Introduction to Euclidean Geometry

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In the vast tapestry of mathematical exploration, few realms evoke a sense of elegance and timeless beauty quite like Euclidean geometry. With its origins deeply rooted in ancient Greece, this branch of mathematics has captivated the hearts and minds of scholars and thinkers for centuries.

1 Motivation

Given

- $\angle DAC = 30°$
- $\angle CDB = 40°$
- $\angle ABD = 50°$
- $DB \perp AC$

What angles can you find?

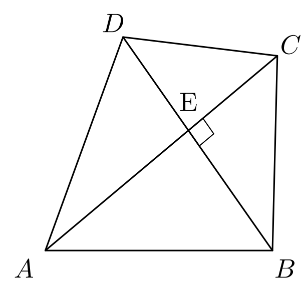

Figure 1: Find the angles!

2 Circle

2.1 Inscribed Angle Theorem

Theorem 1 (Inscribed Angle Theorem)**.** *Let O denotes the center of circle, A and B be any two points on the circle, then $\angle AOB = 2\angle ACB$.*

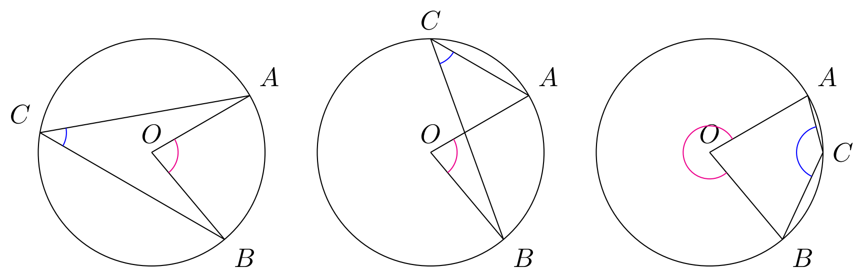

Figure 2: Inscribed Angle Theorem

Proof.
For the first case, we have $OA = OC = OB$. Let $\angle OCB = \angle OBC = a$ and $\angle OCB = \angle OBC = b$, then $\angle ACB = a + b$ and $\angle AOB = 2a + 2b$.
For the second case, we have $OA = OC = OB$. Let $\angle OCB = \angle OBC = a$ and $\angle OCB = \angle OBC = b$, then $\angle ACB = b - a$ and $\angle AOB = 2b - 2a$.
For the third case, we have $OA = OC = OB$. Let $\angle OCB = \angle OBC = a$ and $\angle OCB = \angle OBC = b$, then $\angle ACB = a + b$ and reflex $\angle AOB = 2a + 2b$.

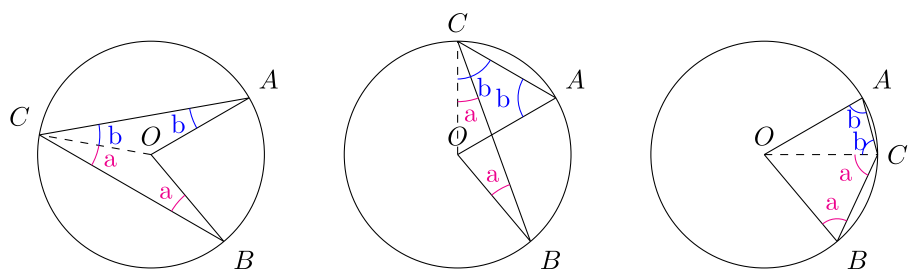

Figure 3: Proof of Inscribed Angle Theorem

□

Corollary 1.1 (angle in semi-circle)**.** *Let AB be a diameter of circle, C be any point on a circle. Then $\angle ACB = 90^o$ (because the angle at centre is 180^o).*

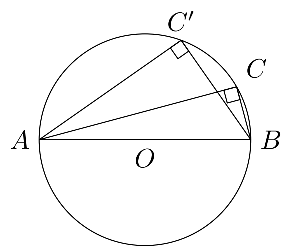

Figure 4: Angle in semi-circle

Corollary 1.2 (angle at circumference $\propto$ arc length)**.** *The arc is proportional to the angle at circumference (center).*

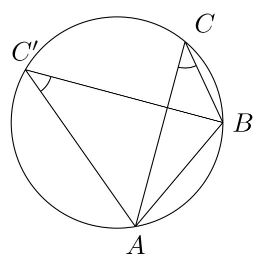

Figure 5: Angle at circumference $\propto$ arc length

2.2 The Extended Law of Sine

By convention, in ΔABC, the opposite side to angle A is named a (similarly for B and C), $\mathcal{R}$ denotes the circumradius of ΔABC, and r denotes the inradius of ΔABC.

