Length and Ratio

T Yeung

THMSS

2024

Outline

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Trigonometric Identities

Trigonometric identities simplify complex expressions.

$$\begin{aligned}
1 &= \sin^2\theta + \cos^2\theta \\
\sin(-\theta) &= -\sin\theta \\
\cos(-\theta) &= \cos\theta \\
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta
\end{aligned}$$

We also have the product-to-sum identities

$$\begin{aligned}
2\cos\alpha\cos\beta &= \cos(\alpha-\beta) + \cos(\alpha+\beta) \\
2\sin\alpha\sin\beta &= \cos(\alpha-\beta) - \cos(\alpha+\beta) \\
2\sin\alpha\cos\beta &= \sin(\alpha-\beta) + \sin(\alpha+\beta)
\end{aligned}$$

that directly follows from the expansion of compound angle formula.

Trigonometric Identities (cont)

By substituting $\frac{\alpha+\beta}{2}$ in α and $\frac{\alpha-\beta}{2}$ in β, we have the sum-to-product identities.

$$\cos(a)\cos(b) = \frac{1}{2}(\cos(a+b) + \cos(a-b)$$
$$\sin(a)\sin(b) = \frac{1}{2}(\cos(a-b) - \cos(a+b))$$
$$\sin(a)\cos(b) = \frac{1}{2}(\sin(a+b) + \sin(a-b))$$

The Extended Law of Sine

Recall the Extended Law of Sine we derived:

Theorem (The Extended Law of Sine)

Given a triangle ABC, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2\mathcal{R}$$

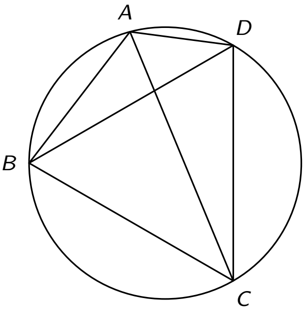
We can use this result to prove the Ptolemy's Theorem.

Ptolemy's Theorem

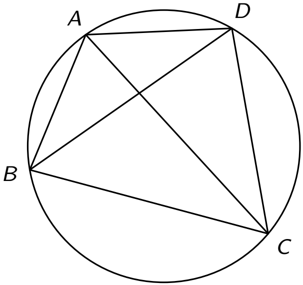
Theorem (Ptolemy's Theorem)

Let $ABCD$ be a cyclic quadrilateral. Then

$$AB \cdot CD + BC \cdot DA = AC \cdot BD$$



Ptolemy's Theorem (proof)



1. WLOG, we set $\mathcal{R} = \frac{1}{2}$ as the radius of $(ABCD)$

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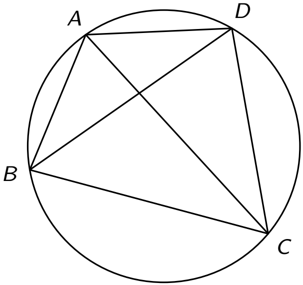

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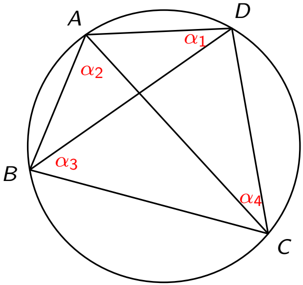

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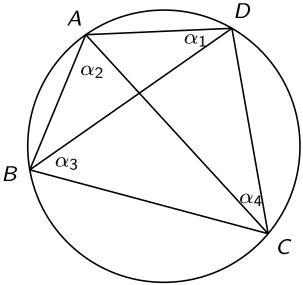

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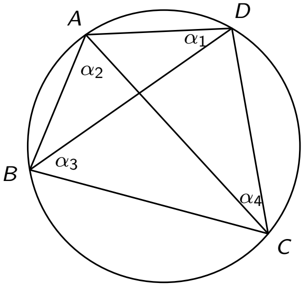

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5. $AB = \sin \alpha_1,\ BC = \sin \alpha_2,$ $CD = \sin \alpha_3,\ DA = \sin \alpha_4$

