Details of the Black-Litterman Approach

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Based on papers on Black-Litterman

Overview

Motivation

- Limits of Mean-Variance (Markowitz) Portfolio Optimization

Black-Litterman

- Obtaining CAPM Prior using Reverse Optimization
- Views on Assets or Portfolios with Uncertainty
- Updating Estimates using Bayes Rule

Comments

- Personal Take on Black-Litterman

Motivation

Mean-Variance Optimization

- Framework for actually constructing a portfolio of assets (Harry Markowitz, 1952)
- Key assumption is that investors are risk averse
  - They will take on increased risk only if they are compensated by higher expected returns
- We focus on single-period analysis for now:
  - Assume m risky assets $(i = 1, 2, \dots m)$
  - Assume single-period returns for the m risky assets $(\boldsymbol{R} = [R_1, R_2, \dots, R_m]')$
  - Assume that we have *expected* returns $E[R] = \boldsymbol{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_m]$ and the covariance matrix of returns $\boldsymbol{\Sigma}$
  - The portfolio is essentially a vector of weights corresponding to each risky asset, which can be expressed as $\boldsymbol{w} = [w_1, w_2, \dots, w_m]'$
  - The realized return of the portfolio: $\boldsymbol{w}'\boldsymbol{R}$
  - The expected return of the portfolio: $\boldsymbol{w}'\boldsymbol{\alpha}$
  - The expected variance of the portfolio: $\boldsymbol{w}'\boldsymbol{\Sigma}\mathbf{w}$

Example 1: Risk Minimization

$$\begin{gathered} Minimize \ \ \frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \\ s.t \\ \boldsymbol{w}'\boldsymbol{\alpha} = \alpha_0, \quad \boldsymbol{w}'\mathbf{1} = 1 \end{gathered}$$

- Lagrangian: $L(\boldsymbol{w}, \lambda_1, \lambda_2) = \frac{1}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\mathbf{w} + \lambda_1(\alpha_0 - \boldsymbol{w}'\boldsymbol{\alpha}) + \lambda_2(1 - \boldsymbol{w}'\mathbf{1})$
- Taking the first order conditions, we have
  - $\boldsymbol{\Sigma}\boldsymbol{w} - \lambda_1\boldsymbol{\alpha} + \lambda_2\mathbf{1} = 0$
  - $\alpha_0 - \boldsymbol{w}'\boldsymbol{\alpha} = 0$
  - $1 - \boldsymbol{w}'\mathbf{1} = 0$
- We can derive the optimal weights:
  - $\boldsymbol{w}_{\mathbf{0}} = \lambda_1\boldsymbol{\Sigma}^{-\mathbf{1}}\boldsymbol{\alpha} + \lambda_2\boldsymbol{\Sigma}^{-\mathbf{1}}\mathbf{1}$

Example 2: Expected Return Maximization

$$Maximize\ \boldsymbol{w}'\boldsymbol{\alpha}$$
$$s.t$$
$$\boldsymbol{w}'\boldsymbol{\Sigma}\mathbf{w} = \sigma_0^2 \quad \boldsymbol{w}'\mathbf{1} = 1$$

- Lagrangian: $L(\boldsymbol{w}, \lambda_1, \lambda_2) = \boldsymbol{w}'\boldsymbol{\alpha} + \lambda_1(\sigma_0^2 - \boldsymbol{w}'\boldsymbol{\Sigma}\mathbf{w}) + \lambda_2(1 - \boldsymbol{w}'\mathbf{1})$
- Taking the first order conditions, we have
  - $\boldsymbol{\alpha} - 2\lambda_1\boldsymbol{\Sigma}\mathbf{w} + \lambda_2\mathbf{1} = 0$
  - $\sigma_0^2 - \boldsymbol{w}'\boldsymbol{\Sigma}\mathbf{w} = 0$
  - $1 - \boldsymbol{w}'\mathbf{1} = 0$
- We can derive the optimal weights:
  - $2\lambda_1\boldsymbol{\Sigma}\boldsymbol{w}_{\mathbf{0}} = \boldsymbol{\alpha} + \lambda_2\mathbf{1} \rightarrow \boldsymbol{w}_{\mathbf{0}} = \frac{1}{2\lambda_1}\left(\boldsymbol{\Sigma}^{-\mathbf{1}}\boldsymbol{\alpha} + \lambda_2\boldsymbol{\Sigma}^{-\mathbf{1}}\mathbf{1}\right)$

