Random Walk in Random Environment Chapter 4

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1 $G(x) = H(x)$

We give the proof about $G(x) = H(x)$.

$$\begin{aligned}\int_{-x}^{x} \sum_{k=-\infty}^{+\infty} (-1)^k \exp\left(-\frac{1}{2}(u-2kx)^2\right) du &= \sum_{k=-\infty}^{+\infty} (-1)^k \int_{(2k-1)x}^{(2k+1)x} e^{-\frac{1}{2}u^2}\, du \\ &= \int_{-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} (-1)^k \mathbf{1}_{[(2k-1)x,(2k+1)x]}(u) e^{-\frac{1}{2}u^2}\, du.\end{aligned}$$

It is obvious that $\sum_{k=-\infty}^{+\infty}(-1)^k \mathbf{1}_{[(2k-1)x,(2k+1)x]}(u)$ is a $4x$-periodic function and is even(consider function graph).

$$\sum_{k=-\infty}^{+\infty} (-1)^k \mathbf{1}_{[(2k-1)x,(2k+1)x]}(u) = a_0 + \sum_{n=1}^{+\infty} a_n \cos\left(\frac{2n\pi}{4x}u\right),$$

where

$$a_0 = \frac{1}{4x}\int_{-2x}^{2x} \left(-\mathbf{1}_{[-2x,-x]}(u) + \mathbf{1}_{[-x,x]}(u) - \mathbf{1}_{[x,2x]}(u)\right) du = 0$$

and

$$\begin{aligned}a_n &= \frac{2}{4x}\int_{-2x}^{2x} \left(-\mathbf{1}_{[-2x,-x]}(u) + \mathbf{1}_{[-x,x]}(u) - \mathbf{1}_{[x,2x]}(u)\right) \cdot \cos\left(\frac{2n\pi}{4x}u\right) du \\ &= \frac{4}{n\pi}\sin\left(\frac{1}{2}n\pi\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k=-\infty}^{+\infty} (-1)^k \mathbf{1}_{[(2k-1)x,(2k+1)x]}(u) &= \sum_{n=1}^{+\infty} \frac{4}{n\pi}\sin\left(\frac{1}{2}n\pi\right)\cos\left(\frac{2n\pi}{4x}u\right) \\ &= \frac{4}{\pi}\sum_{k=0}^{+\infty} \frac{1}{n}\sin\left(\frac{1}{2}(2k+1)\pi\right)\cos\left(\frac{(2k+1)\pi}{2x}u\right) \\ &= \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\cos\left(\frac{(2k+1)\pi}{2x}u\right).\end{aligned}$$

Now, we have

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{-x}^{x}\sum_{k=-\infty}^{+\infty}(-1)^k \exp\left(-\frac{1}{2}(u-2kx)^2\right)du \\
=&\int_{-\infty}^{+\infty}\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\left(\frac{(2k+1)\pi}{2x}u\right)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}\,du \\
=&\mathbb{E}\left[\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\left(\frac{(2k+1)\pi}{2x}\mathbf{X}\right)\right] \\
=&\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\mathbb{E}\left[\cos\left(\frac{(2k+1)\pi}{2x}\mathbf{X}\right)\right]
\end{aligned}$$

where $\mathbf{X}\sim N(0,1)$. Note that

$$\begin{aligned}
&\mathbb{E}\left[\cos\left(\frac{(2k+1)\pi}{2x}\mathbf{X}\right)+i\sin\left(\frac{(2k+1)\pi}{2x}\mathbf{X}\right)\right] \\
=&\ \mathbb{E}\left[\exp\left(i\frac{(2k+1)\pi}{2x}\mathbf{X}\right)\right]=\exp\left(-\frac{(2k+1).\pi^2}{8x^2}\right)
\end{aligned}$$

We proved that $G(x)=H(x)$.

2 Borel-Cantelli Lemma and almost sure convenience

The proofs of almost all strong theorem are based on different forms of the Borel-Cantelli Lemma and those of the Markov inequality. *The main idea of Borel-Cantelli Lemma is to construct a series to control the probability of evens.*

Lemma 2.1. *If* $\sum_{n=1}^{\infty}\mathbb{P}(A_n)<\infty$, *then* $\mathbb{P}(\limsup_{n\to\infty}A_n):=\mathbb{P}(A_n\ i.o.)=0$.

Proof. Define general r.v. $\xi:=\sum_{n=1}^{\infty}\mathbf{1}_{A_n}$, it is obvious ξ is not negative. By $\mathbb{E}[\xi]=\sum_{n=1}^{\infty}\mathbb{P}(A_n)<\infty$, we have $\xi<\infty$ a.s., which is due to

$$\mathbb{P}(\xi=\infty)\le\mathbb{P}(\xi\ge N)\le\frac{1}{N}\mathbb{E}[\xi].$$

By $\xi<\infty$ a.s., we have $\mathbb{P}(A_n\ i.o.)=0$. □

Proof.

$$\mathbb{P}(\limsup_{n\to\infty}A_n)=\lim_{n\to\infty}\mathbb{P}(\cup_{k\ge n}A_k)\le\lim_{n\to\infty}\sum_{k=n}^{\infty}\mathbb{P}(A_k)=0.$$

□

Corollary 2.2. *If*

(i) $\sum_{n=1}^{\infty}\mathbb{P}(A_n|B_n)<\infty$,

(ii) B_n *occurs a.s. if* n *is large enough,*

then A_n *occurs a.s. only finitely many times.*

Proof. By Lemma 2.1, we have

$$\sum_{n=1}^{\infty}\mathbb{P}(A_n\cap B_n)\le\sum_{n=1}^{\infty}\mathbb{P}(A_n|B_n)\mathbb{P}(B_n)\le\sum_{n=1}^{\infty}\mathbb{P}(A_n|B_n).$$

Therefore, A_nB_n occurs a.s. finitely many times. By (ii), we complete the proof since $B_n=\Omega$ if n is large enough. □

The converse of the Borel-Cantelli lemma is trivially false.

