Chapter 17

Credit Risk and Futures options

Credit risk is the possibility that a borrower fails to repay a loan (”defaults” on the loan). Defaults on property loans have been one of the major underlying reasons for the current financial crisis. However, the major market for loans are in the form of bonds. Bonds are issued by companies, or even governments, as a method of raising capital. The fact that they request additional external capital to support the business rather than from within the company the indicates that there is an inherent risk that the company may not be able to readily liquidate resources to repay the bond holders. Hence, there is a risk of default on the debt. Bonds have a face value - the amount payable on maturity and the holder can also receive a form of dividend (*coupon payments*). Credit risk refers to the possibility of the bond holder not receiving the redemption value of their holding. The often used quote attributed to Will Rogers (1930) goes ‘It is not the return on my investment that I am concerned about. It is the return of my investment!’. Thus when bonds are bought and sold - the prices they are sold for depends not only on the face value of the bond, but the likelihood that the company will be able to meet the obligation to make that payment on the *redemption date* (maturity).

Credit agencies, such as *Standard and Poor’s* and *Moody’s* provide a rating scheme that classifies the creditworthiness of these bonds. This rating in turn affects the value of the bond, and vice versa. There is a very strong correlation between *bond value* , *bond yield*, dividends (coupon payments), *credit rating* and *risk of default* . Thus typically if one knows the credit rating of a bond, one knows something about its value and its spread/yield as well as the risk of default. For example. the lowest grade of bonds (in terms of credit rating) are the highest yielding bonds. Similarly, bond values increasing/decreasing often presage the downgrading/upgrading of its credit rating. It is hardly surprising that quantifying value is a focus of all the major bond/fixed-income investors: governments, investment banks, pension funds, insurance companies, hedge funds and even some large private investors.

Just to add to this mountain of jargon and synonyms, in the bond market one also talks about *spread*, which is the same as *yield*, which is the same as *rate of return*.

The highest-rated bonds are called *investment grade bonds* while the lowest are known as *junk bonds*. An example of the highest grade bonds is given the the table 17.1.

17.1 Merton model

Let us consider a company value to consist of three variables: *equity*, *assets* and *debts*. We consider the debts as the bond payments on maturity, the assets are the amount of money the company holds, buildings, machinery etc. that it owns and could sell for cash. Then we shall define the *equity* as the difference between the assets and debt.

$$\mathsf{EQUITY} = \mathsf{ASSETS} - \mathsf{DEBTS}$$

BONDS - GLOBAL INVESTMENT GRADE

Apr 14	Red date	Coupon	Ratings S*	M*	F*	Bid price	Bid yield	Day's chge yield	Mth's chge yield	Spread vs Govts
US$										
Misc Capital	07/14	6.13	BBB	Baa2	-	100.96	1.44	0.27	0.21	1.39
BNP Paribas	06/15	4.80	A-	Baa2	A	104.20	1.22	-0.02	-0.19	1.13
GE Capital	01/16	5.00	AA+	A1	-	107.47	0.64	0.02	-0.06	0.27
Erste Euro Lux	02/16	5.00	A+	-	-	101.82	3.94	0.01	0.12	3.52
Credit Suisse USA	03/16	5.38	A	A1	A	108.30	0.90	-0.02	-0.13	0.28
SPI E&G Aust	09/16	5.75	A-	A3	A-	108.87	1.96	0.07	-0.04	1.59
Abu Dhabi Nt En	10/17	6.17	A-	A3	-	114.75	1.82	-0.03	-0.15	0.73
Swire Pacific	04/18	6.25	A-	A3	A	114.65	2.39	0.06	0.04	0.78
ASNA	11/18	6.95	A-	Baa2	A	119.25	2.44	0.04	0.05	0.84
Codelco	01/19	7.50	AA-	A1	A+	120.92	2.76	0.01	-0.11	1.15
Bell South	10/31	6.88	A-	WR	A	119.74	5.15	0.02	-0.04	2.51
GE Capital	01/39	6.88	AA+	A1	-	133.73	4.58	-0.02	-0.20	1.10
Goldman Sachs	02/33	6.13	A-	Baa1	A	116.85	4.76	0.01	-0.08	2.12

Figure 17.1: Example of investment-grade bonds. The table shows (in the left-most column) the name of the company issuing the bond, for example, BNP Paribas. The redemption date - the date of maturity at which time the bond must be paid to the investor is given in the second column. The coupon payment (in %), that is the annual rate of interest is shown in the next column - this is usually paid at 6 month intervals. Then follows the credit rating by the three agencies, S&P, Moody's and Fitch - the more As that a company bond has, the better *credit rating* and higher price. The bid price for the bond - the price at which it is sold. The last column shows how the yield compares with a low-risk investment government bond.

Since the value of the assets of a company is unpredictable, we can model this as a random process. As a simple measure of the degree of risk we can calculate the ratio of the debt to the value of the assets.

Suppose the debt is F, payable at a time T in the future. The current $(t = 0)$ value of this debt is discounted to Fe^{-rT}: the current value of this debt. Suppose the asset value now $(t = 0)$ is A_0, then Fe^{-rT}/A_0 is called the "leverage". When "highly leveraged" $Fe^{-rT} \gg A_0$ the debt is much larger than the assets of the company. Such a loan has high risk - it is unlikely to be repayable. (In fact, this case represents accounting insolvency). The riskier the loan, the cheaper it becomes and hence the more profitable it becomes (that is the bond yield - coupon payments as a fraction of what it cost to buy are high).

