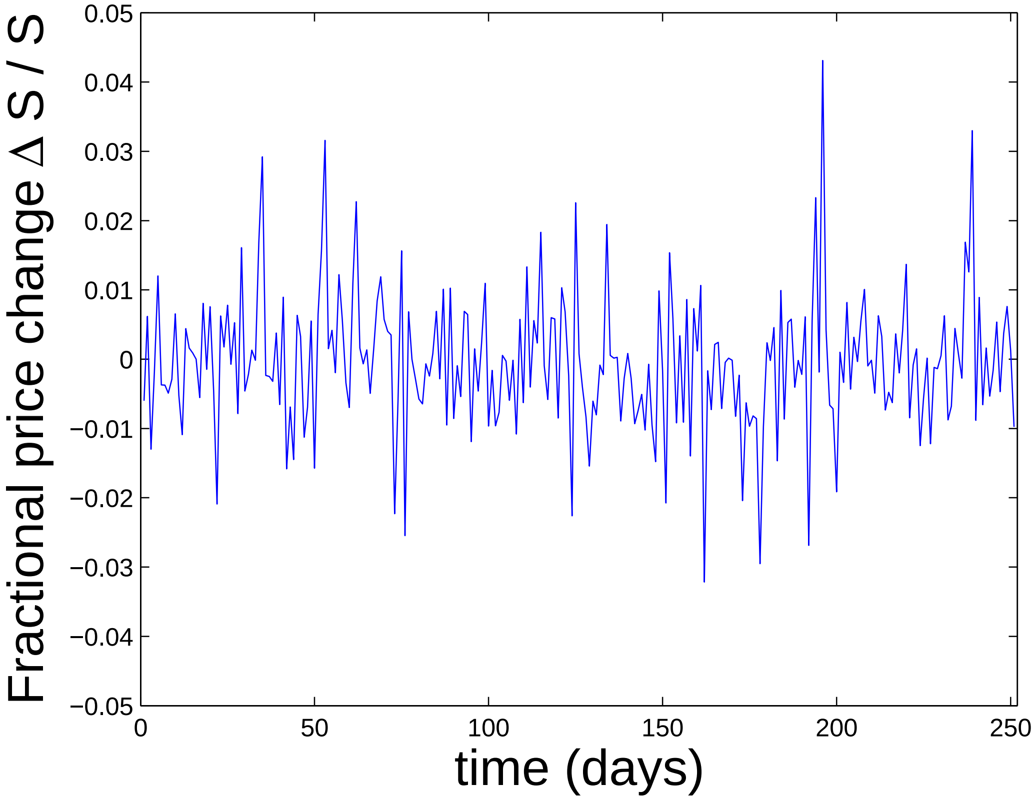

Figure 16.3: The fractional (daily) price change for Wal-Mart Stores Inc. from the New York Stock Exchange showing the volatility of the asset. The volatility is a measure of the amplitude of the variation. The y-axis is the day-to-day fractional price change $(S_{i+1} - S_i)/S_i$, the x-axis is time in units of days. The starting date was 1 Feb 2012 and the end date 1 Feb 2013 giving a total of 252 trading days.

For large ν, we can use Stirling's approximation to derive:

$$\frac{\Gamma\left[\frac{1}{2}(1+\nu)\right]}{\Gamma\left[\frac{1}{2}\nu\right]} \approx \sqrt{\frac{\nu}{2}} \quad . \tag{16.28}$$

and this means that for $\nu \to \infty$, the Student distribution tends towards the normal distribution,

$$\lim_{\nu\to\infty} f(\nu, x) = n(x) \quad . \tag{16.29}$$

The 'fat' tails shown in figure (16.5) mean that the variance for the Student distribution is infinite for $\nu \leq 2$. For $\nu > 2$,

$$m_2 = \frac{\nu}{\nu - 2} \quad . \tag{16.30}$$

The kurtosis is:

$$\beta_2 = 3 + \frac{6}{\nu - 4} \quad , \quad \nu > 4 \quad , \tag{16.31}$$

whereas it is infinite for $\nu \leq 4$.

