MATH 4.1EL
Assignment 2
(Quadratic Equations in One Unknown)

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Answer the questions in the spaces provided on the question sheets. If you do not know how to answer a certain question, write down where you get stuck. Answers can be corrected to 3 significant figures if necessary.

Name, class, class no.: _____
Tutor's name: _____

1 Factorization of Polynomials

1. Expand the following polynomials.

   (a) $(x+y)^2$
   (b) $(x-y)^2$
   (c) $(x+y)(x-y)$
   (d) $(x+y)(x^2-xy+y^2)$
   (e) $(x-y)(x^2+xy+y^2)$

2. Expand the following polynomials

   (a) $(3x+2y)^2$
   (b) $(2x-3y)^2$
   (c) $(3x+2y)(3x-2y)$
   (d) $(2x+y)(4x^2-2xy+y^2)$
   (e) $(2x-3y)(4x^2+6xy+9y^2)$

3. Factorize the following polynomials

   (a) x^2+9x+8
   (b) $4x^2+20x+25$
   (c) x^3-3x^2-10x
   (d) $(p+q)^2-9$
   (e) $m(x+y)^2-4m(x-y)^2$
   (f) $(x-1)^2+16(x-1)+64$

2 Solving Quadratic Equations

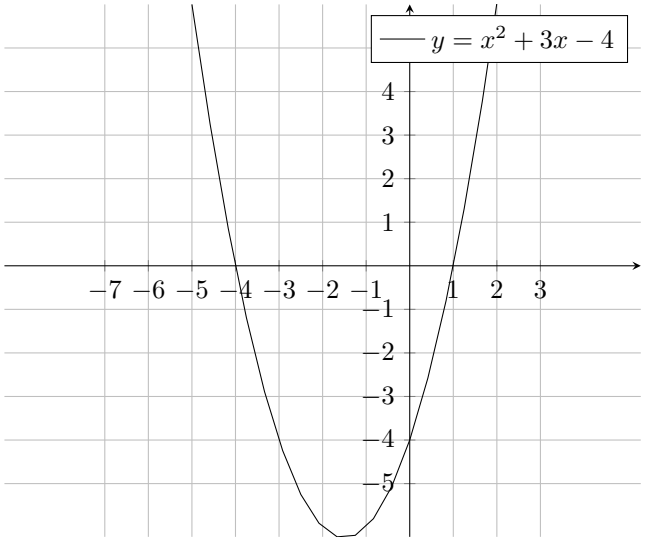
Quadratic equations are polynomials with the degree 2. To solve a quadratic equation, we rely on the following fact heavily:

Theorem: Product being 0 implies one of its factors must be 0

For any real numbers m and n, if $mn=0$, then $m=0$ or $n=0$ (or both).

4. Solve the quadratic equation $(x+8)(x-1)=0$.
5. Solve the quadratic equation $2x^2-x-3=0$.
6. Solve the quadratic equation $x^2-6x=-9$.
7. Solve the quadratic equation $81x^2=9$.
8. Solve the quadratic equation $(7x-1)(3x-4)=5(7x-1)$.

2.1 Graphical Method



9. Given the above graph, how should we solve $x^2+3x-4=0$, how about $\begin{cases} y=x^2+3x-4 \\ y=\frac{1}{4}x-4 \end{cases}$? Explain your reasoning behind your method.
10. According to the graphical method, when will the equation $ax^2+bx+c=0$ have no solution? How about one and two solutions? (a, b and c are constants) Explain your answer in terms of the shape of the graph.

2.2 Quadratic Formula

Theorem: Quadratic formula

The roots of the quadratic equation $ax^2+bx+c=0$ (where $a \neq 0$) are given by the formula below

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

11. Use the quadratic formula to solve the following quadratic equations.

(a) $x^2-4x-5=0$
(b) $6x^2+7x+2=0$
(c) $5x^2+6x-7=0$
(d) $3(2x+4)=3-5x^2$
(e) $(x+3)(x-3)=x(3x+4)$
(f) $5x^2+\frac{2}{5}=2\sqrt{2}x$