Theorem 2 (the Extended Law of Sine)**.** *Given a triangle ABC, we have*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2\mathcal{R}$$

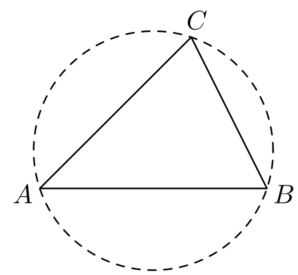

Figure 6: the Extended Law of Sine

Proof. Without loss of generality, we only prove $\frac{a}{\sin A} = 2\mathcal{R}$. Move A to A' such that $A'B$ is the diameter of the circle. Note that $\triangle ACB$ is a right-angled triangle and $\angle BA'C = \angle BAC$. We have $\frac{a}{\sin BAC} = \frac{CB}{\sin \angle BA'C} = A'B = 2\mathcal{R}$ $\square$

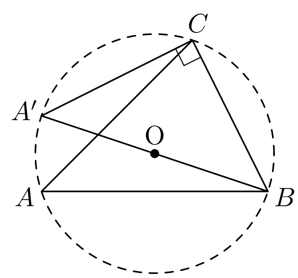

Figure 7: Proof of the Extended Law of Sine

2.3 Relationship between Circumradius and Area

Let $[ABC]$ denotes the area of triangle.

Theorem 3 (Area of a triangle)**.**

$$[ABC] = \frac{1}{2}ab\sin C$$

.

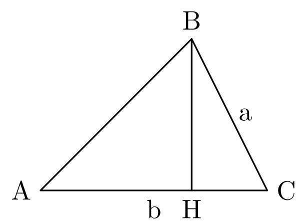

Figure 8: $[ABC] = \frac{1}{2}ab\sin C$

Proof. It suffices to see that $BH = a\sin C$, $AC = b$. Note that the formula also works for C greater than 90^o too. The proof is left as an exercise. $\square$

Theorem 4 (Circumradius and Area)**.**

$$[ABC] = \frac{abc}{4\mathcal{R}}$$

Proof. It follows from the extended law of sine. $\square$

2.4 Relationship between Circumradius and Side Lengths

Let s denotes the semi-perimeter of triangle, i.e. $\frac{a+b+c}{2}$. We state the Heron's formula without proof:

Theorem 5 (Heron's formula)**.** *In $\triangle ABC$, we have*

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$$

.

Together with the previous result, we can find the circumradius of a triangle if we know all 3 side lengths:

Theorem 6 (Circumradius and Side Lengths)**.**

$$\mathcal{R} = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

3 Cyclic Quadrilateral

Is it always possible to find a circle passing through a triangle? How about through a quadrilateral?

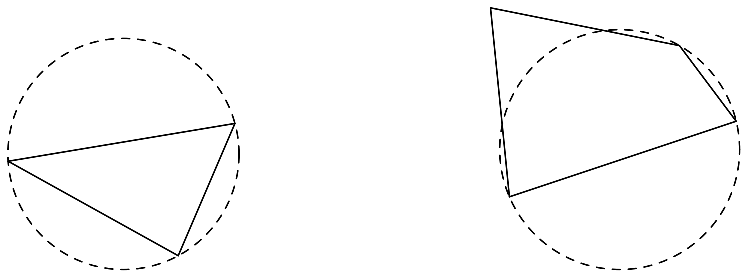

Figure 9: Finding a circle passing through a triangle/quadrilateral?

The answer to the first question is yes but the answer to the second question is no.

