**Economics 681** **Steven A. Matthews**
**Fall 2017** **University of Pennsylvania**

## Problem Sets

Today's Date: November 26, 2017

### Problem Set 1 (Due Wednesday, 9/13)

1. A competitive market has $n$ buyers, each one with the same downward-sloping demand function $d(p)$. Thus, the market demand function is $nd(p)$. The market supply function is $S(p)$, an upward-sloping function. Assume the functions $d$ and $S$ are both differentiable. Let $p^*(n)$ and $x^*(n)$ be the equilibrium price and quantity given a fixed $n$. Determine their comparative static properties. What further assumptions do you need to make?

JR Exercises 1.2, 1.4, 1.7, 1.9

2. For $n = 1, 2$, define $\succsim_n$ on $\mathbb{R}^n_+$ by $x \succsim y$ iff $x \geq y$.[1] Determine whether $\succsim_1$ and $\succsim_2$ are complete, transitive, continuous, convex, and strictly convex.

### Problem Set 2 (Due Wednesday 9/27)

1. Show that if $u : \mathbb{R}^2_+ \to \mathbb{R}$ is a $C^2$ quasiconcave utility function, with $u_1 > 0$ and $u_2 > 0$, then its indifference curves slope down and exhibit (weakly) diminishing marginal rates of substitution. [Hint: Show that $g' < 0$ and $g'' \geq 0$, where the function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is defined, for a given $\bar{u}$, by $u(x_1, g(x_1)) = \bar{u}$.]

2. Preferences are called *homothetic* if they satisfy the following property:

$$x \succsim y \Rightarrow \alpha x \succsim \alpha y \quad \forall \alpha \geq 0$$

   Suppose $\succsim$ is a complete, transitive, strictly monotonic, continuous preference relation on $\mathbb{R}^L_+$. Show that $\succsim$ is homothetic if and only if there exists a utility representation $u$ such that $u(\alpha x) = \alpha u(x)$ for all $\alpha \geq 0$.

3. JR Exercise 1.29

4. Find the demand and indirect utility functions for these utility functions:

   (a) $u(x) = x_1 + x_2$
   (b) $u(x) = \ln x_1 + \ln x_2$
   (c) $u(x) = e^{x_1 x_2}$
   (d) $u(x) = \sqrt{x_1} + x_2$

5. JR Exercise 1.47

6. JR Exercise 1.54.

[1] Recall our convention for vector inequalities: $x \geq y$ means $x_i \geq y_i$ $\forall i$; $x > y$ means $x \geq y$ and $x \neq y$; and $x > y$ means $x_i > y_i$ $\forall i$.

**Problem Set 3 (Due Wednesday 10/18)**

1. JR Exercise 1.63 (Assume the demand function is $C^1$)

2. A consumer in a three-good economy with wealth level $y > 0$ is maximizing locally non-satiated preferences on $\mathbb{R}^3$ and has demand functions for goods 1 and 2 given by:

$$x_1(p,y) = 100 - 5\frac{p_1}{p_3} + \beta\frac{p_2}{p_3} + \delta\frac{y}{p_3}$$
$$x_2(p,y) = \alpha - 5\frac{p_1}{p_3} + \beta\frac{p_2}{p_3} + \gamma\frac{y}{p_3}.$$

   (a) Calculate the demand for good 3.

   (b) Verify that $x_1$ and $x_2$ are homogeneous of degree 0.

   (c) What conditions on $\alpha$, $\beta$, $\delta$ and $\gamma$ are implied by demand theory?

3. Three candidates are running for mayor of Bucolia. The big issue is how will Bucolia collect revenue to pay for the operation of its new fitness center. *H. Economicus*, a citizen of Bucolia, is deciding how to cast his vote. All three candidates have proposed combinations of *mandatory* once-only membership fees (head taxes) and per-use fees. The proposals of candidates $A$, $B$, and $C$, respectively, are as follows:

   (a) \$0 membership fee, \$5 use fee.

   (b) \$20 membership fee, \$3 use fee.

   (c) \$40 membership fee, \$2 use fee.

   H. Economicus would use the facility 10 times if $A$ wins; 15 times if $B$ wins, and 20 times if $C$ wins. Assuming H. Economicus is a sincere voter, for whom will he vote? How does he rank the remaining two candidates? Could any candidate make him worse off than if the facility is not provided at all? Prove your answers.

4. JR Exercise 4.20

**Problem Set 4 (Due Monday 10/30)**

1. JR Exercise 2.19

2. JR Exercise 2.24

3. JR Exercise 2.26, restricting attention to values of $w$ and $b$ satisfying $w < b$.

4. JR Exercise 2.27

5. JR Exercise 2.36

6. My vNM utility function is strictly increasing and satisfies $u(0) = 0$, $u(\$300) = \frac{1}{2}$, and $\lim_{w\to\infty} u(w) = 1$. Consider a gamble $g = (\frac{1}{2} \circ 0, \frac{1}{2} \circ x)$, where $x$ is a prize in dollars. How large must $x$ be in order for me to prefer this gamble to one in which I receive \$400 for sure?

