Introduction to Euclidean Geometry

T Yeung

THMSS

2024

Outline

Outline

Outline

Outline

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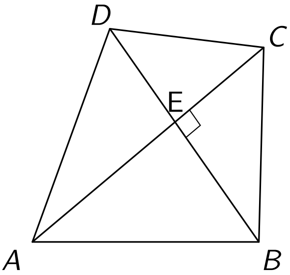
Outline

A brain teaser

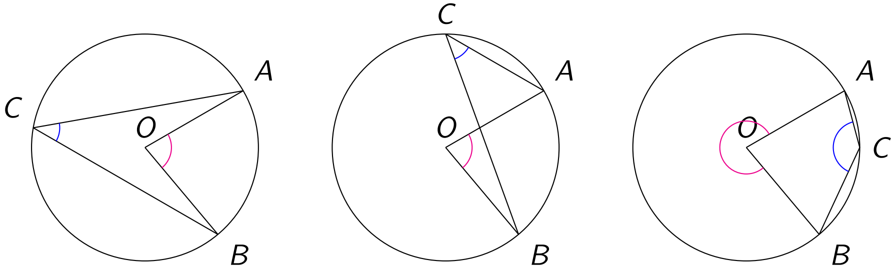


Given:

- $\angle DAC = 30^o$
- $\angle CDB = 40^o$
- $\angle ABD = 50^o$
- $DB \perp AC$

What angles can you compute?

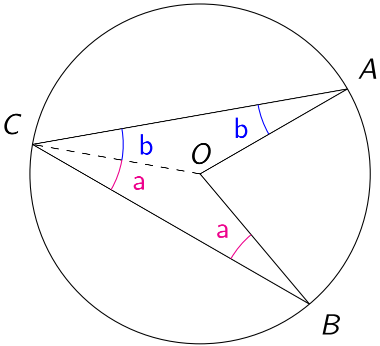
Inscribed angle theorem



Theorem (Inscribed Angle Theorem)

Let O denotes the center of circle, A and B be any two points on the circle, then $\angle AOB = 2\angle ACB$.

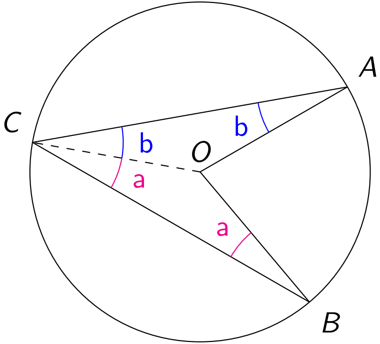
Inscribed Angle Theorem (Proof for case I)



Proof:

1. $OA = OC = OB$

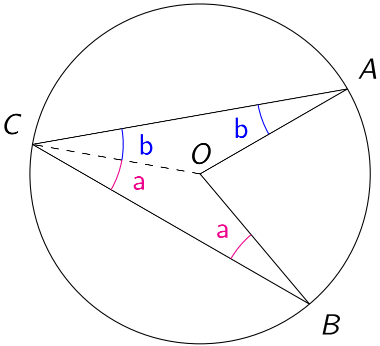
Inscribed Angle Theorem (Proof for case I)



Proof:

1. $OA = OC = OB$
2. $\angle OAC = \angle ACO = a$
3. $\angle OCB = \angle OBC = b$

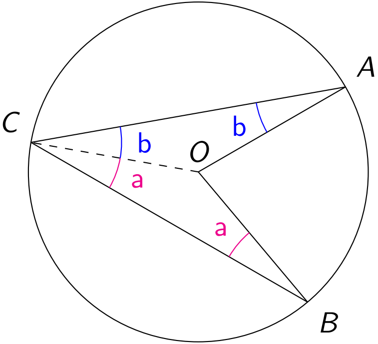
Inscribed Angle Theorem (Proof for case I)



Proof:

1. $OA = OC = OB$
2. $\angle OAC = \angle ACO = a$
3. $\angle OCB = \angle OBC = b$
4. $\angle ACB = a + b$

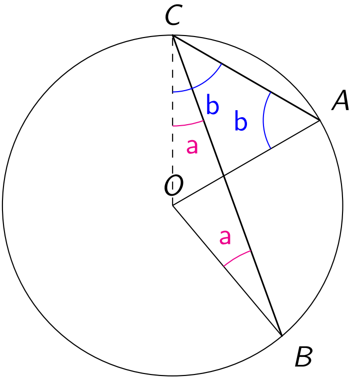
Inscribed Angle Theorem (Proof for case I)



Proof:

1. $OA = OC = OB$
2. $\angle OAC = \angle ACO = a$
3. $\angle OCB = \angle OBC = b$
4. $\angle ACB = a + b$
5. $\angle AOB = 2a + 2b$

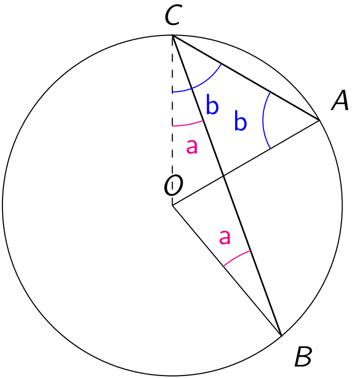
Inscribed Angle Theorem (Proof for case II)



Proof:

1. $OA = OC = OB$

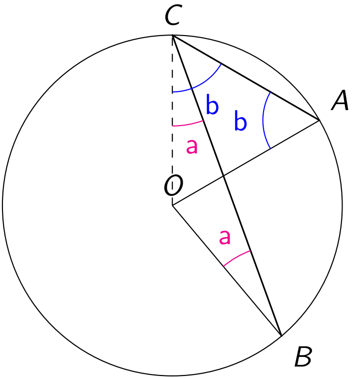
Inscribed Angle Theorem (Proof for case II)



Proof:

1. $OA = OC = OB$
2. $\angle OAC = \angle ACO = a$
3. $\angle OCB = \angle OBC = b$

Inscribed Angle Theorem (Proof for case II)



Proof:

1. $OA = OC = OB$
2. $\angle OAC = \angle ACO = a$
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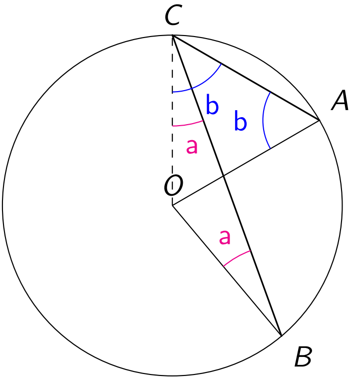
4. $\angle ACB = a - b$

