Exam

100 points, 105 minutes. Closed books, notes, calculators.
Indicate your reasoning, using clearly written words as well as math.

DO JUST *FOUR* OF THE FIVE PROBLEMS.
(Only 1-4 will be graded if you do them all.)

1. (25 pts) Consider a differentiable demand function $x(p,m)$ generated by a strictly increasing $u : \mathbb{R}^n_+ \to \mathbb{R}$ that is homogeneous of degree 1.

   (a) (10 pts) Show that for any $p \in \mathbb{R}^n_{++}$ and $m > 0$, $x(p,m) = mx(p,1)$.

   (b) (15 pts) Show that x satisfies the Law of Reciprocity.

2. (25 pts) An investor has wealth w and can choose to invest any amount $x \geq 0$ in a risky asset. The rate of return on the investment is r, so that the income the investor will consume is

$$y = w - x + (1+r)x = w + rx.$$

   The consumer's Bernoulli utility function for income, $u : \mathbb{R} \to \mathbb{R}$, satisfies $u' > 0$ and $u'' < 0$.

   Both the investor's wealth and the asset's rate of return are random at the time the investment decision is made. Denoting these random variables as $\tilde{w}$ and $\tilde{r}$, assume $\mathbb{E}\tilde{r} > 0$ and that they are perfectly and linearly correlated: $w_0, \beta \in \mathbb{R}$ exist such that

$$\tilde{w} = w_0 + \beta\tilde{r}.$$

   Fix w_0, and assume that for any $\beta \in \mathbb{R}$, the investor has a finite optimal investment level. Denote it as $x^*(\beta)$.

   (a) (5 pts) Give a verbal, "intuitive" (virtually no math!) argument for whether $x^*(\beta)$ is positive or zero when $\beta < 0$.

   (b) (20 pts) Find an expression for $x^*(\beta)$ in terms of β and $x^*(0)$, valid for all $\beta \in \mathbb{R}$.

3. (25 pts) Let $C = \{1, \ldots, N\}$ be a set of consequences, with $N \geq 3$. Let $\succeq$ be a binary relation on the set $\Delta(C)$ of simple lotteries.

   (a) (5 pts) State the independence axiom.

   (b) (10 pts) Show that if $\succeq$ satisfies independence, then $\succeq$ is convex.

   (c) (10 pts) Suppose $\succeq$ is represented by the median function m, where for $L = (p_1, \ldots, p_N)$,

$$m(L) := \min\left\{c \in C : \sum_{k \leq c} p_k \geq .5\right\}.$$

   State and prove whether $\succeq$ satisfies independence.

4. (25 pts) A competitive firm uses two inputs to produce one output according to a strictly concave and strictly increasing C^2 production function, $q = f(x, z)$. The input prices, $(w_x, w_z) \in \mathbb{R}^2_{++}$, are held fixed in this problem and so not written as arguments in functions. The firm has conditional demand functions $x^*(q)$ and $z^*(q)$, which give rise to a *long-run cost function* $c(q)$.

   In the short run, input z is fixed at $\bar{z}$, and only x is variable. The *short-run cost function* is
$$c_S(q, \bar{z}) = \min_{x \geq 0} w_x x + w_z \bar{z} \text{ such that } f(x, \bar{z}) \geq q.$$
   Suppose $\bar{z} = z^*(\bar{q})$ for some $\bar{q} > 0$.

   (a) (5 pts) For $q \neq \bar{q}$, how does $c(q)$ compare to $c_S(q, \bar{z})$? For $q = \bar{q}$?

   (b) (10 pts) Show that at $q = \bar{q}$, the short-run and long-run *marginal* cost curves cross, and the slope of the short-run curve is greater than that of the long-run curve.

   (c) (10 pts) Let $q^*(p)$ and $q_S(p, \bar{z})$ denote the firm's long-run and short-run supply functions. Let $\bar{p}$ satisfy $q^*(\bar{p}) = \bar{q}$. Use the result of (b) to sketch an argument that at $\bar{p}$, the firm's price elasticity of supply is lower in the short run than it is in the long run.

5. (25 pts) Consider a society $N = \{1, \ldots, n\}$ and a finite set X of alternatives. Assume $n \geq 2$ and $\#X \geq 3$. Let $\mathfrak{R}$ be the set of all complete and transitive binary relations on X. For each preference profile $\vec{R} \in \mathfrak{R}^n$, define a binary relation $F(\vec{R})$ on X by
$$\forall x, y \in X : \; xF(\vec{R})y \;\Leftrightarrow\; xR_1y \text{ or } xR_2y.$$
   Answer the following questions, and prove your answers:

   (a) (5 pts) Is F dictatorial?

   (b) (5 pts) Does F satisfy Unanimity?

   (c) (5 pts) Does F satisfy Independence of Irrelevant Alternatives?

   (d) (5 pts) Is $C(B, \vec{R}) = \{x \in B : xF(\vec{R})y \; \forall y \in B\}$ nonempty for all $\vec{R} \in \mathfrak{R}^n$ and nonempty sets $B \subseteq X$?

   (e) (5 pts) Is F an (Arrow) Social Welfare Function?