MATH 5.1ER Quiz 2
(Inequalities in One Unknown)
Time Limit: (50 minutes)

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> Answer the questions in the spaces provided on the question sheets. If you do not know how to answer a certain question, write down where you get stuck. Answers can be corrected to 3 significant figures if necessary.

Name, class, class no.: ______________________________
Tutor's name: ______________________________

1. (4 marks) If the quadratic curve $y = x^2 + kx + 8$ intersects the straight line $y = 4x - 1$ at two distinct points, find the range of possible values of k.

Answer:

$$\begin{cases} y = x^2 + kx + 8 & (1) \\ y = 4x - 1 & (2) \end{cases}$$

Substitute (2) into (1),

$$\begin{aligned} 4x - 1 &= x^2 + kx + 8 \\ x^2 - (4 - k)x + 9 &= 0 \end{aligned}$$

$\because$ There are two intersections

$$\begin{aligned} \Delta &> 0 \\ (4 - k)^2 - 4(9) &> 0 \\ k^2 - 8k - 20 &> 0 \\ (k - 10)(k + 2) &> 0 \\ -2 < k &< 10 \end{aligned}$$

$\therefore -2 < k < 10$

2. (4 marks) If $x^2 + k + kx = 3$ is always positive for all real values of k, find the range of possible values of k.

Answer:

$\because x^2 + kx + (k-3) > 0$
$\therefore \Delta < 0$

$$\begin{aligned} \Delta &< 0 \\ k^2 - 4k + 12 &< 0 \\ (k-6)(k+2) &< 0 \\ -2 &< k < 6 \end{aligned}$$

$\therefore -2 < k < 6$

3. (3 marks) Solve $(2x-3)(3x+1) \geq 4x(2x-3)$

Answer:

$$\begin{aligned} (2x-3)(3x+1) &\geq 4x(2x-3) \\ 6x^2 - 7x - 3 &\geq 8x^2 - 12x \\ 2x^2 - 5x + 3 &\geq 0 \\ (2x+1)(x-3) &\geq 0 \\ x \leq -\tfrac{1}{2} \text{ or } x &\geq 3 \end{aligned}$$

4. (10 marks) α and β are the roots of the quadratic equation $x^2+(p+1)x+(p-1)=0$, where p is real.
   (a) (3 marks) Show that α and β are real and distinct.
   (b) (3 marks) Show that $(\alpha-2)(\beta-2)=3p+5$.
   (c) (4 marks) Given that $\beta<2<\alpha$,
      i. Using the result of (b), show that $p<-\frac{5}{3}$.
      ii. If $(\alpha-\beta)^2<24$, find the range of possible values of p. Hence write down all possible integral value(s) of p.

Answer:

(a)

$$\begin{aligned}\Delta &= (p+1)^2-4(p-1)\\ &= p^2+2p+1-4p+4\\ &= p^2-2p+5=(p-1)^2+4(>0)\end{aligned}$$

$\therefore$ There are two distinct real solutions.

(b) By sum of roots and product of roots formula, we have

$$\begin{cases}\alpha\beta=p-1\\ \alpha+\beta=-(p+1)\end{cases}$$

$$\begin{aligned}(\alpha-2)(\beta-2) &= \alpha\beta-2(\alpha+\beta)+4\\ &= p-1-2[-(p+1)]+4\\ &= p-1+2p-2+4\\ &= 3p+5\end{aligned}$$

(c) i.

$$\begin{aligned}\beta<2&<\alpha\\ (\alpha-2)(\beta-2)&<0\\ 3p+5&<0\\ p&<-\tfrac{5}{3}\end{aligned}$$

ii.

$$\begin{aligned}(\alpha-\beta)^2&<24\\ (\alpha+\beta)^2-4\alpha\beta&<24\\ [-(p+1)]^2-4(p-1)&<24\\ p^2-2p-19&<0\\ 1-2\sqrt{10}<p&<1+2\sqrt{10}\\ -5.32<p&<7.32\end{aligned}$$

$\because -5.32<p<-1.66$
$\therefore p$ can be $-5,-4,-3,-2$

5. Given $x^2 - 2(1+a)x + (3a^2 + 4ab + 4b^2 + 2) = 0$, where a and b are real.
   (a) Show that the discriminant of the equation is $-4[(a-1)^2 + (a-2b)^2]$
   (b) Find a and b if the equation has equal real roots.

Answer:

(a)

$$\begin{aligned}\Delta &= [-2(1+a)]^2 x - 4(3a^2 + 4ab + 4b^2 + 2)\\ &= 4 + 8a + 4a^2 - 12a^2 - 16ab - 16b^2 - 8\\ &= -8a^2 - 16ab - 16b^2 + 8a - 4\\ &= -4[a^2 - 2a + 1 + a^2 - 4ab + 4b^2]\\ &= -4[(a-1)^2 + (a-2b)^2]\end{aligned}$$

(b)

$$\begin{aligned}\Delta &= 0\\ -4[(a-1)^2 + (a-2b)^2] &= 0\\ a = 1 \text{ and } a &= 2b\\ a = 1 \text{ and } b &= \tfrac{1}{2}\end{aligned}$$

$\therefore a = 1$ and $b = \dfrac{1}{2}$