Chapter 11

The chain rule and Itô's Lemma

The fair value of a derivative depends on the asset price. In this chapter, we will examine how we can set up a differential equation for a function which depends on an underlying stochastic quantity.

11.1 Stochastic differential equations

Brownian motion can be described with the following equation:

$$S_t = S_0 + \mu_d t + \sigma W_t \quad . \tag{11.1}$$

where S_t is the asset value at time t, S_0 is the starting value, μ_d is the drift parameter, W_t is a Wiener process, and σ is the volatility of the asset.

The equivalent differential equation follows from the incremental change;

$$S_{t+dt} = S_0 + \mu_d(t + dt) + \sigma(W_t + dW_t) = S_t + \mu_d dt + \sigma dW_t \quad . \tag{11.2}$$

Thus:

$$\boxed{dS_t = \mu_d dt + \sigma dW_t} \quad . \tag{11.3}$$

This is an example of a *stochastic differential equation*, for which we already know the integral (solution), since each differential has a constant prefactor only.

$$\int_{S_0}^{S_t} dS_t = \int_0^t \mu_d dt + \int \sigma dW_t \quad . \tag{11.4}$$

Then we have, for a small change:

$$\delta s = \mu_d \delta t + \sigma W_{\delta t} \tag{11.5}$$

However, in order to evaluate changes we also have to consider the next term in a Taylor series.

$$(\delta s)^2 = (\mu_d \delta t)^2 + 2\mu_d \delta t \sigma W_{\delta t} + \sigma^2 (W_{\delta t})^2 \quad . \tag{11.6}$$

For small δt the first two terms on the RHS are of order $(\delta t)^2$ and $(\delta t)^{3/2}$, respectively. The leading order term is the last, and we have:

$$(\delta s)^2 = \sigma^2 \delta t, \tag{11.7}$$

and we find the assertion we made regarding the diffusion equation (9.16)

$$\lim_{\delta s, \delta t} \frac{(\delta s)^2}{\delta t} = \sigma^2 \tag{11.8}$$

11.1.1 Diffusion revisited

Let's revisit the derivation of the diffusion equation - and this time do it properly. Instead of assuming the binomial model, we consider the case that the jumps have a general distribution. The probability density $u(s,t)$ for the asset price having the value s at a time t, is given by

$$u(s,t+\delta t) = \int f(\delta s) u(s-\delta s,t) d(\delta s) \qquad , \tag{11.9}$$

which is an exact expression, in which $f(\delta s)$ is the probability density that the price changes by an amount δs in a time δt. For the Wiener process, we know this function:

$$f(\delta s) = \frac{1}{\sigma\sqrt{2\pi\delta t}} e^{-(\delta s)^2/(2\sigma^2\delta t)} \tag{11.10}$$

Even if we don't know the distribution, we can still proceed in the following way. We introduce a Taylor series expansion for $u(s-\delta s,t)$ up to second order:

$$u(s,t+\delta t) \approx \int_{-\infty}^{+\infty} f(\delta s)\left[u(s,t) - \delta s\frac{\partial}{\partial s}u(s,t) + \frac{(\delta s)^2}{2}\frac{\partial^2}{\partial s^2}u(s,t)\right] d(\delta s) \qquad , \tag{11.11}$$

This then leads to:

$$u(s,t+\delta t) \approx u(s,t)\int_{-\infty}^{+\infty} f(\delta s)d(\delta s) - \frac{\partial u(s,t)}{\partial s}\int_{-\infty}^{+\infty} f(\delta s)\delta s d(\delta s) + \frac{\partial^2 u(s,t)}{\partial s^2}\int_{-\infty}^{+\infty} f(\delta s)\frac{(\delta s)^2}{2}d(\delta s) \qquad . \tag{11.12}$$

For the first term, we know that, for any probability distribution:

$$\int_{-\infty}^{+\infty} f(\delta s)d(\delta s) = 1 \qquad . \tag{11.13}$$

The second term on the right is the mean.

$$\int_{-\infty}^{+\infty} f(\delta s)\delta s d(\delta s) = E(\delta s) \qquad . \tag{11.14}$$

For the Wiener process (11.10), we have

$$\int_{-\infty}^{+\infty} f(\delta s)\delta s d(\delta s) = 0 \qquad . \tag{11.15}$$

The third term is the expected value of the square of δs:

$$\int_{-\infty}^{+\infty} f(\delta s)(\delta s)^2 d(\delta s) = \mathrm{var}(\delta s) + (E(\delta s))^2 \qquad . \tag{11.16}$$

