AMA3007: FINANCIAL MATHEMATICS

Stochastic calculus applied to derivative pricing

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Chapter 1

Financial Markets

This course of lectures is about the mathematical theory behind buying and selling (trading) in capital markets. Like any branch of applied mathematics this will require learning quite a bit of jargon that is used in the financial services industry, and a glossary is provided to help with the definition of certain terms mentioned in the text.

To be successful in trading in a market, the aim is rather simple: buy something which is cheap (undervalued) and sell it when it is expensive (overvalued). Let's call this the first law of a *rational trader*: one aims to make a profit, or under less favourable conditions minimise a loss. In this sense, one can define a *trading strategy* as corresponding to a mathematical problem of *optimisation* : getting the maximum value from your investment. The problem is estimating the *value* of the product one is buying or selling. One should always make the distinction between the price of a product (how much it costs to buy) and its value (what something is worth). Price and value are not the same thing.

The definition of a market is a place where *buyers* meet *sellers* to trade. In order for a trade to occur, the seller requires a buyer willing to buy at the price on offer. Therefore the price depends on what the buyer is willing to pay - not its intrinsic value. So this price is arrived at through a negotiation. It is a subjective opinion of the two parties as to what the price is.

The financial markets are no different to a farmers' market in that respect, buyers seek out the cheapest prices while the sellers always aim to get the highest possible price for their products. The price at which the buyer and seller agree is called the *market price*. Unlike farmers' markets, the goods being traded are not consumables, but often obscure intangible products, the values of which are not well defined. However, in spite of the uncertainty in these values, in fact as a consequence of this uncertainty, there is a huge market in such trades. This is where traders are forever trying to outwit each other with the aim of making a profit for their investors. In estimating the price, mathematical methods are often used.

This course won't teach you how to become a trader. However, you will learn something about the kinds of products that are bought and sold by traders. In particular, we focus on a class of products called *derivatives*. In this course we will see how mathematics gives the answer to the question as to the *fair price* of these *derivatives* based on *fair value*. The advantage of knowing the *fair price* is allowing one to determine if a product is under/over-valued, and this could be the basis for making a profit.

1.1 Assets and Instruments

1.1.1 Shares

An example of an investment would be buying *shares* in a company. While the price of this investment is transparent, the value is not. Shares are called *stocks* in the US, and buying shares is called an *equity investment.*

Of course the share price is really determined by how much someone is willing to pay for it. Naturally, this is based on some information or data, such as the company financial reports. This still leaves a great deal of uncertainty. There is further uncertainty in terms of how this value will change in the future. Since the future is unpredictable, so is the value of the asset, and thus there is risk in making such an investment.

Financial products (or *instruments*) were created to address this problem of risk; offering a kind of *insurance* against this unpredictability. The prices of these *instruments* are *derived* from the asset prices on which they are based. For this reason these instruments are generically called *derivatives.* We can consider the price of a derivative as a *function* of the asset price, and other variables. It turns out that the price/value of these derivatives, within certain modelling assumptions, are described by sets of partial-differential equations. This is where Mathematics comes in: the modelling and solution of these equations, and thus the accurate pricing of *derivatives.*

While *derivatives* are often described as instruments that reduce risk (insurance policies), they can also used to take risks and make huge profits and losses. In these lectures we will see how one can compensate (*hedge*) risk in holding an asset by simultaneously buying/selling a derivative based on the asset. *Hedge funds* are companies that trade in these derivatives and aim to use advanced mathematics to out-smart other investors by correctly valuing investments. The balancing act of trading, simultaneously, a combination of assets and derivatives requires an accurate understanding of their values. This series of lectures will not provide the answer to that question, but it will provide an introduction to the type of mathematics that is used to calculate the values of these *derivatives.*

1.2 The Stock Market

The *stock market* is a *primary* capital market which companies use to acquire money (*capital*) to run their business. This is done by selling *shares* in the company to *investors*[1]. Shares are called *stocks* in the USA and one also uses the term *equities* just to make things confusing.

