List-recovery Lower Bound for Folded Linear Codes

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Abstract

In this note, we generalize the previous list-recovery lower bound to folded linear codes.

1 Introduction

In [CZ25], the authors prove a list-recovery lower bound for (folded) Reed–Solomon codes.

Theorem 1.1 ([CZ25, Corollary 3.2])**.** *Let $s \geq 1$, $2 \leq \ell \leq L, q \geq \ell$, generator γ of $\mathbb{F}_q^\times$ and $m = \lceil \log_\ell (L+1) \rceil > 1$. Suppose $R = \frac{k}{sn} \leq \frac{m-1}{m}$ and $\frac{k-1}{s} \geq m$. If a folded Reed–Solomon code $\mathsf{FRS}_{n,k}^{(s,\gamma)}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of rate R with appropriate evaluation points in $\mathbb{F}_q$ is (ρ, ℓ, L) list-recoverable, then*

$$\rho \leq \frac{L+1-\ell}{L+1}\left(1 - \frac{mR}{m-1}\right) + \frac{5}{n}.$$

This bound is then proved for random linear codes by a different approach by [LMS25], and generalized to any linear code by [LS25].

A recent survey [RV25, Question 4.9] proposes an open question to generalize this lower bound for additive codes (folded linear codes). In this note, we resolve it when m is not too large. The proof is a simple generalization of [CZ25].

2 Our Result

First we need a preliminary result known as "Hall's theorem for linear spaces".

Theorem 2.1 ([Mos96])**.** *Fix $m \geq 1$. For any m linear subspaces $U_1, \ldots, U_m \subseteq U$ within some ground linear space U, if for any $S \subseteq [m]$, there is $\dim(\sum_{i \in S} U_i) \geq |S|$, then there exists a sequence of linearly independent vectors $u_1, \ldots, u_m$ such that $u_i \in U_i$ for all $i \in [m]$.*

Actually Theorem 2.1 is a lot stronger than what we actually need in our proof, but we still use it for convenience.

Theorem 2.2. *Let $s \geq 2, \ell, m \geq 2, R \in (0,1), n \geq mk/(s(m-1))$ and $L = \ell^m - 1$. For any s-folded linear code $\mathcal{C} \subseteq (\mathbb{F}_q^s)^n$ with rate $R = k/(sn)$, if $\mathcal{C}$ is (ρ, ℓ, L) list-recoverable, then there must be*

$$\rho < \frac{L+1-\ell}{L+1}(1 - \frac{mR}{m-1}) + \frac{m}{n}.$$

Proof. For simplicity we assume $s(m-1) \mid k$. Let $A_1, \ldots, A_m \subseteq [n]$ be pairwise disjoint subsets of coordinates such that $|A_i| = \frac{k}{s(m-1)} - 1, \forall i \in [m]$. For each $i \in [m]$ we define the linear subspace $U_i \in \mathbb{F}_q^k$ as follows.

$$U_i = \{f \in \mathbb{F}_q^k \colon \mathcal{C}(f)[j] = 0, \forall j \in (A_1 \cup \cdots \cup A_m) \setminus A_i\}.$$

It follows that $\dim(U_i) \geq k - s|(A_1 \cup \cdots \cup A_m) \setminus A_i| = s(m-1) \geq m$. Therefore, by Theorem 2.1 we can pick linearly independent $u_1, \ldots, u_m \in \mathbb{F}_q^k$ such that $u_i \in U_i, \forall i \in [m]$. Let $Q = \{a_1, \ldots, a_\ell \in \mathbb{F}_q\}$ be an arbitrary set of ℓ distinct field elements. We define a list of messages as follows

$$\mathsf{LIST} = \{\sum_{i=1}^{m} b_i u_i \colon b_i \in Q, \forall i \in [m]\}$$

Since $u_1, \ldots, u_m$ are linearly independent, we know $|\mathsf{LIST}| = \ell^m = L + 1$.

For any $j \in [m], i \in A_j$, we define $S_i = \{t\mathcal{C}(u_j)[i] \colon t \in Q\}$. There is $|S_i| \leq \ell$. For any $f = \sum_{t=1}^{m} f_t u_t \in \mathsf{LIST}$, $f_1, \ldots, f_m \in Q$, we know

$$\mathcal{C}(f)[i] = \sum_{t=1}^{m} f_t \mathcal{C}(u_t)[i] = f_j \mathcal{C}(u_j)[i] \in S_i.$$

For any other coordinate $i \in [n] \setminus (A_1 \cup A_2 \cdots \cup A_m)$, we "uniformly" pick $f_{i,1}, \ldots, f_{i,\ell} \in \mathsf{LIST}$ such that $S_i = \{\mathcal{C}(f_{i,t})[i] \colon t \in [\ell]\}$.

For any $f \in \mathsf{LIST}$, we know that

$$\sum_{i=1}^{n} [\mathcal{C}(f)[i] \in S_i] = \sum_{i \in A_1 \cup \cdots \cup A_\ell} 1 + \sum_{i \notin A_1 \cup \cdots \cup A_\ell} \frac{\ell}{L+1} \geq \frac{mk}{s(m-1)} - m + \frac{\ell}{L+1}(n - \frac{mk}{s(m-1)})$$

This means

$$\mathsf{dist}(\mathcal{C}(f), S_1 \times \cdots \times S_n) = n - \sum_{i=1}^{n} [\mathcal{C}(f)[i] \in S_i] \leq \frac{L+1-\ell}{L+1}(n - \frac{mk}{s(m-1)}) + m.$$

Therefore, if $\rho \geq \frac{L+1-\ell}{L+1}(1 - \frac{mR}{(m-1)}) + \frac{m}{n}$, it is impossible that $\mathcal{C}$ is (ρ, ℓ, L) list-recoverable. $\square$

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