700 Prelim Questions & Solutions
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1. (Spring 2012) (20 pts) A strictly increasing utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ gives rise to a demand function $\mathbf{x}(\mathbf{p}, y) = (x_1(\mathbf{p}, y), \ldots, x_n(\mathbf{p}, y))$ defined on $\mathbb{R}^{n+1}_{++}$. Assume it and any other functions you use to answer this question are twice continuously differentiable.

   (a) (8 pts) State all the properties this demand function must necessarily satisfy.
   Soln:

   i. (2 pt) Homogeneous of degree zero:

   $$\mathbf{x}(t\mathbf{p}, ty) = \mathbf{x}(\mathbf{p}, y) \ \forall t > 0, \ (\mathbf{p}, y) \in \mathbb{R}^{n+1}_{++}.$$

   ii. (2 pt) Satisfies the budget constraint with equality:

   $$\mathbf{p} \cdot \mathbf{x}(\mathbf{p}, y) = y \ \forall (\mathbf{p}, y) \in \mathbb{R}^{n+1}_{++}.$$

   iii. (4 pts) The Slutsky matrix $[s_{ij}(\mathbf{p}, y)]$ is negative semidefinite and symmetric at any $(\mathbf{p}, y)$, where

   $$s_{ij}(\mathbf{p}, y) := \frac{\partial x_i(\mathbf{p}, y)}{\partial p_j} + x_j(\mathbf{p}, y)\frac{\partial x_i(\mathbf{p}, y)}{\partial y}.$$

   ■

   (b) (12 pts) For each property you listed in (a), sketch a proof of why it must be satisfied.
   Soln:

   i. (1 pt) Multiplying all prices and income by the same positive constant does not change the constraint set in the consumer problem, and even more obviously does not change its objective function, u. Thus, $\mathbf{x}$ is a solution for the budget $(\mathbf{p}, y)$ iff it is a solution for the budget $(t\mathbf{p}, ty)$ for any $t > 0$.

   ii. (2 pts) Fix $(\mathbf{p}, y) \in \mathbb{R}^{n+1}_{++}$, and let $\mathbf{x} = \mathbf{x}(\mathbf{p}, y)$. Since $\mathbf{x}$ solves the consumer problem, it satisfies its constraint, i.e., $\mathbf{p} \cdot \mathbf{x} \leq y$. Suppose this inequality holds strictly. Then $\bar{\mathbf{x}} \gg \mathbf{0}$ exists such that $\mathbf{p} \cdot (\mathbf{x} + \bar{\mathbf{x}}) \leq y$. Hence, $x' = \mathbf{x} + \bar{\mathbf{x}}$ is feasible for the consumer problem, and so $u(\mathbf{x}') \leq u(\mathbf{x})$. Since $\mathbf{x}' \gg \mathbf{x}$, this contradicts the assumption that u is strictly increasing.

   iii. (9 pts) Recall the expenditure function,

   $$e(\mathbf{p}, \bar{u}) := \min_{\mathbf{x} \geq 0} \mathbf{p} \cdot \mathbf{x} \text{ such that } u(\mathbf{x}) \geq \bar{u}.$$

   The solution to this minimization problem is the Hicksian demand function, $\mathbf{x}^h(\mathbf{p}, \bar{u})$. Since $e(\cdot, \bar{u})$ is the lower envelope of a bunch of affine functions of $\mathbf{p}$, it is a concave function of $\mathbf{p}$. Hence, since we've been told all functions in this problem are C^2, the matrix of cross-partials,

   $$\left[\frac{\partial^2 e(\mathbf{p}, \bar{u})}{\partial p_i \partial p_j}\right],$$

is both negative semidefinite and symmetric. By the envelope theorem, $\partial e(\mathbf{p}, \bar{u})/\partial p_i = x_i^h(\mathbf{p}, \bar{u})$. Hence,

$$\left[\frac{\partial x_i^h(\mathbf{p}, \bar{u})}{\partial p_j}\right] = \left[\frac{\partial^2 e(\mathbf{p}, \bar{u})}{\partial p_i \partial p_j}\right],$$

which tells us that $\left[\dfrac{\partial x_i^h(\mathbf{p}, \bar{u})}{\partial p_j}\right]$ is negative semidefinite and symmetric at any $(\mathbf{p}, \bar{u})$. Since the Slutsky equation is in fact the equation

$$s_{ij}(\mathbf{p}, y) = \frac{\partial x_i^h(\mathbf{p}, u^*)}{\partial p_j},$$

where $u^* = u(\mathbf{x}(\mathbf{p}, y))$, we conclude that $[s_{ij}(\mathbf{p}, y)]$ is indeed negative semidefinite and symmetric.

