Chapter 7

Put-call parity and binomial model

The model described in the previous chapter is a crude model, as it concerns only a single time step. We can improve on this by subdividing time into smaller segments. However, before we do this, we first need highlight a couple of important findings that can already be observed from the crude model.

7.1 Put-Call Parity

At expiry, the value of a European put option (with exercise price X and expiry date T) is:

$$p_T = \max\left(X - S_T, 0\right) \qquad , \tag{7.1}$$

while the value of a call option, with exercise price X is:

$$c_T = \max\left(S_T - X, 0\right) \qquad . \tag{7.2}$$

For any variable, a, the following relations hold:

$$\max\left(X, Y\right) + a = \max\left(a + X, Y + a\right) \quad . \tag{7.3}$$

Thus,

$$p_T + S_T = \max\left(X - S_T + S_T, S_T\right) = \max\left(X, S_T\right) \quad . \tag{7.4}$$

Similarly,

$$c_T + X = \max\left(S_T - X + X, X\right) = \max\left(S_T, X\right)$$

Then it follows that,

$$p_T + S_T = c_T + X \qquad . \tag{7.5}$$

But these pay-off functions can reflect investment portfolios at an earlier time, $t < T$.

- The RHS of (7.5) can be obtained by a portfolio containing (long) a European call (strike price X, expiry date T) which has a value c_t, and a bond with value $Xe^{-r(T-t)}$. The currect value of this portfolio (total investment) amounts to:

$$c_t + Xe^{-r(T-t)} \quad . \tag{7.6}$$

- The LHS of (7.5) can be achieved by a portfolio, at t, containing long one unit of the asset and long a put option for the asset at strike price X. This portfolio would cost $S_t + p_t$ to put together at time t, and would have a value at expiry of $S_T + p_T = \max(X, S_T)$.

These portfolios have the same pay-off (7.5), even though these values are *unpredictable*. Hence the initial investments must be the same. But this leads to the conclusion that:

$$\boxed{c_t + Xe^{-r(T-t)} = p_t + S_t} \quad . \tag{7.7}$$

The prices of put and call options for the same strike price and expiry date are thus related to each other through X, S_t, r and $T-t$. This balance between the price of put and call options price is known as the *put-call parity* relation. Once we know the price of a (European) call option, we can determine the price of a (European) put option via this relationship.

The relation can be extended to assets that pay dividends or carry costs.

7.1.1 Put-call parity for dividend paying assets

In chapter 4, we discussed how dividend or costs-to-carry affected the fair strike price for a forward. The same principle can be applied to option prices. The key quantity of interest is the discounted present-value of future dividends or costs-to-carry.

Let D_t be the *present* value of any future dividends payable on an asset, between the current time t and the expiry time T of any options of this asset. This extra benefit to the holder of the asset must be taken into account in option pricing. As in *forward* pricing, one can show (left as an exercise) that this extra value should be discounted to the present and subtracted from the asset spot price.

Then the put-call parity has the expression; for options priced at $t=0$ with expiry at $t=T$ and having the *same* strike price X:

$$\boxed{c_t + Xe^{-r(T-t)} = p_t + S_t - D_t} \quad . \tag{7.8}$$

Conversely, if the holder of the asset incurs *costs to carry* (e.g. storage) which are worth U_t at (discounted) present values. This additional cost must be included in holding the asset.

$$\boxed{c_t + Xe^{-r(T-t)} = p_t + S_t + U_t} \quad . \tag{7.9}$$

Note that both D_t and U_t are assumed to be positive.

If prices violate the put-call parity equation, arbitrage could be exploited to make a risk-free profit. Suppose we had, a zero-dividend asset, and market prices for options c, p, at $t=0$ such that

$$c(\text{market}) + Xe^{-rT} < p(\text{market}) + S_0 \quad . \tag{7.10}$$

Then, one would *buy* the LHS and *sell* the RHS. By this we mean, take a long position in the *call* and invest Xe^{-rT} in bonds, while simultaneously shorting the asset and shorting the put option. The profit at pay-off would be the difference between the two sides of the equation, with added interest, that is:

$$\mathsf{profit} = \left[-c(\text{market}) - Xe^{-rT} + p(\text{market}) + S_0\right] e^{rT} \quad . \tag{7.11}$$

If the inequality was reversed, that is:

$$c(\text{market}) + Xe^{-rT} > p(\text{market}) + S_0 \quad . \tag{7.12}$$

then one would reverse the strategy, namely: *buy* the RHS and *sell* the LHS. In practice, short the *call* and borrow Xe^{-rT} in cash, while simultaneously longing the asset and a put option. We again simply follow the rule of selling the overpriced portfolio and buying the underpriced portfolio.