Conversely when $Fe^{-rT}/A_0 \ll 1$ we say that the loan has "low leverage" or the debt is much smaller than the value of the companies assets. This is likely to be repaid and represents a low risk, and consequently the value of the bond is higher. This would be typical of government bonds in which there would be little risk. However, little risk means stability but little reward. Large pensions funds typically are attracted to this kind of long-term stable investment.

Now suppose that the leverage is low and there is no risk with the asset. This would be the case for government bonds, for example, where we assume that these are zero risk, Then the current value of the debt is simply the discounted future value:

$$\boxed{V_0 = Fe^{-rT}} \quad . \tag{17.1}$$

However, this is in general not the case when there is a risk involved. So, let us use our favourite model for a risky asset. So, we use geometric Brownian motion to describe the change in asset value:

$$dA_t = \mu_A A_t dt + \sigma_A A_t dW_t \qquad , \tag{17.2}$$

where μ_A is some (unknown) constant of the model (the drift rate) and σ_A is the volatility of the asset.

For the lender, the loan is repaid if $A_T \geq F$. That is, the company can "liquidate" (sell-off) assets to get enough money to repay the loan.

Conversely, if $A_T < F$ the company is unable to pay the debt in full. The company will be put into liquidation (only individuals go bankrupt in the UK) and all the assets sold off to try to repay the lenders. The amount the lender will get depends on the sale value of the assets, and the lender's repayment priority level. For simplification, we will assume that there is only one lender, and that there are no insolvency administration fees, so that the lender obtains A_T in case of insolvency.

So the value of the loan at maturity is

$$V_T = \begin{cases} A_T & , \quad A_T < F \\ F & , \quad A_T \geq F \end{cases} \tag{17.3}$$

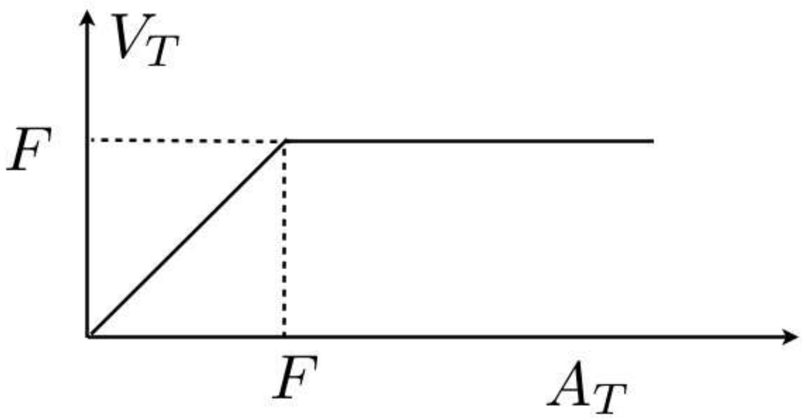

Figure 17.2: Pay-off function for the holder of a bond for which the face value is F. Our notation V_T indicates the value of the debt at the expiry time, T. If the values of the assets at maturity is less than the debt to be repaid ($A_T < F$) the company defaults (and goes into liquidation) while the bond holder receives a fraction of the amount due. If the asset are such that $A_T \geq F$, then the full amount can be repaid.

We note that this pay-off has a hockey - stick shape (similar to an option pay -off) as shown in figure (17.2).

This can be written as:

$$V_T = \min(F, A_T) \qquad , \tag{17.4}$$

but since $\min(x, y) = -\max(-x, -y)$, then

$$V_T = -\max(-A_T, -F) \quad , \tag{17.5}$$

and then it follows that,

$$\boxed{V_T = F - \max(F - A_T, 0)} \qquad . \tag{17.6}$$

So the value of the loan/debt at maturity resembles the formula for a put option at expiry. The link is even stronger since A_t has geometric Brownian motion, so in this case F corresponds to a strike price.

Since A_t has a geometric Brownian motion, it follows that:

$$A_T = A_0 e^{(\mu_A - \frac{1}{2}\sigma_A^2)T + \sigma_A W_T} \quad , \tag{17.7}$$

where W_T is a Wiener process.

We know that the present value of a derivative is given by the discounted risk-neutral expectation of its value at maturity. So in this case we have;

$$V_0 = e^{-rT}\mathbb{E}^*(V_T) \quad . \tag{17.8}$$

That is,

$$V_0 = e^{-rT}\mathbb{E}^*(F - \max(F - A_T, 0)) \quad . \tag{17.9}$$

This simplifies to,

$$V_0 = e^{-rT}F - e^{-rT}\mathbb{E}^*(\max(F - A_T, 0)) \quad . \tag{17.10}$$

The first term on the right-hand-side is simply the risk-free value of the debt (17.1). Clearly, the second term - which is always positive and owes its origin to the riskiness of the asset - reduces the value of the debt. That is, from the viewpoint of the bond holder, the bond value is lowered. That is,

$$V_0 \leq e^{-rT}F \quad . \tag{17.11}$$

Now, the structure of the pay-off given in equation (**??**) contains the main loan minus a term which clearly resembles an option pay-off. We can thus use the (Black-Scholes) mathematics that we have developed for the evalation of option values to determine the expected value arising from the option term. Using the Black-Scholes formula to evaluate the put option, we obtain the expression,

$$\boxed{V_0 = e^{-rT}F - e^{-rT}FN(-d_2^\dagger) + A_0 N(-d_1^\dagger)} \quad . \tag{17.12}$$