For the Wiener process, given by (11.10), this has the value $\sigma^2\delta t$. So (11.12) thus becomes

$$u(s,t+\delta t) \approx u(s,t) + \frac{\sigma^2\delta t}{2}\frac{\partial^2 u(s,t)}{\partial s^2} \qquad . \tag{11.17}$$

Then we have:

$$\frac{u(s,t+\delta t) - u(s,t)}{\delta t} = \frac{\sigma^2}{2}\frac{\partial^2 u(s,t)}{\partial s^2} \qquad . \tag{11.18}$$

And as $\delta t \to 0$, gives:

$$\frac{\partial}{\partial t}u(s,t) = \frac{\sigma^2}{2}\frac{\partial^2 u(s,t)}{\partial s^2} \qquad . \tag{11.19}$$

11.2 Itô's Lemma

Consider a function of time and a stochastic variable: $f(t, S_t)$. Let us assume that S_t follows a Brownian motion:

$$dS_t = \mu_d dt + \sigma dW_t \qquad . \tag{11.20}$$

Then, the differential equation for f is:

$$\boxed{df = \left[\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial S_t^2}\right] dt + \frac{\partial f}{\partial S_t} dS_t} \qquad . \tag{11.21}$$

PROOF: According to Taylor's theorem, to second-order

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S_t}dS_t + \frac{1}{2}\left(\frac{\partial^2 f}{\partial S_t^2}\right)(dS_t)^2 + \left(\frac{\partial^2 f}{\partial S_t \partial t}\right)(dS_t)(dt) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial t^2}\right)(dt)^2 \qquad . \tag{11.22}$$

Now let us discard the terms on the right-hand-side that vanish faster than δt as $(dS, dt) \rightarrow 0$

$$(dS_t)(dt) = (\mu_d dt + \sigma dW_t)dt = \mu_d (dt)^2 + \sigma(dW_t)(dt) \sim (dt)^{3/2} \qquad . \tag{11.23}$$

This term vanishes faster than first-order. On the other hand, the second-order term:

$$(dS_t)^2 = \mu_d (dt)^2 + 2\mu\sigma(dt)(dW_t) + \sigma^2 (dW_t)^2 \qquad , \tag{11.24}$$

contains a first-order term. Thus, up to order δt, we have:

$$(dS_t)^2 = \sigma^2 (dW_t)^2 = \sigma^2 dt \qquad . \tag{11.25}$$

Taken together this gives:

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S_t}dS_t + \frac{1}{2}\left(\frac{\partial^2 f}{\partial S_t^2}\right)\sigma^2 dt \qquad , \tag{11.26}$$

or

$$df = \left[\frac{\partial f}{\partial t}dt + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial S_t^2}\right] dt + \left(\frac{\partial f}{\partial S_t}\right) dS_t \qquad , \tag{11.27}$$

as required.

11.2.1 Example

So suppose,

$$f(t, S_t) = S_t^2 \qquad , \tag{11.28}$$

and that $\mu_d = 0$, and $\sigma = 1$, so that:

$$S_t = W_t \qquad , \tag{11.29}$$

which implies $dS_t = dW_t$ is the stochastic differential equation for S_t. Then according to Itô's lemma we have

$$df = \left[\frac{\partial f}{\partial t} + \frac{1}{2} \times 1 \times \frac{\partial^2 f}{\partial S^2}\right] dt + \frac{\partial f}{\partial S} dS \qquad . \tag{11.30}$$

which gives

$$df = \{0 + \frac{1}{2} \times 2\}dt + (2S)dS \qquad . \tag{11.31}$$

so

$$df = d(S_t^2) = d(W_t^2) = dt + 2W_t dW_t \qquad . \tag{11.32}$$

as found previously.

11.2.2 The chain rule

The chain rule is used when we want to calculate the derivative for a function of a function. That is, take $f(u)$ where u is a function of x and we want to know the change in f with respect to a change in x. To evaluate df/dx, we use:

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} \tag{11.33}$$

which can also be written in the form,

$$\boxed{df = \frac{df}{du} \cdot \frac{du}{dx} \cdot dx\ ,} \tag{11.34}$$

which we can call the Leibniz version of the differential. The chain rule is also called the *function of a function rule*. However, this relation does not take this form when x is a stochastic variable.

Let $f(u)$ be a (doubly) differentiable function of u, and suppose u is function of a stochastic variable. For simplicity we take the Wiener process, W_t. That is, $u(W_t)$.

