<insert class>
Length and Ratio

T Yeung

<insert date>

1 Trigonometric Identities

Trigonometric identities simplify complex expressions.

$$\begin{aligned} 1 &= \sin^2\theta + \cos^2\theta \\ \sin(-\theta) &= -\sin\theta \\ \cos(-\theta) &= \cos\theta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha \\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{aligned}$$

By replacing $+$ with $-$ in $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$, you get the other two identity. The four identities are collectively called the *compound angle formula*. The proof of them is based on geometric construction.

By summing up the compound angle formulas in different ways, we obtain the product-to-sum identities.

$$\begin{aligned} 2\cos\alpha\cos\beta &= \cos(\alpha-\beta) + \cos(\alpha+\beta) \\ 2\sin\alpha\sin\beta &= \cos(\alpha-\beta) - \cos(\alpha+\beta) \\ 2\sin\alpha\cos\beta &= \sin(\alpha-\beta) + \sin(\alpha+\beta) \end{aligned}$$

You can also derive the sum-to-product formulas by replacing α with $\frac{\alpha+\beta}{2}$ and β with $\frac{\alpha-\beta}{2}$ in the above formulas.

$$\begin{aligned} \cos(a)\cos(b) &= \frac{1}{2}(\cos(a+b) + \cos(a-b) \\ \sin(a)\sin(b) &= \frac{1}{2}(\cos(a-b) - \cos(a+b)) \\ \sin(a)\cos(b) &= \frac{1}{2}(\sin(a+b) + \sin(a-b)) \end{aligned}$$

2 The Extended Law Of Sine

Recall the Extended Law of Sine we derived:

Theorem 2.1 (The Extended Law of Sine)**.** *Given a triangle ABC, we have*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2\mathcal{R}$$

We can use this result to prove the Ptolemy's Theorem.

Theorem 2.2 (Ptolemy's Theorem)**.** *Let $ABCD$ be a cyclic quadrilateral. Then*

$$AB \cdot CD + BC \cdot DA = AC \cdot BD$$

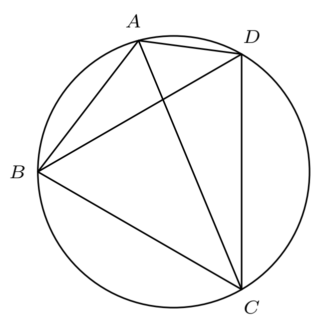


Figure 1: Statement of Ptolemy's Theorem

Proof. Without loss of generality, we set $\mathcal{R} = \frac{1}{2}$ as the radius of $(ABCD)$. Let $AB = \sin\alpha_1$, $BC = \sin\alpha_2$, $CD = \sin\alpha_3$, $DA = \sin\alpha_4$, $AC = \sin\angle ABC = \sin(\alpha_3 + \alpha_4)$, $BD = \sin\angle DAB = \sin(\alpha_2 + \alpha_3)$. Note that by product-to-sum identities, we have

$$\begin{aligned}
\sin\alpha_1 \sin\alpha_3 &= \frac{1}{2}(\cos(\alpha_1 - \alpha_3) - \cos(\alpha_1 + \alpha_3)) \\
\sin\alpha_2 \sin\alpha_4 &= \frac{1}{2}(\cos(\alpha_2 - \alpha_4) - \cos(\alpha_2 + \alpha_4)) \\
\sin(\alpha_2 + \alpha_3)\sin(\alpha_3 + \alpha_4) &= \frac{1}{2}(\cos(\alpha_2 - \alpha_4) - \cos(\alpha_2 + 2\alpha_3 + \alpha_4))
\end{aligned}$$

Since $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 180^o$, we also have

$$\cos(\alpha_1 + \alpha_3) + \cos(\alpha_2 + \alpha_4) = 0$$

Also note that

$$\cos(\alpha_2 + 2\alpha_3 + \alpha_4) = \cos(180^o - \alpha_1 + \alpha_3) = -\cos(\alpha_1 - \alpha_3)$$

The rest is trivial. :)