7. A consumer may invest in a risky asset that has a random *gross* return $\tilde{r}$, with $\mathbb{E}\tilde{r} = 1$. Her expected utility when she invests $x$ in the asset is

$$\mathbb{E}u(w + \tilde{r}x - x).$$

Show, without using calculus, that if the consumer is risk averse, she will not invest in the asset.

8. When the consumer has non-random wealth $w$, define his risk premium, $\pi(w)$, for a gamble $\tilde{x}$ by

$$\mathbb{E}u(\tilde{x} + w) = u\left(\mathbb{E}\tilde{x} + w - \pi(w)\right).$$

Thus, the consumer is willing to pay at most $\pi(w)$ to exchange the gamble for its expected value $\mathbb{E}\tilde{x}$. Assume $u$ is $C^2$, with $u' > 0$ and $u'' < 0$. Show that if $u$ exhibits DARA, then the risk premium decreases in wealth.

9. A consumer has wealth $w$ that she must consume over two periods. The only way to transfer wealth to or from period 2 is through buying or selling a risky asset with returns $\theta\tilde{r}$, where $\theta > 0$, $\mathbb{E}\tilde{r} = 0$, and $\mathbb{E}\tilde{r}^2 > 0$. Her expected utility when she chooses to save an amount $x$ is

$$u(w - x) + \mathbb{E}v(\theta\tilde{r}x).$$

Suppose $x$ can be any real number, and that $u$ and $v$ are $C^3$ with strictly positive first derivatives and strictly negative second derivatives. Let $x^* = x^*(w, \theta)$ be her optimal savings function.

   (a) Does $x^*$ increase or decrease in $w$, or can it do either?

   (b) Is $x^*$ always positive, always negative or neither?

   (c) Sign the derivative $x^*_\theta$.

**Problem Set 5 (Due Monday 11/6)**

1. JR Exercise 3.4

2. Prove this part of JR's Theorem 3.4: If $f$ is homogeneous of degree $\alpha > 0$, then

$$c(w, y) = y^{1/\alpha}c(w, 1), \quad x(w, y) = y^{1/\alpha}x(w, 1).$$

3. JR Exercise 3.25

4. JR Exercise 3.34 (in (c), "shares" should be "cost shares," $s_i = w_i x_i(w, y)/c(w, y)$)

5. JR Exercise 3.36

6. A competitive firm has a $C^2$ production function $f(x_1, x_2)$ for which, at any $x \in \mathbb{R}^2_+$, $\nabla f(x) \gg 0$ and $J(x) := [f_{ij}(x)]$ is negative definite. Let $x(p, w)$ and $y(p, w)$ be the firm's demand and supply functions. Using the first-order conditions for profit maximization, show that at any $(p, w) \gg 0$ for which the firm's input demands are positive, we have

   (a) $\partial y/\partial p > 0$ (Strict Law of Supply),

   (b) $\partial x_1/\partial p > 0$ or $\partial x_2/\partial p > 0$, and

(c) $\partial x_i / \partial w_i < 0$ (Strict Law of Demand).

[Hint: The inverse of a ND matrix is also ND.]

**Problem Set 6 (Not Graded)**

1. JR Exercise 3.55

2. JR Exercise 4.3

3. An industry has the demand curve $D(p) = A - p$. Each of a very large number of potential firms has the same cost function in the long run as in the short run, and it is

$$c(q) = \begin{cases} q + q^2 + 9 & \text{if } q > 0 \\ 0 & \text{if } q = 0. \end{cases}$$

   (a) For $A = 28$, find the long-run competitive equilibrium price, output per operating firm, and number of operating firms.

   (b) Now the demand curve shifts up in the sense that $A$ increases to 67. In the short run, the number of firms is fixed at the number you found in (a). Find the new short-run equilibrium price and per-firm output.

   (c) Now find the new long-run equilibrium for $A = 67$.

4. JR Exercise 4.7, (a) and (b), assuming $a > 0$ and $b > 0$ (not what the book says!)

5. A monopoly has cost function $c(q) = 6q$. The output $q$ is consumed only by consumers $a$ and $b$. Their demand functions for $q$ are $D_a(p) = 10 - p$ and $D_b(p) = 20 - p$, respectively.

   (a) Find the industry demand function $D(p)$; the inverse demand function $P(Q)$; the revenue function $R(Q)$; and the marginal revenue function $R'(Q)$.

   (b) Find the monopoly output $Q^M$ and price $p^M$.

   (c) Suppose now that the firm can practice price discrimination, i.e., charge a price $p_a$ to consumer $a$ and a price $p_b$ to consumer $b$. Find the firm's optimal prices, $p_a^*$ and $p_b^*$, and quantitities $q_a^*$ and $q_b^*$. Who is better off, and who is worse off, relative to the uniform-price solution in (b)?

**Problem Set 7 (Due Monday, 12/11)**

1. A monopoly has cost function $c(q) = 6q$. There is only one price-taking (!) consumer in the market for this good. Her utility function is $u(q) + x$, where $x$ is the amount of income she spends on all other goods – view $x$ as a single good with price 1. Assume also that $u(0) = 0$, and that the consumer's income, $y$, is large enough that for any relevant price $p$ of $q$, her Marshallian demand for good $x$, $x(p, 1, y)$, is positive. Her demand function for $q$ is given by $D(p) = 10 - p$ for $p < 10$, and by $D(p) = 0$ for $p \geq 10$.

