

# Introduction to Euclidean Geometry

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THMSS

2024

# Outline

## 1 Motivation

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## 2 Circle

- Inscribed Angle Theorem
- The Extended Law of Sine
- Relationship between Circumradius and Area
- Relationship between Circumradius and Side Lengths

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- Properties for Cyclic Quadrilateral
- Test for Cyclic Quadrilateral

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## 6 Example Problems

- Prelim 2020 Q6
- Prelim 2022 Q16

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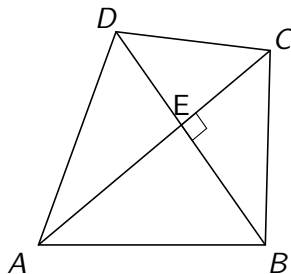
## 6 Example Problems

- Prelim 2020 Q6
- Prelim 2022 Q16

## 7 Practice Problems



# A brain teaser

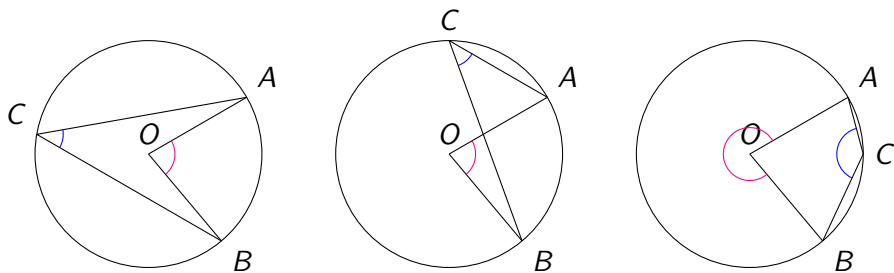


Given:

- $\angle DAC = 30^\circ$
- $\angle CDB = 40^\circ$
- $\angle ABD = 50^\circ$
- $DB \perp AC$

What angles can you compute?

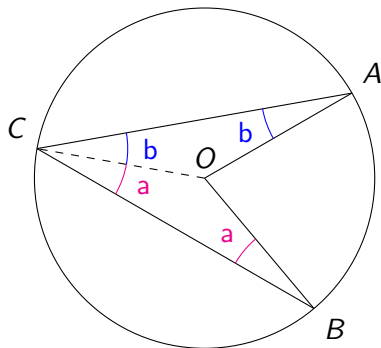
# Inscribed angle theorem



## Theorem (Inscribed Angle Theorem)

Let  $O$  denotes the center of circle,  $A$  and  $B$  be any two points on the circle, then  $\angle AOB = 2\angle ACB$ .

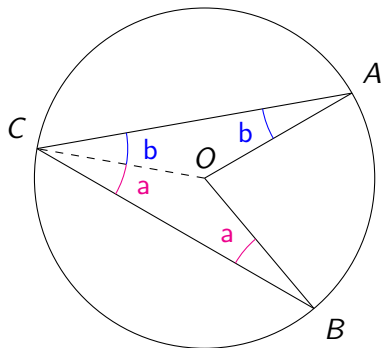
# Inscribed Angle Theorem (Proof for case I)



Proof:

①  $OA = OC = OB$

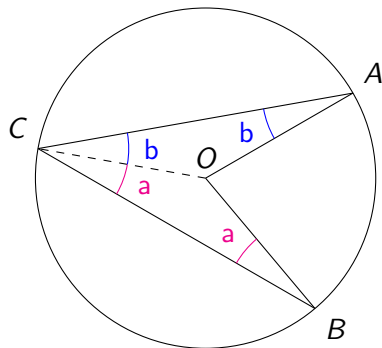
# Inscribed Angle Theorem (Proof for case I)



Proof:

- ①  $OA = OC = OB$
- ②  $\angle OAC = \angle ACO = a$
- ③  $\angle OCB = \angle OBC = b$

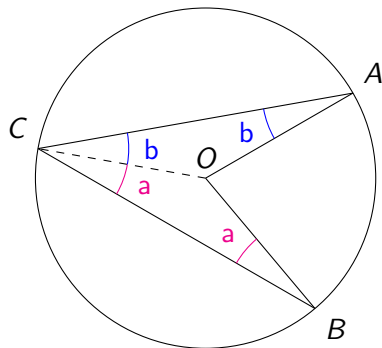
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- ①  $OA = OC = OB$
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- ④  $\angle ACB = a + b$

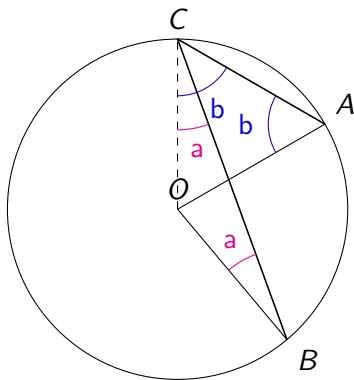
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- ⑤  $\angle AOB = 2a + 2b$

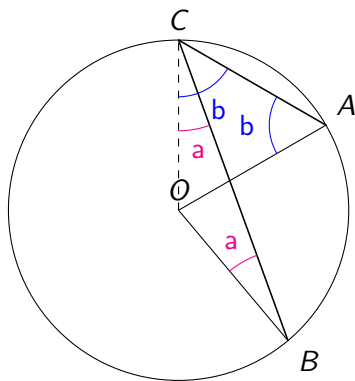
# Inscribed Angle Theorem (Proof for case II)



Proof:

①  $OA = OC = OB$

# Inscribed Angle Theorem (Proof for case II)

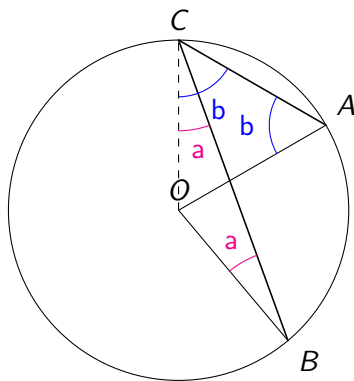


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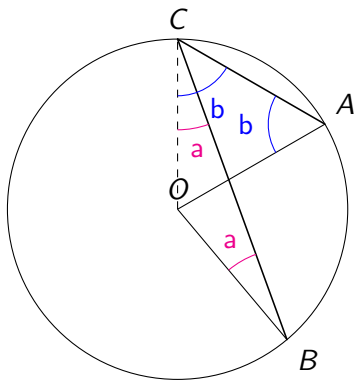
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- ①  $OA = OC = OB$
- ②  $\angle OAC = \angle ACO = a$
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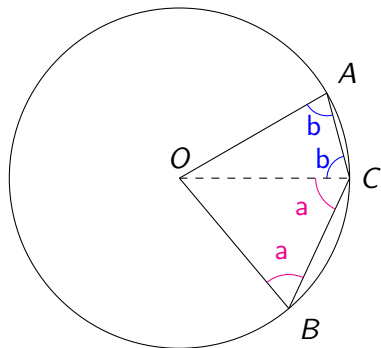
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- ①  $OA = OC = OB$
- ②  $\angle OAC = \angle ACO = a$
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- ④  $\angle ACB = a - b$
- ⑤  $\begin{aligned}\angle AOB &= \angle COB - \angle COA \\ &= (180^\circ - 2b) \\ &\quad - (180^\circ - 2a) \\ &= 2a - 2b\end{aligned}$

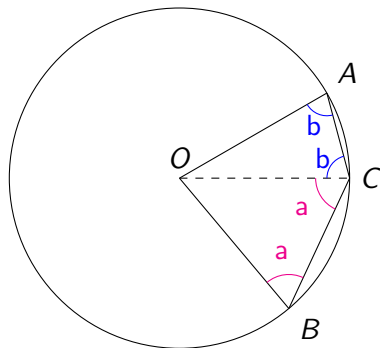
# Inscribed Angle Theorem (Proof for case III)



Proof:

①  $OA = OC = OB$

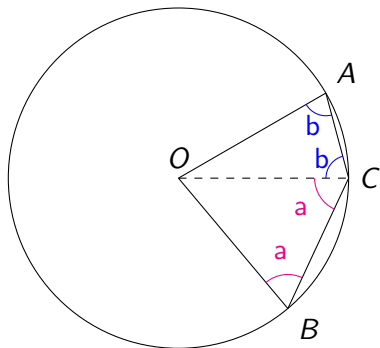
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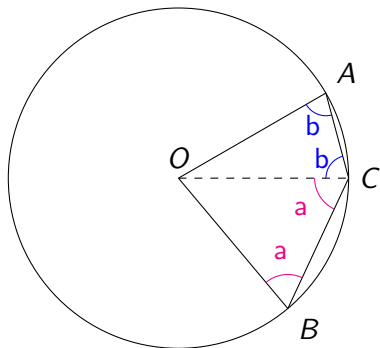
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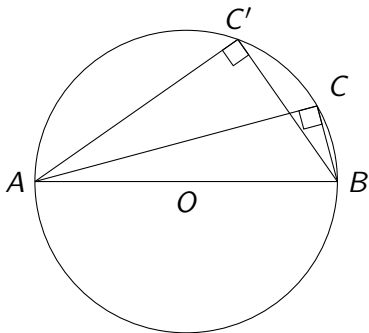
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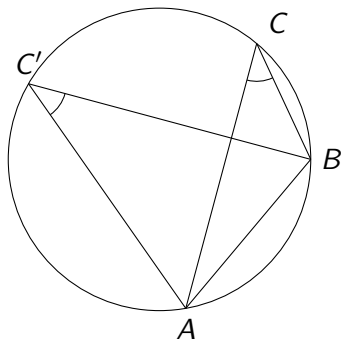
# Inscribed Angle Theorem (Corollary I)



## Corollary ( $\angle$ in semi circle)

*Let  $AB$  be a diameter of circle,  $C$  be any point on a circle. Then  $\angle ACB = 90^\circ$  (because the angle at centre is  $180^\circ$ ).*

# Inscribed Angle Theorem (Corollary II)

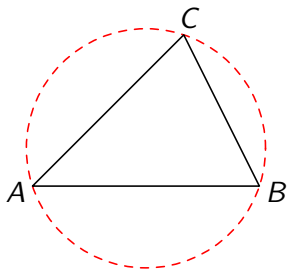


Corollary (angle at circumference  $\propto$  arc length)

*The arc is proportional to the angle at circumference (center).*



# The Extended Law of Sine



## Naming Convention

By convention, in  $\triangle ABC$ , the opposite side to angle  $A$  is named  $a$  (similarly for  $B$  and  $C$ ),  $\mathcal{R}$  denotes the circumradius of  $\triangle ABC$ , and  $r$  denotes the inradius of  $\triangle ABC$ .

## Theorem (The Extended Law of Sine)

Given a triangle  $ABC$ , we have

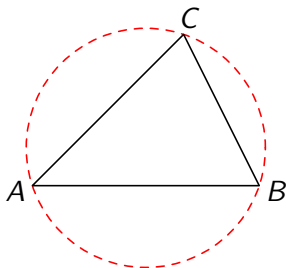
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2\mathcal{R}$$

# The Extended Law of Sine (Proof)

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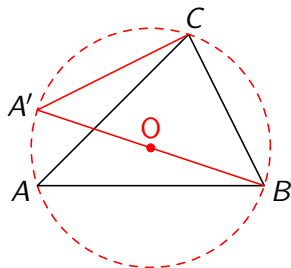
- 1 Without loss of generality, we only prove  $\frac{a}{\sin A} = 2\mathcal{R}$

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Proof:

- 1 Without loss of generality, we only prove  $\frac{a}{\sin A} = 2\mathcal{R}$
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# The Extended Law of Sine (Proof)

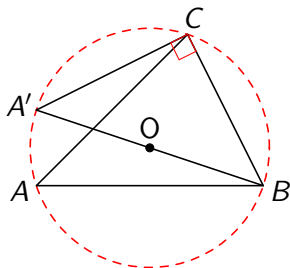
## Theorem (The Extended Law of Sine)

Given a triangle  $ABC$ , we have

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- 2 Move  $A$  to  $A'$  such that  $A'B$  is the diameter of the circle.
- 3 Note that  $\triangle ACB$  is a right-angled triangle and  $\angle BA'C = \angle BAC$ .

