CIS 502

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1 Ch2

Question *(Fall 2014-I, 20 Points) You are given a binary tree which satisfies the (min) heap property: the parent is smaller than both the children. It is not a search tree and need not be balanced. The number of elements in the heap is n. At each node, you can only see the value at the node and follow the left or right pointers to the respective children.*

You want to know if the k^{th} smallest element in the heap is less than the value y. Give a worst case $O(k)$ time algorithm. The algorithm cannot depend on n. Your algorithm only needs to output yes/no.

You do not need to output that element.
You need to give reasons why your answer is correct.

Answer *To count the number of nodes with value less than y. Apply BFS with extra rules:*

1. *if confronting a node with element larger than y, stop searching that branch since its descendants have elements larger than y);*
2. *if* `counter`$\geq k$ *return FALSE (and stop searching);*
3. *return TRUE.* ♢

2 Ch4 Greedy algorithm

Answer (HW1, 4-6, refers to [3]) *Let the contestants be numbered $1, \ldots, n$, and let s_i, b_i, r_i denote the swimming, biking, and running times of contestant i. Here is an algorithm to produce a schedule:*

> *Arrange the contestants in order of decreasing $b_i + r_i$, and send them out in this order.*

We claim that this order minimizes the completion time.

Proof *Suppose not, that is there exists an optimal schedule S not using this order. Then optimal schedule S must contain two contestants i and j so that j is sent out directly after i, but $b_i + r_i < b_j + r_j$. We will call such a pair (i, j) an* inversion. *Consider the schedule S' obtained by swapping the orders of i and j.*

In this schedule S' from swaping $i\&j$ in S, j completes earlier than he/she used to. Also, in the swapped schedule, i gets out of the pool in S' when j got out of the pool in S; but since $b_i + r_i < b_j + r_j$, i finishes sooner in S' than j finished in S. Hence S' does not have a greater completion time, and so it too is optimal.

Continuing in this way, we can eliminate all inversions without increasing the completion time. At the end of this process, we will have a schedule in the order produced by our algorithm, whose completion time is no greater than that of the original optimal order we considered. Thus the order produced by our algorithm must also be optimal. ♢

Answer (4-11, refers to [3]) *Label the edges arbitrarily as* $e_1, \dots, e_m$ *with the property that* $e_{m-n+1}, \dots, e_m$ *belong to* T. *Let* δ *be the minimum difference between any two non-equal edge weights; subtract* $\delta i/n^3$ *from the weight of edge* i. *Note that all edge weights are now distinct, and the sorted order of the new weights is the same as some valid ordering of the original weights. Over all spanning trees of* G, T *is the one whose total weight has been reduced by the most; thus, it is now the unique minimum spanning tree of* G *and will be returned by Kruskal's algorithm on this valid ordering.* ♢

Answer (HW1, 4-12) *(a) False by referring* $(0,1),(6000,2)$ *with* $r = 3000$.
(b) Order the streams by ascent $\frac{b_i}{t_i}, i = 1, \dots, n$.

Proof *Suppose not. Consider the schedule of OPT not following the greedy order since it is feasible and greedy is not. Then there exists two adjacent jobs* i, j *in OPT which were scheduled in the order not preferred by greedy. Suppose OPT started transmitting* i *first at time* t_0 *and had already sent* b_0 *bits by then. Since OPT is feasible then*

$$b_0 \le rt_0, b_0 + b_i \le r(t_0 + t_i), b_0 + b_i + b_j \le r(t0 + t_i + t_j),$$

and $b_i t_j > b_j t_i$.

Swap order of i *and* j *also provides a feasible solution.* ♢

Proof *Suppose not, that is,* $b_0 + b_j > r(t_0 + t_j)$. *Then*

$$b_i = (b_0 + b_i + b_j)) - (b0 + bj) < r(t_0 + t_i + t_j) - r(t_0 + t_j) = rt_i,$$

then $\frac{b_j}{t_j} < \frac{b_i}{t_i} < r$ *implies* $b_0 + b_j < r(t_0 + t_j)$. *A contradiction.*

Answer (HW1, 4-13) *Below is to derive the sort order from the equations (of course this would work easily only in cases where the sort order is fixed - but that is the case here).*

Consider some sort order and we have a pair of adjacent jobs i, j *in the solution of OPT which violate the order. If job* i *went first and started at time* t *then the contribution of the two jobs to the total weighted completion time is*

$$w_i(t + t_i) + w_j(t + t_i + t_j);$$

If we performed a swap to j, i (keeping everything else the same) then the contributions of all other jobs would remain the same, and the contribution of these two would be

$$w_j(t + t_j) + w_i(t + t_i + t_j);$$

The second term would be smaller or equal (this is the condition needed to prove that the swap does not make things worse) when $w_j(t + t_j) + w_i(t + t_i + t_j) < w_i(t + t_i) + w_j(t + t_i + t_j)$, which turns out to be just $w_i t_j < w_j t_i$. Which means schedule in the nonincreasing order of $wj = tj$. ♢

Answer (HW1, 4-15)

1. *Sort everyone by the (nondecreasing) finish time. This is sequence F;*
2. *Also Sort everyone by the (nondecreasing) start time. This is sequence S;*
3. *Find the First finish time in F, say it is fi corresponding to job i. A committee member must have started his/her shift by then.*
4. *Consider all the jobs in S which start by f_i. Among these choose the person who finishes last say $\sigma(i)$. Choose this person in the committee. Note that if the optimum chose differently then this person can oversee the same set of persons overseen by the committee member chosen by the optimum. Remember the last seen element of S so far, say s.*
5. *Continue along F and mark every person who finishes before $\sigma(i)$ as overseen.*
6. *repeat*
   (a) *Consider the next person in F (who is not yet overseen) say this is f_j .*
   (b) *Starting after s in Step 4, find all persons who starts before f_j in S. This person may have already been overseen. Remember the person who finishes last (say $\sigma(j)$) and the new value of s.*
   (c) *Continue along F and mark every person who finishes before $\sigma(j)$ as overseen.*
7. *until $F = \emptyset$.*

Running time $O(n \log n)$ as well. ♢

Question (HW2,4-17) *Consider the following variation on the Interval Scheduling Problem. You have a processor that can operate 24 hours a day, every day. People submit request to run daily jobs on the processor. Each such job comes with a start time and an end time; if the job is accepted to run on the processor, it must run continuously, every day, for the period between its start and end times. (Note that certain jobs can begin before midnight and end after midnight; this makes for a type of situation different from what we saw in the Interval Scheduling Problem.)*

Given a list of n such jobs, your goal is to accept as many jobs as possible (regardless of their length), subject to the constraint that the processor can run at most one job at any given point in time. Provide an algorithm to do this with a running time that is polynomial in n. You may assume for simplicity that no two jobs have the same start or end times.

Answer (refers to [3]) *Here is the algorithm where T_{all} represent the 24–hour time.*

1. *Input $I_1, \dots, I_N$, N intervals/ requests of time, represented by $I_j \doteq (t_{start}, t_{end})$;*
2. *For $j = 1, \dots, N$*
   - (a) *Denote by S_j, the set of intervals/ requests selected; by A_j, the set of intervals/ requests still available;*
     (initialization) $S_j \leftarrow \{I_j\}$, $A_j \leftarrow \{I_k : k = 1, \dots, N, I_k \cap I_j = \emptyset\}$;
   - (b) *Notice $T_{all} \setminus I_j$ can be regarded as a line when rest of the intervals are to be scheduled selectively. According to classic interval/request scheduling problem, S, A can be modified by first-finish-first schedule routine.*
3. *Compare $\#S_j$ and select $j_0 \in arg \max_{j=1,\dots,N} \#S_j$, namely the index j_0 for which we can schedule with as many intevals/ requests as possible;*
4. *Output S_{j_0}.*

Running time: $\diamondsuit$

Question (HW2,4-20) *Every September, somewhere in a far-away mountainous part of the world, the county highway crews get together and decide which roads to keep clear through the coming winter. There are n towns in this county and the road system can be viewed as a (connected) graph $G = (V, E)$ on this set of towns, each representing a road joining two of them. In the winter, people are high enough up in the mountains that they stop worrying about the length of roads and start worrying about their altitude – this is really what determines how difficult the trip will be.*

So each road – each edge e in the graph – is annotated with a number a_e that gives the altitude of the hightest point on the road. We will assume that no two edges have exactly the same altitude value a_e. The height of a path P in the graph is then the maximum of a_e over all edges e on P. Finally, a path between towns i and j is declared to be winter-optimal if it achieves the minimum possible height over all paths from i to j.

The highway cres are going to select a set $E' \subset E$ of the roads to keep clear through the winter; the rest will be left unmaintained and kept off limits to travellers. They all agree that whichever subset of roads E' they decide to keep clear, it should have the property that (V, E') should be no greater than it is in the full graph $G = (V, E)$. We will say that (V,E') is a minimum-altitude connected subgraph if it has this property.

Given that they are going to maintain this key property, however, they otherwise want to keep as few roads clear as possible. One year, they hit upon the following conjecture:

- *The minimum spaning tree of G, with respect to the edge weights a_e, is a minimum-altitude connected subgraph.*

(In an earlier problem, we claimed that there is a unique minimum spanning tree when the edge weights are distinct. Thus, thanks to the assumption that all a_e are distinct, it is okay for us to speak of the minimum spanning tree.)

Initially, this conjecture is somewhat counterintuitive, since the minimum spanning tree is trying to minimize the sum of the values a_e, while the goal of minimizing altitude seems to be asking for a fairly different thing. But lacking an argument to the contrary, they begin considering an even bolder second conjecture:

- *A subgraph (V, E') is a minimum–altitude connected subgraph if and only if it contains the edges of the minimum spanning tree.*

Note that this second conjecture would immediately imply the first one, since a minimum spanning tree contains its own edges.

So here is the question.

1. *Is the first conjecture true, for all choices of G and distinct altitudes a_e? Give a proof or a counterexample with explanation.*
2. *Is the second conjecture true, for all choices of G and distinct altitudes a_e? Give a proof or a counterexample with explanation.*

Answer 1. *True. By exercise 4-8, the minimal spanning tree is unique and can be obtained by Kruskal's algorithm. The thing to show is that Kruskal's algorithm also produces a minimum-altitude connected subgraph. It suffices to show counterpart of lemma 4.17:*

Lemma 1 (4.17') *Let S be any subset of nodes that is neighter empty nor equal to all of V and let edge $e = (v, w)$ be the minimum–cost edge with one end in S and the other in $V - S$. There exists a minimum spanning tree containing the edge e.*

Proof *Suppose not, that is any spanning tree T containing e cannot be a minimal–altitude subgraph.*

Given ang minimal–altitude subgraph T_1. In order to show there exists a tree T_2 containing e that has at most the same altitude, an exchange argument is applied:

The crux is therefore to find an edge that can be successfully exchanged with e. Recall that the ends of e are $v \in S, w \in V - S$. That T_1 is connected infers there exists a path P in T from v to w. Starting at v, suppose first node in $V - S$ of P in sequence $v \to w$ is w' and $v' \in S$ be previous node of w'. Denote $e' = (v', w')$, an edge of T_1 with one end in S and the other in $V - S$.

Exchange of e and e' produces $T_2 = (T_1 - \{e'\}) \cup \{e\}$. T_2 is also a spanning subgraph since:

(a) *Clearly it is connected since (V, T_1) is connected and that any path in (V, T_2) that used the edge e' can now be "rerouted" in (V, T_2);*
(b) *It is also acyclic since the only cycle in $(V, T_2 \cup \{e'\}$ is the one composed of e and the path P and this cycle is not present in (V, T_2) due to the deletion of e'.*

Note that $a_e \leq a_{e'}$ implies altitude of (V, T_2) is less equal to that of (V, T_1); specifically, (V, T_2) is also a minimal–altitude connected subgraph and contains e. This leads to a contradiction!

Therefore, there exists a minimum spanning tree containing edge e. $\diamondsuit$

2. *True. We show that*

> *every edge added by Kruskal's algorithm has to be added to the min-altitude subgraph.*

Proof *Suppose not. Let (u, v) be the first time we add an edge in Kruskal's algorithm which is not in our min-altitude subgraph. Consider the cut defined by the edge (u, v) say $S, V \setminus S$ (where S corresponds to the nodes in the subtree connected to u). By (4-17), every other edge that spans the cut has altitude more than that of (u, v). So the minimum altitude path between u and v itself must be the edge (u, v). Thus we have a contradiction to the min-altitude subgraph property for this pair.* $\diamondsuit$

We now show that

> *the min-altitude subgraph is a tree, provided that the weights are distinct.*

Proof *Suppose not. Consider a cycle – the highest altitude edge can be deleted without affecting the connectivity in any way. Therefore the min-altitude subgraph is exactly the minimum spanning tree.* $\diamondsuit$

Question (HW2, 4-21) *Let us say that a graph $G = (V, E)$ is a near-tree if it is connected and has at most $n+8$ edges, where $n = |V|$. Give an algorithm with running time $O(n)$ that takes a near-tree G with costs on its edges, and returns a minimum spanning tree of G. You may assume that all the edge costs are distinct.*

Answer *Use the cycle property: that given any cycle there exists a minimum spanning tree that does not use the heaviest edge in that cycle.*

1. *ensure that the graph is connected in $O(n + n + 8) = O(n)$ time;*
2. *use favourite algorithm to find a cycle (BFS/DFS) and can find a cycle in $O(n + m) = O(n + n + 8) = O(n)$ time;*
3. *delete one edge. We now have $n + 7$ edges.*

Repeat this step for 8 more steps, we will have $n - 1$ edges left.At this point we must have a tree since the graph is connected. The total running time is $9 \cdot O(n) = O(n)$. ♢

Answer (HW2,4-28)

Question (HW3,4-31, variant of Kruskal's algorithm) *Since Minimum Spanning Tree Problem is somehow related with shortest $u - v$ path, following variant of Kruskal's algorithm is designed:*

1. *Sort all the edges in order of increasing length (assume all edge lengths are distinct except that $d_{uv} = \infty$ once they are disconnected in certain graph);*
2. *construct a subgraph $H = (V, F)$ by considering each edge in order (from short to long);*
3. *when it comes to edge $e = (u, v)$, add e once $3l_e < d_{uv}$.*

Speaking of subgraph $H = (V, F)$ it produced:

1. *Prove that for every pair of nodes $u, v \in V$, the length of the shortest $u - v$ path in H is at most three times the length of the shortest $u - v$ path in G;*
2. *Despite its ability to approximately preserve shortest–path distances, the subgraph H cannot be too dense. Let $f(n)$ denote the*

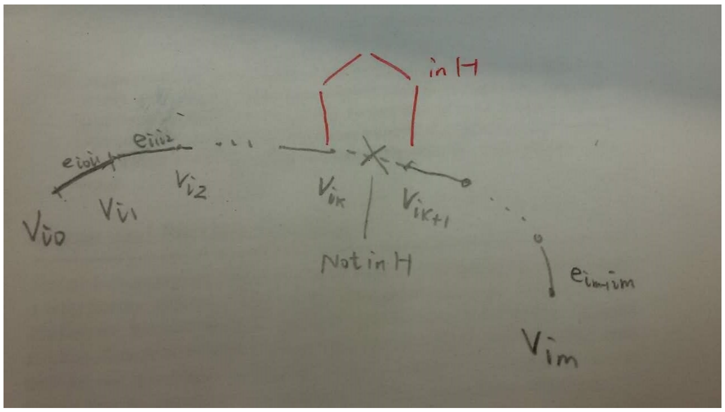


Figure 1: P_{12}, one of shortest paths between v_1, v_2 in $G = (V, E)$.

maximum number of edges that can possibly be produced as the output of this algorithm, over all n− node input graphs with edge lengths. Prove that $\lim_{n\to\infty} \frac{f(n)}{n^2} = 0$.

Answer 1. *Denote a shortest path in* $G = (V, E)$ *between* $v_1, v_2 \in V$ *by*

$$P_{12} \doteq e_{i_0 i_1} e_{i_1 i_2} \ldots e_{i_{m-1} i_m} \xlongequal{i_0=1, i_m=2} e_{1 i_1} e_{i_1 i_2} \ldots e_{i_{m-1} 2},$$

where $\{e_{i_k i_{k+1}} \doteq (v_{i_k}, v_{i_{k+1}})\}_{k=0}^{m-1} \subset E, \{v_{i_k}\}_{k=0}^{m} \subset V$ *and* $\sum_{k=0}^{m-1} l_{e_{i_k i_{k+1}}}$ *is the length of* P_{12}.

Speaking of any $e_{i_k i_{k+1}} \in E \setminus F$ *(or alternatively,* $\in G \setminus H$*) , there exists* $P'_{i_k i_{k+1}} \subset F$*, another path (red notation in figure 1 which as a matter of fact, could be shortest path between* $v_{i_k}, v_{i_{k+1}}$ *in* H*). In all, whatever* $e_{i_k i_{k+1}} \in F$ *or not, there exists a path in* H *such that its length is smaller than* $3 \sum_{k=0}^{m-1} l_{e_{i_k i_{k+1}}}$.

2. ***(refers to [3])***

Lemma 2 *Cycle in H has at least 5 edges.*

Proof *Suppose not and C is such a cycle while $e = (u, v)$ is the last edge added to it. Then at the moment e was considered, there was a u-v path $Q_{uv} \in H = (V, F)$ of at most three edges, on which each edge had length at most ℓ_e. Thus ℓ_e is not less than a third the length of Q_{uv}, and so it should not have been added.* $\diamondsuit$

Lemma 3 *An n-node graph H with no cycle of length ≤ 4 must contain a node of degree at most $\sqrt{n}$.*

Proof *Suppose not and consider any node v of H. Let $S(v) \doteq \{u \in V : (u, v) \in F\}$, set of neighborhood of v and $\#S(v) = |S(v)| > \sqrt{n}$. Notice that there is no edge joining two nodes of $S(v)$, or we would have a cycle of length 3. Now let $N(S(v)) \doteq \{u \in V : S(u) \cap S(v) \neq \emptyset\}$, set of all nodes with a neighbor in $S(v)$. Since H has no cycle of length 4, each node in $N(S)$ has exactly one neighbor in $S(v)$. But $|S(v)| > \sqrt{n}$, and each node in S has $\geq \sqrt{n}$ neighbors other than v, so we would have $|N(S(v))| > n$, a contradiction.* $\diamondsuit$

Proof (1, for 2^{nd} part of statement; refers to [3]) *Now, if we let $g(n)$ denote the maximum number of edges in an n-node graph with no cycle of length 4, then $g(n)$ satisfies the recurrence $g(n) \leq g(n-1) + \sqrt{n}$ by deleting the lowest-degree node and lemma 3, and so we have $g(n) \leq n^{3/2} = o(n^2)$.* $\diamondsuit$

Proof (2, for 2^{nd} part of statement; refers to [3]) *Suppose not. H has at least εn^2 edges, then the sum of all degrees is $2\varepsilon n^2$, $\xRightarrow{lemma\ ??}$ there is a set S of at least εn nodes each of whose degrees is at least εn. Now, consider the set Q of all pairs of edges (e, e') such e and e' each have an end equal to the same node in S. We have $|Q| \geq \varepsilon n \binom{\varepsilon n}{2}$, since there are at least εn nodes in S, and each contributes at least $\binom{\varepsilon n}{2}$ such pairs. For each edge pair $(e, e') \in Q$, they have one end in common; we* label *(e, e') with the pair of nodes at their other ends. Since $|Q| > \binom{n}{2}$ for sufficiently large n, the pigeonhole principle implies that some two pairs of edges $(e, e'), (f, f') \in Q$ receive the same label. But then $\{e, e', f, f'\}$ constitutes a four-node cycle. For a second proof, we observe that an n-node graph H with no cycle of length ≤ 4 must contain a node of degree at most $\sqrt{n}$. For suppose not, and consider any node v of H. Let S denote the set of neighbors of v.*

Notice that there is no edge joining two nodes of S, or we would have a cycle of length 3. Now let $N(S)$ denote the set of all nodes with a neighbor in S. Since H has no cycle of length 4, each node in $N(S)$ has exactly one neighbor in S. But $|S| > \sqrt{n}$, and each node in S has $\geq \sqrt{n}$ neighbors other than v, so we would have $|N(S)| > n$, a contradiction. Now, if we let $g(n)$ denote the maximum number of edges in an n-node graph with no cycle of length 4, then $g(n)$ satisfies the recurrence $g(n) \leq g(n-1) + \sqrt{n}$ (by deleting the lowest-degree node), and so we have $g(n) \leq n^{3/2} = o(n^2)$. $\diamondsuit$

Remark *1. Suppose not. Denote a shortest path in $H = (V, F)$ between $v_1, v_2 \in V$ by*

$$p_{12} = e_{i_0 i_1} e_{i_1 i_2} \dots e_{i_{m-1} i_m} \xlongequal{i_0=1, i_m=2} e_{1 i_1} e_{i_1 i_2} \dots e_{i_{m-1} 2},$$

where $\{e_{i_k i_{k+1}} \doteq (v_{i_k}, v_{i_{k+1}})\}_{k=0}^{m-1} \subset F \subset E, \{v_{i_k}\}_{k=0}^{m} \subset V$ but $d_{v_1 v_2} = \sum_{k=0}^{m-1} l_{e_{i_k i_{k+1}}}$, the length of p_{12} is more than three times of the shortest-path length between v_1, v_2 in $G = (V, E)$.

Notice $p_{i_j i_k} \doteq e_{i_j i_{j+1}} e_{i_{j+1} i_{j+2}} \dots e_{i_{k-1} i_k} (j < k)$ is also one of the shortest paths between v_{i_j}, v_{i_k}. Then above background infers: $\exists k_0 \in [m]$ such that $l_{e_{i_{k_0} i_{k_0+1}}}$ is more than the shortest-path lengths between $v_{i_{k_0}}$, $v_{i_{k_0+1}}$ in G. $\diamondsuit$

Question (HW3,4-32, variant of minimum-cost arborescence algorithm) *Consider a directed graph $G = (V, E)$ with a root $r \in V$ and nonnegative costs on the edges. In this problem we consider variants of the minimum-cost arborescence (MCA) algorithm.*

1. *MCA algorithm discussed in section 4.9 of [2] works as follows. We modify the costs, consider the subgraph of zero-cost edges, look for a directed cycle in this subgraph, and contract it (if one exists). Argue briefly that instead of looking for cycles, we can instead identify and contract strong components[a] of this subgraph;*

2. *After defining y_v to be the minimum cost of an edge entering v and we modified the costs of all edges e entering node v to be $c''_e \doteq \max(0, c_e - 2y_v)$ instead of $c'_e \doteq c_e - y_v$. This new change is likely to turn more edges to 0 cost. Suppose now we find an*

arborescence T of 0 cost. Prove that this T has cost at most twice the cost of the MCA in the original graph;

3. *Assume you do not find an arborescence of 0 cost. Contract all $0-$ cost strong components and recursively apply the same procedure on the resulting graph until an arborescence is found. Prove that this T has at most twice the cost of the MCA in the original graph.*

[a] From [1], a graph is said to be strongly connected if every vertex is reachable from every other vertex.

Answer 1. *Counterpart of (4.38) of [2] needs to be presented:*

Lemma 4 *Let D be a strong component in G consisting of edges of cost 0, such that $r \notin D$. Then there is an optimal arborescence rooted at r that has exactly one edge entering D.*

Proof *Is that true that a strong component might contain a circle?* $\diamondsuit$

(refers to [3]) *Let (V, F) and (V, F') be distinct arborescences rooted at r. Let*

$$\vec{e} \doteq arg \min_{\vec{e} \in F \ominus F'} d_r(\vec{e}), \tag{1}$$

where $F \ominus F' \doteq (F \setminus F') \cup (F' \setminus F)$ and $d_r(\vec{e})$ is the distance from edge e to root r in its arborescence. Suppose $\vec{e} = (u, v) \in F'$. In (V, F), there is some other edge (w, v) entering v.