When a company is profitable the *shareholders* benefit by receiving a partial share of these profits by payments called *dividends.* Shareholders also see the value of their investment change as the *share price* changes over time. Shareholders can sell their holding on the stock market, and thus *liquidate* (convert to cash) their investment

The share price is determined by the *market price.* That is, the *value* of a share is worth whatever someone is willing to pay to acquire it. Of course, this *value*, one would expect, is closely related to the value of the company. In fact, a measure of the value of a company is its *market capitalisation*[2] being the total number of shares that the company has, multiplied by price of each share.

[1] Investors can be members of the public, or large institutions (*banks*, *pension funds*, *hedge funds*, *insurance companies*)
[2] or 'mar. cap.' for short

1.2.1 Stocks and Shares

For an investor (which is often the viewpoint we will take) we can consider our investments as composed of cash or debt (in a bank deposit account) and a *portfolio* (collection) of shares and derivatives. We will often refer to the shares as *assets*, to mean investments with unpredictable future values. We can see how unpredictable these assets are, by referring to figure (1.1). This is an example of the data provided to investors. In this case, it is the share price information (publicly available) taken from the FT (Financial Times) web-site (ft.com) for the oil company BP (British Petroleum) during December 2011. At this point we won't go into detail as to what all the figures mean. Simply note the dark-red jagged graph that shows how the price has changed over the 5-day period (Monday 19th December 2011 to Friday 23rd December 2011).

Figure 1.1: Data from the Financial Times from the **share price** of British Petroleum (BP) plc from Monday 19th December 2011 to Friday 23rd December 2011. In the figure the dark red line plots the mid-price on the London Stock Exchange against time. The irregular motion over time is due to trading (buying/selling). The table of figures contains data on the latest trading day (Friday). **Open** is the mid-price at 08:00 (the start of the trading day) on Friday 23rd. Shares in BP are also traded on the New York Stock Exchange (as well as Buenos Aires, Mexico, Germany, France, Switzerland and Sweden). This explains the jump in price from the previous day (**Previous close**) 450.30 to the **Open** 458.90 is to match the equivalent price in New York. **Day High/Day Low** - the highest (lowest) value of the mid-price on 23/12/11. The **closing price** (under 'summary') was 459.70. Trading closed early for the Christmas holiday. The **shares outstanding** refer to the shares held in public ownership, and the market capitalisation is (approximately) this number multiplied by the **share price**.

This irregular behaviour in the price is the result of *trading* (buying and selling) shares in BP. In column 2 of the data, the 'Market cap' is quoted as 85.39bn GBP. 1 billion (bn) means 10^9. One trillion corresponds to 10^{12}, that is a million million, and one quadrillion means 10^{15}, that is a thousand million million. As discussed above, this is proportional to the share price. The price data is given in column 1, gives the 'open' price for Friday 23rd December. This was the price at the start of the day when trading opened. The price is in pence (GBX), so a single share would have cost you (without commission charges) £4.5890. The 'day high' was the highest value of the price that day, and the 'day low' the lowest. The *average volume* tells us how many shares were traded (on average) per day over the last 30 days. For BP, this is an astonishing 25.46 million. This company has a high liquidity - that is, the volume of trading is such that it is comparatively easy to sell any shares in a portfolio, that is convert your assets (shares) into cash (liquid asset).

1.2.2 Bid-offer spread

The list of prices and the *volume* (quantity) of shares on bid or offer is called the *order book* (see table 1.1). The orders for buying and selling are handled by intermediaries called *market makers*.

An important aspect about buying and selling is that the prices are not quite the same. The *bid price* is the current highest (best) price that buyers are prepared to pay (459.65 pence per share), and the *offer price* is the lowest (best) price that sellers are prepared to offer (459.70 pence per share). The *offer* price is called the *ask* price in the US. The gap between the prices is called the *bid-offer spread* (*bid-ask* US). The average of the *bid* and *offer* price is called the *mid-price* and for convenience this is the price given. So the price for BP at the end of trading on Friday would be 458.675. However, shares are traded on a discrete scale of values, and their prices are thus rounded to the nearest value, or *tick*, and thus quoted as 458.70. The *tick size* is the smallest possible price increment, the *tick values* are the prices. So the BP tick size is 0.05 pence (the smallest possible price difference).

