Chapter 12

Black-Scholes equation

In the previous chapter we considered two different models for asset prices in continuous time: Brownian motion and geometric Brownian motion. The aim of this chapter is to find a differential equation that determines the value/price of an option within these models.

These are the assumptions we make.

- The asset follows a geometric Brownian motion: $dS_t = \mu S_t dt + \sigma S_t dW_t$.
- The asset is perfectly liquid - one can buy/sell the asset at any time and in any amount.
- There are no transaction costs - no bid/ask spread and no taxes or commission costs.
- The interest rate is constant in time.

As a consequence of these assumptions, we will expect real option prices to differ from the values predicted by the model!

We anticipate that the option (a European call option) should depend on the time at which it is sold and on the asset price at that time. Let $c(t, S_t)$ denote the (unknown) price of a call option at a time, t, on an asset (spot price) S_t which expires at a time T. Consider how the price might change as time advance from t to $t + dt$.

Since S_t is stochastic, and we are assuming it has a geometric Brownian motion, then according to Itô's lemma we have

$$dc = \left[\frac{\partial c}{\partial t} + \frac{1}{2}(\sigma S_t)^2 \frac{\partial^2 c}{\partial S_t^2}\right] dt + \frac{\partial c}{\partial S_t} dS_t \qquad . \tag{12.1}$$

where

$$dc = c(t + dt, S_{t+dt}) - c(t, S_t) \qquad . \tag{12.2}$$

is the change in the value of c over a short interval dt.

Previously, in the simple Bernoulli process, we used the principles of a perfectly hedged portfolio and the absence of arbitrage to determine the price of the option. We will use the same principles here.

Suppose we have Portfolio 1:
(a) *short* a European call option, strike price X, which expires at time T.
(b) long Δ units of the asset.

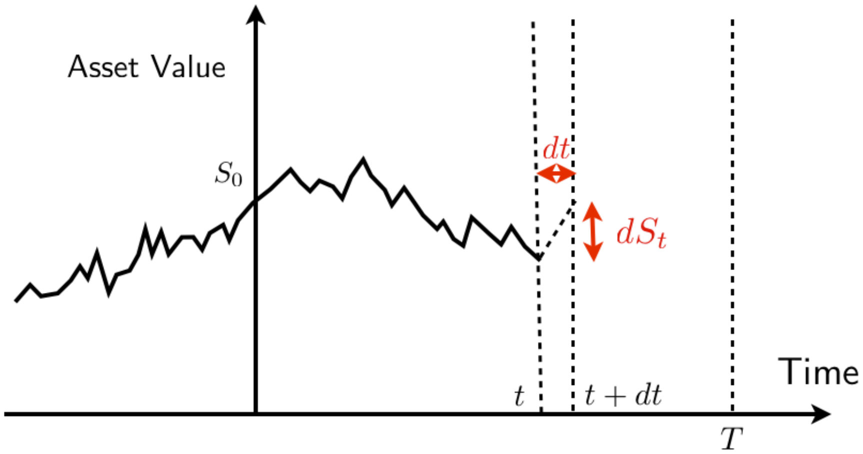

Figure 12.1: Asset price change according to a geometric Brownian motion. The present time is t, the expiry time for an option on this asset is T. We are interested in the connection between the change in the option price as a function of the change in the (underlying) asset.

The current time is t, and we denote the value of the portfolio by $\Pi_t^{(1)}$ where

$$\Pi_t^{(1)} = -c_t + \Delta S_t \qquad . \tag{12.3}$$

As before, holding Δ units of the asset hedges against the changes in the option price/value.