The symbols $d_{1,2}^\dagger$ are just as before, with the appropriate change in notation:

$$d_1^\dagger = \frac{\ln(A_0/F) + (r + \frac{1}{2}\sigma_A^2)T}{\sigma_A\sqrt{T}} \quad . \tag{17.13}$$

and

$$d_2^\dagger = \frac{\ln(A_0/F) + (r - \frac{1}{2}\sigma_A^2)T}{\sigma_A\sqrt{T}} \quad . \tag{17.14}$$

Hence credit risk can be evaluated using a formula similar to the one for an option price, but in this case A_0 is the current value of the company's assets, and F is the debt payable.

$$V_0 \leq e^{-rT}F \quad . \tag{17.15}$$

Using the fact that $1 - N(x) = N(-x)$, we can write

$$\boxed{V_0 = e^{-rT}FN(d_2^\dagger) + A_0 N(-d_1^\dagger) \quad .} \tag{17.16}$$

The riskiness of a bond will depend on the riskiness of the underlying assets. If a company's assets are highly volatile then the company may go under and the bond will never be repaid in full. On the other hand, if the company's assets have a low volatility, the bond is almost certain to be paid at the redemption date when it is due.

These notions can be tested with the Merton model. Suppose that $\sigma \to 0$, that is one has a low-risk asset. Then

$$\lim_{\sigma\to 0} d_1^\dagger \to +\infty \quad , \quad \lim_{\sigma\to 0} d_2^\dagger \to +\infty \qquad . \tag{17.17}$$

Hence:

$$V_0 \to e^{-rT}F \quad . \tag{17.18}$$

This is as expected. A (near) certain pay-off of the amount F at a time T in the future would be worth this amount.

On the other hand, when $\sigma$is large,

$$\lim_{\sigma\to +\infty} d_1^\dagger \to +\infty \quad , \quad \lim_{\sigma\to +\infty} d_2^\dagger \to -\infty \qquad . \tag{17.19}$$

Hence, using expression (17.15), one gets

$$V_0 \to 0 \quad . \tag{17.20}$$

Again, this makes sense in financial terms, that a high-risk bond would be almost worthless, since there is very little chance of receiving the payment F at the redemption date, T.

For the speculator, another relevant quantity is the yield: profitability of the bond. This is defined as

$$Y \equiv \frac{1}{T}\ln\frac{F}{V_0} \qquad , \tag{17.21}$$

which, for a low-risk bond, $\sigma \to 0$, and using (17.17) is just the risk-free interest:

$$Y \approx r \qquad . \tag{17.22}$$

One can show that $Y \geq r$ and for this yield, also called the *spread.* In table (17.1), the *yield spread vs govt.* is simply $Y - r$ expressed in % per year. It shows how advantageous the yield is with respect to a risk-free bond.

From the perspective of the shareholder, their interest is in the "equity" of a company (net value)

$$E_0 = A_0 - V_0 \qquad . \tag{17.23}$$

and in general,

$$E_t = A_t - V_t \qquad . \tag{17.24}$$

$$E_0 = A_0 - e^{-rT}FN(-d_2^\dagger) + A_0N(-d_1^\dagger) \qquad . \tag{17.25}$$

$$\boxed{E_0 = A_0N(d_1^\dagger) - e^{-rT}FN(-d_2^\dagger)} \qquad . \tag{17.26}$$

Merton's model assumes that the process of a company getting into difficulty is a gradual but irregular process. In practice, the company can have large jumps in their value due to circumstances. For example, an oil company might discover a new reserve that will increase its value. On the other hand, an accident at an oil well might adversely affect its value. For this reason, the model has its weaknesses. The bond spreads/yields as indicated by market prices seem to be larger than those predicted by Merton's model based on the asset price jumps. This reflects increased uncertainty due to the unforeseen external circumstances.