3.1 Properties of Cyclic Quadrilateral

Theorem 7 (Supplementary opposite angles)**.** *Opposite angles inside a cyclic quadrilateral adds up to* 180^o*.*

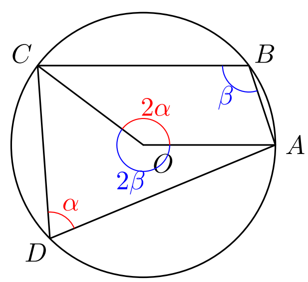

Figure 10: Opposite angles add up to 180^o

Proof. Let $\angle COA = 2\alpha$, reflex $\angle COA = 2\beta$, then $2\alpha + 2\beta = 360^o$. This implies $\alpha + \beta = 180^o$. By Inscribed Angle Theorem, we have $\angle CDA + \angle CBA = \alpha + \beta =$ 180 □

Corollary 7.1. *Exterior angle equals to the opposite interior angle inside a cyclic quadrilateral.*

Theorem 8 (Angles subtended by the same arc)**.** *Angles subtended by the same arc are equal.*

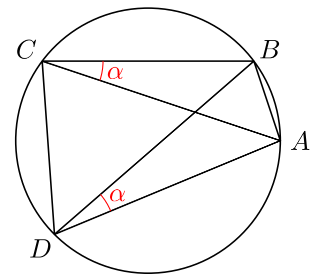

Figure 11: Angle subtended by the same arc are equal

Proof. Follows directly form Theorem 5. □

3.2 Test for Cyclic Quadrilateral

It turns out that the mentioned 3 properties are also tests for cyclic quadrilateral.

Theorem 9 (Test for Cyclic Quadrilateral)**.**

- *Opposite angles adds up to* 180^o.
- *Exterior angle equals the opposite interior angle.*
- *Angles subtended by the same* ***side*** *are equal.*

This means that if ***any*** *of the above* 3 *statement is true, then the quadrilateral is a cyclic quadrilateral.*

The proof is omitted here. Now you should have enough to solve the original problem. :)

4 Power Chord Theorem and its Converse

The key observation in this section is: if you have a point P inside a cyclic quadrilateral $ABCD$. We have $\triangle PCD \sim \triangle PBA$. This motivates the power chord theorem.

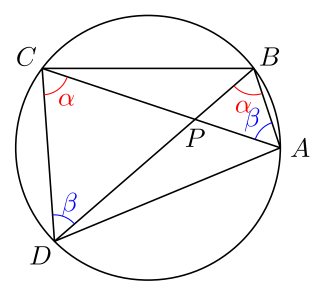

Figure 12: $\triangle PCD \sim \triangle PBA$

Theorem 10. *If you have a point P **inside** a circle and a chord XY passing through P, then $PX \cdot PY$ is the same regardless of what chord XY is picked. The value $PX \cdot PY$ is called the power of P with respect to the circle ω, denoted* $\text{Pow}_\omega(P)$.

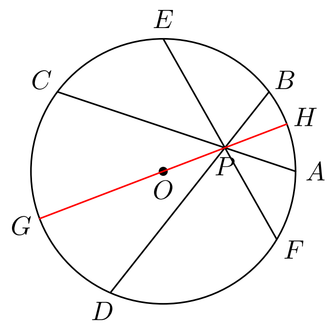

Figure 13: $PC \cdot PA = PE \cdot PF = PB \cdot PD$

Proof. This follows from the similar triangles we just discovered. For example, note that $\triangle PEC \sim PAF$ and $\triangle PCG \sim \triangle PAF$. The former set gives $PE \cdot PF = PA \cdot PC$ and the later set gives $PA \cdot PC = PG \cdot PH$. $\square$

Theorem 11. *For any point P **inside** ω.*

$$\text{Pow}_\omega(P) = \mathcal{R}^2 - OP^2$$

A different configuration motivates the power of a point P outside a circle. By convention, the power of such a point is taken to be negative. This will prove convenient when discussing the radical lemma.

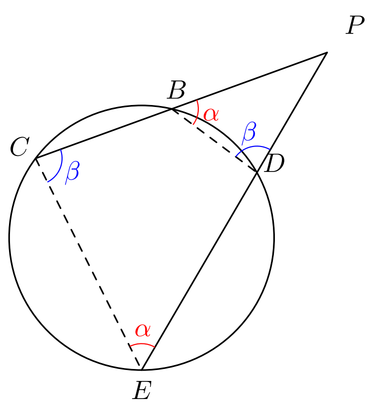

Figure 14: $\triangle PBD \sim \triangle PEC$

Theorem 12. *If you have a point P* ***outside*** *a circle and a chord XY passing through P, then $PX \cdot PY$ is the same regardless of what chord XY is picked. The value $-|PX \cdot PY|$ is called the power of P with respect to the circle ω, denoted* $\mathrm{Pow}_\omega(P)$.

