MATH4.1 Trigonometry

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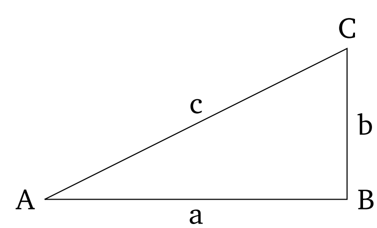
May 5, 2023

1 Motive

We learned about $\sin\theta$, $\cos\theta$ and $\tan\theta$ as well as some trigonometrics identities in form 2 & 3.



However, we have restricted ourselves to talk about them for $0 \leq \theta \leq 90^o$.

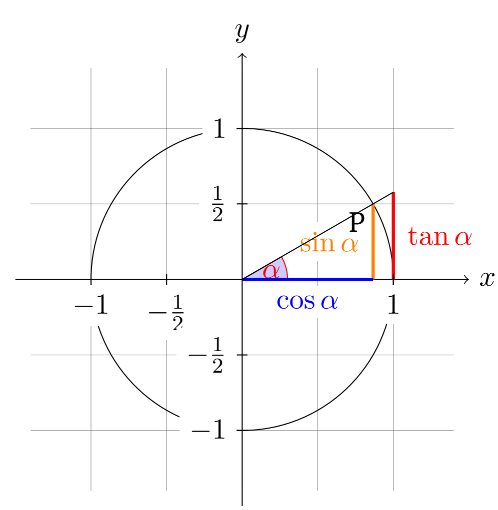
This restriction is due to the previous definition that you learnt about trigonometric identities that it's the ratio of two sides in a **triangle**, for example, $\sin\theta = \frac{\text{opposite side}}{\text{adjacent side}}$.

It turns out that this is an unnecessary restriction. We can extend our definition of $\sin\theta, \cos\theta$ and $\tan\theta$ to any real number θ (positive or negative).

2 Signs of trigonometric functions

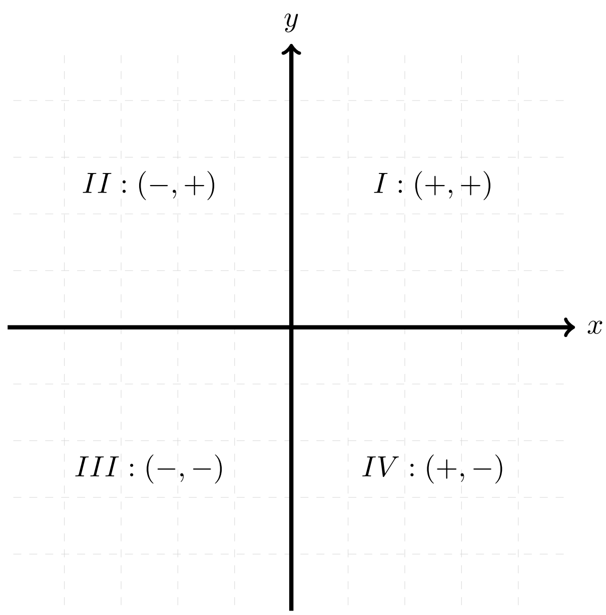
Consider the below circle with radius 1, and an angle α rotating anticlockwise from positive x-axis:

We define $\sin\alpha$ and $\cos\alpha$ to be the the y-coordinate and x-coordinate of P respectively (and ignore $\tan\alpha$ for now). We can observe serveral interesting properties of this definition regarding the sign of the trigonometric functions.



1. For $0 < \alpha < 90^o$, the values for $\sin\alpha, \cos\alpha$ are exactly the **same (including the sign)** as our previous definitions because the radius of the circle is 1.
2. For $90^o < \alpha < 180^o$, the values for $\cos\alpha$ becomes negative, while $\sin\alpha$ remains positive.

3. For $180^o < \alpha < 270^o$, both the values for $\sin\alpha$ and $\cos\alpha$ become negative.
4. For $270^o < \alpha < 360^o$, the value for $\sin\alpha$ is negative and the value for $\cos\alpha$ is positive.
5. For $\alpha > 360^o$, what will happen?
6. For $\alpha < 0^o$, what will happen?



Do we need to remember these facts? No, because it's just about the signs of x-coordinate and y-coordinate in different quadrants.

