

COMP6212 Computational Finance

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Part I: Portfolio Optimization

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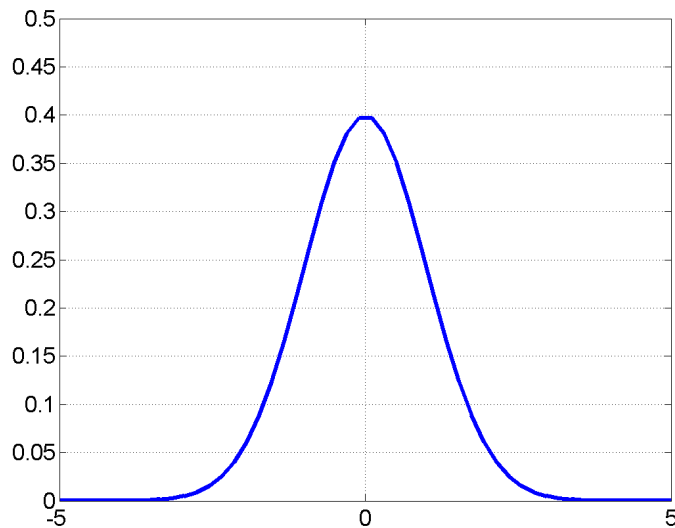
Part I: Portfolio Optimization

- $r_i(t)$ Return on asset i at time t ; i.e. invest at time $t-1$, what have you earned at time t ?
- We think of this as a random variable, say Gaussian distributed with mean μ_i and variance σ_i^2
- The mean is what we expect (on average) to gain by investing
- We think of variance in return as *risk*
- When we look at more than one asset, we can think of how returns on them are correlated: σ_{ij}^2
- A portfolio (investment in N assets) with relative weights π_i
- Return on the portfolio: $r_p = \sum_{i=1}^N \pi_i r_i = \boldsymbol{\pi}^t \mathbf{r}$
- Returns on the portfolio has a multivariate Gaussian distribution

$$\mathbf{r} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{pmatrix} \quad \boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{pmatrix} \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12}^2 & \dots & \sigma_{1N}^2 \\ \sigma_{12}^2 & \sigma_2^2 & \dots & \sigma_{2N}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1N}^2 & \sigma_{2N}^2 & \dots & \sigma_N^2 \end{pmatrix}$$

Gaussian Density

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left\{ -\frac{(x-m)^2}{2\sigma^2} \right\}$$

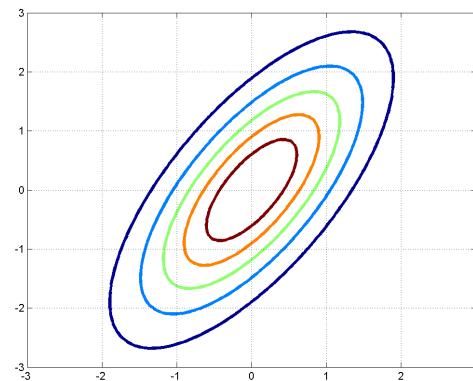
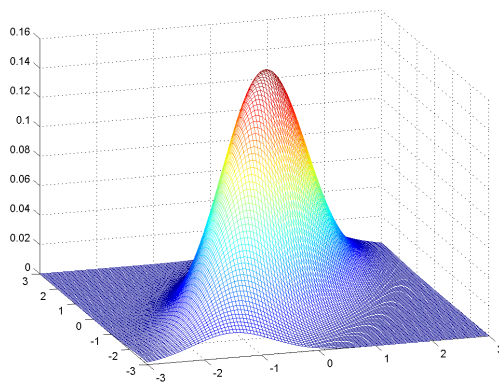


```
x = linspace(-5,5,50);  
m = 0;  
s = 1;  
y = normpdf(x,m,s);  
figure(1), clf  
plot(x,y,'LineWidth',3);  
grid on
```

Multivariate Gaussian Distribution

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} \exp \left\{ -\frac{1}{2}(\mathbf{x} - \mu)^T \Sigma^{-1}(\mathbf{x} - \mu) \right\}$$

- Parameters: mean vector μ covariance matrix Σ



- How do these shapes change with μ and Σ ?