Then the *chain rule* takes the form:

$$\boxed{df = \frac{df}{du}\left[\frac{du}{dW_t}dW_t + \frac{1}{2}\frac{d^2u}{dW_t^2}dt\right] + \frac{1}{2}\frac{d^2f}{du^2}\left(\frac{du}{dW_t}\right)^2 dt} \tag{11.35}$$

The first term on the RHS is the usual Leibniz chain rule expression (11.34), with the other terms stochastic corrections.

PROOF:

By definition, over a short time dt the change in f is given by:

$$df = f(u(W_t + dW_t)) - f(u(W_t)) \qquad , \tag{11.36}$$

Then using Taylor's theorem for (11.36) we get:

$$df = f(u + du) - f(u) = \frac{df}{du}du + \frac{1}{2}(du)^2\frac{d^2f}{du^2}, \tag{11.37}$$

where we now need to go to second order in du.

Now we use Taylor's theorem for u. With $(dW_t)^2 = dt$, we have

$$u(W_t + dW_t) = u(W_t) + \frac{du}{dW_t}dW_t + \frac{1}{2}\frac{d^2u}{dW_t^2}dt + \cdots$$

so the change in u is:

$$du = u(W_t + dW_t) - u(W_t) \approx \frac{du}{dW_t}dW_t + \frac{1}{2}\frac{d^2u}{dW_t^2}(dW_t)^2 = \frac{du}{dW_t}dW_t + \frac{1}{2}\frac{d^2u}{dW_t^2}dt.$$

This can be inserted in the first term in (11.37). For the second term, we need to evaluate $(du)^2$. When we square the expression for du, only the square of the first term survives:

$$(du)^2 = \left(\frac{du}{dW_t}\right)^2 dt \qquad . \tag{11.38}$$

Now putting everything together we obtain the chain rule:

$$\boxed{df(u) = \frac{df}{du}\left[\frac{du}{dW_t}dW_t + \frac{1}{2}\frac{d^2u}{dW_t^2}dt\right] + \frac{1}{2}\frac{d^2f}{du^2}\left(\frac{du}{dW_t}\right)^2 dt} \tag{11.39}$$

Example 1

So, if

$$f = e^{aW_t}$$

where a is a constant, then, taking $u = aW_t$, we have

$$f = e^u \quad , \quad \frac{df}{du} = e^u \quad , \quad \frac{d^2 f}{du^2} = e^u \quad , \quad \frac{du}{dW} = a \quad , \quad \frac{d^2 u}{dW^2} = 0 \qquad .$$

Then according to (11.35):

$$df = e^{aW_t}\left(a\ dW_t + \frac{1}{2}a^2 dt\right) \qquad .$$

Example 2

On the other hand, suppose the stochastic function has the form:

$$f = e^{aW_t^2} \qquad . \tag{11.40}$$

Then we can apply (11.35) taking $u = aW_t^2$. This gives,

$$f = e^u \quad , \quad \frac{df}{du} = e^u \quad , \quad \frac{d^2 f}{du^2} = e^u \quad , \quad \frac{du}{dW} = 2aW \quad , \quad \frac{d^2 u}{dW^2} = 2a \quad .$$

This leads to the result:

$$df = e^{aW_t^2}\left(2aW_t dW_t + a\ dt + 2a^2 W_t^2\ dt\right) \quad . \tag{11.41}$$

11.2.3 General form of Itô's lemma

Consider the stochastic function, $f(t, S_t)$. Suppose that S_t follows the stochastic differential equation:

$$dS_t = a(t, S_t)dt + b(t, S_t)dW_t \qquad . \tag{11.42}$$

then

$$\boxed{df = \left[\frac{\partial f}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 f}{\partial S^2}\right]dt + \left(\frac{\partial f}{\partial S}\right)dS} \qquad . \tag{11.43}$$

Note that, although S_t is highly irregular as a function of t, the function f is a smooth function of S_t. This means in practice that f will also be highly irregular. Itô's lemma simply tells us how the deterministic term, inside the square brackets in (11.43), and the stochastic term, the second term on the right-hand-side of (11.43), contribute to the total change in f.

