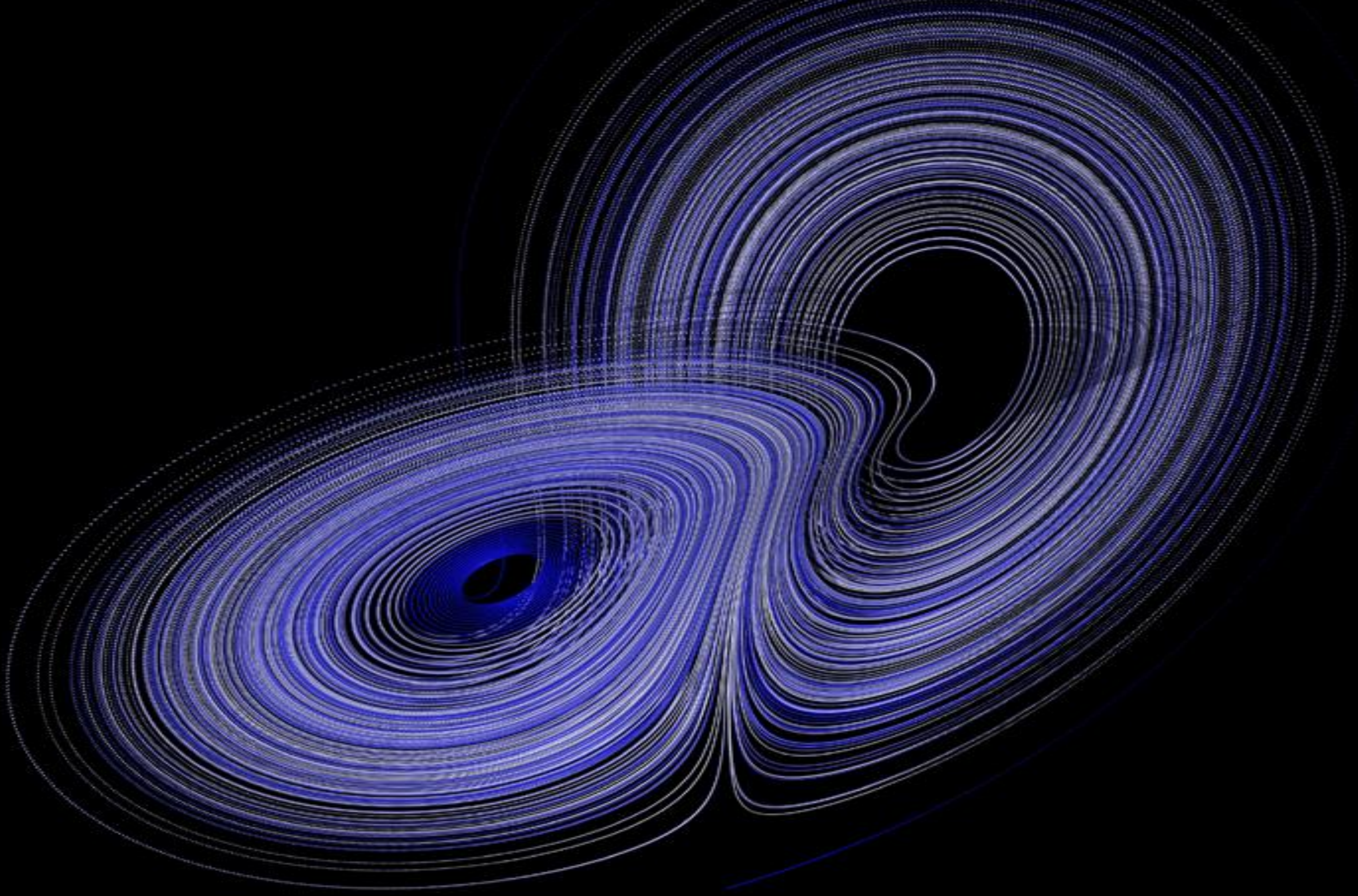
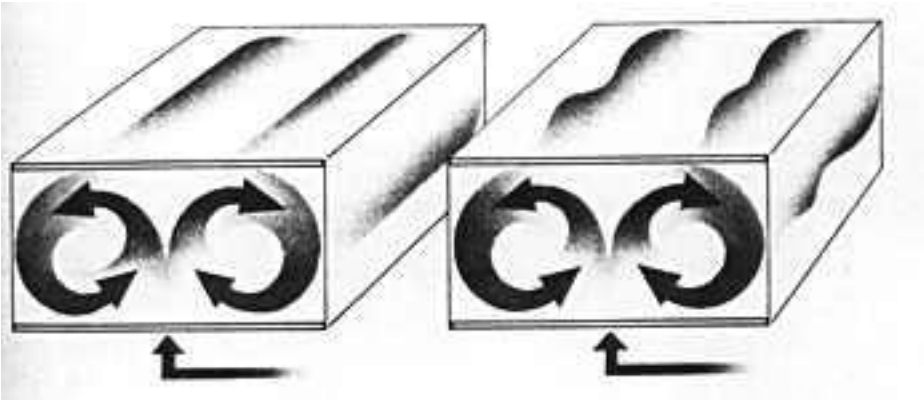
The Lorenz system



