CIS 502 - Algorithms

Fall 2015 Homework 8 Solutions

Problem 1 *Question 8, page 788.*

Solution: We choose a random subset of vertices of size k. Let X_{uv} be the indicator variable that both u, v were chosen.

$$\mathbf{E}[X_{uv}] = \mathbf{Pr}[X_{uv} = 1] = \frac{\binom{k}{2}}{\binom{n}{2}} = \frac{k(k-1)}{n(n-1)}$$

Now the number of edges we see is $X = \sum_{(u,v) \text{ an edge}} X_{uv}$. Now

$$\mathbf{E}[X] = \mathbf{E}\left[\sum_{(u,v) \text{ an edge}} X_{uv}\right] = \sum_{(u,v) \text{ an edge}} \mathbf{E}[X_{uv}] = \sum_{(u,v) \text{ an edge}} \frac{k(k-1)}{n(n-1)} = \frac{mk(k-1)}{n(n-1)}$$

Therefore in expectation we get the current answer. We now analyse the running time (number of repetitions required because in a single try we may get a smaller value). Let p be the probability that the number of edges exceed the expectation.

Let $K = \lceil \frac{mk(k-1)}{n(n-1)} \rceil$. Now consider the following transformation (**for the sake of analysis only**): everytime we get a value less than K we pretend we got $K-1$. Every time we get something something larger or equal K we pretend we got m. This transformation clearly increases the expectation (because we only see integral values as output of the random trial). Therefore:

$$\frac{mk(k-1)}{n(n-1)} \leq (K-1)(1-p) + m \cdot p$$

Therefore

$$\frac{mk(k-1)}{n(n-1)} - (K-1) \leq (m-K+1)p \leq mp$$

Now $\frac{mk(k-1)}{n(n-1)}$ differs from $K-1$ by at least $\frac{1}{n(n-1)}$. Which implies that $p \geq 1/(mn(n-1))$. Therefore the excpected number of trials before I see K or larger is $mn(n-1)$.

Problem 2 *Question 10, page 789.*

Solution: The value b_i is going to update the bids if and only if b_i is the largest value in $1, \ldots, i$ which happens with probability $1/i$.

Define X_i to be indicator variable that the bids are updated at i. Since X_i is an indicator variable:

$$\mathbf{E}[X_i] = \mathbf{Pr}[X_i = 1] = 1/i$$

$X = \sum_i X_i$ is the number of times we update the bids.

$$\mathbf{E}[X] = \mathbf{E}\left[\sum_i X_i\right] = \sum_i \mathbf{E}[X_i] = \sum_i 1/i = H(n)$$

by linearity of expectation (the index i ranges from 1 to n).

Problem 3 *Question 15, page 791.*

Solution: Let $|S'| = t$. We will estimate t.

We want to upper bound the probability of two bad events. The first bad event is that we choose a very large value as our median which has rank more than $\frac{n}{2} + \epsilon n$ in S. The second is that we choose a very small value as our median which has rank less than $\frac{n}{2} - \epsilon n$ in S.

To estimate the first: define X_i to be the indicator variable that the i^{th} draw in S' had rank more than $\frac{n}{2} + \epsilon n$ in S. Then the bad event that the median of S' already has rank more than $\frac{n}{2} + \epsilon n$ in S implies $\sum_i X_i \geq t/2$.

Now X_i are i.i.d. random variables – since we are sampling with replacement, nothing is changing between the i^{th} and the $(i+1)^{st}$ draw. Now $\mathbf{E}[X_i] = \mathbf{E}[X_1] = \frac{1}{2} - \epsilon$; since we must choose the top $\frac{n}{2} - \epsilon n$ in S. Therefore

$$\mu = \sum_i \mathbf{E}[X_i] = \frac{t}{2} - \epsilon t \leq \sum_i X_i - \epsilon t \leq \sum_i X_i - 2\epsilon\mu$$

. Therefore the probability of our bad event is upper bounded by

$$\mathbf{Pr}\left[\sum_i X_i - \mu \geq 2\epsilon\mu\right] \leq e^{-\frac{4\epsilon^2\mu}{3}}$$

Observe that to reduce the right hand side to less than 0.005 we would need $4\epsilon^2\mu \geq 3\ln 200$. This lets us calculate t since $\epsilon = 0.5$.

Problem 4 *Question 18, page 792–793.*

Solution: Part (a) is easy (in comparison to the rest). Consider one edge where one vertex has weight 1 and the other vertex has weight $2c+1$ (for any given $c > 1$). Then the expected solution of the algorithm is $\frac{1}{2} + \frac{1}{2}(2c+1) = c+1 > c$. So the statement is false.

For part (b): Let X_e be the indicator variable that the edge is e is chosen as an uncovered edge by the algorithm. Observe that $p_e = E[X_e]$, where p_e is the probability that an edge is chosen as an uncovered edge by the algorithm.

Now consider:

$$E\left[\sum_{e \text{ incident to } v} X_e\right]$$

Once we pick an edge adjacent to v then with probability 1/2 we choose v in the cover and all the remaining $X_e = 0$ for edges e adjacent to v. Therefore the probability that $\sum_{e \text{incident to} v} X_e$ equals k is at most $1/2^{k-1}$. Thus

$$E\left[\sum_{e \text{ incident to } v} X_e\right] \leq 1 + \frac{1}{2} + \frac{1}{4} \cdots = 2$$

Therefore **by linearity of expectation** for every v:

$$\sum_{e \text{ incident to } v} p_e \leq 2$$

And note that the expected solution $E[S] = E[\sum_e X_e] = \sum_e p_e$ because for every uncovered edge we pick a vertex. Now we have

$$\begin{array}{ll} \sum_e \frac{p_e}{2} = E[S]/2 & \\ \sum_{e \text{ incident to } v} \frac{p_e}{2} \leq 1 & \text{for every } v \\ \frac{p_e}{2} \geq 0 & \text{for every } e \end{array}$$

But consider the primal and dual LP formulations of minimum vertex cover. The primal is

$$\begin{array}{ll} \text{Primal} = \min \sum_v z_v & \\ z_v + z_u \geq 1 & \text{for every } e = (u, v) \\ z_v \geq 0 & \text{for every } e \end{array}$$

And the dual is

$$\begin{array}{ll} \text{Dual} = \max \sum_e y_e & \\ \sum_{e \text{ incident to } v} y_e \leq 1 & \text{for every } v \\ y_e \geq 0 & \text{for every } e \end{array}$$

We have just shown that setting $y_e = p_e/2$ gives us a feasible solution of value $E[S]/2$ to the dual and therefore Dual $\geq E[S]/2$. But then Primal = Dual $\geq E[S]/2$. Which means that $E[S]/2$ is a lower bound on the fractional minimum vertex cover; and thus of any integral cover as well. This proves the theorem for $c = 2$.