PROOF: As before, according to Taylor's theorem, to second-order

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \frac{1}{2}\left(\frac{\partial^2 f}{\partial t^2}\right)(dt)^2 + \left(\frac{\partial^2 f}{\partial t \partial S}\right)(dt)(dS) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial S^2}\right)(dS)^2 \qquad . \tag{11.44}$$

now $(dt)^2 \to 0$ and $dtdS = dt(adt + bdW) \to 0$ hence

$$(dS)^2 = a^2(dt)^2 + 2ab(dt)(dW) + b^2(dW)^2 = b^2 dt \qquad . \tag{11.45}$$

which implies

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \frac{1}{2}b^2\left(\frac{\partial^2 f}{\partial S^2}\right)dt \qquad . \tag{11.46}$$

which gives

$$df = \left[\frac{\partial f}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 f}{\partial S^2}\right]dt + \left(\frac{\partial f}{\partial S}\right)dS \qquad . \tag{11.47}$$

as required.

11.3 Geometric Brownian Motion

A widely used model for asset price movement is the assumption that the *fractional price change* follows a Brownian motion. That is:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \qquad . \tag{11.48}$$

This can be written as

$$\boxed{dS_t = \mu S_t.dt + \sigma S_t.dW_t} \qquad . \tag{11.49}$$

This stochastic differential equation defines *geometric Brownian motion*, also sometimes called exponential Brownian motion. In this section, we will solve (integrate) this equation and obtain an expression for S_t.

The naive approach of separation of variables gives

$$\int \frac{dS_t}{S_t} = \int \mu dt + \int \sigma dW_t \qquad . \tag{11.50}$$

which leads to,

$$\ln S_t - \ln S_0 = \mu t + \sigma W_t. \tag{11.51}$$

This can be rewritten as:

$$S_t = S_0 e^{\mu t + \sigma W_t} \qquad . \tag{11.52}$$

This is **the wrong answer!** We have assumed that

$$d(\ln\ S_t) = \frac{dS_t}{S_t} \qquad , \tag{11.53}$$

without checking that this is correct.

Let us work our way back and consider the derivative of the logarithm. Let,

$$f = \ln S_t \qquad , \tag{11.54}$$

Since S_t has a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \qquad , \tag{11.55}$$

then $a = \mu S_t$ and $b = \sigma S_t$. Application of Itô's lemma (11.43) gives:

$$df = \left[\frac{\partial f}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 f}{\partial S^2}\right]dt + \left(\frac{\partial f}{\partial S}\right)dS \qquad . \tag{11.56}$$

this gives,

$$\frac{\partial f}{\partial S} = \frac{\partial}{\partial S}\ln S = \frac{1}{S} \quad , \quad \frac{\partial^2 f}{\partial S^2} = -\frac{1}{S^2} \qquad . \tag{11.57}$$

Thus

$$df = \left[0 + \frac{1}{2}(\sigma^2 S^2)(-\frac{1}{S^2})\right]dt + \frac{1}{S}\left(\mu S dt + \sigma S dW_t\right) \qquad , \tag{11.58}$$

or

$$df = d\ln S_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t \qquad . \tag{11.59}$$

The derivative of the log function should thus contain a stochastic correction:

$$d\ln S_t = \frac{dS_t}{S_t} - \frac{1}{2}\sigma^2 dt \qquad . \tag{11.60}$$

Equation (11.59) can be integrated without concern since this only has constant coefficients and this gives,

$$f_t - f_0 = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t \qquad . \tag{11.61}$$

$$\ln\ S_t - \ln\ S_0 = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t \qquad . \tag{11.62}$$

or

$$\boxed{S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}} \qquad . \tag{11.63}$$

We see that (11.63) is the correct integration of (11.49), rather that (11.52).

This model of asset price motion has the Wiener process as an exponential, and is sometimes called *exponential Brownian motion*. Note that $S_t > 0$.

Although S_t is unpredictable, we can calculate the expected value and variance of an asset price with geometric Brownian motion:

$$\mathbb{E}\left(S_t\right) = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t}\mathbb{E}\left(e^{\sigma W_t}\right) \tag{11.64}$$

And since $W_t = Z\sqrt{t}$, this can be written as:

$$\mathbb{E}\left(S_t\right) = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}z^2 + \sigma\sqrt{t}z} dz \qquad . \tag{11.65}$$

In tutorials we show that this integral gives:

$$\boxed{\mathbb{E}\left(S_t\right) = S_0 e^{\mu t}} \qquad . \tag{11.66}$$

Furthermore, it can be shown that:

$$\boxed{\mathrm{var}(S_t) = S_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)} \qquad . \tag{11.67}$$

11.3.1 Log-normal distribution

Given that the asset price is given by (11.63) one can easily find the probability density for S_t. It is called the log-normal distribution because the log of the asset price has a normal distribution. Taking the log of (11.63) we get:

$$\ln S = \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma\sqrt{t}Z \qquad . \tag{11.68}$$