Inscribed Angle Theorem (Proof for case II)

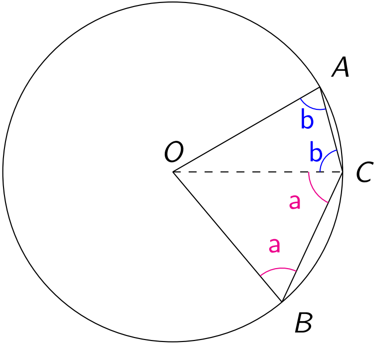


Proof:

1. $OA = OC = OB$
2. $\angle OAC = \angle ACO = a$
3. $\angle OCB = \angle OBC = b$
4. $\angle ACB = a - b$
5. $\angle AOB = \angle COB - \angle COA$

$$= (180^o - 2b)$$
$$- (180^o - 2a)$$
$$= 2a - 2b$$

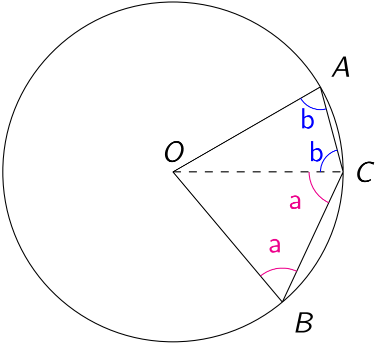
Inscribed Angle Theorem (Proof for case III)



Proof:

1. $OA = OC = OB$

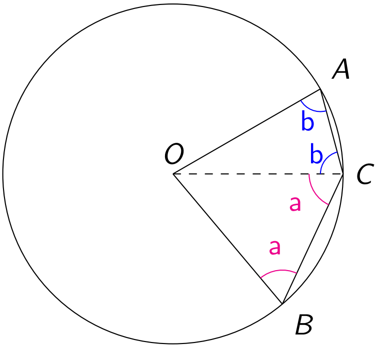
Inscribed Angle Theorem (Proof for case III)



Proof:

1. $OA = OC = OB$
2. $\angle OAC = \angle ACO = a$
3. $\angle OCB = \angle OBC = b$

Inscribed Angle Theorem (Proof for case III)



Proof:

1. $OA = OC = OB$
2. $\angle OAC = \angle ACO = a$
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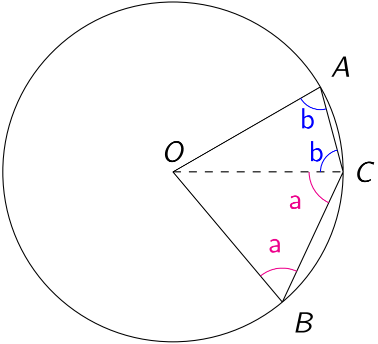
4. $\angle ACB = a + b$

Inscribed Angle Theorem (Proof for case III)



Proof:

1. $OA = OC = OB$
2. $\angle OAC = \angle ACO = a$
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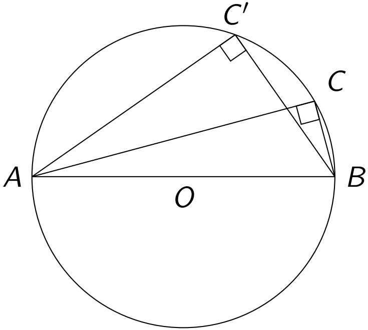
4. $\angle ACB = a + b$
5. $\angle AOB = 2a + 2b$

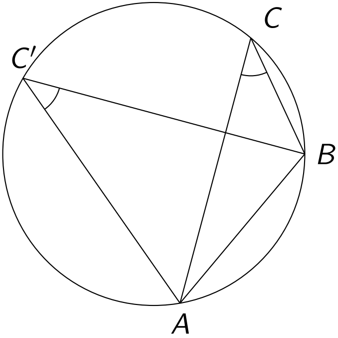
Inscribed Angle Theorem (Corollary I)



Corollary ($\angle$ in semi circle)

Let AB be a diameter of circle, C be any point on a circle. Then $\angle ACB = 90^o$ (because the angle at centre is 180^o).

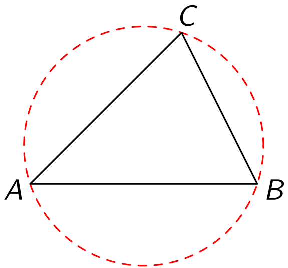
Inscribed Angle Theorem (Corollary II)



Corollary (angle at circumference $\propto$ arc length)

The arc is proportional to the angle at circumference (center).

The Extended Law of Sine



Naming Convention

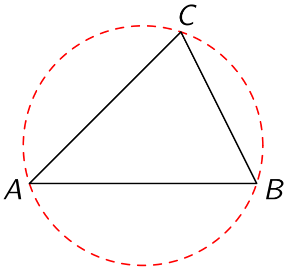
By convention, in ΔABC, the opposite side to angle A is named a (similarly for B and C), $\mathcal{R}$ denotes the circumradius of ΔABC, and r denotes the inradius of ΔABC.

Theorem (The Extended Law of Sine)

Given a triangle ABC, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2\mathcal{R}$$

The Extended Law of Sine (Proof)



Theorem (The Extended Law of Sine)

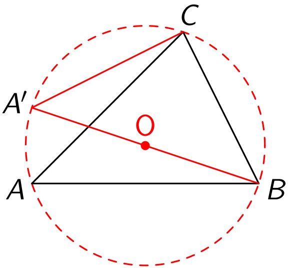
Given a triangle ABC, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2\mathcal{R}$$

Proof:

1. Without loss of generality, we only prove $\frac{a}{\sin A} = 2\mathcal{R}$

The Extended Law of Sine (Proof)



Theorem (The Extended Law of Sine)

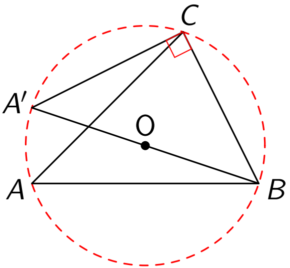
Given a triangle ABC, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2\mathcal{R}$$

Proof:

1. Without loss of generality, we only prove $\frac{a}{\sin A} = 2\mathcal{R}$
2. Move A to A' such that $A'B$ is the diameter of the circle.

The Extended Law of Sine (Proof)



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Proof:

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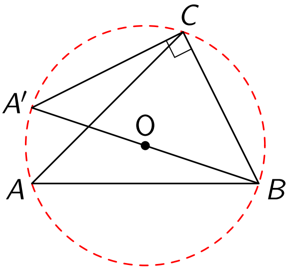
2. Move A to A' such that $A'B$ is the diameter of the circle.
3. Note that $\triangle ACB$ is a right-angled triangle and $\angle BA'C = \angle BAC$.

