Chapter 9

The Binomial model and diffusion processes

An alternative to the application of the central-limit theorem for the random asset price movement over time is to study the behaviour of its probability density as it changes over time. Let $u(s,t)$ be the probability density for the asset price at a time t. Formally it is defined as

$$P\left(s \leq S_t \leq s + \delta s\right) = u(s,t)ds \qquad . \tag{9.1}$$

Now we want to consider how $u(s,t)$ changes over time when the price follows a binomial process. The approach mirrors that used by Einstein in his 1905 paper on Brownian motion.

We now consider $u(s,t)$ in this two-dimensional space.

The binomial model for S_t (see figure 9.1) tells us that the probability density at C depends on the probability density at A and B. To get to C we must pass via A or B. Mathematically this relation is expressed by the *partition rule*, see section 5.4:

$$P(C) = P(C|A)P(A) + P(C|A^c)P(A^c) \qquad . \tag{9.2}$$

which in this case is just:

$$P(C) = P(C|A)P(A) + P(C|B)P(B) \qquad . \tag{9.3}$$

This we can write as,

$$P(C) = P(A)P(A \to C) + P(B)P(B \to C) \qquad , \tag{9.4}$$

where $P\left(C\right)$ is the probability mass at C, which in the continuous limit becomes the probability density. Then $P\left(A\right)$ the probability that the asset is at A (has the value $s + \delta s$ at time t) and $P(A \to C)$ is the *transition probability* that is the probability that the price jumps down to C from A at time $t + \delta t$. Similarly, $P\left(B\right)$ is the probability (density) at B: the probability the asset has the value $s - \delta s$, at the time t. Then $P(B \to C)$ is the probability that the price jumps up from B to C.

From the binomial model one knows that the probability of a price decrease over a time δt is:

$$P(A \to C) = q \quad , \tag{9.5}$$

while the probability of a price increase is:

$$P(B \to C) = p \qquad . \tag{9.6}$$

Then the equation (9.4) can be written:

$$u(s, t + \delta t) = u(s - \delta s, t)p + u(s + \delta s, t)q \qquad . \tag{9.7}$$

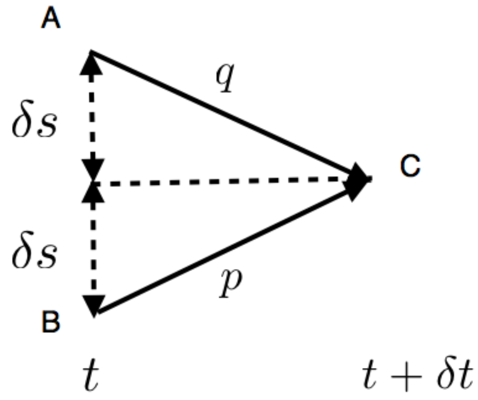

Figure 9.1: The price-time $(s-t)$ diagram for the price-change probability density, $u(s,t)$ in which the asset price follows a binomial process. The price probability at the point C, coordinates, $(s, t+\delta t)$, depends on the probabilities from the earlier time t and the points A, coordinates $(s+\delta s, t)$ and B, with coordinates $(s-\delta s, t)$.

If we now take the limit for δs, $\delta t \to 0$, this equation can be rewritten as a partial differential equation relating infinitisemal changes in s to infinitisemal changes in t.

To derive this partial differential equation, we expand u in a Taylor series around s and t:

$$u(s, t+\delta t) \approx u(s,t) + \delta t \frac{\partial u(s,t)}{\partial t} + \frac{1}{2}(\delta t)^2 \frac{\partial^2 u(s,t)}{\partial t^2} + \dots \qquad . \tag{9.8}$$

$$u(s \pm \delta s, t) \approx u(s,t) \pm \delta s \frac{\partial u(s,t)}{\partial s} + \frac{1}{2}(\delta s)^2 \frac{\partial^2 u(s,t)}{\partial s^2} + \dots \qquad . \tag{9.9}$$

The LHS of equation (9.8) can be written as:

$$u(s, t+\delta t) \approx u(s,t) + \delta t \frac{\partial u}{\partial t} + \dots \qquad . \tag{9.10}$$

The RHS of equation (9.8) is then:

$$u(s \pm \delta s, t) \approx p\left[u - \delta s \frac{\partial u}{\partial s} + \frac{1}{2}(\delta s)^2 \frac{\partial^2 u}{\partial s^2} + \dots\right] + q\left[u + \delta s \frac{\partial u}{\partial s} + \frac{1}{2}(\delta s)^2 \frac{\partial^2 u}{\partial s^2} + \dots\right] \qquad . \tag{9.11}$$

recalling that $p+q=1$, then we have, for the RHS:

$$u(s \pm \delta s, t) \approx u - (p-q)\delta s \frac{\partial u}{\partial s} + \frac{1}{2}(\delta s)^2 \frac{\partial^2 u}{\partial s^2} + \dots \qquad . \tag{9.12}$$