7.2 Price boundaries for options

7.2.1 Options have positive value

We know that, at expiry, $t = T$, the values of the call and put options are:

$$c_T = \max(S_T - X, 0) \quad , \quad p_T = \max(X - S_T, 0) \quad . \tag{7.13}$$

Now one can use a *rational* argument - not a mathematical proof. Anything with *certain* non-negative value in the future must cost something in the present. Therefore we must have, at $t = 0$,

$$\boxed{c_0 \geq 0} \quad , \quad \boxed{p_0 \geq 0}. \tag{7.14}$$

Next, since, at expiry of a call option,

$$-c_T + S_T = \max(X, S_T) \geq 0 \quad , \tag{7.15}$$

then this portfolio (short call plus long asset), must have a non-negative value now, so

$$-c_0 + S_0 \geq 0, \tag{7.16}$$

or,

$$\boxed{c_0 \leq S_0} \quad . \tag{7.17}$$

Similarly, the portfolio consisting of short a put option and long a bond yielding its strike price, has a pay-off:

$$-p_T + X = -\max(X - S_T, 0) + X = \min(S_T - X, 0) + X = \min(S_T, X) \geq 0 \tag{7.18}$$

The portfolio (LHS) would cost at $t = 0$, $-p_0 + Xe^{-rT}$. That is, we would be short a put and long a bond. Therefore, this must be also positive since it has a definite positive pay-off:

$$\boxed{p_0 \leq Xe^{-rT},} \quad . \tag{7.19}$$

giving an upper bound on p_0.

The put-call parity relation, which applies to European options, states that at $t = 0$:

$$c_0 + Xe^{-rT} = p_0 + S_0 \quad , \tag{7.20}$$

Now using (7.14) we have:

$$c_0 - S_0 + Xe^{-rT} = p_0 \geq 0 \quad , \tag{7.21}$$

That is:

$$\boxed{c_0 \geq S_0 - Xe^{-rT}} \quad . \tag{7.22}$$

Finally, again using put-call parity we have that:

$$p_0 + S_0 - Xe^{-rT} = c_0 \geq 0 \quad , \tag{7.23}$$

Then we have the inequality:

$$\boxed{p_0 \geq Xe^{-rT} - S_0} \quad . \tag{7.24}$$

The combination of the inequalities (7.14 - 7.24) are sketched in the diagrams, figure 7.1 and figure 7.2.

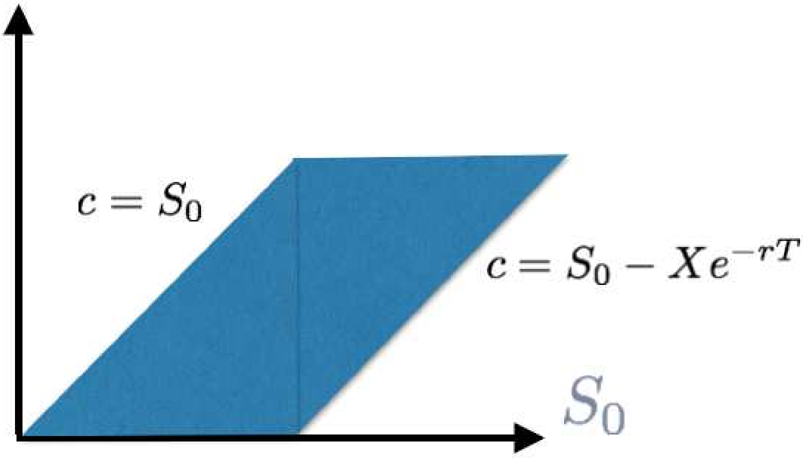

Figure 7.1: The price boundaries for a European call option as a function of the spot price S_0, for a given strike price, X and expiry time T. The shaded area is the *rational* price region: $c \geq 0, c \leq S_0, c \geq S_0 - Xe^{-rT}$. That is, the call option price must lie between these limits irrespective of how the asset price behaves.

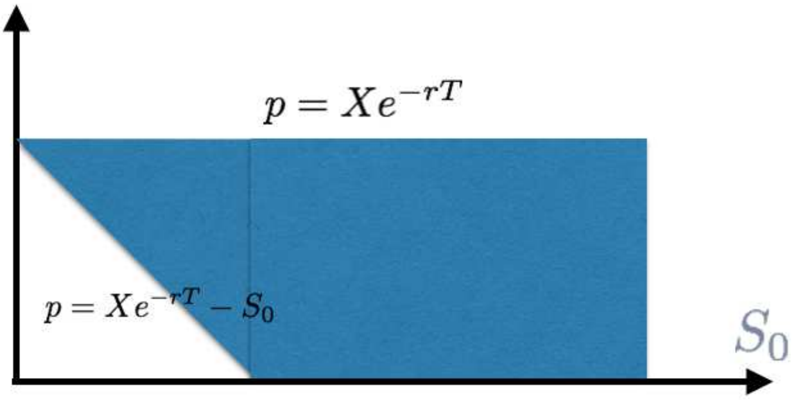

Figure 7.2: The price boundaries for a European put option as a function of the spot price S_0, for a given strike price, X and expiry time T. The shaded region is the *rational* price region: $p \geq 0, p \leq Xe^{-rT}, p \geq Xe^{-rT} - S_0$.

