MATH 4.1ER Assignment 1
Number System

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Answer the questions in the spaces provided on the question sheets. If you do not know how to answer a certain question, write down where you get stuck. Answers can be corrected to 3 significant figures if necessary.

Name, class, class no.: ____________________

Tutor's name: ____________________

1 Natural numbers and Integers

Definition: Natural numbers $\mathbb{N}$

Natural numbers contain $1, 2, 3, \ldots$ but *does not* contain 0.

Definition: Integers $\mathbb{Z}$

Integers include positive integers (natural numbers), 0 (zero) and negative integers.

i.e., $\overbrace{\ldots, -3, -2, -1}^{\text{negative integers}}, \overbrace{0}^{\text{zero}}, \overbrace{1, 2, 3, \ldots}^{\text{positive integers}}$

Determine whether the following statements concerning integers are true. If they are not true, suggest a counterexample.

1. ____ If we pick any two distinct natural numbers a and b, there must be an natural number c satisfying $a < c < b$.
2. ____ If we pick any two natural numbers a and b, $a + b$ must also be a natural number.
3. ____ If we pick any two natural numbers a and b, $a - b$ must also be a natural number.
4. ____ If we pick any two natural numbers a and b, $a \times b$ must also be a natural number.
5. ____ If we pick any two natural numbers a and b, $\frac{a}{b}$ must also be a natural number.
6. ____ If we pick any two integers a and b, $a + b$ must also be an integer.
7. ____ If we pick any two integers a and b, $a - b$ must also be an integer.
8. ____ If we pick any two integers a and b, $a \times b$ must also be an integer.
9. ____ If we pick any two integers a and b, $\frac{a}{b}$ must also be an integer.

2 Rational numbers

Definition: Rational numbers $\mathbb{Q}$

A number that can be expressed as a ratio of two integers *i.e.*, $\frac{p}{q}$, where p and q are integers with $q \neq 0$, is called a rational number. e.g., $\frac{1}{4}, -1 - \frac{2}{3}, \dots$

Determine whether the following statements concerning rational numbers are true. If they are not true, suggest a counterexample.

10. ____ If we pick any two distinct rational numbers a and b, there must be a rational number c satisfying $a < c < b$.
11. ____ If we pick any two rational numbers a and b, $a + b$ must also be a rational number.
12. ____ If we pick any two rational numbers a and b, $a - b$ must also be a rational number.
13. ____ If we pick any two rational numbers a and b, $a \times b$ must also be a rational number.
14. ____ If we pick any two rational numbers a and b, $\frac{a}{b}$ must also be a rational number.

Common mistake: Dividing by 0

For any number n (belonging to any of the following sets $\mathbb{N}/\mathbb{Z}/\mathbb{Q}/\mathbb{R}/\mathbb{C}$), $\frac{n}{0}$ is **always undefined**! ($\frac{0}{0}$ is also undefined).

2.1 Recurring decimals

Definition: Non-terminating/terminating decimals

A **non-terminating decimals** is a number with a **non-ending** decimals. For example,

- $\pi = 3.14159265358\dots$
- $\frac{1}{3} = 0.33333333333\dots$
- $\frac{1}{7} = 0.14285714285\dots$
- $\sqrt{2} = 1.4142135623\dots$

On the other hand, a **terminating decimals** is a number with a decimal part that can be represented with a **finite** number of digits. For example,

- $\frac{1}{5} = 0.2$
- $\sqrt{1.21} = 1.1$
- 2.1234

15. ____ 0.2357 is a terminating-decimal.
16. ____ π is a terminating-decimal.
17. ____ The difference of two terminating decimal must also be a terminating decimal or an integer.
18. ____ The difference of two non-terminating decimal must also be a non-terminating decimal or an integer.

19. ____ The solution(s) to $x^2 - 2 = 0$ are rational numbers.

20. Convert 0.12345 into a fraction.

20. ______________

Definition: Recurring decimals

The notation $0.\overline{12345}$ (or $0.\dot{1}234\dot{5}$) denotes a **recurring decimal** $0.123451234512345\ldots$ repeating indefinitely. The decimal is **non-terminating** and **recurring** (meaning repeating the same sequence of numbers again and again). We can show:

$$\begin{aligned} x &= 0.\overline{12345} \\ 100000x &= 12345.\overline{12345} \\ 100000x &= 12345 + 0.\overline{12345} \\ 100000x &= 12345 + x \\ 99999x &= 12345 \\ x &= \tfrac{12345}{99999} \\ &= \tfrac{4115}{33333} \end{aligned}$$