Edward Lorenz

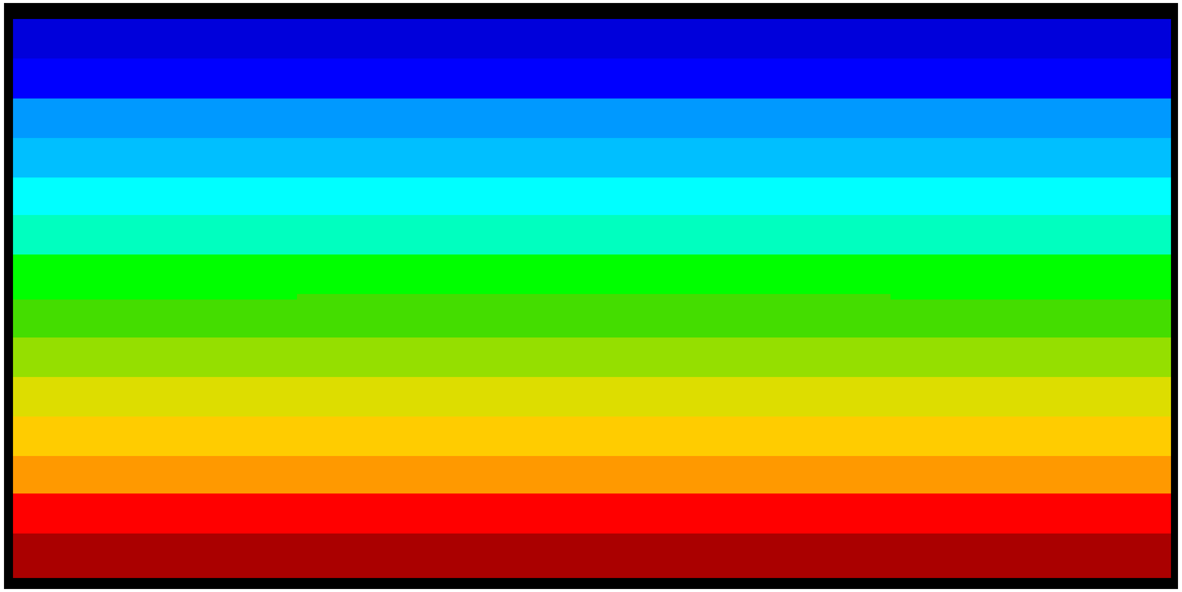
- Professor of Meteorology at the Massachusetts Institute of Technology
- In 1963 derived a three dimensional system in efforts to model long range predictions for the weather
- The weather is complicated! A theoretical simplification was necessary