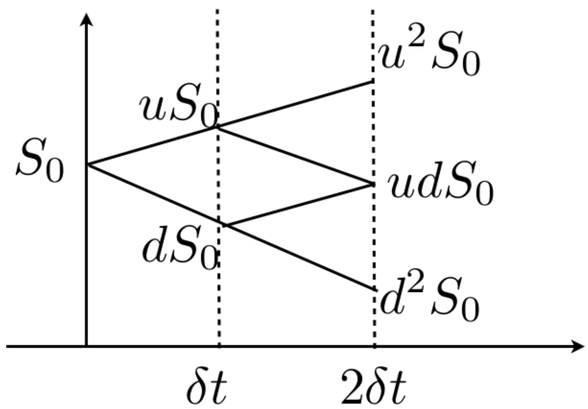

Figure 7.3: Model of asset price change after two Bernoulli trials. This is called a two-step binomial tree for the asset price. For example the probability of the process: $S_0 \to uS_0 \to u^2S_0$, is $p \times p = p^2$

7.3 Binomial model

In the previous chapter we proposed the simplest model of random asset behaviour in which the price goes up or down, with only two possible values, over a time δt. We can repeat this process of tossing a coin for multiple steps in time δt. The basic principles of this *binomial model* are easiest to study in a two-step process, as shown in figure (7.3).

We assume the second step is *independent* of the first. We call such a random process *Markovian*. A *Markov process* is a process in which past behaviour has no bearing on the future. Only the present value of the asset is important, not the history of how the price got there. We say the process has "no memory", or is memoryless. Note that this assumption is completely in conflict with the idea of *regression*. Regression relies on using the past history or trends to extrapolate to the future.

If two random events, A and B, are independent, then by definition we have:

$$\mathcal{P}(A \text{ and } B) = \mathcal{P}(A) \times \mathcal{P}(B) \quad . \tag{7.25}$$

Thus if I toss a coin twice and the probability of HEADS on any toss is 0.5, then the probability of heads on the first toss AND heads on the second toss is $0.5 \times 0.5 = 0.25$.

If it is assumed that the price change is random and memoryless, then the change in asset price in each step can be considered on its own. If the same probability distribution applies each time step, then the process is called a *time-homogenous* Markov process.

7.4 Two-step tree

We consider a two-step process, where the second step, like the first, is random and independent from the first. So, if the asset price after one step is:

$$S_u = uS_0$$

that is the price has gone up, then it will go up again with a probability $0 \leq p \leq 1$ to a value:

$$S_{uu} = u \times uS_0 = u^2 S_0 \quad ,$$

while the probability of it going down from uS_0 to $d \times uS_0$ is $1 - p$, figure (7.3).

Suppose that the *expiry date* of a European call option with strike price X on this asset was $T = 2\delta t$. Then the values of a call option at that expiry time would be

$$c_T = \max\left(S_T - X, 0\right) \tag{7.26}$$

so in this case:

$$c_{uu} = \max\left(u^2 S_0 - X, 0\right) \tag{7.27}$$

$$c_{ud} = \max\left(udS_0 - X, 0\right) \tag{7.28}$$

$$c_{dd} = \max\left(d^2 S_0 - X, 0\right) \tag{7.29}$$

Since each step is independent, we can treat each branch point like an individual Bernoulli process. Thus at each stage in the tree, we will have a p^* given by

$$p^* = \frac{e^{r\delta t} - d}{(u - d)}, \tag{7.30}$$

where the same p^* applies throughout the entire process, since

$$\frac{S_{uu}}{S_u} = \frac{u^2 S_0}{u S_0} = u \tag{7.31}$$

and

$$\frac{S_{ud}}{S_u} = \frac{udS_0}{uS_0} = d \tag{7.32}$$

Since we now the pay-offs at expiry, we can work out the current price by working our way back down the tree from $t = T$. Then

$$c_u = [p^* c_{uu} + (1 - p^*) c_{ud}]\, e^{-r\delta t} \tag{7.33}$$

$$c_d = [p^* c_{ud} + (1 - p^*) c_{dd}]\, e^{-r\delta t} \tag{7.34}$$

and

$$c_0 = [p^* c_u + (1 - p^*) c_d]\, e^{-r\delta t} \tag{7.35}$$

This leads to the formula:

$$\boxed{c_E = \left[p^{*\,2} c_{uu} + 2p^*(1 - p^*) c_{ud} + (1 - p^*)^2 c_{dd}\right] e^{-r2\delta t}} \quad . \tag{7.36}$$

This expression can be expanded and written as:

$$c_0 = \left[p^{*\,2} \max(u^2 S_0 - X, 0) + 2p^*(1 - p^*) \max(duS_0 - X, 0) + (1 - p^*)^2 \max(d^2 S_0 - X, 0)\right] e^{-r2\delta t} \quad . \tag{7.37}$$

To find the price of the corresponding call option, we can use put-call parity.

