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Solutions to Exam 1

1. (20 pts) The primitives of the preference-based decision theory that we studied were a set X and a complete and transitive binary relation on X. We could instead have started with X and a binary relation $\succ$ on X satisfying

   (Asymmetry) For all x and y, if $x \succ y$ then not $y \succ x$, and

   (Negative Transitivity) For all x, y, and z : not $x \succ y$ and not $y \succ z \Rightarrow$ not $x \succ z$.

   The two approaches are equivalent. Show one direction of this equivalence by proving the following:

   Proposition 1 *Each asymmetric and negatively transitive $\succ$ is the strict preference relation derived from some complete and transitive $\succeq$.*

   Soln: From the given $\succ$ we must define $\succeq$, show that this $\succeq$ is complete and transitive, and then show that $\succ$ is the strict part of $\succeq$.

   (a) (Define $\succeq$) Given $\succ$, define $\succeq$ by

   $$x \succeq y \text{ iff not } y \succ x. \tag{*}$$

   It will be useful to write an equivalent statement. In set notation, (*) is: $(x, y) \in \succeq$ iff $(y, x) \notin \succ$. The contrapositive of this is: $(x, y) \notin \succeq$ iff $(y, x) \in \succ$. Thus, (*) is equivalent to

   $$\text{not } x \succeq y \text{ iff } y \succ x. \tag{**}$$

   (b) (Complete) Suppose not $x \succeq y$. Then from (**) we obtain $y \succ x$. Hence, by the asymmetry of $\succ$, we have not $x \succ y$. This and (*) yield $y \succeq x$. This proves $\succeq$ is complete.

   (c) (Transitivity) To prove transitivity, we suppose $y \succeq x$ and $z \succeq y$ and prove $z \succeq x$. Given (*), from $y \succeq x$ we obtain not $x \succ y$, and from $z \succeq y$ we obtain not $y \succ z$. The negative transitivity of $\succ$ now implies not $x \succ z$. This and (*) yield $z \succeq x$.

   (d) ($\succ$ is the strict part of $\succeq$) The strict part of $\succeq$ is a binary relation, say $\succ^*$, defined by

   $$x \succ^* y \text{ iff } x \succeq y \text{ and not } y \succeq x.$$

   Since $\succeq$ is complete, this equivalent to

   $$x \succ^* y \text{ iff not } y \succeq x.$$

   This statement is the same as (**), with the x and y reversed. Since both statements are true for all x,$y \in X$, we obtain the desired result:

   $$x \succ^* y \text{ iff } x \succ y.$$

   ■

2. (20 pts) A consumer in a two-good world demands $x = (1, 2)$ at $(p, m) = (2, 4, 10)$, and he demands $x' = (2, 1)$ at $(p', m') = (6, 3, 15)$. Is he maximizing a locally nonsatiated utility function? Explain.

Soln: No. Observe that $p' \cdot x = 12 < 15 = m'$, and $p \cdot x' = 8 < 10 = m$. Thus, x' is revealed preferred to x and x is revealed preferred to x'. This is not possible because a demand function derived from a locally nonsatiated utility function must satisfy WARP.

Proof 1. Suppose x and x' maximize a locally nonsatiated utility function u at (p, m) and (p', m'), respectively. Then, because x' is chosen when x is affordable, we have $u(x') \geq u(x)$. By continuity, there exists a neighborhood N of x' such that $p \cdot y < 10$ for all $y \in N$. By local nonsatiation, N contains a point y such that $u(y) > u(x') \geq u(x)$. Since y is affordable at (p, m), this contradicts the assumption that x maximizes utility at (p, m). ■

Proof 2. Suppose x and x' maximize a locally nonsatiated utility function u at (p, m) and (p', m'), respectively. Because x' and x are each revealed preferred to the other, we must have $u(x') = u(x)$. Hence, x' also maximizes utility at (p, m). This violates Walras' law, which we know is satisfied by any demand correspondence arising from a locally nonsatiated preferences, since $p \cdot x' = 8 < 10 = m$. ■ ■

3. (20 pts) A consumer's preferences are strictly convex, locally nonsatiated, and give rise to a C^1 Marshallian demand function $x : \mathbb{R}^{L+1}_{++} \to \mathbb{R}^L_+$.

   (a) (10 pts) Fix $(p, m) \in \mathbb{R}^{L+1}_{++}$. Under what further assumptions, if any, is it true that for any differentiable utility function representing the consumer's preferences, her marginal utility of income must be positive at (p, m)?

   Soln: There are no assumptions under which this statement is true.[1] That is, so long as there exists a differentiable utility function representing the consumer's preferences, we can find a differentiable representation $\hat{u}$ that gives rise to an indirect utility function $\hat{v}$ satisfying $\partial \hat{v}/\partial m = 0$ at (p, m).

