Chapter 10

Wiener processes

We now need to make the link between our stochastic process and the diffusion equation. The approach to do so is through so-called Wiener processes.

10.1 Definition

(a) The Wiener process is a function of time with a normal distribution

$$W_t = \sqrt{t}Z \quad . \tag{10.1}$$

where Z is a standard normal random variable with mean 0 and variance 1.

(b) Changes in the value of W from *disjoint* (non-overlapping) time intervals are independent. That is,

$$W_t - W_\tau \quad \text{and} \qquad W_\tau - W_0 \quad , \tag{10.2}$$

are independent random variables for $t \ \geq \ \tau \ \geq 0$

(c) Increments (changes) are time-homogeneous:

$$W_{t+\tau} - W_\tau = W_t - W_0 = W_t \quad \text{for} \;\; \text{all} \quad \tau \geq 0 \quad . \tag{10.3}$$

From this definition, it is immediately clear that our description of Brownian motion can be considered as a Wiener process. Equation (8.54) can be written in the form:

$$\boxed{S_t = \mu_d t + \sigma W_t} \qquad . \tag{10.4}$$

10.2 Properties

It then follows that since $\mathbb{E}(Z) = 0$, $\text{var}(Z) = 1$, we have

(i)

$$W_0 = 0 \quad . \tag{10.5}$$

(ii)

$$\mathbb{E}(W_t) = \mathbb{E}\left(\sqrt{t}Z\right) = \sqrt{t}\mathbb{E}(Z) = 0 \quad . \tag{10.6}$$

(iii)

$$\mathrm{var}(W_t) = \mathrm{var}\left[\sqrt{t}Z\right] = t\ \mathrm{var}(Z) = t \quad . \tag{10.7}$$

We can calculate the probability density for W_t via

$$P(W_t \leq w) = P(\sqrt{t}\ Z \leq w) = P(Z \leq w/\sqrt{t}) \quad , \tag{10.8}$$

giving the probability distribution

$$P(W_t \leq w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{w/\sqrt{t}} e^{-\frac{1}{2}u^2}\ du \quad . \tag{10.9}$$

Then setting $v = u\sqrt{t}$, we get

$$P(W_t \leq w) = \frac{1}{\sqrt{2\pi\ t}} \int_{-\infty}^{w} e^{-\frac{v^2}{2t}}\ dv \quad . \tag{10.10}$$

and

$$f_{W_t}(w) = \frac{d}{dw} P(W_t \leq w) \quad . \tag{10.11}$$

which gives

$$f_{W_t}(w) = \frac{1}{\sqrt{2\pi\ t}} e^{-\frac{w^2}{2t}} \quad . \tag{10.12}$$

Thus W_t has a normal distribution with mean 0 and variance t.

10.3 Differentials

We define the incremental change in W_t as follows, as time changes $t \to t + dt$,

$$dW_t = W_{t+dt} - W_t \quad . \tag{10.13}$$

Then using the property (c) we have:

$$\boxed{dW_t = W_{dt}} \quad . \tag{10.14}$$

Now the Wiener process is continuous everywhere but nowhere differentiable. We can prove the continuity using the epsilon-delta definition, and we can show that it is impossible to differentiate the function. Consider the definition of the derivative of a function applied to a Wiener process:

$$\lim_{h\to 0} \frac{W_{t+h} - W_t}{h} = \lim_{h\to 0} \frac{W_h}{h} = Z \lim_{h\to 0} \frac{\sqrt{h}}{h} \to \infty \quad . \tag{10.15}$$

The sharpness of a Wiener process increases as it shrinks, and the gradient is infinite and random. However, as we shall see shortly, the process W_t^2 *is* differentiable.