Now, as to the question of how the price moves. Consider a simple (schematic) order book (Table 1.1). Bob is an investor willing to take some risk. We call him a *speculator*. Bob decides he wants to invest in the company Gamma plc, and decides to buy 5,000 shares. But he wants to pay 100p for each share. The order book shows that there are only 3,000 shares on offer at that price. So Bob buys the 3,000 shares and leaves in a bid for a further 2,000 at the same price 100p. The effect on the order book is shown in Table 1.1. Bob is now top of the bidding table, and as a result of his buying action he has moved the mid-price up. Bob may even decide he wants the investment badly enough to buy the 1,500 shares on offer at 102 - in which case the price goes up even higher.

BEFORE				AFTER			
BUY (BID)		SELL (ASK/OFFER)		BUY (BID)		SELL (ASK/OFFER)	
Volume	Price (pence)	Price (pence)	Volume	Volume	Price (pence)	Price (pence)	Volume
				2,000	100		
10,000	98	100	3,000	10,000	98		
6,000	96	102	1,500	6,000	96	102	1,500
8,000	94	104	2,000	8,000	94	104	2,000
2,200	92	106	1,700	2,200	92	106	1,700

Table 1.1: Schematic version of an *order book* for shares in a fictitious company Gamma plc. The table shows bid and offer prices and volumes (quantity of shares no bid/offer), before Bob's trade. The initial mid-price is 99 pence - this is defined as half-way between the highest bid price and lowest offer price. Bob placing an order for 5,000 shares at 100 pence has moved the mid-price up to 101p. His action to buy shares affected the price.

As a result of buying, the price moves up. Conversely, when investors sell the stock it moves down in value. For example, Carol might be a large share holder in Gamma plc and wants to liquidate some of her holding, say 4,500 shares to be sold. Carol looks at the order book (Table 1.1) and decides to sell to the bidders at 100 (Bob is fairly happy now that he has found a seller!). So from her 4,500 shares on offer Bob will take 2,000 and she leaves an offer order of 2,500 shares for any other buyers. As a result the order book looks like Table 1.2, with Carol's order top of the offer list.

BUY (BID)		SELL (ASK/OFFER)	
Volume	Price (in pence)	Price (in pence)	Volume
10,000	98	100	2,500
6,000	96	102	1,500
8,000	94	104	2,000
2,200	92	106	1,700

Table 1.2: Schematic version of an *order book* for shares in Gamma plc, showing bid and offer prices and volumes, after Bob's trade (buy) and after Carol's trade (sell). The current mid-price is 99 pence - back to where it was before Bob entered the market.

So all of the irregular structure in share price variation over time is entirely due to trading activity. It is caused by humans.

1.2.3 The efficient market

One cannot trade an asset without having someone to trade with. For every seller there must be a buyer, and thus both parties must be happy with the price of the trade. The buyer and seller would view this as a *fair price*, otherwise why would they enter into the trade. However, to what extent does the share price correctly value the company? This price is simply what traders are prepared to pay, so the price is an expression of the *market* rather than based on a deep analysis of the company's prospects. The market has very limited information about companies, so this really is a rough guess. Let's assume that there is a true (theoretical) value/price of a share which could be found by a complex analysis of every aspect of the company. We say that the market is *efficient* if the market price is a correct approximation to this true price. There is a more extreme version of this idea, called the *efficient market hypothesis* in which it is asserted/assumed that the market price *is* the correct price. But no one really believes that!

1.3 Bears and Bulls, longs and shorts

We talk of a *bull market* when the values of shares, on the whole, are increasing, and speak of a *bear market* when the values are falling. A broad view of the market can be obtained by a share price *index*: a (weighted) average of the share prices of a range of companies: FTSE 100 (London), Dow Jones, S&P 500 and Nasdaq Composite (New York), Dax (Frankfurt), CAC 40 (Paris), Nikkei 225 (Tokyo), ISEQ (Dublin), SPI 200 (Sydney), AEX (Amsterdam), S&P/TSX (Toronto).

