MATH5.1EL L2
More on Counting

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Answer the questions in the spaces provided on the question sheets. If you do not know how to answer a certain question, write down where you get stuck. Answers can be corrected to 3 significant figures if necessary.

Name, class, class no.: ______________________________

Tutor's name: ______________________________

1 Permutation of sets

Definition 1.1. An r-permutation of n objects is a linearly ordered selection of r objects from a source (set S) of n objects. The number of r-permutations of n objects is denoted by

$$P(n, r)$$

An n-permutation of n objects is just called a permutation of n objects. The number of permutations of n objects is denoted by $n!$, read "n factorial".

Theorem 1.1. *The number of r-permutations of an n-set equals*

$$P(n, r) = n(n-1)\dots(n-r+1) = \frac{n!}{(n-r)!}$$

Example 1.1. Find the number of words that can be formed by rearranging letters in the word "LOOT".

Example 1.2. Find the number of 7-digit numbers in base 10 such that all digits are nonzero, distinct, and the digits 8 and 9 do not appear next to each other.

1.1 Circular Permutation

Definition 1.2. A **circular r-permutation** of a set S is an ordered r objects of S arranged as a circle.

Theorem 1.2. *The number of circular r-permutations of an n-set equals*

$$\frac{P(n, r)}{r} = \frac{n!}{(n-r)!r}$$

.

Corollary 1.2.1. The number of circular permutations of an n-set is

$$(n-1)!$$

Example 1.3. Twelve people, including two who do not wish to sit next to each other, are to be seated at a round table. How many circular seating plans can be made?

Example 1.4. We are to seat 5 men, 5 women, and 1 dog in a circular arrangement around a round table. In how many ways can this be done if no man is to sit next to a man and no woman is to sit next to a woman.

Example 1.5. In how many ways can eight rooks be arranged in an 8x8 checkerboard so that no rooks attack each other?

2 Combination of sets

Definition 2.1. A **combination** is a collection of objects (order is immaterial) from a given set. An r-**combination** of an n-set S is an r-subset of S. We denote by $\binom{n}{r}$ the number of r-combinations of an n-set, read "n choose r".

Theorem 2.1. *The number of r-combinations of an n-set equals*

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{P(n,r)}{r!}$$

.

Example 2.1. Assume 6A has 30 students, 6B has 31 students, 6C has 32 students. Find the number of ways to pick 2 students from each class.

Example 2.2. How many 8-letter words can be constructed from 26 letters of the alphabets if each word contains 3, 4, or 5 vowels? There is no restriction of the number of times a letter can be used in a word.

Corollary 2.1.1.

$$\binom{n}{r} = \binom{n}{n-r}$$

Theorem 2.2. *(Binomial Expansion) For non-negative integer n,*

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n = \sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i$$

Example 2.3. Expand $(x+y)^5$.

Example 2.4. Expand $(2x + \frac{y}{3})^4$.

Example 2.5. Find the last 2 digits of 11^{9999} by binomial theorem.

Example 2.6. (IMO prelim 2018) Find the last 4 digits of $2^{27653} - 1$.

Theorem 2.3. *The number of subsets of an n-set S equals*

$$\binom{n}{0} + \binom{n}{1} + \ldots \binom{n}{n} = 2^n$$

Theorem 2.4. *The number of ways to place n distinct objects into k distinct boxes, so that the 1st, 2nd, $\ldots$, kth boxes contain $n_1, n_2, \ldots, n_k$ objects, respectively, equals*

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!n_2!\ldots n_k!}$$

Example 2.7. How many ways can 5 distinct objects be put into three distinct boxes so that the 1st, 2nd, and 3rd boxes contain 2, 2, and 1 objects respectively?

Theorem 2.5. *(Multinomial theorem)*

$$(x_1 + \ldots + x_k)^n = \sum_{\substack{n_1+\ldots+n_k=n \\ n_1\geq 0,\ldots,n_k\geq 0}} \binom{n}{n_1, \ldots, n_k} x_1^{n_1} \ldots x_k^{n_k}$$

Example 2.8. Expand $(x + y + z)^3$.

3 Permutations of multisets

Let M be a multiset of n elements of k types, having 1st, 2nd, ..., kth type of objects repeated $n_1, n_2, \ldots, n_k$ times, respectively. We say M is a **multiset of type** $(n_1, \ldots, n_k)$ with $n = n_1 + \ldots + n_k$ objects.

Theorem 3.1. *Let M be a multiset of k distinct types where each type has infinitely many elements. Then the number of r-permutations of M equals*

$$k^r$$

This is also the number of linearly ordered selections of r objects from a source of k distinct objects with repetition allowed.

Theorem 3.2. *The number of permutations of an n-multiset M of type $(n_1, n_2, \ldots, n_k)$ is*

$$\binom{n}{n_1, \ldots, n_k} = \frac{n!}{n_1! \ldots n_k!}$$

Example 3.1. How many ways are there to arrange the word "MISSISSIPPI"? How about "MATHEMATICS"?

Example 3.2. Find the number of 8-permutations of the multiset

$$M = \{a, a, a, b, b, c, c, c, c\} = \{3a, 2b, 4c\}$$

4 Practice Problems

Problem 4.1. Find the number of diagonals in a 100-side convex polygon.

Problem 4.2.

How many odd positive 3-digit integers are divisible by 3 but do not contain the digit 3?

Problem 4.3. The eighth grade class at Lincoln Middle School has 93 students. Each student takes a math class or a foreign language class or both. There are 70 eighth graders taking a math class, and there are 54 eighth graders taking a foreign language class. How many eighth graders take only a math class and not a foreign language class?

Problem 4.4. A 10-digit arrangement $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ is called beautiful if (i) when read left to right, $0, 1, 2, 3, 4$ form an increasing sequence, and $5, 6, 7, 8, 9$ form a decreasing sequence, and (ii) 0 is not the leftmost digit. For example, 9807123654 is a beautiful arrangement. Determine the number of beautiful arrangements.

Problem 4.5. The expression

$$(x + y + z)^{2006} + (x - y - z)^{2006}$$

is simplified by expanding it and combining like terms. How many terms are in the simplified expression?