Example 2.3. Let $\Omega=(0,1)$, $\mathcal{F}=\mathscr{B}((0,1))$ and $\mathbb{P}=\lambda$. If $A_n=(0,a_n)$ where $a_n\to 0$ as $n\to\infty$, then $\limsup A_n=\emptyset$, but if $a_n=\frac{1}{n}$, we have $\sum a_n=\infty$.

Corollary 2.4. *Let $S_n:=\sum_{k=1}^{n}X_k$, where $X_k\ge 0$. If $\mathbb{E}[S_n]\to\infty$, $\sup_{n\ge1}\mathbb{E}[X_n]<\infty$ and we can find $C,\delta>0$ such that for any $n\in\mathbb{N}_+$,*

$$\mathbf{Var}(S_n)\le C\cdot(\mathbb{E}[S_n])^{2-\delta} \tag{1}$$

then

$$\lim_{n\to\infty}\frac{S_n}{\mathbb{E}[S_n]}=1\ a.s..$$

Proof. We can assume $0<M:=\sup_{n\ge1}\mathbb{E}[X_n]\le 1$. Note that $0\le\mathbb{E}[X_n]\le 1$ and $\mathbb{E}[S_n]\to\infty$, it is easy to see the integer part of $\{E(n):=\mathbb{E}[S_n]\}_{n\ge1}$ can take all natural numbers. Therefore, we can find a subsequence $\{n_k\}_{k\ge1}$, such that

$$k^{\frac{2}{\delta}}\le E(n_k)\le k^{\frac{2}{\delta}}+1,\quad\forall\, k\ge 1.$$

By Markov's inequality, and (1), we have

$$\mathbb{P}\left(\left|\frac{S_{n_k}}{E(n_k)}-1\right|\ge\varepsilon\right)\le\frac{\mathbf{Var}(S_{n_k})}{\varepsilon^2\cdot E(n_k)^2}\le\frac{C}{\varepsilon^2\cdot k^2},\quad\forall k\ge 1,\varepsilon>0.$$

By Borel-Cantelli's lemma, we have

$$\lim_{k\to\infty}\frac{S_{n_k}}{E(n_k)}=1\ a.s..$$

For n large enough, there exista k large enough such that $n\in[n_k,n_{k+1})$. In this time, utilize the monotonicity of S_n and $E(n)$, we have

$$\frac{E(n_k)}{E(n_{k+1})}\cdot\frac{S_{n_k}}{E(n_k)}\le\frac{S_n}{E(n)}\le\frac{E(n_{k+1})}{E(n_k)}\cdot\frac{S_{n_{k+1}}}{E(n_{k+1})}.$$

Since $\frac{E(n_{k+1})}{E(n_k)}\to 1$ when $k\to\infty$, we complete the proof. □

Lemma 2.5. *If $\{A_n\}_{n\ge1}$ are independent evens, then*

$$\sum_{n=1}^{\infty}\mathbb{P}(A_n)=\infty\Rightarrow\mathbb{P}(A_n\ i.o.)=1.$$

Proof. Since $\mathbb{P}(\liminf_{n\to\infty}) = \lim_{n\to}\mathbb{P}(\cup_{k\geq n}A_k^c)$, by the independence of $\{A_n\}_{n\geq 1}$, we have

$$\mathbb{P}(\cap_{k=n}^m A_k^c) = \Pi_{k=n}^m \mathbb{P}(A_k^c) = (1-\mathbb{P}(A_k)) \leq \Pi_{k=n}^m \exp(-\mathbb{P}(A_k)) = \exp\left(-\sum_{k=n}^m \mathbb{P}(A_k)\right) \to 0(m\to\infty).$$

Therefore,

$$\mathbb{P}(\liminf_n A_n^c) = \lim_{n\to\infty}\mathbb{P}(\cap_{k=n}^\infty A_k^c) = \lim_{n\to\infty}\lim_{m\to\infty}\mathbb{P}(\cap_{k=n}^m) = 0.$$

□

In the following, we use Corollary 2.4 to prove

Lemma 2.6. *If $\{A_n\}_{n\geq 1}$ are pairwise independent evens, then*

$$\sum_{n=1}^\infty \mathbb{P}(A_n) = \infty \Rightarrow \mathbb{P}(A_n \ i.o.) = 1.$$