16.2.1 Kolmogorov-Smirnov test

The previous section described how one would go about *estimating* parameters for a predetermined model using the data. This section focusses more on whether the model chosen is credible, based on

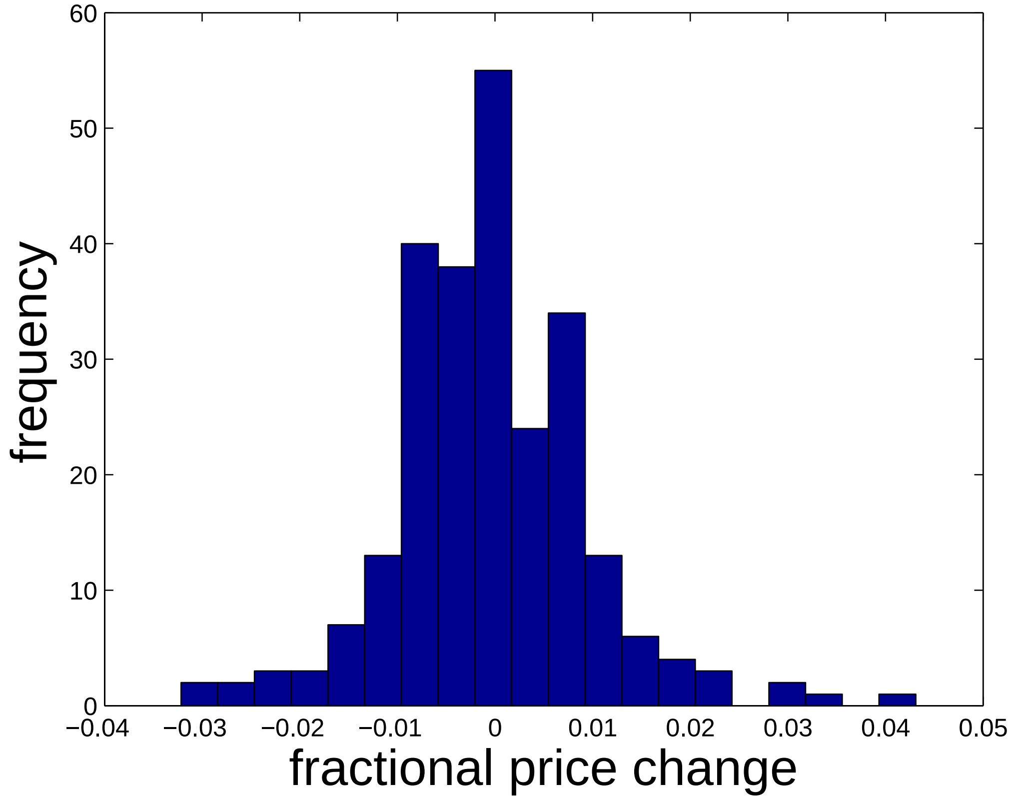

Figure 16.4: The frequency histogram for the fractional (daily) price change for Wal-Mart shares as illustrated in figure 16.3. The average of this distribution is -0.4545×10^{-3} with a standard deviation 0.9897×10^{-2}. To obtain the yearly averages, we need to multiply the daily change by $N = 252$, the number of trading days, for μ, and $\sqrt{N}$ for the standard deviation. The distribution is irregular (not smoothly varying) but has a strong central clustering: most price changes are small and the most probably price change (the mode of the distribution) is zero. The *skewness* (16.23) is positive with a value 0.4207 which means that large positive values (w.r.t. μ) are more prominent than large negative values. The *kurtosis* (16.24) has the value 5.4488 which means that the distribution has fat tails: large price jumps are more frequent than predicted by a similar normal distribution (which has kurtosis of 3).

the data. This is termed the goodness-of-fit of a model.

One simple method for goodness of fit is the Kolmogorov-Smirnov (KS) test. This statistical test is attractive in that it gives us a single number to tell us with what probability the model chosen is a good fit. However, just having a single number for a goodness-of-fit is also a weakness in that it does not give any information why and where the distribution succeeds or fails.