21. Write down the number $1.\overline{3}$ up to 6 decimal places.

21. ______________

22. Write down the number $12.34\overline{56}$ in 10 significant figures.

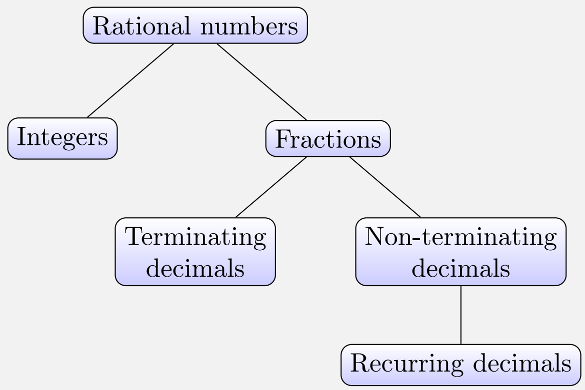
22. ______________

23. Write down the number $0.\overline{2}$ as a fraction by using the method illustrated above.

24. Write down the number $12.34\overline{56}$ as a fraction by using the method illustrated above.

Note: Classification of rational numbers

All integers, terminating decimal, and recurring decimals are rational numbers. This is because all of them can be written in $\frac{p}{q}$, where p and q are integers and $q \neq 0$. Check q.19, q.22, q.23 if you feel uncertain about this.
The figure below should help you understand this subject better.



Determine whether the following statements concerning irrational numbers are true.

25. ____ Given a rational number n is a non-terminating decimal, it must also be a recurring decimal.

26. ____ All integers are rational numbers.

27. ____ $0.\overline{314}$ is a rational number.

28. ____ $\frac{\sqrt{9}}{2}$ is a rational number.

29. ____ If $\frac{p}{q}$ is rational, then p and q are both rational numbers.

3 Irrational numbers

Definition: Irrational numbers

Irrational numbers are numbers that cannot be expressed as a ratio of two integers (i.e. $\frac{p}{q}$, where p and q are integers with $q \neq 0$) is called an irrational number.

Determine whether the following statements concerning irrational numbers are true. If they are not true, suggest a counterexample.

30. ____ A real number is either rational or irrational.

31. ____ If we pick any two irrational numbers a and b, $a + b$ must also be a irrational number.

32. ____ If we pick any two irrational numbers a and b, $a - b$ must also be a irrational number.

33. ____ If we pick any two irrational numbers a and b, $a \times b$ must also be a irrational number.

34. ____ If we pick any two irrational numbers a and b, $\frac{a}{b}$ must also be a irrational number.

35. ____ At least one of $\pi + e$ or $\pi - e$ is irrational.

36. ____ $\pi = \frac{22}{7}$

37. ____ $0.\overline{3}$ is irrational.

38. ____ $1.\overline{2345}$ is irrational.

39. ____ 0 is irrational.

Note: Classification of irrational numbers

All irrational numbers can be expressed as non-terminating and non-recurring decimals.

40. Put tick in the appropriate places in the table.

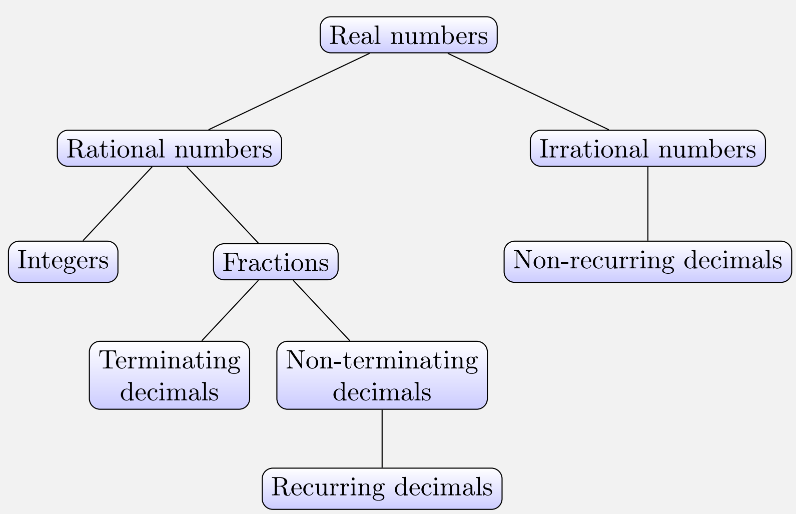
	4	0	-6.2	$\sqrt{15}$	$1.\overline{701}$	$\frac{\pi}{3}$	$-\sqrt{9}$
Natural Number							
Integer							
Terminating decimal							
Recurring decimal							
Irrational number							

4 Real numbers $\mathbb{R}$

Definition: Real numbers $\mathbb{R}$

Real numbers include all rational and irrational numbers. Each real number can be represented by a point on a number line.

Note: Classification of real numbers



Determine whether the following statement is correct

41. ____ The solution(s) to $x^2 + 1 = 0$ are real numbers.