Portfolio Return

- Linear transform of multivariate Gaussian

$$\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}); \mathbf{y} = \mathbf{A}\mathbf{x} \implies \mathbf{y} \sim \mathcal{N}(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^t)$$

- Return on our portfolio is a linear transform of the vector of returns

$$r_P = \boldsymbol{\pi}^T \mathbf{r}$$

- We can immediately write down the distribution of the return on the portfolio

$$r_P \sim \mathcal{N}(\boldsymbol{\pi}^T \boldsymbol{\mu}, \boldsymbol{\pi}^T \boldsymbol{\Sigma} \boldsymbol{\pi})$$

- Mean return $M = \boldsymbol{\pi}^T \boldsymbol{\mu}$ and the variance on it $V = \boldsymbol{\pi}^T \boldsymbol{\Sigma} \boldsymbol{\pi}$
- When $\boldsymbol{\pi}$ changes, M and V change — how?

Efficient Portfolio

- As we change $\boldsymbol{\pi}$ (i.e. invest in different proportions), M and V change
- Not all M and V are realizable.
- We can formulate constrained optimization problems
- For a given risk we tolerate, what is the highest return we can expect

$$\max_{\boldsymbol{\pi}} \boldsymbol{\pi}^T \boldsymbol{\mu} \quad \text{subject to } \boldsymbol{\pi}^T \boldsymbol{\Sigma} \boldsymbol{\pi} = \sigma_0$$

- If we hope for (expect) a given return, at what minimum risk can we achieve it?

$$\min_{\boldsymbol{\pi}} \boldsymbol{\pi}^T \boldsymbol{\Sigma} \boldsymbol{\pi} \quad \text{subject to } \boldsymbol{\pi}^T \boldsymbol{\mu} = r_0$$

- Other constraints possible: $\sum_{i=1}^N \pi_i = 1, \pi_i \geq 0, \alpha \leq \pi_i \leq \beta$

- We estimate μ and Σ from historic data and apply optimization to allocate assets.
- We hope the past might be a good reflection of future!
- Estimation:

$$\begin{aligned}\hat{\mu} &= \frac{1}{T} \sum_{t=1}^T \mathbf{r}(t) \\ \hat{\Sigma} &= \frac{1}{T} \sum_{t=1}^T (\mathbf{r}(t) - \hat{\mu})(\mathbf{r}(t) - \hat{\mu})^T\end{aligned}$$

Solving quadratic program in MATLAB

$$\min_{\mathbf{x}} \mathbf{x}^T \mathbf{H} \mathbf{x} + \mathbf{f}^T \mathbf{x} \text{ such that } \begin{cases} \mathbf{A} \mathbf{x} \leq \mathbf{b} \\ \mathbf{A}_{eq} \mathbf{x} = \mathbf{b}_{eq} \\ \mathbf{lb} \leq \mathbf{x} \leq \mathbf{ub}. \end{cases}$$

```
x = quadprog(H, f, A, b, Aeq, beq, lb, ub, x0)
```

Do `>doc quadprog` in MATLAB and read more.

For the portfolio optimization problem, we might have:

```
pi = quadprog(Sigma, [], [], [], mu', rMax, 0, 1, [])
```

Map the problem variables to the function in the tool.

Efficient Frontier

- Given μ and Σ
- What portfolio has the highest return, unconstrained by risk?

$$\max \pi^T \mu \quad \text{subject to} \quad \sum_{i=1}^N \pi_i = 1, \quad \text{and} \quad \pi_i \geq 0$$

Linear Programming:

$$\min_{\mathbf{x}} \mathbf{f}^T \mathbf{x} \quad \text{such that} \quad \begin{cases} \mathbf{A} \mathbf{x} \leq \mathbf{b} \\ \mathbf{A}_{eq} \mathbf{x} = \mathbf{b}_{eq} \\ \mathbf{lb} \leq \mathbf{x} \leq \mathbf{ub} \end{cases}$$

```
x = linprog( f, A, b, Aeq, beq, lb, ub )  
w1 = linprog(-mu, [], [], ones(1,N), 1, 0, 0);  
r1 = w1 * mu;
```

Efficient Frontier (cont'd)

- What portfolio has lowest variance (unconstrained by expectation)?

$$\min \pi^T \Sigma \pi \quad \text{subject to} \quad \sum_{i=1}^N \pi_i = 1$$

```
w2=quadprog(mu,zeros(N,1),[],[],ones(1,N),1,zeros(N,1),[],[]);  
r2 = w2' * mu;
```

- Portfolios on the efficient frontier will have returns in range $r1$ to $r2$
- Now, select a number of points (returns we need), and find the minimum variance portfolios that will offer these returns!

```
M = linspace(r1, r2, p)  
for j=1:p  
    ret = M(j);  
    w = quadprog(...);  
    V(j) = w' * Sigma * w;  
end
```