$\sin\alpha$ is only concerned with y-coordinate, so we can look at the sign of y-coordinate. In other words, it is positive in the I and II quadrant and negative in the III and IV quadrant. This means that $\sin\alpha$ is positive if $0^o < \alpha < 180^o$. and negative if $180^o < \alpha < 360^o$

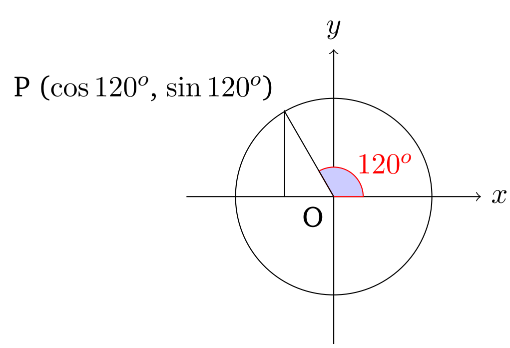
$\cos\alpha$ is only concerned with x-coordinate, so we can look at the sign of x-coordinate. In other words, it is positive in the I and IV quadrant and negative in the II and III quadrant. This means that $\cos\alpha$ is positive if $0^o < \alpha < 90^o$ or $270^o < \alpha < 360^o$ and negative if $90^o < 270^o$.

For $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$, as we already know the sign of $\sin\alpha$ and $\cos\alpha$ in all quadrants, we also know the sign of $\tan\alpha$. $\tan\alpha$ is positive in *I/II/III/IV (circle options that apply)* quadrant and negative in *I/II/III/IV (circle options that apply)* quadrant. This means that $\tan\alpha$ is positive when ______________________ and negative when ______________________

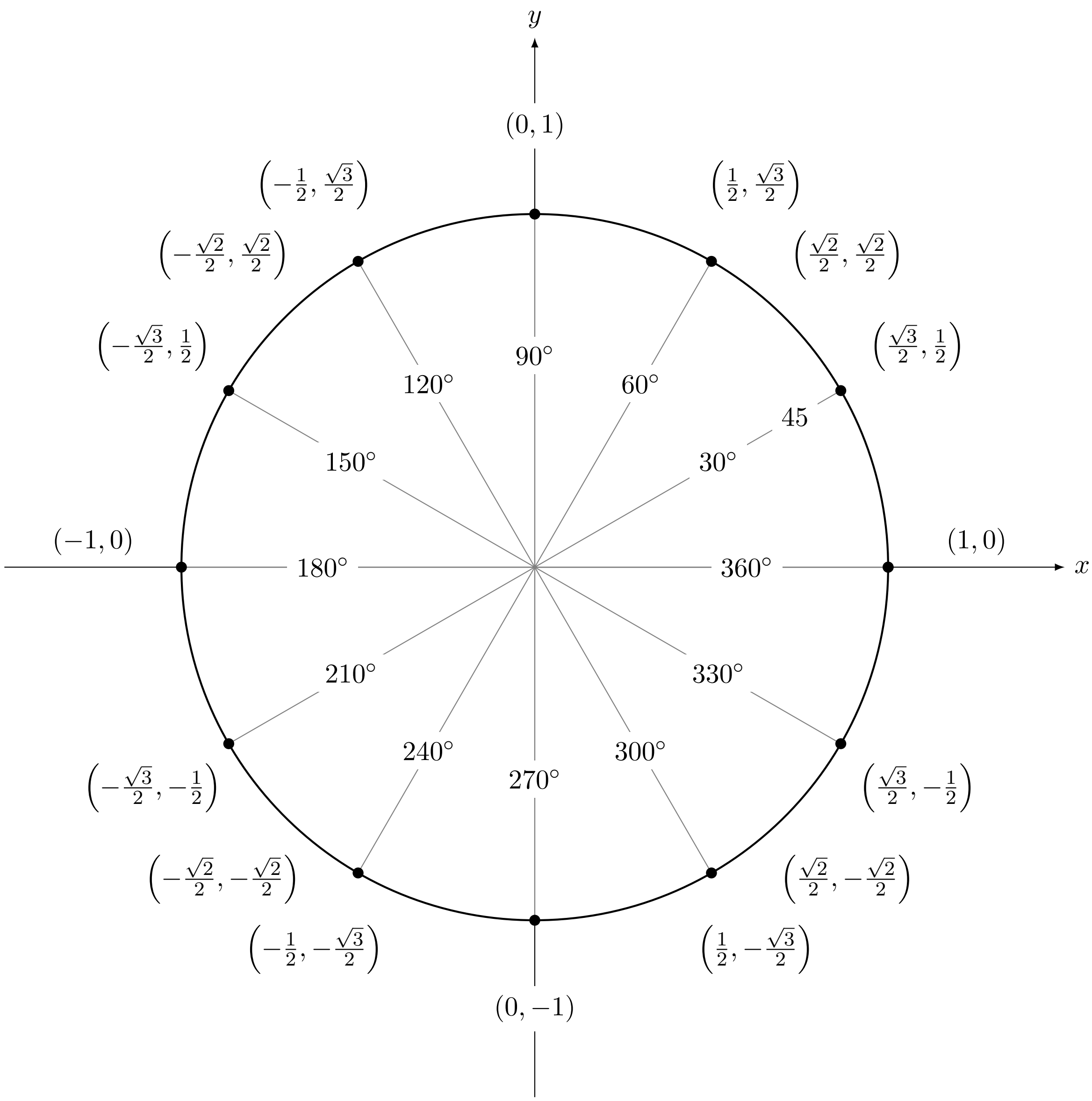
Now check your result against the line marked $\tan\alpha$. First, do you see why the signed length of the line denotes the value of $\tan\alpha$? Second, does the sign of it match what you write?