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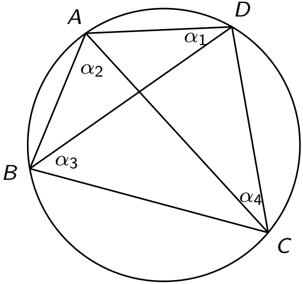

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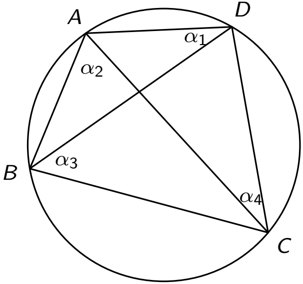
5. $AB = \sin \alpha_1,\ BC = \sin \alpha_2,$ $CD = \sin \alpha_3,\ DA = \sin \alpha_4$
6. $AC = \sin \angle ABC = \sin(\alpha_3 + \alpha_4),$

Ptolemy's Theorem (proof)



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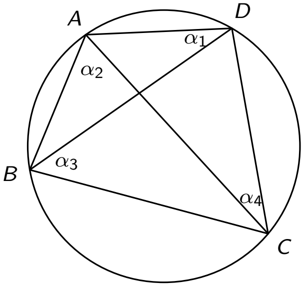

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6. $AC = \sin \angle ABC = \sin(\alpha_3 + \alpha_4),$ $BD = \sin \angle DAB = \sin(\alpha_2 + \alpha_3)$
7. Now what we want to show is $\sin \alpha_1 \sin \alpha_3 + \sin \alpha_2 \sin \alpha_4 = \sin(\alpha_3 + \alpha_4) \sin(\alpha_2 + \alpha_3)$

Ptolemy's Theorem (proof cont.)

8. Note that by product-to-sum identities, we have

$$\sin\alpha_1 \sin\alpha_3 = \frac{1}{2}(\cos(\alpha_1 - \alpha_3) - \cos(\alpha_1 + \alpha_3))$$
$$\sin\alpha_2 \sin\alpha_4 = \frac{1}{2}(\cos(\alpha_2 - \alpha_4) - \cos(\alpha_2 + \alpha_4))$$
$$\sin(\alpha_2 + \alpha_3)\sin(\alpha_3 + \alpha_4) = \frac{1}{2}(\cos(\alpha_2 - \alpha_4) - \cos(\alpha_2 + 2\alpha_3 + \alpha_4))$$

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\end{aligned}$$

9. Since $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 180^o$, we also have

$$\cos(\alpha_1 + \alpha_3) + \cos(\alpha_2 + \alpha_4) = 0$$

Ptolemy's Theorem (proof cont.)

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10. Also note that

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11. The rest is trivial. :)

Stewart's Theorem

Corollary (Stewart's Theorem)

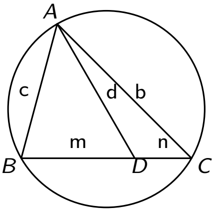
Let ABC be a triangle. Let D be a point on $\overline{BC}$ and let $m = DB$, $n = DC$, $d = AD$. Then

$$a(d^2 + mn) = b^2m + c^2n$$

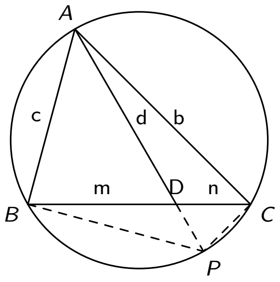
Often this is written in the form

$$man + dad = bmb + cnc$$

as a mnemonic - "a *man* and his *dad* put a *bomb* in the *sink*".



Stewart's Theorem (proof)



1. Let AD meet (ABC) again at P.

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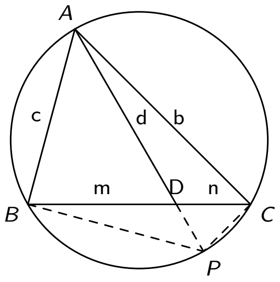
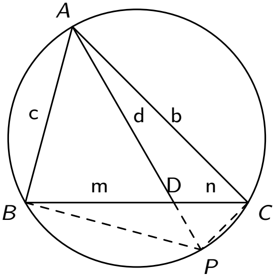