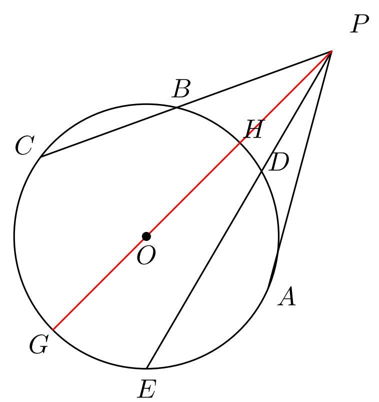

Figure 15: $PB \cdot PC = PD \cdot PE = PA^2$

Proof. Similar to the one above. □

Theorem 13. *For any point P* ***outside*** *ω.*

$$\mathrm{Pow}_\omega(P) = \mathcal{R}^2 - OP^2$$

Theorem 14. $\mathrm{Pow}_\omega(P)$ ***on*** *the circle ω is defined to be* 0*. (Why does this makes sense?)*

Theorem 15. *For any point P on ω.*

$$\mathrm{Pow}_\omega(P) = \mathcal{R}^2 - OP^2$$

Hence $\mathrm{Pow}_\omega(P) = \mathcal{R}^2 - OP^2$ always hold no matter whether P lies inside, on, or outside the circle, thanks to the power outside circle being negative.

4.1 Converse of Power Chord Theorem

In fact, the converse of power chord theorem is also true.

Theorem 16 (Converse of the Power Chord Theorem)**.** *Let A, B, X, Y be four distinct points in the plane and let lines AB and XY intersect at P. Suppose that either P lies in both of the segments $\overline{AB}$ and $\overline{XY}$, or in neither segment. If $PA \cdot PB = PX \cdot PY$, then A, B, X, Y are concyclic.*

This serves as another test for cyclic quadrilateral. The proof is omitted here.

4.2 Another proof of the Pythagoras Theorem

Consider the following figure. We know that radius of a circle OA is perpendicular to the tangent through A.

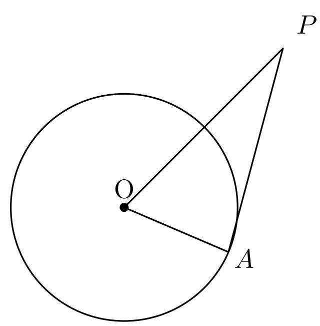

Figure 16: Another proof of the Pythagoras Theorem

Then two ways of calculating the power of P with respect to the circle: $\mathrm{Pow}_\omega(P) = \mathcal{R}^2 - OP^2 = AP^2$ gives rise to the Pythagoras Theorem.

5 Properties of Tangent to Circle

5.1 Angle in Alternate Segment

Theorem 17 (Angle in Alternate Segment)**.** *In any circle, the angle between a chord and a tangent through one of the end points of the chord is equal to the angle in the alternate segment.*

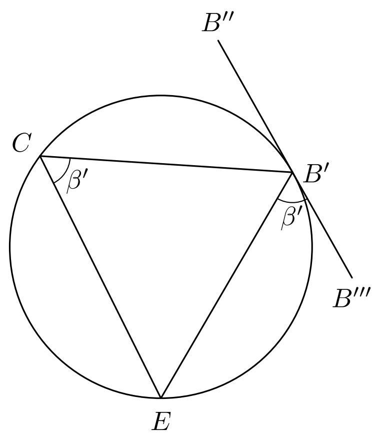

Figure 17: Angle in Alternate Segment

Proof. Recall the configuration in Theorem 14. Imagine if B gets increasingly close to D at B' such that $B'D$ becomes the tangent to the circle at D. Label the new line $B''B'''$ as shown in the second figure. Then $\angle B''DP = \angle B'CE$ since exterior angle of cyclic quadrilateral is the same as the opposite interior angle. Finally, $\angle B''DP = \angle DEB'''$ concludes the proof.