3 Values of trigonometric functions

Then we move on to find the value of $\sin\alpha$ where $\alpha = 120^o$ as an example. We draw a line with length 1 making an angle of 120^o with the positive x-axis like so:



The coordinate of P is $(\cos 120^o, \sin 120^o)$ by definition. We can find the value of it by considering the right angled triangle in the II quadrant. Since the radius of the circle is 1 and the angle that OP makes with the negative x-axis is 60^o. We find that $\cos 120^o = -\cos 60^o$ and $\sin 120^o = \sin 60^o$



Based on the given coordinates, let's try to find the following values.

Problem 1. Find the values of $\sin 60^o$, $\cos 60^o$ and $\tan 60^o$.

Problem 2. Find the values of $\sin 120^o$, $\cos 120^o$ and $\tan 120^o$.

Problem 3. Find the values of $\sin 240^o$, $\cos 240^o$ and $\tan 240^o$.

Problem 4. Find the values of $\sin 390^o$, $\cos 390^o$ and $\tan 390^o$.

Problem 5. Find the values of $\sin 600^o$, $\cos 600^o$ and $\tan 600^o$.

Problem 6. Find the values of $\sin -30^o$, $\cos -30^o$ and $\tan -30^o$.

Problem 7. Find the values of $\sin 180^o$, $\cos 180^o$ and $\tan 180^o$.

Problem 8. Find the values of $\sin 90^o$, $\cos 90^o$ and $\tan 90^o$.

We now consider more arbitary angles.

Problem 1. Given that $\sin 49^o = 0.754$ correct to 3 significant figures, find the values of $\sin 229^o$, $\cos 229^o$ and $\tan 229^o$.

Problem 2. Given that $\sin 29^o = 0.485$ correct to 3 significant figures, find the values of $\sin(-29^o)$, $\cos(-29^o)$ and $\tan(-29^o)$. Verify the two trigonometric identities that you learnt in the past holds for : $\tan(-29^o) = \frac{\sin(-29^o)}{\cos(-29^o)}$ and $\sin^2(-29^o) + \cos^2(-29^o) = 1$

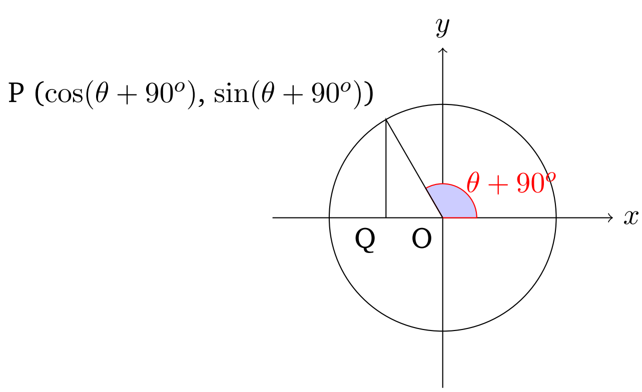
4 Problems of type $\sin(90^o/180^o/270^o \pm \theta)$

To kick off this chapter, let me present to you one identity that may seem shocking at first:

$$\sin(\theta + 90) = \cos(\theta)$$

To see why this is true, we only consider $0 < \theta < 90^o$ first. Then $\theta + 90^o$ lies in the II quadrant.

By doing some angle chasing, we find that $\angle OPQ = \theta$ and hence the $OQ = \sin\theta$, which means the y-coordinate of $P = \sin(\theta + 90^o)$ is actually just $\cos\theta$ in disguise. Note that we have discussed in previous sections that $\sin(\theta + 90^o)$ will be positive since $\theta + 90^o$ is in the II quadrant.



Although this derivation only proves that the identity is true for $0^o < \theta < 90^o$, **the identity is actually true for all** θ. This is because of the symmetry that lies in trigonometric functions. Hence, when we treat expression of the form $\sin(90^o/180^o/270^o \pm \theta)$ and we want to simplify it, we can always **assume** θ **is in the I quadrant** and derive a simplier expression from the original one.

If we consider the x-coordinate of P instead, you will see that since OQ $= \sin\theta$ and and $\cos(\theta + 90^o)$ lies in the negative x-axis, we get another identity $\cos(\theta + 90^o) = -\sin\theta$.