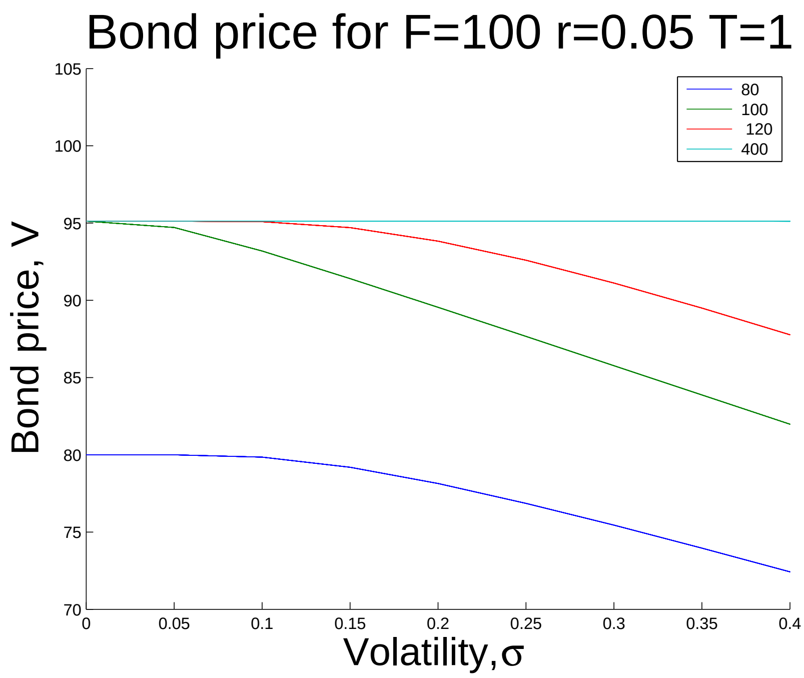

Figure 17.3: The variation in bond price with volatility according to the Merton model. The graph shows bond prices calculated using (17.15) where the redemption date is $T = 1$ (year) the interest rate is $r = 0.05$, per annum. The face value of the (zero-coupon) bond is $F = 100$. In the graph we consider different *leveraging* (ratio of debt to asset value). Various leveragings are shown. That is different values of the current (spot) asset value A_0 =80,100,120 and 140, with respect to the debt. $F = 100$. When the assets are with much more than the debt $A_0 = 400$, then the debt has a low leverage and the price is relatively insensitive to the volatility of the asset. In all cases the higher the volatility (the higher the risk), the lower the value of the bond.

17.2 Futures Options

So far the discussion has been about derivatives derived from an underlying asset such as a share in a company. In this section we consider derivatives of derivatives - that is, where the "underlying" is itself a derivative.

The simplest derivative we considered in this course was the "forward contract" and the closely related "futures contract". The *forward contract* is a legally-binding obligation to buy/sell a certain asset at a certain price, at a certain date in the future. This price, at the time at which the sale takes place, is decided at the time of the contract. Since it is agreed directly between two parties, in what is called an *over-the-counter* arrangement.

When going long in the forward contract, this is an undertaking to *buy* the asset. Short in the forward contract is the obligation to *sell* the asset. To return to our illustration, Alice decides to go long in a forward contract with Bob to buy gold in October at \$600 per ounce. Bob is obliged to sell Alice the gold at this date at this price, and he is short in the contract.

In both cases there is *no* exchange of money between the parties when the forward is set up. Bob does not sell the forward and Alice does not buy the forward. Of course both Alice and Bob are taking a risk in the sense that they cannot predict the price of the asset in the future.

In order that a self-financing risk-free profit could *not* by either the party in the long or short position, the correct (fair) price that should be agreed in the contract now, F_0, should be determined by the underlying asset spot price S_0 and the risk-free interest rate r. The formula is:

$$F_0 = S_0 e^{rT} \qquad . \tag{17.27}$$

In general at any time t (current time), with a futures contract expiring at time T, we have for the fair strike price of the futures:

$$\boxed{F_t = S_t e^{r(T-t)} = S_t e^{r\tau}} \qquad . \tag{17.28}$$

where $\tau = T - t$ is the time to expiry/delivery.

We showed that, since a contract costs nothing to set up, it should be worthless at that time. This was our definition of fair price driven by the market. Since, if the strike price was in any way biased then one of the parties (long or short) would pay a premium. It is this fact that would allow a trader to be extremely speculative, since there is no need to pay up front. However, if the trader is operating on a futures exchange, then it is required that a *margin account* held by the trader on the exchange is able to cover the possible losses.

Of course, as time progresses it becomes clear whether the long or short position is more favourable. If the asset price starts to fall then the person long in the contract is looking at a possible loss, while if the asset starts to rise in value, then the long position in the contract struck earlier seems like a winner. In a futures exchanges one can sell the futures contract at any time and thus cash in on the profit, or cut one's losses by selling the contract.

Consider Alice long in a futures contract for gold, maturing in October with an agreed delivery price (per ounce) agreed back in April to be $F_0 = \$680.00$. This was the fair price in April based upon the formula (17.26). As before, there is no cost in taking out the futures contract - Alice is not out of pocket. Suppose it is now May and the spot price of gold has fallen since April. Alice would be worried by such a development since it looks like she might lose money later in October. Let's say that a futures taken out now, with the same maturity (October) would be \$560.00. That is \$120.00 less than in April. Then for Alice will need to make a payment from her margin account to (temporarily) cover the drop in value. This *margin call* would cost her \$120.00 from her account.

On the other hand if gold had gone up in value in May, so that the equivalent futures price for Gold with expiry in October would be, say, \$750.00, then a (temporary) payment would be made to her margin account of \$70.00. Trading in futures alone is even more highly leveraged than options, since there is zero set-up cost. Although investing in futures costs nothing upfront it starts to cost/make money over time.

So the value of the futures contract, at some time $t \geq 0$, to the party long in the futures would be:

$$f_t = F_t - F_0 \qquad . \tag{17.29}$$

where F_0 was the strike price initially agreed in the contract at $t = 0$, and F_t would be the strike price of the futures contract if taken out now, at time t. Let us repeat the argument as to the correct price F_0.

Now, obviously $F_T = S_T$, that is the correct price to agree to buy an asset forward at expiry since it would have to equal its spot price.

Thus we know the final value of this contract (its pay-off) is just:

$$f_T = F_T - F_0 = S_T - F_0 \qquad . \tag{17.30}$$

If one uses the *technique* of valuation by discounted risk-neutral expectation, then we have:

$$f_0 = e^{-rT}\mathbb{E}^*\left(f_T\right) = e^{-rT}\mathbb{E}^*\left(S_T - F_0\right) \qquad . \tag{17.31}$$

Then according to the fundamental theorem of finance, that the asset price is a martingale under discounted risk-neutral measure, then

$$e^{-rT}\mathbb{E}^*\left(S_T\right) = S_0 \qquad ,$$

and it follows that,

$$f_0 = S_0 - F_0 e^{-rT} \qquad . \tag{17.32}$$

As mentioned many times, the *value* of a futures at the time it is longed/shorted is zero. Then, setting $f_0 = 0$ then we get the result (17.26) which tells us the correct value for the strike price of a forward taken out now, $F_0 = S_0 e^{rT}$. There is nothing new here whatsoever, but it is reassuring to check that one is on the right track.