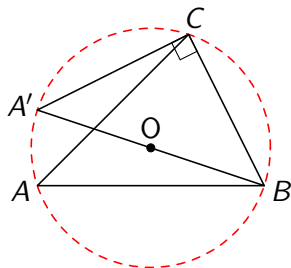


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- 2 Move  $A$  to  $A'$  such that  $A'B$  is the diameter of the circle.
- 3 Note that  $\triangle ACB$  is a right-angled triangle and  $\angle BA'C = \angle BAC$ .
- 4 We have

$$\frac{a}{\sin BAC} = \frac{CB}{\angle BA'C} = A'B = 2\mathcal{R}$$

# Relationship between Circumradius and Area

## Naming Convention

$[ABC]$  denotes the area of  $ABC$ .

## Theorem (Area of a triangle)

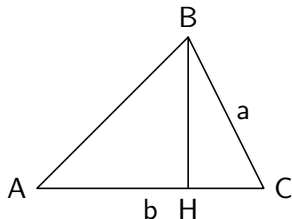
$$[ABC] = \frac{1}{2}ab \sin C.$$

Proof:

- ①  $BH = a \sin C$ ,  $AC = b$
- ②  $[ABC] = \frac{1}{2}BH \cdot AC = \frac{1}{2}ab \sin C$ .

## Theorem (Circumradius and Area)

*By the extended law of sine, we also have  $\sin C = \frac{c}{2R}$ , hence,  $[ABC] = \frac{abc}{4R}$*



# Relationship between Circumradius and Side Lengths

## Naming Convention

$s$  denotes the semi-perimeter of  $\triangle ABC$ , i.e.  $\frac{a+b+c}{2}$ .

We state the Heron's formula without proof:

## Theorem (Heron's formula)

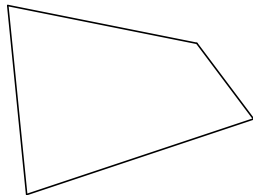
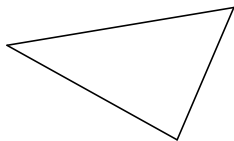
In  $\triangle ABC$ , we have  $[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$ .

Together with the result in the previous slide, we can find the circumradius of a triangle if we know all 3 side lengths:

## Theorem (Circumradius and Side Lengths)

$$[ABC] = \frac{abc}{4\mathcal{R}}$$
$$\mathcal{R} = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

# Cyclic Quadrilateral



## Question 1

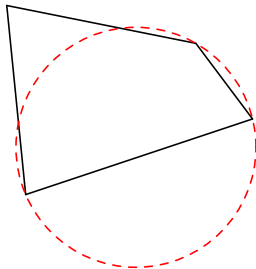
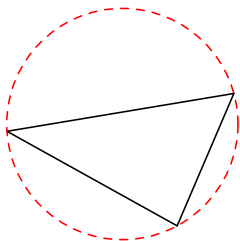
Is it always possible to find a circle passing through a triangle?

## Question 2

Is it always possible to find a circle passing through a quadrilateral?



# Cyclic Quadrilateral



## Question 1

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## Question 2

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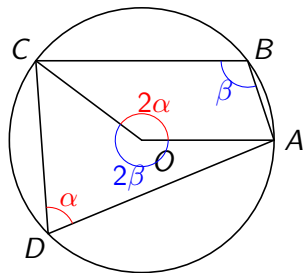
## Answers

Yes, a circumcentre always exists for a triangle;

No, not possible if a point does not lie on the circumcentre formed by the other three points.

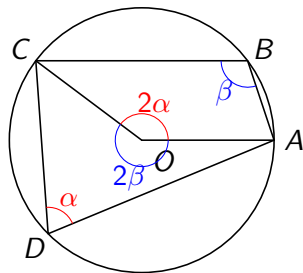
# Properties of Cyclic Quadrilateral

- 1 Let  $\angle COA = 2\alpha$ ,  
Reflex  $\angle COA = 2\beta$



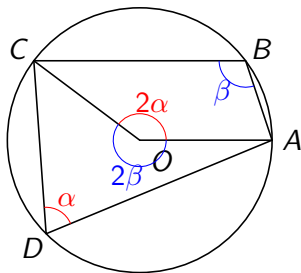
# Properties of Cyclic Quadrilateral

- 1 Let  $\angle COA = 2\alpha$ ,  
Reflex  $\angle COA = 2\beta$
- 2  $2\alpha + 2\beta = 360^\circ \implies \alpha + \beta = 180^\circ$



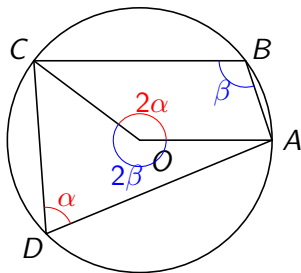
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- 3  $\angle CDA + \angle CBA = \alpha + \beta = 180^\circ$



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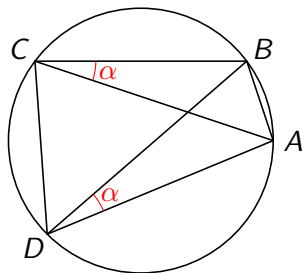
**Theorem (Supplementary opposite angles)**

*Opposite angles inside a cyclic quadrilateral adds up to  $180^\circ$ .*

**Corollary**

*Exterior angle equals to the opposite interior angle inside a cyclic quadrilateral.*

# Properties for Cyclic Quadrilateral



Theorem (Angles subtended by the same arc)

*Angles subtended by the same arc are equal.*

# Test for Cyclic Quadrilateral

## Theorem (Test for Cyclic Quadrilateral)

*It turns out that the mentioned 3 properties are also tests for cyclic quadrilateral.*

