Economics 681 **Steven A. Matthews**
Fall 2017 **University of Pennsylvania**

Suggested Solutions to Problem Set 5
Today's Date: November 10, 2017

1. JR Exercise 3.4

   Soln: Since x^1 is on the same ray from the origin as is x^0, $t \geq 0$ exists such that $x^1 = tx^0$. We must prove $t = f^{-1}(y)/f^{-1}(1)$. Now, since $f(g(x^0)) = 1$, we have

   $$g(x^0) = f^{-1}(1). \tag{1}$$

   Similarly, since $f(g(tx^0)) = y$, we have

   $$g(tx^0) = f^{-1}(y).$$

   This and the linear homogeneity of g imply

   $$tg(x^0) = f^{-1}(y).$$

   Lastly, this and (1) yield the desired result:

   $$t = \frac{f^{-1}(y)}{g(x^0)} = \frac{f^{-1}(y)}{f^{-1}(1)}.$$

   ■

2. Prove this part of JR's Theorem 3.4: If f is homogeneous of degree $\alpha > 0$, then

   $$c(w, y) = y^{1/\alpha} c(w, 1), \quad x(w, y) = y^{1/\alpha} x(w, 1).$$

   Soln: We obtain the first equation as follows:

   $$\begin{aligned}
   c(w, y) &= \min_x w \cdot x \text{ such that } \ f(x) \geq y \\
   &= \min_x w \cdot x \text{ such that } y^{-1} f(x) \geq 1 \\
   &= \min_x w \cdot x \text{ such that } f\left(y^{-1/\alpha} x\right) \geq 1 \ (\because \text{homogeneity}) \\
   &= \min_z w \cdot y^{1/\alpha} z \text{ such that } f(z) \geq 1 \left(\text{change variable to } z := y^{-1/\alpha} x\right) \\
   &= y^{1/\alpha} \left\{\min_z w \cdot z \text{ such that } f(z) \geq 1\right\} \\
   &= y^{1/\alpha} c(w, 1).
   \end{aligned}$$

   We obtain the second equation from Hotelling's lemma: for any i,

   $$x_i(w, y) = \frac{\partial c(w, y)}{\partial w_i} = \frac{\partial y^{1/\alpha} c(w, 1)}{\partial w_i} = y^{1/\alpha} x_i(w, 1).$$

   ■

3. JR Exercise 3.25

   Soln: We assume f has positive first derivatives, and consider prices (p, w) such that $y := y(p, w) > 0$. The FOC for maximizing profit using the cost function is then

   $$p = mc(y).$$

   Letting $x := x(p, w)$, the FOC for profit maximization using the production function directly is, for each i,

   $$pMP_i(x) \leq w_i \ (= \text{ if } x_i > 0).$$

   Substituting $mc(y)$ for p in this FOC yields the desired result:

   $$mc(y) \leq \frac{w_i}{MP_i(x)} \ (= \text{ if } x_i > 0).$$

   Think about the interpretation of this result. ∎

4. JR Exercise 3.34 (in (c), "shares" should be "cost shares," $s_i = w_i x_i(w, y)/c(w, y)$).

   (a) **Soln:** By Euler's theorem (JR page 564), $c(w, y)$ is homogeneous of degree 1 in w iff for all (w, y),

   $$\sum_k c_k(w, y) w_k = c(w, y).$$

   where $c_k = \partial c/\partial w_k$. Simplify notation by using ˆ for ln, as in $\hat{w}_k = \ln(w_k)$. Then, differentiating the expression that defines c with respect to w_k yields

   $$\frac{c_k(w, y)}{c(w, y)} = \frac{\alpha_k}{w_k} + \sum_j \gamma_{kj} \frac{\hat{w}_j}{w_k}.$$

   Hence, using the given restriction that $\sum_k \gamma_{kj} = 0$ for all j, we have

   $$\begin{aligned} \sum_k c_k(w, y) w_k &= \left(\sum_k \alpha_k + \sum_k \sum_j \gamma_{kj} \hat{w}_j \right) c(w, y) \\ &= \left(\sum_k \alpha_k + \sum_j \hat{w}_j \sum_k \gamma_{kj} \right) c(w, y) = \left(\sum_k \alpha_k \right) c(w, y). \end{aligned}$$