Proof. Let $S_n := \sum_{k=1}^n \mathbf{1}_{A_k}$, we compute the variation of S_n, for any $n \in \mathbb{N}_+$,

$$\begin{aligned}\mathbf{Var}(S_n) &= \sum_{k=1}^n \mathbf{Var}(\mathbf{1}_{A_k}) + 2\sum_{1\leq i\leq j\leq n}\mathbf{Cov}(\mathbf{1}_{A_i},\mathbf{1}_{A_i})\\ &= \sum_{k=1}^n \mathbb{P}(A_k) - \sum_{k=1}^n \mathbb{P}^2(A_k) \leq \mathbb{E}[S_n].\end{aligned}$$

Due to $\mathbb{E}[S_n]\to\infty$, by Corollary 2.4, we have $S_n\to\infty$ a.s.. □

Proof. We only need to prove $\mathbb{P}(S_\infty \leq a) = 0\ \forall\, a > 0$. For any $a > 0$, take $N \geq 1$ large enough, such that $\mathbb{E}[S_N] \geq a$. Then for any $n \geq N$, we have

$$\begin{aligned}\mathbb{P}(S_\infty \leq a) &\leq \mathbb{P}(S_n \leq a)\\ &\leq \mathbb{P}(-(S_n - \mathbb{E}[S_n]) \geq \mathbb{E}[S_n] - a)\\ &\leq \frac{\mathbb{E}[|S_n - \mathbb{E}[S_n]|^2]}{|\mathbb{E}[S_n] - a|^2} = \frac{\mathbf{Var}(S_n)}{|\mathbb{E}[S_n]-a|^2}\\ &\leq \frac{\mathbb{E}[S_n]}{|\mathbb{E}[S_n]-a|^2} \to 0,\end{aligned}$$

when $n\to\infty$, since $\mathbb{E}[S_n]\to\infty$. □

The ideas to prove a.s. convergence by Borel-Cantelli lemma:

Theorem 2.7. *Let $\{T_n\}_{n\geq 1}$ be r.v. such that*

$$\sum_{n=1}^\infty \mathbb{P}(|T_n| > \varepsilon) < \infty$$

for each $\varepsilon > 0$. Then $T_n \to 0$ a.s..

Proof. For each $k \geq 1$,

$$\sum_{n=1}^{\infty} \mathbb{P}(|T_n| > 2^{-k}) < \infty.$$

Hence, by the Borel-Cantelli lemma(use $[\limsup_n A_n]^c$), for each $k \geq 1$, $|T_n| \leq 2^{-k}$ for all n sufficiently large, except on a null event N_k. It follows that

$$T_n(\omega) \to 0 \quad \text{for all } \omega \notin \cup_{k=1}^{\infty} N_k.$$

Since $\cup_{k=1}^{\infty} N_k$ is a null event, $T_n \to 0$ a.s. follows. □

Theorem 2.8. *Let $\{T_n\}_{n\geq 1}$ be r.v. such that*

$$\sum_{n=1}^{\infty} \mathbb{P}(|T_n| > \varepsilon_n) < \infty$$

for positive constant $\varepsilon_n \to 0$. Then $T_n \to 0$ a.s..

Proof. Applying Borel-Cantelli lemma to events $\{|T_n| > \varepsilon_n\}, n \geq 1$. □

We need to estimate $\mathbb{P}(|T_n| > \varepsilon)$, it just the Chebyshev inequality

$$\mathbb{P}(|X| > \varepsilon) \leq \mathbb{E}[|X|]/\varepsilon$$

$$\mathbb{P}(|X - \mathbb{E}[X]| > \varepsilon) \leq \mathbf{Var}(X)/\varepsilon^2$$

and

$$\mathbb{P}(X > \varepsilon) \leq \exp(-t\varepsilon)\mathbb{E}[\exp(tX)]$$

for each $\varepsilon > 0$ and real t.

The ideas to prove a.s. convergence. The moment estimate also is useful.

Theorem 2.9. *Suppose that*

$$\sum_{n=1}^{\infty} \mathbb{E}[|T_n|^p] < \infty,$$

for some $p > 0$, then $T_n \to 0$ a.s..

Proof. By $\sum_{n=1}^{\infty} \mathbb{E}[|T_n|^p] < \infty$, we have $\mathbb{E}\left[\sum_{n=1}^{\infty} |T_n|^p\right] < \infty$. Moreover, $\sum_{n=1}^{\infty} |T_n|^p < \infty$ a.s. and hence that $T_n \to 0$ a.s..

□

The ideas to prove a.s. convergence by extracting subsequence.

Lemma 2.10. *Let $\{X_n\}_{n\geq 1}$ is r.v. sequence, if there exists a subsequence $\{n_k\}_{k\geq 1}$ such that*

$$X_{n_k} \to X \text{ a.s..}$$

and

$$\max_{n_{k-1} < n \leq n_k} \left| X_n - X_{n_{k-1}} \right| \to 0 \text{ a.s..}$$

Then,

$$X_n \to X \text{ a.s..}$$

Proof. For n large enough, there exist a unique k such $n_{k-1} < n \le n_k$, then

$$|X_n - X| \le \left|X_{n_{k-1}} - X\right| + \max_{n_{k-1}<m\le n_k} \left|X_m - X_{n_{k-1}}\right| \to 0 \ a.s..$$

$\square$

Theorem 2.11.