1. Let AD meet (ABC) again at P.
2. By similar triangle, we have $\frac{BP}{m} = \frac{b}{d}$ and $\frac{CP}{n} = \frac{c}{d}$

Stewart's Theorem (proof)



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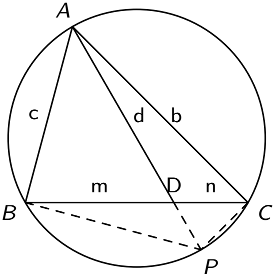

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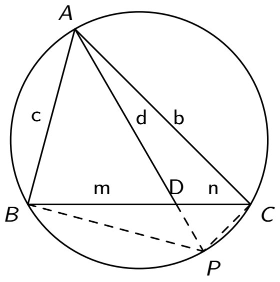
4. By Ptolemy's Theorem, we have $BC \cdot AP = AC \cdot BP + AB \cdot CP$

Stewart's Theorem (proof)



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3. By Power Chord Theorem, we know that $DP = \frac{mn}{d}$
4. By Ptolemy's Theorem, we have $BC \cdot AP = AC \cdot BP + AB \cdot CP$
5. Hence, $a \cdot (d + \frac{mn}{d}) = b \cdot \frac{bm}{d} + c \cdot \frac{cn}{d}$ which is the Stewart's Theorem.

Prelim 2019 Q13

Another application of the Extended Law of Sine:

Prelim 2019 Q13

A, B, C are three points on a circle while P and Q are two points on AB. The extensions of CP and CQ meet the circle at S and T respectively. If $AP = 2, AQ = 7$, $AB = 11, AS = 5$ and $BT = 2$, find the length of ST.

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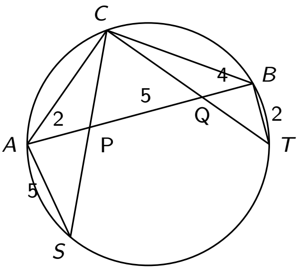

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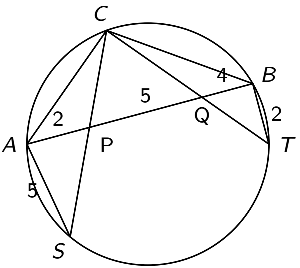
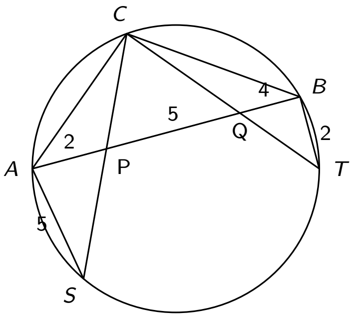
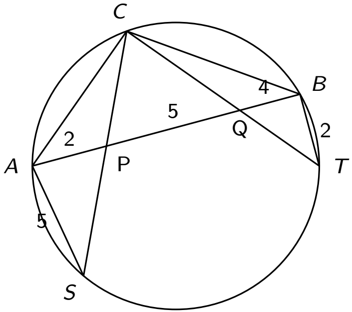


The ratio is not easy to get right. . .

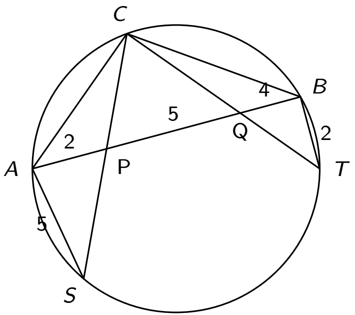
Prelim 2019 Q13 (Solution)





1. The key observation is that $\dfrac{AS}{\sin\angle ACS} = \dfrac{ST}{\sin\angle SCT}$

Prelim 2019 Q13 (Solution)



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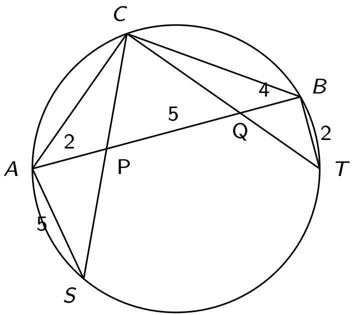