2.3 Discriminant

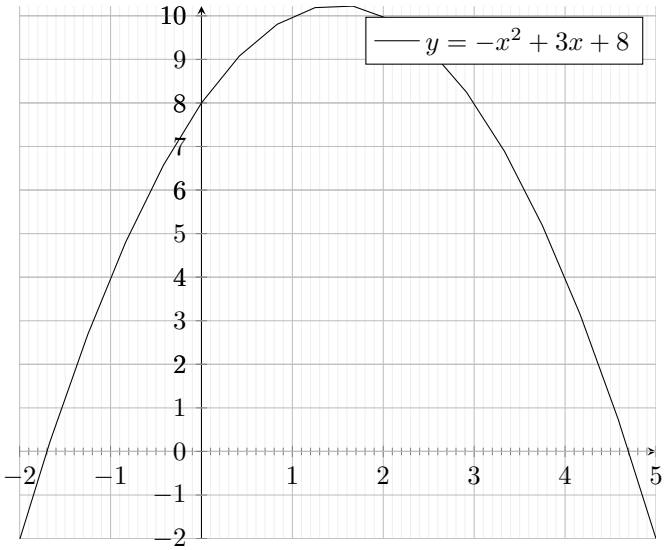
Theorem: Discriminant

The discriminant, often denotes as Δ is the expression inside the surd in the quadratic formula.

$$\Delta = b^2 - 4ac$$

Value of Δ	$\Delta > 0$	$\Delta = 0$	$\Delta < 0$
Number of real roots	2	1	0

12. Explain the why Δ has a relationship with the number of real roots.
13. If the quadratic equation $kx^2 + 70x - 25 = 0$ has two equal real roots, find the value of k.
14. If the quadratic equation $7x^2 - 3(x + 1) = k - x$ intersect with the x-axis, find the range of values of k.
15. The base radius and the height of a solid right circular cylinder are $(9 + x)$ cm and $(12 - x)$ cm respectively.
    (a) Express the curved surface area of the cylinder in terms of x.
    (b) If the curved surface area of the cylinder is $200\pi\text{cm}^2$, use the graph of $y = -x^2 + 3x + 8$ to find the value(s) of x, correct to 1 decimal place.



16. The following shows some patterns.

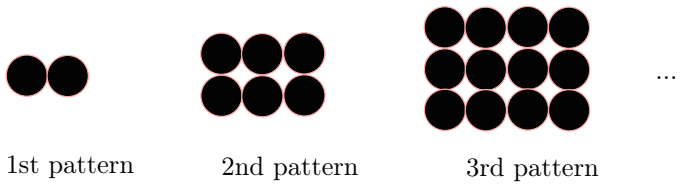


Figure 1: A pattern with dots

    (a) Deduce a quadratic formula in n for the number of dots in the n^{th} figure.
    (b) If the number of dots in the m^{th} pattern is 506, find the value of m.
    (c) If the total number of dots in the k^{th} and $(k + 2)^{\text{th}}$ pattern is 422, find the value of k
    (d) Is there a pattern with 307 dots? Explain your answer.

17. Given that the graph of $y = 16px^2 - 8(3p+2)x - 1$ touches the x-axis.
    (a) Find the two values of p
    (b) For each of the values of p, solve the equation $16px^2 - 8(3p+2)x - 1 = 0$.

18. If the sum of two numbers is 36 and the sum of their reciprocals is $\frac{1}{5}$, find the two numbers.

Note: Reciprocals

The reciprocal of a number x is $\frac{1}{x}$, for example, the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.

19. (a) Solve the equation $2x^2 + 3x + 1 = 0$.
    (b) Hence, solve the equation $\frac{1}{y^2} + \frac{3}{y} + 2 = 0$

3 Relations between Roots and Coefficients

Theorem: Sum of roots and Product of roots

For a quadratic equation $ax^2 + bx + c = 0$ with roots α and β,

$$\text{Sum of roots} = \alpha + \beta = -\frac{b}{a}$$
$$\text{Product of roots} = \alpha\beta = \frac{c}{a}$$

Determine whether the following statements are correct.

20. ____The theorem is true for all complex roots α and β.

21. ____The theorem is true for quadratic equations with $\Delta < 0$.

22. ____$\alpha \neq \beta$

23. ____If $a > 0$ and $c > 0$, then $\alpha\beta > 0$

24. Find the sum of roots and product of roots of the quadratic equation $3x^2 + 3x - 5$. Also, find the solutions α and β to verify the result.