$$\lim_{n\to\infty} \frac{S_n}{n} = 0 \quad a.s.. \tag{2}$$

Proof. Since $\mathbb{E}[\frac{S_n}{n}] = 0$ and $\mathbb{E}[\frac{S_n^2}{n^2}] = \frac{1}{n}$, by Chebyshev inequality, for any $\varepsilon > 0$, we have

$$\mathbb{P}\left(\left|\frac{S_n}{n}\right| \ge \varepsilon\right) \le \frac{1}{n\varepsilon^2},$$

Therefore, we have $\frac{S_{n^2}}{n^2} \to 0$ a.s. when $n \to \infty$. Now we have to estimate the value of S_k for the k lying in the gap. If $n^2 \le k < (n+1)^2$, then

$$\begin{aligned}\left|\frac{S_k}{k}\right| &= \left|\frac{S_{n^2}n^2}{kn^2} + \frac{S_k - S_{n^2}}{k}\right| \\ &\le \left|\frac{S_{n^2}}{n^2}\right| + \left|\frac{k-n^2}{k}\right| \le \left|\frac{S_{n^2}}{n^2}\right| + \left|\frac{(n+1)^2-n^2}{k}\right| \to 0 \quad a.s..\end{aligned}$$

$\square$

Proof. Using $f(t) := \mathbb{E}[e^{tS_n}] = \left(\frac{e^t+e^{-t}}{2}\right)^n$, $\mathbb{E}[S_n^4] = f^{(4)}(t)|_{t=0}$. we have

$$\mathbb{E}\left[\frac{S_n^4}{n^4}\right] = n^{-3} + 6C_n^2 n^{-4} = O(n^{-2}).$$

By Theorem 2.9, we complete the proof. $\square$

Remark 2.12. *By Bernstein inequality,*

$$\mathbb{P}\left(\left|\frac{S_n}{n}\right| \ge \varepsilon\right) \le 2\exp\left(-\frac{2n\varepsilon^2}{(1+2\varepsilon)^2}\right),$$

it is obvious that (2) *holds true.*

3 Between LLN and LIL

By (2), we have $|S_n| = o(n)$ $a.s.$, it is natural to ask whether a better rate can obtained, in fact we have.

Theorem 3.1. *For any $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{S_n}{n^{\frac{1}{2}+\varepsilon}} = 0 \ a.s..$$

Proof. For any a position integer, by $\mathbb{E}[S_n^{2K}] = f^{(2K)}(t)|_{t=0}$, we have

$$\mathbb{E}[S_n^{2K}] = O(n^K).$$

Note that for $2\varepsilon K > 1$, we have

$$\mathbb{E}\left[\left|\frac{S_n^{2K}}{n^{K+2\varepsilon K}}\right|\right] \lesssim \frac{1}{n^{2\varepsilon K}}.$$

By Theorem 2.9, we complete the proof. $\square$

By Borel-Cantelli lemma, we can obtain

Theorem 3.2.

$$\limsup_{n\to\infty} \frac{|S_n|}{n^{\frac{1}{2}}\log n} \leq 1 \ a.s..$$

Proof. By $\mathbb{E}[e^{tS_n}] = \left(\frac{e^t+e^{-t}}{2}\right)^n$, we have

$$\mathbb{E}\left[\exp\left(n^{-\frac{1}{2}}S_n\right)\right] \to e^{1/2}.$$

Hence,

$$\mathbb{P}(S_n \geq (1+\varepsilon)n^{\frac{1}{2}}\log n) = \mathbb{P}\left(\exp(n^{-\frac{1}{2}}S_n) \geq n^{1+\varepsilon}\right) \lesssim \frac{1}{n^{1+\varepsilon}}$$

Therefore,

$$\limsup_{n\to\infty} \frac{S_n}{n^{\frac{1}{2}}\log n} \leq 1 \ a.s..$$

By the symmetry of S_n i.e. S_n equal to $-S_n$ in law, we complete the proof. □

Theorem 3.3. *For any $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{\mathbf{S}_n}{\sqrt{n(\log n)^{1+\varepsilon}}} = 0 \ a.s..$$

Proof. First, we prove Kolmogorov's maximal inequality. Let $X_1, X_2, \cdots$ be independent, mean-zero and $\mathbb{E}[X_k^2] < \infty$ $\forall k \in \mathbb{N}_+$. Then

$$\mathbb{P}\left(\sup_{1\leq k\leq n} |S_k| > \lambda\right) \leq \frac{\mathbb{E}[S_n^2]}{\lambda^2} = \frac{1}{\lambda^2}\sum_{k=1}^{n} \mathbf{Var}(X_k).$$

We partition $A^* := \{\sup_{1\leq k\leq n} S_n > \lambda\}$ into the events $A_k := \{|S_k| > \lambda$ and $|S_j| \leq \lambda$ for all $j < k\}$, then we have

$$\begin{aligned}
\mathbb{E}[S_n^2] \geq \mathbb{E}[S_n^2\,\mathbf{1}_{A^*}] &= \sum_{k=1}^{n} \mathbb{E}[S_n^2\,\mathbf{1}_{A_k}] \\
&= \sum_{k=1}^{n} \left(\mathbb{E}[S_k^2\,\mathbf{1}_{A_k}] + 2\mathbb{E}[S_k(S_n - S_k)\,\mathbf{1}_{A_k}] + \mathbb{E}[(S_n - S_k)^2\,\mathbf{1}_{A_k}]\right) \\
&\geq \sum_{k=1}^{n} \mathbb{E}[S_k^2\,\mathbf{1}_{A_k}] \geq \sum_{k=1}^{n} \lambda^2 \mathbb{P}(A_k) = \lambda^2\mathbb{P}(A^*).
\end{aligned}$$