The first question is what should Δ be to ensure *a risk-free portfolio* over a short time dt. As time advances, the value of the asset will change, and so will the value of the call option.

$$\Pi_{t+dt}^{(1)} - \Pi_t^{(1)} = d\Pi_t^{(1)} = -dc_t + \Delta dS_t \qquad . \tag{12.4}$$

12.1 Risk-Free Hedging

Recall that we are assuming a geometric Brownian motion for the asset value:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \qquad . \tag{12.5}$$

Then, using Itô's lemma, we can derive an expression for the change in the portfolio:

$$d\Pi_t^{(1)} = -\left[\frac{\partial c}{\partial t} + \frac{1}{2}(\sigma S_t)^2 \frac{\partial^2 c}{\partial S_t^2}\right] dt - \frac{\partial c}{\partial S_t} dS_t + \Delta dS_t \qquad . \tag{12.6}$$

The change in value of the portfolio has two distinct components. The deterministic component is the coefficient of dt. The stochastic component is the uncertain term proportional to dS_t.

As in the binomial model, one can arrange the value of Δ such that the change $d\Pi^{(1)}$ is independent of the change in asset value: remove the unpredictability arising from dS_t. This can be achieved by choosing Δ such that

$$-\frac{\partial c}{\partial S_t} dS_t + \Delta dS_t = 0 \qquad , \tag{12.7}$$

or,

$$\boxed{\Delta = \frac{\partial c}{\partial S_t}} \qquad . \tag{12.8}$$

We can see that this is completely analogous to the discrete (binomial) model where the Δ was given by the difference in call prices divided by the asset prices:

$$c = \frac{c_u - c_d}{uS_0 - dS_0} \qquad .$$

However, we don't know c, so we can't determine Δ yet. The valuation of c is the next step.

12.2 No Arbitrage Pricing

The Δ hedging makes Portfolio 1 risk-free. Now consider an alternative risk-free portfolio in which the investment is purely in bonds. An investment in such a portfolio, at time t, of $\Pi_t^{(2)}$, would grow in value by continuous compound interest to the value:

$$\Pi_{t+dt}^{(2)} = \Pi_t^{(2)} e^{rdt} \qquad . \tag{12.9}$$

Then the change in value of this portfolio would be:

$$d\Pi_t^{(2)} = \Pi_{t+dt}^{(2)} - \Pi_t^{(2)} = r\Pi_t^{(2)} dt \qquad , \tag{12.10}$$

where we discard higher order terms in dt.

The principle of *no arbitrage* states - that: two risk-free portfolios with the same final value must have the same initial value (investment).

Let us consider investing the same amount in both portfolios at time, t.

$$\Pi_t^{(1)} = \Pi_t^{(2)} \tag{12.11}$$

Then, the principle of no arbitrage states that:

$$\Pi_{t+dt}^{(1)} = \Pi_{t+dt}^{(2)} \tag{12.12}$$

In other words,

$$\boxed{d\Pi_t^{(1)} = d\Pi_t^{(2)}} \qquad . \tag{12.13}$$

Combining these relations we can write:

$$d\Pi_t^{(2)} = r\Pi_t^{(2)} dt = r\Pi_t^{(1)} dt = r(-c_t + \Delta S_t)dt \qquad . \tag{12.14}$$

But the change in value of the Δ-hedged portfolio is given by

$$d\Pi_t^{(1)} = -\left[\frac{\partial c}{\partial t} + \frac{1}{2}(\sigma S)^2 \frac{\partial^2 c}{\partial S^2}\right] dt \qquad . \tag{12.15}$$

Since these changes in potfolio value must be equal, by the principle (12.13), we can equate (12.15) and (12.14).

$$r(-c_t + \Delta S_t)dt = -\left[\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2}\right] dt \qquad . \tag{12.16}$$

With $\Delta = \frac{\partial c}{\partial S}$, one can write:

$$r\left[-c + \frac{\partial c}{\partial S_t} S_t\right] = -\left[\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2}\right] \qquad . \tag{12.17}$$

This leads to the following partial-differential equation, known as the Black-Scholes equation:

$$\boxed{\frac{\partial c}{\partial t} + rS_t\frac{\partial c}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 c}{\partial S_t^2} - rc = 0} \quad . \tag{12.18}$$

This PDE along with the boundary conditions uniquely determines the value of c.