The Extended Law of Sine (Proof)



Theorem (The Extended Law of Sine)

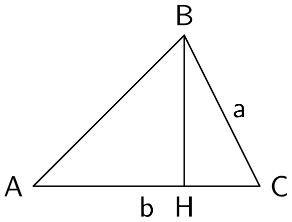
Given a triangle ABC, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2\mathcal{R}$$

Proof:

1. Without loss of generality, we only prove $\frac{a}{\sin A} = 2\mathcal{R}$
2. Move A to A' such that $A'B$ is the diameter of the circle.
3. Note that $\triangle ACB$ is a right-angled triangle and $\angle BA'C = \angle BAC$.
4. We have $\frac{a}{\sin BAC} = \frac{CB}{\angle BA'C} = A'B = 2\mathcal{R}$

Relationship between Circumradius and Area



Naming Convention

$[ABC]$ denotes the area of ABC.

Theorem (Area of a triangle)

$[ABC] = \frac{1}{2}ab\sin C$.

Proof:

1. $BH = a\sin C$, $AC = B$
2. $[ABC] = \frac{1}{2}BH \cdot AC = \frac{1}{2}ab\sin C$.

Theorem (Circumradius and Area)

By the extended law of sine, we also have $\sin C = \frac{c}{2\mathcal{R}}$, *hence,* $[ABC] = \frac{abc}{4\mathcal{R}}$

Relationship between Circumradius and Side Lengths

Naming Convention

s denotes the semi-perimeter of $\triangle ABC$, i.e. $\frac{a+b+c}{2}$.

We state the Heron's formula without proof:

Theorem (Heron's formula)

In $\triangle ABC$, we have $[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$.

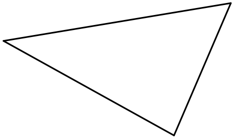
Together with the result in the previous slide, we can find the circumradius of a triangle if we know all 3 side lengths:

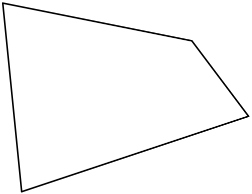
Theorem (Circumradius and Side Lengths)

$$[ABC] = \frac{abc}{4\mathcal{R}}$$

$$\mathcal{R} = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

Cyclic Quadrilateral





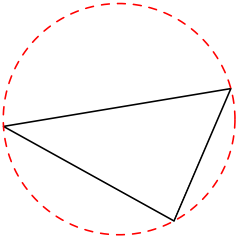
Question 1

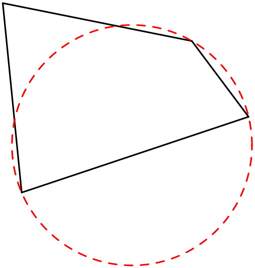
Is it always possible to find a circle passing through a triangle?

Question 2

Is it always possible to find a circle passing through a quadrilateral?

Cyclic Quadrilateral





Question 1

Is it always possible to find a circle passing through a triangle?

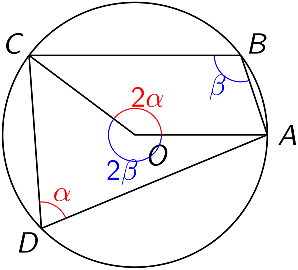
Question 2

Is it always possible to find a circle passing through a quadrilateral?

Answers

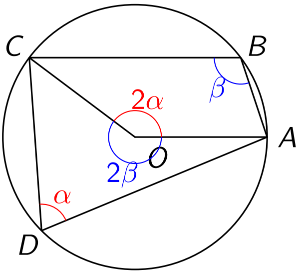
Yes, a circumcentre always exists for a triangle;
No, not possible if a point does not lie on the circumcentre formed by the other three points.

Properties of Cyclic Quadrilateral



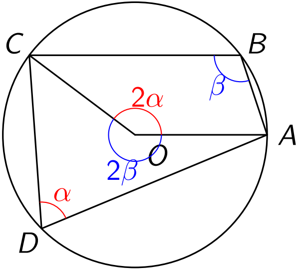
1. Let $\angle COA = 2\alpha$, Reflex $\angle COA = 2\beta$

Properties of Cyclic Quadrilateral



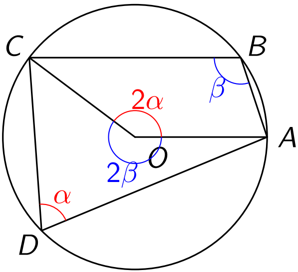
1. Let $\angle COA = 2\alpha$, Reflex $\angle COA = 2\beta$
2. $2\alpha + 2\beta = 360^o \implies \alpha + \beta = 180^o$

Properties of Cyclic Quadrilateral



1. Let $\angle COA = 2\alpha$, Reflex $\angle COA = 2\beta$
2. $2\alpha + 2\beta = 360^o \implies \alpha + \beta = 180^o$
3. $\angle CDA + \angle CBA = \alpha + \beta = 180^o$

Properties of Cyclic Quadrilateral



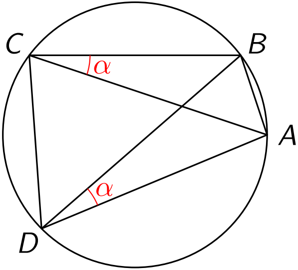
1. Let $\angle COA = 2\alpha$, Reflex $\angle COA = 2\beta$
2. $2\alpha + 2\beta = 360^o \implies \alpha + \beta = 180^o$
3. $\angle CDA + \angle CBA = \alpha + \beta = 180^o$

Theorem (Supplementary opposite angles)

Opposite angles inside a cyclic quadrilateral adds up to 180^o.

Corollary

Exterior angle equals to the opposite interior angle inside a cyclic quadrilateral.



Theorem (Angles subtended by the same arc)

Angles subtended by the same arc are equal.