Grading Notes

- A derivation of the Slutsky equation is not required for this question, although having knowledge of it is required.
- I subtracted roughly 2 points for such statements to the effect that $x(p, y)$ had to be increasing in y, or decreasing in p ("Law of Demand"), or quasiconvex or quasiconcave in (p, y).
- Scores were mostly very low, under 5. There were four perfect or nearly perfect answers (17-20), and one 14 and 10.

■

2. (Spring 2012) (20 pts) A consumer lives for two periods. In period 2 she will purchase a commodity bundle $x = (x_1, x_2)$ to maximize her utility $u(x) = x_1^\alpha x_2^\alpha$ subject to her budget constraint $p_1x_1 + p_2x_2 \leq y$. These prices are fixed positive constants, known even in period 1.

   In period 1 the consumer invests in a risky asset that returns $(1+\tilde{r})z$ in period 2 if she invests an amount z. Her wealth is $w > 0$, and she is restricted to choosing $z \in [0, w]$. Her income in period 2 when she invests z will thus be the random variable $\tilde{y} = w + \tilde{r}z$. The asset has a positive expected return: $\mathbb{E}\tilde{r} > 0$. She chooses z to maximize the expected utility she will ultimately obtain in period 2.

   For each $\alpha > 0$, determine whether the consumer's optimal investment in the asset, $z^*(p, w)$, is decreasing, constant, or increasing in w.

   Soln: $z^*(p, w)$ strictly increases in w for all $\alpha > 0$.

   Proof. (5 pts) Solve the period 2 consumer problem to find the demand functions:

$$x_i(p, y) = \frac{y}{2p_i} \text{ for } i = 1, 2.$$

   Substitute these into the utility function to find the indirect utility function:

$$v(p, y) = (2p_1p_2)^{-\alpha}\, y^{2\alpha}.$$

   An optimal investment is in the set

$$\text{(P)} \qquad \arg \max_{0 \leq z \leq w} \mathbb{E}v(w + \tilde{r}z).$$

   In view of the form of v, the optimal investment is independent of p, and so can be written as $z^*(w)$

   (2 pts) Note that $z^*(w) > 0$, since

$$\tfrac{\partial}{\partial z}\mathbb{E}v(p, w + \tilde{r}z)\big|_{z=0} = (\mathbb{E}\tilde{r})\, v'(w) > 0.$$

   (5 pts) If $\alpha > \frac{1}{2}$, then the objective function in (P) is strictly convex (the consumer is risk loving). In this case the solution of (P) is the upper corner, $z^*(w) = w$, which is obviously strictly increasing in w. The same is true if $\alpha = \frac{1}{2}$, since then the objective function is linear with slope $(\mathbb{E}\tilde{r})\, v'(w) > 0$.

   (8 pts) Suppose now that $\alpha \in (0, \frac{1}{2})$. The objective function in (P) is then strictly concave in z, and so the FOC is necessary and sufficient, and $z^*(w)$ is a continuous function. The indirect utility function satisfies DARA, since

$$R(p, y) := -\frac{v_{yy}(p, y)}{v_y(p, y)} = \frac{1 - 2\alpha}{y}$$

   implies $R_y(p, y) < 0$. If $z^*(w)$ is an interior solution, DARA implies that $z^*(w)$ strictly increases in w, as shown in lecture (and almost all textbooks). If it is a corner solution, then it must be $z^*(w) = w$, which again is strictly increasing in w.

   Alternative Direct Proof. As above, to the derivation of (P) and the conclusion that $z^*(w) > 0$. Note that (P) is equivalent to

$$\max_{0 \leq z \leq w} \mathbb{E}(w + \tilde{r}z)^{2\alpha}.$$

Let $z = tw$, and change the choice variable to t :

$$\max_{0 \le z \le w} \mathbb{E}(w + \tilde{r}tw)^{2\alpha} = \max_{0 \le t \le 1} \mathbb{E}w^{2\alpha} (1 + \tilde{r}t)^{2\alpha} .$$

From this we see that the optimal t, say t^*, does not depend on w, and so the optimal investment function is linear:

$$z^*(w) = t^* w.$$

Since $z^*(w) > 0$, we have $z^{*\prime}(w) = t^* > 0$.

