Solutions to Exam 2

60 points, 75 minutes. Closed books, notes, calculators.
Show your reasoning. ***Read all before answering any.***

1. (10 pts) A risk averse consumer satisfies the expected utility hypothesis. She has initial income w, and will suffer a monetary loss $0 < D < w$ with probability $\pi \in (0,1)$. She can buy insurance that costs q dollars for each dollar of coverage.

   Show, without using calculus, that she will demand full insurance if the insurance rate is actuarily fair. (No credit will be given for proofs that use calculus.)

   Soln: Letting u be her Bernoulli utility function, it must be concave because she is risk averse. Let α be a coverage she demands, i.e., a coverage that solves the problem
   $$\max_{\alpha} \pi u(w - D - q\alpha + \alpha) + (1 - \pi)u(w - q\alpha).$$
   As the insurance rate is actuarily fair, $q = \pi$, so the problem is
   $$\max_{\alpha} \pi u(w - D - \pi\alpha + \alpha) + (1 - \pi)u(w - \pi\alpha).$$
   Since u is concave, Jensen's inequality tells us that for any α,
   $$\begin{aligned} \pi u(w - D - \pi\alpha + \alpha) &+ (1 - \pi)u(w - \pi\alpha) \\ &\leq u\left[\pi(w - D - \pi\alpha + \alpha) + (1 - \pi)(w - \pi\alpha)\right] \\ &= u(w - \pi D). \end{aligned}$$
   She achieves this upper bound if she buys full coverage, $\alpha = D$, since
   $$\pi u(w - D - \pi D + D) + (1 - \pi)u(w - \pi D) = u(w - \pi D).$$
   Thus, $\alpha = D$ solves her expected utility maximization problem. ■

2. (10 pts) Let $f : \mathbb{R}^K_+ \to \mathbb{R}_+$ be a concave production function. Show that the associated cost function $c(w, q)$ is convex in q.

   Soln: Let $q_1, q_2 \in \mathbb{R}$, $\lambda \in [0, 1]$, and for each i,
   $$z_i \in \arg\min_{z \geq 0} w \cdot z \text{ such that } f(z) \geq q_i.$$
   Since f is concave, we have
   $$\begin{aligned} f(\lambda z_1 + (1 - \lambda)z_2) &\geq \lambda f(z_1) + (1 - \lambda)f(z_2) \\ &\geq \lambda q_1 + (1 - \lambda)q_2. \end{aligned}$$
   That is, the input vector $\lambda z_1 + (1 - \lambda)z_2$ is in the feasible set of the problem of minimizing cost subject to producing at least $\lambda q_1 + (1 - \lambda)q_2$. Hence,
   $$\begin{aligned} c(w, \lambda q_1 + (1 - \lambda)q_2) &\leq w \cdot (\lambda z_1 + (1 - \lambda)z_2) \\ &= \lambda(w \cdot z_1) + (1 - \lambda)(w \cdot z_2) \\ &= \lambda c(w, q_1) + (1 - \lambda)c(w, q_2). \end{aligned}$$
   So c is convex in q. ■

3. (20 pts) A consumer of L goods has a locally non-satiated utility function. She has a C^1 Marshallian demand function $x(p,m)$, and all her income-expansion paths are upward-sloping rays.

   (a) (10 pts) Show that her demand function takes the form $x(p,m) = m\hat{x}(p)$.

   Soln: For any p, let $\hat{x}(p) = x(p,1)$. The income-expansion path through the point $\hat{x}(p) \in \mathbb{R}^L_+$ is a ray. The definition of a ray implies *(only)* that for any $m \geq 0$, there exists $\lambda(m,p) \in \mathbb{R}_+$ such that $x(p,m) = \lambda(m,p)\hat{x}(p)$. Walras' law holds because of local non-satiation. Hence, since $\hat{x}(p) = x(p,1)$, we have

$$\begin{aligned}\lambda(m,p) &= \lambda(m,p)\left(p \cdot \hat{x}(p)\right) \\ &= p \cdot \left(\lambda(m,p)\hat{x}(p)\right) \\ &= p \cdot x(p,m) = m.\end{aligned}$$

   Thus, $x(p,m) = m\hat{x}(p)$ for all $m \geq 0$ ■

   (b) (10 pts) Show that $x(p,m)$ satisfies the Law of Reciprocity.

   Soln: From the Slutsky equation, the symmetry of the cross-partials of the Hicksian demand function, and (a), we obtain

$$\begin{aligned}\frac{\partial x_i}{\partial p_j} &= \frac{\partial h_i}{\partial p_j} - x_j \frac{\partial x_i}{\partial m} \\ &= \frac{\partial h_j}{\partial p_i} - \left(m\hat{x}_j\right)\left(\hat{x}_i\right) \\ &= \frac{\partial h_j}{\partial p_i} - \left(m\hat{x}_i\right)\left(\hat{x}_j\right) \\ &= \frac{\partial h_j}{\partial p_i} - x_i \frac{\partial x_j}{\partial m} = \frac{\partial x_j}{\partial p_i},\end{aligned}$$

   where the argument of the x terms is (p,m), that of the h terms is $(p, v(p,m))$, and that of the $\hat{x}$ terms is p. ■

4. (20 pts) Consider a choice structure $\langle \mathfrak{B}, C \rangle$ on a set X, where $\mathfrak{B}$ contains all subsets of X of size three.

   (a) (5 pts) Define the Weak Axiom of Revealed Preference (WARP).

   Soln: $\langle \mathfrak{B}, C \rangle$ satisfies WARP iff

$$\begin{gathered}\text{for all sets } A, B \in \mathfrak{B},\ x \in C(A) \text{ and } y \in A, \\ \text{if } y \in C(B) \text{ and } x \in B, \text{ then } x \in C(B).\end{gathered}$$

   ■

   (b) (5 pts) Define the revealed preference relation $\succeq^*$.

   Soln: For all $x, y \in X$,

$$x \succeq^* y \text{ iff for some } B \in \mathfrak{B},\ x \in C(B) \text{ and } y \in B.$$

   (That's MWG's definition. One could add to it, as I did in class, the reflexivity requirement: $x \succeq^* x$ for all $x \in X$.) ■

(c) (10 pts) Show that $\succeq^*$ is transitive if $\langle \mathfrak{B}, C \rangle$ satisfies WARP.

Soln: Suppose that for some x, y, z we have $x \succeq^* y$ and $y \succeq^* z$. We need to show $x \succeq^* z$. Let $B = \{x, y, z\}$. Then $x \succeq^* z$ if $x \in C(B)$, which we now show.

By assumption, $B \in \mathfrak{B}$, so $C(B)$ is well defined. By our definition of a choice structure, $C(B)$ is nonempty. So one of the following cases must hold.

Case 1: $x \in C(B)$. In this case we are done.

Case 2: $y \in C(B)$. In this case $x \succeq^* y$ and WARP imply $x \in C(B)$.

Case 3: $z \in C(B)$. In this case $y \succeq^* z$ and WARP imply $y \in C(B)$. So Case 2 holds, and again $x \in C(B)$.

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