   (a) Find the function $u$.

(b) A two-part tariff consists of a fixed fee $f$ the consumer must pay in order to purchase the good $q$, and a price $p$ that must be paid per unit purchased. So, if she purchases an amount $q > 0$, she pays $pq + f$. Find the firm's profit-maximizing two-part tariff, $(f, p)$.

2. Alice and Bob are the sole consumers in two competitive markets for goods 1 and 2. Their utility functions are

$$u^A(x_1^A, x_2^A) = \ln x_1^A + x_2^A \quad \text{and} \quad u^B(x_1^B, x_2^B) = \ln x_1^B + x_2^B,$$

respectively. Their endowments are $\omega^A = (2, 0)$ and $\omega^B = (1, 2)$.

(a) Find an equation for the part of the contract curve that is in the interior of the Edgeworth box, and graph the entire contract curve.

(b) Find the Walrasian equilibrium allocation $x$ and price vector $(p_1, 1)$ (good 2 is the numeraire).

3. JR Exercise 5.11

4. JR Exercise 5.15

5. Jane will live $T$ years. Her lifetime consumption bundle is $x = (x_1, ..., x_T)$, where $x_t = (x_{t1}, ..., x_{tn})$ and $n$ is the number of goods. Her utility function for lifetime consumption is $u(x)$. Jane has inherited a computer company that can generate any profit stream $y = (y_1, ..., y_T)$ contained in some compact feasible set $Y \subset \mathbb{R}_+^T$. Each year Jane uses the profits $y_t$ to purchase goods and to add or subtract from her savings account, which collects interest at the annual rate $r$. In year $t$ her savings are $s_t = (1 + r)s_{t-1} + y_t - p_t \cdot x_t$ (negative savings indicate borrowing). (Set $s_0 = 0$.) Jane has *perfect foresight,* meaning that she knows what all future prices will be at the time she is born. She plans her entire life at the moment she is born, choosing $x$, $y$, and $s = (s_1, ..., s_T)$ subject to $y \in Y$ and the constraint that she not die in debt: $s_T \geq 0$.

(a) Jane does not want to operate the company herself. Instead, she wants to tell a manager a rule for choosing $y$ without revealing any private details about herself, such as her utility function. Can she do this and still maximize her lifetime utility? If so, what rule should she tell her manager? Prove your answer.

(b) Let $n = 1$ and $T = 2$. Derive a Slutsky-Hicks expression for $\partial x_1^*/\partial r$ in terms of the first-year savings, $s_1^*$. Explain the intuition.

6. Consider an $I$-person, one-good, $S$-state exchange economy. Consumer $i$ has the utility function $U^i(x^i) = \sum_{s=1}^S \pi_s u_i(x_s^i)$, where $(\pi_1, \ldots, \pi_S) \gg 0$ is the probability vector reflecting their common beliefs about the states. The aggregate endowment of the good in state $s$ is $\bar{\omega}_s$, and these endowments are ordered $\bar{\omega}_s > \bar{\omega}_{s+1}$ for all $s = 1, \ldots, S-1$. Assume the functions $u_i$ have derivatives $u_i' > 0$ and $u_i'' < 0$.

(a) Prove the *co-monotonicity property*: in any interior Pareto efficient allocation, each consumer obtains more of the good in states in which the aggregate endowment is greater.

(b) For $I = S = 2$, what does this result imply about the location of the contract curve in the Edgeworth box? Graph it.

7. Two farmers must decide how many cattle to graze on the common. Each cow can be sold for $p$ at the end of the season. Each farmer's cost of raising cows increases with the number of cows he raises, and also with the number of cows the other farmer raises because of the lower density of grass (food) on the common due to the grazing of the other farmer's cows. In particular, their cost functions are

$$C_1(q_1, q_2) = (2q_1 + q_2)\, q_1 \quad \text{and} \quad C_2(q_1, q_2) = (q_1 + 2q_2)\, q_2,$$

where $q_i$ is the number of cows raised by farmer $i$ (we allow for fractional cows).

(a) What is the Nash equilibrium $(q_1^*, q_2^*)$ of this game?

(b) What outputs would maximize total profit?

8. Each of two players, $i = 1, 2$, can contribute any any amount $x_i \geq 0$ of a private good to the production of a public good $y$. Given a contribution vector $x = (x_1, x_2)$, the amount of public good produced is given by the production function, $y = 2\sqrt{x_1 + x_2}$. Player $i$'s resulting utility is $u_i(y, x_i) = iy - x_i$.The players make their contributions simultaneously.

(a) Find the Nash equilibrium, $x^* = (x_1^*, x_2^*)$, of this game.

(b) Show that for any $a > 1$, there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$, players 1 and 2 will be made better off, relative to the equilibrium you just found, if they are forced to increase their contributions above $x^*$ by $\varepsilon$ and $a\varepsilon$, respectively.