

# MATH4.1EL

## Introduction to Euclidean Geometry

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### 1 Introduction

In the vast tapestry of mathematical exploration, few realms evoke a sense of elegance and timeless beauty quite like Euclidean geometry. With its origins deeply rooted in ancient Greece, this branch of mathematics has captivated the hearts and minds of scholars and thinkers for centuries.

### 2 Motivation

Given

- $\angle DAC = 30^\circ$
- $\angle CDB = 40^\circ$
- $\angle ABD = 50^\circ$
- $DB \perp AC$

What angles can you find?

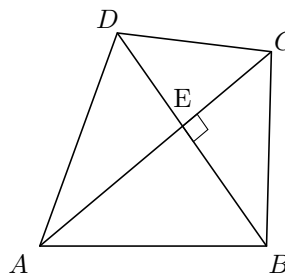
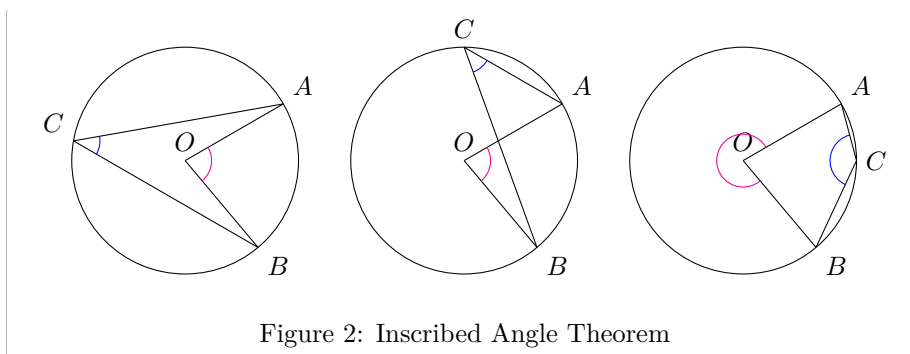


Figure 1: Find the angles!

### 3 Circle

#### 3.1 Inscribed Angle Theorem

**Theorem 3.1** (Inscribed Angle Theorem). *Let  $O$  denotes the center of circle,  $A$  and  $B$  be any two points on the circle, then  $\angle AOB = 2\angle ACB$ .*



*Proof.*

For the first case, we have  $OA = OC = OB$ . Let  $\angle OCB = \angle OBC = a$  and  $\angle OCA = \angle OAB = b$ , then  $\angle ACB = a + b$  and  $\angle AOB = 2a + 2b$ .

For the second case, we have  $OA = OC = OB$ . Let  $\angle OCB = \angle OBC = a$  and  $\angle OCA = \angle OAB = b$ , then  $\angle ACB = b - a$  and  $\angle AOB = 2b - 2a$ .

For the third case, we have  $OA = OC = OB$ . Let  $\angle OCB = \angle OBC = a$  and  $\angle OCA = \angle OAB = b$ , then  $\angle ACB = a + b$  and reflex  $\angle AOB = 2a + 2b$ .

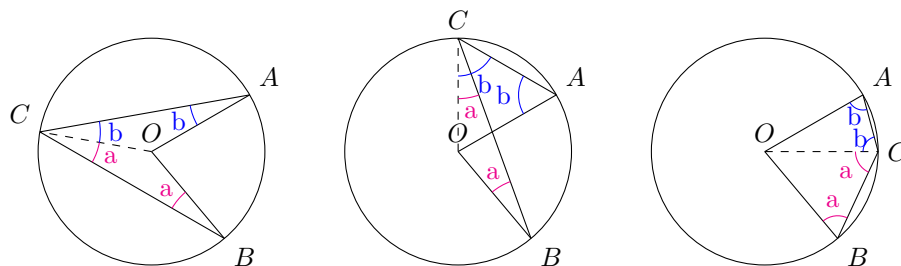


Figure 3: Proof of Inscribed Angle Theorem

□

**Corollary 3.1.1** (angle in semi-circle). *Let  $AB$  be a diameter of circle,  $C$  be any point on a circle. Then  $\angle ACB = 90^\circ$  (because the angle at centre is  $180^\circ$ ).*