Second, let $X_1, X_2, \cdots$ be independent, mean-zero and $\mathbb{E}[X_k^2] < \infty$ $\forall k \in \mathbb{N}_+$, then

$$\sum_{i=1}^{\infty} \mathbf{Var}(X_i) < \infty \Rightarrow \sum_{i=1}^{\infty} X_i < \infty \ a.s..$$

By the assumptions about $\{X_i\}_{i\geq 1}$, we see $\{S_n\}_{n\geq 1}$ is a the Cauthy sequence in $L^2(\Omega)$ space. Therefore, there exist a $S_\infty \in L^2(\Omega)$ such that $S_n \to S_\infty$ in $L^2(\Omega)$. Moreover, there exist a subsequence $\{n_k\}_{k\geq 1}$ such that $S_{n_k} \to S_\infty$ $a.s.$..

For any $k \geq 0$ (let $n_0 := 0, S_0 = 0$), by Kolmogorov inequality, we have

$$\mathbb{P}\left(\max_{n_k < p \leq n_{k+1}} |S_p - S_{n_k}| \geq \varepsilon\right) \leq \frac{1}{\varepsilon^2}\mathbb{E}[|S_{n_{k+1}} - S_{n_k}|^2].$$

Note that

$$\sum_{k=1}^{\infty} \mathbb{E}[\left|S_{n_{k+1}} - S_{n_k}\right|^2] = \sum_{n=1}^{\infty} \mathbb{E}[X_n^2] < \infty.$$

By Borel-Cantelli lemma, we obtain that

$$\max_{n_k < p \le n_{k+1}} |S_p - S_{n_k}| \to 0 \ a.s..$$

Combining $S_{n_k} \to S_\infty$ *a.s.*, we can obtain $S_n \to S_\infty$ *a.s.*. which called **Extract Subsequence Method**. In fact, for n large enough, there exists a unique k large enough such that $n_{k-1} < n \le n_k$, then

$$|S_n - S_\infty| \le \left|S_{n_{k-1}} - S_\infty\right| + \max_{n_{k-1} < m \le n_k} \left|S_m - S_{n_{k-1}}\right| \to 0 \ a.s..$$

Kronecker's lemma Let $\{a_n\}_{n\ge1}$ is a sequence of real number, and suppose $b_n \uparrow \infty$. If $\sum_i \frac{a_i}{b_i} < \infty$, then $\frac{\sum_{k=1}^n a_k}{b_n} \to 0$.

Finally, let $a_n = \mathbf{X}_n(\omega)$ and $b_n = \sqrt{n(\log n)^{1+\varepsilon}} \uparrow \infty$, it suffices to show that

$$\sum_{k=1}^{\infty} \frac{\mathbf{X}_k}{\sqrt{n(\log n)^{1+\varepsilon}}} < \infty \ a.s..$$

We only need to check

$$\sum_{i=1}^{\infty} \mathbf{Var}\left(\frac{\mathbf{X}_k}{\sqrt{n(\log n)^{1+\varepsilon}}}\right) = \sum_{i=1}^{\infty} \frac{\mathbf{Var}(\mathbf{X}_k)}{n(\log n)^{1+\varepsilon}} = \sum_{i=1}^{\infty} \frac{1}{n(\log n)^{1+\varepsilon}} < \infty.$$

Let $f(x) = \frac{1}{x(\log x)^\alpha}$, for x large enough, we have $f(x) > 0$ and f is a monotonically decreasing continuous function about x. Define $F(x) = \log\log x$ if $\alpha = 1$, $F(x) = \frac{1}{1-\alpha}(\log x)^{1-\alpha}$ if $\alpha \ne 1$, then we have $F'(x) = f(x)$ for x large enough. Therefore,

$$\int_N^\infty f(x)\,dx = \begin{cases} \infty, & \text{if } \alpha \ge 1, \\ \frac{1}{\alpha-1}(\log N)^{1-\alpha}, & \text{if } \alpha > 1. \end{cases}$$

□

Similarly, we can obtain that for any $\varepsilon > 0$, $k \in \mathbb{N}_+$

$$\lim_{n\to\infty} \frac{\mathbf{S}_n}{\sqrt{n \log n \log^{(2)} n \cdots (\log^{(k)} n)^{1+\varepsilon}}} = 0 \ a.s..$$

The best possible rate was obtained by Khinchine which is called Law of Iterated Logarithm,