Let us now consider a speculator, a trader in the stock market, let's call him Bob, who buys and sells assets on behalf of his investors/clients. We term the set of investments (cash, bonds, shares/stocks, derivatives) a *portfolio*, and since the value of the assets is unpredictable, the value of the portfolio is unpredictable. However, and this is important, these random variables are *not* independent. Thus Bob can choose the degree of risk he takes by the combination of these investments.

1.3.1 Long

Suppose Bob decides to behave as a *speculator*. He suspects that the share price of company Z will increase over the next few months. Perhaps he has spotted an upward trend in the price, or knows some information that will be good news to the profitability of the company, or he just thinks that they have great products and these will sell really well in the near future. Bob decides to buy some shares in company Z, which at present ($t = 0$) have the *spot price* (current market value) S_0 (pence per share). So he converts some of his cash into stock by buying (let's say) 5,000 shares. We say that Bob's *nominal* investment (amount of actual money invested) is £50 $\times$ S_0.

In terms of Bob's investment portfolio, we say Bob is *long* in this asset (shares in company Z). Which means to say he is holding some shares. We can also say that Bob has taken an *bull position* on this asset. This means the same thing - but it just sounds more sophisticated!

At a later time $t = T$, the share price has changed and its new *spot price* is S_T. It turns out that $S_T > S_0$ and Bob was right. He decides to *take his profit* and sells his shares to the market, *liquidates his holding* or *closes his position* (in the parlance), and thereby has made a profit (per unit), $S_T - S_0$ which, when multiplied by 5,000 tells us how much extra cash he has (in pennies) as a result.

What if the contrary were true. Bob looks at company Y and believes this company is headed for trouble. Can he make money with a share price that is falling ? The answer, surprisingly, is yes! He can do this by *short selling*.

1.3.2 Short

Short selling means selling shares that you do not own, and it works as follows. Although he does not have any shares in Y, he approaches Alice and offers to sell her, say 5000 shares at the current (spot) price S_0 (per share). Let's say S_0 is measured in £.

Alice, conversely, thinks that company X is doing fine, perhaps is slightly undervalued, and agrees to buy the shares from Bob, at the price S_0. Bob is taking a *bear position* on this asset - he believes its value will fall. Alice, on the other hand takes the *bull position*; in her opinion the price will go up. Now clearly, they can't both be right - so someone will win and someone will lose over this difference in opinion.

Alice transfers, $\pounds(50 \times S_0)$ to Bob, reminding him that he owes her 5000 shares. Bob is *selling short* - he does not own ANY shares in company Y. This is called *naked short selling* if Bob really has no shares, or simply *short selling* if Bob has borrowed the shares on a temporary basis (and which have to be returned or paid for at some point).

At some time in the future, he will need to deliver the shares to Alice to fulfil their agreement. So he will need to get his hands on 5000 shares in the near future. We say Bob is *short* 5000 units of the asset.

After a few days, $t = T$, the price of a share in Y has fallen to $S_T < S_0$. Bob was right after all, and Alice was wrong. Bob promptly uses Alice's money $\pounds 50 \times S_0$ and buys 5000 shares at the spot price S_T and calmly waits for Alice's transfer request. Thus Bob makes a profit of $\pounds 50 \times (S_0 - S_T)$. This profit is at the expense of Alice who is now stuck with 5000 shares that are worth a lot less than she paid for them. Alternatively, if Bob was wrong and it transpired that the price steadily increased ($S_T > S_0$) then when Alice asks for her shares, he will need to go to the market, and pay the spot price S_T for the shares and then pass these to Alice. Bob will have to take the loss: $\pounds 50 \times (S_T - S_0)$, and Alice is pleased that her investment at the price S_0 is now worth S_T and her gain is: $\pounds 50 \times (S_T - S_0)$.

We have seen that buying and selling changes the asset price. Selling drives prices down, and short selling does exactly the same. And, of course, this encourages more short selling because one can make money on a falling market in this way. The downside to this is that it may create problems as shareholders see the value of their investments collapse due to trading of this kind. This may encourage investors to sell up to avoid further losses, and this compounds the problem as prices continue to fall. In 2008, the US and UK imposed a three-week ban on the practice soon after the collapse of Lehman Brothers. However, shares continued to tumble during this time as investors sold their stocks. During the Euro-crisis in 2011 France, Spain, Belgium and Italy introduced short-selling restrictions on over 60 financial businesses.