Let's define this test first in general terms. Firstly, we need to convert our discrete data into a continuous distribution. We can do this by defining the *experimental* or *empirical* probability distribution of a set of data as follows. For N data points, $\{z_1, \ldots, z_i, \ldots, z_N$, not necessarily in any order (ascending, descending or time ordering):

$$\boxed{P_E(z) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}\{z \geq z_i\}} \quad . \tag{16.32}$$

where the *indicator function* is defined as:

$$\mathbb{1}\{z \geq z_i\} = \begin{cases} 1 & z \geq z_i \\ 0 & z < z_i \end{cases} \tag{16.33}$$

So that:

$$\tag{16.34}$$

We aim to compare this empirical distribution with a theoretical distribution (our model), which may or may not be parametric. Then this would have the form:

$$P_T(x) = P(X \leq x) \qquad . \tag{16.35}$$

We define the discrepancies between the functions (16.32) and (16.35). The maximum distance D between the distributions is defined as:

$$D = \max_x |P_T(x) - P_E(x)| \qquad . \tag{16.36}$$

The notation $\max\limits_x f(x)$ means the maximum value of the function f with respect to the variable x. So the larger the value of D the more dissimilar the distributions. The Kolmogorov-Smirnov test quantifies the dissimilarity in terms of a p-value.

If D is the maximum distance between the experimental and theoretical distributions, then the probability, p, that the fit is *not* correct is given by:

$$p = 1 - 2\sum_{n=1}^{\infty}(-1)^{n-1}e^{-2n^2ND^2} \tag{16.37}$$

So clearly if D is large then $p \to 1$. So we can make $1-p$ the confidence level that the fit is correct.

In figure 16.6 we have plotted a standardized value of the fractional jump distribution. We change from Y_i to:

$$Z_i = \frac{Y_i - \mu_Y}{\sigma_Y} \qquad , \tag{16.38}$$

so that Z has a zero (sample) mean and a unit (sample) variance. This puts the distribution on the same footing as the standard normal. In figure 16.6, we also plot the standard normal and Student distribution (with $\nu = 10$) for comparison. For this set of data we had already mentioned the presence of fat tails, and this is exhibited by $P_E(z)$ being higher that $P_N(z)$ on the left and lower on the right. In fact, the Student distribution with $\nu = 10$ appears to describe the long-range tail well, whereas for smalll changes in asset price, the standard normal distribution appears to be a better fit.

16.3 Implied Volatility

In pricing an option, the unknown parameter is the *volatility* σ. In the previous section, we made the *assumption* that our model was geometric Brownian motion and by making a fit to the historical data, we could estimate the volatility. The estimate for σ when this is done is called the *historic volatility.* In this way we could estimate prices for options for any given expiry dates and strike prices.

However this assumes that the model of geometric Brownian motion is correct. Another way in which we could test this hypothesis would be to examine the *market prices* for options. That is, given the the option price, and assuming that the price is correct one could invert the process (reverse engineering) and find the value of volatility that the formula implies. This is called the *implied volatility.*

So, beginning with the market price, for a given S, X, r, T, for the European call option,