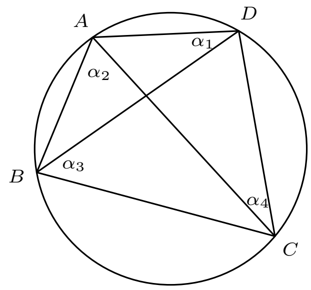


Figure 2: Proof of Ptolemy's Theorem

□

Corollary 2.2.1 (Stewart's Theorem)**.** *Let ABC be a triangle. Let D be a point on $\overline{BC}$ and let $m = DB$, $n = DC$, $d = AD$. Then*

$$a(d^2 + mn) = b^2m + c^2n$$

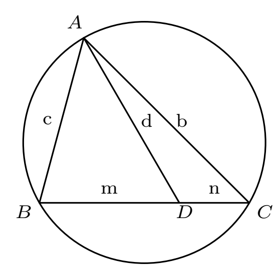


Figure 3: Stewart's Theorem

Often this is written in the form

$$man + dad = bmb + cnc$$

as a mnemonic - "a *man* and his *dad* put a *bomb* in the *sink*".

Proof. Let AD meet (ABC) again at P. By similar triangle, we have $\frac{BP}{m} = \frac{b}{d}$ and $\frac{CP}{n} = \frac{c}{d}$. By Power Chord Theorem, we know that $DP = \frac{mn}{d}$. By Ptolemy's Theorem, we have $BC{\cdot}AP = AC{\cdot}BP + AB{\cdot}CP$. Hence, $a{\cdot}(d + \frac{mn}{d}) = b{\cdot}\frac{bm}{d} + c{\cdot}\frac{cn}{d}$ which is the Stewart's Theorem.

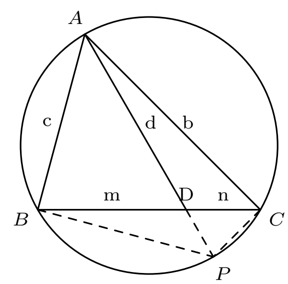


Figure 4: Proof of Stewart's Theorem

□

Take a look at the following problem for another application of the Extended Law of Sine.

Question 1. (Prelim 2019 Q13) A, B, C are three points on a circle while P and Q are two points on AB. The extensions of CP and CQ meet the circle at S and T respectively. If $AP = 2$, $AQ = 7$, $AB = 11$, $AS = 5$ and $BT = 2$, find the length of ST.

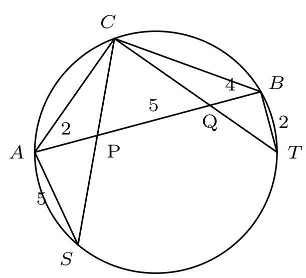
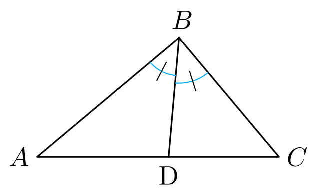


Figure 5: A not-to-scale diagram that is not provided in the contest

3 Angle Bisector Theorem

Theorem 3.1 (Angle Bisector Theorem)**.** *Let BD be the angle bisector of $\angle ABC$, then $AB : BC = AD : DC$.*



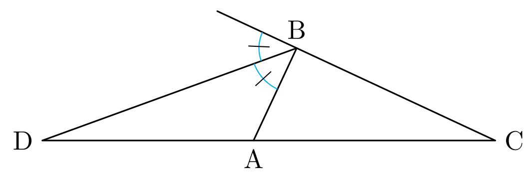
Proof. By the Extended Law of Sine, we have $AB \cdot \sin \angle ABD \cdot DC = BC \cdot \sin \angle CBD \cdot AD$. Rearranging yields the Angle Bisector Theorem. □

Question 2. (Prelim 2022 Q20) Let $ABCD$ be a cyclic quadrilateral and E be the intersection of AC and BD. P and Q are two points on AC such that the points A, E, Q, P, C lie on the same straight line in this order, and that BP bisects $\angle ABC$ whereas DQ bisects $\angle ADC$. If $AE = 4$, $EQ = 2$, and $QP = 3$, find the length of PC.