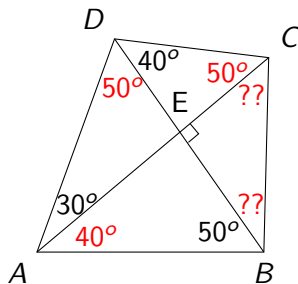
- *Opposite angles adds up to  $180^\circ$ .*
- *Exterior angle equals the opposite interior angle.*
- *Angles subtended by the same **side** are equal.*

*This means that if **any** of the above 3 statement is true, then the quadrilateral is a cyclic quadrilateral.*

The proof is omitted here. Now you should have enough to solve the original problem. :)

# Rerouting to our original problem...

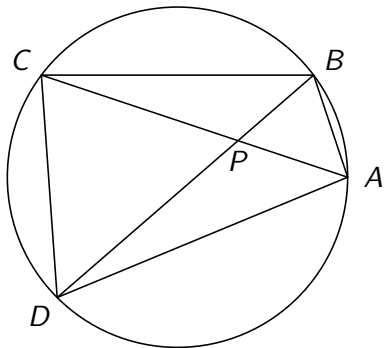
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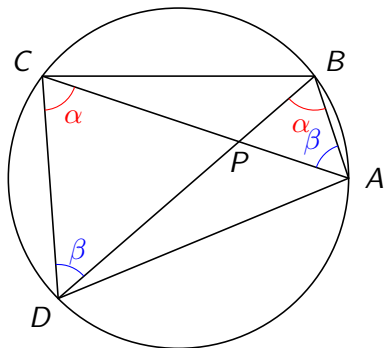
Find the remaining angles!



# Power Chord Theorem (I)



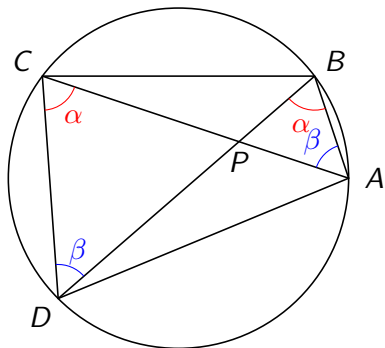
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①  $\angle DCA = \angle DBA$

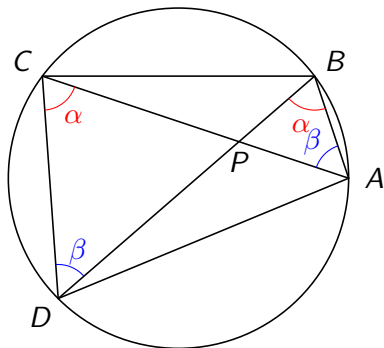
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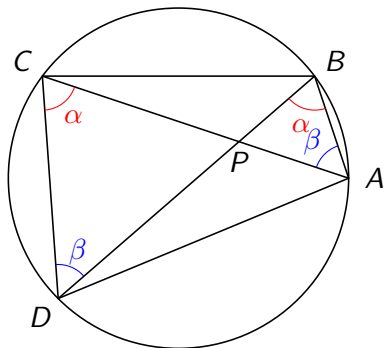
- ①  $\angle DCA = \angle DBA$
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- ③  $\triangle PCD \sim \triangle PBA$
- ④  $\frac{PC}{PB} = \frac{PD}{PA}$

# Power Chord Theorem (I)



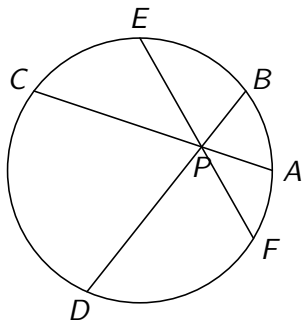
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- ④  $\frac{PC}{PB} = \frac{PD}{PA}$
- ⑤  $PC \cdot PA = PB \cdot PD$

# Power Chord Theorem (I)

In fact, if we have any chord  $XY$  passing through  $P$ ,  $PX \cdot PY$  is always fixed.

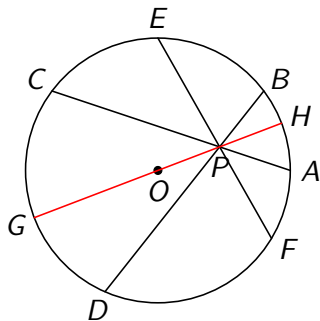
## Power of a Point **inside** the circle

$\text{Pow}_\omega(P)$  with respect to the circle  $\omega$  is defined to be  $|PX \cdot PY|$  for any chord  $XY$  passing through  $P$ .



$$PC \cdot PA = PE \cdot PF = PB \cdot PD$$

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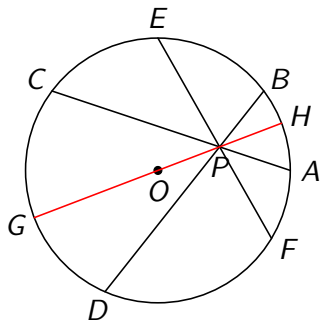
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$$\text{Power} = \mathcal{R}^2 - OP^2 > 0$$

- 1  $\mathcal{R}$  is the radius of circumcircle.