The advantage of an exchange is that one can trade futures contracts, just as one would with shares on the stock market. Given that a futures contract has some risk inherent, and that there is a market then it follows that the market would offer insurance instruments to guard against losses. Indeed one can also trade options on futures contracts. That is, one has a derivative (option) of a derivative (futures contract) of an underlying asset.

17.2.1 Futures options

Consider the current time $t = 0$, and a futures that expires at $t = T$, with some strike price. A *futures call option* gives the holder the right (not obligation) to acquire a long position in a futures contract at the time of expiry of the option, $0 \leq T_E \leq T$.

The *strike price*, X, in this case would relate to the futures price at expiry of the option. In general the pay-off for a *futures call option* at that time, T_E, at a strike price X, is:

$$\max(F_{T_E} - X, 0) \qquad , \tag{17.33}$$

where, of course, F_{T_E} depends on the spot asset price at the time of option expiry, and thus:

$$F_{T_E} = S_{T_E} e^{r(T-T_E)} \qquad . \tag{17.34}$$

The pay-off for a *futures put option*, with strike X and expiry T_E is:

$$\max(X - F_{T_E}, 0) \qquad , \tag{17.35}$$

Our question is, what is the value of a futures option? That is, how much should one pay to hold a call or put option in a futures contract?

To answer this question, let us assume that price of the asset underlying the futures undergoes geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \qquad . \tag{17.36}$$

Then, as we have argued before:

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \tag{17.37}$$

So, since

$$F_t = S_t e^{r(T-t)} \qquad , \tag{17.38}$$

It follows that:

$$F_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} e^{r(T-t)} = S_0 e^{rT} e^{(\mu - r - \frac{1}{2}\sigma^2)t + \sigma W_t} \qquad . \tag{17.39}$$

That is:

$$\boxed{F_t = F_0 e^{(\mu - r - \frac{1}{2}\sigma^2)t + \sigma W_t}} \qquad . \tag{17.40}$$

We can derive this result from the stochastic differential equation directly. Given that,

$$F_t = S_t e^{r(T-t)} \quad , \tag{17.41}$$

we can use the product rule for differentials,

$$dF_t \equiv F_{t+dt} - F_t = S_{t+dt} e^{r(T-t-dt)} - S_t e^{r(T-t)} \tag{17.42}$$

This can be written as

$$dF_t = (S_t + dS) e^{r(T-t)} e^{-r\ dt} - S_t e^{r(T-t)} \tag{17.43}$$

and this simplifies, after taking $e^{-r\ dt} \approx 1 - r\ dt + O(dt^2)$,

$$dF_t = dS_t e^{r(T-t)} - r dt S_t e^{r(T-t)} + O(dt^2) \quad . \tag{17.44}$$

Further, using (17.35), we have:

$$\boxed{dF_t = (\mu - r) F_t dt + \sigma F_t dW_t} \quad . \tag{17.45}$$

That is the futures price also has a geometric Brownian motion with the same volatility as the asset but with a different rate of drift. Nonetheless, it is straightforward to integrate and gives:

$$F_t = F_0 e^{(\mu - r - \frac{1}{2}\sigma^2)t + \sigma W_t} \quad . \tag{17.46}$$

As expected, this result agrees with equation (17.39).

In order to calculate the option price for the underlying futures, which follows the geometric Brownian motion (17.44), we can simply follow the recipe adopted before, and apply the discounted risk-neutral expectation.

So for the call option, expiry at T_E we have,

$$f_c = e^{-rT_E} \mathbb{E}^* \left(\max(F_{T_E} - X, 0) \right) \quad , \tag{17.47}$$

where f_c is the value/price of the call option on a futures contract (which itself has expiry at time $T \geq T_E$) and the expiry of the call option will be at time T_E. In this case the strike price, X, is the price specified in the option for the futures contract which expires at T.

According to the recipe, the asterisk means we replace μ by r. This leads to the following expression:

$$f_c = \frac{e^{-rT_E}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}z^2}\ dz\ \max\left(F_0 e^{-\frac{1}{2}\sigma^2 T_E + \sigma\sqrt{T_E} z} - X, 0 \right) \quad . \tag{17.48}$$

This integral is familiar from the previous chapter. There is a cut-off in the integrand when

$$F_0 e^{-\frac{1}{2}\sigma^2 T_E + \sigma\sqrt{T_E} z} - X \leq 0 \quad . \tag{17.49}$$

that is when

$$z \leq -\frac{[\ln(F_0/X) - \frac{1}{2}\sigma^2 T_E]}{\sigma\sqrt{T_E}} \quad . \tag{17.50}$$

Let's call the right hand side $-d_2'$:

$$d_2' \equiv \frac{[\ln(F_0/X) - \frac{1}{2}\sigma^2 T_E]}{\sigma\sqrt{T_E}} \quad . \tag{17.51}$$

Then

$$f_c = \frac{e^{-rT_E}}{\sqrt{2\pi}} \int_{-d_2'}^{+\infty} e^{-\frac{1}{2}z^2}\ dz\ \left(F_0 e^{-\frac{1}{2}\sigma^2 T_E + \sigma\sqrt{T_E} z} - X \right) \quad , \tag{17.52}$$

giving

$$f_c = e^{-rT_E} F_0[1 - N(-d_1')] - e^{-rT_E} X[1 - N(-d_2')] \quad . \tag{17.53}$$

where

$$d_1 = \frac{[\ln(F_0/X) + \frac{1}{2}\sigma^2 T_E]}{\sigma\sqrt{T_E}} \quad . \tag{17.54}$$

Then finally we have

$$\boxed{f_c = e^{-rT_E}[F_0 N(d_1') - XN(d_2')]} \quad . \tag{17.55}$$

This is called the Black[1] formula for the call option.