Now define $F'' = F \setminus \{(w, v)\} \cup \{\vec{e}\}$. We claim that (V, F'') is also an arborescence rooted at r:

(a) *Clearly F'' has exactly one edge entering each node;*

(b) *To verify that there is an r-x path for every node x, we notice:*

   i. *for those x such that the r-x path in (V, F) does not use v, the same r-x path exists in F'';*

   ii. *for x whose r-x path in (V, F) does use v, let $Q \in (V, F')$ denote the r-u path, and let $P \in (V, F)$ denote the v-x path. Note that $P \cap E$[1] $\subseteq F''$, since they all belong to F and (w, v) is not among them. But we also have $Q \cap E \subseteq F \cap F'$ due to definition (**??**). Thus $Q \cap E \subseteq F''$ since $(w, v) \notin Q$. Hence the concatenated path $(Q \cdot \vec{e} \cdot P) \cap E \subseteq F''$, and so there is an r-x path in (V, F'').*

[1] all edges of $P \in (V, E)$.

Notice $F'' \cap F' = (F \cap F') \cup \{\vec{e}\}$. A sequence of these operations transform (V, F) into (V, F') one edge at a time. But each of these operations changes the cost of the arborescence by at most 1 (since all edges have cost 0 or 1). So if we let (V, F) be a minimum-cost arborescence (of cost a) and we let (V, F') be a maximum-cost arborescence (of cost b), then if $a \leq k \leq b$, there must be an arborescence of cost exactly k.

Remark *Is that true that a strong component might contain a circle?* ♢

2. *If $e_1, \ldots, e_{n-1}$ happened to be an arborescence with $c''_{e_i} = 0, \forall i \in [n-1]$, $c_{e_i} \leq 2y_{v_i}, i \in [n-1]$ had we denote end point of e_i by v_i. This indicates*

$$\sum_{i=1}^{n-1} c_{e_i} \leq 2\sum_{i=1}^{n-1} y_{v_i} \leq 2S((\text{by (4.36) of [2]}),$$

when S is the sum of edge lengths in a MCA[2]. Therefore proof is done;

3.

Question (HW3,5-1) *While analyzing some hard-to-obtain data from two separate databases. Each database contains n numerical values – so there are $2n$ values total – and you may assume that no two values are the same. You'd like to determine the median of this set of $2n$ values, which we will define here to be the n^{th} smallest value.*

Howerver, the only way you can access these values is through queries to the databases. In a single query, you can specify a value k to one of the two data bases, and the chosen database will return the k^{th} smallest value that it contains. Since queries are expensive, you would like to compute the median using as few queries as possible.

Give an algorithm that finds the median value using at most $O(\log n)$ queries.

Question (Fall 2014-I, 30 Points) *You have an undirected graph with weights $w_{e'}$ on edges e'. All weights $w_{e'} \geq 0$. You have already computed an MST T for the graph. You are now told that the weight of some single edge e should actually be $w'_e \geq 0$. For each of the four cases below give the most efficient (as low running time as possible) algorithm*

[2] Minimum-cost arborescence.

for computing the MST under the new weights. Give a short argument for correctness for each.

1. $w'_e < w_e$ *and* $e \in T$ *(10 pts);*
2. $w'_e < w_e$ *and* $e \notin T$ *(5 pts);*
3. $w'_e > w_e$ *and* $e \in T$ *(10 pts);*
4. $w'_e > w_e$ *and* $e \notin T$ *(5 pts).*

Answer

1. *no change;*
2. (a) *add the edge e to* T *and find the unique cycle (using favorite algorithm, say DFS). The running time is* $O(n)$ *because a tree has* $n-1$ *edges;*

   (b) *delete the maximum edge along this cycle and that is the new tree.*

   Follows from the cycle property.
3. *We delete the edge e from* T *and find the two (connected) components* $S, V \setminus S$ *of the tree* T*. This is* $O(n)$*. We now find the minimum edge across this cut in time* $O(m)$*. Based on Kruskal's algorithm, this is the new MST.*
4. *no change.* ♢

3 Ch5 Divide and conquer

Answer (refers to [3]) *Say* A *and* B *are the two databases and* a_i, b_i *are* i^{th} *smallest elements of* A, B.

1. ***Compare the medians of the two databases.*** $k \leftarrow \lceil \frac{1}{2} n \rceil$*, then* a_k *and* b_k *are the medians of the two databases. Without loss of generality, suppose* $a_k < b_k$*. Then one can see that*

$$b_k > a_k > a_j, b_k > b_j, j = 1, \ldots, k-1,$$

   Therefore b_k *is at least* $2k^{th}$ *element in the combine databases.*

   (a) *Since* $2k \geq n$*, all elements that are greater than* b_k *are greater than the median and we can eliminate* $\{b_j, j = k+1, \ldots, n\} \subset B$. $B' \leftarrow \{b_j, j = 1, \ldots, k\}$*, the half of* B.

(b) *Similarly, the first* $\lfloor \frac{1}{2}n \rfloor$ *elements of* A *are less than* b_k*, and thus, are less than the last* $n-k+1$ *elements of* B*. Also they are less than the last* $\lceil \frac{1}{2}n \rceil$ *elements of* A*.So they are less than at least* $n-k+1+\lceil \frac{1}{2}n \rceil = n+1$ *elements of the combine database. It means that they are less than the median and we can eliminate them as well.*

$A' \leftarrow \{a_j, j = k+1, \ldots, n\} = \{a_j, j = \lfloor \frac{1}{2}n \rfloor + 1 \ldots, n\}$*, the remaining parts of* A *.*

2. *Now we eliminate* $\lfloor \frac{1}{2}n \rfloor$ *elements that are less than the median, and the same number of elements that are greater than median. It is clear that the median of the remaining elements is the same as the median of the original set of elements.*

   We can find a median in the remaining set using recursion[3] *for* A', B' *instead of* A, B*.*

[3] formally address the algorithm by writing recursive function: `median(`n`,`$StartA$`,`$StartB$`)`*:*

1. *input: take integers* n*,* $StartA$ *and* $StartB$*;*
2. *output: return the median of the union of the two segments* $A[StartA+1; StartA+n]$ *and* $B[StartB+1; StartB+n]$*;*
3. *codes:* `median`*(*n*,* $StartA$*,* $StartB$*)*

Code 1 (a) *if* $1 == n$ *then return* $\min(a_{StartA+k}, b_{StartB+k})$*;*

(b) $k \leftarrow \lceil \frac{1}{2}n \rceil$*;*

(c) *if* $a_{StartA+k} < b_{StartB+k}$ *then*
return `median(`k`,` $StartA + \lfloor \frac{1}{2}n \rfloor$ `,`$StartB$`);`
else return `median(`k`,` $StartA$`,` $StartB + \lfloor \frac{1}{2}n \rfloor$`).`

To find median in the whole set of elements we evaluate `median(`n`,0,0).`

Let $T(n)$ *be the number of queries asked by our algorithm to evaluate* `median(`n`,`$StartA$`,`$StartB$`).` *Then it is clear that* $T(n) = T(\lceil \frac{1}{2}n \rceil) + 2$*. Therefore* $T(n) = 2\lceil \log n \rceil = O(\log n)$*.* ♢

[3] Note that we can't delete elements from the databases. However, we can access a'_i, b'_i, i^{th} smallest elements of A' and B' separately since $a'_i = a_{i+\lfloor \frac{1}{2}n \rfloor}, b'_i = b_{i+\lfloor \frac{1}{2}n \rfloor}$.

Question (HW3,5-2) *Recall the problem of finding the number of inversions. As in the text, we are given a sequence of n numbers* $a_1, \dots, a_n$ *which we assume are all distinct, and we define an inversion to be a pair* $i < j$ *such that* $a_i > a_j$.

We motivated the problem of counting inversions as a good measure of how different two orderings are. However, one might feel that this measure is too sensitive. Let's call a pair a significant inversion if $i < j$ *and* $a_i > 2a_j$. *Give an* $O(n \log n)$ *algorithm to count the number of significant inversions between two orderings.*

Answer *functions* `Merge-and-Count`, `Merge-and-Count2`, `Sort-and-Count` *are involved while*

- *function* `Merge-and-Count` *is the classic function for classic "counting-inversion" from page 324 of [2];*
- `Merge-and-Count` *is the actual merging process for the problem, calling function* `Merge-and-Count`*;*
- *function* `Sort-and-Count2` *differs from* `Sort-and-Count` *on page 325 of [2] only on one line – that is it calls* `Merge-and-Count2` *instead of* `Sort-and-Count`.

[r,L]=`Merge-and-Count`*(A,B) (classic "*`Merge-and-Count`*" function for classic "counting-inversion", from page 324 of [2]):*

Code 2

1. *Maintain a current pointer into each list, initialized to point to the front elements;*
2. *Maintain a variable Count for the number of inversions, initialized to* 0*;*
3. *While both lists are nonempty:*
   - (a) *Let* a_i *and* b_j *be the elements pointed to by the Current pointer;*
   - (b) *Append the smaller of these two to the output list;*
   - (c) *If* b_j *is the smaller element then: increment Count by the number of elements remaining in A;*
   - (d) *Advance the Current pointer in the list from which the smaller element was selected.*

4. *Once one list is empty, append the remainder of the other list to the output;*

5. *Return Count and the merged list.*

[r,L]=`Merge-and-Count2`*(A,B) (actual "*`Merge-and-Count`*" function for the problem):*

Code 3 1. $[r, L1]$ =`Merge-and-count`*(2A,B) where* $2A \doteq \{2a \in \mathbf{R} : a \in A\}$ *(actually only need to find r);*

2. $[r1, L]$ =`Merge-and-count`*(A,B) (actually only need to merge two lists);*

3. *return r,L.*

[r,(sorted) L]=`Sort-and-Count`*(L) (different from the one in classic "counting-inversion" algorithm)*

Code 4 1. *If the list has one element then: return* $0, L$*;*

2. *divide the list into two halves: A contains the first* $\lfloor \frac{1}{2} n \rfloor$ *elements, B contains the remaining elements;*

3. $[r_A, A]$=`Sort-and-Count2`*(A);*

4. $[r_B, B]$=`Sort-and-Count2`*(B);*

5. $[r, L]$=`Merge-and-Count2`*(A,B); (the only major as classic "counting-inversion" algorithm)*

6. *Return* $r = r_A + r_B + r$ *and the sorted list L.*

Time cost: *Since time cost of* `Merge-and-Sort2` *is also* $O(n)$*, (5.1) and (5.2) of [2] also infer the whole algoritm to be* $O(n \log n)$ *with respect to time cost.* $\diamondsuit$

Question (HW4, 5-4) *You have been working with some physicists who need to study, as part of their experimental design, the interactions among large numbers of very small charged particles. Basically, their setup works as follows. They have an inert lattice structure, and they use this for placing charged particles at regular spacing along a straight line. Thus we can model their structure as consisting of the points*

$\{1, 2, \ldots, n\}$ on the real line; and at each of these points j, they have a particle with charge q_j. (Each charge can be either positive or negative.)

They want to study the total force on each particle, by measuring it and then comparing it to a computational prediction. This computational part is where they need your help. The total net force on particle j, by coulomb's Law, is equal to

$$F_j \doteq \sum_{i<j} \frac{Cq_iq_j}{(j-i)^2} - \sum_{i>j} \frac{Cq_iq_j}{(j-i)^2},$$

They have written the following simple program to compute F_j for all j:

Code 5 *For $j = 1, 2, \ldots, n$;*

1. *Initialize $F_j \leftarrow 0$;*
2. *For $i = 1, 2, \ldots, n$*
   (a) *If $i < j$ then Add $\frac{Cq_iq_j}{(j-i)^2}$ to F_j;*
   (b) *Else if $i > j$ then Add $-\frac{Cq_iq_j}{(j-i)^2}$ to F_j.*
   (c) *Endif*
3. *Output F_j.*

It is not hard to analyze the running time of this program: each invocation of the inner loo, over i, takes $O(n)$ time, and this inner loop is invoked $O(n)$ times total, so the overall running time is $O(n^2)$.

The trouble is, for the large values of n , the program takes several minutes to run. On the other hand, their experimental setup is optimized so that they can throw down n particles, perform the measurements, and be ready to handle n more articles within a few seconds. So they had really like it if there were a way to compute all the forces F_j much more quickly, so as to keep up with the rate of the experiment.

Help them out by designing an algorithm that computes all the forces F_j in $O(n \log n)$ time.

Answer *Define*

$$Q(x) \doteq \sum_{j=0}^{n-1} q_{j+1} x^j,$$
$$R(x) \doteq x^{n-1}\left(-\sum_{j=1}^{n-1}\frac{x^{-j}}{j^2} + \sum_{j=1}^{n-1}\frac{x^j}{j^2}\right),$$

and

$$\hat{Q}(y) \doteq \sum_{s=0}^{2n-1} Q(\omega^{-s})y^s, \hat{R}(y) \doteq \sum_{j=0}^{2n-1} R(\omega^{-s})y^s, \textit{where } 1+\omega+\ldots+\omega^{2n-1} = 0.$$

[4]

Code 6 1. *Use fast fourier transform to calculate convolution of P, Q:*

   (a) *calculate $\hat{Q}(y)$ from $Q(x)$ –$O(n\log n)$;*

   (b) *calculate $\hat{R}(y)$ from $R(x)$ –$O(n\log n)$;*

   (c) *calculate* $\hat{P}(y) \doteq \sum_{s=1}^{2n-1} Q(\omega^{-s})R(\omega^{-s})y^s$ – $O(n)$;

   (d) *calculate* $P(x) = \sum_{j=1}^{2n-1} p_j x^j \doteq \sum_{j=0}^{2n-1} \frac{\hat{P}(\omega^j)}{2n} x^j$ –$O(n\log n)$.

[4] here we keep along with section **conversion and fast Fourier transform**. On the other hand, we can also choose ω as $1+\omega+\ldots+\omega^{2n-2} = 0$ and correspondingly

$$\hat{Q}(y) \doteq \sum_{s=0}^{2n-2} Q(\omega^{-s})y^s, \hat{R}(y) \doteq \sum_{j=0}^{2n-2} R(\omega^{-s})y^s, \text{where } 1+\omega+\ldots+\omega^{2n-2} = 0.$$

Notice now

$$\begin{aligned}
p_{n-1} &= -\frac{q_n}{(n-1)^2} - \ldots - \frac{q_2}{1^2} \\
&= -\sum_{i>1}^{n} \frac{q_i}{(1-i)^2}, \\
p_n &= -\frac{q_n}{(n-2)^2} - \ldots - \frac{q_3}{(1^2} + \frac{q_1^2}{1^2} \\
&= \frac{q_1^2}{1^2} - \sum_{i>2}^{n} \frac{q_i}{(2-i)^2}, \\
p_{n+1} &= -\frac{q_n}{(n-3)^2} - \ldots - \frac{q_4}{1^2} + \frac{q_2^2}{1^2} + \frac{q_1^2}{2^2} \\
&= \sum_{i<3}^{n} \frac{q_i}{(3-i)^2} - \sum_{i>3}^{n} \frac{q_i}{(3-i)^2}, \\
\text{(in all) } p_{n-2+j} &= \sum_{i<j}^{n} \frac{q_i}{(j-i)^2} - \sum_{i>j}^{n} \frac{q_i}{(j-i)^2} = \frac{F_j}{q_j},
\end{aligned}$$

2. *Compute* $F_j \leftarrow p_{n-2+j} q_j, j = 1, \ldots, n$ – $O(n)$.

Running time in all $O(n \log n)$. ♢

Question (HW4,5-7) *Suppose now that you are given an* $n \times n$ *grid graph* G.

We use some of the terminology of the previous question. Again, each node v *is labeled by a real number* x_v*; you may assume that all these labels are distinct. Show how to find a local minimum of* G *using only* $O(n)$ *probes to the nodes of* G*. (Note that* G *has* n^2 *nodes.)*

Answer (refers to [3]) *Denote the set of nodes on the* **border** *of the grid* G *by*

$$\begin{aligned}
B &\doteq \{(1,k) \in G : k = 1 \ldots, n\} \cup \{(k,1) \in G : k = 1 \ldots, n\} \\
&\cup \{(n,k) \in G : k = 1 \ldots, n\} \cup \{(k,n) \in G : k = 1 \ldots, n\}.
\end{aligned}$$

Say that G *has* **Property** *(∗)*

If it contains a node $v \notin B$ that is adjacent to a node in B and satisfies $x_v < x_p, \forall p \in B$. ♢

Note that in a grid G with Property $(*)$, the global minimum *does not occur on the border B (since the global minimum is no larger than v, which is smaller than B) — hence G has at least one local minimum that does not occur on the border. We call such a local minimum an* **internal local minimum**.

Code 7 *1. Find the node p on the border B of minimum value;*

2. If p is a corner node return v;

3. (Denote has a unique neighbor of p not on B by v)
If $x_p < x_v$, then return v;

4. (G satisfies Property $(*)$ since v is smaller than every node on B)
a recursive algorithm:

(a) Denote the union of the nodes in the middle row and middle column of G, not counting the nodes on the border by

$$C \doteq \{(n/2, k) \in G : k = 1, \ldots, n\} \cup \{(k, n/2) \in G : k = 1, \ldots, n\} \setminus B.$$

Deleting $B \cup C$ from G divides up G into four sub-grids.

(b) (let T be all nodes adjacent to S.)
Find the node $u = arg \min_{u \in S \cup T} x_u$ – $O(n)$ probes;
(Notice that $u \notin B$, since $v \in S \cup T$ and $v \prec B$.)

(c) If $u \in C$, then return u (since all of the neighbors of u are in $S \cup T$, and u is smaller than all of them.);

(d) otherwise $u \in T$. Let G' be the sub-grid containing u, together with the portions of S that border it. (Notice G' satisfies Property $(*)$, since u is adjacent to the border of G' and is smaller than all nodes on the border of G'.)
run the recursive algorithm on G' (since G' has an internal local minimum, which is also an internal local minimum of G.)

If $T(n)$ denotes the number of probes needed by the algorithm to find an internal local minimum in an $n \times n$ grid, we have the recurrence $T(n) = O(n) + T(n/2)$, which solves to $T(n) = O(n)$.

In all, the running time is $O(n)$. $\diamondsuit$

4 Dynamic Programming

Answer (6-1c) *1. $MWwithEndPoint(r)$: maximum weight of independent sets with end point v_r inside of path $v_1 v_2 \dots v_r$;*
$MWwithoutEndPoint(r)$: maximum weight of independent sets without end point v_r inside of path $v_1 v_2 \dots v_r$;

2. *final answer:* $\max \left\{ \begin{array}{c} MWwithEndPoint(n), \\ MWwithoutEndPoint(n). \end{array} \right.$ *;*

3. $MWwithEndPoint(r+1) = w_{r+1} + MWwithoutEndPoint(r)$*;*
$MWwithoutEndPoint(r+1) = \max \left\{ \begin{array}{l} MWwithoutEndPoint(r), \\ MWwithoutEndPoint(r). \end{array} \right.$

4. $MWwithoutEndPoint(1) = 0, MWwithEndPoint(1) = w_1$*;*

5. *Running time: $O(n)$ since each recurrence needs only constant time $O(1)$ to evaluate subproblems.* $\diamondsuit$

Answer (6-4c) *1. $\mathtt{MC}[j, \delta]$ minimum cost from during first j months with j^{th} month at city δ ($\delta = 1$ when NY; $\delta = 1$ when SF);*

2. *final:* $\max\{\mathtt{MC}[n, 0], \mathtt{MC}[n, 1]\}$*;*

3. $$\mathtt{MC}[j+1, \delta] \leftarrow \min \left\{ \begin{array}{c} c(\delta, j+1) + \mathtt{MC}[j, \delta], \\ c(1-\delta, j+1) + M + \mathtt{MC}[j, 1-\delta] \end{array} \right.$$
where $c(0, j+1) = N_{j+1}, c(1, j+1) = S_{j+1}$;

4. $\mathtt{MC}[1, 0] = N_1$, $\mathtt{MC}[1, 1] = S_1$*;*

5. *running time: $O(n)$.* $\diamondsuit$

Answer (HW4,6-5) *1. Denote $BQ(m)$, the best quality of a long string $x_1 \dots x_n$ with x_i characters;*

2. *Final answer: $BQ(n)$ with characters $x_1, \dots, x_n$ inputed;*

3. $BQ(m+1) \doteq \max_{0 \le j \le m} \{BQ(j) + quality(x_{j+1} \ldots x_{m+1})\}$. *We are trying all ways instantiating last words;*

4. *Initialization* $BQ(0) = 0$ *while* $quality(y_1 \ldots y_l)$ *are predetermined;*

5. *Running time: since recurrence step takes* $O(n)$ *to evaluate* $O(n)$ *subproblems, total running time is* $O(n^2)$. ♢

Answer (HW4,6-6) 1. *Denote* $V(m)$ *the minimum of the sum of the squares of the slacks of all lines consisting of the first* m *words of the list each with number of characters* $c_1, c_2, \ldots, c_m$*;*

2. *Final answer:* $V(n)$ *where* $n = \#$ *of words in list;*

3. $$V(m+1) \doteq \min_{\substack{1 \le j \le m-1 \\ L \ge \sum_{k=j+1}^{m+1}(c_k+1)-1}} \left\{ V(j) + \left(L + 1 - \sum_{k=j+1}^{m+1} (c_k + 1) \right)^2 \right\}.$$
We are trying all possible lines that can be packed in the last line, which can be $c_{j+1}, \ldots, c_{m+1}$*;*

4. *Initialization:* $V(m) = 0$*;*

5. *Running time: since evaluation* $O(n)$ *subproblems each of which take* $O(n)$ *time, so the total running time is* $O(n^2)$. ♢

Answer (6-7) 1. $m(j) \doteq \min_{k=1,\ldots,j} p(k)$,*temporarily set* $k_0 = \min\{k \in \{1, 2, \ldots, j\} : p(k) = m(j)\}$*;*
$M(j) \doteq \max_{k=k_0,\ldots,j} p(k)$*;*
$d(j)$*: maximum profit can be made amongst days* $1, 2, \ldots, j$*;*

2. *final:* $d(n)$*;*

3.
$$\begin{aligned} m(j+1) \quad & \leftarrow \min\{m(j), p(j+1)\}; \\ & \textit{if } m(j+1) \textit{ is equal to } m(j) \textit{ then } M(j+1) \leftarrow p(j+1); \\ d(j+1) \quad & \leftarrow \max\{d(j), M(j+1) - m(j)\}. \end{aligned}$$

4. *initialization:* $m(1) \leftarrow p(1), m(1) \leftarrow p(1), d(1) \leftarrow 0$*;*

5. *running time:* $O(n)$. ♢

Answer (Midterm 2015, 6-7, Sudipto Guha)

Answer (Midterm 2013, 6-8) 1. $\texttt{opt}[i,j]$ *denotes maximal # of robots killed during time* $[i,n]$ *at time* i *there are* j *second passed since the EMP was last used;*

2. *final:* $\texttt{opt}[1,0]$;

3. $\texttt{opt}[i,j] = \max\left\{\begin{array}{c}\min\{x_i, f(j)\} + \texttt{opt}[i+1,0]; \\ \texttt{opt}[i+1,j+1].\end{array}\right.$;

4. *initial:* $\texttt{opt}[n,j] = \min\{x_n, f(j)\}$;

5. *running time: since it takes* $O(1)$ *time to solve subproblems, the total running time is* $O(n)$. ♢

Answer (6-10c) 1. $\texttt{MV}[j,\delta] \doteq \max$ *value from during first* j *minutes with* j^{th} *minute on machine* δ *(*$\delta = 1$ *when on A;* $\delta = 1$ *when on B);*

2. *final:* $\max\{\texttt{MV}[n,0], \texttt{MV}[n,1]\}$;

3.
$$\texttt{MV}[j+1,\delta] \leftarrow \max\left\{\begin{array}{c}c(\delta, j+1) + \texttt{MV}[j,\delta], \\ \texttt{MV}[j-1, 1-\delta]\end{array}\right.$$
where $c(0,j+1) = N_{j+1}, c(1,j+1) = S_{j+1}$;

4. $\texttt{MV}[1,0] = a_1$, $\texttt{MV}[1,1] = b_1$; $\texttt{MV}[0,1]$=$\texttt{MV}[0,1] = 0$;

5. *running time:* $O(n)$. ♢

Answer (6-13) *By denoting* $w_{ij} \leftarrow -\ln r_{ij}$, *we are actually to find a negative cycle which is in section 6-10 of [2]. It is an algorithm using* $o(N)$ *space and running in* $O(mn)$ *time in the worst case according to (6.36) of [2].* ♢

Answer (6-14) *(a) If such single path* P_0 *exists, maintaining* P_0 *may lead* $\texttt{change}[\texttt{P}_0, \texttt{P}_0, \ldots, \texttt{P}_0] = \texttt{0}$. *By introducing* $\tilde{G} \doteq \langle V, \bigcap_{j=0}^{b} E_j \rangle$, *we can apply Dijkstra's algorithm in section 4.4 of [2] with running time* $O\left(\left|\bigcap_{j=0}^{b} E_j\right| \log |V|\right)$.