   We conclude that c is linearly homogeneous in w iff $\sum_k \alpha_k = 1$. ∎

   (b) **Soln:** The sufficient condition for c to take the Cobb-Douglas form is that $\gamma_{ij} = 0$ for all i, j. For, if this holds we have

   $$c(w, y) = \exp\left(\alpha_0 + \sum_i \alpha_i \hat{w}_i + \hat{y} \right) = \exp(\alpha_0) y \prod_i w_i^{\alpha_i},$$

   which is of the Cobb-Douglas form. ∎

(c) **Soln:** The input cost share for good i is given by

$$\begin{aligned} s_i &= \frac{w_i x_i(w,y)}{c(w,y)} \\ &= \frac{\partial c(w,y)}{\partial w_i}\frac{w_i}{c(w,y)} \; (\because \text{Hotelling's Lemma}) \\ &= \frac{\partial \ln(c(w,y))}{\partial \ln(w_i)} \\ &= \alpha_i + \sum_j \gamma_{ij}\ln(w_j) \; (\because \gamma_{ij} = \gamma_{ji}), \end{aligned}$$

which is linear in logs of input prices (and, trivially, output). ■

5. JR Exercise 3.36

Soln: Wlog, by appropriately choosing the units in which output is measured, we can assume the Cobb-Douglas production function is $f(x) = x_1^\alpha x_2^{1-\alpha}$ for some $\alpha \in (0,1)$. The long-run cost function is

$$C^L(w,y) = \min_{x_1,x_2 \geq 0} w_1x_1 + w_2x_2 \text{ such that } y \leq x_1^\alpha x_2^{1-\alpha}.$$

Assuming $y > 0$, the constraint implies that the solution satisfies $x \gg 0$, and the constraint clearly binds (as otherwise costs could be lowered by lowering each x_i). Hence, the FOCs are

$$\begin{aligned} w_1 &= \lambda\alpha x_1^{\alpha-1}x_2^{1-\alpha}, \\ w_2 &= \lambda(1-\alpha)x_1^\alpha x_2^{-\alpha}, \end{aligned}$$

where λ is the Lagrangian multiplier on the constraint. We thus have

$$\frac{w_1}{w_2} = \frac{\alpha}{1-\alpha}\frac{x_2}{x_1},$$

or $x_2 = \frac{1-\alpha}{\alpha}\frac{w_1}{w_2}x_1$. Substituting this into $y = x_1^\alpha x_2^{1-\alpha}$ yields

$$y = x_1^\alpha\left(\frac{1-\alpha}{\alpha}\frac{w_1}{w_2}x_1\right)^{1-\alpha} \Leftrightarrow x_1(w,y) = \left(\frac{1-\alpha}{\alpha}\frac{w_1}{w_2}\right)^{\alpha-1} y,$$

and so

$$x_2(w,y) = \left(\frac{1-\alpha}{\alpha}\frac{w_1}{w_2}\right)^{\alpha} y.$$

Thus,

$$C^L(w,y) = w_1x_1(w,y) + w_2x_2(w,y) = Kw_1^\alpha w_2^{1-\alpha}y,$$

where $K = (1-\alpha)^{\alpha-1}\alpha^{-\alpha}$. The long-run average and marginal cost functions are thus

$$\begin{aligned} lac(w,y) &= \frac{C^L(w,y)}{y} = Kw_1^\alpha w_2^{1-\alpha}, \\ lmc(w,y) &= \frac{\partial C^L(w,y)}{\partial y} = Kw_1^\alpha w_2^{1-\alpha}. \end{aligned}$$

We see that lac and lmc are constant in y and equal to each other.

The short-run cost function, for any fixed $x_2 = \bar{x}_2 > 0$, is

$$C^S(w, \bar{x}_2, y) = \min_{x_1 \geq 0} w_1 x_1 + w_2 x_2 \text{ such that } y \leq x_1^{\alpha} \bar{x}_2^{1-\alpha}.$$

Again we see that the constraint binds, and so $x_1 = \left(\bar{x}_2^{\alpha-1} y\right)^{1/\alpha}$. Substituting this into the objective function yields

$$C^S(w, \bar{x}_2, y) = w_1 \bar{x}_2^{1-\frac{1}{\alpha}} y^{1/\alpha} + w_2 \bar{x}_2.$$