Grading Notes

Students basically failed or aced this question. I gave twelve scores in the 0-3 range, and seven in the 15-20 range. ■

3. (Fall 2012) A competitive firm uses two inputs, x_1 and x_2, to make one output, y. Consider the following possible cost function, where the exponents a and b are positive constants:

$$c(y, \mathbf{w}) = \left(\tfrac{1}{2}w_1^a + \tfrac{1}{2}w_2^b + \sqrt{w_1 w_2}\right) y.$$

(a) (5 pts) For what values of a and b is c truly a cost function? Prove your answer.

Soln: $a = b = 1$.

Proof. To be a cost function, c must be homogeneous of degree one in $\mathbf{w}$. Hence, for all $t > 0$ and $\mathbf{w} \in \mathbb{R}^2_{++}$ and $y > 0$,

$$\begin{aligned} \left(\tfrac{1}{2}t^a w_1^a + \tfrac{1}{2}t^b w_2^b + t\sqrt{w_1 w_2}\right) y &= t\left(\tfrac{1}{2}w_1^a + \tfrac{1}{2}w_2^b + \sqrt{w_1 w_2}\right) y \\ &\Leftrightarrow \\ t^a w_1^a + t^b w_2^b &= t w_1^a + w_2^b \\ &\Leftrightarrow \\ \left(t^a - t\right) w_1^a &= \left(t^b - t\right) w_2^b. \end{aligned}$$

If $t^a \neq t$, then the LHS varies with w_1, but the RHS does not. Hence, $t^a = t$ for all $t > 0$. This implies $a = 1$. The symmetrical argument proves $b = 1$.

(With $a = b = 1$, the function c is homogeneous of degree one in $\mathbf{w}$, strictly increasing in w for $y > 0$, concave in $\mathbf{w}$, and continuous. It is therefore a true cost function, i.e., there is a production function from which it derives. The answer to (d) below is a constructive proof of this.) ■

For the remaining questions, assume a and b satisfy the restrictions you just identified.

(b) (5 pts) Find the firm's conditional factor demand functions, $\hat{x}_1(y, \mathbf{w})$ and $\hat{x}_2(y, \mathbf{w})$.

Soln: By the envelope theorem,

$$\hat{x}_1(y, \mathbf{w}) = \frac{\partial c(y, \mathbf{w})}{\partial w_1} = \tfrac{1}{2}\left[1 + \left(\frac{w_2}{w_1}\right)^{1/2}\right] y,$$

$$\hat{x}_2(y, \mathbf{w}) = \frac{\partial c(y, \mathbf{w})}{\partial w_2} = \tfrac{1}{2}\left[1 + \left(\frac{w_2}{w_1}\right)^{-1/2}\right] y.$$

■

(c) (5 pts) Find the firm's supply function, $y^*(p, \mathbf{w})$.

Soln: The firm's profit as a function of y,

$$py - \left(\tfrac{1}{2}w_1 + \tfrac{1}{2}w_2 + \sqrt{w_1 w_2}\right) y,$$

is linear in y. It's maximizer is

$$y^*(p, \mathbf{w}) = \left\{\begin{array}{c} 0 \\ [0, \infty) \\ \infty \end{array}\right\} \text{ for } p \left\{\begin{array}{c} < \\ = \\ > \end{array}\right\} \tfrac{1}{2}w_1 + \tfrac{1}{2}w_2 + \sqrt{w_1 w_2}.$$

■

(d) (5 pts) Find a production function f for which c is the corresponding cost function.

Soln: Replacing $\hat{x}_i(y, \mathbf{w})$ by x_i and y by $f(\mathbf{x})$ in the expressions derived in (b) yields

$$x_1 = \tfrac{1}{2}\left[1 + \left(\frac{w_2}{w_1}\right)^{1/2}\right] f(\mathbf{x}),$$

$$x_2 = \tfrac{1}{2}\left[1 + \left(\frac{w_2}{w_1}\right)^{-1/2}\right] f(\mathbf{x}).$$

Reducing these to one equation by eliminating $\frac{w_2}{w_1}$ yields

$$f(\mathbf{x}) = \frac{2x_1x_2}{x_1 + x_2}.$$

■

4. (Fall 2012) Mr. A has a complete and transitive preference ordering $\succeq_A$ over monetary lotteries $\tilde{x}$ that is monotone in the following sense: $\delta_x \succ_A \delta_y$ for all $x > y$, where δ_x and δ_y are the degenerate lotteries that put probability one on the amounts x and y, respectively.