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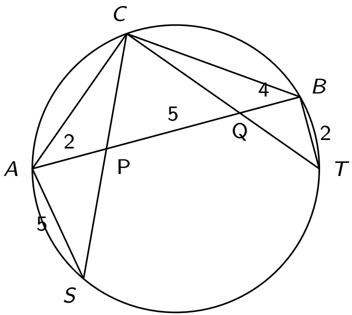
3. Consider $\triangle ACQ$, we know that $\frac{QC}{AC} = 2$

Prelim 2019 Q13 (Solution)



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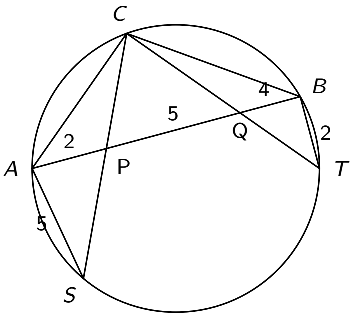
4. We also have $\frac{PQ}{\sin\angle PCQ} = \frac{QC}{\sin\angle CPQ} = \frac{QC}{\sin\angle CPA} = \frac{QC}{AC}\cdot\frac{AC}{\sin\angle CPA} = 2\cdot\frac{AP}{\sin\angle ACP}$

Prelim 2019 Q13 (Solution)



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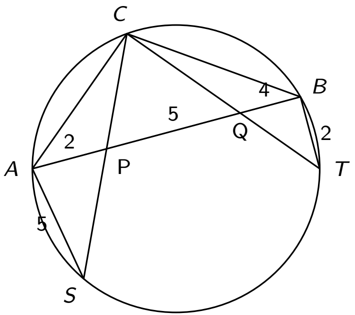
5. Equating the expressions at the ends, we have $\dfrac{\sin\angle ACP}{\sin\angle PCQ} = 2\cdot\dfrac{AP}{PQ} = \dfrac{4}{5}$



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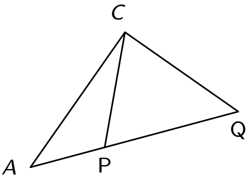
6. Using the first result, we have $ST = \frac{25}{4}$

Angle Bisector Theorem

The third application of the Extended Law of Sine is in proving the Angle Bisector Theorem. In fact, the derivation is exactly the same as a part of the previous question. Recall in $\triangle AQC$, we derived that $\frac{\sin \angle ACP}{\sin \angle PCQ} = \frac{QC}{AC} \cdot \frac{AP}{PQ}$. Rearranging this gives

$$AC \cdot PQ \cdot \sin \angle ACP = QC \cdot AP \cdot \sin \angle PCQ$$

which is generally true.



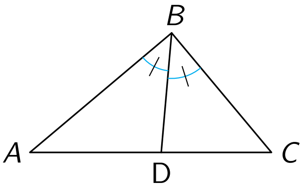
When CP is an angle bisector, we have the angle bisector theorem.

Angle Bisector Theorem

The Angle Bisector Theorem is a special case of the above equality.

Theorem (Angle Bisector Theorem)

Let BD be the angle bisector of $\angle ABC$, then $AB : BC = AD : DC$.



Prelim 2022 Q20

Prelim 2022 Q20

Let $ABCD$ be a cyclic quadrilateral and E be the intersection of AC and BD. P and Q are two points on AC such that the points A, E, Q, P, C lie on the same straight line in this order, and that BP bisects $\angle ABC$ whereas DQ bisects $\angle ADC$. If $AE = 4, EQ = 2$, and $QP = 3$, find the length of PC.

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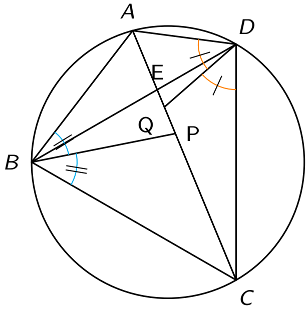
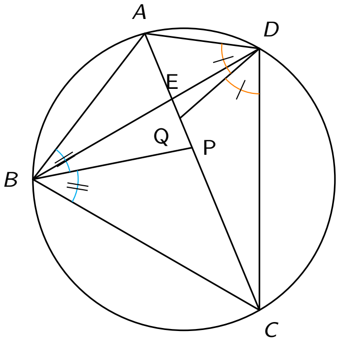
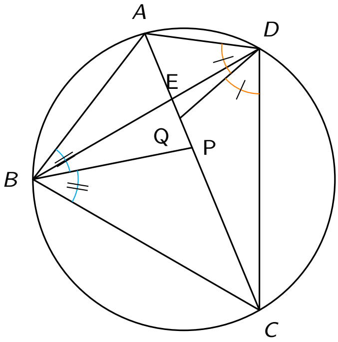


