CIS 502 - Algorithms

Fall 2015 Homework 6 Solutions

Problem 1 *Problem 23, page 428–429, of textbook.*

Solution: Compute a maxflow (using any algorithm). Compute the residual graph. Now consider the set of all nodes reachable from s with capacity > 0 edges in the *residual* graph. Let this set be S. Consider also the set of all nodes which can reach t with capacity > 0 edges in the *residual* graph, let this set be T. Every node in S is upstream, every node in T is downstream. The remaining nodes are central.

Proof: Fix any minimum cut C (there can be multiple ones), $s \in C$. Note that for every edge e that leaves C we must have $f_e = c_e$. And for every edge coming in to C we will have $f_e = 0$. Therefore in the residual graph, there can be no path from any vertex in C to any vertex in the complement of C.

Now observe that S and $V \setminus T$ are both mincuts. Therefore all upstream vertices are a subset of S and all downstream vertices are a subset of T. Now suppose that $v \in S$ was not an upstream vertex. Then there exists a cut S' such that $v \notin S'$ and S' is a mincut. But that would imply that we cannot reach v from S' in the residual graph (argued above, setting $C = S'$). But that is a contradiction since by construction we reached v from s using > 0 capacity edges in the residual.

Likewise if there is a $v \in T$ which is not a downstream vertex then there is a cut S'' separating v and t. But then starting from any vertex in S'' we cannot reach $V \setminus S''$ in the residual graph. But by construction, starting from v we reached t. Contradiction.

Problem 2 *Problem 24, page 429, of textbook. (Not asked, but worth knowing.)*

First Solution: (test if there is another mincut with different outgoing edges) We first find a mincut of G, say C (using maxflow and the residual graph). Suppose C is not the unique mincut then there is some other cut C' which has the same capacity and differ by at least one edge $(u, v) \in C$. Suppose we raise the capacity of only that edge (u, v) by any positive amount δ; and compute a mincut again then we would find another mincut (which may be C' or yet another cut of the same capacity). Otherwise, (if C were the unique mincut) and we raised the capacity of (u, v), then the total flow would increase.

Therefore our algorithm is: for each $(u, v) \in C$, raise only the capacity of (u, v) and compute another mincut. If the flow increases for every (u, v) then the cut C is the unique minimum cut.

Alternate Solution: (test if there is another mincut with different set of vertices) We first find a maxflow f_e of G. We then find the mincut $C = \{u|\ s \text{ can reach } u \text{ in the residual } G_f\}$.

We then find another cut $Z' = \{u|\ u \text{ can reach } t \text{ in the residual } G_f\}$. Note that $s \notin C'$ (otherwise we can augment) and $t \in Z'$. Moreover for each edge $(u, v) \in G$ such that $u \notin Z', v \in Z'$ we must have $f_{uv} = c_{uv}$ or else in the residual G_f, u can reach t. Likewise for each edge $(u, v) \in G$ such that $u \in Z', v \notin Z'$ we must have $f_{uv} = 0$ or else in the residual G_f, u can reach t. Therefore the edges coming in to Z' are saturated — or $V - Z'$, the complement of Z' is a mincut.

Therefore the mincut C is unique iff $C = V - Z'$. We can therefore check this in $O(n+m)$ time (to compute the two reachability sets) after finding the maxflow. Note that this algorithm uses one maxflow computation whereas the previous one used many.

Problem 3 *Problem 31, page 434, of textbook.*

Solution: First see the discussion on circulation with demands and lower bounds on flow in page 383. Note that ultimately we will use Ford Fulkerson and will have an integral flow (7.5.2).

We will reduce our problem to that problem. In particular we will ask "can we pack the boxes such that only k boxes are visible" for values of $k = 1$ through the number of boxes. The smallest value of k for which we answer yes is the solution. Note that given the lengths, at most one box fits inside any other box (without nesting).