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \log\log n}} = 1 \ a.s..$$

Lemma 3.4. *For any positive integer N, we have*

$$\mathbb{P}\left(S_n \ge k\right) \le e^{-\frac{k^2}{2n}}$$

Proof. Since

$$\mathbb{P}\left(S_n \geq k\right) \leq \frac{\mathbb{E}[e^{tS_n}]}{e^{tk}} = \frac{\left(\mathbb{E}[e^{tX_1}]\right)^n}{e^{tk}}$$

and

$$\mathbb{E}[e^{tX_1}] = \frac{e^t + e^{-t}}{2} \leq e^{\frac{t^2}{2}}.$$

By taking $t = \frac{k}{n}$, we have

$$\mathbb{P}\left(S_n \geq k\right) \leq \frac{e^{\frac{nt^2}{2}}}{e^{tk}} = e^{-\frac{k^2}{2n}}.$$

□

Lemma 3.5 (Reflection principle)**.** *For any positive integer m, we have*

$$\mathbb{P}(M_n^+ \geq m, S_n = s) = \begin{cases} \mathbb{P}(S_n = s), & \text{if } s \geq m, \\ \mathbb{P}(S_n = 2m - s), & \text{if } s < m, \end{cases}$$

and

$$\mathbb{P}(M_n^+ \geq m) = \mathbb{P}(S_n \geq m) + \sum_{s=-\infty}^{m-1} \mathbb{P}(S_n = 2m - s) = \mathbb{P}(S_n = m) + \sum_{k=m+1}^{\infty} 2\mathbb{P}(S_n = k)$$

and thus

$$\mathbb{P}(M_n^+ \geq m) = 2\mathbb{P}(S_n \geq m + 1) + \mathbb{P}(S_n = m) \leq 2\mathbb{P}(S_n \geq m).$$

Lemma 3.6.

$$\mathbb{P}(S_n = k) \sim \frac{e^{-\frac{k^2}{2n}}}{\sqrt{\pi n}}.$$

Proof. Recall Stirling's approximation

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Then, we have

$$\begin{aligned} \mathbb{P}\left(S_{2n} = 2k\right) &= \frac{(2n)!}{(n+k)!(n-k)!} 2^{-2n} \\ &\sim \frac{1}{\sqrt{\pi n}} \frac{1}{\left(1+\frac{k}{n}\right)^{n+k}\left(1-\frac{k}{n}\right)^{n-k}} \frac{1}{\sqrt{\left(1+\frac{k}{n}\right)\left(1-\frac{k}{n}\right)}} \\ &= \frac{1}{\sqrt{\pi n}} \frac{\left(1-\frac{k}{n}\right)^k}{\left(1-\frac{k^2}{n^2}\right)^{n+\frac{1}{2}}\left(1+\frac{k}{n}\right)^k} \end{aligned}$$

Note that we need $(n-k) \to \infty$ to use Stirling's formula. We choose $k = \left\lfloor x\sqrt{\frac{n}{2}}\right\rfloor$ so that $\frac{2k}{\sqrt{2n}} \to x$. It is not hard to see that if $x_n \to 0$, and $y_n \to \infty$ such that $x_n y_n \to t$, then $(1+x_n)^{y_n} \to e^t$. Therefore,

$$\mathbb{P}(S_{2n} = 2k) \sim \frac{1}{\sqrt{\pi n}}\left(1 - \frac{x^2}{2n}\right)^{-n}\left(1 - \frac{x}{\sqrt{2n}}\right)^{x\frac{\sqrt{n}}{\sqrt{2}}}\left(1 + \frac{x}{\sqrt{2n}}\right)^{-x\frac{\sqrt{n}}{\sqrt{2}}} \to \frac{1}{\sqrt{\pi n}}e^{-\frac{x^2}{2}}.$$

□

Lemma 3.7. *Let $k > n^{\frac{1}{2}}$, then there exist a constant C such that*

$$\mathbb{P}(S_n \geq k) \geq C \cdot \frac{n^{\frac{1}{2}}}{k} e^{-\frac{k^2}{2n}}.$$

Proof. It is easy to see

$$\mathbb{P}(S_n \geq k) \geq \mathbb{P}(k \leq S_n \leq k + \frac{n}{k}) \geq C \cdot n^{-\frac{1}{2}} \sum_{m=k}^{k+\frac{n}{k}} e^{-\frac{m^2}{2n}}.$$

For $k \leq m \leq k + \frac{n}{k}$, we have

$$e^{-\frac{k^2}{2n}} \geq e^{-\frac{m^2}{2n}} \geq e^{-\frac{(k+\frac{n}{k})^2}{2n}} = \exp\left(-\frac{k^2}{2n} - 1 - \frac{n}{2k^2}\right) \geq C \cdot e^{-\frac{k^2}{2n}}.$$

Therefore, we have

$$\mathbb{P}(S_n \geq k) \geq C \cdot \frac{n^{\frac{1}{2}}}{k} e^{-\frac{k^2}{2n}}.$$