The short-run average cost function is

$$sac(w, \bar{x}_2, y) = w_1 \bar{x}_2^{1-\frac{1}{\alpha}} y^{\frac{1}{\alpha}-1} + w_2 \bar{x}_2 y^{-1}.$$

Letting $y^{\min}$ be the y that minimizes $sac(w, \bar{x}_2, y)$, we find $y^{\min}$ from the FOC:

$$\begin{aligned}
\frac{\partial sac(w, \bar{x}_2, y)}{\partial y} = 0 \;\Leftrightarrow\; & \left(\frac{1}{\alpha} - 1\right) w_1 \bar{x}_2^{1-\frac{1}{\alpha}} y^{\frac{1}{\alpha}-2} - w_2 \bar{x}_2 y^{-2} = 0 \\
\Leftrightarrow\; & \left(\frac{1}{\alpha} - 1\right) w_1 \bar{x}_2^{1-\frac{1}{\alpha}} y^{\frac{1}{\alpha}} - w_2 \bar{x}_2 = 0 \\
\Leftrightarrow\; & y = \alpha^{\alpha}(1-\alpha)^{-\alpha} w_1^{-\alpha} w_2^{\alpha} \bar{x}_2 =: y^{\min}.
\end{aligned}$$

To complete the problem, we need only show that $sac(w, \bar{x}_2, y^{\min}) = lac(w, y^{\min})$:

$$\begin{aligned}
sac(w, \bar{x}_2, y^{\min}) &= w_1 \bar{x}_2^{1-\frac{1}{\alpha}} \left\{\alpha^{\alpha}(1-\alpha)^{-\alpha} w_1^{-\alpha} w_2^{\alpha} \bar{x}_2\right\}^{\frac{1}{\alpha}-1} + w_2 \bar{x}_2 \left\{\alpha^{\alpha}(1-\alpha)^{-\alpha} w_1^{-\alpha} w_2^{\alpha} \bar{x}_2\right\}^{-1} \\
&= \alpha^{1-\alpha}(1-\alpha)^{\alpha-1} w_1^{\alpha} w_2^{1-\alpha} + \alpha^{-\alpha}(1-\alpha)^{\alpha} w_1^{\alpha} w_2^{1-\alpha} \\
&= \left\{\alpha\alpha^{-\alpha}(1-\alpha)^{\alpha-1} + \alpha^{-\alpha}(1-\alpha)^{\alpha-1}(1-\alpha)\right\} w_1^{\alpha} w_2^{1-\alpha} \\
&= \alpha^{-\alpha}(1-\alpha)^{\alpha-1} \left\{\alpha + (1-\alpha)\right\} w_1^{\alpha} w_2^{1-\alpha} \\
&= \alpha^{-\alpha}(1-\alpha)^{\alpha-1} w_1^{\alpha} w_2^{1-\alpha} \\
&= K w_1^{\alpha} w_2^{1-\alpha} = lac(w, y^{\min}).
\end{aligned}$$

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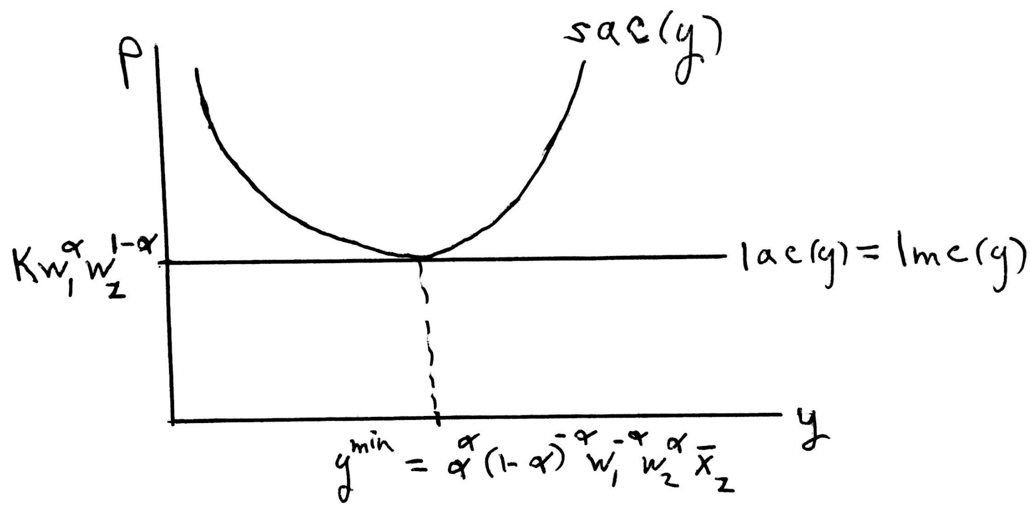
The figure below is for $\alpha \leq 1/2$ (for $a > 1/2$, $sac(y)$ becomes concave as $y \to \infty$).