We can make one further check to see that the price is consistent. Suppose the futures option and the futures contract have the same expiry date: $T = T_E$. Then we can state that:

$$F_{T_E} = F_T = S_0 e^{rT} \tag{17.56}$$

Then

$$d_1' = \frac{\ln(S_0 e^{rT}/X) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln(S_0/X) + rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = d_1 \tag{17.57}$$

That is, the same expression as the *asset call option*. Indeed, we have:

$$f_c = S_0 N(d_1) - e^{-rT} XN(d_2) \quad , \quad T_E = T \quad . \tag{17.58}$$

That is, given the futures price of an asset is the same as its spot price (at the maturity of the futures), a futures option must be worth the same as the corresponding option on the underlying asset.

As usual, if the price would be anything else, we could use arbitrage to make a risk-free profit. Consider a time $t = 0$ and an asset with spot price S_0. Suppose that, there is a call option price c, expiry T, strike price X on the is asset. Furthermore, there is a call option on a futures contract (long) on the asset, with expiry T, strike price X. The price of this option is, f_c.

Suppose that $f_c < c$, then the trader (Alice) would immediately (at $t = 0$) long the call option on the futures and short the asset call option (sell this to Bob). This would leave Alice with extra cash to the value $(c - f_c)$ which could be invested in bonds.

Consider the expiry date $t = T$. Suppose for example the asset has a value such that $S_T > X$. Then the following sequence of trades transpires. Bob will, of course, exercise his call option and offer Alice an amount of cash X and request the asset, now with value S_T, in return. Alice, in turn, will now execute her call option on the futures contract. This gets her a legally binding futures contract which requires her to buy the asset at a price X. She uses her cash to do this, and with the asset in hand transfers this to Bob. So Alice makes no money at the expiry; all her profit was made at $t = 0$. We can see the the process is more complicated since the asset is nested within a futures which is nested within an option on the futures.

In the case in which $S_T < X$, Bob will not exercise his call option. It would not be rational to do so, since he would lose money. Similarly Alice will not exercise her call option on the futures, as this would put her at a disadvantage. At expiry, all positions are covered and Alice can retain her profit.

17.2.2 Put-call parity

The corresponding put option price is obtained from exactly the same arguments. We use the risk-neutral expectation:

$$f_p = e^{-rT_E}\mathbb{E}^*\left(\max(X - F_{T_E}, 0)\right) \quad , \tag{17.59}$$

[1] after Fisher Black

which reduces to

$$\boxed{f_p = e^{-rT_E}[XN(-d_2') - F_0N(-d_1')]} \quad . \tag{17.60}$$

Using these expressions, then one can derive the *put-call parity* relation for these options.

In this case has the form:

$$\boxed{f_p + F_0e^{-rT_E} = f_c + Xe^{-rT_E}} \quad . \tag{17.61}$$

In fact, this relation, just like its equivalent for options on assets is an arbitrage relation. This means that, unless this relation is satisfied, one can construct a risk-free profit-making portfolio. The relation must be true regardless of the mathematical model specifying the motion of the asset price.

One way to see why this relation is valid, is to recognise that the term

$$F_0e^{-rT_E} = S_0e^{r(T-T_E)} \quad .$$

So the left-hand side implies the following investment at $t = 0$: long one put option for a futures, expiry T_E. If exercised, then a futures contract, expiry $T \geq T_E$ would be acquired with a strike price X in the contract. At the same time one buys (longs) a quantity $e^{r(T-T_E)}$ of the asset, at $t = 0$. Now, at $t = T_E$ the value of the left-hand-side portfolio is:

$$f_p(T_E) + e^{r(T-T_E)}S_{T_E} \quad .$$

The first term is simply

$$f_p(T_E) = \max(X - F_{T_E}, 0) \quad , \tag{17.62}$$

while the second term is:

$$e^{r(T-T_E)}S_{T_E} = F_{T_E} \quad . \tag{17.63}$$

Thus the full portfolio has a value:

$$\max(F_{T_E}, X) \quad .$$

The right-hand side represents a long call option in the futures, plus a bond investment of Xe^{-rT_E} at $t = 0$. The net value of this portfolio, at $t = T_E$ is:

$$\max(F_{T_E} - X, 0) + X = \max(F_{T_E}, X) \quad . \tag{17.64}$$

The two portfolios have the same value at $t = T_E$ and consequently must have the same value at $t = 0$, and hence the equality.

17.2.3 Black equation

The equation for the futures options price can also be derived from the hedging argument. In this case we end up with a partial-differential equation equivalent to the Black-Scholes equation which can then be solved by the method of Green functions already described.

So consider a portfolio in which we aim to hold a futures contract and insurance (a futures option) at the same time. The aim is to use the correlation in their prices to create a riskless portfolio.

