Portfolios of Two Risky Assets

- The return on portfolio P, which combines the risky assets A and B, is

$$r_P = w_A r_A + (1 - w_A) r_B, \tag{1}$$

where w_A is the fraction of the portfolio invested in asset A.

- Assume, initially, that

$$0 \leq w_A \leq 1. \tag{2}$$

That is, neither asset can be sold short. (This will be relaxed later.)

- The expected return (mean) of portfolio P is given by

$$E_P = w_A E_A + (1 - w_A) E_B \tag{3}$$

- The portfolio's variance is given by

$$\sigma_P^2 = w_A^2 \sigma_A^2 + (1 - w_A)^2 \sigma_B^2 + 2 w_A (1 - w_A) \underbrace{\sigma_A \sigma_B \rho_{AB}}_{\text{COV}\{r_A, r_B\}} \tag{4}$$

Perfectly Correlated Assets

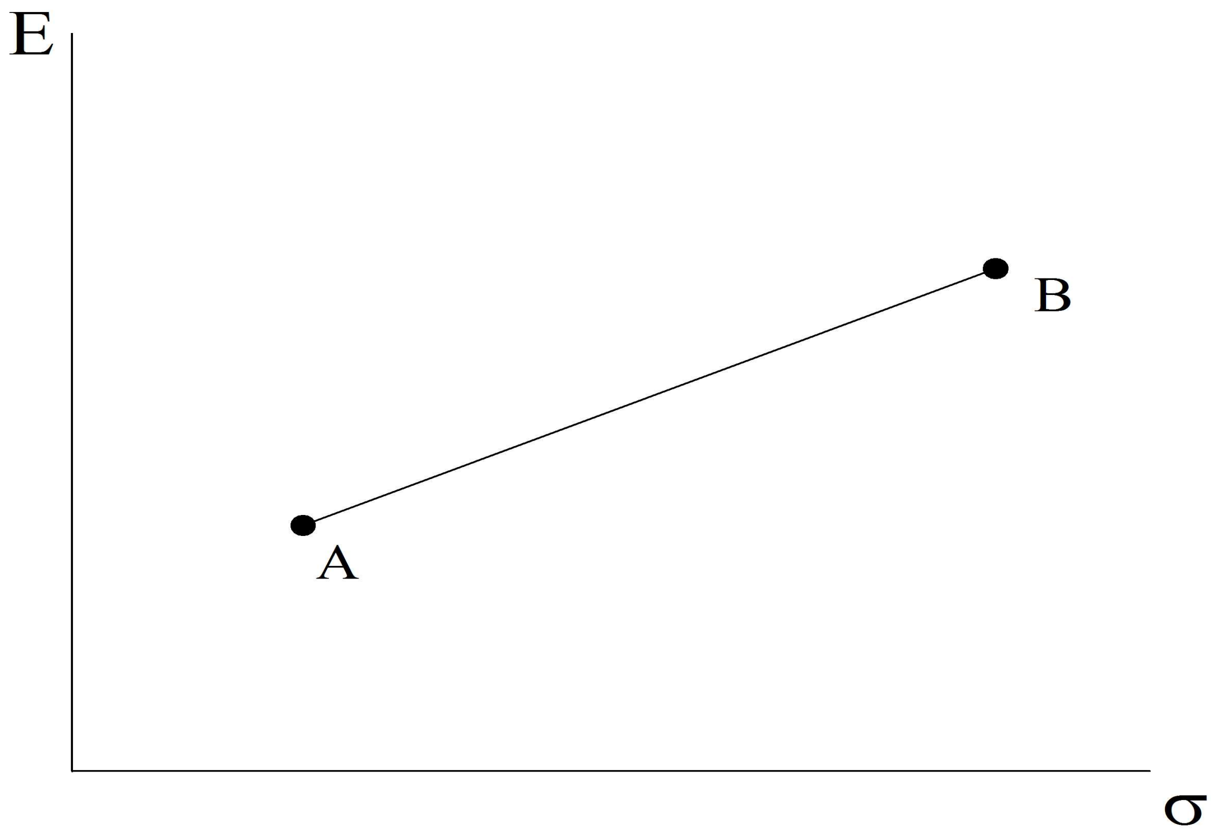
- With perfect positive correlation, or $\rho_{AB} = 1$:
- The variance of r_P becomes

$$\begin{aligned}\sigma_P^2 &= w_A^2\sigma_A^2 + (1-w_A)^2\sigma_B^2 + 2w_A(1-w_A)\sigma_A\sigma_B \\ &= [w_A\sigma_A + (1-w_A)\sigma_B]^2 \end{aligned} \tag{5}$$