(b)

1. $V[j] \doteq \min_{P_0, P_1, \ldots, P_j} \texttt{cost}(P_0, P_1, \ldots, P_j)$;
2. *final:* $V[b]$;
3. $$V[j] \leftarrow \min \begin{cases} (j+1)\ell(0,j), & \textit{special case: no change at all;} \\ k + \min\limits_{1 \le i < j} \{V[i] + (j-i)\ell(i+1,j)\} & \end{cases}$$

   where

   (a) *last changeover amongst* $P_0, \ldots, P_j$ *is between graphs* G_i *and* G_{i+1} *since it seems most useful to think about where the last changeover occurs;*

   (b) $\ell(i+1, j)$ *represents the length of the shortest path from* s *to* t *in graph* $\bigcap_{k=i+1}^{j} G_k = \langle V, \bigcap_{k=i+1}^{j} E_k \rangle$.

   Every time we are dealing with last path;

4. *initialization:* $V[0] = 0$;
5. *running time: at most* $O(b|E_j| \log |V| \le O(|V|^2 \log |V|)$:

   (a) *# of* (i, j) *pairs of subproblems is* $O(b^2)$;

   (b) *evaluating subproblems takes time* $O(j) = O(b)$;

   (c) *each subproblem may take time*

   $$O(|E_j| \log |V| \le O(|V|^2 \log |V|) \xlongequal{n=|V|})(n^2 \log n)$$

   by applying Dijkstra's algorithm in section 4.4 of [2].

   Total running time is $O(b^3 n^2 \log n)$. ♢

Answer (6-14, refers to [3]) *We are given graphs* $G_0, \ldots, G_b$. *While trying to find the last path* P_b, *we have several choices. If* G_b *contains* P_{b-1}, *then we may use* P_{b-1}, *adding* $l(P_{b-1})$ *to the cost function (but not adding the cost of change* K.*) Another option is to use the shortest* s*-*t *path, call it* S_b, *in* G_b. *This adds* $l(S_b)$ *and the cost of change* K *to the cost function. However, we may want to make sure that in* G_{b-1} *we use a path that is also available in* G_b *so we can avoid the change penalty* K. *This effect of* G_b *on the earlier part of the solution is hard to anticipate in a greedy-type algorithm, so we'll use dynamic programming. We will use subproblems* $Opt(i)$

to denote minimum cost of the solution for graphs $G_0, \ldots, G_i$. *To compute* $Opt(n)$ *it seems most useful to think about where the last changeover occurs. Say the last changeover is between graphs* G_i *and* G_{i+1}. *This means that we use the path* P *in graphs* $G_{i+1}, \ldots, G_b$, *hence the edges of* P *must be in every one of these graphs. Let* $G(i, j)$ *for any* $0 \le i \le j \le b$ *denote the graph consisting of the edges that are common in* $G_i, \ldots, G_j$; *and let* $\ell(i, j)$ *be the length of the shortest path from* s *to* t *in this graph (where* $\ell(i, j) = \infty$ *if no such path exists). If the last change occurs between graphs* G_i *and* G_{i+1} *then then we get that* $Opt(b) = Opt(i) + (b - i)\ell(i + 1, b) + K$. *We have to deal separately with the special case when there are no changes at all. In that case* $Opt(b) = (b + 1))\ell(0, b)$. *So we get argued that* $Opt(b)$ *can be expressed via the following recurrence:*

$$Opt(b) = \min\{(b + 1))\ell(0, b), \min_{1 \le i < b} Opt(i) + (b - i)\ell(i + 1, b) + K\}.$$

Our algorithm will first compute all $G(i, j)$ *graphs and* $\ell(i, j)$ *values for all* $1 \le i \le j \le b$. *There are* $O(b^2)$ *such pairs and to compute one such subgraph can take* $O(n^2 b)$ *time, as there are up to* $O(n^2)$ *edges to consider in each of at most* b *graphs. We can compute the shortest path in each graph in linear time via BFS. This is a total of* $O(n^2 b^3)$ *time, polynomial but really slow. We can speed things up a bit to* $O(b^2 n^2)$ *by computing the graphs* $G(i, j)$ *and* $\ell(i, j)$ *for a fixed value of* i *in order of* $j = i \ldots b$. ♢

Answer (6-15b, refers to [4])

1. $N[j]$: *size of the maximum viewable subset of the first* i *events (including the* i^{th}*);*
2. *final:* $N[n]$;
3. $$N[i] = \begin{array}{ll} 1 + \max_{1 \le j < i, |d_i - d_j| \le i - j} N[j] \textit{HSpace2em if } \{j \mid |d_i - d_j| \le i - j\} \ne \emptyset; \\ 1 \textit{HSpace2em else if } |d_i| \le i; \\ 0 \textit{HSpace2em otherwise} \end{array}$$

   namely, it can be achieved through observing some previous events such that the telescope can catch event i *in time thereafter*
4. *initial condition* $N[0] = 0$;
5. *running time:* $O(n^2)$ *since that evluating subproblems takes time* $O(i)$. ♢

Based on this equation, we use an array `OPT[1..n]` *to store the values of* $OPT(i)$ *and design the algorithm as follows.*

```
for i from 1 to n do
HSpace2emif |d_i| > i then
HSpace4emOPT[i] = 0
HSpace2emelse
HSpace4emOPT[i] = 1
HSpace4emfor j from 1 to i-1 do
HSpace5emif OPT[j]≠0 and |d_i - d_j| ≤ i - j and OPT[j]+1>OPT[i] then
HSpace7emOPT[i] = OPT[j]+1
HSpace5emend
HSpace4emend
HSpace2emend
end
```

Answer (6-18, refers to [3]; not attempted) *Consider the directed acyclic graph* $G = (V, E)$ *constructed in class, with vertices* s *in the upper left corner and* t *in the lower right corner, whose* s*-*t *paths correspond to global alignments between* $A = a_1 \ldots a_m$ *and* $B = b_1 \ldots b_n$*. For a set of edges* $F \subset E$*, define* $c(F) \doteq \sum_{e \in F} w_e$*. If* P *is a path in* G*, let* $\Delta(P)$ *denote the set of diagonal edges in* P *(i.e. the* matches *in the alignment).*

Let Q *denote the* s*-*t *path corresponding to the given alignment. Let* E_1 *denote the horizontal or vertical edges in* G *(corresponding to indels),* E_2 *denote the diagonal edges in* G *that do not belong to* $\Delta(Q)$*, and* $E_3 = \Delta(Q)$*. Note that* $E = E_1 \cup E_2 \cup E_3$*.*

Let $\varepsilon = 1/2n$ *and* $\varepsilon' = 1/4n^2$*. We form a graph* G' *by subtracting* ε *from the cost of every edge in* E_2 *and adding* ε' *to the cost of every edge in* E_3*. Thus,* G' *has the same structure as* G*, but a new cost function* c'*.*

Now we claim that path Q *is a minimum-cost* s*-*t *path in* G' *if and only if it is the unique minimum-cost* s*-*t *path in* G*. To prove this, we first observe that*

$$c'(Q) = c(Q) + \varepsilon' |\Delta(Q)| \leq c(Q) + \frac{1}{4},$$

and if $P \neq Q$*, then*

$$c'(P) = c(P) + \varepsilon' |\Delta(P \cap Q)| - \varepsilon |\Delta(P - Q)| \geq c(P) - \frac{1}{2}.$$

Now, if Q *was the unique minimum-cost path in* G*, then* $c(Q) \leq c(P) + 1$ *for every other path* P*, so* $c'(Q) < c'(P)$ *by the above inequalities, and hence* Q

is a minimum-cost s-t path in G'. To prove the converse, we observe from the above inequalities that $c'(Q) - c(Q) > c'(P) - c(P)$ for every other path P; thus, if Q is a minimum-cost path in G', it is the unique minimum-cost path in G.

Thus, the algorithm is to find the minimum cost of an s-t path in G', in $O(mn)$ time and $O(m+n)$ space by the algorithm from class. Q is the unique minimum-cost A-B alignment if and only if this cost matches $c'(Q)$. $\diamondsuit$

Answer (6-19, Sudipto Guha)

1. `Feasible`$[i, a, b]$ *(boolean variable): whether it is feasible to epress $s_1, \ldots, s_i$ (first i biths of $\vec{s}$) as an interleaving of x and y such that*
   (a) *in the last occurrence of x we have seen $x_1, \ldots, x_a$;*
   (b) *in the last occurrence of y we have seen $y_1, \ldots, y_b$.*
2. $\bigvee_{a=1,\ldots,|x|;b=1,\ldots,y.}$`Feasible`$[|s|, a, b]$;
3. $$\texttt{Feasible}[i,a,b] \leftarrow \begin{cases} \texttt{Feasible}[i-1,a-1,b] \wedge \{Is\ s_i = x_a\ and\ a \neq 1\}; \\ \texttt{Feasible}[i-1,a,b-1] \wedge \{Is\ s_i = y_b\ and\ b \neq 1\}; \\ \texttt{Feasible}[i-1,|x|,b] \wedge \{Is\ s_i = x_a\ and\ a = 1\}; \\ \texttt{Feasible}[i-1,a,|y|] \wedge \{Is\ s_i = y_b\ and\ b = 1\}. \end{cases}$$
   where $|x|, |y|$ are lengths of $\vec{x}, \vec{y}$, respectively;
4. *initialize* `Feasible`$[0, a, b] = TRUE$ *for $a = |x|$ or $b = |y|$; otherwise return $FALSE$.*
5. *running time: $O(|s||x||y|)$.* $\diamondsuit$

Answer (6-19, refers to [3]) *Let's suppose that s has n characters total. To make things easier to think about, let's consider the repetition x' of x consisting of exactly n characters, and the repetition y' of y consisting of exactly n characters. Our problem can be phrased as: is s an interleaving of x' and y'? The advantage of working with these elongated strings is that we don't need to "wrap around" and consider multiple periods of x' and y' — each is already as long as s.*

Let $s[j]$ denote the j^{th} character of s, and let $s[1 : j]$ denote the first j characters of s. We define the analogous notation for x' and y'. We know that if s is an interleaving of x' and y', then its last character comes from either x' or y'. Removing this character (wherever it is), we get a smaller recursive problem on $s[1 : n-1]$ and prefixes of x' and y'.

1. *Let* $M[i,j] = true$ *(boolean type: 0=false; 1= true) if* $s[1:i+j]$ *is an interleaving of* $x'[1:i]$ *and* $y'[1:j]$*;*

2. *final:* $\max_{i+j=n} M[i,j]$*;*

3. *Consider sub-problems defined by prefixes of* x' *and* y'*. If there is such an interleaving, then the final character is either* $x'[i]$ *or* $y'[j]$*, and so we have the following basic recurrence:*

   $$M[i,j] \leftarrow \left[\left(M[i-1,j]\&\&s[i+j] = x'[i]\right) \,||\, \left(M[i,j-1]\&\&s[i+j] = y'[j]\right)\right].$$

4. *M[0,0] = yes;*

5. *Running time: There are* $O(n^2)$ *values* $M[i,j]$ *to build up, and each takes constant time to fill in from the results on previous sub-problems; thus the total running time is* $O(n^2)$*.*

   We can build these up via the following loop.

   Code 8 *M[0,0] = yes;*
   For $k = 1, 2, \ldots, n$
   For all pairs (i,j) *so that* $i + j = k$
   If $M[i-1,j] = yes$ *and* $s[i+j] = x'[i]$ *then*
   $M[i,j] = yes$
   Else if $M[i,j-1] = yes$ *and* $s[i+j] = y'[j]$ *then*
   $M[i,j] = yes$
   Else
   $M[i,j] = no$
   Endfor
   Endfor
   Return "yes" if and only there is some pair (i,j) *with* $i+j = n$ *so that* $M[i,j] = yes$*.*

Answer (6-20) 1. $\texttt{opt}[t,j]$ *maximum grades one can obtain amongst* t *hours from* j *course projects;*

2. *final:* $\texttt{opt}[H,n]$*;*

3. $\texttt{opt}[t,j+1] = \max_{h=0,1,\ldots,t} \{\texttt{opt}[t-h,j] + f_{j+1}(h)\}$*. We are always scheduling appropriate time* h *for project* $j+1 (j = 0, 1, \ldots, n-1)$*;*

4. $\texttt{opt}[0, j] = 0, j = 0, 1, \ldots, n$;

5. *running time: there are* $O(Hn)$ *entities for* $\texttt{opt}[t, j]$ *and during each recurrence step, evaluation subproblems takes time* $O(t) \leq O(H)$. *So the total running time is* $O(H^2 n)$. $\diamondsuit$

Answer (6-21) 1. $\texttt{opt}[t, m]$: *maximum return from at most* m *buy-sell transactions amongst first* t *days;*
$\texttt{Single}[s, t]$: *maximum return from at most* 1 *buy-sell transactions amongst* s^{th} *day to* t^{th} *day, (analogous to* $d(j)$ *in question 6-7 with running time* $O(n)$*);*

2. *final answer:* $\texttt{opt}[n, k]$;

3. $\texttt{opt}[t, m] \leftarrow \max_{1 \leq s \leq t} \{\texttt{opt}[t - s - 1, m - 1] + \texttt{Single}(s, t)\}$;

4. *initialization:* $\texttt{Single}(s, s) = 0$; $\texttt{opt}[t, 0] = 0$, $\texttt{opt}[t, 1] = \texttt{Single}(1, t)$;

5. *running time:*

   (a) $\texttt{Single}(s, t)$ *can be computed beforehand with time cost* $O(n^2)$ *by applying (3);*

   (b) *since there are* $O(nk)$ *entities for* $\texttt{opt}[t, m]$ *and during each recurrence step, evaluation subproblems takes time* $O(t) \leq O(n)$, *running time of this part is* $O(n^2 k)$.

   the total running time is $O(n^2 k)$. $\diamondsuit$

Answer (6-21, refers to [3]) *Introduce an* m-exact strategy *to be one with* exactly m *non-overlapping buy-sell transactions.*

1. *By* transaction (i, j), *we mean the single transaction that consists of buying on day* i *and selling on day* j;
$\texttt{Single}[i, j]$: *maximum profit obtainable by executing a single transaction somewhere in the interval of days between* i *and* j *(c.f. problem 6-7);*
$M[m, d]$: *maximum profit obtainable by an* m*-exact strategy on days* $1, \ldots, d$, *for* $0 \leq m \leq k$ *and* $0 \leq d \leq n$.

2. (a) *Note that the transaction achieving the maximum in* $\texttt{Single}[i, j]$ *is either the transaction* (i, j), *or else it fits into one of the intervals* $[i, j - 1]$ *or* $[i + 1, j]$. *Thus we have*

$$\texttt{Single}[i,j] \leftarrow \max \begin{cases} 1000 \cdot [p(j) - p(i)], \\ \texttt{Single}[i, j-1], \\ \texttt{Single}[i+1, j] \end{cases} \tag{2}$$

Using this formula, we can build up all values of $\texttt{Single}[i,j]$ *in time* $O(n^2)$[5]*;*

(b) *final:* $\max_{1 \le m \le k} M(m, n)$*;*

(c) *In the optimal* m*-exact strategy on days* $1, \ldots, d$*, the final transaction occupies an interval that begins at* i *and ends at* j*, for some* $1 \le i < j \le d$*; and up to day* $i-1$ *we then have an* $(m-1)$*-exact strategy. Thus we have*

$$M[m, d] = \max_{1 \le i < j \le d} \texttt{Single}[i,j] + M[m-1, i-1];$$

3. *initialize* $M[m, 0] = M[0, d] = -\infty, \forall m = 1, \ldots, k, \forall d = 1, \ldots, n$*, where* $-\infty$ *denotes the profit obtainable if there isn't room in days* $1, \ldots, d$ *to execute* m *transactions. (E.g. if* $d < 2m$*.)*

4. *running time:*

   (a) $\texttt{Single}[i,j]$*.can be computed beforehand with time cost* $O(n^2)$*;*

   (b) *Since there are* $O(kn)$ *entries for* $M[m,d]$ *and that the time spent per entry is* $O(n)$*, the total time is therefore* $O(kn^2)$*.*

   (c) *the optimal* k*-shot strategy is, by definition, an* m*-exact strategy for some* $m \le k$*; thus, the optimal profit from a* k*-shot strategy is*

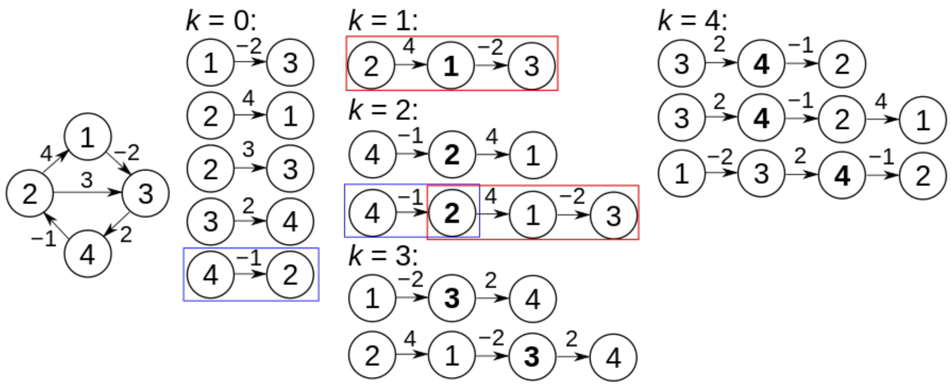
$$\max_{0 \le m \le k} M[m, n].$$

   So the total running time is $O(k^2 n^2)$*.*

We can determine the strategy associated with each entry by maintaining a pointer to the entry that produced the maximum, and tracing back through the dynamic programming table using these pointers. $\diamondsuit$

Answer (6-22, Floyd-Warshall algorithm from [1] and the lecture) *Label* $V \setminus \{s, t\}$ *as* $\{v_1, \ldots, v_{n-2}\}$*.*

[5] By going in order of increasing $i + j$, spending constant time per entry. However, different from mine, that is $O(n)$



$\diamondsuit$

Figure 2: An example of Floyd-Warshall Algorithm from [1].

1. $\texttt{shortestPath}(i, j, k)$*: returns the shortest possible path from i to j using vertices only from the set 1,2,...,k as intermediate points along the way;*
2. *final:* $\texttt{shortestPath}(s, t, n-2)$*;*
3. $\texttt{shortestPath}(i, j, k{+}1) = \min \left\{ \begin{array}{c} \texttt{shortestPath}(i, j, k), \\ \texttt{shortestPath}(i, k+1, k) + \texttt{shortestPath}(k+1, j, k) \end{array} \right\}.$
4. *initialize* $\texttt{shortestPath}(i, j, 0) = w_{i \to j}$*;*
5. *running time: # of entries for* $\texttt{shortestPath}(i, j, k)$ *is* $O(n^3)$ *and evaluating subproblems takes constant time. So the total running time is* $O(n^3)$*.*

Answer (6-23) *(a) For all* $w \in V$*, check whether* $d(w)$ *is equal to* $\min\limits_{v \to w}\{d(v) + c_{v \to w}\}$*. This involves time cost* $O(m+n)$*.*

(b) Introduce new weight to the edges $c'_{v \to w} \doteq c_{v \to w} - d(v) + d(w)$*. Notice*

1. $c'_{v \to w} \geq 0$*;*
2. *for every path* P *beginning at* v *and ending at* w*, denote cost of paths with respect to two weights of edges (denote two graphs by* $\langle G, \{c_{\vec{e}}\}\rangle$*,* $\langle G, \{c'_{\vec{e}}\}\rangle$*, respectively) by* $\ell(P), \ell'(P)$*, then* $\ell'(P) = \ell(P) - d(v) + d(w)$*.*

In this case we

1. *run Dijkstra's algorithm on* $\langle G, \{c'_{\vec{e}}\}\rangle$ *with time cost* $O(m \log n)$*and obtain* $\tilde{d}'(v)$ *shortest path* $v \to t'$ *in* $\langle G, \{c'_{\vec{e}}\}\rangle$ *due to observation (1);*
2. $d'(v) \leftarrow \tilde{d}'(v) + d(v) - d(t')$*, which turns to be correct distances* $v \to t'$ *in* $\langle G, \{c_{\vec{e}}\}\rangle$ *due to observation (2).* ♢

Answer (HW 6.5, 6-24, Sudipto Guha) *Let the number of voters in precinct* P_i *for the two parties be* s_i, t_i *respectively (with* $s_i + t_i = m$*). Assume party 's' has majority overall (therefore the gerrymandering can only help party 's').*

1. $\mathtt{A}[i-1, k-1, a, b]$*: Is it possible to partition the precincts* $P_1 \dots P_i$ *such that the first part has k precincts and party 's' has a votes in first part and b votes in second part?*

   Boolean value returned with 1 *true,* 0 *false;*
2. $$\mathtt{A}[i,k,a,b] = \max \begin{cases} \mathtt{A}[i-1, k-1, a-s_i, b], & P_i \text{ is in first part;} \\ \mathtt{A}[i-1, k, a, b-s_i], & P_i \text{ is in second part.} \end{cases}$$
3. *Initialization:*
4. *Final:* $$\max_{\frac{mn}{2} < a < \sum_{j=1}^{n} s_j - \frac{mn}{2};\, a+b = \sum_{j=1}^{n} s_j} \mathtt{A}[n, \tfrac{n}{2}, a, b];$$
5. *Running time by evaluating # of entities for* $\mathtt{A}$*:* $O(n \cdot n/2 \cdot mn \cdot mn) = O(m^2 n^4)$. ♢

Answer (HW 6.5, 6-24, Sudipto Guha, solution 2) *Let again the number of voters in precinct* P_i *for the two parties be* s_i, t_i *respectively (with* $s_i + t_i = m$*). Assume again party 's' has majority overall.*

Observe that $a + b = \sum_{j=1}^{i} s_j$ *for any feasible partition of* $P_1; \dots Pi$*. Therefore the coordinate b is superfluous.*

1. $\mathtt{A}[i-1, k-1, a]$*: Is it possible to partition the precincts* $P_1 \dots P_i$ *such that the first part has k precincts and party 's' has a votes in first part and* $\sum_{j=1}^{i} s_j - a$ *votes in second part?*

   Boolean value returned with 1 *true,* 0 *false;*

2.

$$\mathtt{A}[i,k,a] = \max \begin{cases} \mathtt{A}[i-1,k-1,a-s_i], & P_i \text{ is in first part;} \\ \mathtt{A}[i-1,k,a], & P_i \text{ is in second part.} \end{cases}$$

3. *Initialization:* $\mathtt{A}[0,0,0]=1$; $\mathtt{A}[i,k,a]=0$, *if* $i<k$ *or* $k>\frac{n}{2}$;

4. *Final:* $\max\limits_{\frac{mn}{2} < a < \sum_{j=1}^{n} s_j - \frac{mn}{2}} \mathtt{A}[n, \frac{n}{2}, a]$;

5. *Running time by evaluating # of entities for* A: $O(n \cdot n/2 \cdot mn) = O(mn^3)$. ♢

Answer (6-28) **(a)** *(Just the Greedy thought.) Let J be the optimal subset. By definition all jobs in J can be scheduled to meet their deadline. Now consider the problem of scheduling to minimize the maximum lateness from class, but consider the jobs in J only. We know by the definition of J that the minimum lateness is* 0 *(i.e., all jobs can be scheduled in time)by applying greedy algorithm with the ordering of their deadline. Hence ordering the jobs in J by the deadline generates a feasible schedule for this set of jobs.*

(b) *The problem is analogous to the subset selection problem we considered in section 6.1 of [2]: subproblems analogous to the ones in section 6.1 of [2].*

Consider subproblems using a subset of jobs $\{1,\ldots,m\}$. Order the jobs by increasing deadline, and assume that they are numbered this way, i.e., we have that $d_1 \leq \ldots \leq d_n = D$. To solve the original problem we consider two cases: either the last job n is accepted or it is rejected. If job n is rejected, then the problem reduces to the subproblem using only the first $n-1$ items. Now consider the case when job n is accepted. By part (a) we know that we may assume that job n will be scheduled last. In order to make sure that the machine can finish job n by its deadline D, all other jobs accepted by the schedule should be done by time $D-t_n$. We will define subproblems so that this problem is one of our subproblems.