7.5 Binomial tree

A sequence of repeated identical Bernoulli trials as above gives rise to a binomial distribution. Consider the sequence of coin tosses on which the probability of heads is p on each and every toss in a sequence of n tosses. We can think of each toss being the random (independent) price change (heads=up, tails=down) from one day to the next, for example. Taken over several steps (days) this gives rise to a *probability tree.*

So consider HEADS as being the probability of the price increasing:

$$\mathcal{P}(\text{heads on any toss}) = p \qquad 0 \le p \le 1 \quad . \tag{7.38}$$

and thus, the other side of the coin would be the price decrease with a probability:

$$\mathcal{P}(\text{tails on any toss}) = q = 1 - p \quad . \tag{7.39}$$

Then in this series of n tosses, the probability of exactly i heads, $i \in \{0, 1, \ldots, n-1, n\}$ is:

$$\mathcal{P}(X = i) = \frac{n!}{i!(n-i)!} p^i q^{n-i} \quad , \tag{7.40}$$

where $p^i q^{n-i}$ is the probability of exactly i heads and the binomial coefficient

$$^{n}C_i = \left(\begin{array}{c} n \\ i \end{array} \right) = \frac{n!}{i!(n-i)!} \quad . \tag{7.41}$$

Equivalently, ${}^nC_i$ is the number of ways of choosing i distinct objects from n distinct objects, without regard to ordering. So, for example, the number of ways to get 2 heads from 4 tosses is:

$$ {}^4C_2 = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \frac{4!}{2!(4-2)!} = 6 \quad . \tag{7.42}$$

The two-step Bernoulli model can be generalised to an n-step model. Suppose we have an asset with a known (current) spot price S_0. In the future, a time $T = n\delta t$ the price can vary as shown in figure (7.4). It can have a range of values, from the lowest $d^n S_0$ to the highest $u^n S_0$, with corresponding probabilities from the *binomial distribution*:

$$\boxed{\mathcal{P}(S_{n\delta t} = u^i d^{n-i} S_0) = \binom{n}{i} p^i (1-p)^{n-i}} \quad . \tag{7.43}$$

Then the expected value of the asset price in the future is:

$$\mathbb{E}\left(S_{n\delta t}\right) = \sum_{i=0}^{n} u^i d^{n-i} S_0 \mathcal{P}(S_{n\delta t} = u^i d^{n-i} S_0) \tag{7.44}$$

$$\mathbb{E}\left(S_{n\delta t}\right) = \sum_{i=0}^{n} \binom{n}{i} [pu]^i [(1-p)d]^{n-i} S_0 \quad . \tag{7.45}$$

Using the binomial theorem, we can sum the series to get:

$$\mathbb{E}\left(S_{n\delta t}\right) = [pu + (1-p)d]^n S_0 \tag{7.46}$$

Recall that we only use this *expected value* as a rough estimate for our best guess at what the asset value might be. Similarly we can find the variance for the asset price. The square-root of the variance is the standard deviation which is a measure of the *dispersion*, an estimate of the error in this guess. By definition:

$$\mathrm{var}\left[S_{n\delta t}\right] \equiv \mathbb{E}\left(S_{n\delta t}^2\right) - \left[\mathbb{E}\left(S_{n\delta t}\right)\right]^2 \quad . \tag{7.47}$$

It is not difficult, using the binomial theorem, to obtain the result:

$$\mathrm{var}\left[S_{n\delta t}\right] = \left\{ \left[pu^2 + (1-p)d^2\right]^n - [pu + (1-p)d]^{2n} \right\} S_0^2 \quad . \tag{7.48}$$

Suppose we want to calculate the call or put option price. Then we know that this is defined via the discounted risk-neutral expectation. That is, for an n-step binomial tree, the price of a *call option* expiry $n\delta t$, with strike price X, has the form:

$$c_0 \equiv e^{-rn\delta t}\mathbb{E}^*\left(c_{n\delta t}\right) = e^{-rn\delta t}\mathbb{E}^*\left(\max(S_{n\delta t} - X, 0)\right) \tag{7.49}$$