The result (12.18) was published in 1973 by Fischer Black (Chicago) and Myron Scholes (MIT) [1], although the original manuscript was submitted to the journal at the end of 1970!

Around the same time, Robert Merton (Harvard) expanded the understanding of the result[2], and for that reason it is also called the Black-Scholes-Merton equation. The derivation of the equation led to the award of the Nobel prize in Economics in 1997. The prize cannot be awarded posthumously, and Black had died some time before this, so the prize was collected by Merton and Scholes.

The citation from the Royal Bank of Sweden reads, *Robert C. Merton and Myron S. Scholes have, in collaboration with the late Fischer Black, developed a pioneering formula for the valuation of stock options. Their methodology has paved the way for economic valuations in many areas. It has also generated new types of financial instruments and facilitated more efficient risk management in society.*

12.2.1 Boundary condition

In order to solve this partial differential equation, it is necessary to know the boundary conditions. These are determined by the known value of the call option at expiry, $t = T$:

$$c_T = \max(S_T - X, 0) \quad . \tag{12.19}$$

The interesting aspect of this boundary condition is that it is in the future, $t = T$. Thus we must work backwards in time to calculate the present option price, from the future option price.

12.3 Arithmetic Brownian motion

It is interesting to compare the corresponding partial-differential equation when the asset price has a *arithmetic* Brownian motion rather than a *geometric* Brownian motion. This model of asset price was the original (Bachelier 1905) model of which Bachelier obtained an option price.

The definition of an arithmetic Brownian motion is the stochastic differential equation:

$$dS = \mu_B dt + \sigma_B dW_t \quad , \tag{12.20}$$

where we use the labels μ_B and σ_B to distinguish from the geometric Brownian motion. In the context of signal processing in electrical engineering, the geometric process is called *multiplicative noise* and the arithmetic process is mown as *additive noise*.

To derive the option price equation, we repeat the steps as above. That is begin with the hedging and then apply the arbitrage principle.

$$\Pi_t^{(1)} = -c_t + \Delta S_t \tag{12.21}$$

and so,

$$d\Pi_t^{(1)} = -\left[\frac{\partial c}{\partial t} + \frac{1}{2}\sigma_B^2\frac{\partial^2 c}{\partial S_t^2}\right]dt - \frac{\partial c}{\partial S_t}dS_t + \Delta dS_t \quad . \tag{12.22}$$

[1] Journal of Political Economy **81** pp 637-654
[2] Bell Journal of Economics and Management Science **4** 141-183

Again, using hedging to remove the risk, we have:

$$\Delta = \frac{\partial c}{\partial S} \qquad . \tag{12.23}$$

The use of the no-arbitrage argument then leads to,

$$\boxed{\frac{\partial c}{\partial t} + rS_t \frac{\partial c}{\partial S_t} + \frac{1}{2}\sigma_B^2 \frac{\partial^2 c}{\partial S_t^2} - rc = 0} \qquad , \tag{12.24}$$

which we will call the Bachelier equation. The same boundary conditions apply:

$$c_T = \max(S_T - X, 0) \qquad . \tag{12.25}$$

Note that in the Bachelier equation, as in the Black-Scholes equation, μ_B is absent.

12.4 Solution of Black-Scholes Equation

The Black-Scholes equation is a second-order linear PDE, and it can easily be solved by numerical methods on a computer. However, the equation can be converted, through a change of variable, to the diffusion equation and thereby solved analytically. In this section, we describe the steps involved.

However, before we start, it is important to realise that the Black-Scholes equation is not a stochastic differential equation anymore. Thus, we can use our standard mathematical tools again.