For a time $0 \leq d \leq D$ and $m=0,\ldots,n$ let $\mathtt{opt}(d,m)$ denote the the maximum subset of requests in the set $\{1,\ldots,m\}$ that can be satisfied by the deadline d. What we mean is that in this subproblem the machine is no longer available after time d, so all requests either have to be scheduled to be done by deadline d, or should be rejected (even if the deadline d_i of the job is $d_i > d$). Now we have the following statement.

1. $\texttt{opt}(d, m)$: *maximum number of jobs amongst first m jobs in first d time units;*

2. *final:* $\texttt{opt}(d, n)$*;*

3. $$\texttt{opt}(d, m) = \max \left\{ \begin{array}{c} \texttt{opt}(m-1, d), \\ \texttt{opt}(m-1, d-t_m)+1 \} \end{array} \right.$$

   that is, selecting between whether or not job m is in the optimal solution $\texttt{opt}(d, m)$*;*

4.

5.

-

This suggests the following code.

Code 9 `Select-Jobs(n,D)`*:*

1. *Array* $M[0 \ldots n, 0 \ldots D]$
2. *Array* $S[0 \ldots n, 0 \ldots D]$
3. *For* $d = 0, \ldots, D$
   (a) $M[0, d] = 0$
   (b) $S[0, d] = \phi$
4. *Endfor*
5. *For* $m = 1, \ldots, n$
6. *For* $d = 0, \ldots, D$
   (a) *If* $M[m-1, d] > M[m-1, d-t_m] + 1$ *then*
   $M[m, d] = M[m-1, d]$, $S[m, d] = S[m-1, d]$*;*
   (b) *Else*
   $M[m, d] = M[m-1, d-t_m] + 1$, $S[m, d] = S[m-1, d-t_m] \cup \{m\}$.
   (c) *Endif*
7. *Endfor*
8. *Endfor*

9. *Return* $M[n, D]$ *and* $S[n, D]$

The correctness follows immediately from the statement 3. The running time of $O(n^2D)$ *is also immediate from the for loops in the problem, there are two nested* `for` *loops for for* m *and one for* d*. This means that the internal part of the loop gets invoked* $O(nD)$ *time. The internal part of this* `for` *loop takes* $O(n)$ *time, as we explicitly maintain the optimal solutions. The running time can be improved to* $O(nD)$ *by not maintaining the* S *array, and only recovering the solution later, once the values are known.*

Question (Fall 2014-I, 25 points) *A subsequence is defined to be palindromic if it is the same when it is read left-to-right or right-to-left. A sequence can have many palindromic subsequences. For example*

$$a; b; a; f; a; b; a; c; d; e; d; c$$

Has several such subsequences $aba; aa; baab; cdedc$*, etc. Give an efficient (as low running time as possible) algorithm to find the longest palindromic subsequence of a sequence* $a1; a2; \ldots; an$*. For full credit give an* $O(n^2)$ *time algorithm. A subsequence need not be a substring (contiguous) as the example shows.*

Answer (1, SUDIPTO GUHA)

1. $LCS[i; j]$*: (length of the) longest common subsequence of* $X_1; \ldots; X_i$ *and* $Y_1; \ldots; Y_j$*.*

2. *Final answer is* $LCS[|X|; |Y|]$*;*

3. $$LCS[i; j] = \max \begin{cases} LCS[i-1; j] \\ LCS[i; j-1] \\ 1 + LCS[i-1; j-1] & \text{(provided } X_i = Y_j\text{,} \\ & \text{otherwise this option does not exist)} \end{cases}$$

4. $LCS[i; 0] = LCS[0; j] = 0, \forall i, j$*;*

5. *The size of the table is* $O(n^2)$ *and each entry is updated in* $O(1)$ *time.* $\diamondsuit$

Answer (2, SUDIPTO GUHA)

1. $PAL[i; j]$*: (length of the) longest palindromic sequence.*

2. *Final answer is* $PAL[1; n]$*;*

3. $LCS[i;j] = \max \begin{cases} PAL[i+1;j] \\ PAL[i;j-1] \\ 1 + PAL[i+1;j-1] & \text{(provided } X_i = Y_j, \\ & \text{otherwise this option does not exist)} \end{cases}$

4. $PAL[i;j] = 0, PAL[i;i] = 1, \forall i > j$;

5. *running time* $O(n^2)$. ♢

Question (2015 midterm 2) *Given n numbers* $x_1, \ldots, x_n$*, the numbers* $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ *define a sub-sequence if* $1 \le i_1 < i_2 < \ldots < i_k \le n$. *Length of the sub-sequence is* k. *For example given* $1.1, -2, 0, -6, 17.2, 13$ *the numbers* $1.1, 17.2, 13$ *define a sub-sequence of length* 3.

1. *Given* $x_1, \ldots, x_n$ *as above, give a polynomial time DP algorithm that finds the length of the maximum length increasing sub-sequence. A sub-sequence above is of length* 3*, either* $-2, 0, 17.2$ *or* $-2, 0, 13$. *Just finding the length will suffice.*

2. *Define a sub-sequence* $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ *to be "maybe increasing" if* $x_{i_a} - 7 < x_{i_b}$ *for any* $1 \le i_a < i_n \le k$. *That is, the sub-sequence never decreases by more than* 7. *for example the sub-sequence* $-2, 0, -6, 17.2, 13$ *is "maybe increasing" but the entire sequence is not. given* $x_1, \ldots, x_n$ *as above. Given* $x_1, \ldots, x_n$ *as above, find the length of the maximum length "maybe increasing" sub-sequence.*

3. *Define a sub-sequence* $x_{i_1}, \ldots, x_{i_k}$ *to be "surely increasing" if* $x_{i_1} + 7 < x_{i_2}, x_{i_2} + 7 < x_{i_3}$*, etc., that is, the numbers are not only increasing but also differ by at least* 7. *For example the sub-Given* $x_1, \ldots, x_n$ *as above, find the length of the maximum length "surely increasing" sub-sequence.*

Answer (2015 midterm 2) 1. (a) *Define* $x_{n+1} = \infty, \forall x \in \mathbb{R}, x < x_{n+1} = \infty$; *define* $\texttt{opt}[j,k]$: *max length of "maybe increasing" sub-sequence of* $\{x_1, x_2, \ldots, x_j\}$ *such that every element is smaller than* x_k $(j = 1, 2, \ldots, n, k = j + 1 j + 2, \ldots, n + 1$;

(b) $\texttt{opt}[j,k] \leftarrow \max \begin{cases} 1 + \texttt{opt}[j-1, j], & \text{the option is available iff } x_j < x_k; \\ \texttt{opt}[j-1, k] & x_j \text{ is not selected.} \end{cases}$

(c) *Final* $\texttt{opt}[n, n+1]$;

(d) *Initialization:* $\texttt{opt}[1,k] = \begin{cases} 1, & x_1 < x_k + 7; \\ 0, & \text{otherwise} \end{cases}, k = 2, \ldots, n+1$;

   (e) *Running time* $O(n^2)$.

2. (a) *Define* $x_{n+1} = \infty, \forall x \in \mathbb{R}, x < x_{n+1} = \infty, \infty + 7 = \infty$; *define* $\texttt{opt}[j,k]$: *max length of "maybe increasing" sub-sequence of* $\{x_1, x_2, \dots, x_j\}$ *such that every element is smaller than* x_k $(j = 1, 2, \dots, n, k = j+1, j+2, \dots, n+1$;

   (b) $\texttt{opt}[j,k] \leftarrow \max \begin{cases} 1 + \texttt{opt}[j-1, j], & \text{the option is available iff } x_j < x_k; \\ \texttt{opt}[j-1, k] & x_j \text{ is not selected.} \end{cases}$

   (c) *Final* $\texttt{opt}[n, n+1]$;

   (d) *Initialization:* $\texttt{opt}[1,k] = \begin{cases} 1, & x_1 < x_k + 7; \\ 0, & \text{otherwise} \end{cases}, k = 2, \dots, n+1$;

   (e) *Running time* $O(n^2)$.

3. *same as 1 with running time* $O(n^2)$. ♢

Question *You are given an array* $A[1], \dots, A[n]$ *which contain both negative and positive values. Assume* $A[0] = A[n+1] = 0$.

1. ***(10 points)*** *Find an interval* $[i, j]$ *such that* $\sum_{t=i}^{j} A[t]$ *is maximized. An* $O(n^2)$ *solution will not get any credit.*

2. ***(10 points)*** *Find two intervals* $[i, j]$ *and* $[k, l]$ *where* $i \le j < k \le l$ *such that* $\sum_{t=i}^{j} A[t] + \sum_{t=k}^{j} A[t]$ *is maximized. An* $O(n^4)$ *solution will not get any credit.*

More credit will be provided to a solution which is more efficient.

Answer 1. (a) $B[j] = \max$ *sum of any interval of* $1, \dots, j$ *ending in* j;

   (b) $B[j] \leftarrow \max\{B[j-1] + A[j], A[j]\}$,
   since if we include $j-1$ *in that interval then we should include the largest interval ending in* $j-1$ – *adding* $A[j]$ *with* $B[j-1]$, *sum of any interval of* $1, \dots, j-1$ *ending in* $j-1$;

   (c) *Final* $B[n]$;

   (d) *Initialization:* $B[0] = 0$;

   (e) *Running time* $O(n)$.

2. (a) $C[i] = \max$ *sum of any interval starting from* i;
   $C'[i] = \max_{s \ge i} C[s]$ *which means the largest interval sum starting after* i. *This can be computed in* $O(n)$ *time;*

(b) $C[i] \leftarrow \max\{B[i+1]+A[i], A[i]\}$[6];

(c) Final $\max_j\{B[j]+C'[j+1]\}$;

(d) Intialization;

(e) Running time $O(n)$.

Here we finish. ◊

5 Ch7 Network flow

Question (HW5, 7-10) *Suppose you are given a directed graph $G = (V, E)$, with $c_e > 0, \forall e \in E$. a designated source $s \in V$, and a designated sink $t \in V$. You are also given a maximum s-t flow in G, defined by a flow value f_e on each edge e. The flow $\{f_e\}$ is* acyclic*: there is no cycle in G on which all edges carry positive flow.*

Now, suppose we pick a specific edge $\vec{e}^* \in E$ and reduce its capacity by 1 unit. Show how to find a maximum flow in the resulting capacitated graph in time $O(m)$, where m is the number of edges in G.

Answer (HW5, 7-10) *Denote capacity of directed edge $\vec{e} \in E_2$ by $c_{\vec{e}}$ in the last residual graph $G_2 = \langle V, E_2\rangle$ with $\{c_{\vec{e}} \in \mathcal{N}_{\geq 0} : \vec{e} \in E_2\}$. Denote $\vec{e}^* = (u, w)$.*

1. *If $c_{(w,u)} = 0$, then the maximum flow does not change since $\vec{e}^*$ must have not been take advantage of to ship any flow to form the maximum flow;*

2. *Else,*

   (a) *use DFS to find a $u \to s$ path $P_1 = uv_{k-1}v_{k-2}\dots v_1 s$ and a $t \to w$ path $P_2 = tv_{l-1}v_{l-2}\dots v_{k+2}w$ in the residual graph with $c_{(v_{j+1},v_j)} > 0$ $(s = v_0, u = v_k, w = v_{k+1}, t = v_l)$. These run with $O(m+n)$ time;*

   (b) *Update capacities in the following way:*

      i. $c'_{(v_k,v_{k+1})} \leftarrow c_{(v_k,v_{k+1})}, c'_{(v_{k+1},v_k)} \leftarrow c_{(v_{k+1},v_k)} - 1$;

      ii. $c'_{(v_j,v_{j+1})} \leftarrow c_{(v_j,v_{j+1})} + 1, , c'_{(v_{j+1},v_j)} \leftarrow c_{(v_{j+1},v_j)} - 1,\ j = 0, 1, \dots, l-1, j \neq k$.

   Now we construct new $\{c_{\vec{e}} \in \mathcal{N}'_{\geq 0} : \vec{e} \in E_2\}$ for $G_2 = \langle V, E_2\rangle$. This runs with time $O(m)$.

[6] That is what Sudipto Guha says, "(switching i for j and $n+1$ for 0 and $i+1$ for $j-1$)".

3. *Use DFS to search whether there is a positive $s \to t$ flow. This runs with $O(m+n)$ time. If exists, the value of maximum flow does not change (although the flow does change); if not, the value maximum flow is reduced by 1 unit.*

Total running time is at most $O(m+n)$.
Correctness: ♢

Remark *How about add 1 unit to capacity of the specific edge $e^* \in E$?* ♢

Answer (refers in [5]) *Add the new flow to the residual flow graph in $O(m)$ time. Perform a tree traversal from the source node to detect whether a path now exists to the sink. If so, augment along that path and increase the maximum ow by one. If exists, augment along that path and increase the maximum ow by one.*

The running time is $O(n)$.
Correctness: ♢

> **Question (HW5, 7-12)** *Consider the following problem. You are given a flow network with unit-capacity edges: it consists of a directed graph $G = (V, E)$, a source $s \in V$, and a sink $t \in V$; and $c_e = 1$ for every $e \in E$. You are also given a parameter k.*
>
> *The goal is delete k edges so as to reduce the maxmimum s-t flow in G by as much as possible. In other words, you should find a set of edges $F \subseteq E$ so that $|F| = k$ and the maximum s-t flow in $G' = (V, E - F)$ is as small as possible subject to this.*
>
> *Give a polynomial-time algorithm to solve this problem.*

Answer (HW5, 7-12, refers to [3]) *If the minimum s-t cut has size $\leq k$ (since the flow network with unit-capacity edges), then we can reduce the flow to 0.*

Otherwise, let $f > k$ be the value of the maximum s-t flow. We identify a minimum s-t cut (A, B), and delete k of the edges out of A. The resulting subgraph has a maximum flow value of at most $f - k$ (by (7.9) in [2]).

> **(Claim)** *For any set of edges F of size k, the subgraph $G' = (V, E-F)$ has an s-t flow of value at least $f - k$.* ♢

Proof *Indeed, consider any cut (A, B) of G'. There are at least f edges out of A in G, and at most k have been deleted, so there are at least $f - k$ edges out of A in G'. Thus, the minimum cut in G' has value at least $f - k$, and so there is a flow of at least this value.* ♢

Remark *What if not "with unit-capacity edges"?* ◊

Question (HW5, 7-14) *We define the* escape problem *as follows:*

We are given a directed graph $G = (V, E)$ (picture a network of roads); a certain collection of nodes $X \subset V$ are designated as populated nodes*, and a certain other collection $S \subset V$ are designated as* safe nodes*. (Assume that X and S are disjoint.) In case of an emergency, we want evacuation routes from the populated nodes to the safe nodes. A set of evacuation routes is defined as a set of paths in G so that*

(i) each node in X is the tail of one path,

(ii) the last node on each path lies in S, and

(iii) the paths do not share any edges.

Such a set of paths gives a way for the occupants of the populated nodes to "escape" to S, without overly congesting any edge in G.

(a) Given G, X, and S, show how to decide in polynomial time whether such a set of evacuation routes exists.

(b) Suppose we have exactly the same problem as in (a), but we want to enforce an even stronger version of the "no congestion" condition (iii). Thus, we change (iii) to say "the paths do not share any nodes.*"*

With this new condition, show how to decide in polynomial time whether such a set of evacuation routes exists.

Also, provide an example with a given G, X, and S, in which the answer is "yes" to the question in (a) but "no" to the question in (b).

Answer (HW5, 7-14) *(a) Redistribute nodes in X left side and nodes in S right side as shown in figure 3. Add source s, sink t to V and add directed edges to E. Assign each edge with capacity 1 – this is important to avoid paths sharing edges. In all, denote*

$$E' \leftarrow \{(s, v) : v \in S\} \cup \{(u, t) : u \in X\} \cup E, V' \leftarrow V \cup \{s, t\},$$

and we actually construct a new graph $G' = \langle V', E' \rangle$ (with capacity on each edge) on which can apply algoritm to find maximum flow.

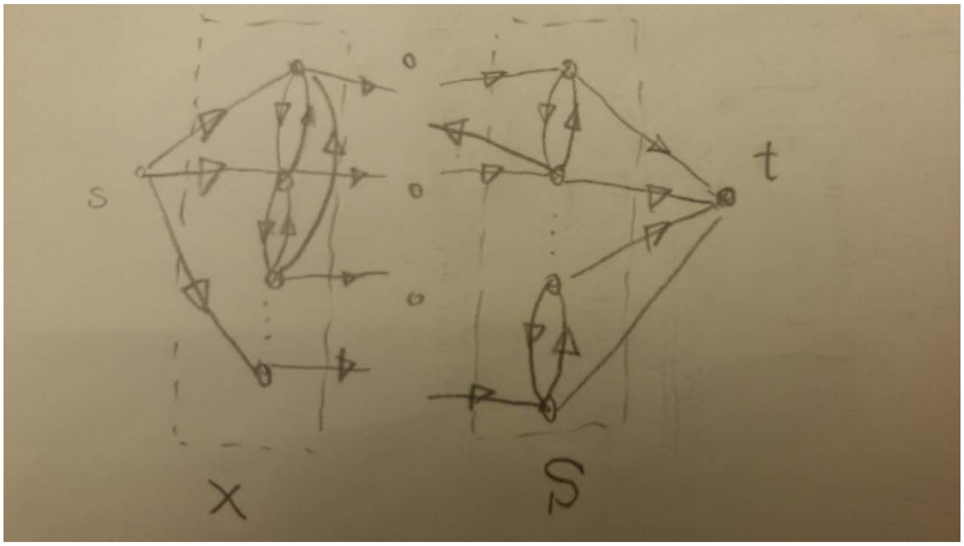


Figure 3: Construction of graph

If apply generic Preflow–Push Maximum–flow Algorithm in section 7.4 of [2], by statement (7.33) of [2] we can obtain it with running time $O\left((m+n)n^2\right) = O(mn^2+n^3)$*; in the version where we always select the node at maximum height runs in* $O(n^3)$ *time.*

A set of evacuation routes exists if and only if value of maximum flow is $|X|$*.*

(b)

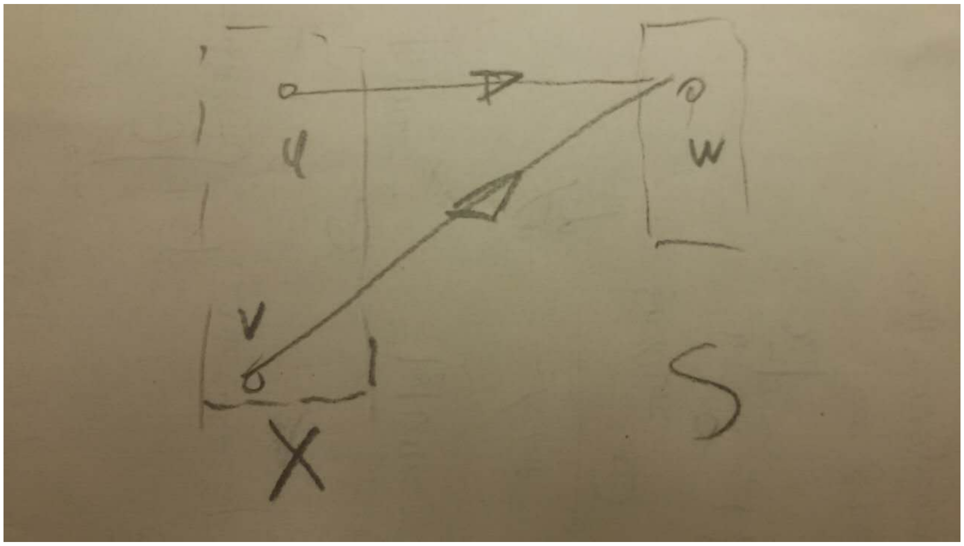
Answer (7-14, Sudipto Guha) *(a) Running time is* $O(|X|(m+n))$ *by Ford-Fulkerson algorithm in section 7.1 of [2].*

(b)split every vertex $v \in V$ *into* 2 *pieces* v_{in}, v_{out} *and denote:*

$$V'' \leftarrow \{v_{in}, v_{out} : v \in V\} \cup \{s_{out}, t_{in}\}, E'' \leftarrow \{(u_{out}, u_{in} : (u,v) \in E'\}, G'' \leftarrow \langle V'', E'' \rangle,$$

Compute the maxflow on G'' *which provides us the escape routes.* $\diamondsuit$

Question (HW5, 7-17) *You've been called in to help some network administrators diagnose the extent of a failure in their network. The network is designed to carry traffic from a designated source node* s *to a designated target node* t*, so we will model it as a directed graph* $G = (V, E)$*, in which the capacity of each edge is* 1*, and in which each*



◇

Figure 4: An example showing (a) "yes" but (b) "no".

node lies on at least one path from s to t.

Now, when everything is running smoothly in the network, the maximum $s-t$ flow in G has value k. However, the current situation – and the reason you're here – is that an attacker has destroyed some of the edges in the network, so that there is now no path from s to t using the remaining (surviving) edges. For reasons that we won't go into here, they believe the attacker has destroyed only k edges, the minimum number needed to separate s from t (i.e. the size of a minimum $s-t$ cut); and we'll assume they're correct in believing this.