Explicitly this is:

$$\boxed{c_0 = e^{-rn\delta t} \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} p^{*i} (1-p^*)^{n-i} \max\left(u^i d^{n-i} S_0 - X, 0\right)} \quad . \tag{7.50}$$

where c_0 is the *fair* price of the call option at $t = 0$, for an expiry time $T = n\delta t$, strike price X. The expression for p^* is given as before (7.30) as the 'up' value:

$$p^* = \frac{e^{r\delta t} - d}{u - d} \quad . \tag{7.51}$$

There are two special cases that lead to simple answers. These have previously been discussed for the Bernoulli process and the same argument hold for the binomial case. Suppose $X > u^n S_0$, then the strike price is *always* above the possible future asset prices. Then, $\max\left(u^i d^{n-i} S_0 - X, 0\right) = 0$, for any i, so every term in the sum (7.50) is zero, and the call option is completely worthless:

$$c_0 = 0 \qquad , \quad X > S_T \quad .$$

On the other hand, when $X < d^n S_0$, the call option is always exercised. The term,

$$\max\left(u^i d^{n-i} S_0 - X, 0\right) = u^i d^{n-i} S_0 - X \qquad .$$

Then the binomial series can be summed without difficulty and after a bit of extra work one obtains:

$$c_0 = S_0 - X e^{-rT} \qquad , \quad X < S_T \quad . \tag{7.52}$$

This result can be easily obtained using the arbitrage argument that led to (6.47).

Suppose, however, that the strike price, X, lies within the range of possible asset values, that is: $d^n S_0 \leq X \leq u^n S_0$. In this case we can not use the binomial theorem to evaluate the sum in (7.50) analytically. Instead we must evaluate the sum numerically, although, anticipating the future chapters on stochastic processes in continuous time, the sum can, in the limit $n \to \infty$, to a good approximation be replaced by an integral.

For $n = 1$, the Bernoulli one-step process, we have:

$$c_0 = e^{-r\delta t}\left[p^* \max\left(uS_0 - X, 0\right) + (1 - p^*)\max\left(dS_0 - X, 0\right)\right] \tag{7.53}$$

This agrees, as it should with (6.28). For $n = 2$, (7.50) gives expression (7.37).

The term ${p^*}^i(1-p^*)^{n-i}$ is the (risk-neutral) probability that the asset goes up i times and down $n-i$ times over n steps. The binomial coefficient is the number of ways this is possible, the external factor is the discounting factor, and finally, $\max\left(u^i d^{n-i} S_0 - X, 0\right)$, is the pay-off at the end of branch i of the tree.

For a put option, we have a different pay-off function, the value of the put at expiry $n\delta t$ is, $\max(X - S_{n\delta t}, 0)$, so that, the discounted risk-neutral expectation is:

$$p_0 = e^{-rn\delta t}\mathbb{E}^*\left(p_{n\delta t}\right) = e^{-rn\delta t}\mathbb{E}^*\left(\max(X - S_{n\delta t}, 0)\right) \tag{7.54}$$

That is, the fair price one should pay for a put option at $t = 0$ is:

$$\boxed{p_0 = e^{-rn\delta t}\sum_{i=0}^{n}\frac{n!}{i!(n-i)!}{p^*}^i(1-p^*)^{n-i}\max\left(X - u^i d^{n-i} S_0, 0\right)} \tag{7.55}$$

where the option has an expiry time T, strike price X.

Again, the binomial formula includes the Bernoulli formula as a special case. For $n = 1$ we have:

$$p_0 = e^{-r\delta t}\left[p^* \max\left(X - uS_0, 0\right) + (1 - p^*)\max\left(X - dS_0, 0\right)\right] \tag{7.56}$$

In figure 7.5 we show the results for a 12 step binomial tree. Each time step is one month, $\delta t = 1/12$ years.

Again, following the arbitrage arguments outlined in section 6.2.3 the price of the call option decreases with increasing X. We also notice that, the further away the expiry date, the more valuable the call option is. This point will be discussed in more depth in the context of the continuous random variable.