Consider now the square of the increment:

$$(dW_t)^2 = (\sqrt{dt}\ Z)^2 = dt.Z^2 \quad , \tag{10.16}$$

which is random, but the expected value of this random variable is:

$$\mathbb{E}\left((dW_t)^2\right) = dt.\mathbb{E}\left(Z^2\right) = dt \quad , \tag{10.17}$$

The variance of $(dW_t)^2$,

$$\mathrm{var}\{(dW_t)^2\} = \mathrm{var}(dt.Z^2) = (dt)^2\mathrm{var}(Z^2) = (dt)^2\{\mathbb{E}\left(Z^4\right) - 1\} = 2(dt)^2 \quad . \tag{10.18}$$

We say a variable is *not* random (deterministic) when there is no uncertainty in its value. The measure of uncertainty is the variance, so a variable with zero variance is considered deterministic.

The uncertainty (variance) in $(dW_t)^2$ is of order $(dt)^2$, which means that, as $dt \to 0$, this vanishes faster than (dt). We assert the following important relation:

$$\boxed{(dW_t)^2 = dt} \quad , \quad dt \to 0 \quad . \tag{10.19}$$

as $dt \to 0$, not just in *expectation*, but also with certainty. That is, $(dW_t)^2$ is not random if we ignore terms of second order in dt. Now this is clearly not right, we have left out a term Z^2. However, in defence of this approximation, we note the following: A random process is not predictable, one can only make deductions and predictions on average. At some stage in the calculation, we will need to average over Z. At that point, the average of Z^2 will be 1 and then (10.19). In this approach we take the average early on rather than deferring it to a later time. It turns out that this is the right thing to do, even though the justification provided is not to the highest mathematical standards.

Equation (10.19) is fundamental: it bridges the gap between stochastic and deterministic calculus. Continuing with this idea, we can neglect anything higher than first order in dt. So this means

$$dt dW_t = (dt)^{3/2} Z \to 0 \quad \text{as } O(dt^{3/2}), \quad dt \to 0 \quad , \tag{10.20}$$

and

$$(dW_t)^3 = (dt)^{3/2} Z^3 \to 0 \quad \text{as } O(dt^{3/2}), \quad dt \to 0 \quad . \tag{10.21}$$

Stochastic calculus differs from ordinary calculus. This can be demonstrated in a very simple case. The quadratic function:

$$y = x^2 \quad . \tag{10.22}$$

is easy to differentiate from first principles. Consider:

$$dy = y(x + dx) - y(x) \tag{10.23}$$

then,

$$dy = (x + dx)^2 - x^2 = 2x dx + (dx)^2 = 2x dx \quad \text{as} \quad dx \to 0 \quad . \tag{10.24}$$

Now we consider,

$$y = W_t^2 \quad . \tag{10.25}$$

$$dy = d(W_t^2) = (W_t + dW_t)^2 - W_t^2 = 2W_t dW_t + (dW_t)^2 \quad . \tag{10.26}$$

However, according to (10.19) the *second-order* term is not negligible, it is first order in time and gives:

$$\boxed{d(W_t^2) = 2W_t dW_t + dt} \quad . \tag{10.27}$$

Thus, we need to revise basic theorems on differentiation, such as *the product rule*, *the quotient rule* and the *chain rule*.

As a second example, consider the differential of W_t^n where n is some positive integer. By the definition of a differential:

$$d(W_t^n) \equiv (W_t + dW_t)^n - W_t^n \quad . \tag{10.28}$$

Using the binomial theorem, this can be written as:

$$d(W_t^n) = \left(W_t^n + nW_t^{n-1} dW_t + \frac{n(n-1)}{2} W_t^{n-2} (dW_t)^2 + \cdots \right) - W_t^n \quad , \tag{10.29}$$

where terms involving $(dW_t)^3$ and higher powers can be neglected, but NOT terms of second order. Then we have, using the identity (10.19)

$$\boxed{d(W_t^n) = nW_t^{n-1}dW_t + \frac{n(n-1)}{2}W_t^{n-2}dt \qquad ,} \tag{10.30}$$

which is consistent with (10.27).