The network administrators are running a monitoring tool on node s, which has the following behavior: if you issue the command ping(v), for a given node v, it will tell you whether there is currently a path from s to v. (So ping(t) reports that no path currently exists; on the other hand, ping(s) always reports a path from s to itself.) Since it's not practical to go out and inspect every edge of the network, they'd like to determine the extent of the failure using this monitoring tool, through judicious use of the ping command.

So here's the problem you face: give an algorithm that issues a se-

quence of ping commands to various nodes in the network, and then reports the full set of nodes that are not currently reachable from s. You could do this by pinging every node in the network, of course, but you'd like to do it using many fewer pings (given the assumption that only k edges have been deleted). In issuing this sequence, your algorithm is allowed to decide which node to ping next based on the outcome of earlier ping operations.

Give an algorithm that accomplishes this task using only $O(k \log n)$ pings.

Answer (HW 5, 7-17, Sudipto Guha) *Observe that if the mincut in the graph was unique then we already knew which edges were cut! So there are multiple min-cuts and the goal is to find which mincut has been destroyed.*

1. *Run Ford-Fulkerson to find a maxflow of k; we then perform a flow decomposition and get k edge disjoint paths from s to t. Each of these steps require $O(mk)$ time.*

   Claim 1 *Every cut contains one of the edges from each of these paths.*

   Proof *Suppose not, it is not separating s and t. Therefore every mincut has exactly one edge from each of the paths – otherwise that cut will have more than k edges and will not be a mincut.* $\diamondsuit$

   Therefore the adversary has destroyed exactly one edge on each of the paths we found. We can now perform binary search to find the specific edge which has been cut.

2. *$O(\log n)$ pings on every paths since each path is of length at most n.*

We need $O(k \log n)$ pings. $\diamondsuit$

Answer (HW 6.5, 7-22) *Bipartite graph whether or not with perfect matching.* $\diamondsuit$

Question (HW6, 7-23) *Suppose you're looking at a flow network G with source s and sink t, and you want to be able to express something like the following intuitive notion: some nodes are clearly on the "source side" of the main bottlenecks; some nodes are clearly on the "sink side" of the main bottlenecks; and some nodes are in the middle. However, G can have many minimum cuts, so we have to be careful in how we try* *making this idea precise.*

Here's one way to divide the nodes of G into three categories of this sort.

- *We say a node v is* upstream *if for all minimum s-t cuts (A, B), we have $v \in A$ — that is, v lies on the source side of every minimum cut.*
- *We say a node v is* downstream *if for all min s-t cuts (A, B), we have $v \in B$ — that is, v lies on the sink side of every min cut.*
- *We say a node v is* central *if it is neither upstream nor downstream; there is at least one min s-t cut (A, B) for which $v \in A$, and at least one min s-t cut (A', B') for which $v \in B'$.*

Give an algorithm that takes a flow network G, and classifies each of its nodes as being upstream, downstream, or central. The running time of your algorithm should be within in a constant factor of the time required to compute a single *max flow.*

Lemma 5 *Given a max flow f and a min cut (A, B) for a $s-t$ flow problem. Two of which might not be required to have any correspondence*[a]*. Then no edge $\in G_f$ can go from A to B.*

[a] In other word, (A, B) is not necessarily obtained from G_f.

Proof *Suppose not, then (7.8) of [2] implies $\nu(f) < c(A, B)$ where*

$$\nu(f) \doteq \textit{value of flow } f, c(A, B) \doteq \sum_{e \textit{ out of } A} c_e.$$

A contradiction by (7.13) of [2]. ♢

Answer (HW6, 7-23, refers to [3]) *Consider the cut (A^*, B^*) found by performing BFS on G_f*[7] *at the end of any* max *flow algorithm.*

(Claim 1) *A node v is upstream iff $v \in A^*$.* ♢

Proof $\Rightarrow$) *If v is upstream, then $v \in A^*$ (otherwise, it lies on the sink-side of the minimum cut (A^*, B^*));*

$\Leftarrow$) *Suppose not, that is, $\exists v \in A^*$ were not upstream. Then $\exists (A', B')$ (a minimum cut) with $v \in B'$:*

[7] Residual graph of flow f.

1. $v \in A^* \Rightarrow \exists P_{s \to v} \in G_{f^*}$;

2. $v \in B' \Rightarrow \exists$ *edge* $(u, w) \in P_{s \to v}$ *with* $u \in A'$ *and* $w \in B'$.

A contradiction since no edge $\in G_{f^*}$ *can go from the source side to the sink side of any minimum cut by lemma.* $\diamondsuit$

A completely symmetric argument shows the following. Let B_* *denote the nodes that can reach* t *in* G_f, *and let* $A_* = V - B_*$. *Then* (A_*, B_*) *is a minimum cut. Then*

(Claim 2) *A node* w *is downstream iff* $w \in B_*$. $\diamondsuit$

Proof $\Rightarrow$) *If* w *is downstream, then* $w \in B_*$ *(otherwise, it lies on the source-side of the minimum cut* (A_*, B_*)*);*

$\Leftarrow$) *Suppose* $w \in B_*$ *were not downstream. Then there would be a minimum cut* (A', B') *with* $w \in A'$*:*

1. $w \in B_* \Rightarrow \exists P_{w \to t} \in G_{f_*}$;

2. $w \in A' \Rightarrow \exists$ *edge* $(u, v) \in P_{w \to t}$ *with* $u \in A'$ *and* $w \in B'$.

A contradiction since no edge in the residual graph can go from the source side to the sink side of any minimum cut by lemma. $\diamondsuit$

Thus, our algorithm is to compute a maximum flow f, *build construct residual graph* G_f, *and use BFS to find the sets* A^* *and* B_*. *These are the upstream and downstream nodes respectively; the remaining nodes are central.* $\diamondsuit$

Answer (HW 6, 7-23, Sudipto Guha, Yezheng Li) *Compute a max flow* f *(then fixed)*[8] ,G_f. *Now consider*

1. $S \doteq \{v \in V : \exists$ *path* $P_{s \to v} \in G_f\}$, *the set of all nodes reachable from* s *with capacity* > 0 *edges in the residual graph.*

2. $T \doteq \{w \in V : \exists$ *path* $P_{w \to t} \in G_f\}$, *the set of all nodes which can reach* t *with capacity* > 0 *edges in the residual graph.*

where notice

[8] Using any algorithm.

Claim 2 *Both* $(S, V-S)$, $(V-T, T)$ *are min cuts of* G.

Proof *In* G_f, *no positive weighted edge* $V - S \to S$ *or* $T \to V - T$. ♢

Then

Claim 3 (Yezheng Li) $S \subset A', T \subset B'$ *for any minimum cut* (A', B').

Proof (Yezheng Li)

1. *Speaking of* $S \subset A'$. *Suppose not, that is,* $\exists v \in S \setminus A' \Rightarrow v \in S \cap B'$. *Then*

   (a) $v \in S \Rightarrow \exists s \to v$ *path* $P_{s \to v} \in G_f$.

   (b) $v \in B' \Rightarrow \exists (u, w) \in P_{s \to v}$ *with* $u \in A', w \in B'$.

   A contradiction since no edge $\in G_f$ *can go from the source side to the sink side of any minimum cut by lemma;*

2. *Likewise, speaking of* $T \subset B'$. *Suppose not, that is,* $\exists w \in T \setminus B' \Rightarrow w \in T \cap A'$. *Then*

   (a) $w \in T \Rightarrow \exists w \to t$ *path* $P_{w \to t} \in G_f$.

   (b) $w \in A' \Rightarrow \exists (u, v) \in P_{w \to t}$ *with* $u \in A', v \in B'$.

   A contradiction by lemma again.

To conclude, the proof is finished. ♢

Claim 4 (Sudipto Guha) *Every node in* S *is upstream, every node in* T *is downstream. The remaining nodes are central.*

Proof (Yezheng Li) *By **Claim 1** we know "every node in* S *is upstream" and "every node in* T *is downstream".*

Furthermore,

1. *If* v *is upstream, then* $v \in S$.

   Proof *Suppose not.* $\exists v \notin S$ *but is upstream.* $v \notin S$ *means no* $s \to v$ *path* $\in G_f$, $\Rightarrow (S, V - S)$ *is a min cut of* G *such that* $v \in V - S \Rightarrow$ *A contradiction by definition of "upstream".* ♢

2. *Analogously, if* w *is downstream, then* $w \in T$.

   Proof *Suppose not.* $\exists w \notin S$ *but is downstream.* $w \notin T$ *means no* $w \to t$ *path* $\in G_f$, $\Rightarrow (V - T, T)$ *is a min cut of* G *such that* $w \in V - T$ $\Rightarrow$ *A contradiction by definition of "downstream".* ♢

To conclude,S is the set of all upstream nodes and T is the set of all downstream nodes. ♢

Answer (7-24) *By 7-23, one can just check whether or not $S = A, T = B$ to answer the question.* ♢

Answer (HW 6.5, 7-27) *Create a bipartite graph where $L = \{D_1, \ldots, D_d\}$ corresponds to the days and $R = \{P_1, \ldots, P_k\}$ corresponds to the people. If person j is present in S_i (the set of people going on day i) then draw a directed arc from $D_i \to P_j$. Now add a source s which connects to each $P_i \in L$ with capacity 1 (meaning we need 1 driver for that day) and a sink t which connects to each j with capacity $\lceil \triangle_j \rceil$.*

Observe that the fractional fair schedule defines a fractional flow of capacity d from s to t – here 1 unit of flow is sent from s to each i and the flow on the $i \to j$ edge is $\frac{1}{|S_j|}$. The total flow arriving at j is $\triangle_j \leq \lceil \triangle_j \rceil = c_{j\to t}$.

(Claim 1) *Max flow is d.* ♢

Proof *According to fractional fair schedule, max flow is at least d. But there is a cut $\langle \{s\}, V - \{s\} \rangle$ of capacity d. Therefore, max flow is d.* ♢

We can now use Ford Fulkerson to find the max flow (it will be an integral flow, in time $O(kd^2)$ because the total number of edges is at most kd. The flow can be decomposed into d paths each carrying one unit flow in time $O(kd^2)$. The flow paths through j correspond to the days j is driving. Observe that the schedule is fair by construction. Note this proves the existence, part (a) and the construction, part (b) simultaneously. ♢

Question (HW6, 7-31) *Some of your friends are interning at the small high-tech company WebExodus. A running joke among the employees there is that the back room has less space devoted to high-end servers than it does to empty boxes of computer equipment, piled up in case something needs to be shipped back to the supplier for maintainence.*

A few days ago, a large shipment of computer monitors arrived, each in its own large box; and since there are many different kinds of monitors in the shipment, the boxes do not all have the same dimensions. A bunch of people spent some time in the morning trying to figure out how to store all these things, realizing of course that less space would be taken up if some of the boxes could be nested *inside others.*

Suppose each box i is a rectangular parallelepiped with side lengths *equal to* (i_1, i_2, i_3)*; and suppose each side length is strictly between half a meter and one meter. Geometrically, you know what it means for one box to nest inside another — it's possible if you can rotate the smaller so that it fits inside the larger in each dimension. Formally, we can say that box* i *with dimensions* (i_1, i_2, i_3) nests *inside box* j *with dimensions* (j_1, j_2, j_3) *if there is a permutation* a, b, c *of the dimensions* $\{1, 2, 3\}$ *so that* $i_a < j_1$*, and* $i_b < j_2$*, and* $i_c < j_3$*. Of course, nesting is recursive — if* i *nests in* j*, and* j *nests in* k*, then by putting* i *inside* j *inside* k*, only box* k *is visible. We say that a* nesting arrangement *for a set of* n *boxes is a sequence of operations in which a box* i *is put inside another box* j *in which it nests; and if there were already boxes nested inside* i*, then these end up inside* j *as well. (Also notice the following: since the side lengths of* i *are more than half a meter each, and since the side lengths of* j *are less than a meter each, box* i *will take up more than half of each dimension of* j*, and so after* i *is put inside* j*, nothing else can be put inside* j*.) We say that a box* k *is* visible *in a nesting arrangement if the sequence of operations does not result in its ever being put inside another box.*

So this is the problem faced by the people at WebExodus: Since only the visible boxes are taking up any space, how should a nesting arrangement be chosen so as to minimize the number *of visible boxes?*

Give a polynomial-time algorithm to solve this problem.

Example. *Suppose there are three boxes with dimensions* $(.6, .6, .6)$*,* $(.75, .75, .75)$*, and* $(.9, .7, .7)$*. Then the first box can be put into either of the second or third boxes; but in any nesting arrangement, both the second and third boxes will be visible. So the minimum possible number of visible boxes is two, and one solution that achieves this is to nest the first box inside the second.*

Answer (HW 6, 7-31, refers to [3] and section 7.9 of [2]) *Reduce the given problem to a max flow problem where units of flow correspond to sets of boxes nested inside one visible box. We construct the following directed graph* $G = \langle V, E \rangle$*:*

- *about nodes* V*:*
  - *For each box* i*, two nodes* $\{u_i, v_i\} \subset V$*;*
  - *A source node* $s \in V$*, corresponding to the back room where boxes are stored;*

    a sink node $t \in V$*, corresponding to nothing inside empty boxes.*

- *about edges E:*
  - *between u_i, v_i that corresponds to this box, an edge $(u_i, v_i) \in E$ with since each box is exactly in one set of boxes nested one in another.;*
  - *$\exists$ an edge $(v_i, u_j) \in E$ for each i and j so that box j can nest inside box i;*
  - *Edges $\{(s, u_i), i = 1, 2, \ldots, n\} \subset E$ since any box can be visible;*
  - *Edges $\{(v_j, t), j = 1, 2, \ldots, n\} \subset E$ since any box can be empty.*

  Each edge $u \to v$ has capacity $c_{u\to v} = 1$ such that $0 \le f_{u\to v} \le c_{u\to v} = 1$.

- *about demands:*
  - *$d_{u_i} = d_{v_i} = 0$;*
  - *$d_s = -k, d_t = k$.*

We claim the following:

> **(Claim)** There is a nesting arrangement with k visible boxes if and only of there is a feasible circulation in G with demand $-k$ in the source node s and demand k in the sink t. $\diamondsuit$

Proof $\Rightarrow$) *Suppose there is a nesting arrangement with k visible boxes. Each sequence of nested boxes inside one visible box i_1, i_2, $\ldots$, i_n defines a path from s to t:*

$$(s, u_{i_1}, v_{i_1}, u_{i_2}, v_{i_2}, \ldots, u_{i_n}, v_{i_n}, t)$$

Therefore we have k paths from s to t. The circulation corresponding to all these paths satisfy all demands, capacity and lower bound.

$\Leftarrow$) *consider a feasible circulation (with integer flow values) in our network. There are exactly k edges going to t that carries one unit of flow. Consider one of such edges (v_i, t). We know that (u_i, v_i) has one unit of flow. Therefore, there is a unique edge into u_i that carries one unit of flow. If this edge is of the kind (v_j, u_i) then put box i inside j and continue with box j. If this edge of the kind (s, u_i), then put the box i in the back room. This box became visible. Continuing in this way we pack all boxes into k visible ones.*

So we can answer the question whether there is a nesting arrangement with exactly k visible boxes. Now to find the minimum possible number of *visible boxes we answer this question for $k = 1, 2, 3$, and so on, until we find a positive answer. The maximum number of this iteration is n, therefore the algorithm is polynomial since we can find a feasible circulation in polynomial time.* ♢

Question (2015, midterm 2) *You are given a directed graph $G = (V, A)$ with integer capacities c_e and an integral flow f_e.*

(c) Define an edge e to be "critical" if increasing c_e by 1 would increase the max flow. Given a max flow f_e on the original graph find a "critical" edge if such exists. The running time should be as low as possible.

Answer (2015, midterm 2) *Use BFS to find*

$$T \doteq \{v \in V : \exists v \to t \text{ path } \in G_f\},$$

and this takes $O(m + n)$ time. Notice $V \setminus T$, ♢

6 Ch8 NP and Computational intractability

Answer (8-1) *(a) Yes; (b)Nope since 3-SAT is NP-complete.* ♢

Answer (8-2) *Independent set.* ♢

Answer (8-3) *Set cover.* ♢

Answer (8-4) ***(a)*** *N-P complete.*

The Resource Reservation *problem (RRP) can be restated as follows. We have a set of m resources, and n processes, each of which requests a subset of the resources. The problem is to decide if there is a set of k processes whose requested resource sets are disjoint.*

(Claim 1) *RRP $\in$ NP.* ♢

Proof *Notice that if we are given a set of k processes, we can check in polynomial time that no resource is requested by more than one of them. The proof is finished.*

(Claim 2) *RRP $\in$NP-complete since* Independent Set $\leq_P$ RRP.♢

Proof *Given an instance of the independent set problem — specified by a graph G and a number k — we create an equivalent RRP. The resources are the edges, the processes correspond to the nodes of the graph, and the process corresponding to node v requests the resources corresponding to the edges incident on v. Note that this reduction takes polynomial time to compute. We need to show two things to see that the resource reservation problem we created is indeed equivalent.* ♢

First, if there are k processes whose requested resources are disjoint, then the k nodes corresponding to these processes form an independent set. This is true as any edge between these nodes would be a resource that they both request.

If there is an independent set of size k, then the k processes corresponding to these nodes form a set of k processes that request disjoint sets of resources. ♢

(b) *For each process v_i, if v_i is processed, whether any other process v_j can be processed; running time $O(n^2)$.*

(c)

(d) ♢

Answer (8-5) *Speaking of Vertex Cover (VC) and Hitting set problem (HS), we actually have*

> **Claim 5** *VC $\leq_P$ HS.*

Proof *For any VC in a (undirected) graph $G = \langle V, E \rangle$, we are to construct a HS: let*

$$\begin{aligned} A &\leftarrow E = \{e \in E\}, \\ \mathcal{B} &\doteq \{B_v \doteq \{e \in E : v \text{ is adjacent to } e.\} : v \in V\}, \end{aligned}$$

then if we can find a HS $\{B_{v_1}, B_{v_2}, \ldots, B_{v_k}\} \subset \mathcal{B}$, of size $\leq k$, we have found a vertex cover $\{v_1, v_2 \ldots, v_k\}$ of size $\leq k$ for $G = \langle V, E \rangle$. ♢

So HS is NP hard. ♢

Remark *Speaking of Set cover (SC) and HS, we have*

> **Claim 6** *HS $\leq_P$ SC.*

Proof *For any HS, $A = \{a_1, a_2, \ldots, a_n\}$ and subsets $\mathcal{B} \doteq \{B_1, B_2, \ldots, B_m\}$ we are to construct a SC: let*

$$\begin{array}{ll} U & \leftarrow \{x_j, \textit{ an element on behalf of } B_j\}, \textit{ the universe}; \\ \mathcal{S} & \doteq \{S_i \doteq \{x_j \in U : a_i \in B_j\}, \textit{ a subset of } U, \textit{ on behalf of } a_i\}, \\ & \quad \textit{a collection of subsets of } U \end{array}$$

then if we can find a SC $\{S_{i_1}, S_{i_2}, \ldots, S_{i_k}\} \subset \mathcal{S}$, of size $\leq k$, we have found a HS $\{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\}$ of size $\leq k$ for $G = \langle V, E\rangle$. $\diamondsuit$

To conclude, above shows: VC $\leq_P$ HS $\leq_P$ SC and simple reductions facilitate, say, design of approximation algorithms. $\diamondsuit$

Remark *As for weighted VC, weighted HS and weighted SC, we also have*

Claim 7 *Weighted VC $\leq_P$ weighted HS $\leq_P$ weighted SC.*

Proof

1. *To show "weighted VC $\leq_P$ weighted HS", consider any (undirected) graph $G = \langle V, E\rangle$ with weights $w_v : v \in V$, we are to construct a HS: let*

$$\begin{array}{ll} A & \leftarrow E = \{e \in E\}, \\ \mathcal{B} & \doteq \{B_v \doteq \{e \in E : v \textit{ is adjacent to } e.\} : v \in V\}, \end{array}$$

   then if we can find a HS $\{B_{v_1}, B_{v_2}, \ldots, B_{v_k}\} \subset \mathcal{B}$, of weights $\leq W$, we have found a vertex cover $\{v_1, v_2 \ldots, v_k\}$ of weights $\leq V$ for $G = \langle V, E\rangle$.

2. *To show "weighted HS $\leq_P$ weighted SC", consider any HS: $A = \{a_1, a_2, \ldots, a_n\}$ with weights $\{w_1, w_2, \ldots, w_n\}$ and subsets $\mathcal{B} \doteq \{B_1, B_2, \ldots, B_m\}$ we are to construct a SC: let*

$$\begin{array}{ll} U & \leftarrow \{x_j, \textit{ an element on behalf of } B_j\}, \textit{ the universe}; \\ \mathcal{S} & \doteq \{S_i \doteq \{x_j \in U : a_i \in B_j\}, \textit{ a subset of } U, \textit{ on behalf of } a_i\}, \\ & \quad \textit{a collection of subsets of } U \end{array}$$

   then if we can find a SC $\{S_{i_1}, S_{i_2}, \ldots, S_{i_k}\} \subset \mathcal{S}$, of size $\leq k$, we have found a HS $\{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\}$ of size $\leq k$ for $G = \langle V, E\rangle$. $\diamondsuit$

Simple reductions facilitate, say, design of approximation algorithms in 85.$\diamondsuit$

Question (8-6) *Consider an instance of the Satisfiability problem, specified by clauses $C_1, \ldots, C_k$ over a set of Boolean variables $x_1, \ldots, x_n$. We say that the instance is* monotone *if each term in each clause consists of a non-negated variable; that is, each term is equal to x_i, for some i, rather than $\overline{x_i}$. Monotone instances of Satisfiability are very easy to solve: they are always satisfiable, by setting each variable equal to* 1*.*

For example, suppose we have the three clauses

$$(x_1 \vee x_2), (x_1 \vee x_3), (x_2 \vee x_3).$$

This is monotone, and indeed the assignment that sets all three variables to 1 *satisfies all the clauses. But we can observe that this is not the only satisfying assignment; we could also have set x_1 and x_2 to* 1*, and x_3 to* 0*. Indeed, for any monotone instance, it is natural to ask how few variables we need to set to* 1 *in order to satisfy it.*

Given a monotone instance of satisfiability, together with a number k, the problem of Monotone Satisfiability with Few True Variables *asks: is there a satisfying assignment for the instance in which at most k variables are set to* 1*? Prove this problem is NP-complete.*

Remark (Refers to [3]) *On the surface,* Monotone Satisfiability with Few True Variables *(MS) is written in the language of SAT problem. But at a technical level, it's not so closely connected to SAT; after all no variables appear negated, and what makes it hard is the constraint that only a few variables can be set to true.*

Indeed, what's going on is that one has to choose a small number of variables, in such a way that each clause contains one of the chosen variables. Phrased this way, it resembles a type of covering problem. ♢

Answer (8-6, refers to [3]) *Reduce MS to Vertex Cover (VC).*

(Claim 1) *VC $\leq_P$ MS.* ♢

Proof *Given a graph $G = (V, E)$ and a number k, we are to decide whether there is a vertex cover in G of size at most k. Create an equivalent instance of MS as follows:*

1. *a variable x_i for each vertex v_i;*
2. *a clause $C_j = (x_a \vee x_b)$ for each edge $e_j = (v_a, v_b)$.*

This is the full instance: we have clauses $C_1, C_2, \ldots, C_m$, one for each edge of G, and we want to know if they can all be satisfied by setting at most k variables to 1.