Test for Cyclic Quadrilateral

Theorem (Test for Cyclic Quadrilateral)

It turns out that the mentioned 3 *properties are also tests for cyclic quadrilateral.*

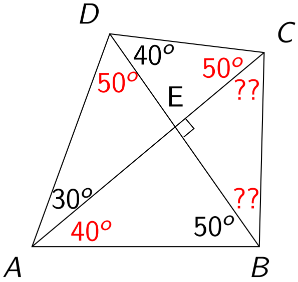
- *Opposite angles adds up to* 180^o.
- *Exterior angle equals the opposite interior angle.*
- *Angles subtended by the same* ***side*** *are equal.*

This means that if ***any*** *of the above* 3 *statement is true, then the quadrilateral is a cyclic quadrilateral.*

The proof is omitted here. Now you should have enough to solve the original problem. :)

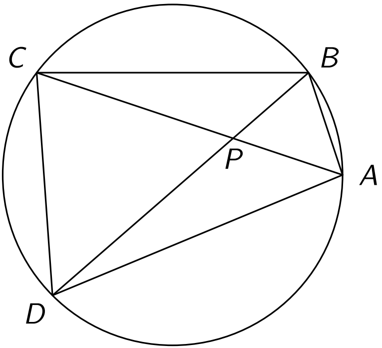
Rerouting to our original problem. . .

Now you should have enough to solve the original problem. :)

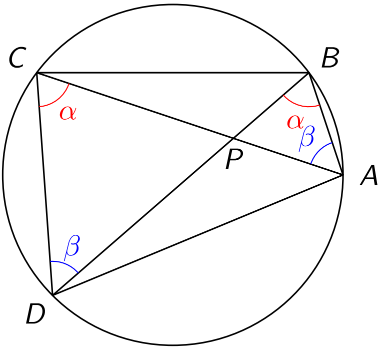


Find the remaining angles!

Power Chord Theorem (I)

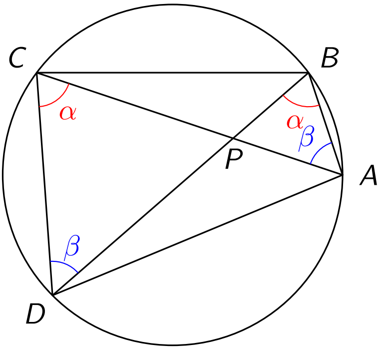


Power Chord Theorem (I)



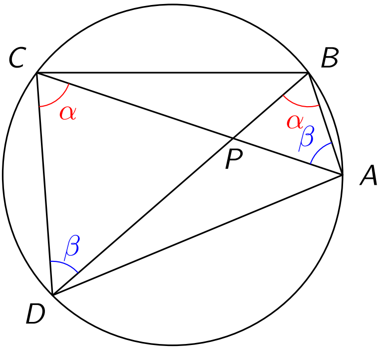
1. $\angle DCA = \angle DBA$
2. $\angle CDB = \angle DAB$

Power Chord Theorem (I)



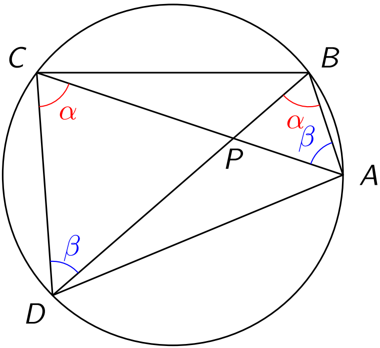
1. $\angle DCA = \angle DBA$
2. $\angle CDB = \angle DAB$
3. $\Delta PCD \sim \Delta PBA$

Power Chord Theorem (I)



1. $\angle DCA = \angle DBA$
2. $\angle CDB = \angle DAB$
3. $\Delta PCD \sim \Delta PBA$
4. $\frac{PC}{PB} = \frac{PD}{PA}$

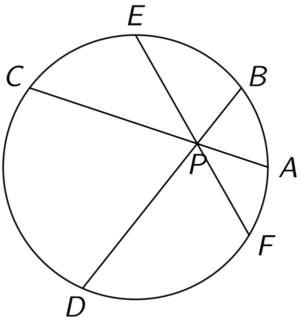
Power Chord Theorem (I)



1. $\angle DCA = \angle DBA$
2. $\angle CDB = \angle DAB$
3. $\Delta PCD \sim \Delta PBA$
4. $\frac{PC}{PB} = \frac{PD}{PA}$
5. $PC \cdot PA = PB \cdot PD$

Power Chord Theorem (I)

In fact, if we have any chord XY passing through P, $PX \cdot PY$ is always fixed.

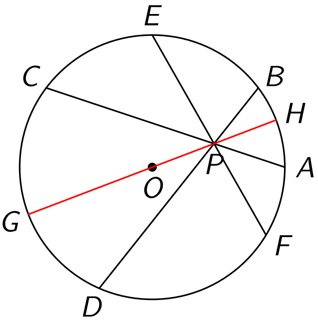


$$PC \cdot PA = PE \cdot PF = PB \cdot PD$$

Power of a Point inside the circle

$\text{Pow}_\omega(P)$ with respect to the circle ω is defined to be $|PX \cdot PY|$ for any chord XY passing through P.

Power Chord Theorem (I)



$$PC \cdot PA = PE \cdot PF = PB \cdot PD$$

In fact, if we have any chord XY passing through P, $PX \cdot PY$ is always fixed.

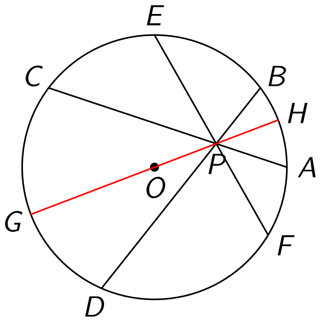
Power of a Point **inside** the circle

$\text{Pow}_\omega(P)$ with respect to the circle ω is defined to be $|PX \cdot PY|$ for any chord XY passing through P.

Power $= \mathcal{R}^2 - OP^2 > 0$

1. $\mathcal{R}$ is the radius of circumcircle.

Power Chord Theorem (I)



$PC \cdot PA = PE \cdot PF = PB \cdot PD$

In fact, if we have any chord XY passing through P, $PX \cdot PY$ is always fixed.

Power of a Point inside the circle

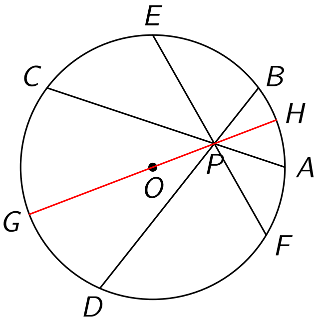
$\text{Pow}_\omega(P)$ with respect to the circle ω is defined to be $|PX \cdot PY|$ for any chord XY passing through P.