# Power Chord Theorem (I)



$$PC \cdot PA = PE \cdot PF = PB \cdot PD$$

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## Power of a Point **inside** the circle

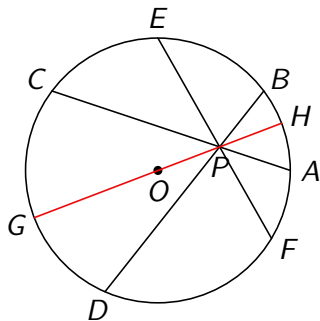
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$$\text{Power} = \mathcal{R}^2 - OP^2 > 0$$

- 1  $\mathcal{R}$  is the radius of circumcircle.
- 2 Draw a diameter  $GH$  through  $P$



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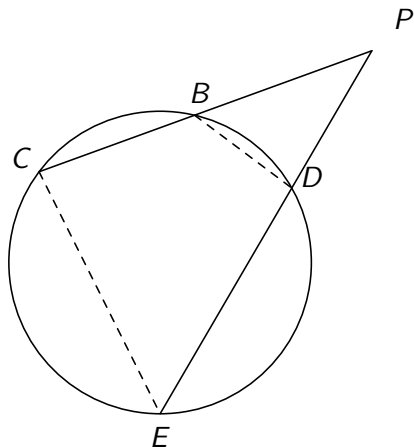
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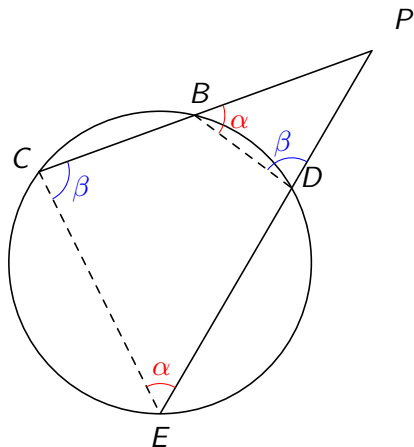
- ①  $\mathcal{R}$  is the radius of circumcircle.
- ② Draw a diameter  $GH$  through  $P$
- ③  $\text{Pow}_\omega(P) = PG \cdot PH$   
 $= (\mathcal{R} + OP)$   
 $\cdot (\mathcal{R} - OP) = \mathcal{R}^2 - OP^2$

# Power Chord Theorem (II)



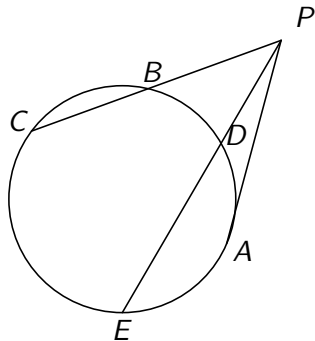
- ①  $\angle PBD = \angle PEC$
- ②  $\angle PDB = \angle PCE$
- ③  $\triangle PBD \sim \triangle PEC$
- ④  $\frac{PB}{PE} = \frac{PD}{PC}$
- ⑤  $PB \cdot PC = PD \cdot PE$

# Power Chord Theorem (II)



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## Power Chord Theorem (II)



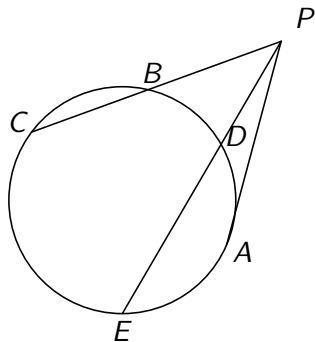
$$PB \cdot PC = PD \cdot PE = PA^2$$

Again, if we have any chord  $XY$  passing through  $P$ ,  $PX \cdot PY$  is always fixed.

### Power of a Point **outside** the circle

$\text{Pow}_\omega(P)$  with respect to the circle  $\omega$  is defined to be  $-|PX \cdot PY|$  for any chord  $XY$  passing through  $P$ .

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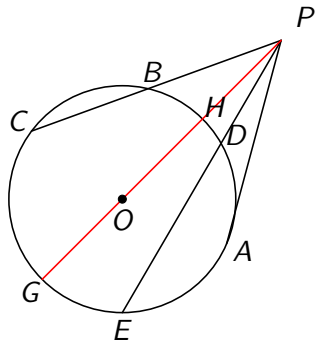
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- 1 Let  $R$  be the radius of circumcircle.

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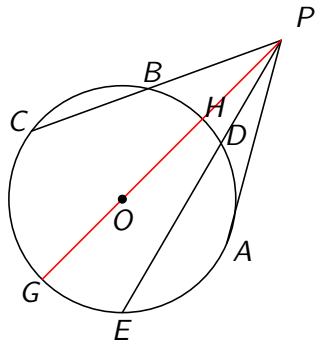
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# Power Chord Theorem (II)



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Again, if we have any chord  $XY$  passing through  $P$ ,  $PX \cdot PY$  is always fixed.

## Power of a Point **outside** the circle

$\text{Pow}_\omega(P)$  with respect to the circle  $\omega$  is defined to be  $-|PX \cdot PY|$  for any chord  $XY$  passing through  $P$ .

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 $= (\mathcal{R} + OP)$   
 $\cdot (\mathcal{R} - OP) = \mathcal{R}^2 - OP^2$

# Power Chord Theorem (III)

Power of a Point **on** the circle

$\text{Pow}_\omega(P)$  with respect to the circle  $\omega$  is equal to 0. (Why does this makes sense?)

$$\text{Power} = \mathcal{R}^2 - OP^2 = 0$$

As expected.



# Converse of Power Chord Theorem

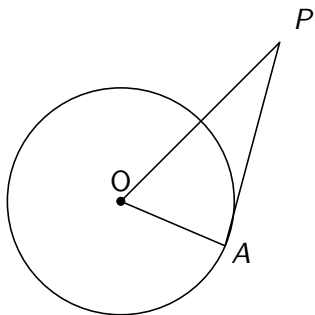
In fact, the converse of power chord theorem is also true.

## Theorem (Converse of the Power Chord Theorem)

*Let  $A, B, X, Y$  be four distinct points in the plane and let lines  $AB$  and  $XY$  intersect at  $P$ . Suppose that either  $P$  lies in both of the segments  $\overline{AB}$  and  $\overline{XY}$ , or in neither segment. If  $PA \cdot PB = PX \cdot PY$ , then  $A, B, X, Y$  are concyclic.*

This serves as another test for cyclic quadrilateral. The proof is omitted here.

# Another proof of the Pythagoras Theorem



## Tangent $\perp$ Radius

Tangent of a circle at  $A$  is perpendicular to  $OA$ . (Why?)