(Claim 2) *The answer to the VC instance is "yes" if and only if the answer to the MS instance is "yes".* ♢

For suppose there is a vertex cover S in G of size at most k, and consider the effect of setting the corresponding variables to 1 (and all other variables to 0). Since each edge is covered by a member of S, each clause contains at least one variable set to 1, and so all clauses are satisfied. Conversely, suppose there is a way to satisfy all clauses by setting a subset X of at most k variables to 1. Then if we consider the corresponding vertices in G, each edge must have at least one end equal to one of these vertices — since the clause corresponding to this edge contains a variable in X. Thus the nodes corresponding to the variables in X form a vertex cover of size at most k.♢

Answer (8-8, refers to [7]) *This problem is essentially the Set Packing Problem, where we are given a set U, which is the set of magnets*[9] *and subsets $S_1 \ldots, S_n$ which represent words formed from the magnets that Madison knows how to spell. Note that if "CAT" was a word in Madisons vocabulary, then both of the sets CA^1T and CA^2T would appear among the S_i.*

We are interested in the maximum number of disjoint sets[10].

(Claim 1) *Independent Set (IS) $\leq_P$ Set Packing.* ♢

Proof *Given an instance of $IS(G, k)$ where $G = \langle V, E \rangle$, set $E \doteq U$. For each vertex v_i, introduce a set $S_i \doteq \{e : e = (v_i, x)\}$ which has one element for each edge incident to v_i.*

(Claim 2) *G has an independent set of size k iff there are k disjoint sets among the S_i.* ♢

Indeed, if I is an independent set of size k then the k sets S_v for $v \in I$ have no common elements. Also, if $\{S_{i_1}, \ldots, S_{i_k}\}$ are k disjoint sets then the vertices $v_{i_1}, \ldots, v_{i_k}$ have no edges between them thus they form an independent set of size. ♢

[9] Each magnet representing a symbol, but we are allowing more copies of the same symbol which we number arbitrarily $1, 2, 3, \ldots$. For example if we had two copies of the symbol A, we would have elements A^1 ,A^2.

[10] They correspond to words in Madisons vocabulary that can be simultaneously spelled out by the magnet pieces

Answer (8-9) *Independent set.* ♢

Question (8-10) *Your friends at WebExodus have recently been doing some consulting work for companies that maintain large, publicly accessible Web sites — contractual issues prevent them from saying which ones — and they've come across the following* STRATEGIC ADVERTISING *problem.*

A company comes to them with the map of a Web site, which we'll model as a directed graph $G = (V, E)$. The company also provides a set of t trails *typically followed by users of the site; we'll model these trails as directed paths $P_1, P_2, \ldots, P_t$ in the graph G. (I.e. each P_i is a path in G.)*

The company wants WebExodus to answer the following question for them: Given G, the paths $\{P_i\}$, and a number k, is it possible to place advertisements on at most k of the nodes in G, so that each path P_i includes at least one node containing an advertisement? We'll call this the STRATEGIC ADVERTISING *problem, with input G, $\{P_i : i = 1, \ldots, t\}$, and k.*

Your friends figure that a good algorithm for this will make them all rich; unfortunately, things are never quite this simple ...

(a) *Prove that* STRATEGIC ADVERTISING *is NP-complete.*

(b) *Your friends at WebExodus forge ahead and write a pretty fast algorithm $\mathcal{S}$ that produces yes/no answers to arbitrary instances of the* STRATEGIC ADVERTISING *problem. You may assume that the algorithm $\mathcal{S}$ is always correct.*

Using the algorithm $\mathcal{S}$ as a black box, design an algorithm that takes input G, $\{P_i\}$, and k as in part (a), and does one of the following two things:

- *Outputs a set of at most k nodes in G so that each path P_i includes at least one of these nodes,* or
- *Outputs (correctly) that no such set of at most k nodes exists.*

Your algorithm should use at most a polynomial number of steps, together with at most a polynomial number of calls to the algorithm $\mathcal{S}$.

Answer (8-10, refers to [3]) (a) *We'll say a set of advertisements is "valid" if it covers all paths in $\{P_i\}$. First,* STRATEGIC ADVERTISING *(SA) is in NP: Given a set of k nodes, we can check in $O(kn)$ time (or better)* *whether at least one of them lies on a path P_i, and so we can check whether it is a valid set of advertisements in time $O(knt)$.*

> **(Claim 1)** VERTEX COVER $\leq_P$ *SA.* ♢

Proof *Given an undirected graph $G = (V, E)$ and a number k, produce a directed graph $G' = (V, E')$ by arbitrarily directing each edge of G. Define paths $P_{\vec{e}}, \vec{e} \in E'$. This construction involves one pass over the edges, and so takes polynomial time to compute.*

> **(Claim 2)** *G' has a valid set of at most k advertisements if and only if G has a vertex cover of size at most k.* ♢

For suppose G' does have such a valid set U; since it meets at least one end of each edge, it is a vertex cover for G. Conversely, suppose G has a vertex cover T of size at most k; then, this set T meets each path in $\{P_i\}$ and so it is a valid set of advertisements. ♢

(b) *Construct the algorithm by induction on k.*

1. *If $k = 1$, simply check whether there is any node that lies on all paths. END.*

2. *($k > 1$) ask the fast algorithm $\mathcal{S}$ whether there is a valid set of advertisements of size at most k.*

   (a) *If it says "no," we simply report this.*

   (b) *If it says "yes", we perform the following test for each node v: we delete v and all paths through it, and ask $\mathcal{S}$ whether, on this new input, there is a valid set of advertisements of size at most $k - 1$.*

   > **(Claim 3)** *There is at least one node v where this test will succeed.* ♢

   For consider any valid set U of at most k advertisements (we know one exists since $\mathcal{S}$ said "yes"): The test will succeed on any $v \in U$, since $U - \{v\}$ is a valid set of at most $k - 1$ advertisements on the new input. INDUCTION[11]*.*

[11] Once we identify such a node, we add it to a set T that we maintain. We are now dealing with an input that has a valid set of at most $k - 1$ advertisements, and so our algorithm will finish the construction of T correctly by induction.

*The **running time** of the algorithm involves $O(n+t)$ operations and calls to $\mathcal{S}$ for each fixed value of k, for a total of $O(n^2+nt)$ operations.* $\diamondsuit$

Question (8-11) *As some people remember, and many have been told, the idea of hypertext predates the World Wide Web by decades. Even hypertext fiction is a relatively old idea — rather than being constrained by the linearity of the printed page, you can plot a story that consists of a collection of interlocked virtual "places" joined by virtual "passages."*[a] *So a piece of hypertext fiction is really riding on an underlying directed graph; to be concrete (though narrowing the full range of what the domain can do) we'll model this as follows.*

Let's view the structure of a piece of hypertext fiction as a directed graph $G=(V,E)$. Each node $u \in V$ contains some text; when the reader is currently at u, they can choose to follow any edge out of u; and if they choose $e=(u,v)$, they arrive next at the node v. There is a start node $s \in V$ where the reader begins, and an end node $t \in V$; when the reader first reaches t, the story ends. Thus, any path from s to t is a valid plot *of the story. Note that, unlike a Web browser, there is not necessarily a way to go back; once you've gone from u to v, you might not be able to ever return to u.*

In this way, the hypertext structure defines a huge number of different plots on the same underlying content; and the relationships among all these possibilities can grow very intricate. Here's a type of problem one encounters when reasoning about a structure like this. Consider a piece of hypertext fiction built on a graph $G=(V,E)$ in which there are certain crucial thematic elements *— love; death; war; an intense desire to major in computer science; and so forth. Each thematic element i is represented by a set $T_i \subseteq V$ consisting of the nodes in G at which this theme appears. Now, given a particular set of thematic elements, we may ask: is there a valid plot of the story in which each of these elements is encountered? More concretely, given a directed graph G, with start node s and end node t, and thematic elements represented by sets $T_1, T_2, \dots, T_k$, the* Plot Fulfillment *problem (PF) asks: is there a path from s to t that contains at least one node from each of the sets T_i?*

Prove that Plot Fulfillment *is NP-complete.*

[a] See e.g. http://www.eastgate.com

Remark *PF looks like a covering problem; in fact, it looks a lot like the Hitting Set problem (HS) from the previous question: we need to "hit" each set*

T_i. However, we have the extra feature that the set with which we "hit" things is a path in a graph; and at the same time, there is no explicit constraint on its size. So we use the path structure to impose such a constraint. ♢

Answer (8-11, refers to [3]) *Thus, we will show that HS $\leq_P$ PF.*

(Claim 1) *PF is $\in$NP.* ♢

Proof *Given an instance of the problem and a proposed $s-t$ path P*[12] *, we can check whether or not P is a valid path in the graph and that it meets each set T_i.* ♢

(Claim 2) *HS $\leq_P$ PF.* ♢

Proof *Specifically, let us consider an instance of HS, with a set $A = \{a_1, \ldots, a_n\}$, subsets $S_1, \ldots, S_m$, and a bound k. We construct the following instance of PF. The graph G will have nodes s, t, and*

$$\{v_{ij} : 1 \leq i \leq k,\ 1 \leq j \leq n\}.$$

There is an edge from s to each v_{1j} ($1 \leq j \leq n$), from each v_{kj} to t ($1 \leq j \leq n$), and from v_{ij} to $v_{i+1,\ell}$ for each $1 \leq i \leq k-1$ and $1 \leq j, \ell \leq n$. In other words, we have a layered graph, *where all nodes v_{ij} ($1 \leq j \leq n$) belong to "layer i", and edges go between consecutive layers. Intuitively the nodes v_{ij}, for fixed j and $1 \leq i \leq k$ all represent the element $a_j \in A$.*

We now need to define the sets T_ℓ in the PF instance. Guided by the intuition that v_{ij} corresponds to a_j, we define

$$T_\ell = \{v_{ij} : a_j \in S_\ell,\ 1 \leq i \leq k\}.$$

Now, we claim that there is a valid solution to this instance of PF if and only if our original instance of HS had a solution. First, suppose there is a valid solution to the PF instance, given by a path P, and let

$$H = \{a_j : v_{ij} \in P \text{ for some } i\}.$$

Notice that H has at most k elements. Also for each ℓ, there is some $v_{ij} \in P$ that belongs to T_ℓ, and the corresponding a_j belongs to S_ℓ; thus, H is a hitting set.

Conversely, suppose there is a hitting set $H = \{a_{j_1}, a_{j_2}, \ldots, a_{j_k}\}$. Define the path $P = \{s, v_{1,j_1}, v_{2,j_2}, \ldots, v_{k,j_k}, t\}$. Then for each ℓ, some a_{j_q} lies in S_ℓ, and the correponding node v_{q,j_q} meets the set T_ℓ. Thus P is a valid solution to the PF instance. ♢

[12] The author Yezheng Li does not think it easy to find every proposed $s-t$ path.

Answer (8-13)

Question (8-14) *We've seen the Interval Scheduling problem in class; here we consider a computationally much harder version of it that we'll call* MULTIPLE INTERVAL SCHEDULING*. As before, you have a processor that is available to run jobs over some period of time. (E.g. 9 AM to 5 PM.)*

People submit jobs to run on the processor; the processor can only work on one job at any single point in time. Jobs in this model, however, are more complicated than we've seen in the past: each job requires a set *of intervals of time during which it needs to use the processor. Thus, for example, a single job could require the processor from 10 AM to 11 AM, and again from 2 PM to 3 PM. If you accept this job, it ties up your processor during those two hours, but you could still accept jobs that need any other time periods (including the hours from 11 to 2).*

Now, you're given a set of n jobs, each specified by a set of time intervals, and you want to answer the following question: For a given number k, is it possible to accept at least k of the jobs so that no two of the accepted jobs have any overlap in time?

Show that MULTIPLE INTERVAL SCHEDULING *is NP-complete.*

Remark *The problem is in NP since, given a set of k intervals, we can check that none overlap.*

Answer (8-14, solution 1, refers to [3]) *Via* INDEPENDENT SET*: here is how we could use an algorithm* **A** *for* MULTIPLE INTERVAL SCHEDULING *to decide whether a graph $G = (V, E)$ has an independent set of size at least k. Let m denote the number of edges in G, and label them $e_1, \ldots, e_m$. As before, we divide time into m disjoint* slices*, $s_1, \ldots, s_m$, with slice s_i intuitively corresponding to edge e_i. For each vertex v, we define a job that requires precisely the slices s_i for which the edge e_i has an endpoint equal to v. Now, two jobs can both be accepted if and only if they have no time slices in common; that is, if and only if they aren't the two endpoints of some edge. Thus one can check that a set of jobs that can be simultaneously accepted corresponds precisely to an independent set in the graph G.* ♢

Answer (8-14, solution 2, refers to [3], not attempted) *Another way to solve this problem is via* 3-DIMENSIONAL MATCHING*. Suppose we had such an algorithm $\mathcal{A}$.*

Answer (8-15) *Given a Set Cover problem* $\langle U, \mathcal{S} \rangle$[13], *construct an NEOP* $\langle \mathcal{F}, \mathcal{L}, \mathbb{S} \rangle$ *as the following:*

- *for every element* $v_i \in U$*, define frequency* $f_i, i = 1, \dots n$ *and* $\mathcal{F} \doteq \{f_a : v_a \in U\}$*;*
- *for every subset* $S_j \in \mathcal{S}$*, define a location* $l_j, j = 1, \dots m$ *and* $\mathcal{L} \doteq \{l_j, j = 1, \dots, m\}$[14]*;*
- *define interference source* i *as the following:* $F_i \doteq \{f_i\}$ *and* $L_i \doteq \{l_j \in \mathcal{L} : v_i \notin S_j\}, i = 1, \dots, n$*. Therefore set of interference sourses:* $\mathbb{S} \doteq \{\langle F_i, L_i \rangle, j = 1, \dots, n\}$*.*

Above construction is of polynomial running time. Assign both problem with same number $k (\leq j)$*.*

> **(Claim 1)** *Answer is yes to SC* $\langle U, \mathcal{S} \rangle$ *with # of subsets* k *iff answer is yes to the NEOP* $\langle \mathcal{F}, \mathcal{L}, \mathbb{S} \rangle$ *with # of locations* k*.* ♢

Proof $\Rightarrow$) *Suppose for SC* $\langle U, \mathcal{S} \rangle$*, we indeed have* k *subsets* $S_1, S_2, \dots, S_k \in \mathcal{S}$ *such that* $\cup_{i=1}^{k} S_i = U$*. Namely,*

$$\forall v_a \in U, \exists i \in [k] \text{ such that } v_a \in S_i;$$

By choosing locations $l_1, \dots, l_k \in \mathcal{L}$ *in NEOP* $\langle \mathcal{F}, \mathcal{L}, \mathbb{S} \rangle$*, since* $v_a \in S_i \xLeftrightarrow{F_i = \{f_i\}} l_a \notin L_i$*, above interpretation of the union of set means*

$$\forall f_a \in \mathcal{F}, \exists i \in [k] \text{ such that } l_a \notin L_i,$$

which means $\forall f_a \in \mathcal{F}$ *can be detected once we choose* $l_1, \dots, l_k$ *as our locations.*

$\Leftarrow$) *Every steps of above interpretations is reversible.* ♢

> **Question (8-16)** *Consider the problem of reasoning about the identity of a set from the size of its intersections with other sets. You are given a finite set* U *of size* n*, and a collection* $A_1, \dots, A_m$ *of subsets of* U*. You are also given numbers* $c_1, \dots, c_m$*. The question is: does there exist a set* $X \subset U$ *so that for each* $i = 1, 2, \dots, m$*, the cardinality of* $X \cap A_i$ *is equal to* c_i*? We will call this an instance of the* Intersection Inference *problem, with input* U*,* $\{A_i\}$*, and* $\{c_i\}$*.*

[13] $\#U = n$, $\#\mathcal{S} = m$.

[14] And unnecessarily define $\mathcal{FO}_j \doteq \{f_a \in \mathcal{F} : v_a \in S_j\}$, set of frequencies from which can be observed.

Prove that Intersection Inference *is NP-complete.*

Answer (8-16, intersection inference problem)

Question (8-17) *You are given a graph $G = (V, E)$ with weights w_e on its edges $e \in E$. The weights can be negative or positive. The* ZERO-WEIGHT-CYCLE *problem is to decide if there is a simple cycle in G so that the sum of the edge weights on this cycle is exactly 0. Prove that this problem is NP-complete.*

Answer (8-17, refers to [9]) *Reduce from Partition problem to Zero-Weighted-Cycle problem (ZWC).*

(Claim 1) *ZWC is in NP.* ♢

Proof *Use a zero cycle as the certificate. Then verification is simply checking that the certificate is actually a cycle of the graph, and that the edge weights sum to zero.* ♢

Given a Partition problem: a multi-set[15] $A = \{a_1, \ldots, a_n\}$*, create a graph* $G = (V, E)$ *for ZWC as follows:*

- $V = \{v_1, \ldots, v_n\}$;
- *define two edges* $v_i \to v_{(i+1)_{mod} n}$*, one of weight a_i, the other of weight* $-a_i$ *for each* $a_i \in A$.

Now G has 2^n possible cycles, all of length n. Each cycle contains exactly one edge $(v_i, v_{(i+1)_{mod} n}), \forall i$.

(Claim 2) *"Yes" to Partition problem iff "Yes" to the ZWC.* ♢

Proof ⇒) *If there exists a partition A', then we can construct a cycle C, using the positive weight edge* $v_i \to v_{(i+1)_{mod} n}, a_i \in A'$*, and the negative weight edge from* $v_i \to v_{(i+1)_{mod} n}, a_i \notin A'$.

Then

$$\sum_{e \in C} w(e) = \sum_{i=1}^{n} w\left(v_i \to v_{(i+1)_{mod} n}\right) = \sum_{x \in A'} x - \sum_{y \in A - A'} y = 0,$$

So there exists a zero-weight cycle.

[15] members of a multi-set are not necessarily different from each other

$\Leftarrow$) *If there exists a zero cycle $C \in G$, then using the positive weight edges, introduce $A^+ = \left\{a_i \in A : w\left((v_i, v_{(i+1)_{mod} n})\right) > 0\right\}$ and $A^- \doteq A - A^+ = \left\{a_i \in A : w\left((v_i, v_{(i+1)_{mod} n})\right) < 0\right\}$. Then "yes" to ZWC means*

$$
\begin{aligned}
& 0 = \sum_{1 \le i \le n} w((v_i, v_{(i+1)_{mod} n}) \in C), \\
\Leftrightarrow \quad & 0 = \sum_{x \in A^+} x - \sum_{y \in A^-} y, \\
\Leftrightarrow \quad & \sum_{x \in A^+} x = \sum_{y \in A^-} y,
\end{aligned}
$$

resulting in "yes" to Partition problem. $\diamondsuit$

Answer (8-17, refers to [8], incorrect) *First, given a simple cycle in G, we can determine whether the sum of its edge weights is zero in polynomial time. Thus Zero-Weight-Cycle (ZWC) $\in$ NP. Then we reduce the Subset Sum [16] Problem to ZWC.*

Question (8-18) *Textbook chapter 8, exercise 18: Youve been asked to help some organizational theorists analyze data on group decision-making. In particular, theyve been looking at a dataset that consists of decisions made by a particular governmental policy committee, and they are trying to decide whether its possible to identify a small set of influential members of the committee. Here is how the committee works. It has a set $M = \{m_1, \ldots, m_n\}$ of n members, and over the past year it has voted on t different issues. On each issue, each member can vote either "yes", "no", or "abstain"; the overall effect is that the committee presents an affirmative decision on the issue if the number of yes votes is strictly greater that the number of "no" votes (the "abstain" votes don't count for either side), and it delivers a negative decision otherwise. Now we have a big table consisting of the vote cast by each committee member on each issue, and we'd like to consider the following definition. We say that a subset of members $M_0 \subset M$ is decisive if, had we looked just at the votes cast by the members of M_0, the committee's decision on every issue would have been the same. (In other words, the overall outcome of the voting among the members of M_0 is the same on every issue as the outcome of the voting by the entire committee.) Such a subset can*

[16] The subset sum problem is: given a set of integers, determine whether the sum of some non-empty subset equal exactly zero.

be viewed as a kind of :inner circle" that reflects the behaviour of the committee as a whole. Here is the question: Given the votes cast by each member on each issue, and given a parameter k, we want to know whether there is a decisive subset consisting of at most k members. We will call this an instance of the Decisive Subset Problem.

Answer (8-18) *There is an NP-complete problem with a polynomial time reduction to DS. We choose Vertex Cover (VC).*

(Claim 1) *Decisive Subset (DS) is in NP.* ♢

Proof *We can verify a certificate is a solution to DS in polynomial time. Given a set of committee members, we can easily check that the size is k, and by iterating over all n members and the k members in the subset for each issue, we can check whether or not the votes of the k members on the issue represent the same votes of all n members. If this is the case for all issues, then the certificate is a valid solution, and if not, then it is not a valid solution. Thus, this can be verified in time* $O((n+k)m)$. ♢

Let $G = (V, E)$ *be an undirected graph, and let* k *be a bound such that we want to know if* G *contains a vertex cover of size at most* k. *For each edge* $e \in E$, *create an issue Ie, and for each node* $v \in V$, *create a committee member* m_v.

If $e = (u, v)$, *then we have that members* m_u *and* m_v *vote yes on issue* I_e, *and all other committee members abstain on issue* I_e. *Note then that the vote of the entire committee on issue* I_e *is thus yes.*

(Claim 2) *There is a decisive set (on the instance generated according to the reduction given above) of size at most* k *on this collection of issues and voting members iff there is a VC of size at most* k. ♢

Proof *Then, for each edge* $e = (u, v)$, *at least one of* u *or* v *must be in the vertex cover. For the set of members corresponding to the nodes in the vertex cover, the voting outcome for each issue Ie will then be yes, so the corresponding set of members is indeed decisive as required. Now, assume there is indeed a decisive set of size at most k. Then, for each issue, we must have that the decisive set voted armatively on the issue. Thus, we must have at least one of the two members mu or mv voting yes on the issue Ie in our decisive set. This corresponds to having at least one of the two endpoints - either u or v - of the edge e = u,v in the vertex cover, and since this is the case across all issues, or edges, the decisive set indeed represents a vertex cover.* ♢

To conclude, DS is NP-complete. ♢

Answer (8-19) ***(a)*** *Apply maximum flow and running time can be strictly polynomial by Preflow–Push algorithm.*

(b) *Reduce from Independent set.* ♢

Answer (HW 6, 8-20, refers to [6]) *In order to reduce this problem to the graph coloring problem, we construct the following graph* $G = \langle V, E, \rangle$*:*

Let $V \doteq \{p_1, p_2, ...p_n\}$ *and* $E \doteq \{(p_i, p_j) \in E | d(p_i, p_j) > B\}$*. Now color this graph with k colors. Each color is a partition set, and obviously, no two points that are "too far" can have the same color (and the are neighbors in the graph!).* ♢

Answer (HW 6, 8-27)

Answer (HW 6.5, 8-28) *Obviously* $\in$ *NP and furthermore, for an independent set problem in* $G = \langle V, E \rangle$*. Subdivide each edge to insert a vertex (remove the original edge and add a path of length 2) and add edges between the newly added vertices. We obtain* $G' = \langle V', E' \rangle$*.*

(Claim 1) *There is an IS of size* k *in* G *iff there is an SIS of size* k*.* ♢

Proof

Therefore,

(Claim 2) *IS* $\leq_P$ *SIS.* ♢

Question (2015, midterm 2) *Consider the problem of MAX-4-CUT where we are asked to partition the vertex set of the graph into four pieces. An edge* (u, v) *is cut if the vertices* u, v *belong to different partitions. We want to* max # *of edges being cut and want to solve the following decision problem: is there a 4-CUT that cuts at least* k *edges?*

Prove that MAX-4-CUT is NP-hard.