Prelim 2022 Q20 Solution



- $AE = 4$, $EQ = 2$, $QP = 3$
- Find PC

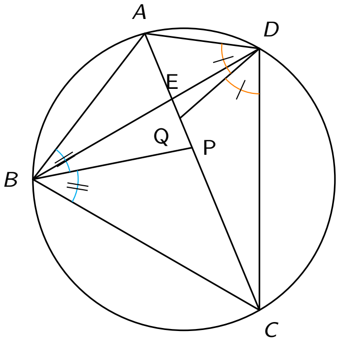
Prelim 2022 Q20 Solution



- $AE = 4, EQ = 2, QP = 3$
- Find PC

1. $\dfrac{AE}{EC} = \dfrac{[ABD]}{[CBD]} = \dfrac{\frac{1}{2}AB\cdot AD\sin\angle BAD}{\frac{1}{2}BC\cdot CD\sin\angle BCD} = \dfrac{AP}{PC}\cdot\dfrac{AQ}{QC}$

Prelim 2022 Q20 Solution



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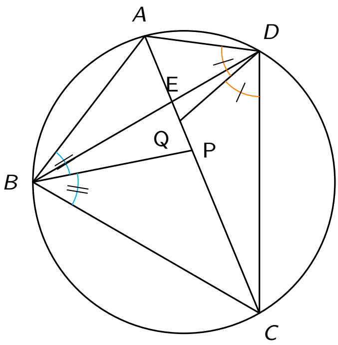
2. Let $PC = x$, we have $\frac{4}{5+x} = \frac{9}{x} \cdot \frac{6}{3+x}$

Prelim 2022 Q20 Solution



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- $AE = 4, EQ = 2, QP = 3$
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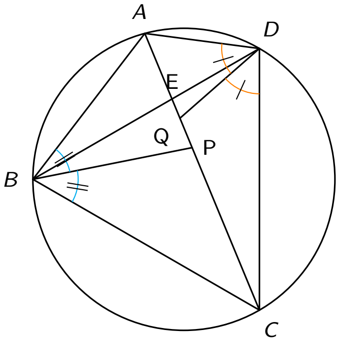
Prelim 2022 Q20 Solution



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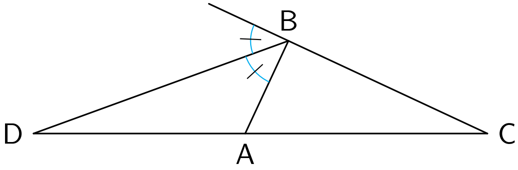
4. Hence $PC = 15$

Extended Angle Bisector Theorem

We have the following extended version of the Angle Bisector Theorem

Theorem (Extended Angle Bisector Theorem)

Let BD be the external angle bisector of $\angle ABC$, then
$AB : BC = AD : DC$.



The proof is left as an exercise. :)

Prelim 2022 Q19

Prelim 2022 Q19 (Modified)

In $\triangle ABC, AB < AC$. The internal bisector of $\angle BAC$ meets BC at D, while the external bisector of $\angle BAC$ meets CB produced at E. If $EB = 10$ and $BD = 5$, find the length of DC.