3.1 Extended Angle Bisector Theorem

We have the following extended version of the Angle Bisector Theorem

Theorem 3.2 (Extended Angle Bisector Theorem)**.** *Let BD be the external angle bisector of $\angle ABC$, then $AB : BC = AD : DC$.*



Proof. Left as an exercise. □

Question 3. Prove the Extended Angle Bisector Theorem.

Question 4. (Prelim 2022 Q19) In $\triangle ABC, AB < AC$. The internal bisector of $\angle BAC$ meets BC at D, while the external bisector of $\angle BAC$ meets CB produced at E. If $EB = 10$ and $BD = 5$, find the length of DC.

4 Cosine's Law

Theorem 4.1 (Cosine's Law)**.** *In* $\triangle ABC$, $c^2 = a^2 + b^2 - 2ab\cos C$. *Equivalently,* $\cos C = \frac{a^2+b^2-c^2}{2ab}$

Proof. Observe that $c^2 = AH^2 + HB^2 = b\sin C^2 + (a - b\cos C)^2 = a^2 + b^2 - 2ab\cos C$.

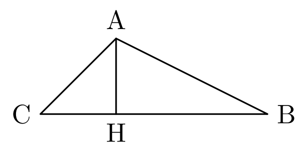


Figure 6: Proof of Cosine's Law

$\square$

Question 5. Prove the Stewart's Theorem with the Cosine's Law.

5 Ceva's Theorem

In a triangle, a *cevian* is a line joining a vertex of a triangle to a point on the interior of the opposite side. A natural question is when three cevians of a triangle concurs. This is answered by Ceva's theorem.

Theorem 5.1 (Ceva's Theorem)**.** *Let* $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$ *be cevians of a triangle* ABC. *They concur if and only if*

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$$

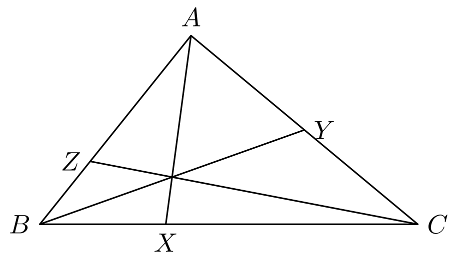


Figure 7: Statement of Ceva's Theorem

Proof. Observe that $\frac{[ABX]}{[AXC]} = \frac{BX}{XC}$ and $\frac{[BPX]}{[CPX]} = \frac{BX}{XC}$. Hence $\frac{[APB]}{[APC]} = \frac{BX}{XC}$. Similarly $\frac{[BPA]}{[BPC]} = \frac{AY}{YC}$ and $\frac{[CPB]}{[CPA]} = \frac{ZB}{ZA}$. Multiplying the above three equations gives $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$.

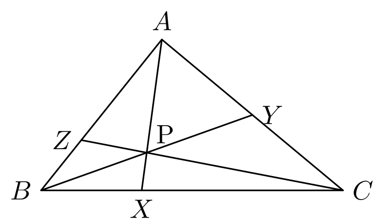


Figure 8: Proof of Ceva's Theorem

□

The above proof only proves the forward direction, i.e., if three points concurs, then the ratio must comply with the equation; The backward direction is also true, i.e., if the ratio complies with the equation, then three points concur. However, the proof is not given here since the technique is not very helpful for short-answer contest.

Theorem 5.2 (Trigonometric Form of Ceva's Theorem)**.** *Let $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$ be cevians of a triangle ABC. They concur if and only if*

$$\frac{\sin\angle BAX \sin\angle CBY \sin\angle ACZ}{\sin\angle XAC \sin\angle YBA \sin\angle ZCB} = 1$$

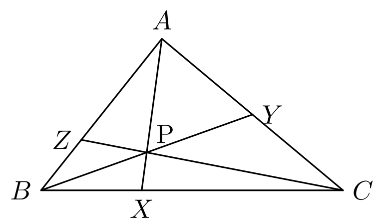


Figure 9: Trigonometric Form of Ceva's Theorem

Proof. Use the Extended Law of Sine. Left as an exercise. □

Question 6. Prove the Trigonometric Form of Ceva's Theorem.