□

Theorem 3.8. *Define $F_n := \sqrt{2n \log\log n}$, then we have*

$$\limsup_{n\to\infty} \frac{S_n}{F_n} = 1 \ a.s..$$

Proof. The proof will be presented in two steps. The first one gives an upper bound of $\limsup_{n\to\infty} \frac{S_n}{F_n}$, i.e. for any $\varepsilon > 0$, we show that

$$\limsup_{n\to\infty} \frac{S_n}{F_n} \leq 1 + \varepsilon \ a.s..$$

The second one gives a lower bound of $\limsup_{n\to\infty} \frac{S_n}{F_n}$, i.e. for $0 < \varepsilon < \frac{1}{2}$,

$$\limsup_{n\to\infty} \frac{S_n}{F_n} \geq 1 - \varepsilon \ a.s..$$

Step 1. Let $\Theta > 1$, $n_k := \lfloor \Theta^k \rfloor$, by reflection principle and Lemma 3.4, we have

$$\begin{aligned}
\mathbb{P}\left(M^+_{n_k} \geq (1+\varepsilon)F_{n_k}\right) &\leq 2\mathbb{P}\left(S_{n_k} \geq (1+\varepsilon)F_{n_k}\right) \\
&\leq 2\exp\left(-\frac{(1+\varepsilon)^2 F^2_{n_k}}{2n_k}\right) \\
&= 2\exp\left(-(1+\varepsilon)^2 \log\log n_k\right) \sim (k\log\Theta)^{-(1+\varepsilon)^2}
\end{aligned}$$

By Borel-Cantelli lemma, we have

$$M^+_{n_k} \leq (1+\varepsilon)F_{n_k} \ a.s.,$$

for all but finitely many k.

Let $n_k \leq n < n_{k+1}$,

$$\frac{S_n}{F_n} \leq \frac{M^+_n}{F_n} = \frac{M^+_{n_{k+1}}}{F_{n_{k+1}}} \frac{F_{n_{k+1}}}{F_n} \frac{M^+_n}{M^+_{n_{k+1}}} \leq (1+\varepsilon)\frac{F_{n_{k+1}}}{F_n} \leq 1 + 2\varepsilon \ a.s.,$$

where $\Theta(\varepsilon, k)$ is close enough to 1.

Step 2. Define $\mathbf{n}_k = n_{k+1} - n_k$, by the definition of $\{S_n\}_{n\geq 1}$, we have $\{S_{n_{k+1}} - S_{n_k}\}_{k\geq 1}$ is mutually independent, $S_{\mathbf{n}_k}$ and $S_{n_{k+1}} - S_{n_k}$ are equal in law. By Lemma 3.6, for large $k(\varepsilon)$ enough, we have

$$\mathbb{P}\left(S_{\mathbf{n}_k} = S_{n_{k+1}} - S_{n_k} \geq (1-\varepsilon)F_{\mathbf{n}_k}\right) \geq C \cdot \frac{\mathbf{n}_k^{\frac{1}{2}}}{(1-\varepsilon)F_{\mathbf{n}_k}} \exp\left(-\frac{(1-\varepsilon)^2 F_{\mathbf{n}_k}^2}{2\mathbf{n}_k}\right)$$
$$\sim \frac{1}{\sqrt{\log\log \mathbf{n}_k}}(\log \mathbf{n}_k)^{-(1-\varepsilon)^2}$$

where the last sim is due to $0 < \varepsilon < \frac{1}{2}$. Note that $\log \mathbf{n}_k \sim k$, we have $\sum_{k=1}^{\infty} \mathbb{P}(S_{\mathbf{n}_k} \geq (1-\varepsilon)F_{\mathbf{n}_k}) = \infty$, by Borel-Cantelli lemma, we have

$$S_{n_{k+1}} \geq S_{n_k} + (1-\varepsilon)F_{\mathbf{n}_k} \quad i.o.\ a.s..$$

By the symmetric of the upper bound, we have

$$\liminf_{k\to\infty} \frac{S_{n_k}}{F_{n_k}} \geq \liminf_{n\to\infty} \frac{S_n}{F_n} \geq -(1+\varepsilon)\ a.s..$$

Therefore, we have

$$\begin{aligned}\frac{S_{n_{k+1}}}{F_{n_{k+1}}} &\geq \frac{S_{n_k}}{F_{n_k}} \frac{F_{n_k}}{F_{n_{k+1}}} + (1-\varepsilon)\frac{F_{\mathbf{n}_k}}{F_{n_{k+1}}} \\ &\geq -(1+\varepsilon)\frac{F_{n_k}}{F_{n_{k+1}}} + (1-\varepsilon)\frac{F_{\mathbf{n}_k}}{F_{n_{k+1}}} \to -\frac{(1+\varepsilon)}{\Theta^{\frac{1}{2}}} + (1-\varepsilon)\left(\frac{\Theta-1}{\Theta}\right)^{\frac{1}{2}}.\end{aligned}$$

Take $\varepsilon \to 0^+$ and $\Theta \to \infty$, we complete the proof. □

Using almost the same method, we can prove the result about Brownian motion. For the convenience of the readers, we provide the detailed proof.