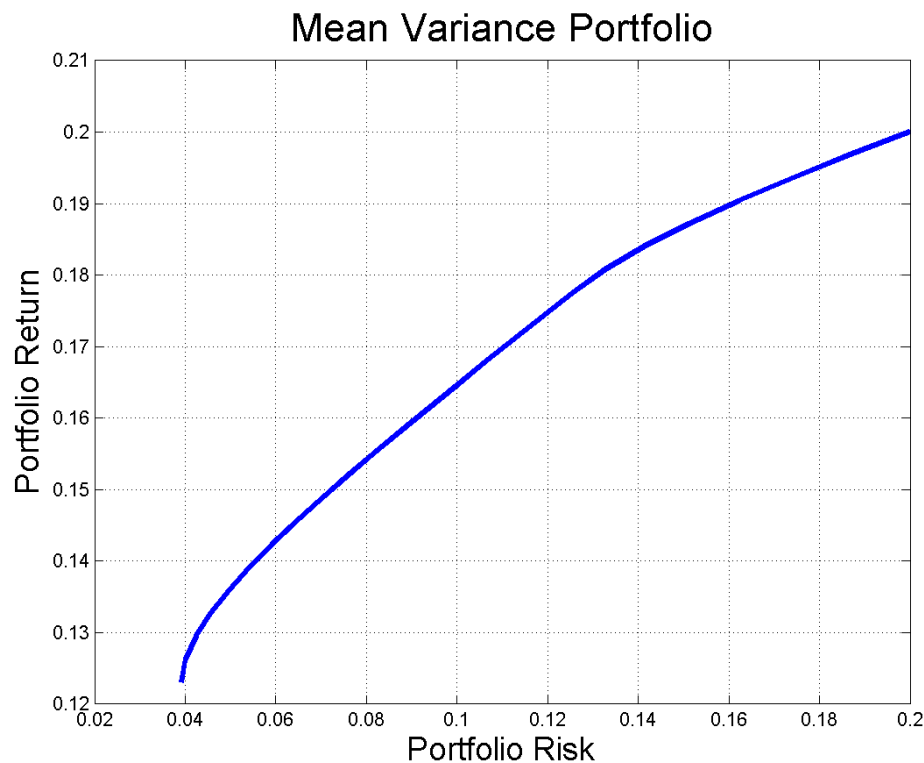
Complete Function to Compute the Efficient Frontier

```
function [PRisk, PRoR, PWts] = NaiveMV(ERet, ECov, NPts)
ERet = ERet(:); % makes sure it is a column vector
NAssets = length(ERet); % get number of assets
% vector of lower bounds on weights
V0 = zeros(NAssets, 1);
% row vector of ones
V1 = ones(1, NAssets);
% set medium scale option
options = optimset('LargeScale', 'off');
% Find the maximum expected return
MaxReturnWeights = linprog(-ERet, [], [], V1, 1, V0);
MaxReturn = MaxReturnWeights' * ERet;
% Find the minimum variance return
MinVarWeights = quadprog(ECov, V0, [], [], V1, 1, V0, [], [], options);
MinVarReturn = MinVarWeights' * ERet;
MinVarStd = sqrt(MinVarWeights' * ECov * MinVarWeights);
% check if there is only one efficient portfolio
if MaxReturn > MinVarReturn
    RTarget = linspace(MinVarReturn, MaxReturn, NPts);
    NumFrontPoints = NPts;
else
    RTarget = MaxReturn;
    NumFrontPoints = 1;
end
```

Complete Function (cont'd)

```
...
% Store first portfolio
PRoR = zeros(NumFrontPoints, 1);
PRisk = zeros(NumFrontPoints, 1);
PWts = zeros(NumFrontPoints, NAssets);
PRoR(1) = MinVarReturn;
PRisk(1) = MinVarStd;
PWts(1,:) = MinVarWeights(:)';
% trace frontier by changing target return
VConstr = ERet';
A = [V1 ; VConstr ];
B = [1 ; 0];
for point = 2:NumFrontPoints
    B(2) = RTarget(point);
    Weights = quadprog(ECov, V0, [], [], A, B, V0, [], [], options);
    PRoR(point) = dot(Weights, ERet);
    PRisk(point) = sqrt(Weights' * ECov * Weights);
    PWts(point, :) = Weights(:)';
end
```

Summary: what have we achieved?