Power $= \mathcal{R}^2 - OP^2 > 0$

1. $\mathcal{R}$ is the radius of circumcircle.
2. Draw a diameter GH through P

Power Chord Theorem (I)

In fact, if we have any chord XY passing through P, $PX \cdot PY$ is always fixed.



$$PC \cdot PA = PE \cdot PF = PB \cdot PD$$

Power of a Point **inside** the circle

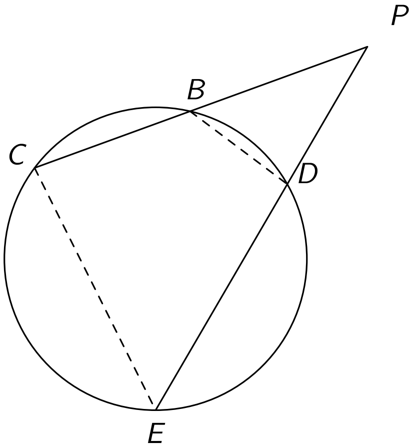
$\mathrm{Pow}_\omega(P)$ with respect to the circle ω is defined to be $|PX \cdot PY|$ for any chord XY passing through P.

Power $= \mathcal{R}^2 - OP^2 > 0$

1. $\mathcal{R}$ is the radius of circumcircle.
2. Draw a diameter GH through P
3. $\mathrm{Pow}_\omega(P) = PG \cdot PH$

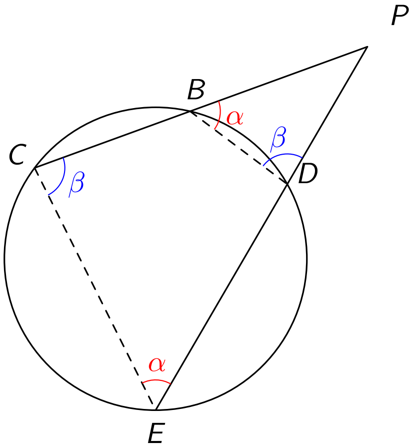
$$= (\mathcal{R} + OP) \cdot (\mathcal{R} - OP) = \mathcal{R}^2 - OP^2$$

Power Chord Theorem (II)



1. $\angle PBD = \angle PEC$
2. $\angle PDB = \angle PCE$
3. $\Delta PBD \sim \Delta PEC$
4. $\frac{PB}{PE} = \frac{PD}{PC}$
5. $PB \cdot PC = PD \cdot PE$

Power Chord Theorem (II)



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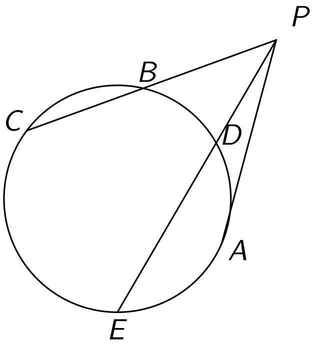
3. $\Delta PBD \sim \Delta PEC$
4. $\frac{PB}{PE} = \frac{PD}{PC}$
5. $PB \cdot PC = PD \cdot PE$

Power Chord Theorem (II)



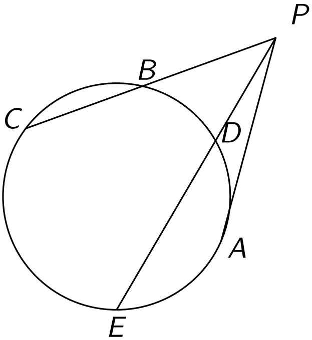
$PB \cdot PC = PD \cdot PE = PA^2$

Again, if we have any chord XY passing through P, $PX \cdot PY$ is always fixed.

Power of a Point outside the circle

$\mathrm{Pow}_\omega(P)$ with respect to the circle ω is defined to be $-|PX \cdot PY|$ for any chord XY passing through P.

Power Chord Theorem (II)



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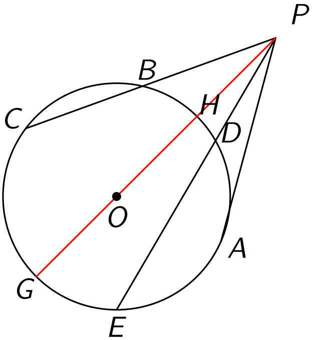
Power $= \mathcal{R}^2 - OP^2 < 0$

1. Let R be the radius of circumcircle.

Power Chord Theorem (II)



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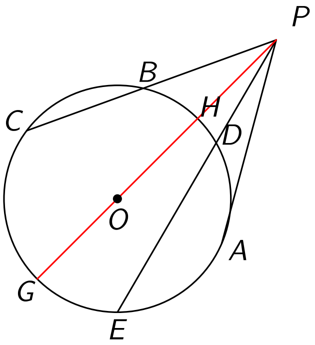

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Power of a Point outside the circle

$\text{Pow}_\omega(P)$ with respect to the circle ω is defined to be $-|PX \cdot PY|$ for any chord XY passing through P.

Power $= \mathcal{R}^2 - OP^2 < 0$

1. Let R be the radius of circumcircle.
2. Draw a diameter GH through P
3. $\text{Pow}_\omega(P) = -|PG \cdot PH|$
   $= (\mathcal{R} + OP)$
   $\cdot (\mathcal{R} - OP) = \mathcal{R}^2 - OP^2$

Power Chord Theorem (III)

Power of a Point **on** the circle

$\mathrm{Pow}_\omega(P)$ with respect to the circle ω is equal to 0. (Why does this makes sense?)

Power $= \mathcal{R}^2 - OP^2 = 0$

As expected.

Converse of Power Chord Theorem

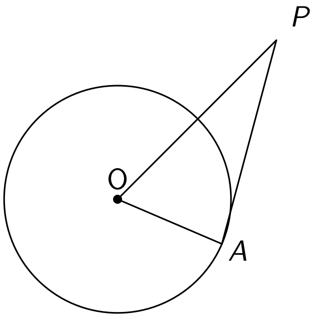
In fact, the converse of power chord theorem is also true.

Theorem (Converse of the Power Chord Theorem)

Let A, B, X, Y be four distinct points in the plane and let lines AB and XY intersect at P. Suppose that either P lies in both of the segments $\overline{AB}$ and $\overline{XY}$, or in neither segment. If $PA \cdot PB = PX \cdot PY$, then A, B, X, Y are concyclic.

This serves as another test for cyclic quadrilateral. The proof is omitted here.

Another proof of the Pythagoras Theorem



Tangent $\perp$ Radius

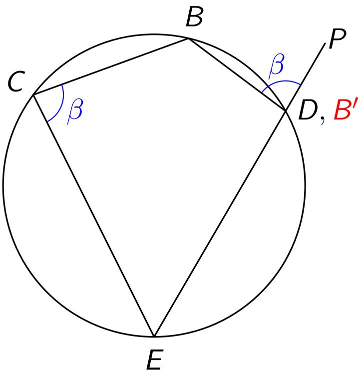
Tangent of a circle at A is perpendicular to OA. (Why?)