Prelim 2022 Q19 (Modified)

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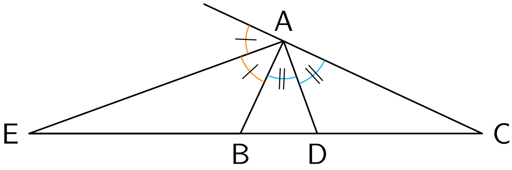
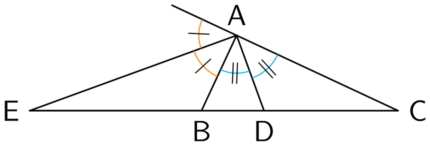
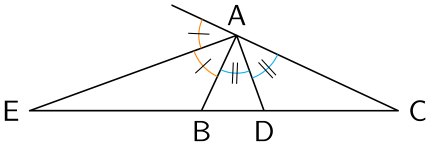


Prelim 2022 Q19 (Solution)



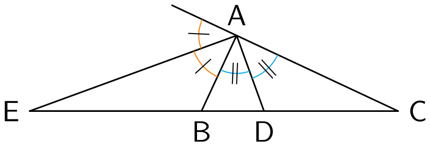
- $EB = 10, \ BD = 5$
- Find DC



- $EB = 10,\ BD = 5$
- Find DC

1. Let $DC = x$

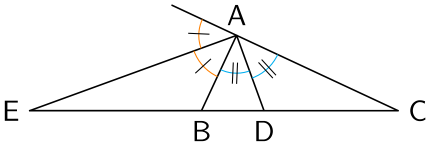
Prelim 2022 Q19 (Solution)



- $EB = 10,\ BD = 5$
- Find DC

1. Let $DC = x$
2. $\frac{EB}{EC} = \frac{AB}{BC} = \frac{BD}{DC}$

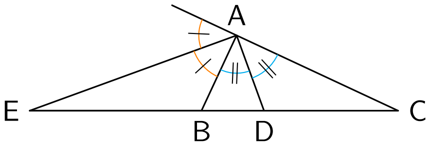
Prelim 2022 Q19 (Solution)



- $EB = 10,\ BD = 5$
- Find DC

1. Let $DC = x$
2. $\frac{EB}{EC} = \frac{AB}{BC} = \frac{BD}{DC}$
3. $\frac{10}{15+x} = \frac{5}{x}$

Prelim 2022 Q19 (Solution)



- $EB = 10,\ BD = 5$
- Find DC

1. Let $DC = x$
2. $\frac{EB}{EC} = \frac{AB}{BC} = \frac{BD}{DC}$
3. $\frac{10}{15+x} = \frac{5}{x}$
4. Solving yields $x = 15$.

Cosine's Law

Theorem (Cosine's Law)

In $\triangle ABC$, $c^2 = a^2 + b^2 - 2ab\cos C$. *Equivalently,* $\cos C = \frac{a^2+b^2-c^2}{2ab}$

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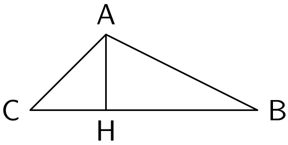
Proof:



Observe that

$c^2 = AH^2 + HB^2 = b\sin C^2 + (a - b\cos C)^2 = a^2 + b^2 - 2ab\cos C$

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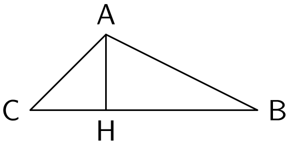

Observe that

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Exercise

Prove the Stewart's Theorem with the Cosine's Law

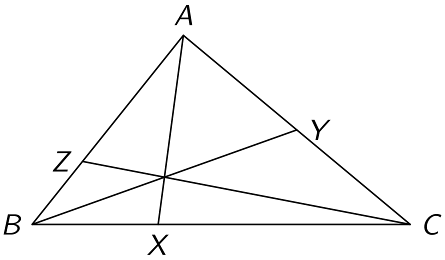
Ceva's Theorem

In a triangle, a *cevian* is a line joining a vertex of a triangle to a point on the interior of the opposite side. A natural question is when three cevians of a triangle concurs. This is answered by Ceva's theorem.