That is,

$$Z = \frac{\ln S - \ln S_0 - \left(\mu - \frac{1}{2}\sigma^2\right) t}{\sigma\sqrt{t}} \qquad . \tag{11.69}$$

Then the probability density is given by:

$$g(s)ds = P(s \leq S \leq s + ds) \tag{11.70}$$

Therefore, using equation (8.29):

$$g(s) = \frac{dZ}{dS} f(z) = \frac{1}{s\sigma\sqrt{t}} \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \tag{11.71}$$

That is,

$$g(s) = \frac{1}{s\sigma\sqrt{2\pi t}} \exp\left[-\frac{1}{2}\left(\frac{\ln s - \ln s_0 - \left(\mu - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right)^2\right] \qquad , 0 < s < +\infty \quad . \tag{11.72}$$

11.4 Ornstein-Uhlenbeck equation

An equation that is half-arithmetic and half-geometric Brownian motion, is the *Ornstein-Uhlenbeck* process. This is occasionally used in financial mathematics and in that context it is called the Vasicek model.

The stochastic differential equation defining this process can be written as:

$$\boxed{dX_t = -\kappa X_t dt + \sigma dW_t} \qquad . \tag{11.73}$$

This equation originated in a physics problem in which $X(t)$ is the *velocity* of a particle affected by a random force, the second term on the right in (11.73), and at the same time, damped by some kind of viscous force which is proportional to the velocity: the first term [1]. If we had $\sigma = 0$, the solution is simple exponential decay

$$X_t = X_0 e^{-\kappa t} \qquad , \qquad \sigma \to 0 \quad , \tag{11.74}$$

while, in the absence of damping, the solution is also simple Brownian motion (without drift):

$$X_t = X_0 + \sigma W_t \qquad , \qquad \kappa \to 0 \quad . \tag{11.75}$$

Let us consider now the general case. The equation can be simplified by changing the dependent variable so that

$$X_t = e^{-\kappa t} Y_t \tag{11.76}$$

where Y_t has to be determined. This is analogous to the use of an *integrating factor* in the solution of first-order ordinary differential equations.

Then the equation for Y_t is:

$$e^{-\kappa t} dY_t - \kappa e^{-\kappa t} Y_t dt = -\kappa e^{-\kappa t} Y_t dt + \sigma dW_t \qquad . \tag{11.77}$$

Then we have,

$$dY_t = e^{\kappa t} \sigma dW_t \qquad , \tag{11.78}$$

which can be integrated to give:

$$Y_t - Y_0 = \sigma \int_0^t e^{\kappa t} dW_t \qquad . \tag{11.79}$$

That is:

$$\boxed{X_t = e^{-\kappa t} X_0 + \sigma e^{-\kappa t} \int_0^t e^{\kappa t'} dW_{t'}} \qquad . \tag{11.80}$$

And we can check this result in the limit, $\kappa \to 0$, in which case we have the Brownian motion (with zero drift), equation (11.75). On the other hand, if the stochastic term is zero, $\sigma \to 0$, then (11.80) gives the deterministic exponential decay, as it should: equation (11.74).

[1] G. E. Uhlenbeck and L. S. Ornstein, Phys. Rev. 36, 823 (1930)

Similar to geometric Brownian motion, one can find analytic results for the expected value and the variance. We use the Itô definition for the integral, dividing the interval $0 \le t' \le t$ into n equally wide intervals of width h. The let t_i be at the lower end of the interval: $t_i = (i-1)h$, with $1 \le i \le n$.

$$X_t = e^{-\kappa t} X_0 + \sigma e^{-\kappa t} \sum_{i=1}^{n} e^{\kappa t_{i-1}} (W_{t_i} - W_{t_{i-1}}) \quad . \tag{11.81}$$

In this sum, each of the terms has the form of a mini-jump: $W_{t_i} - W_{t_{i-1}} = W_h$. However, each of these terms is random and certainly *independent* of all the other jumps. So the expectation of each of these individually is $\mathbb{E}(W_h) = 0$. Thus:

$$\mathbb{E}\left(e^{\kappa t_{i-1}} (W_{t_i} - W_{t_{i-1}})\right) = e^{\kappa t_{i-1}} \mathbb{E}(W(h)) = 0 \quad . \tag{11.82}$$

Therefore

$$\boxed{\mathbb{E}(X_t) = e^{-\kappa t} X_0} \quad . \tag{11.83}$$

For the variance, we need to evaluate the term:

$$\mathbb{E}\left(\left[\int_0^t e^{\kappa t'} dW_{t'}\right]^2\right) \quad . \tag{11.84}$$