Proof of Pythagoras Theorem

Rearranging $\mathrm{Pow}_\omega(P) = PA^2 = OP^2 - OA^2$ gives

$$PA^2 + OA^2 = OP^2$$

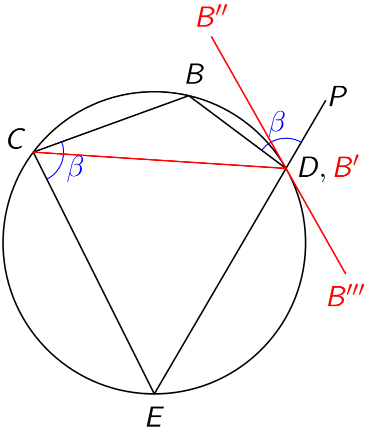
Recall the proof to Power chord theorem (II)



Angle in alternate segment

1. Imagine if B gets increasingly close to D as B'.

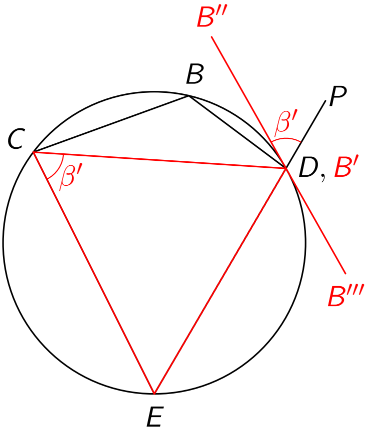
Recall the proof to Power chord theorem (II)



Angle in alternate segment

1. Imagine if B gets increasingly close to D as B'.
2. DB' is arbitrarily close to the tangent to the circle at D.
3. Let $B''B'''$ denotes the tangent.

Recall the proof to Power chord theorem (II)

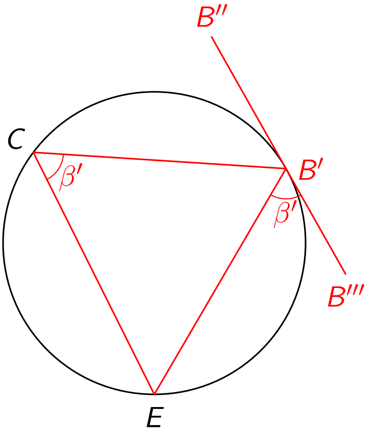


Angle in alternate segment

1. Imagine if B gets increasingly close to D as B'.
2. DB' is arbitrarily close to the tangent to the circle at D.
3. Let $B''B'''$ denotes the tangent.
4. $\angle B''DP = \angle DCE$

A little digression: Angle in Alternate Segment

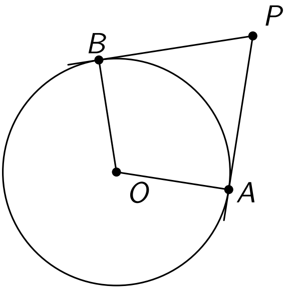
Recall the proof to Power chord theorem (II)



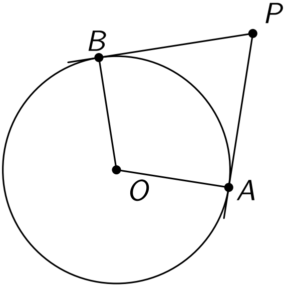
Angle in alternate segment

1. Imagine if B gets increasingly close to D as B'.
2. DB' is arbitrarily close to the tangent to the circle at D.
3. Let $B''B'''$ denotes the tangent.
4. $\angle B''DP = \angle DCE$
5. $\angle B'''DE = \angle DCE$

Other Properties of Tangents



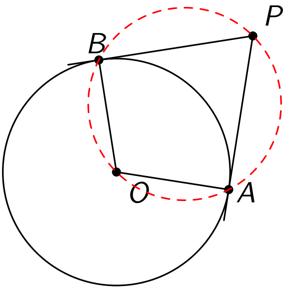
Other Properties of Tangents



$$\begin{aligned} PA = PB &= OP^2 - \mathcal{R}^2 \\ &= \mathrm{Pow}_\omega(P) \end{aligned}$$

Equal Tangents

Given a point P outside ω circle, there are two tangents of equal length from P to ω.



$$PA = PB = OP^2 - \mathcal{R}^2$$
$$= \text{Pow}_\omega(P)$$

Equal Tangents

Given a point P outside ω circle, there are two tangents of equal length from P to ω.

$OABP$ forms a cyclic quadrilateral

$OABP$ is a cyclic quadrilateral since $\angle OBP + \angle OAP = 180^o$. Hence we have $\angle BOA + \angle BPA = 180^o$.

Example Problem 1

Question 1

(Prelim 2020 Q6) In $\Delta ABC, AB = 6, BC = 7, CA = 8$. Let D be the mid-point of minor arc AB on the circumcentre of ΔABC. Find AD^2.

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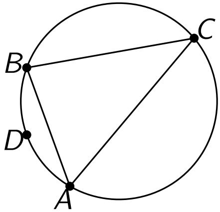
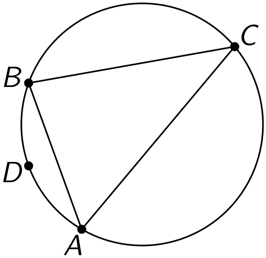


Example Problem 1 (Solution)



- $AB = 6,\ BC = 7,\ CA = 8$
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- Find AD^2

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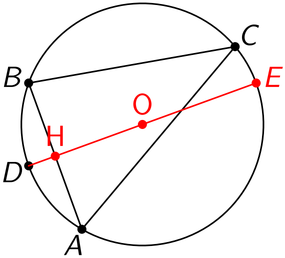

- $AB = 6, \; BC = 7, \; CA = 8$
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1. Drop a perpendicular line from D to BA, that the line passes through O.