The asset, we will assume, follows a geometric Brownian motion of the form:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad , \tag{17.65}$$

and as shown above, this means the fair price for a futures contract has the geometric Brownian motion.

$$dF_t = (\mu - r)F_t dt + \sigma F_t dW_t \quad . \tag{17.66}$$

Portfolio 1, will be balanced with *short* a call option and long a quantity of futures, to hedge the option.

Suppose the portfolio is set up at time t. At this time the asset has a spot price S_t and the fair strike price of a futures, expiry at time T is $F_t = e^{r(T-t)}S_t$. Recall this price is not the value of the futures contract to the holder. At this time, the contract is worthless. however any change in the asset value will affect the contract price and, consequently the option price for the futures. It is precisely for this reason that both are being balanced.

At this time the portfolio is *short* one unit of a call with an unknown value: f_c, and Δ units long in a futures contract. There is no inherent value in these contracts, so the present value of the portfolio is

$$\Pi^1_t = -f_c(t, F_t) \quad . \tag{17.67}$$

We will see that $f_c \geq 0$, so in fact this portfolio is in the red, this is reflected by the liability we could face if the call goes against this investor.

Now, consider a short time in the future $t + dt$, the value of the portfolio now has the value:

$$\Pi^1_{t+dt} = -f_c - df_c + \Delta dF_t \quad . \tag{17.68}$$

The change in value of the portfolio is, using Ito's lemma, given by:

$$d\Pi^1_t = -df_c + \Delta dF_t = -\left(\frac{\partial f_c}{\partial t} + \frac{1}{2}\sigma^2 F_t^2 \frac{\partial^2 f_c}{\partial F_t^2}\right) dt - \frac{\partial f_c}{\partial F_t} dF_t + \Delta dF_t \quad . \tag{17.69}$$

The usual hedging argument applies: by choosing the amount of futures (Δ) so that:

$$\Delta = \frac{\partial f_c}{\partial F_t} \tag{17.70}$$

then one can eliminate any uncertainty (due to dF_t) in the change in portfolio value. A portfolio hedged in this way then has the change in value:

$$d\Pi^1_t = -\left(\frac{\partial f_c}{\partial t} + \frac{1}{2}\sigma^2 F_t^2 \frac{\partial^2 f_c}{\partial F_t^2}\right) dt \quad . \tag{17.71}$$

Note that the negative sign might mean this portfolio is *losing* money. This does not matter. It matters that the portfolio is risk-free and predictable. That is, knowing its value at time t, we know its value at time $t + dt$. In arbitrage, money can be made if one knows for certain that one portfolio is losing money faster than another: one shorts the portfolio that loses most. Arbitrageurs, in general, can make money on a market whether prices go up or down, as long as there is mispricing in the market.

Consider a second risk-free (predictable) portfolio based on a bond investment. Placing an investment Π^2_t in bonds grows in value to

$$d\Pi^2_t = r\Pi^2_t dt \quad . \tag{17.72}$$

According to the principle of arbitrage, equal investments in two different risk-free portfolios must have the final value. That is the change in value, must be the same.

So if:

$$\Pi^1_t = \Pi^2_t \tag{17.73}$$

then

$$d\Pi^1_t = d\Pi^2_t \tag{17.74}$$

This mean that, given

$$d\Pi^2_t = r\Pi^2_t dt = r\Pi^1 dt = d\Pi^1_t \quad . \tag{17.75}$$

$$r(-f_c)dt = -\left(\frac{\partial f_c}{\partial t} + \frac{1}{2}\sigma^2 F_t^2 \frac{\partial^2 f_c}{\partial F_t^2}\right) dt \quad . \tag{17.76}$$

This gives the following equation, called the *Black equation*:

$$\boxed{\frac{\partial f_c}{\partial t} + \frac{1}{2}\sigma^2 F_t^2 \frac{\partial^2 f_c}{\partial F_t^2} - r f_c = 0 \quad .} \tag{17.77}$$

We have the boundary condition, that at the option expiry time T_E:

$$f_c(T_E, F_{T_E}) = \max(F_{T_E} - X, 0) \qquad . \tag{17.78}$$

17.2.4 Solution of the Black equation

The Black equation can be converted to the convection-diffusion equation by the following changes of variables:

$$\begin{aligned} y &= \ln F_t \\ \tau &= T_E - t \\ g &= e^{-r\tau} f \end{aligned}$$

Then the Black equation (17.76) takes the form:

$$\frac{\partial g}{\partial \tau} = -\frac{1}{2}\sigma^2 \frac{\partial g}{\partial y} + \frac{1}{2}\sigma^2 \frac{\partial^2 g}{\partial y^2} \tag{17.79}$$

The convection term is eliminated by the usual Galilean transformation:

$$x = y - \frac{1}{2}\sigma^2 \tau \qquad , \tag{17.80}$$

which gives, the pure diffusion equation:

$$\frac{\partial g}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 g}{\partial x^2} \tag{17.81}$$

which has the Green function solution:

$$g = G(x, \tau) = \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-x^2/(2\sigma^2\tau)} \tag{17.82}$$