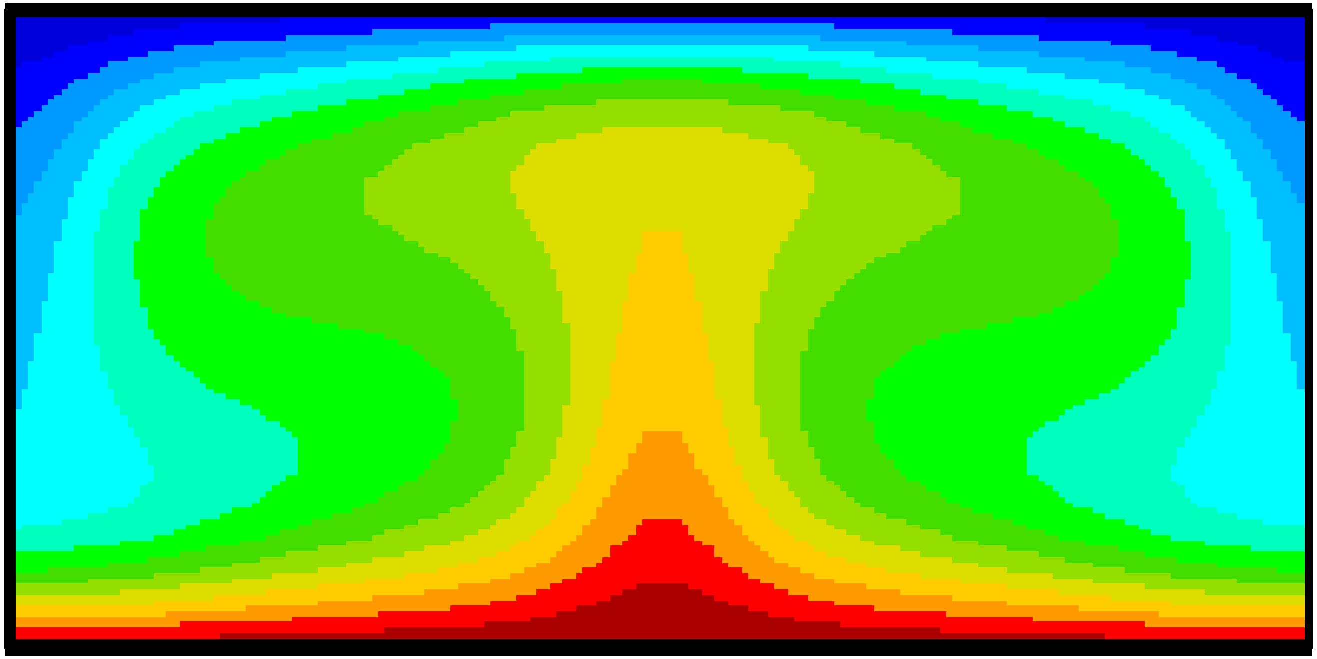
The Lorenz system



- Lorenz-rendszer egy folyadék mozgását írja le két réteg között különböző hőmérsékleten. Pontosabban, a folyadékot egyenletesen melegítik alulról és egyenletesen hűtik felülről.

D. Gulick, Encounters with Chaos, Mc-Graw Hill, Inc., New York, 1992.





- Az új változók jelentése:
- X a dimenziótlan sebességet,
- Y a dimenziótlan hőmérséklet különbséget a fel és le áramlások között és
- Z a hővezetési egyensúlytól vett eltérés dimenziótlanított jellemzője

The equations

- σ - a folyadék típusától függ
        Összefügg a viszkozitás és a hőátadással az áramlás közben.
- b geometriai, fizikai arány
- r - Rayleigh-számtól, hő vezetőképesség
- (Prandtl szám, Rayleigh szám és fizikai arány)

$$\frac{dx}{dt} = s\left(y - x\right)$$

$$\frac{dy}{dt} = -xz + rx - y$$

$$\frac{dz}{dt} = xy - bz$$

σ, r, b   Are positive parameters

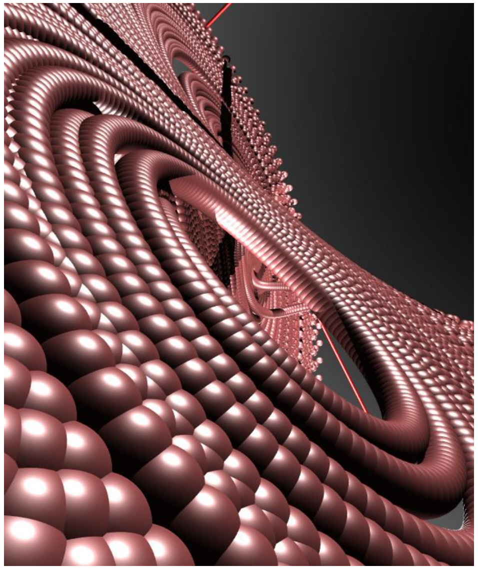
σ=10, b=8/3, r varies in [0,30]