Example 3: Risk Aversion Optimization

$$Maximize\ \boldsymbol{w}'\boldsymbol{\alpha} - \frac{1}{2}\lambda\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$$
$$s.t$$
$$\boldsymbol{w}'\mathbf{1} = 1$$

- Lagrangian: $L(\boldsymbol{w}, \lambda_1) = \boldsymbol{w}'\boldsymbol{\alpha} - \frac{1}{2}\lambda\boldsymbol{w}'\boldsymbol{\Sigma}\mathbf{w} - \lambda_1(1 - \boldsymbol{w}'\mathbf{1})$
- Taking the first order conditions, we have
  - $\boldsymbol{\alpha} - \lambda\boldsymbol{\Sigma}\mathbf{w} = 0$
  - $1 - \boldsymbol{w}'\mathbf{1} = 0$
- We can derive the optimal weights:
  - $\boldsymbol{w}_{\mathbf{0}} = \frac{1}{\lambda}\boldsymbol{\Sigma}^{-\mathbf{1}}\boldsymbol{\alpha}$

Drawbacks of Mean-Variance

- Biased optimal portfolios
  - Parameters (mean expected returns, covariance matrix) are estimated with uncertainty, and the optimizer will end up maximizing this error
  - This often leads to unintuitive results and unbalanced portfolios with high turnover
- Confidence in expected returns
  - The mean-variance setting implicitly assumes 100% confidence in the expected returns views
- Requires expected returns to be specified for the entire universe
- Traditional solutions circumvent the problem by:
  - More constraints: minimum / maximum allocation limits, trading costs
  - Linear objective functions: solve problem to maximize expected returns without taking variance into account
  - Black-Litterman is simply another way to circumvent this problem with a Bayesian approach

Black-Litterman Approach

Bayesian Method

- Actually quite useful for portfolio construction setting
  - Allows us to impose a prior view and alter our view upon arrival of new data
- Recall the Bayes Formula:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

- Bayesian in Black-Litterman
  1. Equilibrium point is introduced, which in this case is the market portfolio
     = $P(A)$ or the Prior
  2. Investor forms views about the asset returns with confidence assigned to each of them
     = $P(B|A)$ or the Likelihood
  3. Optimal allocation is found as a function of views, equilibrium, and confidence
     = $P(A|B)$ or the Posterior

1. Priors

- Suppose the returns of the m risky assets are
$$\boldsymbol{r} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$
- Now we use CAPM: recall that CAPM states that all investors should hold the market portfolio as their risky asset.
- Thus, starting with the market portfolio, we can reverse optimize and derive the excess returns associated with the market portfolio (see example 3 from previous section):
$$\boldsymbol{\Pi} = \lambda \boldsymbol{\Sigma} \boldsymbol{w_m}$$
where $\boldsymbol{w_m}$ is the vector of asset weights corresponding to the market portfolio.
- Investor is assumed to start with the Bayesian prior, characterized as the following:
$$\boldsymbol{\mu} = \boldsymbol{\Pi} + \boldsymbol{\epsilon_1}$$
where $\boldsymbol{\epsilon_1} \sim \boldsymbol{N}(\boldsymbol{0}, \boldsymbol{\tau\Sigma})$. The idea here is that the *precision* of the estimates is proportional to the variance of the returns.