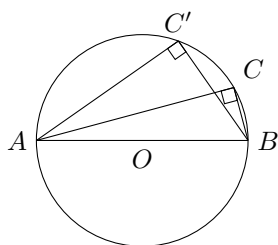
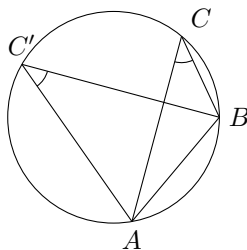


Figure 4: Angle in semi-circle

**Corollary 3.1.2** (angle at circumference  $\propto$  arc length). *The arc is proportional to the angle at circumference (center).*

Figure 5: Angle at circumference  $\propto$  arc length

### 3.2 The Extended Law of Sine

By convention, in  $\triangle ABC$ , the opposite side to angle  $A$  is named  $a$  (similarly for  $B$  and  $C$ ),  $\mathcal{R}$  denotes the circumradius of  $\triangle ABC$ , and  $r$  denotes the inradius of  $\triangle ABC$ .

**Theorem 3.2** (the Extended Law of Sine). *Given a triangle  $ABC$ , we have*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2\mathcal{R}$$

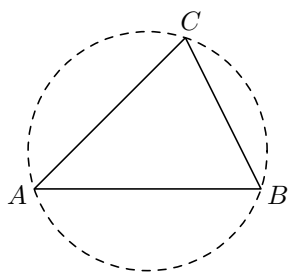


Figure 6: the Extended Law of Sine

*Proof.* Without loss of generality, we only prove  $\frac{a}{\sin A} = 2\mathcal{R}$ . Move  $A$  to  $A'$  such that  $A'B$  is the diameter of the circle. Note that  $\triangle ACB$  is a right-angled triangle and  $\angle BA'C = \angle BAC$ . We have  $\frac{a}{\sin \angle BAC} = \frac{CB}{\sin \angle BA'C} = A'B = 2\mathcal{R}$   $\square$

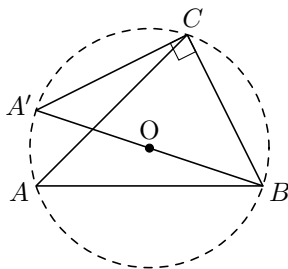


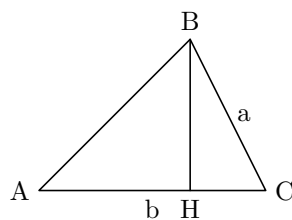
Figure 7: Proof of the Extended Law of Sine

### 3.3 Relationship between Circumradius and Area

Let  $[ABC]$  denotes the area of triangle.

**Theorem 3.3** (Area of a triangle).

$$[ABC] = \frac{1}{2}ab \sin C$$

Figure 8:  $[ABC] = \frac{1}{2}ab \sin C$ 

*Proof.* It suffices to see that  $BH = a \sin C$ ,  $AC = b$ . Note that the formula also works for  $C$  greater than  $90^\circ$  too. The proof is left as an exercise.  $\square$

**Theorem 3.4** (Circumradius and Area).

$$[ABC] = \frac{abc}{4\mathcal{R}}$$

*Proof.* It follows from the extended law of sine.  $\square$

### 3.4 Relationship between Circumradius and Side Lengths

Let  $s$  denotes the semi-perimeter of triangle, i.e.  $\frac{a+b+c}{2}$ . We state the Heron's formula without proof:

**Theorem 3.5** (Heron's formula). *In  $\triangle ABC$ , we have*

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$$

Together with the previous result, we can find the circumradius of a triangle if we know all 3 side lengths:

**Theorem 3.6** (Circumradius and Side Lengths).

$$\mathcal{R} = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

## 4 Cyclic Quadrilateral

Is it always possible to find a circle passing through a triangle? How about through a quadrilateral?