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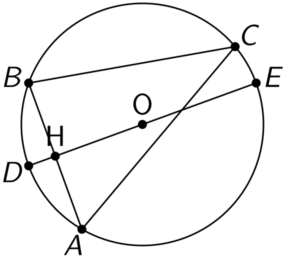

- $AB = 6,\ BC = 7,\ CA = 8$
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1. Drop a perpendicular line from D to BA, that the line passes through O.
2. $\mathrm{Pow}_{\omega}(H) = BH \cdot HA = 9$

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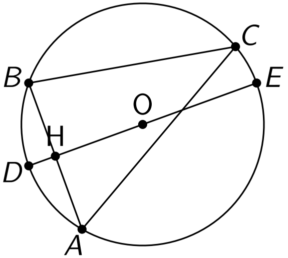

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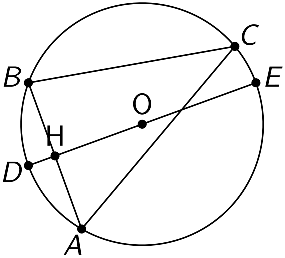

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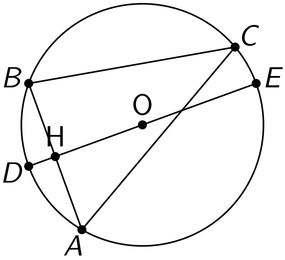
2. $\text{Pow}_\omega(H) = BH \cdot HA = 9$
3. Another way to compute Pow_ω is $DH \cdot HE$.
4. $\mathcal{R} = OD = OE$ is the circumradius of the circle and its given by $\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$, where s is the semi perimeter.

Example Problem 1 (Solution cont.)



- $AB = 6,\ BC = 7,\ CA = 8$
- D is the mid-point of $\overset{\frown}{AB}$
- Find AD^2

5. $\mathcal{R} = \dfrac{6\cdot7\cdot8}{4\sqrt{\frac{21}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}}} = \dfrac{16}{\sqrt{15}}$
6. $$\begin{aligned} Pow_\omega(H) &= (OE + OH) \\ &\quad\cdot (OD - OH) \\ &= \left(\tfrac{16}{\sqrt{15}}\right)^2 - OH^2 \\ &= 9 \end{aligned}$$
   $$OH = \frac{11}{\sqrt{15}}$$
7. $$\begin{aligned} AD^2 &= DH^2 + AH^2 \\ &= \left(\tfrac{5}{\sqrt{15}}\right)^2 + 3^2 \\ &= \tfrac{32}{3} \end{aligned}$$

Example Problem 2

Question 2

(Prelim 2022 Q16) $ABCD$ is a parallelogram with $\angle B$ acute. A circle is tangent to BC, CD and DA. The circle intersects AC at M and N, where M is closer to A than N. If $AM = 9$, $MN = 16$ and $NC = 2$, find the area of $ABCD$.

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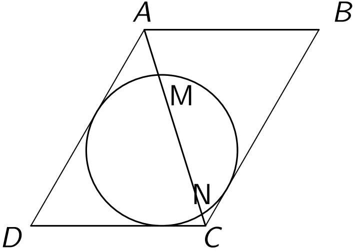

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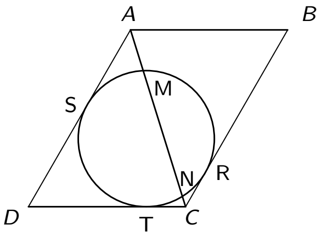
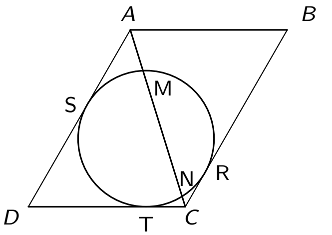


- $AM = 9, MN = 16, NC = 12$
- Circle ω tangent to BC, CD, DA at R, S, T

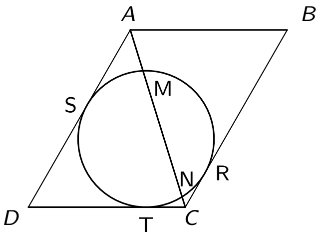
Example Problem 2 (Solution)



- $AM = 9, MN = 16, NC = 12$
- Circle ω tangent to BC, CD, DA at R, S, T

1. $\text{Pow}_\omega(A) = -AM \cdot AN = -AS^2 \Rightarrow AS = 15$

Example Problem 2 (Solution)



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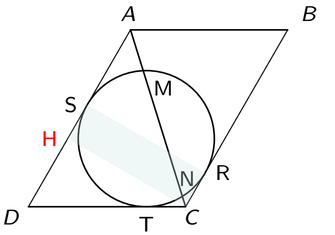
1. $\text{Pow}_\omega(A) = -AM \cdot AN = -AS^2 \Rightarrow AS = 15$
2. $\text{Pow}_\omega(C) = -CN \cdot CM = -CT^2 \Rightarrow CT = CR = 6$

Example Problem 2 (Solution)



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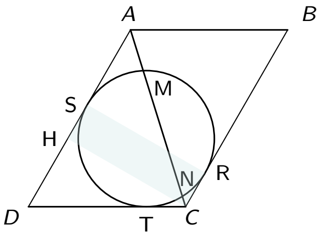
3. Drop a perpendicular line from C to AD at H, note that $SRCH$ forms a rectangle. Hence $SH = CR = 6$

Example Problem 2 (Solution)



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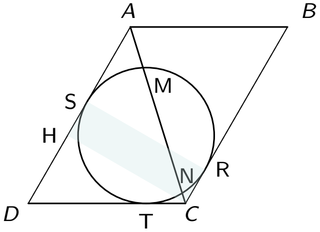
3. Drop a perpendicular line from C to AD at H, note that $SRCH$ forms a rectangle. Hence $SH = CR = 6$
4. Let $SD = DT = x$, in $\triangle DHC$, by Pythagoras Theorem, $(x-6)^2 + (x+6)^2 = AC^2 - AH^2$

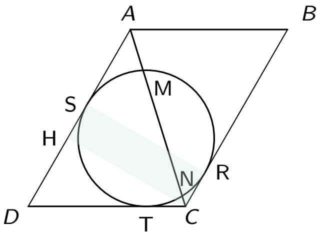
Example Problem 2 (Solution cont.)



$AM = 9, MN = 16, NC = 12,$
$AS = 15, CT = CR = SH = 6$

5. Solving $(x-6)^2 - (x+6)^2 = AC^2 - AH^2 = 27^2 - 21^2 = 288$ gives $x = 12$.

Example Problem 2 (Solution cont.)



$AM = 9, MN = 16, NC = 12,$

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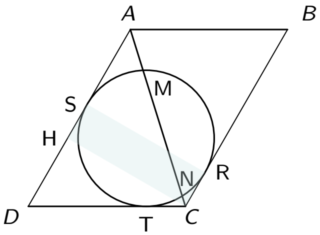
5. Solving $(x-6)^2 - (x+6)^2 = AC^2 - AH^2 = 27^2 - 21^2 = 288$ gives $x = 12$.
6. $[ABCD] = AD \cdot CH = (12+15)\sqrt{288} = 324\sqrt{2}$

Example Problem 2 (Solution cont.)