## Proof of Pythagoras Theorem

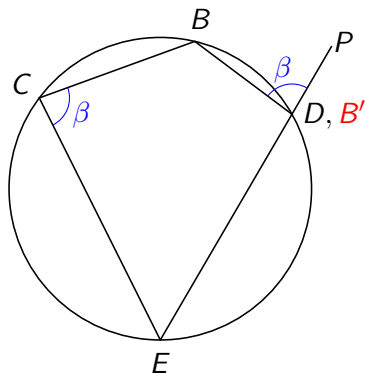
Rearranging

$\text{Pow}_\omega(P) = PA^2 = OP^2 - OA^2$   
gives

$$PA^2 + OA^2 = OP^2$$

# A little digression: Angle in Alternate Segment

Recall the proof to Power chord theorem (II)

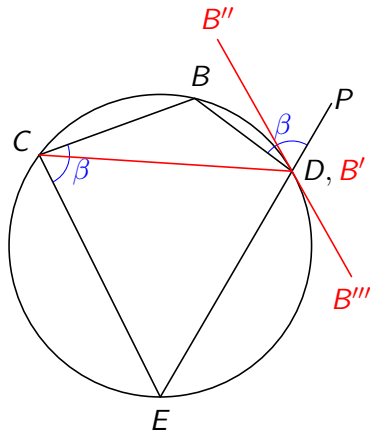


## Angle in alternate segment

- 1 Imagine if  $B$  gets increasingly close to  $D$  as  $B'$ .

# A little digression: Angle in Alternate Segment

Recall the proof to Power chord theorem (II)

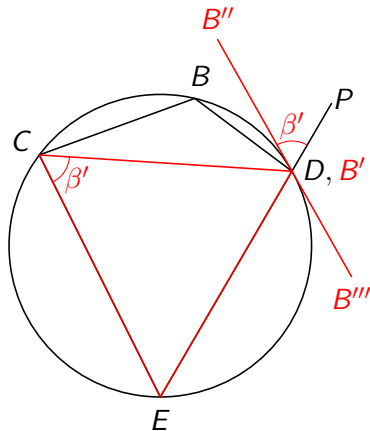


## Angle in alternate segment

- 1 Imagine if  $B$  gets increasingly close to  $D$  as  $B'$ .
- 2  $DB'$  is arbitrarily close to the tangent to the circle at  $D$ .
- 3 Let  $B''B'''$  denotes the tangent.

# A little digression: Angle in Alternate Segment

Recall the proof to Power chord theorem (II)

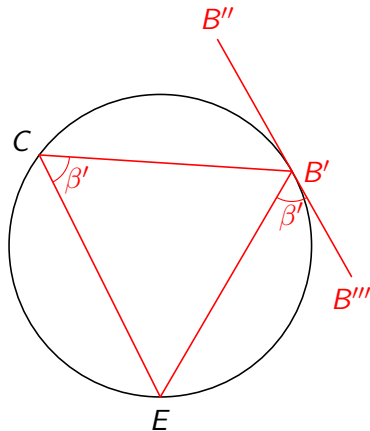


## Angle in alternate segment

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# A little digression: Angle in Alternate Segment

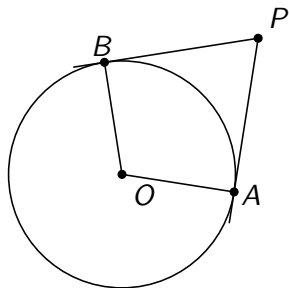
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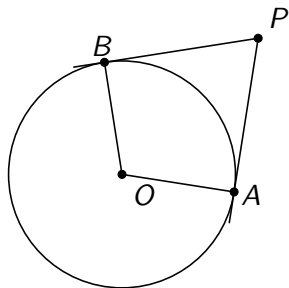
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- 5  $\angle B'''DE = \angle DCE$

# Other Properties of Tangents



# Other Properties of Tangents



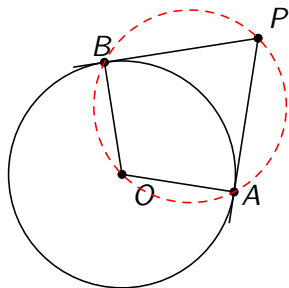
$$\begin{aligned} PA = PB &= OP^2 - \mathcal{R}^2 \\ &= \text{Pow}_\omega(P) \end{aligned}$$

## Equal Tangents

Given a point  $P$  outside  $\omega$  circle, there are two tangents of equal length from  $P$  to  $\omega$ .



# Other Properties of Tangents



$$\begin{aligned} PA &= PB = \sqrt{OP^2 - R^2} \\ &= \text{Pow}_\omega(P) \end{aligned}$$

## Equal Tangents

Given a point  $P$  outside  $\omega$  circle, there are two tangents of equal length from  $P$  to  $\omega$ .

## $OABP$ forms a cyclic quadrilateral

$OABP$  is a cyclic quadrilateral since  $\angle OBP + \angle OAP = 180^\circ$ . Hence we have  $\angle BOA + \angle BPA = 180^\circ$ .

# Example Problem 1

## Question 1

(Prelim 2020 Q6) In  $\triangle ABC$ ,  $AB = 6$ ,  $BC = 7$ ,  $CA = 8$ . Let  $D$  be the mid-point of minor arc  $AB$  on the circumcircle of  $\triangle ABC$ . Find  $AD^2$ .

# Example Problem 1

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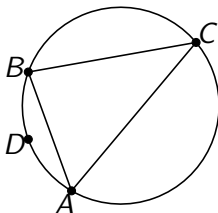
Sad news: in math contest, the geometric diagram is usually not provided since ~~problem-setters are lazy~~ the construction of the diagram is a part of the problem.

# Example Problem 1

## Question 1

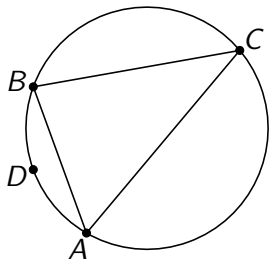
(Prelim 2020 Q6) In  $\triangle ABC$ ,  $AB = 6$ ,  $BC = 7$ ,  $CA = 8$ . Let  $D$  be the mid-point of minor arc  $AB$  on the circumcircle of  $\triangle ABC$ . Find  $AD^2$ .

