Econ 681
Introduction to GE with Dates and States
JR 5.3 and More

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Arrow-Debreu Exchange Economy

The Arrow-Debreu GE model builds on the state-preference consumer model in order to incorporate uncertainty, and it adds a time dimension. Despite incorporating time and uncertainty, it is essentially a static model. All trade occurs at the initial date.

I people, n goods, $S+1$ states, $t=0,1$ dates

$e^i_{\ell s}=$ consumer i's $t=1$ endowment of good ℓ in state s

$e^i_{\ell 0}=$ consumer i's $t=0$ endowment of good ℓ
$e^i=(e^i_0,e^i_1,\dots,e^i_S)\in\mathbb{R}^{n(S+1)}_+$ consumer i's endowment vector

$\bar{e}_s\in\mathbb{R}^n_+$ aggregate endowment vector in state $s=0,\dots,S$

$x^i=(x^i_0,x^i_1,\dots,x^i_S)\in\mathbb{R}^{n(S+1)}_+$ consumer i's consumption bundle

$U^i:\mathbb{R}^{n(S+1)}_+\to\mathbb{R}$ consumer i's utility function.

Definition. For any state s and good ℓ, one unit of the ℓs **state contingent commodity** is a contract promising its owner the delivery of one unit of good ℓ when and if state s is realized.

Each of the Sn state contingent commodities is bought and sold in a competitive market at date $t = 0$. There are no markets at $t = 1$.

- There must be agreement on what the set of states is, and each agent must be able to verify which state is realized. (Complete information.)
- There is no presumption that any agent satisfies the EUH, or if some do, that they agree on the state probabilities.

Definition. An **Arrow-Debreu Equilibrium** is an allocation and a vector of state contingent prices, $x^* \in \mathbb{R}_+^{In(S+1)}$ and $p \in \mathbb{R}_+^{n(S+1)}$, such that

- Consumer maximization: each x^{i*} solves the problem

$$\max_{x^i} U(x^i) \text{ such that } \sum_{\ell,s} p_{\ell s} x^i_{\ell s} \leq \sum_{\ell,s} p_{\ell s} e^i_{\ell s}$$

- Market clearing: For each $s = 0, \ldots, S$,

$$\sum_i x_s^{i*} = \bar{e}_s$$

This is exactly the same object as the Walrasian equilibrium we discussed before. So we have the same Big Three Theorems (ET, FFWT, and SFWT). Similarly, all results extend to production economies.

Efficiency in Arrow-Debreu given EUH, State-Independent Utility, and Common Beliefs

Proposition. In our AD exchange economy, assume the following hold:

- (EUH and common beliefs) A strictly positive probability vector $(\pi_1, \ldots, \pi_S)$ exists such that for each i, U_i takes the form

$$U^i(x^i) = \sum_{s=1}^{S} \pi_s u_i(x_0^i, x_s^i);$$

- (Risk aversion) Each function u_i is strictly concave in its second argument; and

- (No aggregate uncertainty) $\bar{e}_s = \bar{e}_{s'}$ for all $s, s' > 0$.

Then no consumer bears any risk in any Pareto efficient allocation.
(Calculus proof on blackboard, for case $I = S = 2$, $n = 1$.)

Proof. Let x be a feasible allocation such that for some consumer j, it is not true that $x_s^j = x_{s'}^j$ for all s,$s' > 0$. For each consumer i, define

$$y^i = \sum_{s=1}^{S} \pi_s x_s^i \in \mathbb{R}_+^n, \qquad z^i = (x_0^i, y^i, \ldots, y^i) \in \mathbb{R}_+^{n(S+1)}.$$

The absence of aggregate uncertainty implies that $z = (z^1, \ldots, z^I)$ is feasible:

$$\sum_i y^i = \sum_i \sum_{s=1}^{S} \pi_s x_s^i = \sum_{s=1}^{S} \pi_s \sum_i x_s^i \leq \sum_{s=1}^{S} \pi_s \bar{e}_s = \bar{e}_s \ \forall s > 0.$$

Jensen's inequality implies z Pareto dominates x : for each i,

$$\begin{aligned} \sum_{s=1}^{S} \pi_s u_i(x_0^i, y^i) &= u_i(x_0^i, y^i) = u_i(x_0^i, \sum_{s=1}^{S} \pi_s x_s^i) \\ &\geq \sum_{s=1}^{S} \pi_s u_i(x_0^i, x_s^i) = \sum_{s=1}^{S} \pi_s u_i(x_s^i), \end{aligned}$$

and the inequality is strict for $i = j$. So x is not Pareto efficient. ■

Equilibrium in Arrow-Debreu given EUH and Common Beliefs

Example. Find the equilibrium (x, p) in the following case: $I = S = 2$, $n = 1$, $e^1 = (0, 3, 0)$, $e^2 = (0, 0, 3)$, common beliefs (π_1, π_2), and Bernoulli utility functions $u_i(x_1, x_2)$ that are strictly concave with $\partial u_i / \partial x_s > 0$.