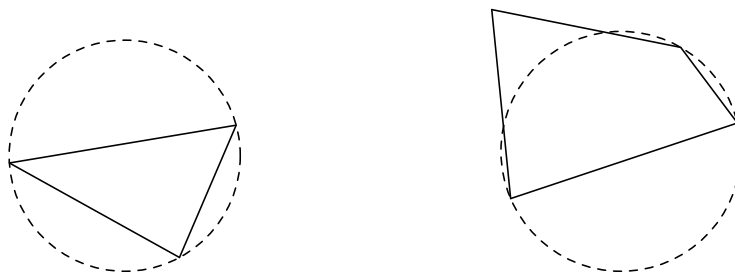
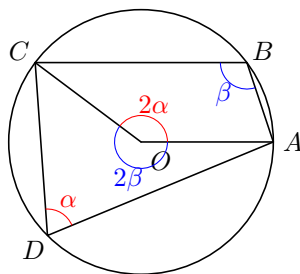


Figure 9: Finding a circle passing through a triangle/quadrilateral?

The answer to the first question is yes but the answer to the second question is no.

#### 4.1 Properties of Cyclic Quadrilateral

**Theorem 4.1** (Supplementary opposite angles). *Opposite angles inside a cyclic quadrilateral add up to  $180^\circ$ .*

Figure 10: Opposite angles add up to  $180^\circ$ 

*Proof.* Let  $\angle COA = 2\alpha$ , reflex  $\angle COA = 2\beta$ , then  $2\alpha + 2\beta = 360^\circ$ . This implies  $\alpha + \beta = 180^\circ$ . By Inscribed Angle Theorem, we have  $\angle CDA + \angle CBA = \alpha + \beta = 180$   $\square$

**Corollary 4.1.1.** *Exterior angle equals to the opposite interior angle inside a cyclic quadrilateral.*

**Theorem 4.2** (Angles subtended by the same arc). *Angles subtended by the same arc are equal.*

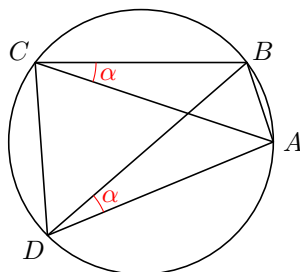


Figure 11: Angle subtended by the same arc are equal

*Proof.* Follows directly from Theorem 5. □

## 4.2 Test for Cyclic Quadrilateral

It turns out that the mentioned 3 properties are also tests for cyclic quadrilateral.

**Theorem 4.3** (Test for Cyclic Quadrilateral).

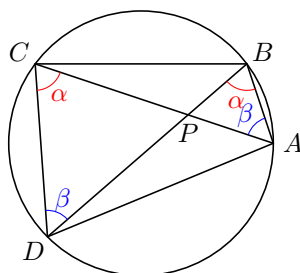
- *Opposite angles add up to  $180^\circ$ .*
- *Exterior angle equals the opposite interior angle.*
- *Angles subtended by the same **side** are equal.*

*This means that if **any** of the above 3 statements is true, then the quadrilateral is a cyclic quadrilateral.*

The proof is omitted here. Now you should have enough to solve the original problem. :)

## 5 Power Chord Theorem and its Converse

The key observation in this section is: if you have a point  $P$  inside a cyclic quadrilateral  $ABCD$ . We have  $\triangle PCD \sim \triangle PBA$ . This motivates the power chord theorem.

Figure 12:  $\triangle PCD \sim \triangle PBA$

**Theorem 5.1.** If you have a point  $P$  *inside* a circle and a chord  $XY$  passing through  $P$ , then  $PX \cdot PY$  is the same regardless of what chord  $XY$  is picked. The value  $PX \cdot PY$  is called the power of  $P$  with respect to the circle  $\omega$ , denoted  $\text{Pow}_\omega(P)$ .

