Analytic Techniques

T Yeung

THMSS

2024

Outline

1. The Cartesian Coordinates
   - The Shoelace Formula

Outline

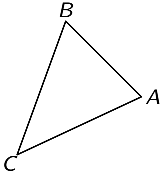
1. The Cartesian Coordinates
   - The Shoelace Formula
2. Mass Point Geometry

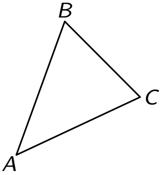
The Shoelace Formula

Consider three points $A = (x_1, y_1)$,$B = (x_2, y_2)$,and $C = (x_1, y_1)$. The **signed** area of ABC is given by the determinant.

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

The signed area is positive if ABC is in a anticlockwise order (left), and negative (right) otherwise.

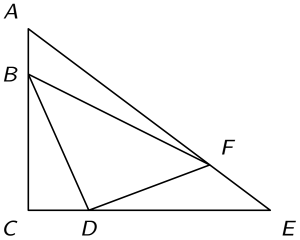




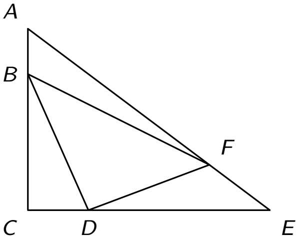
The shoelace formula gives an implicit way to test whether point A, B, and C are collinear. Do you know how?

2004 AMC 10B Q18

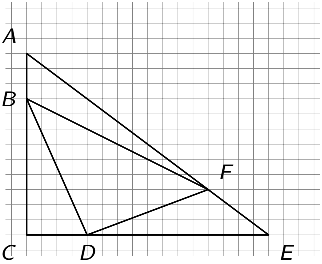
In a right triangle $\triangle ACE$, we have $AC = 12, CE = 16$, and $EA = 20$. Points B,D, and F are located on AC, CE, EA respectively, so that $AB = 3$, $CD = 4$, and $EF = 5$. What is the ratio of the area of $\triangle DBF$ to that of $\triangle ACE$?



2004 AMC 10B Q18 (Solution)

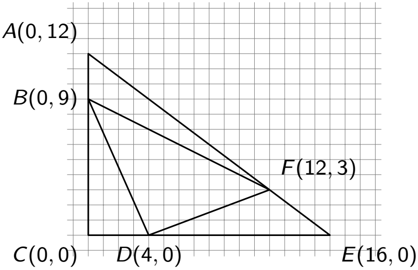


2004 AMC 10B Q18 (Solution)

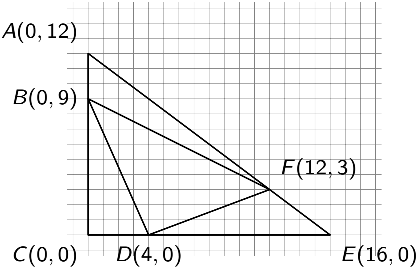


1. Apply grid

2004 AMC 10B Q18 (Solution)



1. Apply grid
2. Find coordinate

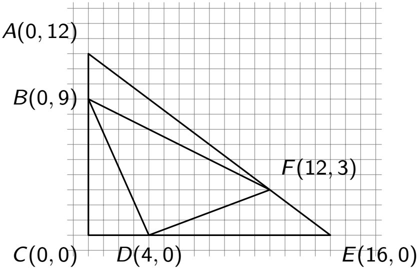


1. Apply grid
2. Find coordinate
3. Apply shoelace formula:

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 0 & 9 & 1 \\ 4 & 0 & 1 \\ 12 & 3 & 1 \end{vmatrix}$$

$$= 42$$

2004 AMC 10B Q18 (Solution)



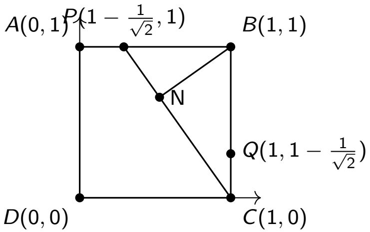
1. Apply grid
2. Find coordinate
3. Apply shoelace formula:

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 0 & 9 & 1 \\ 4 & 0 & 1 \\ 12 & 3 & 1 \end{vmatrix}$$
$$= 42$$

There is an easier way to find the area of this triangle, do you know how?

2022 Prelim Q9

$ABCD$ is a square with side length 1. P and Q are points on AB and BC respectively such that $BP = BQ = \frac{1}{\sqrt{2}}$. N is the foot of perpendicular from B to CP. Find NQ^2.

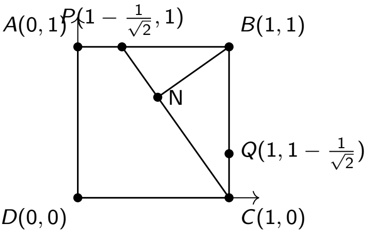


1. Set coordinates

Any smarter way?

While this should work out, there is a better way. Do you know how?

2022 Prelim Q9 (Solution)

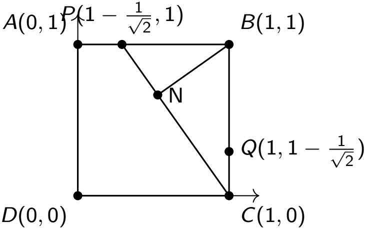


1. Set coordinates
2. Calculate the coordinate of N

Any smarter way?

While this should work out, there is a better way. Do you know how?