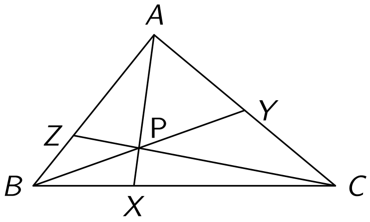
Theorem (Ceva's Theorem)

Let $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$ be cevians of a triangle ABC. They concur if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$$

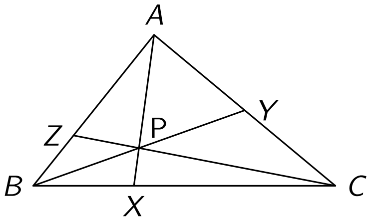


Ceva's Theorem (proof)



I will only prove the forward direction, i.e. if three cevians concur, then the identity $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$
Proof:

Ceva's Theorem (proof)



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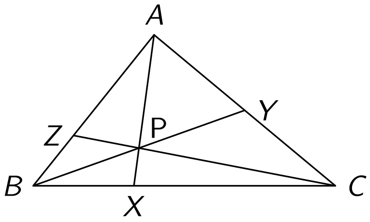
Proof:

1. $\frac{[ABX]}{[AXC]} = \frac{BX}{XC}$ and $\frac{[BPX]}{[CPX]} = \frac{BX}{XC}$.

Ceva's Theorem (proof)



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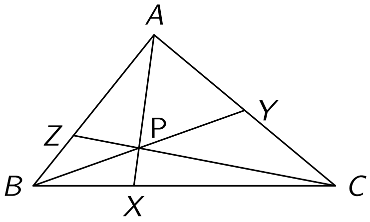
Proof:

1. $\frac{[ABX]}{[AXC]} = \frac{BX}{XC}$ and $\frac{[BPX]}{[CPX]} = \frac{BX}{XC}$.
2. Hence $\frac{[APB]}{[APC]} = \frac{BX}{XC}$

Ceva's Theorem (proof)



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2. Hence $\frac{[APB]}{[APC]} = \frac{BX}{XC}$
3. Similarly $\frac{[BPA]}{[BPC]} = \frac{AY}{YC}$ and $\frac{[CPB]}{[CPA]} = \frac{ZB}{ZA}$.

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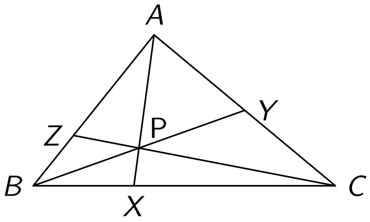

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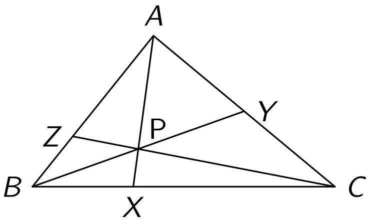
2. Hence $\frac{[APB]}{[APC]} = \frac{BX}{XC}$
3. Similarly $\frac{[BPA]}{[BPC]} = \frac{AY}{YC}$ and $\frac{[CPB]}{[CPA]} = \frac{ZB}{ZA}$.
4. Multiplying the above three equations gives $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$

Trigonometric Form of Ceva's Theorem

Let $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$ be cevians of a triangle ABC. They concur if and only if

$$\frac{\sin\angle BAX \sin\angle CBY \sin\angle ACZ}{\sin\angle XAC \sin\angle YBA \sin\angle ZCB} = 1$$



The proof is a direct application of the law of Sine and is left as an exercise.

Existence of orthocenter, incenter, centroid

Have you ever wondered why the three altitudes, angle bisectors, or medians must concur at the same point?

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For the orthocentre (of acute triangle), we check

$$\frac{\sin(90^o - B)\sin(90^o - C)\sin(90^o - A)}{\sin(90^o - C)\sin(90^o - A)\sin(90^o - B)} = 1$$

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For the incenter, we check

$$\frac{\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}{\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C} = 1$$

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For the centroid, we have

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We no longer have to take the existence of our centers for granted!

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$$\frac{\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}{\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C} = 1$$

For the centroid, we have

$$\frac{1}{1}\frac{1}{1}\frac{1}{1} = 1$$

We no longer have to take the existence of our centers for granted!

Where is our circumcentre?

Why don't we prove the circumcentre case as well?