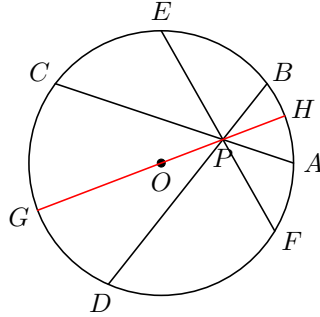


Figure 13:  $PC \cdot PA = PE \cdot PF = PB \cdot PD$

*Proof.* This follows from the similar triangles we just discovered. For example, note that  $\triangle PEC \sim \triangle PAF$  and  $\triangle PCG \sim \triangle PAF$ . The former set gives  $PE \cdot PF = PA \cdot PC$  and the later set gives  $PA \cdot PC = PG \cdot PH$ .  $\square$

**Theorem 5.2.** For any point  $P$  *inside*  $\omega$ .

$$\text{Pow}_\omega(P) = R^2 - OP^2$$

A different configuration motivates the power of a point  $P$  outside a circle. By convention, the power of such a point is taken to be negative. This will prove convenient when discussing the radical lemma.

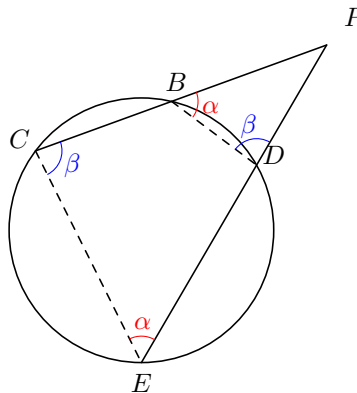


Figure 14:  $\triangle PBD \sim \triangle PEC$



**Theorem 5.3.** If you have a point  $P$  **outside** a circle and a chord  $XY$  passing through  $P$ , then  $PX \cdot PY$  is the same regardless of what chord  $XY$  is picked. The value  $-|PX \cdot PY|$  is called the power of  $P$  with respect to the circle  $\omega$ , denoted  $\text{Pow}_\omega(P)$ .

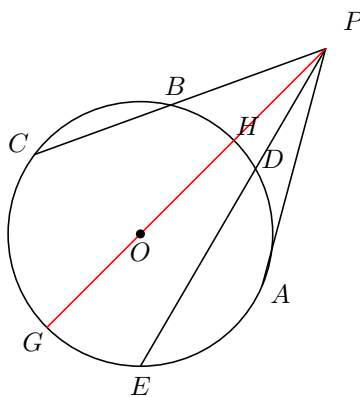


Figure 15:  $PB \cdot PC = PD \cdot PE = PA^2$

*Proof.* Similar to the one above. □

**Theorem 5.4.** For any point  $P$  **outside**  $\omega$ .

$$\text{Pow}_\omega(P) = \mathcal{R}^2 - OP^2$$

**Theorem 5.5.**  $\text{Pow}_\omega(P)$  **on** the circle  $\omega$  is defined to be 0. (Why does this makes sense?)

**Theorem 5.6.** For any point  $P$  on  $\omega$ .

$$\text{Pow}_\omega(P) = \mathcal{R}^2 - OP^2$$

Hence  $\text{Pow}_\omega(P) = \mathcal{R}^2 - OP^2$  always hold no matter whether  $P$  lies inside, on, or outside the circle, thanks to the power outside circle being negative.

### 5.1 Converse of Power Chord Theorem

In fact, the converse of power chord theorem is also true.

**Theorem 5.7** (Converse of the Power Chord Theorem). *Let  $A, B, X, Y$  be four distinct points in the plane and let lines  $AB$  and  $XY$  intersect at  $P$ . Suppose that either  $P$  lies in both of the segments  $\overline{AB}$  and  $\overline{XY}$ , or in neither segment. If  $PA \cdot PB = PX \cdot PY$ , then  $A, B, X, Y$  are concyclic.*

This serves as another test for cyclic quadrilateral. The proof is omitted here.

## 5.2 Another proof of the Pythagoras Theorem

Consider the following figure. We know that radius of a circle  $OA$  is perpendicular to the tangent through  $A$ .