$$c_{\mathsf{market}}(S, X, r, T) \qquad , \tag{16.39}$$

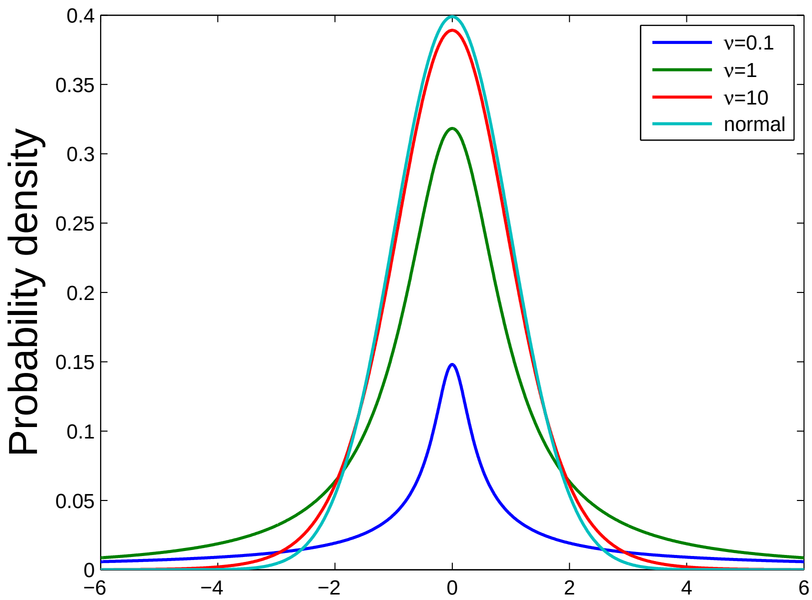

Figure 16.5: The Student distribution (16.25) for different values of ν. For large ν the distribution tends towards the normal distribution. As ν decreases, the distribution has a sharper central peak with broader tails, meaning that large values of x are more probable.

we could set this value equal to the Black-Scholes formula as then invert this to find σ. That is solve the equation:

$$\boxed{c_{\mathsf{market}}(S, X, r, T) = c_{\mathsf{BS}} = SN(d_1) - Xe^{-rT}N(d_2)} \quad , \tag{16.40}$$

for σ_I. If, in this exercise, we found a consistent value for σ_I for all combinations of X and T (S and r are independent of the option), then it would support the validity of the Black-Scholes model in pricing (16.40).

The equation (16.40) cannot be solved in simple terms. Instead we use numerical methods. The numerical problem we have to solve is one of a general class of *root finding*, that is finding the values of x for a given function such that

$$f(x) = 0 \quad . \tag{16.41}$$

In our case we seek the roots of the function:

$$f(\sigma) = c_{\mathsf{BS}}(S, X, r, T, \sigma) - c_{\mathsf{market}}(S, X, r, T) \tag{16.42}$$

There are many excellent numerical methods to solve such problems. One of the simplest and (under certain conditions we won't discuss here) most efficient methods goes back to Newton.

Newton-Raphson method

The Newton-Raphson iteration proceeds as follows. If x_n is a guess at the solution of $f(x) = 0$, then

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad . \tag{16.43}$$

is a better guess, provided we are sufficiently close to the root.

In this case:

$$\sigma_{n+1} = \sigma_n - \frac{c_{\mathsf{BS}}(\sigma_n) - c_{\mathsf{market}}}{\mathcal{V}_c} \tag{16.44}$$