Solution:

- By the FWT and the previous proposition, $x_1^1 = x_2^1 =: y^1$.
- From consumer 1's FOC we obtain

$$\frac{\pi_1 u_1'(x_1^1)}{\pi_2 u_1'(x_2^1)} = \frac{p_1}{p_2} \quad \Rightarrow \quad \frac{\pi_1}{\pi_2} = \frac{p_1}{p_2}$$

  Thus, normalizing prices so that $p_1 + p_2 = 1$, we have $p_s = \pi_s$.
- From her binding budget constraint we have

$$p_1 x_1^1 + p_2 x_2^1 = 3p_1 \quad \Rightarrow \quad y^1 = 3\pi_1$$

- Hence, the equilibrium is $p = (\pi_1, \pi_2)$ together with

$$x^1 = (3\pi_1, 3\pi_1), \quad x^2 = (3 - 3\pi_1, 3 - 3\pi_1) = (3\pi_2, 3\pi_2)$$

Arrow Securities Economy

Arrow gave us a real dynamic GE model: only spot markets and asset markets. The asset markets allow transfers of wealth between states (insurance) and dates (savings). Each spot market takes place at a give date and state.

- Same exchange environment.
- At $t = 0$, assets can be bought and sold. **Arrow security** s is an asset that will pay one unit of good 1 (say) at $t = 1$ if state s is realized, and nothing if any other state is realized.
- $q_s =$ price of Arrow security s
- $z_s^i =$ amount of Arrow security s that consumer i buys; no endowments of securities
- $p_{\ell s} =$ **spot price** of good ℓ in state $s = 0, \ldots, S$
- Consumer problem:

$$\max_{x^i \geq 0, z^i} U(x^i) \text{ such that } (i)\ p_0 \cdot x_0^i + q \cdot z^i \leq p_0 \cdot e_0^i,$$

$$(ii)\ p_s \cdot x_s^i \leq p_s \cdot e_s^i + p_{1s} z_s^i \ \forall s > 0$$

Radner Equilibrium in the Arrow Securities Economy

Definition. (x^*, z^*, p, q) is a **Radner Equilibrium** of the Arrow securities economy iff (x^{i*}, z^{i*}) solves the problem of consumer i for all i given the spot prices p and Arrow security prices q, and the markets clear:

$$\sum_i z_s^{i*} = 0 \text{ for all } s > 0,$$

$$\sum_i x_s^{i*} = e_s \text{ for all } s \geq 0.$$

Theorem. x^* is the allocation achieved by an Arrow-Debreu equilibrium if and only if it is the allocation achieved by a Radner equilibrium of the Arrow securites economy.

Question: Which is less realistic, that all trade occurs at date 0, or that all consumers perfectly foresee the future spot prices p_s?

Example

- $I = S = n = 2$, $e_0^i = 0$ for $i = 1, 2$, and EUH and common beliefs:

$$(\pi_1, \pi_2) = (\tfrac{1}{2}, \tfrac{1}{2})$$

- Consumer 1 has all the endowment in state 1, and consumer 2 has all the endowment in state 2:

$$e_1^1 = e_2^2 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \quad e_2^1 = e_1^2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Note that there is no aggregate uncertainty
- Draw figure

Incomplete Markets

In the Arrow securities model, there is one asset for each state, and their return vectors are linearly independent. Realistic? What if there are fewer than S assets?
Example. $I = S = 2$, $n = 1$. The endowments are

$$e^1 = (1, 2, 1), \quad e^2 = (1, 1, 2).$$

The utility functions are the same:

$$U^i(x_0^i, x_1^i, x_2^i) = \log x_0 + \tfrac{1}{2} \log x_1^i + \tfrac{1}{2} \log x_2^i.$$

There is only one asset, a riskless bond: 1 unit of it pays 1 unit of good 1 in every state $s > 0$. Denote its price as q.

- **Claim 1.** Radner equilibrium allocation is $x = e$.
  Proof. Can normalize one price at each date-state. So can set $p_0 = p_1 = p_2 = 1$.

  $$\max_{z, x_0, x_1, x_2} U_1(x_0, x_1, x_2) \text{ s.t. } x_0 + qz \leq 1,\ x_1 \leq 2 + z,\ x_2 \leq 1 + z$$

  Let $z^1(q)$ be the resulting demand function for the bond. Note that consumer 2's is the same: $z^2(q) = z^1(q)$. So market clearing implies $z^{i*} = 0$ for both i. Hence, $x^* = e$.
- **Claim 2.** x^* is Pareto dominated by $x^1 = x^2 = (1, 1.5, 1.5)$.