Equilibria

$$S1 = (0, 0, 0)$$

$$S2 = \left(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1\right)$$

$$S3 = \left(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1\right)$$



Linear stability of S1

Jacobian

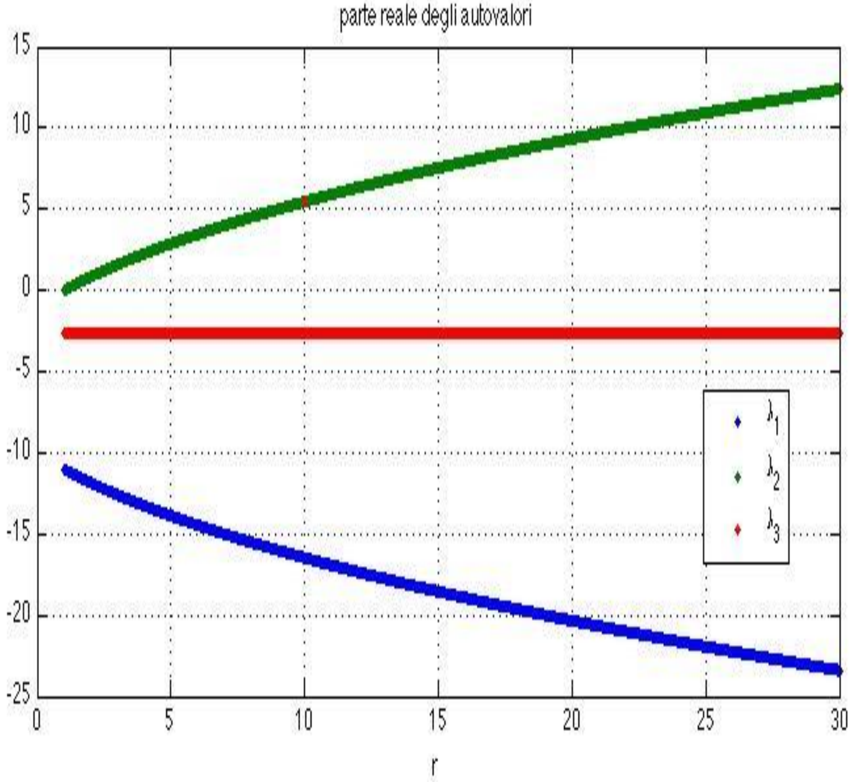
$$J(0,0,0) = \begin{bmatrix} -s & s & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}$$