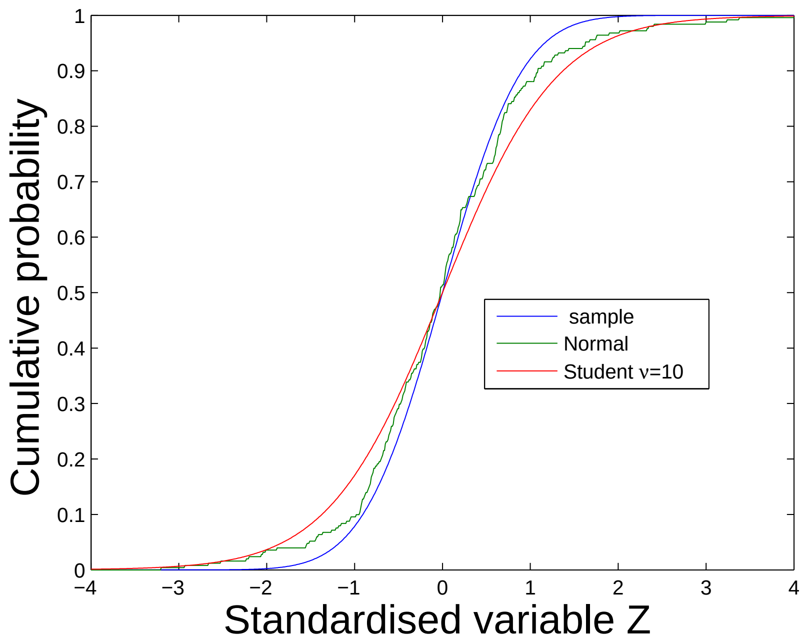

Figure 16.6: Goodness of fit comparisons between the standardized sample distribution $P_E(x)$ given by (16.32) and a standard normal distribution $P_N(x) = N(x)$, along with a Student distribution (16.27) with $\nu = 10$. All three curves have common values at $z = 0, \pm\infty$. The differences can be seen with P_E being asymmetric (skew) and having a fatter (leptokurtic) tail than the standard normal. A Kolmogorov-Smirnov test statistic can be calculated to quantify the goodness of fit. In this case, neither of the two theoretical models is particularly good, but the data set is quite small, $N < 250$, so one cannot make a definitive conclusion based on this result.

where as before:

$$\mathcal{V}_c \equiv \frac{\partial c_{\mathsf{BS}}(\sigma)}{\partial \sigma} \quad . \tag{16.45}$$

Let's take a practical example. The April 2013 values of call options on Apple shares with expiry date in August 2013 are tabulated versus strike price X in table 16.1 and sketched in figure 16.7.

An example of the implied volatility calculation is shown in figure 16.8 which is taken from the data for shares in Apple. In this case we are given a future expiry date for the call option. For this date, option prices for different strike prices are given.

The variation in implied volatility indicated in figure 16.8 is typical of market prices. In general the option prices do *not* obey simple Black-Scholes pricing, although they do to a good approximation. One can then conclude either the market prices are wrong or the Black-Scholes formula is wrong. In practical terms it is not clear which is the case. Given our historical analysis of the asset price variation, we do not have clear evidence for the validity of Brownian motion. However this was based on just one example of one particular asset over one particular timeframe.

It is unreasonable that a model as simple as geometric Brownian motion (GBM) would really explain asset price variation. In particular, geometric Brownian motion is incapable of describing stock market shocks. So it is rather surprising that, in this case, this simple model gives reasonable results for asset pricing.

To some extent, the market has learned from previous failures. Up to Black Monday (19 October 1987), when the stock market fell by 20%, option prices for all strike prices were based on the same volatility.

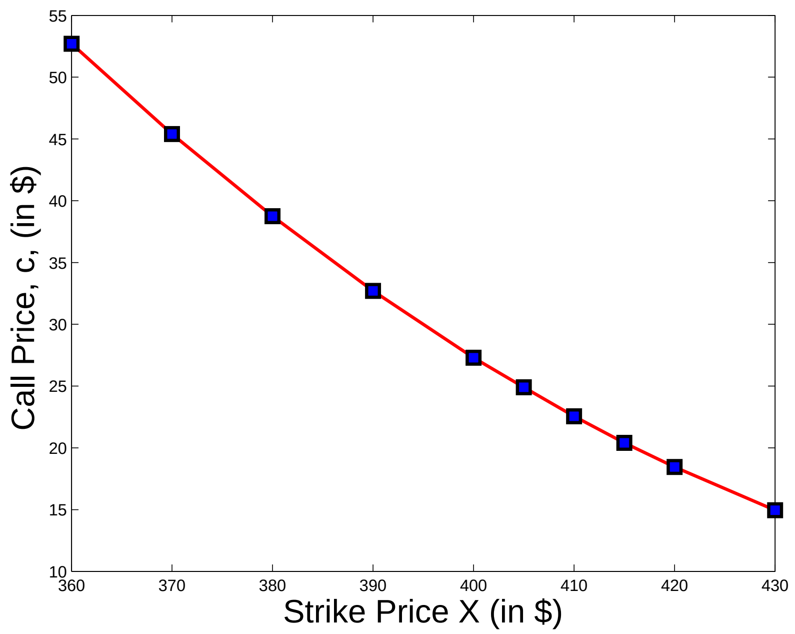

Figure 16.7: Prices (in \$) for European call option prices in shares for Apple versus Strike price X. The April 2013 price of the asset was $S = \$411$, and the option expiry date was August 2013. As expected we note that as X increases, c decreases, that is $\partial c/\partial X < 0$, and we note that $\partial^2 c/\partial X^2 > 0$.