Equating the LHS and RHS we have,

$$u + \delta t \frac{\partial u}{\partial t} + \dots = u - (p-q)\delta s \frac{\partial u}{\partial s} + \frac{1}{2}(\delta s)^2 \frac{\partial^2 u}{\partial s^2} + \dots \qquad . \tag{9.13}$$

To leading order we then have:

$$\frac{\partial u}{\partial t} = -(p-q)\frac{\delta s}{\delta t}\frac{\partial u}{\partial s} + \frac{1}{2}\frac{(\delta s)^2}{\delta t}\frac{\partial^2 u}{\partial s^2} \qquad . \tag{9.14}$$

In the previous chapter, we have already defined

$$\mu_d = (p-q)\frac{\delta s}{\delta t} \qquad . \tag{9.15}$$

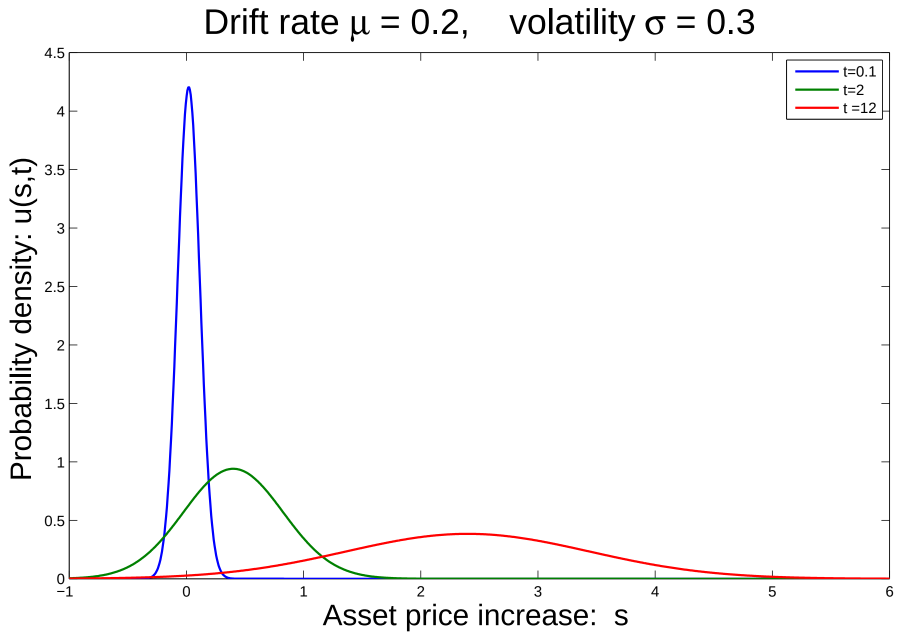

Figure 9.2: Solution of the convection-diffusion equation (9.17) showing the probability density for the change in asset price s with time t. The function $u(s,t)$ is given by equation (9.25). The curves shown are for a drift rate $\mu_d = 0.2$, at the times $t = 0.1$, 2, and 12. The peak (maximum) of probability travels at the drift rate, so that at $t = 12$ the maximum probability is at $s = 0.2 \times 12 = 2.4$.

We will further assert (without any justification at this point):

$$\lim{}_{\delta s, \delta t \to 0} \quad \frac{(\delta s)^2}{\delta t} = \text{constant} = \sigma^2 \qquad . \tag{9.16}$$

Then we have the following partial-differential equation

$$\boxed{\frac{\partial u}{\partial t} = -\mu_d \frac{\partial u}{\partial s} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial s^2}} \qquad . \tag{9.17}$$

This is called the convection-diffusion equation which commonly arises in Physics, Chemistry and Engineering. It is known as the Kolmogorov equation in probability theory.

The reason we need to keep the second-order derivative in ∂s arises from the stochastic term Z in (8.54). This term has a time dependence of $\sqrt{t}$. The first-order term vanishes as the mean of Z is 0. The second-order term scales linearly with t, and therefore remains important. Since the expectation of Z^2 , or the variance of Z, has a value 1, the second-order term needs to be retained in the partial differential equation.

