Economics 701A **Steven A. Matthews**
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Suggested Solutions to the Exam

80 points, 75 minutes. Closed books, notes, calculators.
Indicate your reasoning, using clearly written words as well as math.

1. (40 pts) A consumer with a strictly increasing utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ has the expenditure function

$$e(p, U) = \left(p_1^a + p_2^b + 2p_1^c p_2^c\right) U^2, \text{ where } a, b, c > 0.$$

(a) (10 pts) What can you say about the exponents a, b, and c? Prove your answers.

Soln: $a = b = 1$ and $c = 1/2$.

Proof. We know $te(p, U) = e(tp, U)$ for all $t, p_1, p_2 > 0$. Use the given e and rearrange to obtain

$$(t - t^a)\, p_1^a + (t - t^b) p_2^b + 2(t - t^{2c}) p_1^c p_2^c = 0.$$

As $b, c > 0$, taking $p_2 \downarrow 0$ in this equation yields $(t - t^a)\, p_1^a = 0$. The only way this can hold for all $t, p_1 > 0$ is for $t = t^a$, or rather, $a = 1$. A similar argument yields $b = 1$. Hence, the equation displayed above becomes

$$2(t - t^{2c}) p_1^c p_2^c = 0,$$

and the only way this can hold for all $t, p_1, p_2 > 0$ is for $c = 1/2$. ■

(b) (10 pts) Find the Hicksian demand functions, $h_1(p, U)$ and $h_2(p, U)$.

Soln: By the envelope theorem (here, "Shepard's lemma"),

$$h_1(p, U) = \frac{\partial e(p, U)}{\partial p_1} = (1 + r)\, U^2,$$

where $r = \left(\frac{p_2}{p_1}\right)^{1/2}$. Similarly,

$$h_2(p, U) = \frac{\partial e(p, U)}{\partial p_2} = \left(1 + r^{-1}\right) U^2.$$

■

(c) (10 pts) Find the indirect utility function, $v(p, m)$.

Soln: From the duality identity $e(p, v(p, m)) = m$ we obtain

$$(p_1 + p_2 + 2\sqrt{p_1 p_2})\, v(p, m)^2 = m.$$

Hence,

$$v(p, m) = \sqrt{\frac{m}{p_1 + p_2 + 2\sqrt{p_1 p_2}}} = \frac{\sqrt{m}}{\sqrt{p_1} + \sqrt{p_2}}.$$

■

(d) (10 pts) Find the utility function u.

Soln: For every $x \in \mathbb{R}^2_{++}$, $p \in \mathbb{R}^2_{++}$ exists such that $x = h(p, u(x))$ (an implication of the Supporting Hyperplane Theorem). From (b), again letting $r = \left(\frac{p_2}{p_1}\right)^{1/2}$, this gives us

$$x_1 = (1+r)\, u(x)^2, \quad x_2 = \left(1 + r^{-1}\right) u(x)^2.$$

Reducing these to one equation by eliminating r, and then solving for $u(x)$ yields

$$u(x) = \sqrt{\frac{x_1 x_2}{x_1 + x_2}}.$$

■

2. (40 pts) Urn R contains 49 green balls and 51 blue balls. Urn U contains 100 balls, each of which is green or blue. One ball will be randomly drawn from one urn. The decision maker, DM, will be allowed to choose the urn.

In scenario G, DM will be paid \$1000 if the ball that is drawn is green, and nothing if it is blue. In this scenario DM chooses urn R. That is, referring to the gambles as G_R when she chooses urn R and G_U when she chooses urn U, her preference is $G_R \succ G_U$.

In scenario B, DM will be paid \$1000 if the ball she chooses is blue, and nothing if it is green. In this scenario DM also chooses urn R. That is, referring to these gambles as B_R when she chooses urn R and B_U when she chooses urn U, her preference is $B_R \succ B_U$.

(a) (20 pts) Are DM's choices consistent with maximizing expected utility, with subjective probabilities for urn U and objective ones for urn R? Prove your answer.

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Solution 1. No. Assume they are. Then DM's beliefs regarding the color of the ball if it is drawn from urn R are $p_R(G) = 1 - p_R(B) = .49$, and she holds some belief $p \in [0,1]$ that the ball from urn U is green. We can normalize her Bernoulli utility function so that $u(0) = 0$ and $u(1000) = 1$. Then, since $G_R \succ G_U$, we know

$$\begin{aligned} Eu(G_R) > Eu(G_U) &\Rightarrow .49u(1000) + .51u(0) > pu(1000) + (1-p)u(0) \\ &\Rightarrow .49 > p. \end{aligned}$$

But since $B_R \succ B_U$, we also have

$$\begin{aligned} Eu(B_R) > Eu(B_U) &\Rightarrow .49u(0) + .51u(1000) > pu(0) + (1-p)u(1000) \\ &\Rightarrow .51 > 1 - p \\ &\Rightarrow .49 < p. \end{aligned}$$

This is a contradiction.

Solution 2. (This is the answer one student gave. It is correct and funny, it made grading a little more tolerable. Thanks.) **Yes.**

Make the auxillary assumption that R and U denote the colors of the urns, say Red and Ultramarine. Then the DM's preferences are perfectly consistent with expected utility maximization with respect to any probability beliefs. Just let her Bernoulli utility function be $u(c, w) = 1_{\{c=R\}}$, where $c \in \{R, U\}$ is the color of the chosen urn and $w \in \{0, 1000\}$ is the amount of dollars won. She thus does not care about her winnings, but only about the color of the urn she chooses. Her expected utility from choosing urn R is 1, and from choosing urn U it is 0. ■

(b) (10 pts) Describe a set of states such that each of the four gambles is a Savage "act," i.e., a function from the set of states to a set of consequences. Depict them in a table as we did in lecture.

Soln: A state can be represented as a pair ru, where $r \in \{G, B\}$ is the color of the ball that would be drawn from urn R if DM chooses that urn, and u is the color of the ball that would be drawn from urn U if DM chooses that urn. One can think of a "game master" actually drawing one ball from each earn, and then showing the pair of balls to DM after she makes her choice of urn.

There are thus four states: GB, BG, GG, and BB. We need just two consequences, 1 (for "get \$1000") and 0 (for "get \$0"). The four given acts are depicted in the following table:

		States			
		GB	BG	GG	BB
	B_R	0	1	0	1
Acts	B_U	1	0	0	1
	G_U	0	1	1	0
	G_R	1	0	1	0

For example, the act B_R is the function of states defined by $B_R(GB) = B_R(GG) = 0$ and $B_R(BG) = B_R(BB) = 1$. ■

(c) (10 pts) Are DM's choices consistent with the Sure-Thing Principle? Prove your answer directly from the definition of the principle.

Soln: No.

Let E be the event $E = \{GB, BG\}$, and so $-E = \{GG, BB\}$ is the complimentary event. Depict each act as (x_E, x_{-E}). Then $B_R \succ B_U$ amounts to

$$((0,1),(0,1)) \succ ((1,0),(0,1)).$$

Hence, if DM did satisfy the Sure-Thing Principle, we would have

$$((0,1),(1,0)) \succ ((1,0),(1,0)),$$

or rather, $G_U \succ G_R$. This would contradict the observed $G_R \succ G_U$. ■