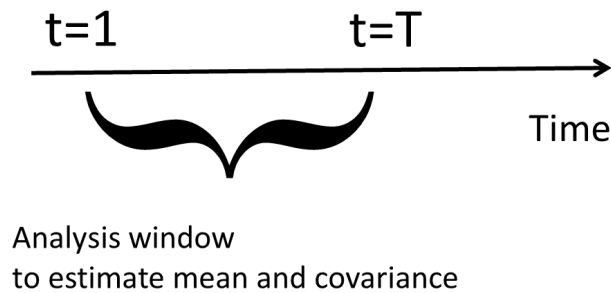
Homework

- Three assets with the following properties:

$$m = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.15 \end{bmatrix}; \quad C = 100 * \begin{bmatrix} 0.005 & -0.010 & 0.004 \\ -0.010 & 0.040 & -0.002 \\ 0.004 & -0.002 & 0.023 \end{bmatrix};$$

- Study the code of function `NaiveMV` and draw the efficient frontier.
- Use the function `frontcon` in MATLAB and draw the efficient frontier.

Estimation of Parameters



- Estimate parameters μ and C from data within a window
- Optimize portfolio, invest and wait
- Periodically re-balance the portfolio (re-estimate parameters)
- Trade-off:
 - Need long window for accurate estimation
 - But relationships may not be stationary over long durations
- Shrinkage in covariance estimates

Advances on the Mean-Variance Portfolio

- Do such portfolios make money?

DeMiguel, J. *et al.* (2009) Optimal versus Naive Diversification: How Inefficient Is the 1/N Portfolio Strategy?, *The Review of Financial Studies* **22**(5): 1915.

- Including transaction costs into the optimization

Lobo, M.S. *et al.* (2007) Portfolio Optimization with Linear and Fixed Transaction Costs, *Annals of Operations Research* **152**: 341.

- Forcing the portfolio to be sparse and stable

Brodie, J. *et al.* (2009) Sparse and stable Markowitz portfolios, *PNAS* **106**(30): 12267
Takeda, A. *et al.* (2013) Simultaneous pursuit of out-of-sample performance and sparsity in index tracking portfolios, *Comput Manag Sci* **10**: 21

- Optimizing the execution of trade

TBC

Portfolio Performance

- Sharpe Ratio: mean to standard deviation of portfolio return

$$S = \frac{m - r}{\sigma}$$

r “risk free” interest rate

- Value at Risk (VAR): Value such that probability of loss exceeding this is 0.01.

$$V \text{ such that } P[-G > V] = 0.01$$

“if a portfolio of stocks has a one-day 5% VaR of 1 million, there is a 0.05 probability that the portfolio will fall in value by more than 1 million over a one day period”

`ValueAtRisk=portvrisk(PortReturn,PortRisk,RiskThreshold,PortValue)`

- cVAR: Conditional Value at Risk (later)

Empirical Evaluation

DeMiguel, J. *et al.* (2009),

- Comparison of a number of portfolio optimization methods
 - $\frac{1}{N}$ with re-balancing
 - Sample based mean-variance
 - Bayesian methods (of shrinking estimates)
 - Constraints
 - Combination of portfolios (model averaging / mixing)
- No method *consistently* beats the naive strategy!

Sparse Portfolios

Brodie *et al.* (2007) PNAS

- N assets; \mathbf{r}_t , return vector at time t
- Expected return and covariance:

$$\mathbf{r}_t = \begin{pmatrix} r_{1,t} \\ r_{2,t} \\ \vdots \\ r_{N,t} \end{pmatrix} \quad \mathbf{E}[\mathbf{r}_t] = \boldsymbol{\mu} \quad \mathbf{E}[(\mathbf{r}_t - \boldsymbol{\mu})(\mathbf{r}_t - \boldsymbol{\mu})^T] = \mathbf{C}$$

- Markowitz portfolio

$$\begin{cases} \min_{\mathbf{w}} \mathbf{w}^T \mathbf{C} \mathbf{w} \\ \text{subject to } \mathbf{w}^T \boldsymbol{\mu} = \rho \text{ and } \mathbf{1}_N^T \mathbf{w} = 1 \end{cases}$$

- Short selling allowed; *i.e.* w_j need not be positive
- Covariance: $\mathbf{C} = \mathbf{E}[\mathbf{r}_t \mathbf{r}_t^T - \boldsymbol{\mu} \boldsymbol{\mu}^T]$

Sparse Portfolios, Brodie *et al.* (2007) PNAS (cont'd)