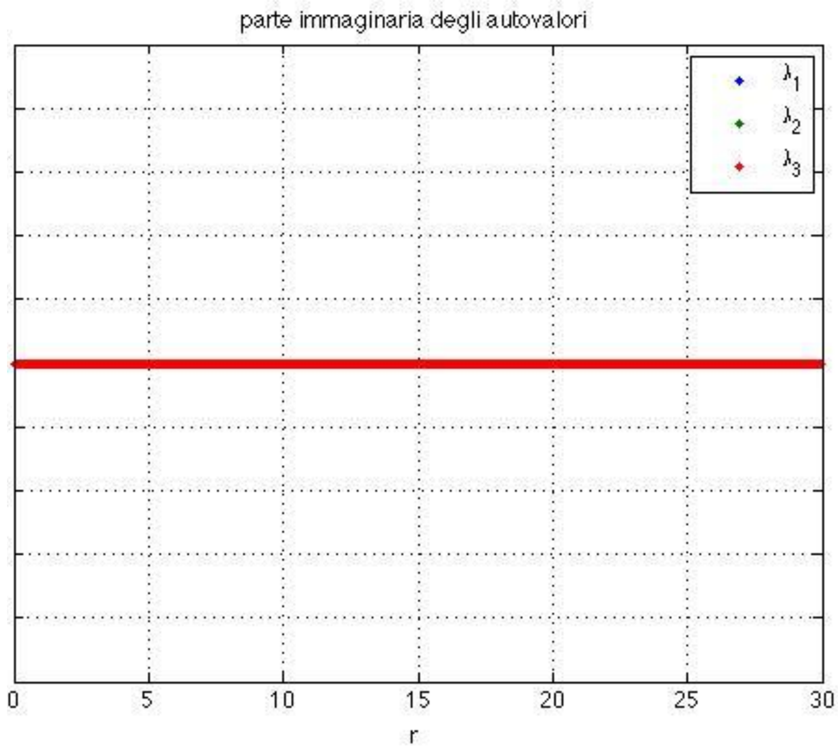
Eigenvalues of J

$$l_{1,2} = \frac{-(s+1) \pm \sqrt{(s-1)^2 + 4sr}}{2}$$

$$l_3 = -b$$

Linear stability of S1



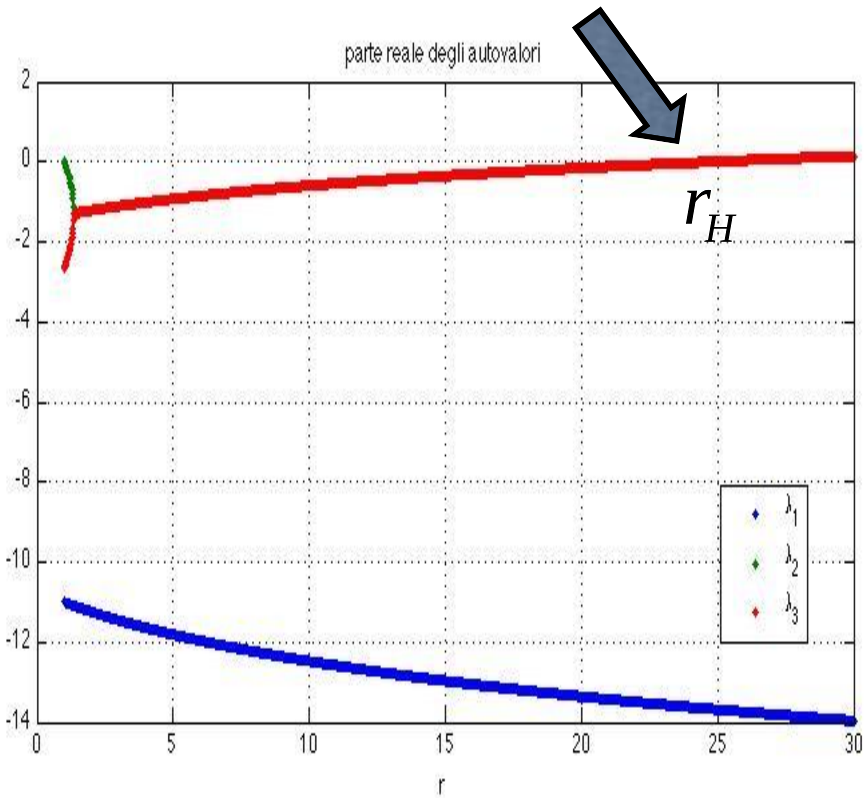


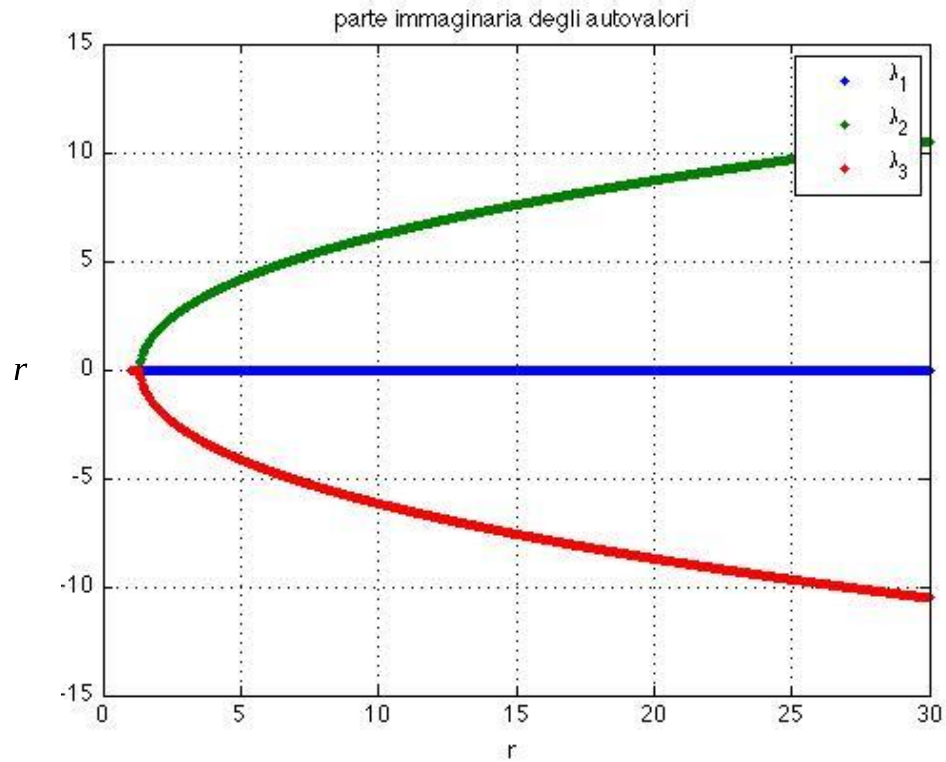
Linear stability of S1

$$l_{1,2} = \frac{-(s+1) \pm \sqrt{(s-1)^2 + 4sr}}{2}$$

Valori di r	Comportamento
$0 < r < 1$	$\sqrt{(s-1)^2 + 4sr} \pounds s + 1$ Negative eigenvalues: asymptotic stability
$r = 1$	$l_1 = 0 \quad l_2 = -(s+1) \quad l_3 = -b$ Marginal stability
$r > 1$	$\sqrt{(s-1)^2 + 4sr} > s + 1 \quad l_1 > 0$ Positive eigenvalues: instability