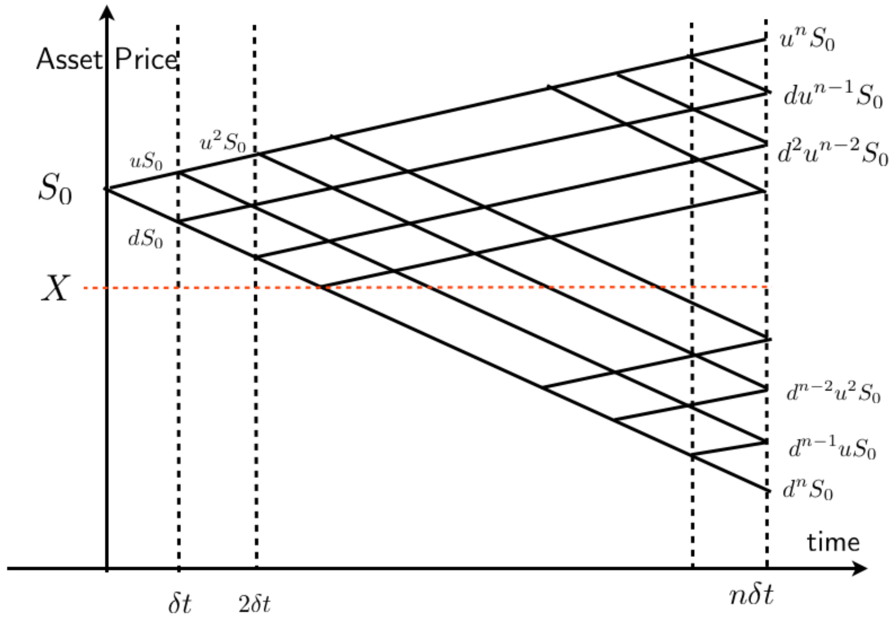

Figure 7.4: Binomial tree for an n-step process. The asset price has value S_0 at present. In the future, after a time $T = n\delta t$, it can have a range of values: $S_T \in \{d^nS_0, d^{n-1}uS_0, d^{n-2}u^2S_0, \ldots, du^{n-1}S_0, u^nS_0\}$. A possible strike price X for an option is also shown for illustration. The criss-crossing lines for all the intermediate times have been omitted for clarity.

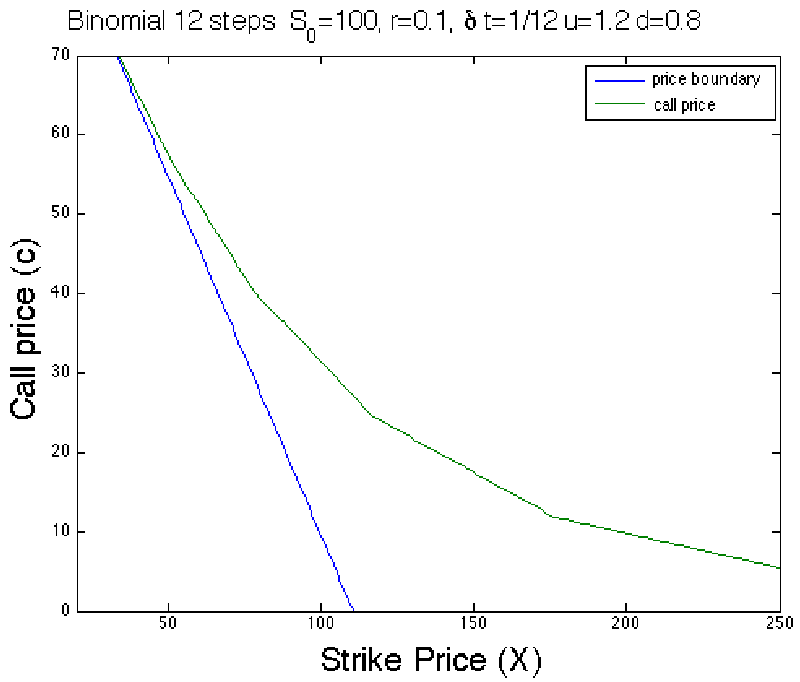

Figure 7.5: The fair price of a European call option, c, as a function of strike price, X, for an asset that follows a binomial process for $n = 12$ steps. The parameters for the asset are $S_0 = 100$, $u = 1.2$ and $d = 0.8$, with $\delta t = 1/12$ and $r = 0.10$. Thus the expiry of the option is one year in the future. The green line shows the option price calculated using the formula (7.50), the blue line is the option price boundary defined by $S_0 - Xe^{-rn\delta t}$, that is in accord with equation (7.22)

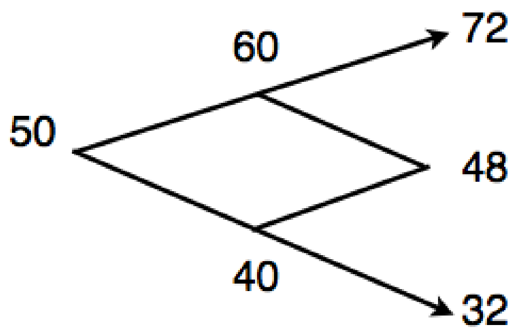

Figure 7.6: Two-step binomial tree for the asset price movement. The asset starts at $S_0 = 50$, then can either move up to $uS_0 = 60$, or down to $dS_0 = 40$ with a given probability. On the second step, the randomness is repeated so the price, after two steps can either be at $S_{2\delta t} = 32, 48$ or 72.