- Mean and covariance estimated from data (expectations as sample averages):
- $\hat{\boldsymbol{\mu}} = \frac{1}{T} \sum_{t=1}^T \mathbf{r}_t$
- \mathbf{R} $T \times N$ matrix with rows as \mathbf{r}_t^T
- Optimization problem rewritten as

$$\begin{cases} \hat{\mathbf{w}} = \min_{\mathbf{w}} \frac{1}{T} \|\rho \mathbf{1}_T - \mathbf{R} \mathbf{w}\|_2^2 \\ \text{subject to } \mathbf{w}^T \hat{\boldsymbol{\mu}} = \rho, \mathbf{w}^T \mathbf{1}_N = 1 \end{cases}$$

- Often there is strong correlation between returns
 - Assets in the same sector respond in similar ways
- Strong correlations make \mathbf{R} ill-conditioned \implies numerically unstable optimization
- Solution: regularization
- Brodie *et al.* suggest l_1 regularizer

$$\begin{cases} \hat{\mathbf{w}} = \min_{\mathbf{w}} [\|\rho \mathbf{1}_T - \mathbf{R} \mathbf{w}\|_2^2 + \tau \|\mathbf{w}\|_1] \\ \text{subject to } \mathbf{w}^T \hat{\boldsymbol{\mu}} = \rho, \mathbf{w}^T \mathbf{1}_N = 1 \end{cases}$$

Index Tracking

Brodie *et al.* (2007) (cont'd)

- Passive investor, wishing to get the same return as stock index (e.g. FTSE100)
- Invest in all 100 stocks of the FTSE?
- Transaction costs very high
- Can we find a small subset of the 100 stocks (say 10), that will approximate the performance of the index?
- Subset selection / cardinality constrained optimization

$$\begin{cases} \min_{\mathbf{w}} [\|\mathbf{y} - \mathbf{R} \mathbf{w}\|_2^2] \\ \text{subject to } \|\mathbf{w}\|_0 = w_0 \end{cases}$$

- 0th norm \rightarrow number of nonzero elements of $\mathbf{w} \rightarrow$ subset of assets
- The above is of combinatorial complexity
- Suboptimal algorithm: greedy search

Brodie *et al.* (2007) (cont'd)

- A convenient proxy to achieve sparsity is *lasso* (l_1 constrained regression)

$$\min_{\mathbf{w}} [\|\mathbf{y} - \mathbf{R} \mathbf{w}\|_2^2 + \tau \|\mathbf{w}\|_1]$$

- Several elements of \mathbf{w} will be zero
- Tune τ to achieve different levels of sparsity
- Can incorporate transaction costs into the optimization

$$\min_{\mathbf{w}} \left[\|\mathbf{y} - \mathbf{R} \mathbf{w}\|_2^2 + \tau \sum_{i=1}^N s_i |w_i| \right]$$

- Transaction costs:
 - Usually have fixed (overhead) part and transaction-dependent part
 - Institutional investors fixed part negligible
 - Small investors can assume fixed cost only

Brodie *et al.* (2007) (cont'd)

Portfolio adjustment (re-balancing)

- We are holding a portfolio \mathbf{w}
- We want to make an adjustment $\Delta_{\mathbf{w}}$, new portfolio $\mathbf{w} + \Delta_{\mathbf{w}}$
- Transaction costs only on the adjustments

$$\begin{cases} \Delta_{\mathbf{w}} = \min_{\Delta_{\mathbf{w}}} [\|\rho \mathbf{1}_T - \mathbf{R}(\mathbf{w} + \Delta_{\mathbf{w}})\|_2^2 + \tau \|\Delta_{\mathbf{w}}\|_1] \\ \text{subject to } \Delta_{\mathbf{w}}^T \hat{\boldsymbol{\mu}} = 0 \text{ and } \Delta_{\mathbf{w}}^T \mathbf{1}_N = 1 \end{cases}$$

Brodie *et al.* (2007) (cont'd)

Homework:

Coursework 1 will involve confirming some claims in Brodie *et al.*'s paper. Please download the paper and start reading.

- We will use the CVX toolbox within MATLAB to implement optimization
- <http://cvxr.com/>
- Download, uncompress, set MATLAB to the cvx directory and do `cvx_setup`
- Take MATLAB back into your working directory

Example of using CVX

```
T = 150; N = 50;
R = randn(T, N);
rho = 0.02;
tau = 1;
mu = rand(N, 1);

cvx_begin quiet
variable w(N)
    minimize( norm(rho*ones(T,1)-R*w) + tau*norm(w,1) )
    subject to
        w'*ones(N,1) == 1;
        w'*mu == rho;
        w > 0;
cvx_end

figure(1), clf, bar(w); grid on
```