Vertical = cost per unit output, Horizontal = output

6. A competitive firm has a C^2 production function $f(x_1, x_2)$ for which, at any $x \in \mathbb{R}^2_+$, $\nabla f(x) \gg 0$ and $J(x) := [f_{ij}(x)]$ is negative definite. Let $x(p,w)$ and $y(p,w)$ be the firm's demand and supply functions. Using the first-order conditions for profit maximization, show that at any $(p,w) \gg 0$ for which the the firm's input demands are positive, we have

   (a) $\partial y/\partial p > 0$ (Strict Law of Supply),

   (b) $\partial x_1/\partial p > 0$ or $\partial x_2/\partial p > 0$, and

   (c) $\partial x_i/\partial w_i < 0$ (Strict Law of Demand).

   [Hint: The inverse of a ND matrix is also ND.]

   Soln: Since $y(p,w) = f(x(p,w))$, we have

$$\frac{\partial y}{\partial p} = f_1 \frac{\partial x_1}{\partial p} + f_2 \frac{\partial x_2}{\partial p} = \nabla f \cdot \frac{\partial x}{\partial p}. \tag{2}$$

   Hence, since $\nabla f(x) \gg 0$, (b) follows immediately from (a). It remains to prove (a) and (b). To do so, we use the FOC conditions,

$$\begin{aligned} pf_1(x(p,w)) &= w_1, \\ pf_2(x(p,w)) &= w_2. \end{aligned}$$

   To prove (a), differentiating with respect to p, and dropping the arguments of the functions to improve readability, yields

$$\begin{aligned} f_1 + pf_{11}\frac{\partial x_1}{\partial p} + pf_{12}\frac{\partial x_2}{\partial p} &= 0, \\ f_2 + pf_{21}\frac{\partial x_1}{\partial p} + pf_{22}\frac{\partial x_2}{\partial p} &= 0, \end{aligned}$$

   or rather,

$$\left[p \begin{matrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{matrix} \right] \begin{pmatrix} \partial x_1/\partial p \\ \partial x_2/\partial p \end{pmatrix} = \begin{pmatrix} -f_1 \\ -f_2 \end{pmatrix},$$

   which can be written even more parsimoniously as

$$pJ\frac{\partial x}{\partial p} = -(\nabla f).$$

   We can solve this system of equations for the vector $\partial x/\partial p$ by dividing by p and left-multiplying both sides by the matrix J^{-1} to obtain

$$\frac{\partial x}{\partial p} = -\frac{1}{p} J^{-1}(\nabla f).$$

   Now multiply both sides on the left by the transpose of ∇f to obtain

$$\nabla f \cdot \frac{\partial x}{\partial p} = -\frac{1}{p}(\nabla f)^T J^{-1}(\nabla f).$$

   Thus, in light of (2), we have

$$\frac{\partial y}{\partial p} = -\frac{1}{p}(\nabla f)^T J^{-1}(\nabla f) > 0,$$

where the inequality holds because J^{-1} is ND, $\nabla f \neq 0$, and $p > 0$.

We prove (c) similarly, and just for $i = 1$ (the proof for $i = 2$ is the same). Differentiating the FOC with respect to w_1 again yields two linear equations that can be solved for $\partial x/\partial w_1 = (\partial x_1/\partial w_1, \partial x_2/\partial w_1)^T$. Doing so yields

$$\frac{\partial x}{\partial w_1} = \frac{1}{p} J^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{p} \begin{pmatrix} a \\ c \end{pmatrix},$$

where a is the $(i, j) = (1, 1)$ element of J^{-1} and c is the $(i, j) = (2, 1)$ element. Hence,

$$\frac{\partial x}{\partial w_1} = \frac{1}{p} a < 0,$$

where the inequality holds because $a < 0$, since J^{-1} is ND, and $p > 0$. ■