STEP 1

Change from t to τ, where

$$\tau = T - t \quad , \tag{12.26}$$

is the *time to expiry*, and change from S to y, where

$$\boxed{y = \ln S} \qquad . \tag{12.27}$$

We then use the chain rule to change the variables:

$$\left(\frac{\partial c}{\partial t}\right)_S = \left(\frac{\partial \tau}{\partial t}\right)_S \left(\frac{\partial c}{\partial \tau}\right)_y = -\left(\frac{\partial c}{\partial \tau}\right)_y \qquad , \tag{12.28}$$

and

$$\left(\frac{\partial c}{\partial S_t}\right)_t = \left(\frac{\partial y}{\partial S}\right)_t \left(\frac{\partial c}{\partial y}\right)_\tau = \frac{1}{S}\left(\frac{\partial c}{\partial y}\right)_\tau \qquad . \tag{12.29}$$

This gives

$$\left(\frac{\partial^2 c}{\partial S^2}\right)_t = \frac{\partial}{\partial S}\left(\frac{\partial c}{\partial S}\right)_\tau = \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial c}{\partial y}\right)_\tau \qquad , \tag{12.30}$$

so that

$$\left(\frac{\partial^2 c}{\partial S^2}\right)_t = -\frac{1}{S^2}\left(\frac{\partial c}{\partial y}\right)_\tau + \left(\frac{1}{S}\right)\left(\frac{1}{S}\right)\left(\frac{\partial}{\partial y}\right)_\tau \left(\frac{\partial c}{\partial y}\right)_\tau \qquad . \tag{12.31}$$

We finally have, and for brevity we will drop the subscript (constant with respect to) for the partial derivatives,:

$$\left(\frac{\partial^2 c}{\partial S^2}\right) = \frac{1}{S^2}\left(-\frac{\partial c}{\partial y} + \frac{\partial^2 c}{\partial y^2}\right) \qquad . \tag{12.32}$$

This changes the Black-Scholes equation (12.18) to the form

$$-\frac{\partial c}{\partial \tau} + rS\left(\frac{1}{S}\frac{\partial c}{\partial y}\right) + \frac{1}{2}\sigma^2 S^2\left(\frac{1}{S^2}\right)\left(-\frac{\partial c}{\partial y} + \frac{\partial^2 c}{\partial y^2}\right) - rc = 0 \quad , \tag{12.33}$$

where now the aim is to determine $c(\tau, y)$. This can be simplified as:

$$\frac{\partial c}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 c}{\partial y^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial c}{\partial y} - rc \quad . \tag{12.34}$$

We continue by changing the dependent variable c. We write

$$c(\tau, y) = e^{-r\tau}w(y, \tau) \quad . \tag{12.35}$$

This is equivalent to the use of the *integrating factor* from Level 1. Then

$$\frac{\partial c}{\partial \tau} = \frac{\partial}{\partial \tau}\left(e^{-r\tau}w\right) = -re^{-r\tau}w + e^{-r\tau}\frac{\partial}{\partial \tau}w = -rc + e^{-r\tau}\frac{\partial}{\partial \tau}w. \tag{12.36}$$

This eliminates the last term of the RHS in equation (12.33) to give

$$\boxed{\frac{\partial w}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 w}{\partial y^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial w}{\partial y}} \quad . \tag{12.37}$$

This is the final result of the changes of variables. We must now solve this equation for $w(\tau, y)$ with the boundary condition at $\tau = 0$, that is at $t = T$:

$$w(0, y_0) = \max(e^{y_0} - X, 0) \tag{12.38}$$

where, according to (12.27), $y_0 = \ln S_T$.

Equation (12.37) is the convection-diffusion equation again! Previously, it was encountered for the probability density for the asset price with Brownian motion, now it describes the option price.

12.5 Solution of the convection-diffusion equation by Green's functions

Writing the PDE (partial-differential equation) (12.37) in the shorthand operator notation:

$$\frac{\partial}{\partial \tau}w(\tau, y) = \mathcal{L}_y w(\tau, y) \quad . \tag{12.39}$$

where the *operator* is defined as:

$$\mathcal{L}_y = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial y^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial y} \quad . \tag{12.40}$$

is a linear partial-differential operator. This class of PDE has two important properties

- *translation invariance*

  If $w_1(\tau, y)$ is a solution of (12.39), then so too is

$$w_1(\tau, y + a) \quad ,$$

where a is any constant.