5.1 Existence of orthocenter, incenter, centroid

If you have two altitudes, they obviously meet at the same point. However, why does the third altitude necessarily pass through the intersection of the first and second altitudes?

This is given by the Ceva's Theorem. We only need to check

$$\frac{\sin(90^o - B)\sin(90^o - C)\sin(90^o - A)}{\sin(90^o - C)\sin(90^o - A)\sin(90^o - B)} = 1$$

Hence, the orthocentre exists.

Similar calculations show that the incenter exists:

$$\frac{\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}{\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C} = 1$$

Alternatively, we can use the normal form of Ceva's Theorem together with the Angle Bisector Theorem to show that the incenter exists.

Lastly, the existence of centroid is trivial with the help of Ceva's Theorem.

$$\frac{1}{1}\frac{1}{1}\frac{1}{1} = 1$$

6 Menelaus's Theorem

Theorem 6.1 (Menelaus's Theorem)**.** *Let X, Y, Z be points on lines BC, CA, AB in a triangle ABC, distinct from its vertices. Then X, Y, Z are collinear (if and only if*

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1$$

where lengths are directed.

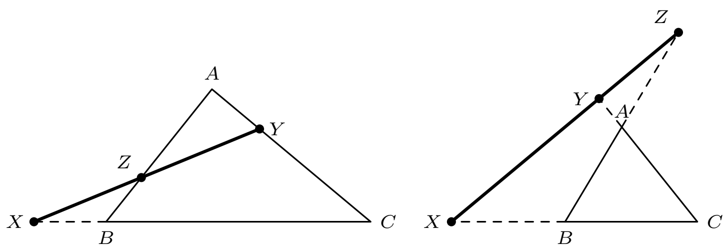


Figure 10: Statement of Menelaus's Theorem

Proof. Drop a perpendicular line from A to A', B to B', C to C' on XY. We have $\frac{CY}{YA} = \frac{CC'}{AA'}$, $\frac{AA'}{BB'} = \frac{AZ}{ZB}$, and $\frac{BX}{XC} = -\frac{BB'}{CC'}$. Multiplying all of them gives the Menelaus's Theorem.

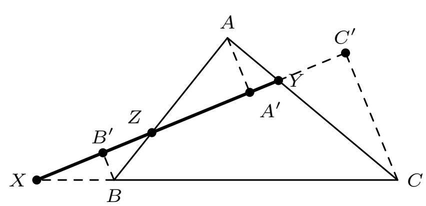


Figure 11: Proof of Menelaus's Theorem

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7 The Centroid Triangle

Theorem 7.1 (The Median Division)**.** *The medians divides the triangle into 6 equal parts.*

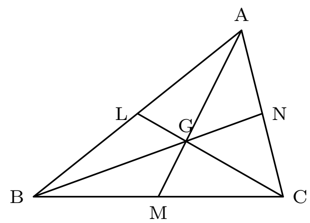

Figure 12: Statement of Median Division

Proof. Left as an exercise (hint: Consider ratio of areas). □

Question 7. Prove that the medians divide the triangle into 6 equal parts.

Question 8. (Centroid Division) Show that $AG = 2GM, GC = 2LG, BG = 2GN$.

8 Practice Problems

Question 9. Point P is on side AB of right angled $\triangle ABC$ with B as the right angled. Point Q is on AC such that PQ is perpendicular to AC. It is given that $BC = 3$ and $BP = PA = 2$. Find the length BQ.

Question 10. (Prelim 2020 Q16) $\triangle ABC$ is right-angled at B, with $AB = 1$ and $BC = 3$. E is the foot of perpendicular from B to AC. BA and BE are produced to D and F respectively such that D,F,C are collinear and $\angle DAF = \angle BAC$. Find the length of AD.

Question 11. (Prelim 2019 Q10) In $\triangle ABC$, $AB < AC$. Let H be the orthocentre of $\triangle ABC$, and D be the foot of the perpendicular from A to BC. If $AH = 4$,$HD = 3$ and $BC = 12$, find the length of BD.