Stabilità lineare di S2 e S3





$$r_H = \frac{s\left(s + b + 3\right)}{s - b - 1} = 24.74$$

Linear stability of S2 and S3

The eigenvalues of J(S2) e J(S3) coincide and the associated eigenvalues are linearly independent

Eigenvalues as functions of r

r	λ_1	λ_2	λ_3
5	-11.809	$-0.92872 + i\ 4.1476$	$-0.92872 - i\ 4.1476$
20	-13.357	$-0.15479 + i\ 8.7087$	$-0.15479 - i\ 8.7087$
24.7368	-13.667	$-1.2725\ 10^{-6} + i\ 9.6245$	$-1.2725\ 10^{-6} - i\ 9.6245$
24.74	-13.667	$9.5435\ 10^{-5} + i\ 9.6251$	$9.5435\ 10^{-5} - i\ 9.6251$
28	-13.855	$0.093956 + i\ 10.195$	$0.093956 - i\ 10.195$

Close to r=24.74 the real part of the eigenvalues are positive From this value of r all the equilibria are unstable

Summary

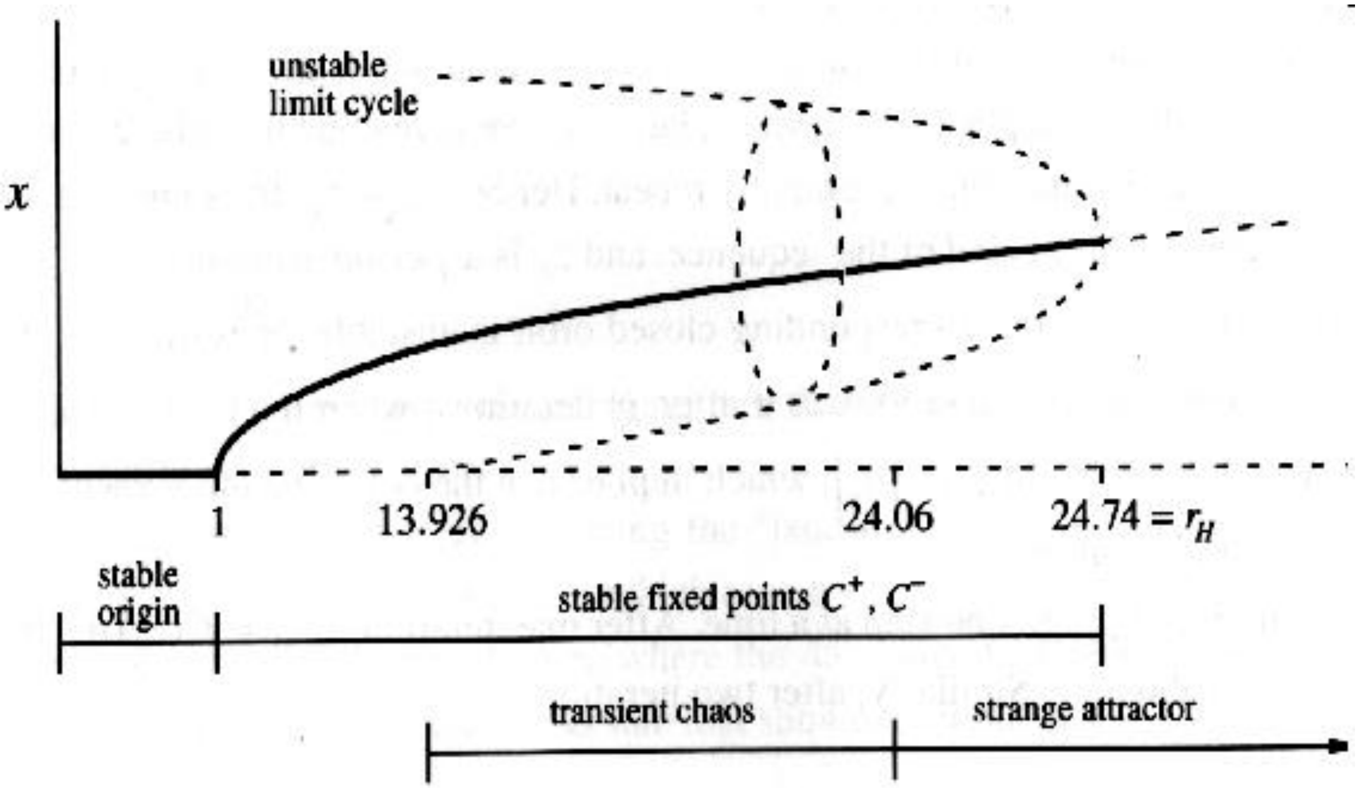
r	S	λ	stabilità
	S_1	$\lambda_1, \lambda_2, \lambda_3 < 0$	attrattiva
0<r<1	S_2	–	non esiste
	S_3	–	non esiste
	S_1	$\lambda_1, \lambda_2, \lambda_3 < 0$	attrattiva
r=1	S_2	$\lambda_1, \lambda_2, \lambda_3 < 0$	attrattiva
	S_3	$\lambda_1, \lambda_2, \lambda_3 < 0$	attrattiva
	S_1	$\lambda_1, \lambda_2 < 0, \lambda_3 > 0$	repulsiva
1<r<24,74	S_2	λ_1, Pε(λ_2), Pε(λ_3) < 0	attrattiva
	S_3	λ_1, Pε(λ_2)), Pε(λ_3) < 0	attattiva
24,74<r<30	λ_1, Pε(λ_2), Pε(λ_3) >0 regime caotico		