1.4 Commodities

The price of financial products are often strongly related to asset prices, and indeed are considered assets in their own right. We often use the term *underlying asset* (or even just 'underlying') when we refer to the basis of these products. However, assets include a much wider range of products such as *commodities* (wheat, oil, copper, etc), foreign currency, art, property, software, music, patents, and so on. These are all *risky* investments.

1.5 Bonds

An important class of investment for us will be *risk-free* investments, such as deposit savings accounts in which (*compounded* or *simple*) *interest* is paid on the money deposited. In capital markets, there is a product analogous to savings accounts: *bonds*. A *bond* is a loan made by investors to a government or large company. Bonds are sold to the *bond holder* for a *face value* (price), with a predetermined *maturity* (date on which the loan/face value) is repaid. In addition to the repayment at maturity, the bond holder (lender) receives *interest payments* over the duration, called *coupon payments*. Again, these future payments would be agreed at the time of purchase. Bonds are labelled as a *fixed-income investment* since the payments to the investor are agreed in advance and fixed at the time of purchase.

For example a 5-year bond with face value £1,000 having a 5% annual coupon means the following. A bond holder, purchasing the bond for £1,000, will receive £50 at the end of each year, for 5 years. At maturity (5 years later) the *principal* of the loan/bond (£1,000) is paid back. Thus the investor receives a total of £1,250 from this investment. Bonds are often issued as a way for a company to raise money.

Government bonds in the UK, and also in Ireland, are called *gilts*, and issued by the Treasury. Bonds issued by large companies are referred to as *corporate bonds*. The term *gilt* or *gilt-edged security* is a reflection of the fact that (up to now) the Government has never failed to make interest or principal payments on gilts. *Conventional gilts* guarantee a coupon payment every six months until *maturity* date, at which point the holder receives the final coupon payment and the return of the *principal* (also called the *face value* or *notional value* of the bond).

The *coupon rate* usually follows the bank interest rates at the time it is issued, although it is slightly higher. For UK gilts, the *coupon* indicates the cash payment per £100 (notional value) that the holder will receive per year. This payment is made in two equal semi-annual payments on fixed dates six months apart. For example, an investor who holds £1,000 of '4% UK Treasury Gilt 2016', will receive two coupon payments of £20 each on 7 March and 7 September, until the maturity on 7th September 2016, when the final coupon payment is paid, and the principal (notional value) £1,000 is repaid to the investor.

There are many types of government bonds with a range of maturity; some bonds have a 30-year duration. The US has Treasury Bills, Treasury Notes and Treasury Bonds. In the UK *Index-linked gilts* have coupon payments and principal adjusted in line with inflation, given by the UK Retail

Prices Index. The US equivalent is called a *savings bond*.

In theory, we consider bonds to be risk-free investments, because the bonds are usually issued by credit-worthy institutions (large companies such as Tesco, Glaxo-Smith-Kline, BT, for example). However, if the borrower (the institution that issued the bonds), whether it is a large prosperous company or country, goes bust, it may fail to repay this loan. We say it has *defaulted* on the loan. Unlike a savings account, however, one cannot cash in bonds before maturity. Instead bonds can be bought and sold in the same way as equities, although there is less choice of products and the market is not as liquid (harder to find buyers and sellers).

The entire investment banking industry has the aim of *out-performing* the risk-free bond investment. It does this by making *risky investments*.

1.6 Derivatives

The next class of financial instruments to consider are *derivatives*. Derivatives, as explained above, owe their name to the fact that the products and their prices are *derived* from the value of an underlying asset. Essentially, these are *insurance products* which aim to minimise the risk associated with large fluctuations in the underlying asset price.

A classic example would be an airline where the necessary purchase of aviation fuel is a large part of its costs. In this case, the price of oil will greatly affect the price of kerosene, and this in turn will affect the operating costs (and hence profits) of the airline. This ia particularly the case for airlines, since flight tickets are sold in advance. A sudden large increase in oil prices would thus be considered a threat to the value of the company and the company would seek to insure against such an event. So the company might choose to purchase a derivative to *hedge* against large price increases in the underlying asset.