To answer the question for a particular k: For each box i create an edge (s_i, t_i). If Box j fits inside box i then add the directed edge (t_i, s_j). Now we add k sources (demand -1) and k sinks (demand 1). Each source is connected to each s_i. Each sink is connected to each t_i.

All edge capacities are 1. We now require a lower bound of flow 1 through (s_i, t_i) for all i.

We now show that if there exists a packing then there exists an integral solution to the flow with demands and lower bounds.

Consider an outer box. Since at most one box fits within another box without nesting, the nesting set of boxes within the outer box correspond to one unit flow which satisfies the lower bound of the edges corresponding the boxes. Moreover we can use one source and one sink to route this flow. Therefore if there exists a packing using exactly k outer boxes then there is an integral flow which satisfies all the demands and lower bounds.

Given an integral flow which satisfies all the demands and lower bounds, this flow is of value k. This can be partitioned into k edge disjoint paths such that each path carries one unit flow.

Consider one such unit flow path. It must start at a source and proceed to some s_i. Then it must go to the corresponding t_i (since there is no other outgoing edge). From t_i the flow must go to some other s_j or to a sink. If the flow went to any s_j it must also visit t_j. The set of (s_i, t_i) edges visited by this unit flow can be nested! Therefore the k paths corresponding to the unit flow give us a packing.

The maxflow instance we will eventually reduce to has a maximum flow of at most k, and the number of edges is for n boxes is $O(n^2)$. So Ford Fulkerson is going to run in time $O(n^2 k)$. Summing this over all k the running time is $O(n^4)$. We can also perform a binary search over k which will be $O(n^3 \log n)$.

Problem 4 *Problem 20, page 515-516, of textbook.*

Solution: We only show how to prove that the problem is NP-Hard. We will reduce the 3 Coloring problem to our problem.

Given a graph G where we wish to decide the 3-colorability, we want to produce a graph G' where we wish to test low diameter clustering. We now describe G'. The vertex set remains the same; $V(G) = V(G')$. We complement the edges; i.e., edge $(u, v) \in E(G')$ if and only if $(u, v) \not inE(G)$. The length of each edge in G' is 1. The distance between any two pair of nodes not connected in G' is computed by the shortest path on G'. Set $B = 1$ and $k = 3$.

If G was 3 colorable then let the vertices are partitioned into V_1, V_2, V_3 where each V_i is an independent set. This means that for $u, v \in V_i$ the edge $(u, v) \notin E(G)$. Which means that for $u, v \in V_i$ the edge $(u, v) \in E(G')$. This means that any two vertices in V_i are at within distance 1 in G'.

If G' has 3 clusters which all have diameter 1; let the clusters be C_1, C_2, C_3. We note that for $u, v \in C_i$ we must have $(u, v) \in E(G')$ since otherwise the distance will be larger than 1. Therefore for $u, v \in C_i$ we must have $(u, v) \notin E(G)$. This means that C_i is an independent set in G. Therefore we can color each C_i using a separate color and we can color G using 3 colors.

Problem 5 *Problem 27, page 518–519, of textbook.*

Solution: The problem is NP hard for $k = 2$ (note that $k = 1$ is trivial). We show that Partition reduces to this problem.

Given an instance of Partition, which are integers $a_i > 0$, we are asked does there exist an S such that

$$\sum_{i \in S} a_i = \sum_{i \notin S} a_i = \frac{1}{2} \sum_i a_i$$

Now observe that $2x^2 + 2y^2 \geq (x + y)^2$ and the equality holds if and only if $x = y$. Therefore if we set $B = \frac{1}{2} \left(\sum_i a_i\right)^2$ then

$$\left(\sum_{i \in S} a_i\right)^2 + \left(\sum_{i \notin S} a_i\right)^2 \leq \frac{1}{2} \left(\sum_i a_i\right)^2 \qquad \Longleftrightarrow \qquad \sum_{i \in S} a_i = \sum_{i \notin S} a_i$$

Therefore there exists a solution for the Partition problem if and only if there exists a solution for the new problem.