16.3.1 Frowns and smiles

The variations in implied volatility mean that the curves can have different shapes. If the implied volatility decreases then increases, with some kind of U-shape, it is said to have a *volatility smile*. On the other hand, if the volatility rises then falls, we have an inverted smile - called a *volatility frown*. If the volatility slopes down as the strike price increases it is called a *volatility reverse skew* or *smirk*. While if the volatility goes upwards it is the *forward skew*.

In figure 16.8 we have an increasing volatility as the strike price increases. How might one interpret this within the GBM model and Black-Scholes formula? We have mentioned that increasing σ means increasing the value of the call option: the Vega of a call is positive. So, by the same token, we could say that increasing call values means increasing volatility, and decreasing call prices mean a lowering of volatility. But lower call prices could also occur through market demand. There may be more demand for put options, raising the price of a put option and, through put-call parity, lowering the price of a call option.

So one possible rationalisation for the lower volatility on the left of figure (16.8) is that in this case the market shows that there is less demand for call options for Apple shares well below the spot prices.

In this way, option prices provide an indication of market opinions regarding assets. Traders can thus use volatility indices to make investment decisions as they can see the assessment of other traders in general. In this way, the Black-Scholes formula is used more as a yard-stick of market sentiment rather than a tool to calculate prices.

Strike Price	Call Price
360	52.70
370	45.40
380	38.75
390	32.70
400	27.30
405	24.90
410	22.55
415	20.40
420	18.45
430	14.95

Table 16.1: Prices for European call option prices in shares for Apple. The spot price of the asset was (April 2013) $S = 411$ the option expiry date was August 2013. The risk-free interest rate r is unfortunately unknown, but likely to be small.

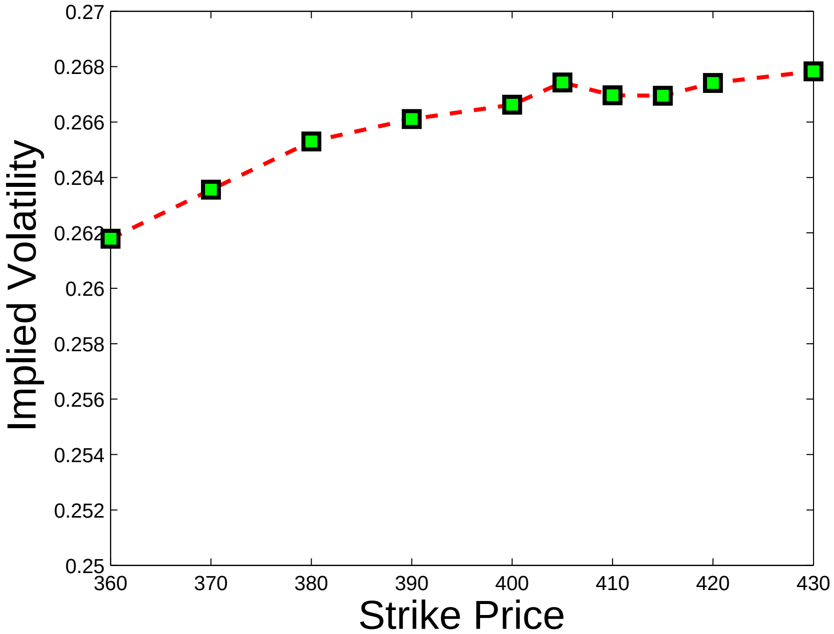

Figure 16.8: Values of the implied volatility for European call option prices in shares for Apple. The spot price of the asset was (April 2013) $S = 411$. We consider the options for strike prices close to this value, with the option expiry date in August 2013. We see that the implied volatility $\sigma \approx 0.265\ \mathrm{yr}^{-1/2}$. Since there is a slight variation this indicates that the option prices do not follow the Black-Scholes formula exactly.