2. Views

- Recall that we define the combination of all views (ex. Asset A will outperform asset B by x%) as the conditional distribution, $P(B|A)$
- Two more constraints before we proceed:
  - Each view is unique and uncorrelated with other views (sparse covariance matrix)
  - Sum of weights in a view is either zero or one
- Portfolio Views
  - P is the $K \times N$ matrix with portfolio weights. Each row represents a view of the portfolio (N assets)
  - Q is the $K \times 1$ vector of views regarding the expected returns of these portfolios.
  - Ω is the $K \times K$ matrix of the covariance of these views. Ω^{-1} is naturally the confidence in the views.
- Practical Example (shamelessly taken from Jay Walters)
  - Row 1: asset 1 will outperform asset 3 by 2% with confidence ω_{11}
  - Row 2: asset 2 will return 3% with confidence ω_{22}

$$P = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \; ; \; Q = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \; ; \; \Omega = \begin{bmatrix} \omega_{11} & 0 \\ 0 & \omega_{22} \end{bmatrix}$$

2. Views

- Once again, the views are assumed to have some errors:

$$\boldsymbol{Q} = \boldsymbol{P\mu} + \boldsymbol{\epsilon_2}$$

where $\boldsymbol{\epsilon_2} \sim N(0, \boldsymbol{\Omega})$.

- Recall from previous section that $\boldsymbol{\Pi} = \boldsymbol{\mu} + \boldsymbol{\epsilon_1}$. We can thus combine the two equations and express them as $\boldsymbol{Y} = \boldsymbol{X\mu} + \boldsymbol{\epsilon}$ where:

$$\boldsymbol{Y} = \begin{pmatrix} \boldsymbol{\Pi} \\ \boldsymbol{Q} \end{pmatrix}; \boldsymbol{X} = \begin{pmatrix} \mathbf{I} \\ \boldsymbol{P} \end{pmatrix}; \boldsymbol{\epsilon} \sim \boldsymbol{N}(\mathbf{0}, \boldsymbol{V}); \boldsymbol{V} = \begin{pmatrix} \boldsymbol{\tau\Sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega} \end{pmatrix}$$

- To get an estimate of $\boldsymbol{\mu}$**,** we employ a GLS regression, by which the formula is given as:

$$\widehat{\boldsymbol{\mu}} = \left(\boldsymbol{X}'\boldsymbol{V}^{-\mathbf{1}}\boldsymbol{X}\right)^{-\mathbf{1}}(\boldsymbol{X}'\boldsymbol{V}^{-\mathbf{1}}\boldsymbol{Y})$$

3. Posterior

- Another way to think about this is to observe the following property:

  If X_1, X_2 are normally distributed as:

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\right)$$

  Then the conditional distribution is given as:

$$X_1 | X_2 \sim N(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(X_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$$

- In both cases, we arrive at the Black-Litterman formula for the posterior distribution of expected returns:

$$E[\boldsymbol{R}|\boldsymbol{Q}] = \left[(\boldsymbol{\tau\Sigma})^{-1} + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right]^{-1}\left[(\boldsymbol{\tau\Sigma})^{-1}\boldsymbol{\Pi} + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\mathbf{Q}\right]$$

$$Var[\mathbf{R}|\boldsymbol{Q}] = \left[(\boldsymbol{\tau\Sigma})^{-1} + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right]^{-1}$$

Usefulness

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Personal Take on Black-Litterman

- One intuitive way to understand the approach is through public vs. private information
  - The "Market" portfolio reflects all public information available to investors.
    - Therefore, by knowing the weights of the market portfolio, we can reverse-calculate the characteristics of individual assets.
  - The investor views reflect all private information that is not incorporated in the market.
- Bayesian updating is more intuitive.
  - It's much more convenient to think about the relative performance of different assets.
  - It's also convenient to have a "benchmark" to which we adjust our views.
- Black-Litterman is not strictly superior to mean-variance.
  - Many (quantitative) asset managers use Black-Litterman, but some hedge funds / asset managers still use a variant of mean-variance portfolio optimization.
  - Black-Litterman doesn't work so well with global portfolios, because the asset weights are less a function of asset returns and variance than they are in the U.S. market.