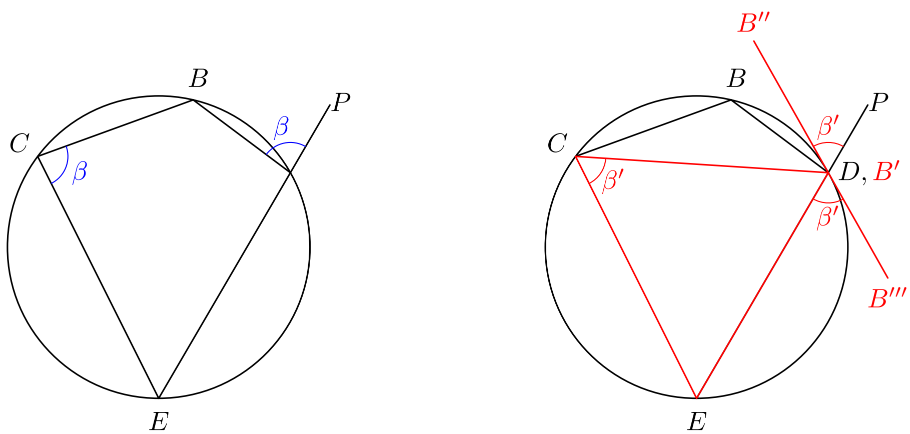

Figure 18: Proof of Angle in Alternate Segment

□

5.2 Other Properties of Tangents

Theorem 18 (Equal Tangents). *The two tangents of a point outside the* *circle P to the circle has equal length.*

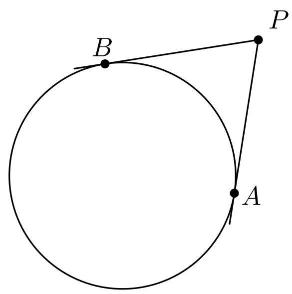

Figure 19: Equal Tangents

Proof. $\text{Pow}_{\omega} = PA^2 = PB^2$ □

Theorem 19. *Let P be a point outside circle and A and B be the two points of tangency from P to the circle, then $OABP$ is concyclic.*

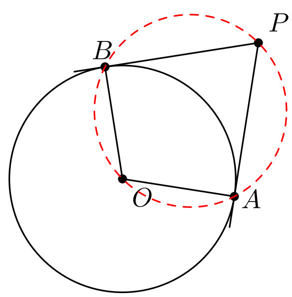

Figure 20: OABP is concyclic

Proof. $\angle OBP + \angle OAP = 180^o$ □

6 Example Problems

Question 1. (Prelim 2020 Q6) In $\triangle ABC, AB = 6, BC = 7, CA = 8$. Let D be the mid-point of minor arc AB on the circumcentre of $\triangle ABC$. Find AD^2.

Question 2. (Prelim 2022 Q16) $ABCD$ is a parallelogram with $\angle B$ acute. A circle is tangent to BC, CD and DA. The circle intersects AC at M and N, where M is closer to A than N. If $AM = 9$, $MN = 16$ and $NC = 2$, find the area of $ABCD$.

7 Practice Problems

Question 3. (Prelim 2023 Q17) $ABCD$ is a square. P is a point inside $ABCD$ such that $\angle APD + \angle BPC = 180^o$ and $\angle BPC$ is acute. If $PB = 3$ and $PC = 4$, find BC.

Question 4. (Prelim 2021 Q12) $OABC$ is a trapezium with $OC \parallel AB$ and $\angle AOB = 37^o$. Furthermore, A, B, C all lie on the circumference of a circle centered at O. The perpendicular bisector of OC meets AC at D.

Question 5. (Prelim 2018 Q13) Let O be the circumcentre of $\triangle ABC$. Suppose $AB = 1$ and $AO = AC = 2$. D and E are points on the extensions of AB and AC respectively such that $OD = OE$ and $BD = \sqrt{2}EC$. Find the value of OD^2.

Question 6. (Prelim 2018 Q16) $ABCD$ is a cyclic quadrilateral with $AC = 56$, $BD = 65$, $BC > DA$ and $\frac{AB}{BC} = \frac{CD}{DA}$. Find the ratio of the area of $\triangle ABC$ to the area of $\triangle ADC$.

Question 7. (Prelim 2016 Q20) In $\triangle ABC$, P and Q are points on AB and AC respectively such that $AP : PB = 8 : 1$ and $AQ : QC = 15 : 1$. X and Y are points on BC such that the circumcircle of $\triangle APX$ is tangent to both BC and CA, while the circumcircle of $\triangle AQY$ is tangent to both AB and BC. Find $\cos BAC$.