   (a) (1 pt) Define what it means for Mr. A to be strictly risk averse.

   Soln: Mr. A is *strictly risk averse* iff for all non-degenerate $\tilde{x}$,

$$\delta_{\mathbb{E}\tilde{x}} \succ_A \tilde{x}. \tag{1}$$

■

   (b) (2 pts) Define Mr. A's certainty equivalent c_A for a given non-degenerate lottery $\tilde{x}$.

   Soln: The *certainty equivalent* c_A for $\tilde{x}$ is the number satisfying

$$\delta_{c_A} \sim_A \tilde{x}. \tag{2}$$

   This is well-defined, since monotonicity and transitivity imply that no more than one number c_A can satisfy this indifference relation. (However, c_A is not guaranteed to exist without further assumptions.) ■

   (c) (3 pts) Assume Mr. A is strictly risk averse. For the $\tilde{x}$ and c_A from (b), what can you say about the relationship between c_A and $\mathbb{E}\tilde{x}$? Prove your answer.

   Soln: $c_A < \mathbb{E}\tilde{x}$.

   Proof. Since $\tilde{x}$ is non-degenerate and Mr. A is strictly risk averse, (1) holds. This, (2), and transitivity imply $\delta_{\mathbb{E}\tilde{x}} \succ_A \delta_{c_A}$. Monotonicity now implies $c_A < \mathbb{E}\tilde{x}$. ■

Now assume Mr. A has a Bernoulli utility function, u_A. Ms. B also has a Bernoulli utility function, u_B. Both functions are twice differentiable, with $u_A'(x) > 0$ and $u_B'(x) > 0$ for all $x \in \mathbb{R}$.

   (d) (1 pts) Define their Arrow-Pratt coefficients of absolute risk aversion, $R_i(x)$ for $i = A, B$.

   Soln: For $i = A, B$ and $x \in \mathbb{R}$,

$$R_i(x) := -\frac{u_i''(x)}{u_i'(x)}.$$

■

   (e) (13 pts) Assume Mr. A is more risk averse than Ms. B, in the sense that $R_A(x) > R_B(x)$ for all $x \in \mathbb{R}$. Letting $\tilde{x}$ be a non-degenerate lottery, and c_A and c_B be their certainty equivalents for it, prove that $c_A < c_B$.

   Soln: Step 1. A strictly concave increasing function $h : u_B(\mathbb{R}) \to \mathbb{R}$ exists such that $u_A = h \circ u_B$.

   Proof. An amount x gives utility $u_B(x)$ to B and utility $u_A(x)$ to A. The function h that maps the former into the latter must satisfy

$$h(u_B(x)) = u_A(x), \tag{3}$$

   or rather, $h(v) := u_A(u_B^{-1}(v))$. This defines h on $u_B(\mathbb{R})$. Now, differentiating (3) w.r.t. x yields

$$h' u_B' = u_A', \tag{4}$$

and hence $h' > 0$. Differentiating again yields

$$h''(u_B')^2 + h'u_B'' = u_A'',$$

which in light of (4) is

$$h''(u_B')^2 + \frac{u_A'}{u_B'}u_B'' = u_A''.$$

Substitute $-R_i u_i'$ for each u_i'' and rearrange to obtain

$$h'' = \frac{u_A'}{\left(u_B'\right)^2}\left(R_B - R_A\right) < 0,$$

which proves h is strictly concave.

Step 2. $c_A < c_B$.