Note: Data random - probably won't work all the time

Portfolio Optimization with Transaction Costs

Lobo *et al.* (2007) *Ann Oper Res* 152:341

- Portfolio weights: $\mathbf{w} = [w_1 \ w_2 \ \dots \ w_n]^T$
- Returns: \mathbf{a} ; $E[\mathbf{a}] = \bar{\mathbf{a}}$; $E[(\mathbf{a} - \bar{\mathbf{a}})(\mathbf{a} - \bar{\mathbf{a}})^T] = \Sigma$
- We consider an adjustment to the portfolio of value \mathbf{x}
- New portfolio: $\mathbf{w} + \mathbf{x}$; Wealth: $\mathbf{a}^T (\mathbf{w} + \mathbf{x})$
- Portfolio return and variance:

$$\begin{aligned} E[W] &= \bar{\mathbf{a}}^T (\mathbf{w} + \mathbf{x}) \\ E[(W - E[W])^2] &= (\mathbf{w} + \mathbf{x})^T \Sigma (\mathbf{w} + \mathbf{x}) \end{aligned}$$

- Transaction cost: $\phi(\mathbf{x})$
- Budget Constraint:

$$\mathbf{1}^T \mathbf{x} + \phi(\mathbf{x}) \leq 0$$

self financing constraint

Lobo *et al.* (2007) (cont'd)

Possible Optimizations:

$$\begin{aligned} &\underset{\mathbf{x}}{\text{maximize}} \quad \bar{\mathbf{a}}^T (\mathbf{w} + \mathbf{x}) \\ &\text{subject to} \quad \mathbf{1}^T \mathbf{x} + \phi(\mathbf{x}) \leq 0 \\ &\quad \quad \quad \mathbf{w} + \mathbf{x} \in \mathcal{S} \end{aligned}$$

\mathcal{S} some feasible set (with other constraints)

$$\begin{aligned} &\underset{\mathbf{x}}{\text{minimize}} \quad \phi(\mathbf{x}) \\ &\text{subject to} \quad \bar{\mathbf{a}}^T (\mathbf{w} + \mathbf{x}) \geq r_{\min} \\ &\quad \quad \quad \mathbf{w} + \mathbf{x} \in \mathcal{S} \end{aligned}$$

Costs are separable (usual assumption):

$$\phi(\mathbf{x}) = \sum_{i=1}^n \phi_i(x_i)$$

$$\phi_i(x_i) = \begin{cases} \alpha_i^+ x_i, & x_i \geq 0 \\ -\alpha_i^- x_i, & x_i \leq 0 \end{cases}$$

- α_i^+ and α_i^- cost rates for buying and selling asset i
- Convex cost function
- $\phi_i = \alpha_i^+ x_i^+ + \alpha_i^- x_i^-$ with $x_i^+ \geq 0$ and $x_i^- \geq 0$

Fixed plus linear transaction costs:

$$\phi_i(x_i) = \begin{cases} 0, & x_i = 0 \\ \beta_i^+ + \alpha_i^+ x_i, & x_i \geq 0 \\ \beta_i^- - \alpha_i^- x_i, & x_i \leq 0 \end{cases} >$$

Diversification Constraints

- Limit the amount of investment in any asset

$$w_i + x_i \leq p_i, \quad i = 1, 2, \dots, n$$

- Limit the fraction of total wealth held in each asset

$$w_i + x_i \leq \mathbf{1}^T (\mathbf{w} + \mathbf{x})$$

- Limit exposure in any small group of assets (say in a sector)

$$\sum_{i=1}^r (w_i + x_i)_{[l]} \leq \mathbf{1}^T (\mathbf{w} + \mathbf{x})$$

< gama(i)

gama(i)<1

< gama(i)

(Tricky, but can show this is convex – see Eqn (11) in paper)

Lobo *et al.* (2007) (cont'd)

- Constraints on short-selling
... on individual asset

$$w_i + x_i \geq -s_i, \quad i = 1, \dots, n$$

...or as bound on the total short position

$$\sum_{i=1}^n (w_i + x_i)_- \leq S$$

- Collateralization:

$$\sum_{i=1}^n (w_i + x_i)_- \leq \gamma \sum_{i=1}^n (w_i + x_i)_+$$

What I have borrowed to sell is smaller than a fraction of what I own

Lobo *et al.* (2007) (cont'd)