2022 Prelim Q9 (Solution)



1. Set coordinates
2. Calculate the coordinate of N
3. Calculate NQ

Any smarter way?

While this should work out, there is a better way. Do you know how?

Mass Point Geometry

Mass point geometry, colloquially known as mass points, is a problem-solving technique in geometry which applies the physical principle of the center of mass to geometry problems involving triangles and intersecting cevians.

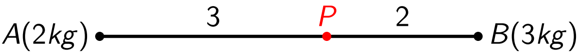
All problems that can be solved using mass point geometry can also be solved using either similar triangles, vectors, or area ratios, but many students prefer to use mass points.

Though modern mass point geometry was developed in the 1960s by New York high school students, the concept has been found to have been used as early as 1827 by August Ferdinand Möbius in his theory of homogeneous coordinates. (Wikipedia, 2024).

Some Physics Background

Given a line segment with masses put on two ends, we can compute the center of mass. For example:

We can find the center of mass P which divides AB in $3:2$.

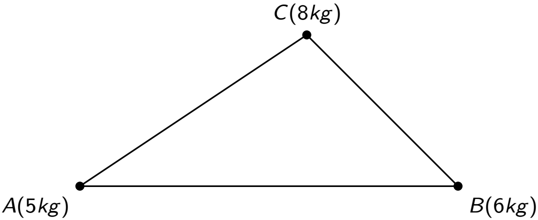


Intuition

The center of mass is the location where a pole can support the rod without the rod tilting.
Since B is heavier than A, the center of mass should be closer to B than A.

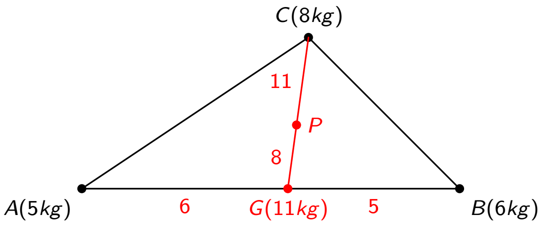
Center of Mass of Triangle

The **center of mass** of a triangle, also called the **centroid** (yes, the centroid you have learnt before) of the triangle, can be compute with a two step process. For example,



Center of Mass of Triangle

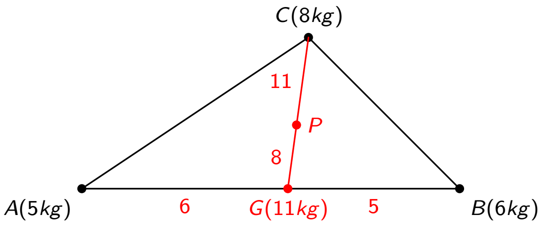
The **center of mass** of a triangle, also called the **centroid** (yes, the centroid you have learnt before) of the triangle, can be compute with a two step process. For example,



We can find the center of mass of A and B first at G. Then we find the center of mass of G and C at P. P is our desired center of mass.

Center of Mass of Triangle

The **center of mass** of a triangle, also called the **centroid** (yes, the centroid you have learnt before) of the triangle, can be compute with a two step process. For example,



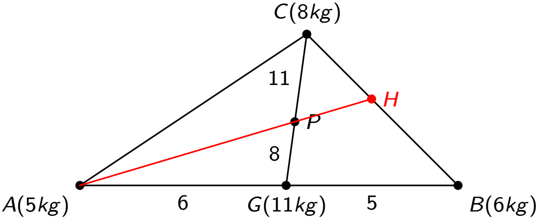
We can find the center of mass of A and B first at G. Then we find the center of mass of G and C at P. P is our desired center of mass.

Centroid

The centroid you learnt before assume equal mass distribution at three points.

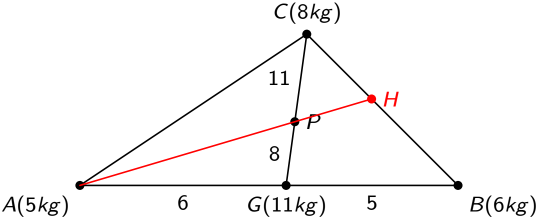
Mass Point Geometry 101

Continuing on our previous example, now we produce PA to meet at BC at H. What is $CH : HB$?



Mass Point Geometry 101

Continuing on our previous example, now we produce PA to meet at BC at H. What is $CH : HB$?



We can utilize the fact that center of mass of anything is unique and there are many ways to compute it. For example, an alternative way of computing the center of mass of this figure is to compute the center of mass of CB first, then then compute the center of mass of the result and A. Hence, H must be the center of mass of BC and $CH : BH = 3 : 4$.

Mass Point Geometry 102

In fact, much of the power of mass point geometry comes from the fact that **center of mass is unique** and **there are different ways of computing center of masses**.
Usually we assign the mass of a triangle based on the side lengths given in a triangle.

Example Problem 1 (From Wikipedia)

In triangle ABC, E is on AC so that $CE = 3AE$ and F is on AB so that $BF = 3AF$. If BE and CF intersect at O and line AO intersects BC at D, compute $\frac{OB}{OE}$ and $\frac{OD}{OA}$.

Example Problem 1 (Solution)

In triangle ABC, E is on AC so that $CE = 3AE$ and F is on AB so that $BF = 3AF$. If BE and CF intersect at O and line AO intersects BC at D, compute $\frac{OB}{OE}$ and $\frac{OD}{OA}$.

The End

Thank You!