School of Electronics and Computer Science
University of Southampton

COMP6212 Computational Finance 2017/18, Assignment (Part I: 50%)

Issue	20 February 2018
Due	7 March 2018 10:00AM)

(a) 25 Marks

1. Consider two assets whose expected returns and covariances on returns are given by $\mathbf{m} = \begin{bmatrix} 0.10 & 0.10 \end{bmatrix}^{\mathbf{t}}$ and $\begin{bmatrix} 0.005 & 0.0 \\ 0.0 & 0.005 \end{bmatrix}$ respectively. Derive the Markowitz efficient frontier for this data, deriving your solution from basic algebra and not using a computer.

2. Consider three securities whose expected returns are

$$\mathbf{m} = \begin{bmatrix} 0.10 & 0.20 & 0.15 \end{bmatrix}^t,$$

and their corresponding covariances are

$$C = \begin{bmatrix} 0.005 & -0.010 & 0.004 \\ -0.010 & 0.040 & -0.002 \\ 0.004 & -0.002 & 0.023 \end{bmatrix}$$

Generate 100 random portfolios and plot a scatter diagram in the $E - V$ space as shown in Fig. 1 of [1].

Use `MATLAB`'s financial toolbox to draw the efficient portfolio frontier for this three-asset model, and for the three two-asset portfolios, taking the assets pair-wise. Draw the graphs and scatter of points using the same scale and briefly comment on what you observe.

3. Set up the `CVX` Convex Programming toolbox in MATLAB and familiarize yourself with it. Work through one or two simple examples given in the documentation.

Replace the two optimization steps in the `NaiveMV` function (*i.e.* calls to `linprog` and `quadprog` by `CVX` and show that similar results (or identical results?) are produced.

4. Obtain daily `FTSE 100` data for the past three years and data for the prices of 30 companies in the `FTSE` index. `Yahoo Finance` may be a convenient source.

Select three stocks at random and estimate the expected returns and covariances from the first half of the time series. Select assets for which approximately equal lengths of data is available and use a simple method to impute any missing values in them. Using the above estimates design an efficient portfolio.

Would the portfolio have performed better than the simple $\frac{1}{N}$ portfolio during the remainder of the period for which you have data (see [2])?

5. Implement one of the enhancements reviewed in [2], selecting the method to implement according to the last digit of your student number as shown in Table 1.

Last digit of student number	Method to implement
$0, 1, 2$	Bayes-Stein shrinkage portfolio (Eqn. (4))
$3, 4, 5$	MacKinlay & Pastor model (Eqn. 7)
$6, 7, 8\&9$	Shortsale constrained portfolio (Eqn. 8)

Table 1: Choice of method to implement

6. Implement and compare two strategies for *Index Tracking* using the returns on the FTSE index and 30 constituent considered in the earlier section : **(a)** a greedy forward selection algorithm that selects about a fifth of the available stocks; and **(b)** sparse index tracking portfolio using l_1 regularization as discussed in [3]. Tune the regularization parameter so that the number of stocks selected by the regularization scheme is similar to the number selected in method (a). Comment on the subsets selected by the two methods.

7. Lobo *et al.* [4] discuss how transaction costs may be included in optimizing adjustments to a portfolio. Study how they impose various constraints, and explain "in your own words" aspects of the conclusions explained in Fig. 2, and the optimization problem formulated in the example of Section 1.6 (what the objective function is, what the constraints are and how you might extend the work you have done so far to implement this). What is needed is an insightful discussion, you need not actually implement this.

(b) 25 Marks

Study how the Black-Scholes model of pricing options was derived before attempting this task. The notation is : K, the strike price; S, the value of the underlying asset; r the risk-free interest rate; T, the time of maturity; t, the current time and $\mathcal{N}(x)$, the cummulative normal distribution.

1. Write the expression for $\mathcal{N}'(x)$ (the derivative of $\mathcal{N}(x)$).
   Show that $S\mathcal{N}'(d_1) = K \exp(-r(T-t))\mathcal{N}'(d_2)$ where d_1 and d_2 were defined as:

$$\begin{aligned} d_1 &= \frac{\log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{(T-t)}} \\ d_2 &= d_1 - \sigma\sqrt{(T-t)} \end{aligned}$$

   Calculate the derivaties $\partial d_1/\partial S$ and $\partial d_2/\partial S$
   With the solution for the call option price given by

$$c = S\mathcal{N}(d_1) - K\exp(-r(T-t))\mathcal{N}(d_2)$$

   show that its derivative with respect to time is

$$\frac{\partial c}{\partial t} = -rK\exp(-r(T-t))\mathcal{N}(d_2) - S\mathcal{N}'(d_1)\frac{\sigma}{2\sqrt{(T-t)}}$$

   Show that $\partial c/\partial S = \mathcal{N}(d_1)$

Differentiating again to get

$$\frac{\partial^2 c}{\partial S^2} = \mathcal{N}'(d_1)\frac{1}{S\sigma\sqrt{(T-t)}},$$

and substituting in the relevant expression, show that the expression for the call option price indeed the solution to the Black-Scholes differential equation.