Once again, we replace the integral by a sum, but now we need two summations with different indices:

$$\begin{aligned} \left[\int_0^t e^{\kappa t'} dW_{t'}\right]^2 &= \sum_{i=1}^{n} e^{\kappa t_{i-1}} (W_{t_i} - W_{t_{i-1}}) \times \sum_{j=1}^{n} e^{\kappa t_{j-1}} (W_{t_j} - W_{t_{j-1}}) && (11.85) \\ &= \sum_{i,j=1}^{n} e^{\kappa (t_{i-1} + t_{j-1})} (W_{t_i} - W_{t_{i-1}})(W_{t_j} - W_{t_{j-1}}) && (11.86) \end{aligned}$$

When the jumps occur at different times $i \neq j$, each term is independent and this gives,

$$\mathbb{E}\left((W_{t_i} - W_{t_{i-1}})(W_{t_j} - W_{t_{j-1}})\right) = \mathbb{E}\left(W_{t_i} - W_{t_{i-1}}\right) \mathbb{E}\left(W_{t_j} - W_{t_{j-1}}\right) = 0 \quad , \quad i \neq j \quad . \tag{11.87}$$

For $i = j$ we have:

$$\mathbb{E}\left((W_{t_i} - W_{t_{i-1}})^2\right) = h \quad . \tag{11.88}$$

Then:

$$\mathbb{E}\left(\left[\int_0^t e^{\kappa t'} dW_{t'}\right]^2\right) = \sum_{i=1}^{n} e^{2\kappa t_{i-1}} h \tag{11.89}$$

Now, we take the limit $h \to 0$ and $n \to \infty$, while maintaining $nh = t$, so that the sum changes back into an integral:

$$\lim_{n \to \infty, h \to 0} \sum_{i=1}^{n} e^{2\kappa t_{i-1}} h = \int_0^t e^{2\kappa t} dt \quad . \tag{11.90}$$

Then we have:

$$\mathbb{E}\left(\left[\int_0^t e^{\kappa t'} dW_{t'}\right]^2\right) = \frac{e^{2\kappa t} - 1}{2\kappa} \tag{11.91}$$

Hence:

$$\boxed{\mathrm{var}(X_t) = \frac{\sigma^2}{2\kappa}\left[1 - e^{-2\kappa t}\right]} \quad . \tag{11.92}$$

Again, we note that in the limit $\kappa \to 0$, the limit of no damping, we get:

$$\mathrm{var}(X_t) \to \sigma^2 t \quad , \quad \kappa \to 0 \quad . \tag{11.93}$$

as we expected from equation (11.75).

One application of this in finance is to model random quantities that tend to revert to a mean. For example, we might consider that the *interest rate* itself is not constant but has some stochastic variation. However, it can be assumed that the interest rate tends to revert back to a long-term average.

The Vasicek model for interest rate variation is the stochastic differential equation:

$$dr_t = a(b - r_t)dt + \sigma dW_t \tag{11.94}$$

where a and b are constants, and r_t is the interest rate. This can be transformed to the Ornstein-Uhlenbeck equation by the following change of variable:

$$r_t \equiv X_t + b(1 - e^{-at}) \qquad , \tag{11.95}$$

The Vasicek model then has the solution:

$$r_t = r_0 e^{-at} + b(1 - e^{-at}) + \sigma e^{-at} \int_0^t e^{at'} dW_{t'} \qquad , \tag{11.96}$$

Then it follows that:

$$\mathbb{E}\left(r_t\right) = r_0 e^{-at} + b(1 - e^{-at}) \tag{11.97}$$

and

$$\mathrm{var}(r_t) = \frac{\sigma^2}{2a}(1 - e^{-2at}) \qquad . \tag{11.98}$$

So, the mean of r_t is a weighted average of the current interest rate and its long-term average.

11.5 Value at Risk

We can also use stochastic differential equations to evaluate what level of risk we are taking with an investment. This is known as Value at Risk. This is important, as this can indicate whether the return on the investment is worth the risk.

Suppose we have an asset which follows a geometric Brownian motion. That is,

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t} \qquad , \tag{11.99}$$

where we *know* μ and σ then we could ask what the range of variation of the asset would be. This would give the holder of the asset an idea of how much he/she would gain or lose.

We already know the expected value and variance of such an asset, which gives us a good idea of the risk inherent in investing in the asset. Another useful quantity is the possible financial loss in holding the asset: the *value at risk* (VaR).