Proof. By the definition of c_A and a Bernoulli utility function, $u_A(c_A) = \mathbb{E}u_A(\tilde{x})$. This and $u_A = h \circ u_B$ (Step 1) yield

$$h(u_B(c_A)) = \mathbb{E}h(u_B(\tilde{x})).$$

The strict concavity of h, the non-degeneracy of $\tilde{x}$, the strict monotonicity of u_B, and Jensen's inequality imply

$$\mathbb{E}h(u_B(\tilde{x})) < h(\mathbb{E}u_B(\tilde{x})).$$

By the definition of c_B,

$$h(\mathbb{E}u_B(\tilde{x})) = h(u_B(c_B))$$

These three displays yield $h(u_B(c_A)) < h(u_B(c_B))$. Hence, since $h \circ u_B$ is strictly increasing, $c_A < c_B$. ■

5. (June 2013) A competitive firm uses two inputs to produce one output according to a strictly increasing production function, $y = f(z_1, z_2)$. The input prices, w_1 and w_2, are constant in this problem, and hence we simplify notation by not writing them as arguments of functions.

   In the "long-run," the firm chooses z_1 and z_2 to maximize profit. Assume this gives rise to C^2 input demand and supply functions, $z_1^L(p)$, $z_2^L(p)$, and $y^L(p)$, defined on $\mathbb{R}_{++}$.

   In the "short-run," the firm chooses only z_2 to maximize profit, because the first input is fixed at some level $\bar{z}_1 > 0$. Assume this gives rise to a C^2 supply function, $y^S(\cdot, \bar{z}_1)$, defined on $\mathbb{R}_{++}$.

   Suppose $\bar{p} > 0$ is an output price such that $z_1^L(\bar{p}) = \bar{z}_1$. Show that at $\bar{p}$, the long-run and short-run supply functions specify the same output. Show also that at $\bar{p}$, the price elasticity of supply is larger, at least weakly, for the long-run supply curve than it is for the short-run supply curve.

   Soln: Consider the long and short-run profit-maximizing programs:

$$\mathrm{P^L}(p): \quad \pi^L(p) = \max_{z_1, z_2 \geq 0} pf(z_1, z_2) - w_1 z_1 - w_2 z_2,$$

$$\mathrm{P^S}(p, \bar{z}_1): \quad \pi^S(p) = \max_{z_1, z_2 \geq 0} pf(z_1, z_2) - w_1 z_1 - w_2 z_2 \quad \text{such that } z_1 = \bar{z}_1.$$

   From the problem statement, we know that $\bar{z} := (\bar{z}_1, z_2^L(\bar{p}))$ solves $\mathrm{P^L}(\bar{p})$, and so $y^L(\bar{p}) = f(\bar{z})$. Program $\mathrm{P^S}(p, \bar{z}_1)$ is the same as $\mathrm{P^L}(\bar{p})$ but with an added constraint. Thus, since $\bar{z}$ is feasible for $\mathrm{P^S}(p, \bar{z}_1)$, it must also solve it. This implies $y^S(\bar{p}, \bar{z}_1) = f(\bar{z})$, and we have the first desired result: $y^S(\bar{p}, \bar{z}_1) = y^L(\bar{p})$.

   To obtain the elasticity result, not from the previous paragraph that $\pi^S(p, \bar{z}_1) - \pi^L(p) \leq 0$ for all p, with equality at $p = \bar{p}$. Thus,

$$\bar{p} \in \arg\max_{p \geq 0} \pi^S(p, \bar{z}_1) - \pi^L(p).$$

   The necessary second-order condition for this (as $\bar{p} > 0$) is

$$\text{(NSOC)} \quad \frac{\partial^2 \pi^S(\bar{p}, \bar{z}_1)}{\partial p^2} \leq \frac{\partial^2 \pi^L(\bar{p})}{\partial p^2}.$$

   From Hotelling's lemma we have

$$\frac{\partial \pi^S(\bar{p}, \bar{z}_1)}{\partial p} = y^S(\bar{p}, \bar{z}_1) \text{ and } \frac{\partial \pi^L(\bar{p})}{\partial p} = y^L(\bar{p}),$$

   and so NSOC implies

$$\frac{\partial y^S(\bar{p}, \bar{z}_1)}{\partial p} \leq \frac{\partial y^L(\bar{p})}{\partial p}.$$

   Thus, since $y^S(\bar{p}, \bar{z}_1) = y^L(\bar{p})$, the short-run elasticity is less than the long-run elasticity:

$$\frac{\partial y^S(\bar{p}, \bar{z}_1)}{\partial p} \cdot \frac{\bar{p}}{y^S(\bar{p}, \bar{z}_1)} \leq \frac{\partial y^L(\bar{p})}{\partial p} \cdot \frac{\bar{p}}{y^L(\bar{p})}.$$