Suppose $w_1(\tau, y)$ is a solution of (12.39) $w_1(\tau, y+a)$ is also a solution of (12.39). This is because, shifting the coordinate $y \to y+a$ does not affect the operator (12.40). Suppose the coordinate is changed to $z = y + a$, then this is just a simple change of variable and:

$$\frac{\partial}{\partial y}_\tau = \frac{\partial}{\partial z}_\tau \qquad . \tag{12.41}$$

It then follows that:

$$\mathcal{L}_y = \mathcal{L}_z \qquad , \tag{12.42}$$

and thus the set of solutions will be the same, whether one uses y or z as the reference coordinate. This makes sense, since a shift in coordinates is just the observing moving along the y axis. This does not change the 'physics' of the convection-diffusion process and the solutions will be the same, just slightly displaced spatially.

Spatial invariance plays a role in financial mathematics through eg. a change in currency.

- *superposition*

  If $w_1(\tau, y)$ and $w_2(\tau, y)$ are both solutions of (12.39), then so is the linear combination (superposition)

  $$a_1 w_1 + a_2 w_2 \quad ,$$

  where a_1 and a_2 are any constants.

  Consider,

  $$w = a_1 w_1 + a_2 w_2 \qquad . \tag{12.43}$$

  Since a_1 and a_2 are constants, then:

  $$\frac{\partial w}{\partial \tau} = a_1 \frac{\partial w_1}{\partial \tau} + a_2 \frac{\partial w_2}{\partial \tau} \qquad . \tag{12.44}$$

  since both w_1 and w_2 are solutions, we have

  $$a_1 \frac{\partial w_1}{\partial \tau} + a_2 \frac{\partial w_2}{\partial \tau} = a_1 \mathcal{L}_y w_1 + a_2 \mathcal{L}_y w_2 \qquad . \tag{12.45}$$

  Since $\mathcal{L}_y$ is linear and differential, we can also write this as:

  $$\mathcal{L}_y \left(a_1 w_1 + a_2 w_2\right) \qquad . \tag{12.46}$$

  Thus:

  $$\frac{\partial w}{\partial \tau} = \mathcal{L}_y w \quad . \tag{12.47}$$

12.5.1 Fundamental solution of the Black-Scholes equation

Returning to the Black-Scholes equation transformed to the convection-diffusion equation, we have

$$\frac{\partial w}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 w}{\partial y^2} + \left(r - \frac{1}{2}\sigma^2\right) \frac{\partial w}{\partial y} \qquad . \tag{12.48}$$

By setting (defining) the drift rate

$$-\mu_d = \left(r - \frac{1}{2}\sigma^2\right) \qquad , \tag{12.49}$$

one special solution (previously found) of (12.48) is,

$$\phi(y,\tau) = \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{(y-\mu_d\tau)^2}{2\sigma^2\tau}} \quad . \tag{12.50}$$

As $\tau \to 0$, this function becomes infinitely high and infinitesimally thin, and we get the Dirac δ-function:

$$\lim_{\tau\to 0} \quad \phi(y,\tau) = \delta(y) \quad , \tag{12.51}$$

and this was identified as the Green's function of the equation.

The importance of the Green's function is that *any* solution, the *general solution*, of equation (12.48) can be constructed by *superpositions* and *translations* of the Green's function. That is, the *general solution* can be written as:

$$\boxed{W(y,\tau) = \int_{-\infty}^{+\infty} g(y')\phi(y-y',\tau)dy'} \quad . \tag{12.52}$$