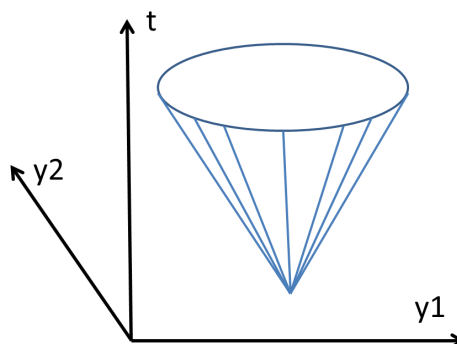
Variance:

$$(\mathbf{w} + \mathbf{x})^T \Sigma (\mathbf{w} + \mathbf{x}) \leq \sigma_{\max}$$

Can also be written as:

$$\| \Sigma^{1/2} (\mathbf{w} + \mathbf{x}) \| \leq \sigma_{\max}$$

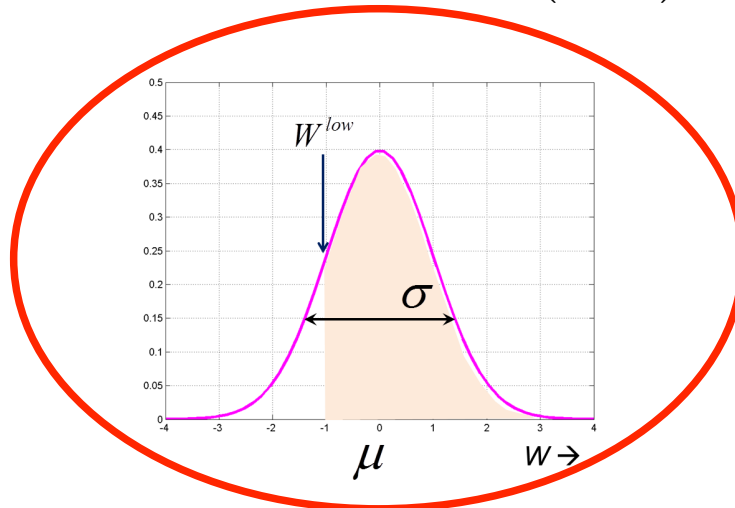
- This is *Second Order Cone* constraint.



Shortfall Risk Constraint:

$$P(W \geq W^{\text{low}}) \geq \eta$$

$$W = \mathbf{a}^T (\mathbf{w} + \mathbf{x}) \sim \mathcal{N}(\mu, \sigma^2)$$



$$P\left(\frac{W - \mu}{\sigma} \leq \frac{W^{\text{low}} - \mu}{\sigma}\right) \leq 1 - \eta$$

Lobo et al. (2007) (cont'd)

- But $(W - \mu)/\sigma \sim \mathcal{N}(0, 1)$
- Hence

$$P\left(\frac{W - \mu}{\sigma} \leq \frac{W^{\text{low}} - \mu}{\sigma}\right) = \Phi\left(\frac{W^{\text{low}} - \mu}{\sigma}\right)$$

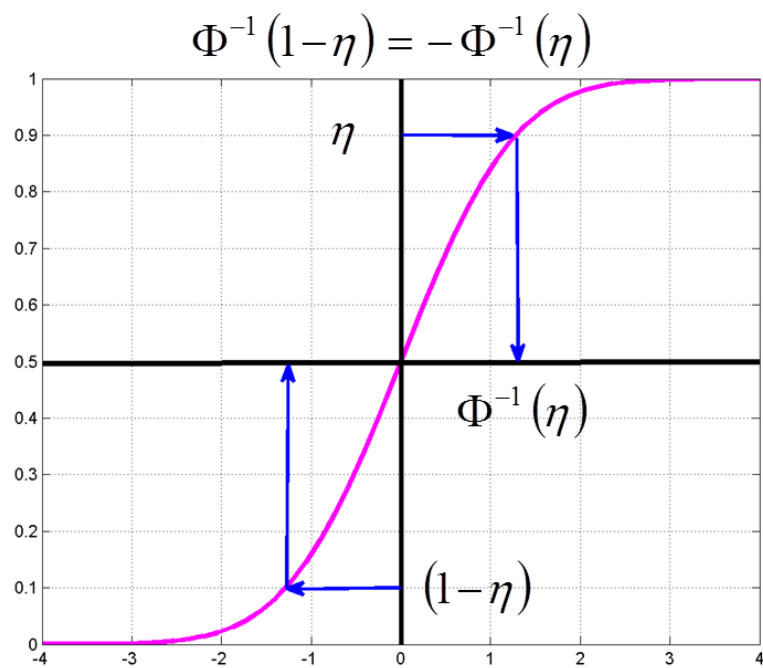
$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z \exp\left\{-\frac{t^2}{2}\right\} dt$$