Recall that, as $\tau \to 0$, we have

$$\lim_{\tau \to 0} G(x, \tau) = \delta(x) \tag{17.83}$$

In the solution of (17.80), we need the boundary condition for the call option. This is the pay-off for the call option on the futures given by (17.77):

$$\lim_{\tau \to 0} g(x, \tau) = \max(e^x - X, 0) \quad . \tag{17.84}$$

Then using the Green function we have the solution of (17.80) as,

$$\begin{aligned} g(x, \tau) &= \int_{-\infty}^{+\infty} g(x', 0) G(x - x', \tau) dx' \\ &= \int_{-\infty}^{+\infty} \max(e^{x'} - X, 0) \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-(x-x')^2/(2\sigma^2\tau)} dx' \end{aligned} \tag{17.85}$$

So the value of a call option now ($t = 0$) which expires at a time T_E, would have $\tau = T_E$ and can be written as:

$$f_c = \frac{e^{-rT_E}}{\sigma\sqrt{2\pi T_E}} \int_{-\infty}^{+\infty} \max(e^{x'} - X, 0) e^{-(x-x')^2/(2\sigma^2 T_E)} dx' \quad . \tag{17.86}$$

Now converting back into the financial variables, we can write $x = \ln F_0 - \frac{1}{2}\sigma^2 T_E$. The integral can be performed after some further simplification. The main steps of this have been covered in the option pricing for assets and are exactly analogous. These are as follows, the expression

$$f_c = \frac{e^{-rT_E}}{\sigma\sqrt{2\pi T_E}} \int_{\ln X}^{+\infty} (e^{x'} - X) e^{-(x-x')^2/(2\sigma^2 T_E)} dx' \quad . \tag{17.87}$$

can be simplified to the two integrals:

$$f_c = I_1 + I_2 \tag{17.88}$$

where

$$I_1 = \frac{e^{-rT_E}}{\sigma\sqrt{2\pi T_E}} \int_{\ln X}^{+\infty} e^{x'} e^{-(x-x')^2/(2\sigma^2 T_E)} dx' \tag{17.89}$$

$$I_2 = -\frac{e^{-rT_E} X}{\sigma\sqrt{2\pi T_E}} \int_{\ln X}^{+\infty} e^{-(x-x')^2/(2\sigma^2 T_E)} dx' \tag{17.90}$$

$$\tag{17.91}$$

Then I_2 can be simplified by changing variable $u = (x' - x)/\sigma\sqrt{T_E}$,

$$I_2 = -Xe^{-rT_E} \times \frac{1}{\sqrt{2\pi}} \int_{-d_2'}^{+\infty} e^{-u^2/2} du = -Xe^{-rT_E} N(d_2') \quad . \tag{17.92}$$

The first integral is more complicated and requires a few extra steps.

$$I_1 = \frac{e^{-rT_E}}{\sigma\sqrt{2\pi T_E}} \int_{\ln X}^{+\infty} \exp\left(x + \frac{1}{2}\sigma^2 T_E - \frac{(x' - x - \sigma^2 T_E)^2}{2\sigma^2 T_E} \right) dx' \tag{17.93}$$

Then letting $u = (x' - x - \sigma^2 T_E)/\sigma\sqrt{T_E}$, we get:

$$I_1 = \frac{e^{-rT_E}}{\sqrt{2\pi}} e^{\ln F_0} \int_{-d_1'}^{+\infty} e^{-u^2/2} du = F_0 e^{-rT_E} N(d_1') \tag{17.94}$$

This leads to the result that was obtained before by the martingale approach (17.54), namely:

$$f_c = F_0 e^{-rT_E} N(d_1') - Xe^{-rT_E} N(d_2') \quad . \tag{17.95}$$

Given that the function $0 \le N(z) \le 1$, it follows that, as $T_E \to +\infty$:

$$\lim_{T_E \to +\infty} f_c = 0 \quad . \tag{17.96}$$

17.2.5 Greeks for futures options

Just like the asset options, one can derive Greeks for the futures options. That is, determine the sensitivity of the option with respect to the parameters. In the same way in which one can hedge assets, one can hedge futures by a combination of the option and futures contract: the simple Δ.

So, if one were short a call option on a futures contract, one could go long Δ units in the futures contract, where:

$$\Delta(\text{futures call}) = \frac{\partial f_c}{\partial F} = e^{-rT_E} N(d_1') \quad . \tag{17.97}$$

From this we can assert that:

$$0 \leq \Delta(\textsf{futures\ \ call}) \leq e^{-rT_E} \tag{17.98}$$

Thus, one can infer that as $T_E \to 0$, $\Delta(\textsf{futures\ \ call}) \to 0$, that is one does not need to hold any futures to balance the option. This makes sense since, in the same limit, the futures itself is worthless, as shown by equation (17.95).

The corresponding Γ is:

$$\Gamma(\textsf{futures\ \ call}) = \frac{\partial^2 f_c}{\partial F^2} = e^{-rT_E} \frac{1}{\sigma F_0 \sqrt{T_E}\sqrt{2\pi}} e^{-(d_1')^2/2} \qquad . \tag{17.99}$$

The rate of change of the option value over time, the Θ for this option is given by:

$$\Theta(\textsf{futures\ \ call}) = \frac{\partial f_c}{\partial t} = -\frac{\partial f_c}{\partial T_E} = r f_c - \frac{\sigma F_0 e^{-rT_E} e^{-d_1^2/2}}{2\sqrt{T_E}\sqrt{2\pi}} \tag{17.100}$$

This result we could have also obtained from the Black equation (17.76) which tells us that:

$$\frac{\partial f_c}{\partial t} = r f_c - \frac{1}{2}\sigma^2 F_0^2 \Gamma \qquad . \tag{17.101}$$