16.4 Extension of Black-Scholes: stochastic volatility

The natural thing to do with a simple model that doesn't quite work, is to add refinements that make it work better, There have been, and continue to be, many different models of this type. There are too many of these models to discuss, and most of these models will require computer codes to derive more accurate option prices. Normally, only the simplest models can be solved analytically.

One popular model, which allows for variation of the volatility, is to allow the volatility itself to be stochastic. Again, there are several flavours of such models, but the simplest is the Heston model. This can be easily summarised by the mathematical relations, firstly a Brownian type motion for the asset:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t \qquad , \tag{16.46}$$

and

$$dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t} dW_t \quad . \tag{16.47}$$

The Wiener processes in equations (16.46) and (16.47) can be correlated in general. The equation (16.47) is known as a Cox-Ingersoll-Ross process after its introduction in a paper in 1985[1]. We see the equations are coupled, but not symmetrically unless their respective Wiener processes are. Equation (16.47) 'feeds in' to (16.46) but not vice versa. So they don't require simultaneous solution.

So, v_t is similar to the square of the volatility, and by necessity $v_t \geq 0$. The equation (16.47) resembles the Ornstein-Uhlenbeck process, at least the first term on the right is the same, and this is the damping term. So, we can deduce the asymptotic (long run) expected-value of the process.

Taking expectation of (16.47) we get,

$$\frac{d}{dt}\mathbb{E}\left(v_t\right) = \kappa(\theta - \mathbb{E}\left(v_t\right)) \quad . \tag{16.48}$$

This ordinary first-order equation is easily solved:

$$\mathbb{E}\left(v_t\right) = v_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}), \tag{16.49}$$

where v_0 is the initial value of v_t. The long-run tendency of the expected value towards a constant is a characteristic of a *new-reverting process*. In this case κ is the rate of reversion and θ the 'level of reversion'.

A further calculation, which follows the same lines as the previous one, but for which we leave out the details, gives

$$\mathrm{var}(v_t) = \frac{\xi^2 v_0}{\kappa}\left(e^{-\kappa t} - e^{-2\kappa t}\right) + \frac{\theta\xi^2}{2\kappa}\left(1 - e^{-\kappa t}\right)^2 . \tag{16.50}$$

However, the downside to these more complicated models, such as (16.46), is that the solution for option pricing is more complicated. Without going into details, it is interesting to see how the Black-Scholes equation is modified. Technically, one can only make the arbitrage argument if one can trade in risk-free volatility: the equivalent of bonds for volatility. So it is stretching reality to develop the theory much further, but if we do, then we end up with a PDE of the following form for a call option:

$$\frac{\partial c}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 c}{\partial S^2} + rS\frac{\partial c}{\partial S} - rc + \frac{1}{2}\xi^2 v\frac{\partial^2 c}{\partial v^2} + \rho\xi vS\frac{\partial^2 c}{\partial S\partial v} + \left[\kappa(\theta - v) - \lambda\right]\frac{\partial c}{\partial v} = 0 \quad . \tag{16.51}$$

The first four terms are the usual Black-Scholes terms, the last three terms are the modifications. The symbol ρ is the correlation coefficient for the two Wiener processes:

$$\rho dt = \mathbb{E}\left(W^s_{dt} W^v_{dt}\right) \quad , \tag{16.52}$$

[1]Cox JC, Ingersoll JE and Ross SA (1985). *A theory of the term structure of interest rates.* Econometrica 53: 385-407.

and λ is a market rate of volatility risk. Although the equation appears complicated, surprisingly, one can find a closed form for the option prices [2].

Since the volatility is not constant, the asset prices no longer follows a GBM, and this feature is appealing since real prices do not strictly follow GBM motions. For details on the solution see the excellent book by Kwok [3].

[2] S. Heston, A closed-form solution for options with stochastic volatility. The Review of Financial Studies (1993).
[3] Y-K Kwok (2008) Mathematical models of financial derivatives.