Problem 1. Simplify $\sin(\theta + 180^o)$, $\cos(\theta + 180^o)$ and $\tan(\theta + 180^o)$.

Problem 2. Simplify $\sin(\theta + 270^o)$, $\cos(\theta + 270^o)$ and $\tan(\theta + 270^o)$.

Problem 3. Simplify $\sin(\theta - 90^o)$, $\cos(\theta - 90^o)$ and $\tan(\theta - 90^o)$.

Problem 4. Simplify $\sin(\theta - 90^o)$, $\cos(\theta - 90^o)$ and $\tan(\theta - 90^o)$.

Problem 5. Simplify $\frac{\sin(270^o - \theta)}{\cos(180^o - \theta)}$

Problem 6. Simplify $\tan(270^o + \theta)\cos(180^o + \theta)$

Problem 7. Simplify $\frac{2 - 2\cos^2(270^o - \theta)}{\sin(180^o + \theta)\tan(270^o - \theta)}$

Problem 8. Find the value of $\cos 1^o + \cos 2^o + ... + \cos 179^o$

Problem 9. Find the value of $\tan 91^o \cdot \tan 179^o + \tan 92^o \cdot 178^o + \tan 93^o \cdot 177^o + ... + \tan 179^o \cdot \tan 91^o$

Problem 10. Given that $\cos(270^o + \theta) = -0.6$ and $\cos\theta > 0$, find the values of $\sin\theta, \cos\theta$ and $\tan\theta$.

Problem 11. Given that $\tan(180^o - \theta) = 3$, find the values of $\sin\theta, \cos\theta$ and $\tan\theta$

Problem 12. Given that $\cos(360^o - \theta) = \frac{2}{5}$, find the value of $\frac{4\sin 210^o \cdot \cos(-\theta)}{1 + \tan^2(180^o + \theta)}$

Problem 13. α, β and θ are the interior angles of a triangle with $\alpha + \beta = 90^o$. Prove the following identities:

(a) $\sin\alpha - \cos\beta = \cos\theta$

(b) $\tan\alpha\tan\beta - \tan(\theta)2 = 0$

(c) $\sin\alpha = \sin\beta\tan(\theta - \beta)$

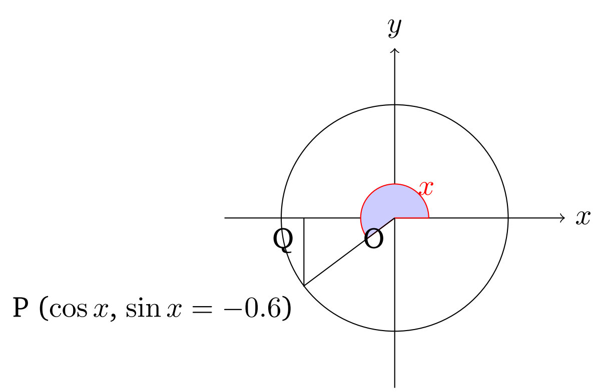
(d) $\cos^2\alpha + \cos^2\beta + \cos 2\theta = 0$

5 Solving trigonometric equations

Example: If $\sin x = -0.6$, find x where $0 \leq x \leq 360^o$.

Solution: To approach this type of problem, we first need to find out which quadrant does x lie in. In this case, only quadrant III and IV are possible since $\sin x$ is negative. Then we can draw the circle that we've already been very familiar with to find the value of x.

We first deal with the angle in quadrant III, we consider the right-angled triangle OQP first (and ignore the signs first), we find that $\angle QOP = \sin^{-1}(0.6)$ and hence $x = 180^o + \angle QOP = 216.869^o$ Then we consider the angle in quadrant IV (not drawn in the figure) , by angle chasing you will be able to find that $\theta = 360^o - \sin^{-1}(0.6) = 323.1^o$



Many students may be tempted to try $\sin^{-1}(-0.6)$. While the calculator can give you a correct x. This approach can only find 1 solution for x when there may exist many (depending on the range of x allowed). You will have to manually find the other solutions anyway.

Problem 1. If $\cos x = 0.2$, find x, where $0^o \leq x \leq 360^o$.

Problem 2. Solve $4 \cos x = 1$, where $0^o \leq x \leq 360^o$.

Problem 3. Solve $5 \tan x = -3$, where $0^o \leq x \leq 360^o$.

Problem 4. Solve $\sqrt{3} \cos x = -1$, where $0^o < x < 180^o$.

Problem 5. Solve $-3 \tan x = 7$, where $0^o \leq x < 360^o$.

Problem 6. Without using a calculator, find θ for each of the following, where $0^o < \theta < 360^o$

(a) $\tan \theta = \tan 42^o$

(b) $\cos \theta = -\cos 87^o$

(c) $\sin \theta = \cos 1^o$

(d) $\cos \theta = -\sin 63^o$