This is a special case of the stochastic *chain rule* that we consider in the next chapter, more commonly known as *Itô's lemma.*

10.4 Stochastic integration

Relation (10.27) can be integrated to give:

$$\int_0^t d(W_t^2) = 2\int_0^t W_t dW_t + \int_0^t dt \quad . \tag{10.31}$$

That is,

$$W_t^2 = 2\int_0^t W_t dW_t + t \quad . \tag{10.32}$$

which can be written as:

$$\boxed{\int_0^t W_t dW_t = \tfrac{1}{2}W_t^2 - \tfrac{1}{2}t} \quad . \tag{10.33}$$

This time, the extra (non-Newtonian) term occurs in the integral. This is called the Itô form of the integral. To understand this result, as in differentiation, we must return to first principles. Our definition of integration is usually taken as the Riemann sum. Consider a closed interval $[a, b]$ for the variable x. Then this is partitioned into n non-overlapping sub-intervals;

$$x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n \qquad ,$$

where $x_0 = a$ and $x_n = b$. Then let x_k^* be some point in the k interval,

$$x_{k-1} \leq x_k^* \leq x_k \qquad .$$

and the width of this k^{th} interval is:

$$\Delta x_k \equiv x_k - x_{k-1}$$

Then, in the limit of many 'strips' $n \to \infty$ and $\max \Delta x_k \to 0$:

$$\int_a^b f(x)dx \approx \sum_{i=1}^n f(x_i^*)\Delta x_i$$

This is the mathematical version of the rectangle rule for numerical integration. However, it leaves the choice of ${x_i}^*$.

So consider the integral:

$$\int_0^a x dx$$

where we take the intervals as equally wide with width h, that is $\Delta x = h = a/n$ and also choose $x_i^* = x_{i-1} = (i-1)h$. Then:

$$\int_0^a x dx \approx \sum_{i=1}^n (i-1)h.h = h^2 \tfrac{1}{2}n(n-1) = \tfrac{1}{2}a^2\left(1 - \tfrac{1}{n}\right)$$

Alternatively, if we choose $x_i^* = x_i = ih$, we find

$$\int_0^a x dx = \sum_{i=1}^{n} ih.h = h^2 \tfrac{1}{2} n(n+1) = \tfrac{1}{2} a^2 \left(1 + \tfrac{1}{n}\right).$$

In the limit for $n \to \infty$, both integrals give the result:

$$\int_0^a x dx = \tfrac{1}{2} a^2 \qquad . \tag{10.34}$$

Now consider the definite integral (10.33) written as such a sum:

$$\int_0^t W_t dW_t = \sum_{i=1}^{n} W_{t_i^*} \left(W_{t_i} - W_{t_{i-1}}\right) \tag{10.35}$$

This integral notation needs to be clearly defined before we play around with the variables too much.

First of all, we note the limits of integration refer to the *time limits* not the limits of W_t, this is one of the quirky notations of stochastic calculus.

What this means is that we are we are integrating over a length of time $[0, t]$, over which the function W_t changes randomly. Now we make the assumption that we can take t^* at the lower limit of the interval, that is *suppose*:

$$\boxed{W_{t_i^*} = W_{i-1}} \qquad , \tag{10.36}$$

where we denote W_{t_i} as W_i.

Then in the sum (10.35)

$$\sum_{i=1}^{n} W_{t_i^*} \left(W_{t_i} - W_{t_{i-1}}\right) = \sum_{i=1}^{n} W_{i-1} \left(W_i - W_{i-1}\right) \tag{10.37}$$

But we can rewrite the term:

$$W_{i-1} \left(W_i - W_{i-1}\right) = \tfrac{1}{2} \left(W_i^2 - W_{i-1}^2\right) - \tfrac{1}{2} \left(W_i - W_{i-1}\right)^2 \tag{10.38}$$

Then:

$$\sum_{i=1}^{n} W_{t_i^*} \left(W_{t_i} - W_{t_{i-1}}\right) = \tfrac{1}{2} \sum_{i=1}^{n} \left(W_i^2 - W_{i-1}^2\right) - \tfrac{1}{2} \sum_{i=1}^{n} \left(W_i - W_{i-1}\right)^2 \tag{10.39}$$

The first series on the RHS is simple to sum, since successive terms cancel each other :

$$\tfrac{1}{2} \sum_{i=1}^{n} \left(W_i^2 - W_{i-1}^2\right) = \tfrac{1}{2} \left(W_n^2 - W_0^2\right) \tag{10.40}$$