Theorem 3.9. *For a Brownian motion B in $\mathbb{R}$, we have*

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} = 1\ a.s..$$

Proof. when $u \to \infty$, we have

$$\int_u^{\infty} e^{-\frac{1}{2}}\,dx \sim u^{-1}\int_u^{\infty} xe^{-\frac{x^2}{2}} = u^{-1}e^{-\frac{u^2}{2}}.$$

In fact,

$$\int_u^{\infty} \frac{1}{x}(xe^{-\frac{x^2}{2}})\,dx = \int_u^{\infty} \frac{1}{x}d(e^{-\frac{x^2}{2}}) = \frac{1}{x}e^{-\frac{x^2}{2}}\Big|_0^{\infty} - \int_u^{\infty} \frac{1}{x^2}e^{-\frac{x^2}{2}}\,dx.$$

Let $M_t^+ = \sup_{0\leq s\leq t} B_s$, then by reflection principle, we have

$$\mathbb{P}(M_t^+ > ut^{\frac{1}{2}}) = 2\mathbb{P}(B_t > ut^{\frac{1}{2}}) \sim \frac{2}{\sqrt{2\pi}}u^{-1}e^{-\frac{u^2}{2}}.$$

Step 1. Define $F_t = \sqrt{2t \log\log t}$, for any $\Theta > 1$ and $1+\varepsilon > 1$, for n large enough, we have

$$\begin{aligned}
&\mathbb{P}(M^+_{\Theta^n} > (1+\varepsilon)F_{\Theta^n}) \\
\leq& 2\mathbb{P}\left(\frac{B_{\Theta^n}}{\sqrt{\Theta^n}} > \frac{(1+\varepsilon)F_{\Theta^n}}{\sqrt{\Theta^n}}\right) \\
\lesssim& \sqrt{\frac{\Theta^n}{(1+\varepsilon)^2 F^2_{\Theta^n}}} \exp\left(-\frac{1}{2}\frac{(1+\varepsilon)^2 F^2_{\Theta^n}}{\Theta^n}\right) \\
\lesssim& \frac{1}{\sqrt{\log\log\Theta^n}} \exp\left(-(1+\varepsilon)^2 \log\log\Theta^n\right) \sim (n\log\Theta)^{-(1+\varepsilon)^2}.
\end{aligned}$$

By the Borel-Cantelli lemma, we obtain that for n large enough,

$$\frac{M^+_{\Theta^n}}{F_{\Theta^n}} \leq (1+\varepsilon) \ a.s..$$

Therefore, for $\Theta^n \leq t < \Theta^{n+1}$, Θ approach 1,

$$\frac{B_t}{F_t} \leq \frac{M^+_t}{F_t} = \frac{M^+_{\Theta^{n+1}}}{F_{\Theta^{n+1}}} \frac{F_{\Theta^{n+1}}}{F_t} \frac{M^+_t}{M^+_{\Theta^{n+1}}} \leq (1+\varepsilon)\frac{F_{\Theta^{n+1}}}{F_t} \leq 1+2\varepsilon \ a.s..$$

Step 2. For $0 < \varepsilon < \frac{1}{2}$, we have

$$\begin{aligned}
\mathbb{P}\left(B_{\Theta^{n+1}} - B_{\Theta^n} > (1-\varepsilon)F_{[\Theta^{n+1}-\Theta^n]}\right) &\geq C \cdot \frac{(\Theta^{n+1}-\Theta^n)^{\frac{1}{2}}}{F_{[\Theta^{n+1}-\Theta^n]}} \exp\left(-\frac{(1-\varepsilon)^2 F^2_{[\Theta^{n+1}-\Theta^n]}}{2(\Theta^{n+1}-\Theta^n)}\right) \\
&\sim \frac{1}{\sqrt{\log\log(\Theta^{n+1}-\Theta^n)}} \left(\log(\Theta^{n+1}-\Theta^n)\right)^{-(1-\varepsilon)^2}
\end{aligned}$$

Since $\log(\Theta^{n+1}-\Theta^n) \sim n$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(B_{\Theta^{n+1}} - B_{\Theta^n} > (1-\varepsilon)F_{[\Theta^{n+1}-\Theta^n]}\right) < \infty.$$

By Borel-Cantelli lemma, we have

$$B_{\Theta^{n+1}} \geq B_{\Theta^n} + (1-\varepsilon)F_{[\Theta^{n+1}-\Theta^n]} \quad i.o. \ a.s..$$

By the symmetric of the upper bound, we have

$$\liminf_{n\to\infty} \frac{B_{\Theta^n}}{F_{\Theta^n}} \geq \liminf_{t\to\infty} \frac{B_t}{F_t} \geq -(1+\varepsilon) \ a.s..$$

Therefore, we have

$$\frac{B_{\Theta^{n+1}}}{\Theta^{n+1}} \geq B_{\Theta^n} + (1-\varepsilon)F_{[\Theta^{n+1}-\Theta^n]} \quad i.o. \ a.s..$$

$$\begin{aligned}
\frac{B_{\Theta^{n+1}}}{\Theta^{n+1}} &\geq \frac{B_{\Theta^n}}{F_{\Theta^n}} \frac{F_{\Theta^n}}{F_{\Theta^{n+1}}} + (1-\varepsilon)\frac{F_{[\Theta^{n+1}-\Theta^n]}}{F_{\Theta^{n+1}}} \\
&\geq -(1+\varepsilon)\frac{F_{\Theta^n}}{F_{\Theta^{n+1}}} + (1-\varepsilon)\frac{F_{[\Theta^{n+1}-\Theta^n]}}{F_{\Theta^{n+1}}} \to -\frac{(1+\varepsilon)}{\Theta^{\frac{1}{2}}} + (1-\varepsilon)\left(\frac{\Theta-1}{\Theta}\right)^{\frac{1}{2}}.
\end{aligned}$$

Take $\varepsilon \to 0^+$ and $\Theta \to \infty$, we complete the proof. □