Suppose we have a quantity of money S_0 that we wish to invest, we are considering two possibilities: (a) investing in risk-free bonds (b) investing in risky assets.

Then, choosing method (a), after a time T the value of the portfolio would be entirely predictable:

$$S_0 e^{rT} \qquad , \tag{11.100}$$

Alternatively, if the same (entire) amount were to be invested in assets, at $t = 0$, that is one takes a speculative view, then the value of this investment at $t = T$ would be, using (11.63):

$$S_T = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T} \tag{11.101}$$

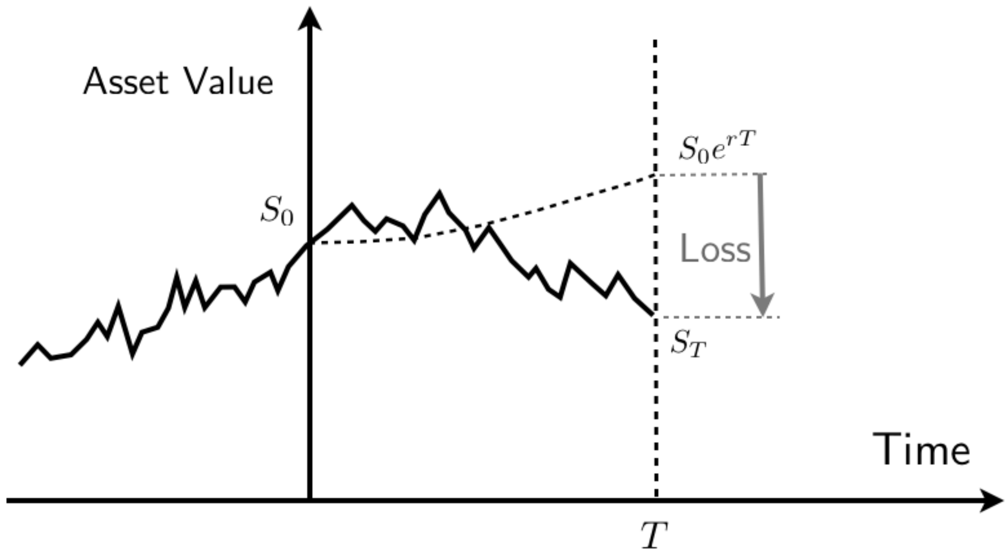

Figure 11.1: The rewards, at time T, in employing two investment strategies: (a) investing in risk-free bonds (b) investing in at $t = 0$, S_0 would lead to $S_0 e^{rT}$ or S_T. Thus the possible loss to speculator (investor in the risky is the difference between these values: $L = S_0 e^{rT} - S_T$.

Then the possible *loss* in making the asset investment rather than the risk-free investment is:

$$\boxed{L = S_0 e^{rT} - S_T = S_0 e^{rT} - S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T}} \quad . \tag{11.102}$$

This is illustrated in figure 11.1.

Now since W_T is random with mean and median 0, then 50% of the outcomes will be $W_T < 0$. The boundary for this loss would be $W = 0$. So 50% of the outcomes give a loss of value:

$$L = S_0 e^{rT} - S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)T} \tag{11.103}$$

or more. This loss value may be positive or negative. Negative losses are profits and negative profits are losses!

The VaR $\mathcal{V}$ is the *maximum loss* to a specified probability p.

$$\boxed{P(L \leq \mathcal{V}) = p} \quad . \tag{11.104}$$

We can see that the VaR is essentially a *percentile*, in the language of statistics. For example, the 90th percentile is the value below which 90 percent of the events occur, that is there is a probability of 0.9 that the event will fall below this value, and in this case it is for the *log-normal distribution.*

Let's convert this implicit equation into an explicit equation for $\mathcal{V}$.

$$P\left(S_0\ e^{rT} - S_0\ e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma Z\sqrt{T}}\ \leq\ \mathcal{V}\right) = p \quad . \tag{11.105}$$

This can be rearranged to give:

$$P\left(Z\ \geq Z_0\right) = p \quad , \tag{11.106}$$

where,

$$Z_0 = \frac{\ln\ \left(\frac{S_0 e^{rT} - \mathcal{V}}{S_0}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \tag{11.107}$$

But Z is a standard normal variable, and thus

$$P(Z \geq Z_0) = 1 - P(Z \leq Z_0) = 1 - N(Z_0) \quad . \tag{11.108}$$

So

$$1 - N(Z_0) = p \quad , \tag{11.109}$$

and thus,

$$Z_0 = N^{-1}(1-p) \quad , \tag{11.110}$$

where N^{-1} means the *inverse* standard normal distribution. That is, Z_0 is the solution of the equation:

$$1 - p = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_0} e^{-\frac{1}{2}z^2} dz \quad , \tag{11.111}$$

This is a standard special function and can be computed without difficulty. One can also use tables if necessary, but this is a bit inconvenient since one uses them in reverse. The inverse function means that we need to search backwards! To save you the trouble, some common values are tabulated below.