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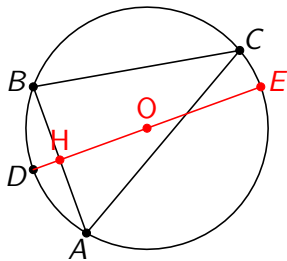
# Example Problem 1 (Solution)

- $AB = 6$ ,  $BC = 7$ ,  $CA = 8$
- $D$  is the mid-point of  $\widehat{AB}$
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# Example Problem 1 (Solution)

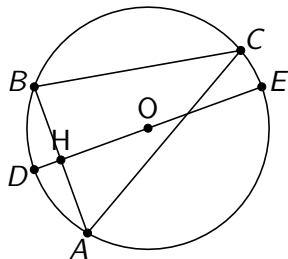
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- $D$  is the mid-point of  $\widehat{AB}$
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## Solution

- 1 Drop a perpendicular line from  $D$  to  $BA$ , that the line passes through  $O$ .

# Example Problem 1 (Solution)

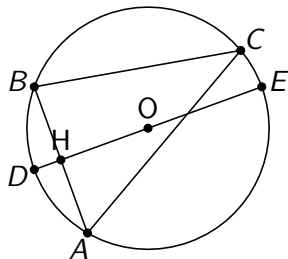


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- 2  $\text{Pow}_\omega(H) = BH \cdot HA = 9$

# Example Problem 1 (Solution)



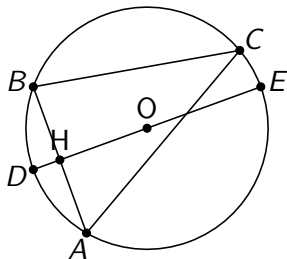
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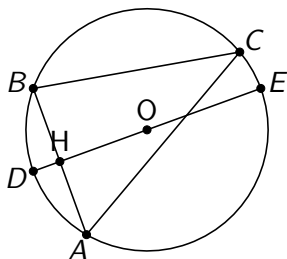


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- 3 Another way to compute  $\text{Pow}_\omega$  is  $DH \cdot HE$ .
- 4  $\mathcal{R} = OD = OE$  is the circumradius of the circle and its given by 
$$\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}},$$
 where  $s$  is the semi perimeter.

## Example Problem 1 (Solution Cont.)



### Solution

$$\textcircled{5} \quad \mathcal{R} = \frac{6 \cdot 7 \cdot 8}{4 \sqrt{\frac{21}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2}}} = \frac{16}{\sqrt{15}}$$

$$\begin{aligned} \textcircled{6} \quad \text{Pow}_\omega(H) &= (OE + OH) \cdot (OD - OH) \\ &= \left(\frac{16}{\sqrt{15}}\right)^2 - OH^2 \\ &= 9 \end{aligned}$$

$$OH = \frac{11}{\sqrt{15}}$$

$$\begin{aligned} \textcircled{7} \quad AD^2 &= DH^2 + AH^2 \\ &= \left(\frac{5}{\sqrt{15}}\right)^2 + 3^2 \\ &= \frac{32}{3} \end{aligned}$$

## Example Problem 2

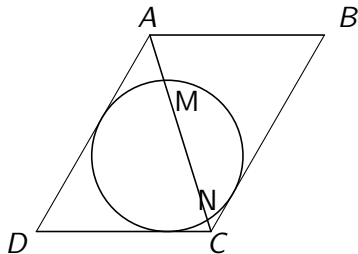
### Question 2

(Prelim 2022 Q16)  $ABCD$  is a parallelogram with  $\angle B$  acute. A circle is tangent to  $BC$ ,  $CD$  and  $DA$ . The circle intersects  $AC$  at  $M$  and  $N$ , where  $M$  is closer to  $A$  than  $N$ . If  $AM = 9$ ,  $MN = 16$  and  $NC = 2$ , find the area of  $ABCD$ .

## Example Problem 2

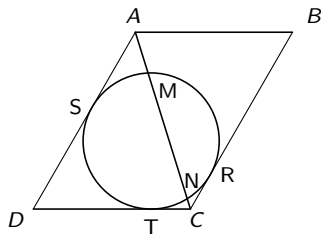
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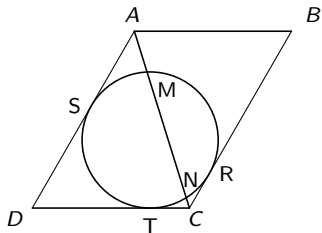
## Example Problem 2 (Solution)

- $AM = 9$ ,  $MN = 16$ ,  $NC = 12$
- Circle  $\omega$  tangent to  $BC$ ,  $CD$ ,  $DA$  at  $R$ ,  $S$ ,  $T$



## Example Problem 2 (Solution)

- $AM = 9, MN = 16, NC = 12$
- Circle  $\omega$  tangent to  $BC, CD, DA$  at  $R, S, T$

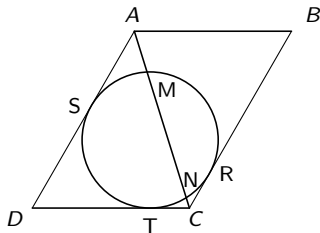


### Solution

$$\textcircled{1} \text{Pow}_{\omega}(A) = -AM \cdot AN = -AS^2 \Rightarrow AS = 15$$

## Example Problem 2 (Solution)

- $AM = 9, MN = 16, NC = 12$
- Circle  $\omega$  tangent to  $BC, CD, DA$  at  $R, S, T$

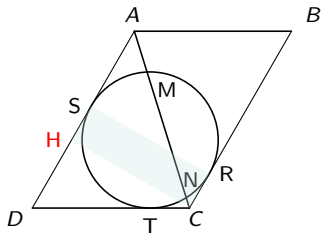


### Solution

- 1  $\text{Pow}_{\omega}(A) = -AM \cdot AN = -AS^2 \Rightarrow AS = 15$
- 2  $\text{Pow}_{\omega}(C) = -CN \cdot CM = -CT^2 \Rightarrow CT = CR = 6$