■

6. (June 2013) An investor can invest any fracton $\alpha \in [0,1]$ of her initial wealth $w > 0$ in a risky asset. Her random wealth when she invests αw is

$$\tilde{y} = (1 + \tilde{r}\alpha)w,$$

where $\tilde{r}$ is the return of the asset. Assume the random variable $\tilde{r}$ is not degenerate, and $\mathbb{E}\tilde{r} > 0$. The investor's Bernoulli utility u satisfies $u' > 0$ and $u'' < 0$. Let $\alpha^*(w)$ be the investor's expected utility maximizing α. Assume that for any w of interest in this problem, $\alpha^*(w) < 1$. (In words, the asset is not so surely profitable that the investor would choose to invest all her wealth in it.)

Kenneth Arrow claimed, as an empirical matter, that as an investor becomes wealthier, she invests a smaller proportion of her wealth in risky assets. Give a sufficient condition for this to be true, in this problem, in terms of the function that measures the investor's *relative risk aversion:*

$$R(y) := -\frac{yu''(y)}{u'(y)}.$$

Soln: *$\alpha^*(w)$ decreases in w if $R(\cdot)$ is increasing, i.e., if u satisfies IRRA.*

Proof. $\alpha^*(w)$ is the solution of

$$\max_{0 \leq \alpha \leq 1} \mathbb{E}u(w + \tilde{r}\alpha w).$$

The derivative with respect to α of the objective function is $\mathbb{E}\tilde{r}u'(w + \tilde{r}\alpha w)$, which is positive at $\alpha = 0$, since $\mathbb{E}\tilde{r} > 0$ and $u'(w) > 0$. Hence, $\alpha^*(w) > 0$. Since the problem states that the other corner is also not a solution, we have the interior FOC:

$$\text{(FOC)} \quad \mathbb{E}w\tilde{r}u'(w + \tilde{r}\alpha^*(w)w) = 0.$$

Divide FOC by w and then differentiate with respect to w to obtain

$$\mathbb{E}\tilde{r}u''(w + \tilde{r}\alpha^*(w)w)[1 + \tilde{r}\alpha^*(w) + \tilde{r}\alpha^{*\prime}(w)w] = 0.$$

Letting $\tilde{y} = w + \tilde{r}\alpha^*(w)w$, this becomes

$$\mathbb{E}\tilde{r}u''(\tilde{y})[\tilde{y}w^{-1} + \tilde{r}\alpha^{*\prime}(w)w] = 0.$$

Hence,

$$\alpha^{*\prime}(w) = \frac{\mathbb{E}\tilde{r}\tilde{y}u''(\tilde{y})}{-w^2\mathbb{E}\tilde{r}^2u''(\tilde{y})} = \frac{N}{(+)}.$$

So $\text{sign}(\alpha^{*\prime}(w)) = \text{sign}(N)$. Note that

$$N = \mathbb{E}\tilde{r}\tilde{y}u''(\tilde{y}) = -\mathbb{E}\tilde{r}R(\tilde{y})u'(\tilde{y}).$$

From FOC we see that $\mathbb{E}\tilde{r}R(w)u'(\tilde{y}) = 0$. Hence,

$$\begin{aligned} N &= \mathbb{E}\tilde{r}R(w)u'(\tilde{y}) - \mathbb{E}\tilde{r}R(\tilde{y})u'(\tilde{y}) \\ &= \mathbb{E}\left\{\tilde{r}\left[R(w) - R(\tilde{y})\right]u'(\tilde{y})\right\} \\ &< 0, \end{aligned}$$

where the inequality holds because $u' > 0$ and the definitions of $\tilde{y}$ and IRRA imply

$$\text{sign}(\tilde{r}) = \text{sign}(\tilde{y} - w) = -\text{ sign}\left[R(w) - R(\tilde{y})\right].$$

■

7. (August 2013) (20 pts) A consumer has wealth w that she must consume over two periods. In period 2 she has a random income shock, $\theta\tilde{y}$, where $\theta > 0$, $\mathbb{E}\tilde{y} = 0$, and $\mathbb{E}\tilde{y}^2 > 0$. Her expected utility when she chooses to save an amount x is

$$u(w - x) + \mathbb{E}v(x + \theta\tilde{y}).$$

She can save any amount, i.e., x can be any real number. The functions u and v are C^3, with strictly positive first derivatives and strictly negative second derivatives. Let $x^*(w, \theta)$ denote her optimal savings function.