Table 11.1: Values of the inverse standard normal distribution: *percentiles* of the distribution. Note that for value of $p > 0.5$, we use, $N^{-1}(p) = 1 - N^{-1}(1-p)$.

p	$N^{-1}(p)$
0.00	$-\infty$
0.05	-1.645
0.10	-1.280
0.15	-1.036
0.20	-0.839
0.25	-0.674
0.50	0.000

$$Z_0 = N^{-1}(1-p) = \frac{\ln\left(\frac{S_0 e^{rT} - \mathcal{V}}{S_0}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \quad . \tag{11.112}$$

Then solving for $\mathcal{V}$ we get

$$\boxed{\mathcal{V} = S_0\ e^{rT} - S_0\ e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}N^{-1}(1-p)}} \quad . \tag{11.113}$$

Consider the extreme limits of this formula when the asset has a high risk or a low risk. Suppose the asset in which you have invested all your money is highly volatile. How much of your investment is at risk? The natural, and correct, response to this is that all of your money would be at risk.

In the formula (11.113) one can test this letting $\sigma \to +\infty$ and we find:

$$\mathcal{V} \approx S_0 e^{rT} \quad . \tag{11.114}$$

That is, all your initial investment S_0, plus all the interest payments you have forgone by putting the investment in a risky asset rather than a risk-free bond which pays off at time T.

On the other hand, when the asset is risk-free we have $\sigma \to 0$, which also means that $\mu \to r$, in which case:

$$\mathcal{V} = 0 \quad . \tag{11.115}$$

So the the VaR depends on the level of confidence p. For example. consider the case $p = 0.5$. For this value (the median) we have $N^{-1}(0.5) = 0$, and thus from (11.113):

$$\mathcal{V} = S_0 \left(e^{rT} - e^{\left(\mu - \frac{1}{2}\sigma^2\right)T} \right) \quad . \tag{11.116}$$

That is, if one considers the investment on a 50:50 basis - the toss of a coin. Then one is 50% certain to lose money when:

$$rT > \left(\mu - \frac{1}{2}\sigma^2 \right) T \quad . \tag{11.117}$$

The value of the loss would not exceed the amount (11.116) again with 50% confidence.

11.5.1 Example

Question: An investor with £ 1,000 to invest in either assets or bonds has the following choice.

(i) Risky asset with drift rate and volatility, $\mu = 22$ % and $\sigma = 20$ % (yr $^{-1/2}$), respectively.

(ii) Risk-free investment with continuous annual interest rate, r =5 % (per annum).

Calculate the VaR $\mathcal{V}$, at 90% probability for investment in the asset over 6 months. That is, we are asked for the maximum loss one would anticipate with 90 % probability if one were to invest all the money into the asset.

Calculate the value-at-risk with 50% probability.

Solution:

Firstly, $p = 0.90$ which implies, the percentile, see table (11.1)

$$N^{-1}(1-p) = N^{-1}(0.1) = -N^{-1}(0.9) = -1.28 \quad . \tag{11.118}$$

therefore

$$\mathcal{V} = 1000\ e^{0.05\times0.5} - 1000\ e^{0.22\times0.5 - 0.5\times0.2^2\times0.5 + 0.2\sqrt{0.5}\times(-1.28)} = £103.14 \quad . \tag{11.119}$$

That is, one is 90 % confident that, by investing £1000 in the risky asset, the loss will NOT exceed £103.14 over the next 6 months.

To calculate the VaR with 50% probability. $p = 0.5$, we note that this is the median value. So $N^{-1}(0.050) = 0$ (see table 11.1).

$$\mathcal{V} = 1000\ e^{0.05\times0.5} - 1000\ e^{0.22\times0.5 - 0.5\times0.2^2\times0.5} = -£79.86 \tag{11.120}$$

That is, one is 50% sure that the loss will NOT exceed £-79.86. In other words, one is 50% confident that the profits will be MORE than $\sim$ £80.