## Example Problem 2 (Solution)

- $AM = 9, MN = 16, NC = 12$
- Circle  $\omega$  tangent to  $BC, CD, DA$  at  $R, S, T$



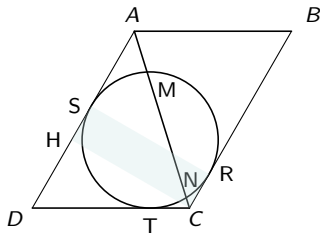
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- 1  $\text{Pow}_{\omega}(A) = -AM \cdot AN = -AS^2 \Rightarrow AS = 15$
- 2  $\text{Pow}_{\omega}(C) = -CN \cdot CM = -CT^2 \Rightarrow CT = CR = 6$
- 3 Drop a perpendicular line from  $C$  to  $AD$  at  $H$ , note that  $SRCH$  forms a rectangle. Hence  $SH = CR = 6$



## Example Problem 2 (Solution)

- $AM = 9, MN = 16, NC = 12$
- Circle  $\omega$  tangent to  $BC, CD, DA$  at  $R, S, T$

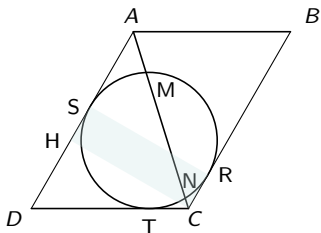


### Solution

- 1  $\text{Pow}_{\omega}(A) = -AM \cdot AN = -AS^2 \Rightarrow AS = 15$
- 2  $\text{Pow}_{\omega}(C) = -CN \cdot CM = -CT^2 \Rightarrow CT = CR = 6$
- 3 Drop a perpendicular line from  $C$  to  $AD$  at  $H$ , note that  $SRCH$  forms a rectangle. Hence  $SH = CR = 6$
- 4 Let  $SD = DT = x$ , in  $\triangle DHC$ , by Pythagoras Theorem,  
 $(x - 6)^2 + (x + 6)^2 = AC^2 - AH^2$

## Example Problem 2 (Solution Cont.)

- $AM = 9$ ,  $MN = 16$ ,  $NC = 12$ ,  $AS = 15$ ,  $CT = CR = SH = 6$

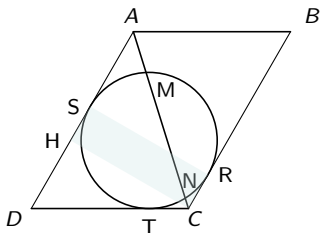


### Solution

- Solving  $(x - 6)^2 - (x + 6)^2 = AC^2 - AH^2 = 27^2 - 21^2 = 288$  gives  $x = 12$ .

## Example Problem 2 (Solution Cont.)

- $AM = 9, MN = 16, NC = 12, AS = 15, CT = CR = SH = 6$

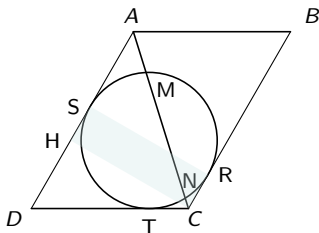


### Solution

- 5 Solving  $(x - 6)^2 - (x + 6)^2 = AC^2 - AH^2 = 27^2 - 21^2 = 288$  gives  $x = 12$ .
- 6  $[ABCD] = AD \cdot CH = (12 + 15)\sqrt{288} = 324\sqrt{2}$

## Example Problem 2 (Solution Cont.)

- $AM = 9, MN = 16, NC = 12, AS = 15, CT = CR = SH = 6$



### Solution

- 5 Solving  $(x - 6)^2 - (x + 6)^2 = AC^2 - AH^2 = 27^2 - 21^2 = 288$  gives  $x = 12$ .
- 6  $[ABCD] = AD \cdot CH = (12 + 15)\sqrt{288} = 324\sqrt{2}$

### Question

Where did we use the parallelogram condition?

# Practice Problems

## Question 1

(Prelim 2023 Q17)  $ABCD$  is a square.  $P$  is a point inside  $ABCD$  such that  $\angle APD + \angle BPC = 180^\circ$  and  $\angle BPC$  is acute. If  $PB = 3$  and  $PC = 4$ , find  $BC$ .

## Question 2

(Prelim 2021 Q12)  $OABC$  is a trapezium with  $OC \parallel AB$  and  $\angle AOB = 37^\circ$ . Furthermore,  $A, B, C$  all lie on the circumference of a circle centered at  $O$ . The perpendicular bisector of  $OC$  meets  $AC$  at  $D$ .

## Question 3

(Prelim 2018 Q13) Let  $O$  be the circumcentre of  $\triangle ABC$ . Suppose  $AB = 1$  and  $AO = AC = 2$ .  $D$  and  $E$  are points on the extensions of  $AB$  and  $AC$  respectively such that  $OD = OE$  and  $BD = \sqrt{2}EC$ . Find the value of  $OD^2$ .

# Practice Problems

## Question 4

(Prelim 2018 Q16)  $ABCD$  is a cyclic quadrilateral with  $AC = 56$ ,  $BD = 65$ ,  $BC > DA$  and  $\frac{AB}{BC} = \frac{CD}{DA}$ . Find the ratio of the area of  $\triangle ABC$  to the area of  $\triangle ADC$ .

## Question 5

(Prelim 2016 Q20) In  $\triangle ABC$ ,  $P$  and  $Q$  are points on  $AB$  and  $AC$  respectively such that  $AP : PB = 8 : 1$  and  $AQ : QC = 15 : 1$ .  $X$  and  $Y$  are points on  $BC$  such that the circumcircle of  $\triangle APX$  is tangent to both  $BC$  and  $CA$ , while the circumcircle of  $\triangle AQY$  is tangent to both  $AB$  and  $BC$ . Find  $\cos BAC$ .

*Thank You!*