   Proof 1. Let u be a differentiable representation of the preferences. Let v be the corresponding indirect utility function. Let $x^* = x(p, m)$. Define another utility function by

$$\hat{u}(x) = [u(x) - u(x^*)]^3 .$$

   Note that $\hat{u}$ represents the same preferences as does u. The indirect utility function for $\hat{u}$ at an arbitrary $(\hat{p}, \hat{m})$ is

$$\hat{v}(\hat{p}, \hat{m}) = [v(\hat{p}, \hat{m}) - u(x^*)]^3 .$$

   For $\hat{v}$, the marginal utility of income at (p, m) is

$$\left.\frac{\partial \hat{v}(\hat{p}, \hat{m})}{\partial m}\right|_{(\hat{p},\hat{m})=(p,m)} = 3\,[v(p, m) - u(x^*)]^2\, \frac{\partial v(p, m)}{\partial m} = 0,$$

   since $v(p, m) = u(x^*)$. ∎

   Proof 2. Same u, x^*, and $\hat{u}$. Then x^* maximizes $\hat{u}$ subject to $p \cdot x \le m$. Hence, $\hat{\lambda} \in \mathbb{R}_+$ exists such that for all i,

$$\text{(FOC)} \quad \frac{\partial u(x^*)}{\partial x_i} \le \hat{\lambda} p_i, \text{ equality if } x_i^* > 0.$$

   Since preferences are locally nonsatiated, Walras' law holds: $p \cdot x^* = m$. Hence, as $m > 0$, $x_i^* > 0$ for some i. For this i the FOC holds with equality. Hence,

$$\frac{\partial \hat{v}(p, m)}{\partial m} = \hat{\lambda} = \frac{1}{p_i}\frac{\partial \hat{u}(x^*)}{\partial x_i} = \frac{3}{p_i}\,[u(x^*) - u(x^*)]^2\, \frac{\partial u(x^*)}{\partial x_i} = 0. \blacksquare$$

∎

   (b) (10 pts) Suppose $u(\cdot)$ represents the consumer's preferences, and the corresponding expenditure function satisfies $\partial^2 e/\partial p_1 \partial u > 0$ for all (p, u) at which it is well defined. What does this tell us about her demand function for good 1?

   Soln: Good 1 is normal, in the sense that $\frac{\partial x_1(p,m)}{\partial m} \ge 0$.

   Proof. We have

$$x_1(p, m) = h_1(p, v(p, m) = \frac{\partial e(p, v(p, m))}{\partial p_1}$$

$$\Rightarrow \quad \frac{\partial x_1(p, m)}{\partial m} = \frac{\partial^2 e(p, v(p, m))}{\partial p_1 \partial u}\frac{\partial v(p, m)}{\partial m}.$$

   Since $\partial^2 e/\partial p_1 \partial u > 0$ and $\partial v/\partial m \ge 0$ (as v increases in m), we conclude that $\partial x_1/\partial m \ge 0$. ∎

[1] Except for an assumption that implies a differentiable representing utility function does not exist (e.g., the preferences are lexicographic), which makes the statement vacuously true.

4. (20 pts) In a two-good world, consider the following possible expenditure function, where a and b are positive exponents:

$$e(p,u) = \left(\tfrac{1}{2}p_1^a + \tfrac{1}{2}p_2^b + \sqrt{p_1 p_2}\right)u.$$

(a) (8 pts) For what values of (a,b) is e truly an expenditure function? Explain.

Soln: $a = b = 1$.

Proof. An expenditure function must be homogeneous of degree one in p. Hence, for all $t > 0$, $p \in \mathbb{R}^2_{++}$ and $u \neq 0$,

$$\begin{aligned}
\left(\tfrac{1}{2}t^a p_1^a + \tfrac{1}{2}t^b p_2^b + t\sqrt{p_1 p_2}\right)u &= t\left(\tfrac{1}{2}p_1^a + \tfrac{1}{2}p_2^b + \sqrt{p_1 p_2}\right)u \\
&\Leftrightarrow \\
t^a p_1^a + t^b p_2^b &= t p_1^a + t p_2^b \\
&\Leftrightarrow \\
(t^a - t)\,p_1^a &= \left(t^b - t\right)p_2^b.
\end{aligned}$$

If $t^a \neq t$, then the LHS varies with p_1, but the RHS does not. Hence, $t^a = t$ for all $t > 0$. This implies $a = 1$. The symmetrical argument proves $b = 1$. (With $a = b = 1$, the function e is homogeneous of degree one in p, strictly increasing in p for $u > 0$, concave in p, and continuous. It is therefore a true expenditure function, i.e., there is a utility function from which it derives. The answer to (c) below is a constructive proof of this.) ■

For (b) and (c), assume a and b satisfy the restrictions you just identified.

(b) (4 pts) Find the corresponding Hicksian demand functions.

Soln: By Shepard's lemma (envelope theorem 1),

$$h_1(p,u) = \frac{\partial e(p,u)}{\partial p_1} = \tfrac{1}{2}\left[1 + \left(\frac{p_2}{p_1}\right)^{1/2}\right]u,$$

$$h_2(p,u) = \frac{\partial e(p,u)}{\partial p_2} = \tfrac{1}{2}\left[1 + \left(\frac{p_2}{p_1}\right)^{-1/2}\right]u.$$

■

(c) (8 pts) Find a utility function for which e is the expenditure function.

Soln: Replacing $h_i(p,u)$ by x_i and u by $u(x)$ in the expressions derived in (b) yields

$$x_1 = \tfrac{1}{2}\left[1 + \left(\frac{p_2}{p_1}\right)^{1/2}\right]u(x),$$

$$x_2 = \tfrac{1}{2}\left[1 + \left(\frac{p_2}{p_1}\right)^{-1/2}\right]u(x).$$

Reducing these to one equation by eliminating $\frac{p_2}{p_1}$ yields

$$u(x) = \frac{2x_1 x_2}{x_1 + x_2}.$$

■