25. If α and β are the roots of the equation $7x^2 - 2x - 1 = 0$, find the value of each of the following expressions, assuming that $\alpha > \beta$

    (a) $(3-\alpha)(3-\beta)$
    (b) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$
    (c) $7\alpha^2 + 2\beta$
    (d) $\frac{1+2\alpha}{\alpha^3} + \frac{1+2\beta}{\beta^3}$
    (e) $\alpha^2 + \beta^2$
    (f) $\alpha - \beta$

Common mistake: $a^2 + b^2 \neq (a+b)^2$

$$a^2 + b^2 \neq (a+b)^2 = a^2 + 2ab + b^2$$

Visualize it!

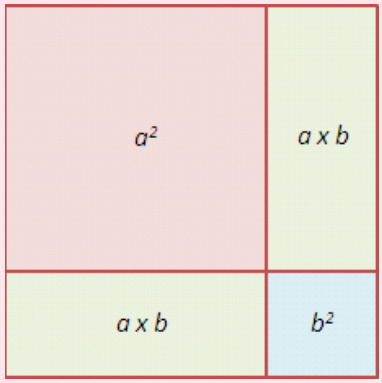


Figure 2: $(a+b)^2 = a^2 + 2ab + b^2$

26. Form a quadratic equation in x with roots $6 + \sqrt{5}$ and $6 - \sqrt{5}$.

27. If α and β are the roots of the equation $3x^2 - 5x - 9 = 0$, form a quadratic equation in x with roots $\alpha + 2$ and $\beta - 2$.

28. α and β are the roots of the equation $x^2 - 4x + 2 = 0$, where $\alpha > \beta$.
    (a) Find the value of $\alpha - \beta$
    (b) Form a quadratic equation in x with roots $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ and $\dfrac{1}{\alpha} - \dfrac{1}{\beta}$. (Leave the radical sign $\surd$ in the answers.)

29. α and β are the roots of the equation $x^2 + 3x - 8 = 0$.
    (a) Find the value of $\dfrac{\beta}{\alpha} + \dfrac{\alpha}{\beta}$.
    (b) Form a quadratic equation in x with roots $\alpha - \dfrac{1}{\alpha}$ and $\beta - \dfrac{1}{\beta}$.

4 Exercises

30. If $\alpha \neq \beta$ and $\begin{cases} 3\alpha = \alpha^2 - 5 \\ 3\beta = \beta^2 - 5 \end{cases}$, then $\alpha\beta =$?
    A. 3
    B. -3
    C. 5
    D. -5

31. If the roots of the quadratic equation $x^2 - kx + 3 = 0$ are α and β, then $\alpha^3 + \beta^3 =$
    A. k^3
    B. $k^3 - 3k$
    C. $k^3 - 9k$
    D. $k^3 - 12k$

32. If α and β are unequal real numbers, and $\begin{cases} \alpha^2 + 12\alpha = 2 \\ \beta^2 + 12\beta = 2 \end{cases}$, then $(3\alpha + 1)(3\beta + 1) =$
    A. $x = 6$
    B. $x = \dfrac{1}{6}$
    C. $x = 6$ or $\dfrac{1}{6}$
    D. $x = -6$

33. Solve the equation $x + \dfrac{1}{x} = 6 + \dfrac{1}{6}$
    A. $x = 6$
    B. $x = \frac{1}{6}$
    C. $x = 6$ or $\frac{1}{6}$
    D. $x = -6$

34. Which of the following quadratic equations can be formed from the roots $-k$ and $\frac{1}{k} (k \neq 0)$
    A. $x^2 + \left(\frac{1}{k} - k\right) x - 1 = 0$
    B. $x^2 + \left(k - \frac{1}{k}\right) x - 1 = 0$
    C. $x^2 + \left(\frac{1}{k} - k\right) x + 1 = 0$
    D. $x^2 + \left(k - \frac{1}{k}\right) x + 1 = 0$

35. Let k be a constant. If the roots of the quadratic equation $x^2 + 4x + 3k = 0$ are α and β, then $4\alpha - \beta^2 =^2$
    A. $3k - 4$
    B. $3k + 4$
    C. $3k - 16$
    D. $3k + 16$

36. If the equation $2x^2 + (4k - 1)x - k + \frac{5}{4} = 0$ has no real roots in x, where k is a constant, find the range of possible values of k.

37. A piece of wire of length 42 cm is cut into two parts and bent into a square and an equilateral triangle respectively. If the area of the square and that of the triangle are in the ratio $4 : \sqrt{3}$, find the length of the side of the square.