r>30	Chaotic regime with periodic windows

Bifurcations

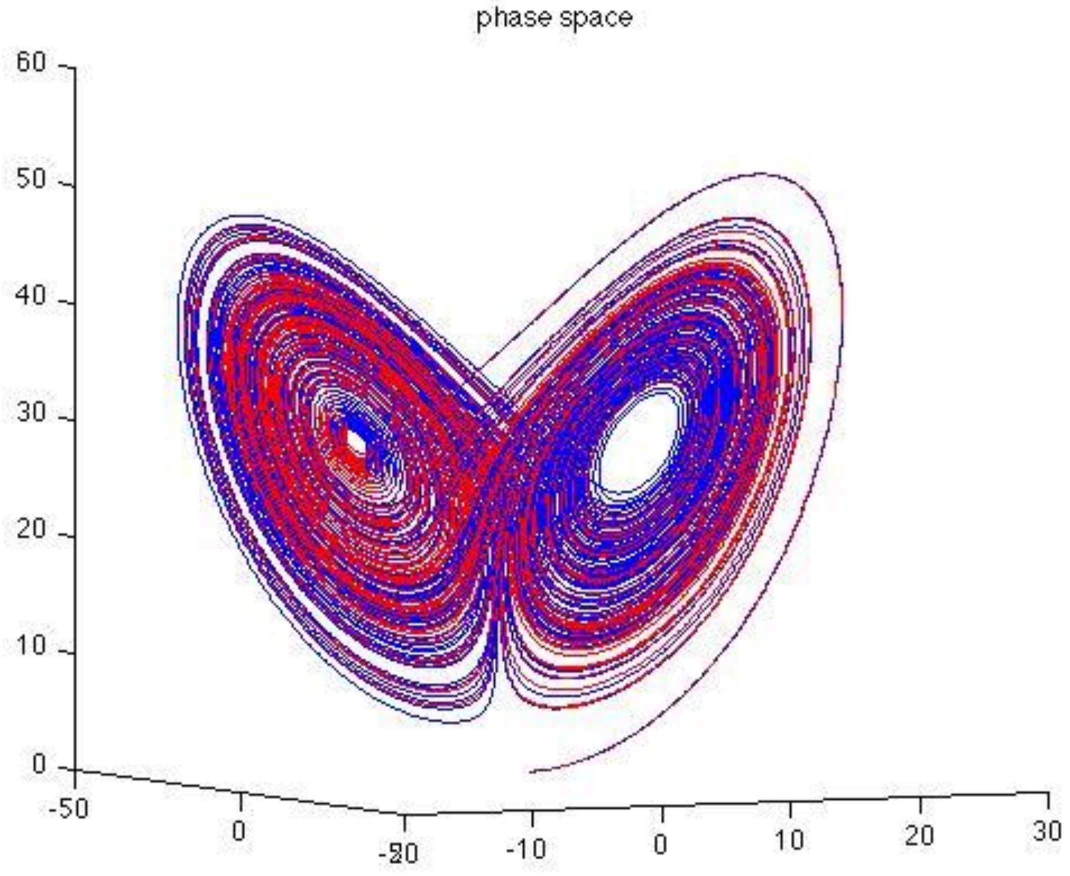
r = 1	S1 become unstable; S2, S3 are stable	Supercritical pitchfork bifurcation
r = 24.74	S2 e S3 become unstable, but no limit cycle is observed	S2 and S3 undergo a subcritical Hopf bifurcation