In the second series, let us assume that the times are equally spaced so that $t_i = (i-1)h$, and in the limit $h \to 0$, then:

$$\left(W_i - W_{i-1}\right)^2 = W_h^2 = h \qquad .$$

This gives the final result:

$$\tfrac{1}{2} \sum_{i=1}^{n} \left(W_i^2 - W_{i-1}^2\right) = \tfrac{1}{2} \left(W_n^2 - W_0^2\right) - \tfrac{1}{2} nh, \tag{10.41}$$

and translating back into an integral with $nh = t$, we have the result:

$$\boxed{\int_0^t W_t dW_t = \tfrac{1}{2} W_t^2 - \tfrac{1}{2} t} \qquad . \tag{10.42}$$

This is called the *Itô form* of the Wiener integral, and it contains an extra term in the integral.

10.5 Itô versus Stratanovich

One of the questions is what would happen if we took a different point of the interval as reference point for the function value. Will we get the same result for the Wiener integral?

Suppose we chose to go for the upper limit of the interval:

$$\boxed{W_{t_i^*} = W_i} \quad , \tag{10.43}$$

Then in the sum (10.35)

$$\sum_{i=1}^{n} W_{t_i^*}\left(W_{t_i} - W_{t_{i-1}}\right) = \sum_{i=1}^{n} W_i\left(W_i - W_{i-1}\right) \tag{10.44}$$

Each term of the series can be rewritten as

$$W_i\left(W_i - W_{i-1}\right) = \tfrac{1}{2}\left(W_i^2 - W_{i-1}^2\right) + \tfrac{1}{2}\left(W_i - W_{i-1}\right)^2 \tag{10.45}$$

Following the same simplification as previously, we get:

$$\sum_{i=1}^{n} W_i\left(W_i - W_{i-1}\right) = \tfrac{1}{2}\sum_{i=1}^{n}\left(W_i^2 - W_{i-1}^2\right) + \tfrac{1}{2}\sum_{i=1}^{n}\left(W_i - W_{i-1}\right)^2 \tag{10.46}$$

This gives the final result:

$$\sum_{i=1}^{n} W_i\left(W_i - W_{i-1}\right) = \tfrac{1}{2}\left(W_n^2 - W_0^2\right) + \tfrac{1}{2}nh \tag{10.47}$$

Since $nh = t$ we have the result (10.33):

$$\boxed{\int_0^t W_t dW_t = \tfrac{1}{2}W_t^2 + \tfrac{1}{2}t} \quad . \tag{10.48}$$

So a seemingly innocuous choice of t^* gives a completely different result for the integral.

We could also consider choosing the average value of the function:

$$\boxed{W_{t_i^*} = \tfrac{1}{2}(W_i + W_{i-1})} \quad , \tag{10.49}$$

This gives us:

$$\begin{aligned}
\sum_{i=1}^{n} W_{t_i^*}\left(W_{t_i} - W_{t_{i-1}}\right) &= \tfrac{1}{2}\sum_{i=1}^{n}\left(W_i + W_{i-1}\right)\left(W_i - W_{i-1}\right) &(10.50)\\
&= \tfrac{1}{2}\sum_{i=1}^{n}\left(W_i^2 - W_{i-1}^2\right) = \tfrac{1}{2}\left(W_n^2 - W_0^2\right), &(10.51)
\end{aligned}$$

which gives:

$$\int_0^t W_t dW_t = \tfrac{1}{2}W_t^2 \quad , \tag{10.52}$$

which is the *usual* answer. This is called the *Stratanovich* form of the Wiener integral.

Thus we have three different answers for the integral depending on how we define the integral. This simply exposes the inherent difficulties in stochastic calculus. None of the choices is incorrect by definition. Ultimately the right choice is more one of convenience given the specific situation. However, once one has decided on a choice, it is important to use it consistently. In these notes we always choose the Itô convention. The reason for this is that the current value of the asset is known, and that its value in the future is affected by stochastic processes.