- $\frac{W^{\text{low}} - \mu}{\sigma} \leq \Phi^{-1}(1 - \eta)$
- $\Phi^{-1}(1 - \eta) = -\Phi^{-1}(\eta)$

$$\mu - W^{\text{low}} \geq \Phi^{-1}(\eta) \sigma$$

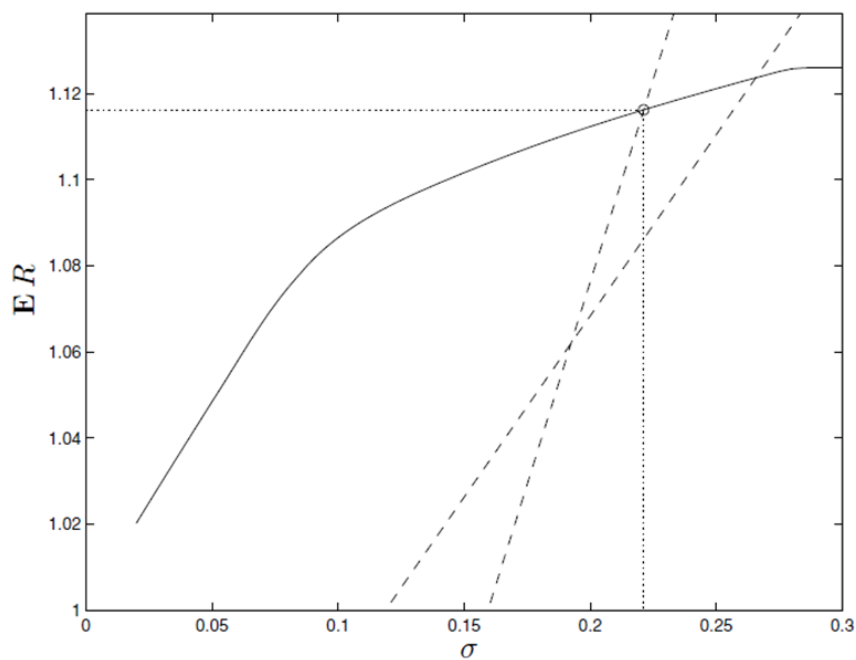
- Using $\mu = \bar{\mathbf{a}}^T (\mathbf{w} + \mathbf{x})$ and $\sigma^2 = (\mathbf{w} + \mathbf{x})^T \Sigma (\mathbf{w} + \mathbf{x})$

$$\Phi^{-1}(\eta) \|\Sigma^{1/2}(\mathbf{w} + \mathbf{x})\| \leq \bar{\mathbf{a}}^T (\mathbf{w} + \mathbf{x}) - W^{\text{low}}$$



Lobo *et al.* (2007) (cont'd)

Shortfall Risk on M-V Space



$$\text{maximize } \bar{\mathbf{a}}^T (\mathbf{w} + \mathbf{x}^+ - \mathbf{x}^-)$$

$$\text{subject to } \mathbf{1}^T (\mathbf{x}^+ - \mathbf{x}^-) + \sum_{i=1}^n (\alpha_i^+ x_i^+ + \alpha_i^- x_i^-) \leq 0$$

$$x_i^+ \geq 0, x_i^- \geq 0, i = 1, 2, \dots, n$$

$$w_i + x_i^+ - x_i^- \geq s_i, i = 1, 2, \dots, n$$

$$\Phi^{-1}(\eta_j) \|\Sigma^{1/2} (\mathbf{w} + \mathbf{x}^+ - \mathbf{x}^-)\| \leq \bar{\mathbf{a}}^T (\mathbf{w} + \mathbf{x}^+ - \mathbf{x}^-) - W_j^{\text{low}}, j = 1, 2$$

Puzzle

```

m1 = [0.15 0.2 0.08 0.1]';
C1 = [ 0.2    0.05   -0.01    0.0
       0.05   0.30    0.015   0.0
      -0.01   0.015   0.10    0.0
       0.0    0.0     0.0     0.0
      ];

m2 = [0.15 0.2 0.08]';
C2 = [ 0.2    0.05   -0.01
       0.05   0.30    0.015
      -0.01   0.015   0.10
      ];

[V1, M1, PWts1] = NaiveMV(m1, C1, 25);
[V2, M2, PWts2] = NaiveMV(m2, C2, 25);

figure(2), clf,
plot(V1, M1, 'b', V2, M2, 'r', 'LineWidth', 3),
title('Mean Variance Portfolio', 'FontSize', 22)
xlabel('Portfolio Risk', 'FontSize', 18)
ylabel('Portfolio Return', 'FontSize', 18);
    
```