Answer (2015, midterm 2) *4-coloring problem (4-COLOR)* $\leq_P$ *MAX-4-CUT.*

Given an instance of 4-COLOR, G *with* m *edges, we set* $k = m$*.*

$\Rightarrow$*) If the graph is 4 colorable, then* $\exists$ *4 partitions (corresponding to the 4 colors) such that at least* $k = m$ *edges are cut.*

$\Leftarrow$*) If* $\exists$ *4 partitions such that every edge has two endpoints in different partitions then we can color all vertices in partition 1 with color 1, ones in partition 2 with color 2, etc, till we color all vertices.* ♢

Question (Final 2014, 20 pts) *Professor Npsrus wanted to use the cluster to solve his favorite problem of "(Minimum) ST with forbidden edge pairs" ("(M)ST with P") where we are given a weighted graph* $G = (V, E)$ *and are given* t *pairs of edges* $P_1, P_2, \ldots, P_t$ *such that for any pair* $P_r = (e_1(r), e_2(r))$ *both the edges* $e_1(r)$ *and* $e_2(r)$ *cannot be in the tree simultaneously*[a]*. Show that it is NP hard to decide if* $\exists$ *a ST with P of weight at most* K.

(Hint: Professor 002 suggests considering the Independent Set problem (IS).)

[a] In other word, it is fine if the tree contains one or none of them.

Answer (Final 2014, 20 pts) *Given an IS instance* $G = \langle V, E\rangle$ *with* $\#V = n$*. Consider the star graph* $G' = \langle V', E'\rangle$ *with* $\#V' = n + 1$*.* $V' = V \cup \{v_0\}$ *and node* v_0 *is at the center and*

$$E' = \left\{e_i^0 = (v_0, v_i), e_i^1 = (v_0, v_i) : v_i \in V\right\}, w(e_i^j) = j, j = 0, 1; i = 1, \ldots, n,$$

namely, v_0 *has two edges to each* $v_i \in V$ *labelled* e_i^0*,* e_i^1 *respectively and* $w(e_i^j) = j, j = 0, 1$*. Now if* $(u, v) \in G$ *we forbid the pair* (e_u^0, e_v^0)*. We set* $K = n - k$*.*

Claim 8 $\exists$ *IS*$\in G$ *with size* $\geq k \Leftrightarrow \exists$ *a "ST with P"* $\in G'$ *of weight at most* $K = n - k$*.*

Proof $\Rightarrow$*) Denote IS to be* $\{v_1, \ldots, v_k\} \subset V$ *in* G*. Pick* $\{e_j^0 : j = 1, \ldots, k\} \cup \{e_j^1 : j = k + 1, \ldots, n\} \subset E'$ *in* G' *and we form a "ST with P" of weight* $K = n - k$*.*

$\Leftarrow$*) Denote "ST with P" of weight* $K = n - k$ *to be* $\{e_j^0 : j = 1, \ldots, k\} \cup \{e_j^1 : j = k + 1, \ldots, n\} \subset E'$ *in* G'*. Then* $\{v_1, \ldots, v_k\} \subset V$ *is an IS of size* k *in* G*.* ♢

The reduction shows that "(M)ST with P" is NP hard. ♢

Remark (Sudipto Guha) *Other constructions with paths of length* k *also exist.* ♢

7 Ch11 Approximation algorithm

Answer (11-1, somehow refers to [3]) *(a) With* $n = 2K$ *and* $w_1 = w_3 = \ldots = w_{2K-1} = K, w_2 = w_4 = \ldots = w_{2K} = 1$*, this algorithm returns a solution with # of trucks* $= 2K$*; however, # of trucks can be* $K + 1$*.*

(b) Assume the algorithm returns a solution with # of trucks $= M(\leq n)$ *and item labeled* $i_{j-1}+1, i_{j-1}+2, \ldots, i_j, j = 1, \ldots, M$ *are in* j^{th} *truck where* $0 = i_0 < i_1 < \ldots < i_M = n$. *Mathematically we have:*

$$\begin{aligned}
w_1 + w_2 + \ldots + w_{i_1} &\leq K, & w_1 + w_2 + \ldots + w_{i_1} + w_{i_1+1} &> K;\\
w_{i_1+1} + w_{i_1+2} + \ldots + w_{i_2} &\leq K, & w_{i_1+1} + w_{i_1+2} + \ldots + w_{i_2} + w_{i_2+1} &> K;\\
w_{i_2+1} + w_{i_2+2} + \ldots + w_{i_3} &\leq K, & w_{i_2+1} + w_{i_2+2} + \ldots + w_{i_3} + w_{i_3+1} &> K;\\
w_{i_{M-2}+1} + w_{i_{M-2}+2} + \ldots + w_{i_{M-1}} &\leq K, & w_{i_{M-2}+1} + w_{i_{M-2}+2} + \ldots + w_{i_{M-1}} + w_{i_{M-1}+1} &> K;\\
w_{i_{M-1}+1} + w_{i_{M-1}+2} + \ldots + w_n &\leq K.
\end{aligned}$$

Then

$$\begin{aligned}
\lfloor \frac{M}{2} \rfloor &= \frac{\lfloor \frac{M}{2} \rfloor K}{K}\\
&< \frac{1}{K} \sum_{j=0}^{\lfloor \frac{M}{2} \rfloor - 1} \left(w_{i_{2j}+1} + w_{i_{2j}+2} + \ldots + w_{i_{2j+1}} + w_{i_{2j+1}+1} \right)\\
&\leq \frac{\sum_{s=1}^{0} w_j}{K} \leq \texttt{opt} \in \mathbb{N},
\end{aligned}$$

or for short, $\lfloor \frac{M}{2} \rfloor < \texttt{opt} \Rightarrow \frac{M}{2} \leq \lceil \frac{M}{2} \rceil \leq \texttt{opt}$, *namely, the desired result holds.* ♢

Remark *What if we make* $w_1 \geq w_2 \geq \ldots \geq w_n$? ♢

Answer (11-2) *(a) Reduce to kind of vertex cover, thus kind of set cover. Then applying method in section 11.3.*

(b) No triangular inequality on the distance functions. ♢

Answer (11-3) *(a)* $B \leftarrow B$, $a_1 \leftarrow 1$, $a_i \leftarrow B, i \geq 2$.

(b)

Claim 9 *Sort* $\{a_i\}$ *such that* $a_1 \geq a_2 \geq a_3 \geq \ldots$,*before the algorithm in (a). Then we can get a 2-approximation.*

Proof *Assume eventually* $S = \{a_1, \ldots, a_m\}$, *then*

$$\begin{aligned}
B &< \sum_{x \in S \cup \{a_{m+1}\}} x = a_{m+1} + \sum_{x \in S} x \leq 2 \sum_{x \in S} x, \left(\text{by } \sum_{x \in S} x \geq a_{m+1} \right)\\
\Rightarrow & \sum_{x \in S} x > \frac{B}{2} > \frac{T_{\texttt{OPT}}}{2}.
\end{aligned}$$

The algorithm satisfies our need and running time is $O(n \log n)$. $\diamondsuit$

Answer (11-4) *By referring to claim 7, we know "weighted HS" $\leq_P$ "weighted SC". Then we have an $H(b)$-approximation algorithm and $H(b) \leq b, \forall b \in \mathbb{N}$.* $\diamondsuit$

Answer (11-5) *Denote the machine M_i the one attains the maximum load T in* `Greedy Balance` *algorithm, When we assigned the last job j to $M_i, i = 1, 2, \ldots, 10$, we have $T_i - t_j \leq \frac{1}{10}\sum_k T_k$; together with $t_j \leq 50$, we have*

$$\begin{aligned} T \quad &= T_i \leq \frac{1}{10}\sum_k T_k + 50 = \frac{\frac{1}{10}\sum_k T_k + 50}{\frac{1}{10}\sum_k T_k} \cdot \frac{1}{10}\sum_k T_k \\ &= \left(1 + \frac{50}{\frac{1}{10}\sum_k T_k}\right)\frac{1}{10}\sum_k T_k \\ &\leq \left(1 + \frac{1}{6}\right)\frac{1}{10}\sum_k T_k \text{ by } \sum_k T_k \geq 3000 \\ &< (1 + 20\%)\,\frac{1}{10}\sum_k T_k, \end{aligned}$$

so the phenomena is explained. $\diamondsuit$

Answer (11-6) *Just notice $t_j \leq 2T^*$.* $\diamondsuit$

Answer (11-7) *(a) Every time show customer j the advertisement A_i which has the lowest total weight. The spread S its find satisfies $S \leq \left(1 + \frac{1}{2m}\right) S_{OPT}$.*

(b) There are advertisements $A_1, A_2, A_3 (, m = 3)$ and customers with values $a, 2, 2, a$ with $2 < a < \frac{7}{6}\frac{2a+4}{4} = \frac{7(a+2)}{12} \Leftrightarrow 2 < a < \frac{14}{5}$. Then algorithm in (a) gives the assignment: $a; 2, a; 2$ with spread $S = 2$ but the optimal solution should be $a; a; 2, 2$ with spread $S_{OPT} = a$.

Answer (11-8) *Correct. Suppose we have an optimal solution $S^{\texttt{OPT}}$. Every time take the last jobs from the server with largest time cost $T_i = \max_{k=1,\ldots,m} T_k$. We obtain an order of jobs $J_n, J_{n-1}, \ldots, J_1$, and $J_1, \ldots, J_{n-1}, J_n$ is the order of jobs that if we input in this order in Greedy-Balance algorithm, we can get the optimal solution $S^{\texttt{OPT}}$.* $\diamondsuit$

Answer (11-9)

Answer (11-10) *(a) Suppose not, that is, $\exists v \in T \setminus S$ such that $\forall v' \in S$ if $(v, v') \in E$, then $w(v) > w(v')$.*

1. *if $\forall v' \in S, (v, v') \notin E$, then $S \cap \{v\}$ is a larger independent set than S, a contradiction;*

2. *if $v' \in S, (v, v') \in E$ and v' is the first node added to S in "heaviest-first" algorithm, then at the time v' is added to S, v as well as neighbors of v has not been added $\Rightarrow$ v should be added to S instead of v'. A contradiction.* $\diamondsuit$

(b) We know that any node v in G has at most 4 neighbors. So for any independent set T returned by the algorithm, up to 4 nodes from $T - S$ can share one node from $S - T$ as their common neighbor, according to the result proven in subproblem (a), we have

$$\begin{aligned}\sum_{v \in T} w(v) &= \sum_{v \in T \cap S} w(v) + \sum_{v \in T - S} w(v) \\ &\leq \sum_{v' \in S \cap T} w(v') + 4 \sum_{v' \in S - T} w(v') \leq 4 \sum_{v' \in S} w(v') \qquad (3)\end{aligned}$$

Therefore the "heaviest-first" greedy algorithm returns an independent set of total weight at least[17] 1/4 times the maximum total weight of any independent set in the grid graph G.

Answer (11-11, 1) *Consider $s_i \leftarrow \lfloor \frac{nw_i}{\epsilon W} \rfloor$. Now consider a knapsack with sizes s_i and value v_i such that the size does not exceed $\lfloor \frac{n}{\epsilon} \rfloor$. If $\exists$ a solution O such that $\sum_{i \in O} w_i \leq W$ and $\sum_{i \in O} v_i \geq V$, then*

$$\sum_{i \in O} s_i = \sum_{i \in O} \lfloor \frac{nw_i}{\epsilon W} \rfloor \leq \sum_{i \in O} \frac{nw_i}{\epsilon W} \leq \frac{n}{\epsilon}.$$

Therefore by dynamic programming, we can find a set A such that $\sum_{i \in A} s_i \leq \lfloor \frac{n}{\epsilon} \rfloor$ and $\sum_{i \in A} v_i \geq V$. Now

$$\begin{aligned}\sum_{i \in A} w_i &= \frac{W\epsilon}{n} \sum_{i \in A} \frac{nw_i}{\epsilon W} \leq \frac{W\epsilon}{n} \sum_{i \in A} \left(\lfloor \frac{nw_i}{\epsilon W} \rfloor + 1 \right) \\ &= \frac{W\epsilon}{n} \sum_{i \in A} (s_i + 1) = \frac{W\epsilon}{n} \sum_{i \in A} s_i + \frac{W\epsilon}{n} \sum_{i \in A} 1 \\ &\leq (1 + \epsilon) W,\end{aligned}$$

which is what was desired. $\diamondsuit$

[17] Why not "at most" here?

Answer (11-11,2) *Consider* $s_i \leftarrow \lceil \frac{nw_i}{\epsilon W} \rceil$. *Now consider a knapsack with sizes* s_i *and value* v_i *such that the size does not exceed* $\lfloor \frac{n}{\epsilon} \rfloor + n$. *If* $\exists$ *a solution* O *such that* $\sum_{i \in O} w_i \leq W$ *and* $\sum_{i \in O} v_i \geq V$, *then*

$$\sum_{i \in O} s_i = \sum_{i \in O} \lceil \frac{nw_i}{\epsilon W} \rceil \leq \sum_{i \in O} \left(\frac{nw_i}{\epsilon W} + 1 \right) \leq \frac{n}{\epsilon} + n.$$

Therefore by dynamic programming, we can find a set A *such that* $\sum_{i \in A} s_i \leq \lfloor \frac{n}{\epsilon} \rfloor + n$ *and* $\sum_{i \in A} v_i \geq V$. *Now*

$$\sum_{i \in A} w_i = \frac{W\epsilon}{n} \sum_{i \in A} \frac{nw_i}{\epsilon W} \leq \frac{W\epsilon}{n} \sum_{i \in A} \lceil \frac{nw_i}{\epsilon W} \rceil = \frac{W\epsilon}{n} \sum_{i \in A} s_i \leq \frac{W\epsilon}{n} \left(\lfloor \frac{n}{\epsilon} \rfloor + n \right) \leq (1+\epsilon) W,$$

which is what was desired. $\diamondsuit$

Question (Final 2014, 20 Points) *Recall that MST*[a] *is defined as follows: given a weighted graph* $G = \langle V, E, w\rangle (w_e \geq 0, \forall e \in E)$*, the weight of any tree is the sum of the weights of the edges in the tree.*

1. ***(5 points)*** *Professor Xcluster has found himself a large cluster that can perform computations in parallel. He immediately implemented the following round based algorithm for computing the MST* $\in G$*: in a round every vertex chooses the smallest weight edge incident to it. After the round is over we merge the connected components into a super vertex and an edge* (u, v) *becomes the edge* $(super(u), super(v))$[b].

   We repeat for several rounds till there is one node. Give a short proof that the algorithm is correct. What property would the proof rely on?

2. ***(15 points)*** *Professor Xcluster, found out that instead of computing the minimum at each node, his code only managed to compute the minimum approximately. Particular, if the weight of the correct minimum weight edge incident to any vertex was* x *the algorithm only guaranteed that the edge output for that vertex was at most* $2x$*. He correctly believed that he found a 2-approximate MST , but had no proof. Provide a proof that Xcluster found a 2-approximate MST.*

(Hint: Denote Xcluster returns tree T*. Professor 002 suggests considering the new graph* G' *which has the same vertex and edge sets as the original* G *but a different set of weights:* $w'_e = \begin{cases} w_e, & e \notin T; \\ \frac{w_e}{2}, & e \in T. \end{cases}$ *You can assume part (a) for this part.)*

[a] Minimum spanning tree. Our goal is to find the MST which contains every vertex in V.

[b] where $super(u) \neq super(v)$ are the connected components containing u and v respectively (we ignore loops).

Answer

1. *Cut property: for any* $S, V \setminus S$*, shortest path between them is included in the MST.*

2. *Suppose part 2 returns a tree* $T \in G$ *of weight* $w(T)$*. Observe that the tree* T *found in part 2 is the correct MST* $\in G'$*. Therefore* $w(T^*)$*, the*

weight of the MST $T^* \in G$ satisfies

$$\begin{aligned}
w(T^*) &= w(T^* \cap T) + w(T^* \setminus T) = 2w'(T^* \cap T) + w'(T^* \setminus T) \\
&\geq w'(T^*) \\
T \text{ is the MST } \in G' \quad &\geq w'(T) = \frac{w(T)}{2}.
\end{aligned}$$

Finishes. ♢

Remark *Two properties: cut property (from Kruskal's algorithm) and circle property (for any circle, largest weight edge of the circle is eliminated – from Prim's algorithm).* ♢

8 Ch13 Randomized algorithm

Answer (13-1) *I think solution in [10] to be wrong – if $G = K_n$, how is it possible to find by the randomized algorithm with $\frac{2}{3}|E|$ edges.* ♢

Answer (13-3) *(a)* $\mathbf{P}[v \in S] = \frac{1}{2}\left(1 - \frac{1}{2}\right)^d = \frac{1}{2^{d+1}}$.

(b)

$$\mathbf{P}[v \in S] = p\,(1-p)^d. \tag{4}$$ ♢

Answer (13-4) *Denote X_j=# of incoming links to node v_j.*

(a)

$$\mathbf{E}X_j = \sum_{k=j}^{k=n-1} \frac{1}{k} \in \left(\ln\frac{n}{j}, \ln\frac{n-1}{j-1}\right) = \left(\ln\left(1 + \frac{n-j}{j}\right), \ln\left(1 + \frac{n-j}{j-1}\right)\right).$$

(b) Notice $\mathbf{P}\left(X_j = 0\right) = \frac{j-1}{j} \cdot \frac{j}{j+1} \cdot \ldots \cdot \frac{n-2}{n-1} = \frac{j-1}{n-1}$. *So*

$$\begin{aligned}
\mathbf{E}\left[\sum_{j=1}^{n} 1_{\{X_j=0\}}\right] &= \sum_{j=1}^{n} \mathbf{E}1_{\{X_j=0\}} = \sum_{j=1}^{n} \mathbf{P}(X_j = 0) \\
&= \sum_{j=1}^{n} \frac{j-1}{n-1} = \frac{1}{n-1} \cdot \frac{n(n-1)}{2} = \frac{n}{2}.
\end{aligned}$$

Answer (13-8, HW8, Sudipto Guha) *"dense subgraph" might be NP hard since CLAUSE is NP hard.* ♢

Answer (13-9) *Reject first $\frac{n}{2}$ buyers and record m, the largest amount of money they offer. Later on, once I see another buyer offers $\geq m$, accept it.* $\mathbf{P} = \frac{\frac{n}{2}}{n} \cdot \frac{n-\frac{n}{2}}{n-1} = \frac{n}{4(n-1)} \geq \frac{1}{4}$. $\diamondsuit$

Answer (13-10, HW8, [11]) *Let X_k indicator function of whether or not bid k updates b^*. Then notice $\mathbf{P}\left(b_k \text{ is the biggest bid so far}\right) = \frac{1}{k}$. So*

$$\begin{aligned} \mathbf{E}\left[\sum_{k=1}^{n} X_k\right] &= \sum_{k=1}^{n} \mathbf{P}\left(b_k \textit{ is the biggest bid so far}\right) \\ &= \sum_{k=1}^{n} \frac{1}{k}. \end{aligned}$$

Answer (13-11)

Answer (13-12)

Answer (13-13)

$$\begin{aligned} \mathbf{P}\left(X_1 - X_2 > c\sqrt{n}\right) & \overset{X_1+X_2=2n}{=\!=\!=} \mathbf{P}\left(X_1 - n > \frac{c}{2}\sqrt{n}\right) \\ &= \mathbf{P}\left(\sum_{j=1}^{2n} \xi_j > n(1+\frac{c}{2\sqrt{n}})n\right) \\ \textit{Chernoff bound} \quad & \leq \left(\frac{e^{\frac{c}{2\sqrt{n}}}}{(1+\frac{c}{2\sqrt{n}})^{1+\frac{c}{2\sqrt{n}}}}\right)^{n} \\ &= \left(\frac{e}{(1+\frac{c}{2\sqrt{n}})^{\left(\frac{2\sqrt{n}}{c}+\frac{c^2}{4}-1\right)}}\right)^{\frac{c\sqrt{n}}{2}} \cdot (1+\frac{c}{2\sqrt{n}})^{-\frac{c\sqrt{n}}{2}} \\ \textit{Suppose } c \geq \sqrt{9} \quad & \leq \left(\frac{e}{\left(1+\frac{c}{2\sqrt{n}}\right)^{\left(\frac{2\sqrt{n}}{c}+2\right)}}\right)^{\frac{c\sqrt{n}}{2}} \cdot \left(1+\frac{c}{2\sqrt{n}}\right)^{-\frac{c\sqrt{n}}{2}} \\ & \leq \left(1+\frac{c}{2\sqrt{n}}\right)^{-\frac{c\sqrt{n}}{2}} \leq \left(1+\frac{c}{2}\right)^{-\frac{c}{2}}. \end{aligned}$$

Answer (13-14, refers to [3]) (a) *Let n be odd, $k = n^2$, and represent the set of basic processes as the disjoint union of n sets $X_1, \ldots, X_n$ of cardinality n each.*

The set of processes P_i associated with job J_i will be equal to $X_i \cup X_{i+1}$, addition taken modulo n.

Claim 10 *No perfectly balanced assignment of processes to machines.*

Proof *Suppose there were, and define*

> $\Delta_i =$# *of processes* $\in X_i$ *assigned to machine* M_1 $-$ # *of processes* $\in X_i$ *assigned to machine* M_2.

By the perfect balance property, we have $\Delta_{i+1} = -\Delta_i, \forall i, \Rightarrow \Delta_i = -\Delta_i$, and hence $\Delta_i = 0, \forall i$. A contradiction since n is odd. $\diamondsuit$

(b) *Consider independently assigning each process i a* label *L_i equal to either 0 or 1, chosen uniformly at random. Thus we may view the label L_i as a 0-1 random variable. Now for any job J_i, we assign each process in P_i to machine M_1 if its label is 0, and machine M_2 if its label is 1.*

Consider the event E_i, that more than $\frac{4}{3}n$ of the processes associated with J_i end up on the same machine. The assignment will be nearly balanced if none of the E_i happen. E_i is precisely the event that $\sum_{t \in J_i} L_t$ either exceeds $\frac{4}{3}$ times its mean (equal to n), or that it falls below $\frac{2}{3}$ times its mean. Thus, we may upper-bound the probability of E_i as follows.

$$\begin{aligned}
\Pr[E_i] &\leq \Pr[\sum_{t \in J_i} L_t < \frac{2}{3}n] + \Pr[\sum_{t \in J_i} L_t > \frac{4}{3}n] \\
&\leq \left(e^{-\frac{1}{2}(\frac{1}{3})^2}\right)^n + \left(\frac{e^{\frac{1}{3}}}{(\frac{4}{3})^{\frac{4}{3}}}\right)^n \\
&\leq 2 \cdot .96^n.
\end{aligned}$$

Thus, by the union bound, the probability that any of the events E_i happens is at most $2n \cdot .96^n$, which is at most $.06$ for $n \geq 200$.

Thus, our randomized algorithm is as follows. We perform a random allocation of each process to a machine as above, check if the resulting assignment is perfectly balanced, and repeat this process if it isn't. Each iteration takes polynomial time, and the expected number of iterations is simply the expected waiting time for an event of probability $1 - .06 = .94$, which is $1/.94 < 2$. Thus the expected running time is polynomial.