42. ____ If n is real number, n must be either integers, terminating decimals or recurring decimals.

5 Complex numbers $\mathbb{C}$

By adding real numbers with imaginary numbers, we can extend the imaginary numbers further.

Definition: Complex numbers $\mathbb{C}$

A number that can be written in the form $a + bi$ is called a **complex number**, where a and b are real numbers with $i = \sqrt{-1}$.

Note: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

The notation $A \subset B$ means that B contains A. Note that the set of complex number ($\mathbb{C}$) contains all numbers.

The number $\sqrt{-1}$ is denoted by i, which is called the **imaginary unit**. Hence $i^2 = -1$ and we say that the equation $x^2 + 1 = 0$ has two solutions $x = \pm i$.

If N is a positive real number, then it is defined that

$$\sqrt{-N} = \sqrt{N}i$$

For example,

- $\sqrt{-2} = \sqrt{2}i$
- $\sqrt{-4} = \sqrt{4}i = 2i$
- $\sqrt{-9} = \sqrt{9}i = -3i$

Common mistake: hi

It is incorrect to say that $\sqrt{-9} = \sqrt{-1}\sqrt{9} = 3i$, even though the result is correct. This is because $\sqrt{AB} = \sqrt{A}\sqrt{B}$ is only valid for positive real numbers A and B.
If the theorem extended to negative real numbers A and B, consider

$$\begin{aligned} \sqrt{9} &= \sqrt{(-3)(-3)} && \text{(valid)} \\ &= \sqrt{-3}\sqrt{-3} && (\textbf{invalid!}) \\ &= \sqrt{3}i\sqrt{3}i && \text{(valid)} \\ &= 3i^2 && \text{(valid)} \\ &= -3 && \text{(valid)} \end{aligned}$$

which concludes a contradiction $\sqrt{9} = -3$. Hence, the new theorem cannot be correct. Hence, the rule that $\sqrt{-N} = \sqrt{N}i$ is what we invoked to say $\sqrt{-2} = \sqrt{2}i$, but not $\sqrt{AB} = \sqrt{A}\sqrt{B}$.

Express each of the following in the form bi, where b is a real number.

43. $\sqrt{-3}$

43. ______________

44. $\sqrt{-16}$

44. ______________

45. $-\sqrt{-21}$

45. ______________

46. $-\sqrt{-25}$

46. ______________

Determine whether the following statement is true.

47. ____ It is always true that $\sqrt{A}\sqrt{B} = \sqrt{AB}$ if A and B are real numbers.

5.1 Real part and imaginary part

Definition: Real part and imaginary part

For a complex number $a + bi$, where a and b are real numbers, a is called the **real part** and b is called the **imaginary part**.

Definition: Purely imaginary number

For a complex number $a + bi$, where a and b are real numbers, when $a = 0$, the complex number becomes bi. We call such complex number **purely imaginary**.

Identify the real part and the imaginary part of each of the following complex numbers. Separate your answer with a semicolon.

48. $1 - 3i$

48. ______________

49. $8 - \pi i$

49. ______________

50. 9

50. ______________

51. $\sqrt{-4} - 8$

51. ______________

52. $6i - \sqrt{-3}$

52. ______________

5.2 Operations of Complex Numbers

The usual arithmetic operations with imaginary numbers are similar to that of monomials. You may treat i as a variable first and apply the definition of $i = \sqrt{-1}$ later in the process.

Note: Division of complex numbers

For complex number of the form $\dfrac{a+bi}{c+di}$, you can simplify it by multiplying it by $\dfrac{c-di}{c-di}$

53. Simplify the following expressions

(a) $2i \times 6i$

(b) $(5i+6i)(5i-6i)$

(c) i^{64}

(d) $\dfrac{18i^6}{6i^2}$

(e) $\sqrt{-25}-\sqrt{-16}$

(f) $(6+i)^2$

(g) $\dfrac{12+9i}{3}$

(h) $\dfrac{1}{2+2i}$

(i) $\dfrac{1+5i}{2-3i}$

(j) $\dfrac{2-i}{i+3}-\dfrac{i-3}{2+i}$

(k) $i(6+2i)^2$

Note: a+bi

For real numbers a, b, c, d, if $a + bi = c + di$, then $a = c$ and $b = d$

54. If $x(3 - i) + (y - xi) = -3 + 4i$, find the values of the real numbers x and y.

55. Draw a tree that relates complex numbers, real numbers, rational numbers, integers, positive integers, zero, and negative integers. Add suitable categories if needed to make your tree complete. Give examples to each category.

56. If $2a + 3i - 2\left(\frac{2 + bi}{1 + i}\right) = (a + 1) + 4i$, find the values of the real numbers a and b.