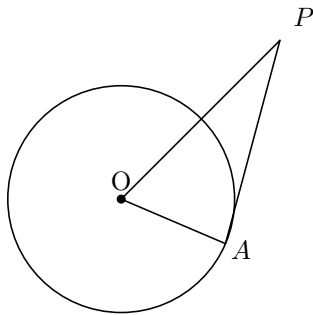


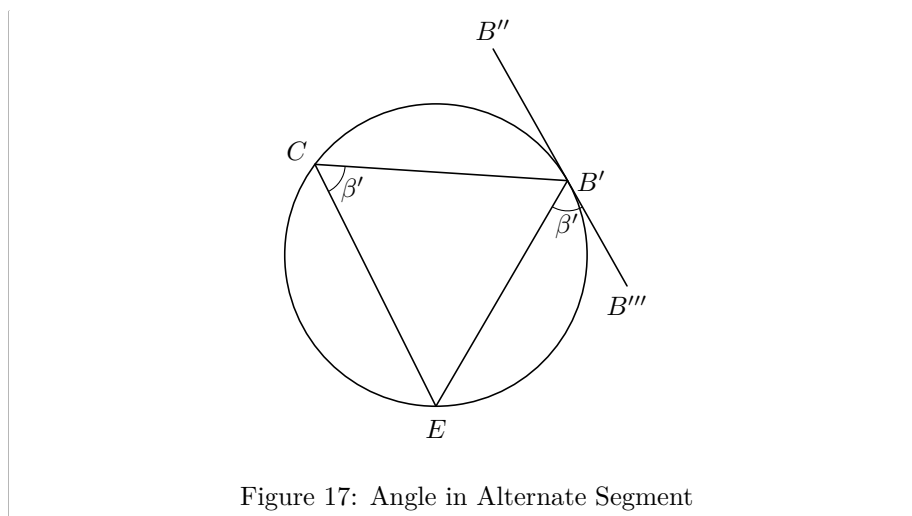
Figure 16: Another proof of the Pythagoras Theorem

Then two ways of calculating the power of  $P$  with respect to the circle:  $\text{Pow}_\omega(P) = R^2 - OP^2 = AP^2$  gives rise to the Pythagoras Theorem.

## 6 Properties of Tangent to Circle

### 6.1 Angle in Alternate Segment

**Theorem 6.1** (Angle in Alternate Segment). *In any circle, the angle between a chord and a tangent through one of the end points of the chord is equal to the angle in the alternate segment.*



*Proof.* Recall the configuration in Theorem 14. Imagine if  $B$  gets increasingly close to  $D$  at  $B'$  such that  $B'D$  becomes the tangent to the circle at  $D$ . Label the new line  $B''B'''$  as shown in the second figure. Then  $\angle B''DP = \angle B'CE$  since exterior angle of cyclic quadrilateral is the same as the opposite interior angle. Finally,  $\angle B''DP = \angle DEB'''$  concludes the proof.

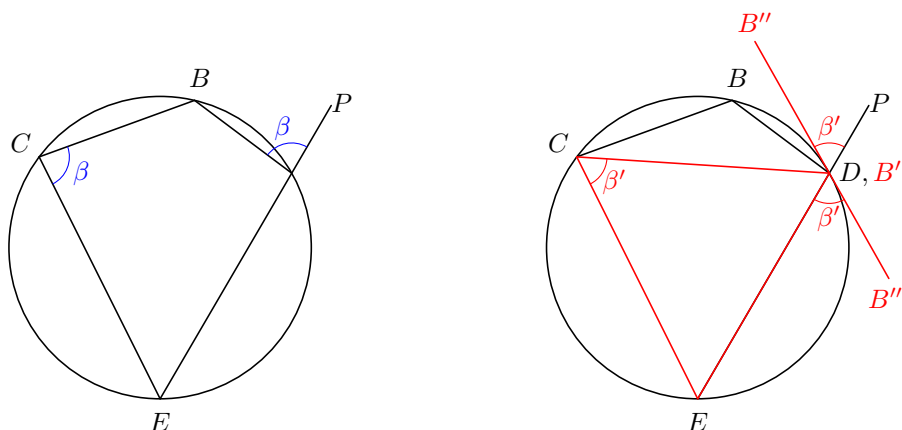


Figure 18: Proof of Angle in Alternate Segment

□

## 6.2 Other Properties of Tangents

**Theorem 6.2** (Equal Tangents). *The two tangents of a point outside the*

circle  $P$  to the circle has equal length.