7.6 American Options

American options differ from European options in one respect only. The can be exercised at (specified) dates prior to expiry, as well as at expiry. This additional flexibility, of having additional exercise dates, might be advantageous to the holder. One would expect that the price of this additional privilege would mean that the *American option* would cost more than the *European option* with the same expiry date.

Consider a possible asset price history for which the investor holds a put option (American) with an intermediate exercise date t_E and expiry T.

Scenario 1 : The holder of an American put has the choice to exercise, but the put is "out of the money", and so the investor would *not* choose to do so (this would lose money).

Scenario 2 : The holder of the put is "in the money" and could exercise the option for profit.

Let us denote p_E = European; p_A = American, put. To illustrate the difference, we will consider an example. Suppose we have an asset that follows a 2 - year binomial tree as shown (figure 7.6).

So the asset prices varies at random and after 2 years can be either 32, 48 or 72. Suppose we take r = 0.05 and the strike price is $X = 52$. We have a European put that can only be exercised at $t = 2$ (years). We also have an American put that can be exercised after 1 year or 2 years. The pay -off for a put is

$$\max\left(X - S_T, 0\right), \tag{7.57}$$

where X is the strike price, S_T is the asset price at exercise time/date. We can calculate the European put by the usual risk-neutral expectation measure expectation (discounted). So start on the right-hand edge of the tree and work left (back in time):

The *put value*. like any derivative, is derived using the risk-neutral expectation: $p = \{p^* p_u + (1 - p^*)p_d\}e^{-r\delta t}$ and the put values at $t = 2$. The risk-neutral probability is

$$p^* = \frac{e^{r\delta t} - d}{u - d} \tag{7.58}$$

and for the 'up' price of the tree we have $u = \frac{72}{60} = 1.2$, while the 'down' price is $d = \frac{48}{60} = 0.8$ which implies

$$p^* = \frac{e^{0.05} - 0.8}{1.2 - 0.8} = 0.6282 \tag{7.59}$$

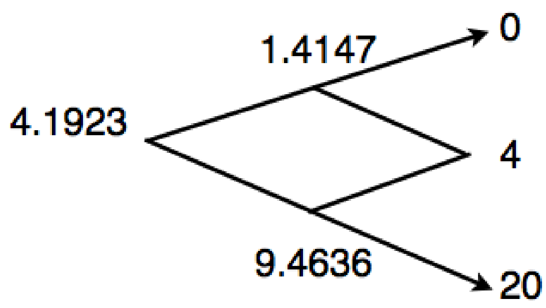

Figure 7.7: Two-step binomial tree for the put option value. The strike price is agreed at $t = 0$ as $X = 52$ and that gives us all the possible values of the put option at pay-off:max$(X - S_T, 0)$, that is $20, 4, 0$. We can work backwards in time along the tree to calculate what the values of the put should be 1 year before expiry. Then finally to the present time $t = 0$.

So the option prices at the two points at $t = 1$ are given by the risk-neutral expectations. The put values at $t = 2$ for an asset price of 72 and 48, namely 0 and 4, create a value:

$$[p^* p_u + (1 - p^*) p_d] \, e^{-r\delta t} \tag{7.60}$$

of

$$[0.6282 \times 0 + 0.3718 \times 4] \, e^{-0.05} = 1.4147 \tag{7.61}$$

for the put option at a time $t = 1$ and an asset price of $S_1 = 60$. Similarly working down the tree we have, the pay-off values at $t = 2$ for an asset price of 48 and 32 of 4 and 20, so that if asset price is valued at $S_1 = 40$, and the option has one year to expiry, the fair price of the option is

$$[0.6282 \times 4 + 0.3718 \times 20] \, e^{-0.05} = 9.4636 \qquad . \tag{7.62}$$

Then finally, the root of the tree has the value:

$$p_E = [0.6282 \times 1.4147 + 0.3718 \times 9.4536] \, e^{-0.05} = 4.1923 \qquad . \tag{7.63}$$

which is the fair price/value of the European put option for an asset with the behaviour shown in figure 7.6.

Now the American put option includes the option of exercise at one year's time or in two year's time, that is at the times $t = 1$ or $t = 2$. The payoff at $t = 2$, holding the option to the end, is as before, 0, 4, or 20 depending on the asset value (which is unpredictable). As before the risk-neutral probability is $p^* = 0.6282$. So we can price the value of holding an American put at the points B and C (figure 7.8). At point B we have the value (to the holder) using the risk-neutral measure.

$$p_A(B) = [0.6282 \times 0 + 0.3718 \times 4] \, e^{-0.05} = 1.4147 \tag{7.64}$$

as before, identical to the European put option.