The superposition corresponds to the integral - recall that integration is just the continuous variable version of a sum. So, in this sense, the function $g(y')$ is equivalent to the constants a_1, a_2. Similarly the change of variable, $y - y'$ is a general type of translation

$$\begin{aligned} \frac{\partial W}{\partial \tau} &= \frac{\partial}{\partial \tau}\int_{-\infty}^{+\infty} g(y')\phi(y-y',\tau)dy' \\ &= \int_{-\infty}^{+\infty} g(y')\frac{\partial}{\partial \tau}\phi(y-y',\tau)dy' \\ &= \int_{-\infty}^{+\infty} g(y')\mathcal{L}_y\phi(y-y',\tau)dy' \\ &= \mathcal{L}_y\int_{-\infty}^{+\infty} g(y')\phi(y-y',\tau)dy' \quad . \end{aligned} \tag{12.53}$$

Then it follows that:

$$\frac{\partial W}{\partial \tau} = \mathcal{L}_y W(y,\tau) \quad . \tag{12.54}$$

That is, we have a solution, whatever the form of the function $g(y')$.

12.5.2 Boundary conditions

Now $g(y')$ is determined by the boundary conditions and thus equation (12.52) gives a unique solution.

At $\tau = 0$ ($t = T$), $c = \max(S_T - X, 0)$ which implies

$$W(y,0) = \max(S_T - X, 0) = \max(e^{y_0} - X, 0) \quad . \tag{12.55}$$

This implies that,

$$\max(e^y - X, 0) = \int_{-\infty}^{+\infty} g(y')\phi(y-y',0)dy' \quad . \tag{12.56}$$

Since $\phi(y-y',0) = \delta(y-y')$, we have:

$$\begin{aligned} \max(e^y - X, 0) &= \int_{-\infty}^{+\infty} g(y')\delta(y-y')dy' \\ &= g(y)\int_{-\infty}^{+\infty} \delta(y-y')dy' \\ &= g(y) \end{aligned} \tag{12.57}$$

Thus this function $g(y)$ is nothing other than the boundary conditions for the solution: $W(y,\tau)$ at $\tau = 0$.

$$g(y) = \max(e^y - X, 0) \qquad . \tag{12.58}$$

Thus, in conclusion, the solution is given by the integral

$$\boxed{W(y,\tau) = \int_{-\infty}^{+\infty} \max(e^{y'} - X, 0)\phi(y - y', \tau)dy'} \qquad . \tag{12.59}$$

12.5.3 Black-Scholes formula for a call option

We could leave the solution in the form (12.59), but we can simplify the result.

The integrand factor $\max(e^{y'} - X, 0)$ means that a limited range of y' values contribute. When $e^{y'} - X \leq 0$ there is no contribution since the integrand is zero. That is, for all values of y'

$$y' \leq \ln X \qquad , \tag{12.60}$$

the function is zero, and this provides us with a lower limit for the integral. For values of $y' \geq \ln X$, then we have:

$$\max(e^{y'} - X, 0) = e^{y'} - X \qquad , \tag{12.61}$$

Then converting to the financial variables,

$$c = e^{-r\tau} W(y,\tau) \qquad . \tag{12.62}$$

$$c = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{\ln X}^{+\infty} \left[e^{y'} - X\right] e^{-\frac{[y-y'+(r-\frac{1}{2}\sigma^2)\tau]^2}{2\sigma^2\tau}} dy' \qquad . \tag{12.63}$$

Multiplying out the bracket we can separate the integral into two terms:

$$c = I_1 + I_2 \qquad , \tag{12.64}$$

where

$$\boxed{I_1 = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{\ln X}^{+\infty} dy' \exp\left\{y' - \frac{[y - y' + (r - \frac{1}{2}\sigma^2)\tau]^2}{2\sigma^2\tau}\right\}} \qquad . \tag{12.65}$$

and

$$\boxed{I_2 = \frac{e^{-r\tau}(-X)}{\sigma\sqrt{2\pi\tau}} \int_{\ln X}^{+\infty} dy' \exp\left\{-\frac{[y - y' + (r - \frac{1}{2}\sigma^2)\tau]^2}{2\sigma^2\tau}\right\}} \qquad . \tag{12.66}$$