There are many varieties (flavours) of derivatives, but these are based upon 4 fundamental types.

1. OPTIONS

   The holder of an *option* has the *right* (but not the *obligation*) to buy/sell an asset in the future at a price agreed *now*.

2. FUTURES

   A *futures* is a contract (legally binding obligation) to buy/sell an asset at an agreed date in the future (the *delivery date*).

   The price for the purchase/sale of the asset is fixed/agreed at the time of the contract, and the payment is made at the delivery date.

   Such contracts are *traded* at an *exchange*.

3. FORWARD

   A *forward contract* is also a legally-binding obligation to buy/sell an asset at an agreed date in the future (the *delivery date*) at a price agreed now.

   The contract is negotiated directly between two parties rather than through an *exchange*.

4. SWAP

   This is also a *private agreement* (contract), also called an *over-the-counter* (OTC), between two parties to exchange (*swap*) assets at a specified future date. We can think of a swap contract as a combination of forward contracts. The parties agree to exchange one cash flow stream for another according to some prearranged rules. The most important *swap* market (in terms of its value) is based on *interest rates*, and is worth more than \$ 100 trillion (that is $\$10^{11}$) per year.

1.6.1 Weapons of Maths destruction

On the whole, derivatives are a good thing for the economy. They encourage the amount of trading in assets (the amount is called the *volume of liquidity*), since they offer insurance to the buyers and sellers against the risks. The prices of the derivatives will reflect the risks involved in these investments. Thus, these derivative prices provide additional information to the market on the uncertainties in these investments. Commodity derivatives actually stabilised commodity prices and benefited farmers across the world. Credit default swaps, for example, made it easier for companies to take on large debts since they were able to take out an insurance against failure.

However, derivatives played a major role in the financial crisis of 2009. For this reason, a good deal of blame has been attached to the mathematicians who devised and priced these products. Derivatives allowed investors (and companies) to take risks rather than avoid risks. In that sense, their existence encouraged investors to dive into markets that were previously considered too dangerous. If derivatives contributed to the current mess, then they did so by discouraging diligence, rather than enabling greed. Traders felt able to take more and more risk by taking out more and more insurance. The problem with this strategy is that, if something goes wrong, then the insurance policy is cashed in, and the insurer liable for these claims is in a difficult position.

If the insurance claims (payments out) exceed the insurance premiums (payments in) then the insurer will go bankrupt. AIG (Americal International Group) was a main insurer of collateralized debt obligations (CDOs), and had to be rescued by the US Government during the financial crisis.

1.7 Trading strategy

The *position* of an investor/trader describes the *state* of the investment. A trader who has bought, and thus *holds* an instrument (asset or derivative) is said to be *long* in the instrument. A trader who has sold an instrument is said to be *short* in the instrument.

We speak of an *investment portfolio* as being the combination of all the investments, cash, assets and derivatives. A trader can balance these investments and adopt strategies of the following kinds:

(a) SPECULATION

A risk-taking approach, in which the investor attempts to guess/predict the future of asset prices, and takes positions accordingly.

(b) ARBITRAGE

A risk-free approach in which a combinations of trades in different instruments are used to exploit pricing errors in the market.

(b) HEDGING

A combination of risky investments in assets with a set of investments in derivatives to off-set (hedge) the risk and thus to minimise the uncertainty in the future value of the portfolio.

A trader can adopt some or all these strategies at different times. Ideally, the trader wants to aim to be an *arbitrageur* (or *arbitrageuse*) ensuring that a zero-risk profit can be made which is better than one could obtain by investing purely in bonds (the simple risk-free investment method). Alternatively, a trader could be a speculator if he/she is almost certain that a particular stock/share is grossly over-valued or under-valued and decides to take a *calculated risk* by trading in the asset alone. Finally, the trader could take a middle road in which the investment in the risky asset is covered by a derivative as an insurance against something anomalous occurring. This is called *hedging*. In the course on these lectures we will consider all three approaches,

We have already discussed a little about speculation in section 1.3, and we will return to this later. But first we describe some essential aspects of Forwards and Options.