2. Some prices of call and put options written on the `FTSE100` Index are available in `http://users.ecs.soton.ac.uk/mn/assignment2data/`. These are daily data during the period February to December 1994. The date of maturity is the day following the last date in the time series. There are ten data files with names `c2925.prn` etc., where the initial letter `c` stands for call option and `p` denotes a put option. The number (`2925`) is the strike price of the option. Each file has three columns: the date, the price of the option and the value of the underlying asset (the FTSE index). The annual interst rate during this period may be assumed to be 6.0%.

   Evaluate how well the option prices in the given data satisfy the Black-Scholes model. For each option, evaluate the price given by Black-Scholes from $T/4+1$ to T (the length of the time series), using volatility estimated using a sliding window of range $t - T/4$ to t. Compare the price obtained by using the formula with the true price of the option. Are there any systematic differences? For estimating volatility from historic data, see for example Hull [5] Section 13.4 or a similar source.

   On a random set of 30 days in the range $T/4+1 : T$, compute the *implied volatilities* and plot them as scatter plot against the corresponding volatilities you estimated from data. For prices on any particular day, is there any systematic variation of implied volatilities computed from options with different strike prices (Hint: Look up the term *volatility smile*)?

3. Do one of the following tasks, selecting as follows: If the last digit of your student number is odd, choose (a); if even, choose (b).

   (a) Compare pricing a call option with European style exercise using the Black-Scholes and Binomial lattice methods. Using *one* random set of values for the parameters (strike price, time to maturity, interest rate and volatility) taken from the data given to you, evaluate how the binomial lattice method approximates Black-Scholes as the step time δt is decreased. Plot a graph of the absolute difference between the two methods as a function of δt.
   Consider pricing an option using a binomial lattice. The code for pricing a put option with American style exercise included the following lines[6]:

```
[...]
for tau=1:N
    for i= (tau+1):2:(2*N+1-tau)
        hold = p_u*PVals(i+1) + p_d*PVals(i-1);
        PVals(i) = max(hold, K-SVals(i));
    end
end
[...]
```

Explain *in your own words* the steps involved in this part of the code. How will this change if you were pricing a call option (with American style exercise)?

(b) Hutchinson *et al.* [7] show that the complex relationship between option prices and the underlying variables can be reasonably approximated by nonparametric neural network type models. In particular, they train a Radial Basis Functions (RBF) model on data simulated from a Black-Scholes model and show that not only good approximations to the options price can be obtained, but also the sensitivity *Delta* ($\Delta = \partial C/\partial S$) can be relaibly extracted from it. On the data used above, and using data simulated from Black-Scholes equations, verify if the claims made in [7] are true.

Marking Scheme

For each part, 18/25 marks will be awarded for correctly doing what you are instructed to do. Additional marks are gained by "going the extra mile", *i.e.* for showing initiative of reading slightly outside what was taught, demonstrating clear understanding, presenting the work to high standards etc.

Report

Write a report of no more than ten pages describing the work you have done, answering any questions above. Pay attention to technical writing to high standards. Do not cut and paste formulae or figures from other sources; you must typeset them or draw them yourself. Neat hand-written sketches and formulae may be included if they are your own work. Figures and tables should have informative captions.

There will be a help session on 21 Feb 2018. Please make sure you use this session to:

- Grab some data (index, assets and options) and plot them
- Set up `CVX` and run a small program using it (see slides of lectures)
- Read through the whole of this sheet and understood what is expected

Thereafter, you will be expected to work on your own and **individually**.

References

[1] H. Markowitz, "Portfolio selection," *The Journal of Finance*, vol. 7, no. 1, pp. 77 – 91, 1952.

[2] V. DeMiguel, L. Garlappi, and R. Uppal, "Optimal versus naive diversification: How inefficient is the 1/n portfolio strategy?" *The Review of Financial Studies*, vol. 22, no. 5, pp. 1915 – 1953, 2009.

[3] J. Brodie, I. Daubechies, C. De Mol, D. Giannone, and I. Loris, "Sparse and stable Markowitz portfolios," *PNAS*, vol. 106, no. 30, pp. 12 267 – 12 272, 2009.

[4] M. Lobo, M. Fazel, and S. Boyd, "Portfolio optimization with linear and fixed transaction costs," *Annals of Operations Research*, vol. 152, no. 1, pp. 341–365, 2007.

[5] J. C. Hull, *Options, Futures and Other Derivatives.* Prentice Hall, 2009.

[6] P. Brandimarte, *Numerical Methods in Finance and Economics.* Wiley, 2006.

[7] J. Hutchinson, A. Lo, and T. Poggio, "A nonparametric approach to pricing and hedging derivative securities via learning networks," *The Journal of Finance*, vol. 49, no. 3, pp. 851–889, 1994.

Mahesan Niranjan
February 2018