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6. $[ABCD] = AD \cdot CH = (12+15)\sqrt{288} = 324\sqrt{2}$

Question

Where did we use the parallelogram condition?

Practice Problems

Prelim 2023 Q17

$ABCD$ is a square. P is a point inside $ABCD$ such that $\angle APD + \angle BPC = 180^o$ and $\angle BPC$ is acute. If $PB = 3$ and $PC = 4$, find BC.

Prelim 2021 Q12

$OABC$ is a trapezium with $OC \parallel AB$ and $\angle AOB = 37^o$. Furthermore, A, B, C all lie on the circumference of a circle centered at O. The perpendicular bisector of OC meets AC at D.

Prelim 2018 Q13

Let O be the circumcentre of $\triangle ABC$. Suppose $AB = 1$ and $AO = AC = 2$. D and E are points on the extensions of AB and AC respectively such that $OD = OE$ and $BD = \sqrt{2}EC$. Find the value of OD^2.

Practice Problems

Prelim 2018 Q16

$ABCD$ is a cyclic quadrilateral with $AC = 56$, $BD = 65$, $BC > DA$ and $\frac{AB}{BC} = \frac{CD}{DA}$. Find the ratio of the area of $\triangle ABC$ to the area of $\triangle ADC$.

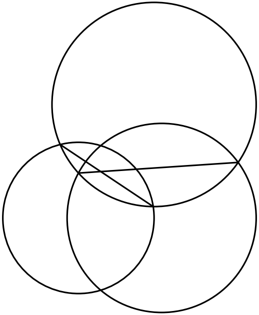
Prelim 2016 Q20

In $\triangle ABC$, P and Q are points on AB and AC respectively such that $AP : PB = 8 : 1$ and $AQ : QC = 15 : 1$. X and Y are points on BC such that the circumcircle of $\triangle APX$ is tangent to both BC and CA, while the circumcircle of $\triangle AQY$ is tangent to both AB and BC. Find $\cos BAC$.

Prelim 2012 Q13

$ABCD$ is a convex quadrilateral in which AC and BD meet at P. Given $PA = 1$, $PB = 2, PC = 6$ and $PD = 3$. Let O be the circumcentre of $\triangle PBC$. If OA is perpendicular to AD, find the circumradius of $\triangle PBC$.

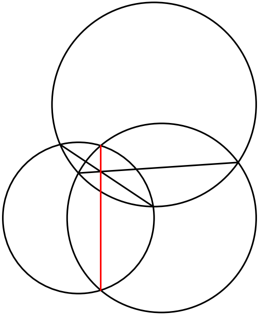
Three circles intersect as in below.



Are the chords necessarily concurrent?

The Radical Axis and Radical Center (Supplementary)

Three circles intersect as in below.



Are the chords necessarily concurrent?

The Radical Axis

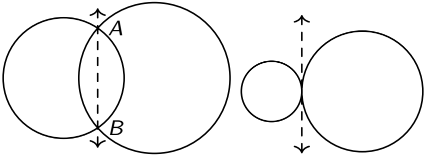
Definition (Radical Axis)

Given two circles ω_1 and ω_2 with distinct centers, the radical axis of the circles is the set of points P such that

$$\mathrm{Pow}_{\omega_1}(P) = \mathrm{Pow}_{\omega_2}(P)$$

Theorem (Radical Axis)

Let ω_1 and ω_2 be circles with distinct centers O_1 and O_2. The radical axis of ω_1 and ω_2 is a straight line perpendicular to $\overline{O_1 O_2}$.
In particular, if ω_1 and ω_2 intersect at two points A and B, then the radical axis is AB.



Proof of the Radical Axis

Suppose that $O_1 = (a, 0)$ and $O_2 = (b, 0)$ in the coordinate plane and the circles have radii r_1 and r_2 respectively. Then for any point $P = (x, y)$ we have

$$\mathrm{Pow}_{\omega_1}(P) = O_1P^2 - r_1^2 = (x-a)^2 + y^2 - r_1^2$$

Similarly,

$$\mathrm{Pow}_{\omega_2}(P) = O_2P^2 - r_2^2 = (x-b)^2 + y^2 - r_2^2$$

Equating the two, we fine the radical axis of ω_1 and ω_2 is the set of points $P = (x, y)$ satisfying

$$\begin{aligned} 0 &= \mathrm{Pow}_{\omega_1}(P) - \mathrm{Pow}_{\omega_2}(P) \\ &= [(x-a)^2 + y^2 - r_1^2] - [(x-b)^2 + y^2 - r_2^2] \\ &= (-2a + 2b)x + (a^2 - b^2 + r_2^2 - r_1^2) \end{aligned}$$

which is a straight line perpendicular to the x-axis (as $-2a + 2b \neq 0$). This implies the result.

Proof of the Radical Axis Theorem (cont.)

The second part is an immediately corollary. The points A and B have equal power to both circles; therefore, both A and B have equal power (zero) to both circles. Therefore, both A and B lie on the radical axis. Consequently, the radical axis must be the line AB itself.

The Radical Axis of non-intersecting circles

The construction of radical axis of intersecting circles is easy.
Bonus: How can you construct the radical axis of non-intersecting circles?
Let's try on Geogebra.

The Standard Equation of Circle: $\mathrm{Pow}_\omega((x, y)) = 0$

As a side remark, the expansion of $\mathrm{Pow}_\omega((x, y)) = 0$ yields $(x - m)^2 + (y - n)^2 - r^2 = 0$. Could you recognize what it is?

Proof of concurrent chords

Let l_{12} be the radical axis of ω_1 and ω_2, and define l_{23} and l_{13} similarly. Let P be the intersection of these two lines. Then

$$P \in l_{12} \Rightarrow \text{Pow}_{\omega_1}(P) = \text{Pow}_{\omega_2}(P)$$
$$P \in l_{23} \Rightarrow \text{Pow}_{\omega_2}(P) = \text{Pow}_{\omega_3}(P)$$

which implies $\text{Pow}_{\omega_1}(P) = \text{Pow}_{\omega_3}(P)$. Hence $P \in l_{31}$. The pairwise radical axes concur at a single point K (except if O_1, O_2, O_3 are collinear), called the **radical center** of the three circles.

Note that the converse is also true.

Theorem (Converse of radical center)

If l_{12} and l_{23} intersect on l_{13} (i.e., O_1, O_2 and O_3 are not collinear), then the quadrilateral formed by the intersections of l_{12} and l_{23} with ω_1 and ω_3 must concyclic.

The End

Thank You!