   (a) (5 pts) Does x^* increase or decrease in w, or can it do either depending on the utility functions? Prove your answer.
   Soln: x^* strictly increases in w.
   Proof. x^* satisfies

$$\text{(FOC)} \quad -u'(w - x^*) + \mathbb{E}v'(x^* + \theta\tilde{y}) = 0.$$

   Differentiating with respect to w yields

$$x_w^*(w, \theta) = \frac{u''(w - x^*)}{D},$$

   where $D = u''(w - x^*) + \mathbb{E}v''(x + \theta\tilde{y}) < 0$. Hence, as $u'' < 0$, we have $x_w^*(w, \theta) > 0$. ■

   (b) (15 pts) Show that x^* strictly increases in θ if v exhibits nonincreasing absolute risk aversion (NIARA).
   Soln: x^* strictly increases in θ.
   Proof. We have been told that $A = -v''/v'$ is a nonincreasing function, which is equivalent here to $A' \leq 0$. Differentiate the equation $v'' = -Av'$ to obtain

$$\begin{aligned} v''' &= -A'v' - Av'' \\ &= -(- \text{ or } 0)(+) - (+)(-) \\ &> 0. \end{aligned}$$

   Now calculate the derivative x_θ^* by differentiating (FOC):

$$x_\theta^*(w, \theta) = \frac{\mathbb{E}v''(x^* + \theta\tilde{y})\tilde{y}}{-D}.$$

   So $x_\theta^*(w, \theta)$ and $\mathbb{E}v''(x^* + \theta\tilde{y})\tilde{y}$ have the same sign.
   Since $\mathbb{E}\tilde{y} = 0$, we have

$$\mathbb{E}v''(x^* + \theta\tilde{y})\tilde{y} = \mathbb{E}\left[v''(x^* + \theta\tilde{y}) - v''(x^*)\right]\tilde{y}.$$

   Because $\theta > 0$ and $v''' > 0$, for all $y \neq 0$ we have

$$\left[v''(x^* + \theta y) - v''(x^*)\right] y > 0.$$

   Hence, since $\mathbb{E}\tilde{y}^2 > 0$, we have $\mathbb{E}v''(x^* + \theta\tilde{y})\tilde{y} > 0$. This proves that $x_\theta^*(w, \theta) > 0$. ■

8. (August 2013) (20 pts) Let $Y \subseteq \mathbb{R}^N$ denote a production set and $\pi : \mathbb{R}^N_+ \to \mathbb{R}$ a profit function.

   (a) (5 pts) Suppose you are given the set Y. Show how the associated profit function π can be derived.

   Soln: Given Y, for each $p \in \mathbb{R}^N_+$ one finds $\pi(p)$ by performing the maximization in the definition of a profit function:

$$\pi(p) := \max_{y \in Y} p \cdot y.$$

■

   (b) (15 pts) Suppose instead that you are given the function π, and told that it is the profit function asociated with a closed and convex Y that satisfies free disposal. Show how Y can be derived, and prove that your method works.

   Soln: Using the given π, calculate the set

$$\hat{Y} := \{y \in \mathbb{R}^N : p \cdot y \leq \pi(p) \text{ for all } p \in \mathbb{R}^N_+\}.$$

   The following result shows that this is the desired set.

   Proposition. $Y = \hat{Y}$.

   Proof. Since π is the profit function associated with Y, the definition of a profit function implies $Y \subseteq \hat{Y}$. To show the reverse, let $y \notin Y$. We must prove $y \notin \hat{Y}$.

   As $y \notin Y$ and Y is closed and convex, a separating hyperplane theorem implies the existence of a nonzero $p \in \mathbb{R}^N$ such that

$$p \cdot y > \max_{x \in Y} p \cdot x = \pi(p).$$

   If any $p_i < 0$, this inequality could not hold, since free disposal would imply

$$\sup_{x \in Y} p \cdot x = \infty.$$

   Hence, $p \in \mathbb{R}^N_+$. This and $p \cdot y > \pi(p)$ imply $y \notin \hat{Y}$. ■