To simplify I_2, we change the variable y' such that:

$$z = \frac{y - y' + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \qquad . \tag{12.67}$$

which implies that

$$dz = -\frac{dy'}{\sigma\sqrt{\tau}} \qquad . \tag{12.68}$$

This change of variable means that the limits of integration also change:

$$y' = \infty \rightarrow z = -\infty \tag{12.69}$$

and

$$y' = \ln X \to z = \frac{y - \ln X + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \qquad . \tag{12.70}$$

Recalling that $y = \ln S$, we can define the variable:

$$\boxed{d_2 = \frac{\ln(S/X) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}} \tag{12.71}$$

Then we can write:

$$I_2 = \frac{e^{-r\tau}(-X)}{\sigma\sqrt{2\pi\tau}}(-\sigma\sqrt{\tau}) \int_{d_2}^{-\infty} e^{-\frac{1}{2}z^2} dz \qquad . \tag{12.72}$$

And we note that

$$-\frac{1}{\sqrt{2\pi}} \int_{d_2}^{-\infty} e^{-\frac{1}{2}z^2} dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{-\frac{1}{2}z^2} dz = N(d_2) \tag{12.73}$$

where $N(z)$ is the standard normal probability distribution. That is:

$$N(z) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}u^2} du \qquad . \tag{12.74}$$

Finally we have:

$$\boxed{I_2 = -Xe^{-r\tau}N(d_2)} \qquad . \tag{12.75}$$

For I_1, we also convert the integral into a version of the normal distribution. Firstly, we note the term inside the exponential is:

$$y' - \frac{[y - y' + (r - \frac{1}{2}\sigma^2)\tau]^2}{2\sigma^2\tau} \qquad . \tag{12.76}$$

Our aim is to choose a change of variable to *complete the square*. That is to convert:

$$y' - \frac{[y - y' + (r - \frac{1}{2}\sigma^2)\tau]^2}{2\sigma^2\tau} = -\frac{1}{2}z^2 + q \tag{12.77}$$

where q is some constant (independent of y').

The appropriate transformation is:

$$z = \frac{y - y' + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \qquad . \tag{12.78}$$

this implies that:

$$y' = \infty \to z = -\infty \tag{12.79}$$

and

$$y' = \ln X \to z = \frac{y - \ln X + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \qquad . \tag{12.80}$$

Recalling that $y = \ln S$, then we can define the variable:

$$\boxed{d_1 = \frac{\ln(S/X) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}} \tag{12.81}$$

Then, after a fair bit of algebra, we get:

$$I_1 = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}(-\sigma\sqrt{\tau}) \int_{d_1}^{-\infty} e^{r\tau + y}\, e^{-\frac{1}{2}z^2} dz = \frac{e^{\ln S}}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-\frac{1}{2}z^2} dz \qquad . \tag{12.82}$$

Then this reduces to the expression:

$$\boxed{I_1 = SN(d_1)} \quad . \tag{12.83}$$

This finally gives a simple expression for (12.63), the Black-Scholes formula

$$\boxed{c = SN(d_1) - Xe^{-r\tau}N(d_2)} \quad . \tag{12.84}$$

Thus we can deduce that c_t is determined by five variables/parameters: S_t the current (spot) price, X the strike price, σ the volatility , r the interest rate, and T the expiry time. Of these five variables, four are known publicly at the outset: r, T, X and S_t. The unknown parameter is the volatility σ. Thus the price of options is strongly linked to the uncertainty in the future asset value and this uncertainty is unknown. In a sense, traders in options are buying and selling uncertainty.

It is important to note that c_t is not determined by μ the drift rate. This is initially surprising since the expected value of the asset *does* depend on μ. We found that (11.66)

$$\mathbb{E}\left(S_T\right) = S_t \; e^{\mu(T-t)} \quad . \tag{12.85}$$

The absence of μ is related to the absence of p in the binomial model for the option price. Options are priced in the risk-neutral world.