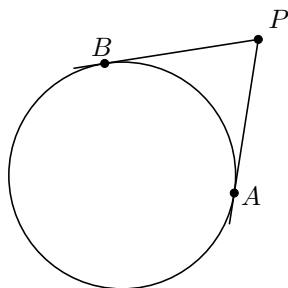


Figure 19: Equal Tangents

*Proof.*  $\text{Pow}_\omega(P) = PA^2 = PB^2$

□

**Theorem 6.3.** Let  $P$  be a point outside circle and  $A$  and  $B$  be the two points of tangency from  $P$  to the circle, then  $OABP$  is concyclic.

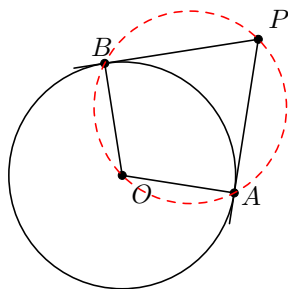


Figure 20: OABP is concyclic

*Proof.*  $\angle OBP + \angle OAP = 180^\circ$

□

## 7 Example Problems

**Question 1.** (Prelim 2020 Q6) In  $\triangle ABC$ ,  $AB = 6$ ,  $BC = 7$ ,  $CA = 8$ . Let  $D$  be the mid-point of minor arc  $AB$  on the circumcircle of  $\triangle ABC$ . Find  $AD^2$ .

**Question 2.** (Prelim 2022 Q16)  $ABCD$  is a parallelogram with  $\angle B$  acute. A circle is tangent to  $BC$ ,  $CD$  and  $DA$ . The circle intersects  $AC$  at  $M$  and  $N$ , where  $M$  is closer to  $A$  than  $N$ . If  $AM = 9$ ,  $MN = 16$  and  $NC = 2$ , find the area of  $ABCD$ .

## 8 Practice Problems

**Question 3.** (Prelim 2023 Q17)  $ABCD$  is a square.  $P$  is a point inside  $ABCD$  such that  $\angle APD + \angle BPC = 180^\circ$  and  $\angle BPC$  is acute. If  $PB = 3$  and  $PC = 4$ , find  $BC$ .

**Question 4.** (Prelim 2021 Q12)  $OABC$  is a trapezium with  $OC \parallel AB$  and  $\angle AOB = 37^\circ$ . Furthermore,  $A, B, C$  all lie on the circumference of a circle centered at  $O$ . The perpendicular bisector of  $OC$  meets  $AC$  at  $D$ .

**Question 5.** (Prelim 2018 Q13) Let  $O$  be the circumcentre of  $\triangle ABC$ . Suppose  $AB = 1$  and  $AO = AC = 2$ .  $D$  and  $E$  are points on the extensions of  $AB$  and  $AC$  respectively such that  $OD = OE$  and  $BD = \sqrt{2}EC$ . Find the value of  $OD^2$ .

**Question 6.** (Prelim 2018 Q16)  $ABCD$  is a cyclic quadrilateral with  $AC = 56$ ,  $BD = 65$ ,  $BC > DA$  and  $\frac{AB}{BC} = \frac{CD}{DA}$ . Find the ratio of the area of  $\triangle ABC$  to the area of  $\triangle ADC$ .

**Question 7.** (Prelim 2016 Q20) In  $\triangle ABC$ ,  $P$  and  $Q$  are points on  $AB$  and  $AC$  respectively such that  $AP : PB = 8 : 1$  and  $AQ : QC = 15 : 1$ .  $X$  and  $Y$  are points on  $BC$  such that the circumcircle of  $\triangle APX$  is tangent to both  $BC$  and  $CA$ , while the circumcircle of  $\triangle AQY$  is tangent to both  $AB$  and  $BC$ . Find  $\cos BAC$ .

**Question 8.** (Prelim 2012 Q13)  $ABCD$  is a convex quadrilateral in which  $AC$  and  $BD$  meet at  $P$ . Given  $PA = 1$ ,  $PB = 2$ ,  $PC = 6$  and  $PD = 3$ . Let  $O$  be the circumcentre of  $\triangle PBC$ . If  $OA$  is perpendicular to  $AD$ , find the circumradius of  $\triangle PBC$ .