The convection-diffusion equation describes, for example, the spread of chemical pollution in a river. Let u be the concentration of the chemical, then it spreads in a river by two processes (i) convection, that is flowing downstream, and (ii) diffusion, spreading out (dilution). We will now see the mathematical form of the solution.

First, we can simplify the equation by removing the convection term. This can be done by a change of variable, more technically a Galilean transformation. Mathematically this means we can remove the flow/convection term in the equation.

Let:

$$\boxed{s' = s - \mu_d t} \qquad , \tag{9.18}$$

be the new coordinate. Then using the chain rule we have:

$$\left(\frac{\partial}{\partial s}\right)_t = \left(\frac{\partial s'}{\partial s}\right)_t \left(\frac{\partial}{\partial s'}\right)_t = \left(\frac{\partial}{\partial s'}\right)_t \quad , \tag{9.19}$$

and

$$\left(\frac{\partial}{\partial t}\right)_s = \left(\frac{\partial}{\partial t}\right)_{s'} + \left(\frac{\partial s'}{\partial t}\right)_s \left(\frac{\partial}{\partial s'}\right)_t = \left(\frac{\partial}{\partial t}\right) - \mu_d \left(\frac{\partial}{\partial s'}\right)_t \quad . \tag{9.20}$$

In fact this type of time derivative is called the *convective derivative* in fluid mechanics. The effect of this change of variable is to simplify equation (9.17), since it now becomes:

$$\left[\left(\frac{\partial}{\partial t}\right)_{s'} - \mu_d \left(\frac{\partial}{\partial s'}\right)_t\right] u = -\mu_d \left(\frac{\partial u}{\partial s'}\right)_t + \frac{1}{2}\sigma^2 \left(\frac{\partial^2 u}{\partial s'^2}\right) \quad , \tag{9.21}$$

resulting in

$$\left(\frac{\partial u}{\partial t}\right) = \frac{1}{2}\sigma^2 \left(\frac{\partial^2 u}{\partial s'^2}\right) \quad . \tag{9.22}$$

where $s' = s - \mu t$. Of course the diffusion equation has an infinite number of possible solutions.

One particularly important solution of this equation is:

$$u(s', t) = \frac{1}{\sqrt{2\pi t \sigma^2}} e^{-s'^2/2\sigma^2 t} \quad . \tag{9.23}$$

The correctness of this solution can be verified by substitution, and can also be derived.One can further verify that:

$$\int_{-\infty}^{+\infty} u(s', t) ds' = 1 \quad . \tag{9.24}$$

so that this is indeed a properly normalised function. Then changing back from s' to s we can write:

$$\boxed{u(s, t) = \frac{1}{\sqrt{2\pi t \sigma^2}} e^{-(s-\mu_d t)^2/2\sigma^2 t}} \quad . \tag{9.25}$$

This particular solution is a special solution. It is called the Green's function or the fundamental solution of the diffusion equation. It is plotted in figure (9.2) in which we see that, as time increases the peak/maximum of the probability density moves at the drift rate μ_d and the probability density spreads as t increases. However we also note that at $t \to 0$, the range of asset prices gets narrower and narrower, since there is much less uncertainty in the asset value soon after the random process begins. Of course, since the integral over the probability density has to equal 1, as the curve gets narrower it also gets taller until, at $t = 0$, the curve will be infinitely high and infinitesimally thin. In fact, we find:

$$\boxed{\lim_{t \to 0} u(s, t) = \delta(s)} \quad . \tag{9.26}$$

Here we use the Dirac delta-function, mentioned in section 8.3 . It is a probability density that is zero everywhere except at one point, yet has an area under the curve equal to 1.

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ +\infty & x = 0 \end{cases} \tag{9.27}$$

with the property,

$$\int_{-\infty}^{+\infty} \delta(x) dx = 1 \quad . \tag{9.28}$$

This explains why *this* solution (9.25) is the solution we want. It satisfies the initial condition that at $t = 0$ we are certain of the price. The delta-function applies to continuous random variables which are not random. So we must have $u(s, 0) = 0$, for all $s \neq 0$.

It is only after the known present, $t = 0$, that we enter the unpredictable future $t > 0$. And this future is more and more uncertain as time progresses, as is shown in the figure (9.2). The broadening of the distribution indicates a growing uncertainty.

The convection-diffusion equation is a simple version of a class of equation called the Fokker-Planck equation. This equation has the form:

$$\frac{\partial}{\partial t} f(x,t) = -\frac{\partial}{\partial x}\left[\mu(x,t) f(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[D(x,t) f(x,t)\right] \qquad , \tag{9.29}$$

and is a widely studied equation arising from its many applications in the study of particle motion under random forces.