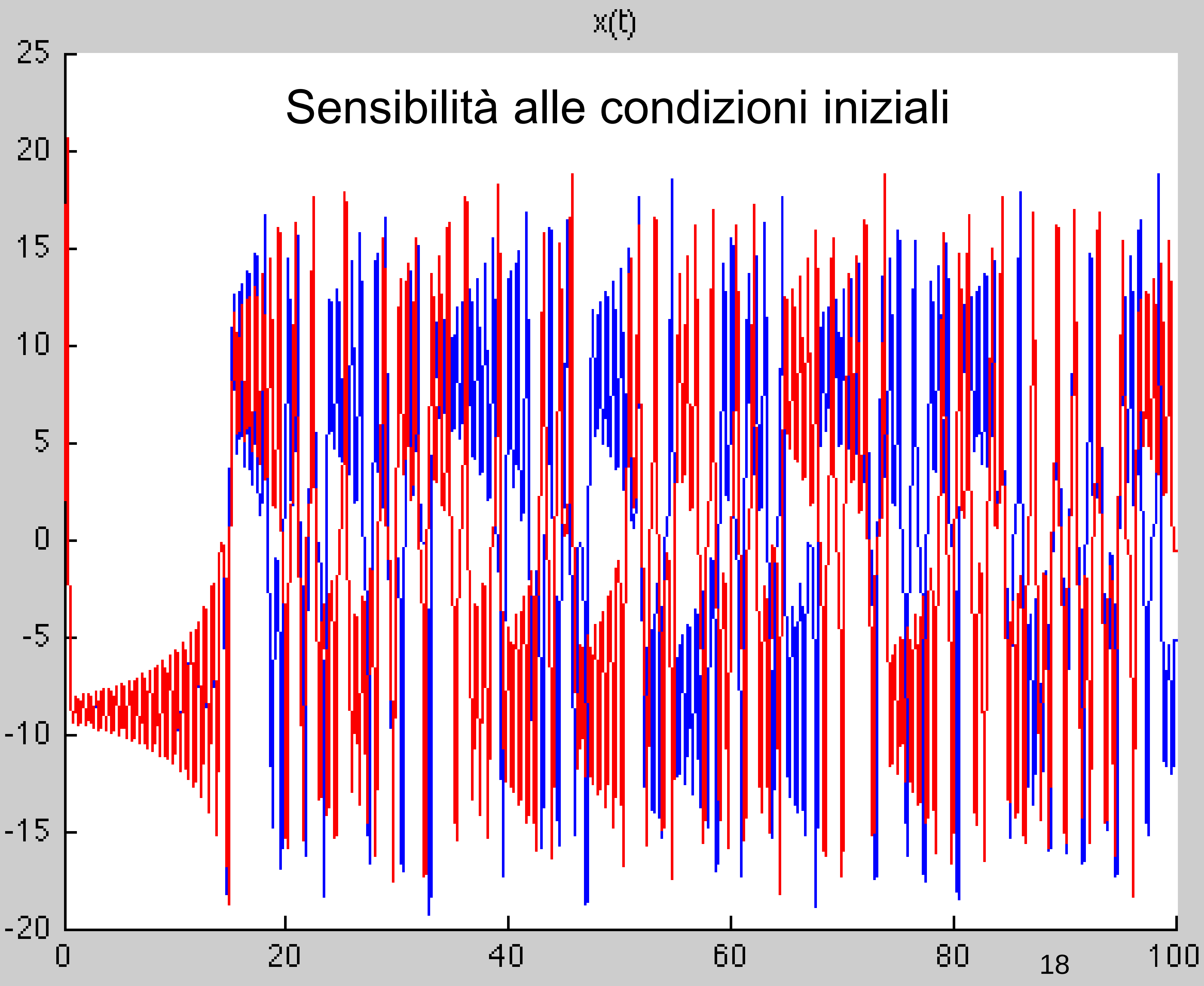
Bifurcation diagram



The Lorenz attractor

- “One meteorologist remarked that if the theory were correct, one flap of a seagull's wings would be enough to alter the course of the weather forever”.




x(t)
Sensibilità alle condizioni iniziali
25
20
15
10
5
0
-5
-10
-15
-20
0
20
40
60
80
100