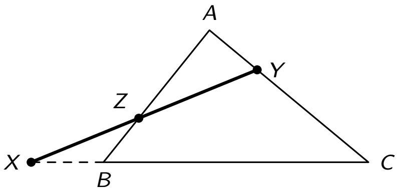
Menelaus's Theorem

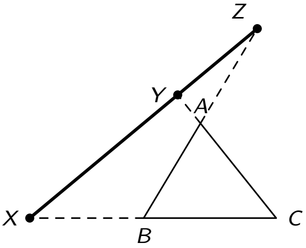
Theorem (Menelaus's Theorem)

Let X, Y, Z be points on lines BC, CA, AB in a triangle ABC, distinct from its vertices. Then X, Y, Z are collinear if and only if

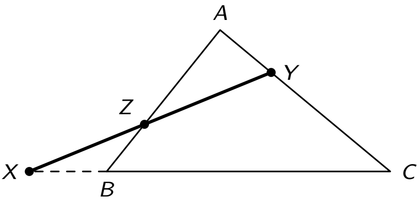
$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1$$

where lengths are directed.



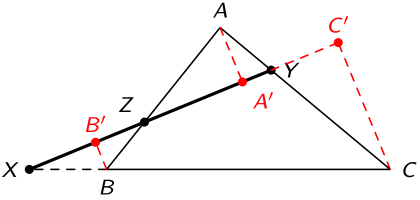


Proof of Menelaus's Theorem



Proof of the first case:

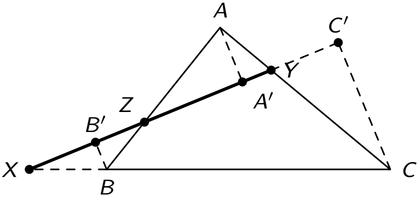
Proof of Menelaus's Theorem



Proof of the first case:

1. Drop a perpendicular line from A to A', B to B', C to C' on XY.

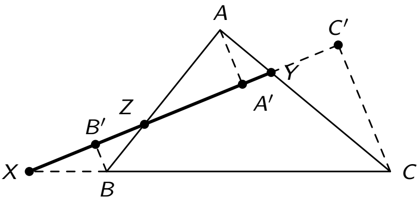
Proof of Menelaus's Theorem



Proof of the first case:

1. Drop a perpendicular line from A to A', B to B', C to C' on XY.
2. We have $\frac{CY}{YA} = \frac{CC'}{AA'}$, $\frac{AA'}{BB'} = \frac{AZ}{ZB}$, and $\frac{BX}{XC} = -\frac{BB'}{CC'}$

Proof of Menelaus's Theorem



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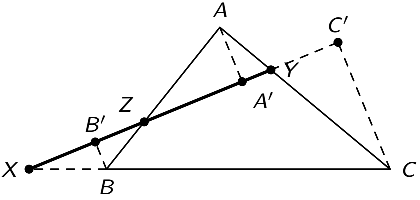
3. Multiplying all of them gives the Menelaus's Theorem.

Proof of Menelaus's Theorem



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3. Multiplying all of them gives the Menelaus's Theorem.

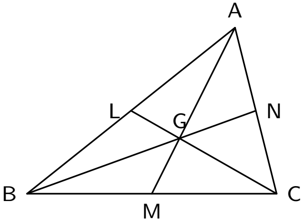
The proof of the second case is identical.

The Centroid Triangle

This slide serves to, yet again, stress the importance of the area ratios.

Theorem (The Centroid Division)

The medians divides the triangle into 6 equal parts.



Exercise

Prove the claim. Hence, show that $AG = 2GM$, $CG = 2LG$, $BG = 2GN$

Practice Problems

Question

Point P is on side AB of right angled $\triangle ABC$ with B as the right angled. Point Q is on AC such that PQ is perpendicular to AC. It is given that $BC = 3$ and $BP = PA = 2$. Find the length BQ.

Prelim 2020 Q16

$\triangle ABC$ is right-angled at B, with $AB = 1$ and $BC = 3$. E is the foot of perpendicular from B to AC. BA and BE are produced to D and F respectively such that D, F, C are collinear and $\angle DAF = \angle BAC$. Find the length of AD.

Prelim 2019 Q10

In $\triangle ABC$, $AB < AC$. Let H be the orthocentre of $\triangle ABC$, and D be the foot of the perpendicular from A to BC. If $AH = 4, HD = 3$ and $BC = 12$, find the length of BD.

The End

Thank You!