Calculating the value of the put option at the point C for a holder of the option

$$p_A(C) = [0.6282 \times 4 + 0.3718 \times 20] \, e^{-0.05} = 9.436 \quad \text{(asbefore)} \tag{7.65}$$

But now we need to consider the value in exercising the option at this point. If the asset price is 60, the option is out-of-the-money, and there is no reason to exercise it. However, if the asset price is $S_1 = 40$, early exercise of the put option with a strike price $X = 52$ gives a pay-off of

$$\max{(52 - 40, 0)} = 12. \tag{7.66}$$

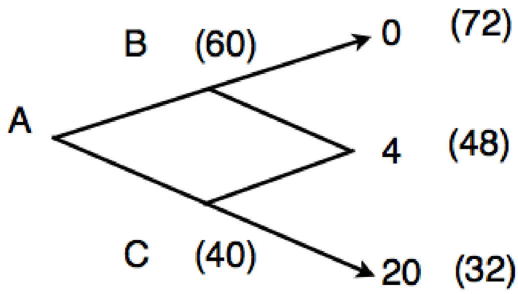

Figure 7.8: Two-step binomial tree for an American put option. The strike price is agreed at $t = 0$ as $X = 52$, but there are two exercise dates $t = 1$ or $t = 2$. The asset values are shown in brackets. The pay-off at $t = 2$ would be: $\max(X - S_T, 0)$, that is $20, 4, 0$. The value of the option at $t = 1$ is given by either the discounted risk-neutral expectation of the value of the put option at $t = 2$, or by the current ($t = 1$) pay-off of the put option. We can work backwards in time along the tree to calculate what the values would be at the present time A.

This is more than the price of the option, and therefore it is advantageous to exercise the option before expiry. Since this is possible with an American put option, the value of the American option is 12 (and *not* 9.4636) at time $t = 1$ and asset price $S_1 = 40$.

In the calculation of the price of the American put at $t = 0$, we need to use the correct value of the American put at point C:

$$p_A(A) = [0.6282 \times 1.4147 + 0.3718 \times 12]\, e^{-0.05} = 5.0894 \tag{7.67}$$

We therefore have that $p_A > p_E$ in this example. This is generally true, as we shall see. American puts are more valuable than European puts.

7.7 Pricing an American call option

An American call option will not be more valuable than a European call option (if the asset does not provide dividends). In fact, we can prove

Theorem: An *American call option* has the same price as a *European call option*,

$$\boxed{c_A = c_E} \quad . \tag{7.68}$$

Proof:

Consider a time $t < T$, before the expiry time T of an option, for which the asset has spot price S_t and the strike price is X. Since the American call option has additional choices over the European option, we must have

$$c_E(t) \le c_A(t), \tag{7.69}$$

At any time, the value of a European call option (7.22) satisfies the inequality:

$$c_E(t) \ge S_t - Xe^{-r(T-t)} \tag{7.70}$$

The value of an American call option, if exercised now, would be:

$$c_{Ax}(t) = \max(S_t - X, 0) \qquad . \tag{7.71}$$

But

$$S_t - Xe^{-r(T-t)} \geq S_t - X \qquad ,$$

therefore,

$$c_E(t) \geq c_{Ax}(t) \qquad . \tag{7.72}$$

Thus, there is no advantage to exercise a call option early - the value of an American option is never more than a European option. Now this is true at all times, but if one never exercises early, then the American option is just a European option, where exercise only takes place at expiry (if at all). Hence they should be priced the same,

$$\boxed{c_A(t) = c_E(t)} \qquad . \tag{7.73}$$

Theorem An American put is worth more than (or equal to) European put:

$$\boxed{p_A \geq p_E} \qquad . \tag{7.74}$$

There are no simple proofs of this result. Clearly the American put is worth at least as much as the European put. The previous example demonstrated that there are cases where early exercise is advantageous, and hence it stands to reason that the American put is more valuable. However, it is not easy to determine the value. Pricing path-dependent options is complicated, and requires numerical methods.

7.8 Price Boundaries for American Options

As a consequence of the above relationships between American and European options, we have no equivalent *put-call parity* equation for American options. Instead we have a *put-call parity* inequality.

In this case, we use the European put-call parity equation:

$$c_E + Xe^{-rT} = p_E + S_0$$

and the fact that, $c_E = c_A$, to get the lower bound,

$$p_A \geq c_{E,A} + Xe^{-rT} - S_0. \tag{7.75}$$

For an upper bound, let us consider the combination of a long European call, value/price c_E and X invested in bonds. Compare this portfolio with a long American put p_A and long one unit of the underlying asset.

$$p_A + S_0 \leq c_{E,A} + X \tag{7.76}$$

and since $c_{E,A} \leq S_0$ then:

$$p_A \leq X \qquad . \tag{7.77}$$

Note the absence in the discount factor for the strike price in the last bound. Early exercise may mean that there is no time for the investment in bonds to grow.