This analysis also proves the existence *of a nearly balanced allocation for any set of jobs.*

(Note that the algorithm can run forever, with probability 0*. This doesn't cause a problem for the expectation, but we can deterministically guarantee termination without hurting the running time very much as follows. We first run k iterations of the randomized algorithm; if it still hasn't halted, we now find the nearly balanced assignment that is guaranteed to exist by trying all* 2^k *possible allocations of processes to machines, in time* $O(n^2 \cdot 2^k)$*. Since this brute-force step occurs with probability at most* $.06^k$*, it adds at most* $O(n^2 \cdot .12^k) = O(n^2 \cdot .12^n) = o(1)$ *to the expected running time.)* $\diamondsuit$

Answer (13-15)

Lemma 6 (Chernoff bounds) *Let* $X_j, j = 1, \ldots, n$ *Bernoulli distribution with parameter* p_j*. For* $\mu_n \geq \sum_{j=1}^{n} p_j = \sum_{j=1}^{n} \mathbf{E} X_j = \sum_{j=1}^{n} \mathbf{P}(X_j = 1)$*. For* $\forall r > 0$*, we have*

$$\mathbf{P}\left(\sum_{j=1}^{n} X_j > (1+r)\mu_n\right) < \left[\frac{e^r}{(1+r)^{(1+r)}}\right]^{\mu_n} \doteq B(r)^{\mu_n},$$

or equivalently in terms of large derivation,

$$\frac{1}{n}\ln \mathbf{P}\left(\sum_{j=1}^{n} X_j > (1+r)\mu_n\right) < \frac{\mu_n}{n}\left[r - (1+r)\ln(1+r)\right] \equiv \frac{\mu_n}{n}\ln B(r).$$

Proof *By Markov inequality with $\phi(x) = e^{tx}$ on $\sum_{j=1}^n X_j$:*

$$\begin{aligned}
\mathbf{P}\left(\sum_{j=1}^n X_j > (1+r)\mu_n\right) \quad & \le e^{-t\mu_n(1+r)}\mathbf{E}\left[e^{t\sum_{j=1}^n X_j}\right] = e^{-t\mu_n(1+r)}\prod_{j=1}^n \mathbf{E}\left[e^{tX_j}\right] \\
& = e^{-t\mu_n(1+r)}\prod_{j=1}^n \left[(1-p_j) + p_j e^t\right] \\
& = e^{-t\mu_n(1+r)}\prod_{j=1}^n \left[1 + p_j(e^t - 1)\right] \\
& \le \exp\left[(e^t-1)\sum_{j=1}^n p_j - t\mu_n(1+r)\right] \\
& \le \exp\left[(e^t - 1 - t(1+r))\mu_n\right] \\
& \overset{t\leftarrow \ln(r+1)}{=\!=\!=} \left[\frac{e^r}{(1+r)^{(1+r)}}\right]^{\mu_n}.
\end{aligned}$$

Here proof finishes. ♢

Answer (13-15, HW8, [3]) *Divide set S into $\epsilon^{-1} = 20$ quantiles $Q_1, \ldots, Q_{\epsilon^{-1}}$, where*

$$\begin{aligned}
Q_i \quad & \doteq \{x \in S : \#\{y \in S : y > x\} \ge \epsilon(\epsilon^{-1} - i)\#S, \#\{y \in S : y < x\} \ge \epsilon(i-1)\#S\} \\
& = \{x \in S : \#\{y \in S : y > x\} \ge \epsilon(\epsilon^{-1} - i)\#S\} \\
& \cap \{x \in S : \#\{y \in S : y < x\} \ge \epsilon(i-1)\#S\}. \qquad (5)
\end{aligned}$$

Sampling S' from S is like throwing a set of numbers at random into bins labeled with $Q_1, \ldots, Q_{\epsilon^{-1}}$.

Suppose we choose $\#S' = 40,000 (< \frac{1}{2}\epsilon^{-1})$ and sample with replacement. Set $r = \frac{1}{10}$ such that $r < \frac{2\epsilon}{1-2\epsilon}$ (from (7)) and r is so large that

$$B(r) \doteq \frac{e^r}{(1+r)^{(1+r)}} \le \left(\frac{\epsilon p}{2}\right)^{\epsilon^{-1}(\#S')^{-1}}, \qquad (6)$$

where $p = 1\%$, the desired probability tha the algorithm fails.

Consider the event

$$\begin{aligned}
\mathcal{E} \quad & \doteq \{(1-r)\epsilon\#S' \le \#(Q_i \cap S') \le (1+r)\epsilon\#S', \forall i\} \\
& = \{1,800 \le \#(Q_i \cap S') \le 2,200, \forall i\}.
\end{aligned}$$

Claim 11 *If $\mathcal{E}$ occurs, then* `median`*(S')*$\in Q_{\frac{1}{2}\epsilon^{-1}} \cup Q_{\frac{1}{2}\epsilon^{-1}+1}$, *and thus will be a ϵ-approximate median of S.*

Proof *If $\mathcal{E}$ occurs, then*

$$
\begin{aligned}
\#\bigcup_{j=1}^{\frac{1}{2}\epsilon^{-1}-1}\left(Q_j \cap S'\right), \#\bigcup_{j=\frac{1}{2}\epsilon^{-1}+2}^{\epsilon^{-1}}\left(Q_j \cap S'\right) &\leq \left(\frac{\epsilon^{-1}}{2}-1\right)\cdot(1+r)\epsilon\#S' \\
&= (1+r)\left(\frac{1}{2}-\epsilon\right)\#S' (= 19,800) \\
(r < \frac{2\epsilon}{1-2\epsilon}) \qquad &< \frac{\#S'}{2}, \qquad (7)
\end{aligned}
$$

indicating `median`*(S')*$\notin \bigcup_{j=1}^{\frac{1}{2}\epsilon^{-1}-1}(Q_j \cap S') \cup \bigcup_{j=\frac{1}{2}\epsilon^{-1}+2}^{\epsilon^{-1}} Q_i$. *Therefore* `median`*($S'$)*$\in Q_{\frac{1}{2}\epsilon^{-1}} \cup Q_{\frac{1}{2}\epsilon^{-1}+1}$. $\diamondsuit$

Following we are to evaluate $\mathbf{P}(\mathcal{E})$.

Claim 12 $\mathbf{P}\left(\#(Q_i \cap S') > (1+r)\epsilon\#S'\right), \mathbf{P}\left(\#(Q_i \cap S') < (1-r)\epsilon\#S'\right) \leq \left[\frac{e^r}{(1+r)^{(1+r)}}\right]^{\epsilon\#S'} < 6.238 \cdot 10^{-5}$.

Proof *Notice* $\#(Q_i \cap S') = \sum_{x\in Q_i} 1_{\{x\in S'\}}$ *so let* $\{X_x = 1_{\{x\in S'\}}, x \in Q_i\}$ *and*

$$
\mu_{\#Q_i} = \mathbf{E}\sum_{x\in Q_i} 1_{\{x\in S'\}} = \sum_{x\in Q_i}\mathbf{P}(x \in S') = \epsilon\#S \cdot \frac{\#S'}{\#S} = \epsilon\#S' = 2,000.
$$

Then

$$
\begin{aligned}
\mathbf{P}\left(\#(Q_i \cap S') > (1+r)\epsilon\#S'\right) &= \mathbf{P}\left(\#(Q_i \cap S') > (1+r)\mu_{\#Q_i}\right) \\
(\textit{Chernoff bound}) \qquad &\leq \left[\frac{e^r}{(1+r)^{(1+r)}}\right]^{\mu_{\#Q_i}} = \left[\frac{e^r}{(1+r)^{(1+r)}}\right]^{\epsilon\#S'} \\
&< 6.238 \cdot 10^{-5}.
\end{aligned}
$$

Likewise,

$$
\mathbf{P}\left(\#(Q_i \cap S') < (1-r)\epsilon\#S'\right) < 6.238 \cdot 10^{-5}.
$$

Here we finish the proof. $\diamondsuit$

Applying the Union Bound over the ϵ^{-1} choices of i,

$$\begin{aligned}
\mathbf{P}(\mathcal{E}^c) &\leq \sum_{i=1}^{\epsilon^{-1}} \left[\mathbf{P}\left(\#(Q_i \cap S') > (1+r)\epsilon\#S'\right) + \mathbf{P}\left(\#(Q_i \cap S') < (1-r)\epsilon\#S'\right)\right] \\
\overset{(11)}{} &\leq 2\epsilon^{-1} \cdot \left[\frac{e^r}{(1+r)^{(1+r)}}\right]^{\epsilon\#S'} \leq p,
\end{aligned}$$

Notice "Algorithm returns ϵ-approximate median" is the same as saying that $\texttt{median}(S') \in Q_{\frac{1}{2}\epsilon^{-1}} \cup Q_{\frac{1}{2}\epsilon^{-1}+1}$ *and that* $\{\texttt{median}(S') \in Q_{\frac{1}{2}\epsilon^{-1}} \cup Q_{\frac{1}{2}\epsilon^{-1}+1}\}$
$\subset \mathcal{E}$. ♢

Remark *Since $B(r), r > 0$ is monotonically decreasing* [18]*, existence of such r requires that $\#S$ such that*

$$B\left(\frac{2\epsilon}{1-2\epsilon}\right) < \left(\frac{\epsilon p}{2}\right)^{\epsilon^{-1}(\#S')^{-1}} \tag{8}$$

$$\Leftrightarrow \qquad \#S' > \cdot \frac{(1-2\epsilon)\ln\left(\frac{2}{\epsilon p}\right)}{\epsilon\left[\ln\left(\frac{1}{1-2\epsilon}\right) - 2\epsilon\right]}, \tag{9}$$

When it comes to $p = 1\%$, $\epsilon = 20\% = 0.05$, a lower bound of $\#S'$ from (9) is $27850.47\ldots$. ♢

[18] Notice $B'(r) = -B(r)\ln(1+r) < 0, \forall r > 0$.

Answer (13-15, HW8, simplified version) *Divide set S into three parts*[19]*:*

$$
\begin{aligned}
J_1 &\doteq \left\{x \in S : \#\{y \in S : y > x\} \geq \epsilon\left(\frac{\epsilon^{-1}}{2}+1\right)\#S\right\} = \bigcup_{i=1}^{\frac{\epsilon^{-1}}{2}-1} Q_i, \\
J_2 &\doteq \left\{x \in S : \#\{y \in S : y < x\} \geq \epsilon\left(\frac{\epsilon^{-1}}{2}+1\right)\#S\right\} = \bigcup_{i=\frac{\epsilon^{-1}}{2}+2}^{\epsilon^{-1}} Q_i,
\end{aligned}
$$

and $S-(J_1\cup J_2) = Q_{\frac{1}{2}\epsilon^{-1}} \cup Q_{\frac{1}{2}\epsilon^{-1}+1}$.

Sampling S' from S is like throwing a set of numbers at random into $J_1, J_2, S-(J_1\cup J_2)$.

Suppose we choose $\#S' = 40{,}000 (< \frac{1}{2}\epsilon^{-1})$ *(from claim 13) and sample with replacement. Set* $r = \frac{1}{10}$ *such that* $r \leq \frac{2\epsilon}{1-2\epsilon}$ *and r is so large that*

$$
B(r) \doteq \frac{e^r}{(1+r)^{(1+r)}} \leq \left(\frac{p}{2}\right)^{\left(\frac{1}{2}-\epsilon\right)^{-1}(\#S')^{-1}}, \tag{10}
$$

where $p = 1\%$, the desired probability that the algorithm fails.

Consider the event

$$
\begin{aligned}
\tilde{\mathcal{E}} &\doteq \{\#(J_1\cap S'), \#(J_2\cap S') \leq (1+r)\left(\frac{1}{2}-\epsilon\right)\#S'\} \\
&= \{\#(J_1\cap S'), \#(J_2\cap S') \leq 19{,}800\}.
\end{aligned}
$$

[19] For example,

$$
\begin{aligned}
J_1 &\doteq \left\{x \in S : \#\{y \in S : y > x\} \geq \epsilon\left(\frac{\epsilon^{-1}}{2}+1\right)\#S\right\} \\
&= \left[\bigcup_{i=1}^{\frac{\epsilon^{-1}}{2}-1} \{x \in S : \#\{y \in S : y > x\} \geq \epsilon(\epsilon^{-1}-i)\#S\}\right] \\
&\cap \left[\bigcup_{i=1}^{\frac{\epsilon^{-1}}{2}-1} \{x \in S : \#\{y \in S : y < x\} \geq \epsilon(i-1)\#S\}\right] \\
&= \bigcup_{i=1}^{\frac{\epsilon^{-1}}{2}-1} Q_i.
\end{aligned}
$$

Claim 13 *If $\tilde{\mathcal{E}}$ occurs, then* `median`$(S')\in S-(J_1\cup J_2)$*, and thus will be a ϵ-approximate median of S.*

Proof *This requires $r<\frac{2\epsilon}{1-2\epsilon}$.* ♢

Following we are to evaluate $\mathbf{P}(\tilde{\mathcal{E}})$.

Claim 14 $\mathbf{P}\left(\#(J_i\cap S')>(1+r)\left(\frac{1}{2}-\epsilon\right)\#S'\right)\le\left[\frac{e^r}{(1+r)^{(1+r)}}\right]^{\left(\frac{1}{2}-\epsilon\right)\#S'}$, $i=1,2$.

Proof *Notice $\#(J_1\cap S')=\sum_{x\in J_1}1_{\{x\in S'\}}$ so let $\{X_x=1_{\{x\in S'\}},x\in J_1\}$ and*

$$\mu_{\#J_1}=\mathbf{E}\sum_{x\in J}1_{\{x\in S'\}}=\sum_{x\in J}\mathbf{P}(x\in S')=\left(\frac{\epsilon^{-1}}{2}-1\right)\epsilon\#S\cdot\frac{\#S'}{\#S}=\left(\frac{1}{2}-\epsilon\right)\#S'=18,000.$$

Then

$$\begin{aligned}\mathbf{P}\left(\#(J_1\cap S')>(1+r)\left(\frac{1}{2}-\epsilon\right)\#S'\right) &=\mathbf{P}\left(\#(J_1\cap S')>(1+r)\mu_{\#J_1}\right)\\ (\textit{Chernoff bound})\quad &\le\left[\frac{e^r}{(1+r)^{(1+r)}}\right]^{\mu_{\#J_1}}=\left[\frac{e^r}{(1+r)^{(1+r)}}\right]^{\left(\frac{1}{2}-\epsilon\right)\#S'}\\ &<6.238\cdot 10^{-5}.\end{aligned}$$

Likewise, the rest three are also hold. ♢

Applying the Union Bound over the ϵ^{-1} choices of i,

$$\begin{aligned}\mathbf{P}(\tilde{\mathcal{E}}^c)\quad &\le\sum_{i=1}^{2}\mathbf{P}\left(\#(J_i\cap S')\right)>(1+r)\left(\frac{1}{2}-\epsilon\#S'\right)=2\cdot\left[\frac{e^r}{(1+r)^{(1+r)}}\right]^{\left(\frac{1}{2}-\epsilon\right)\#S'}\\ (11)\quad &\le p,\end{aligned}$$

Notice "Algorithm returns ϵ-approximate median" is the same as saying that `median`$(S')\notin J_1,J_2$ *and that*
$\{$`median`$(S')\notin J_1,J_2\}\subset\tilde{\mathcal{E}}$.

Remark *Since $B(r), r > 0$ is monotonically decreasing* [20]*, existence of such r requires that $\#S$ satisfies*

$$B\left(\frac{2\epsilon}{1-2\epsilon}\right) < \left(\frac{p}{2}\right)^{\left(\frac{1}{2}-\epsilon\right)^{-1}(\#S')^{-1}} \tag{11}$$

$$\Leftrightarrow \qquad \#S' > \frac{2\ln\left(\frac{2}{p}\right)}{\ln\frac{1}{1-2\epsilon} - 2\epsilon}, \tag{12}$$

When it comes to $p = 1\%$, $\epsilon = 20\% = 0.05$, a lower bound of $\#S'$ from (9) is 1976.7939 .... $\diamondsuit$

Answer (13-16, refers to [3]) *One algorithm is the following.*

1. *For $i = 1, 2, \dots, n$*
   (a) *Receiver j computes $\beta_{ij} = f(\beta_1^* \cdots \beta_{i-1}^*, \alpha_i^j)$.*
   (b) *β_i^* is set to the majority value of β_{ij}, for $j = 1, \dots, k$*[21]*.*
2. *End for*
3. *Output β^**

Let $X_{ij} = \begin{cases} 1, & \alpha_i^j \text{ was corrupted;} \\ 0, & \text{otherwise.} \end{cases}$. If a majority of the bits in $\{\alpha_i^j : j = 1, 2, \dots, k\}$ are corrupted, then $X_i = \sum_j X_{ij} > k/2$. Now, since each bit is corrupted with probability $\frac{1}{4}$, $\mu = \sum_j EX_{ij} = k/4$. Thus, by the Chernoff bound, we have

$$\begin{aligned} \Pr[X_i > k/2] &= \Pr[X_i > 2\mu] \\ &< \left(\frac{e}{4}\right)^{k/4} \\ &\le (.91)^k. \end{aligned}$$

Now, if

$$k \ge 11\ln n > \frac{\ln n - \ln .1}{\ln(1/.91)},$$

then

$$\Pr[X_i > k/2] < .1/n.$$

(So it is enough to choose k to be the smallest odd integer greater than $11\ln n$.) Thus, by the union bound, the probability that any *of the sets $\{\alpha_i^j : j = 1, 2, \dots, k\}$ have a majority of corruptions is at most .1.*

[20] Notice $B'(r) = -B(r)\ln(1+r) < 0, \forall r > 0$.

[21] We'll make sure to choose an odd value of k to prevent ties.

Assuming that a majority of the bits in each of these sets are not corrupted, which happens with probability at least .9, one can prove by induction on i that all the bits in the reconstructed message β^* will be correct. ♢

Answer (13-17, random walk) *Hitting time $\tau = \tau_{-1}$. Notice $\mathbf{E}S_{\tau_1} = -1, \mathbf{E}X_1 = -\frac{1}{3}$. Wald identity implies $\mathbf{E}\tau_{-1} = \frac{\mathbf{E}S_{\tau_{-1}}}{\mathbf{E}X_1} = \frac{-1}{-\frac{1}{3}} = 3$.* ♢

Answer (13-18, HW8) *1. Not true. Suppose only one edge with two endpoints assigning weights $1, 1+2c$ separately.*

2. (a) Basic line is that LP for (weighted) vertex cover is

$$\min\langle \mathbf{w}, \mathbf{x}\rangle = \sum_{v\in V} w_v x_v, \text{ such that}$$
$$x_v + x_u \geq 1, \qquad \forall e = (u,v) \in E,$$
$$x_v \geq 0, \qquad \forall v \in V.$$

and the dual is

$$\max\langle 1_m, \mathbf{y}\rangle = \sum_{e\in E} y_e, \text{ such that}$$
$$\sum_{e \text{ incident to } v} y_e \leq w_v, \qquad \forall v \in V,$$
$$y_e \geq 0, \qquad \forall e = (u,v) \in E.$$

And we know any solution $\{y_e^0, e \in E\}$ satisfying conditions for the dual, we have

$$\langle 1_m, \mathbf{y}^0\rangle \leq \langle 1_m, \mathbf{y}^{LP}\rangle = \langle \mathbf{w}, \mathbf{x}^{LP}\rangle \leq \langle \mathbf{w}, \mathbf{x}^{\mathtt{OPT}}\rangle.$$

(b) Next we are to show one $\mathbf{y}^0$. Let X_e be the indicator variable that the edge is e is chosen as an uncovered edge by the algorithm.

Claim 15 *In case of $\mathbf{w} = 1_n$, $\left\{\frac{\mathbf{E}X_e}{4}, e \in E\right\}$ satisfies the condition in the dual.*

Proof *incomplete* ♢

In this case, $\mathbf{E}$ # of edges being chosen as uncovered edge by the algorithm= $\mathbf{E}$# of nodes in the VC $= \sum_{e\in E} \mathbf{E}X_e \leq 4\langle \mathbf{w}, \mathbf{x}^{\mathtt{OPT}}\rangle = 4\langle 1_n, \mathbf{x}^{\mathtt{OPT}}\rangle$. ♢

Remark (LP)

Question (Final 2014, 20pts, Harsh) *Consider the following problem: we are given a n items from a large universe $[U] = 0, \ldots, U - 1$. We do not know which items are provided to us beforehand and there are only r distinct items. We decided to hash every element of the universe, independently of other values, to a value chosen uniformly at random in the range [0; 1] of real numbers.*

1. ***(5 points)*** *Let $t = \frac{k}{r}$ for some integer $k > 0$. What is the best upper bound of the probability that $\frac{k}{2}$ elements have hash value less than t?*
2. ***(5 points)*** *What is the best upper bound of the probability that $2k$ or more elements have hash value greater than t?*

   For part 1 and 2, we are more interested in knowing how you would bound the probability than the exact value. Provide the definition of the random variables you use to get full credit.
3. ***(10 points)*** *Suppose we knew how to create and store such an hash function. Professor 001 believes that he has an algorithm for estimating r; he wants to iterate over the n items and at any point store only $O(k)$ hash values. At the end of the iteration he wants to output a number r' such that $\frac{r}{2} \leq r' \leq 2r$ with probability $1 - \frac{2}{n}$. What value of k should he choose and which hash values should he maintain in the iteration?*

Hint: Professor 002 recalls

1. Markov Inequality for a nonnegative random variable X to be $\mathbf{P}[X > a\mathbf{E}X] \leq \frac{1}{a}, \forall a > 0$;
2. Chebyshev bound to be $\mathbf{P}[|X - E[X]| \geq a\mathbf{E}X] \leq \frac{\mathrm{Var}X}{(a\mathbf{E}X)^2}, \forall a > 0$;
3. Chernoff bounds, where $X_1, X_2, \ldots, X_h$ are independent 0/1 random variables with $S_h = \sum_{i=1}^{h} X_i$, $\mu_h = \mathbf{E}S_h$ that

$$\mathbf{P}[|S_h - \mu_h| \geq a\mu_h] \leq e^{-\frac{a^2\mu_h}{3}}, \forall a < 1.$$

Answer (Final 2014, 20pts, Harsh) *Define X_i to be the event that the hash of element i was less of equal $\frac{k}{r}$. The $\sum_{i=1}^{r} \mathbf{E}X_i = r \cdot \frac{k}{r} = k$ since there are only r distinct elements.*

1.

$$\begin{aligned}\mathbf{P}\left(\frac{k}{2} \textit{ elements have hash value } < \frac{k}{r}\right) &= \mathbf{P}\left(\sum_{i=1}^{r} X_j \geq \frac{k}{2}\right) \\ &= \mathbf{P}\left(S_r \geq \frac{1}{2}\mu_r\right) \leq \mathbf{P}\left(|S_r - \mu_r| \geq \frac{1}{2}\mu_r\right) \\ &\leq \exp\left(-\frac{\mu_r}{12}\right).\end{aligned}$$

2.

$$\begin{aligned}\mathbf{P}\left(2k \textit{ elements have hash value } < \frac{k}{r}\right) &= \mathbf{P}\left(\sum_{i=1}^{r} X_j \geq 2k\right) \\ &= \mathbf{P}\left(S_r \geq 2\mu_r\right) \leq \mathbf{P}\left(|S_r - \mu_r| \geq \mu_r\right) \\ &\leq \exp\left(-\frac{\mu_r}{3}\right)\end{aligned}$$

3. *Not attempted*

Finished. ♢

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