- Taking the square-root then gives

$$\sigma_P = w_A\sigma_A + (1-w_A)\sigma_B \tag{6}$$

- From (3) and (6), it is clear that portfolio opportunities are described by a line segment:



E
B
A
σ

Perfectly Negatively Correlated Assets

- With perfect negative correlation, $\rho_{AB} = -1$:
- The variance of r_P becomes

$$\begin{aligned} \sigma_P^2 &= w_A^2\sigma_A^2 + (1-w_A)^2\sigma_B^2 - 2w_A(1-w_A)\sigma_A\sigma_B \\ &= [w_A\sigma_A - (1-w_A)\sigma_B]^2 \end{aligned} \tag{7}$$

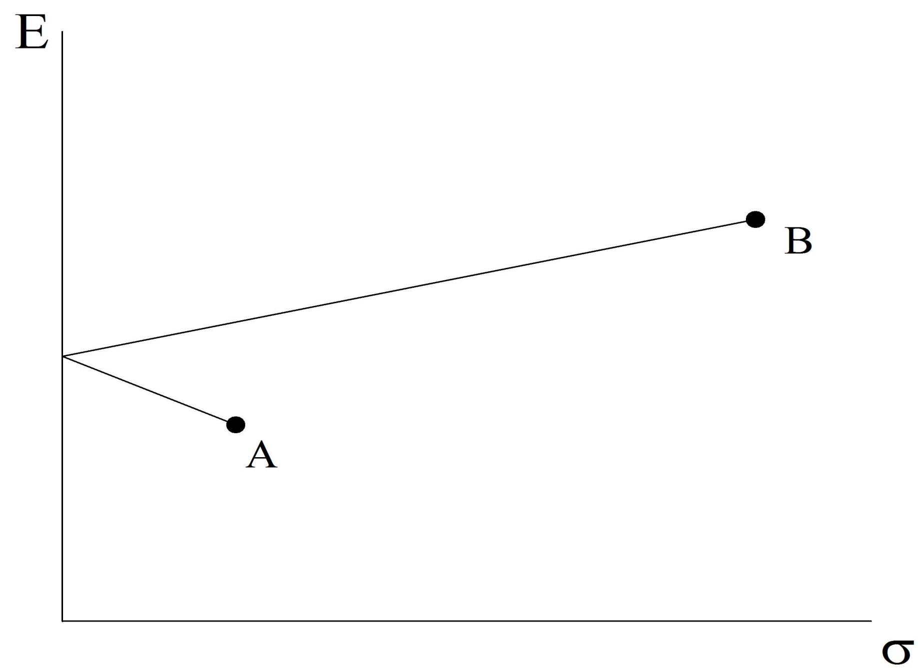
- Taking the square-root then gives

$$\begin{aligned} \sigma_P &= |w_A\sigma_A - (1-w_A)\sigma_B| \\ &= \begin{cases} w_A\sigma_A - (1-w_A)\sigma_B & \text{for } w_A \geq \frac{\sigma_B}{\sigma_A+\sigma_B} \\ -w_A\sigma_A + (1-w_A)\sigma_B & \text{for } w_A < \frac{\sigma_B}{\sigma_A+\sigma_B} \end{cases} \end{aligned} \tag{8}$$

- A riskless return is created by setting w_A equal to the hedge ratio

$$w^* = \frac{\sigma_B}{\sigma_A + \sigma_B}. \tag{9}$$

- The portfolio opportunities are represented by a pair of line segments:



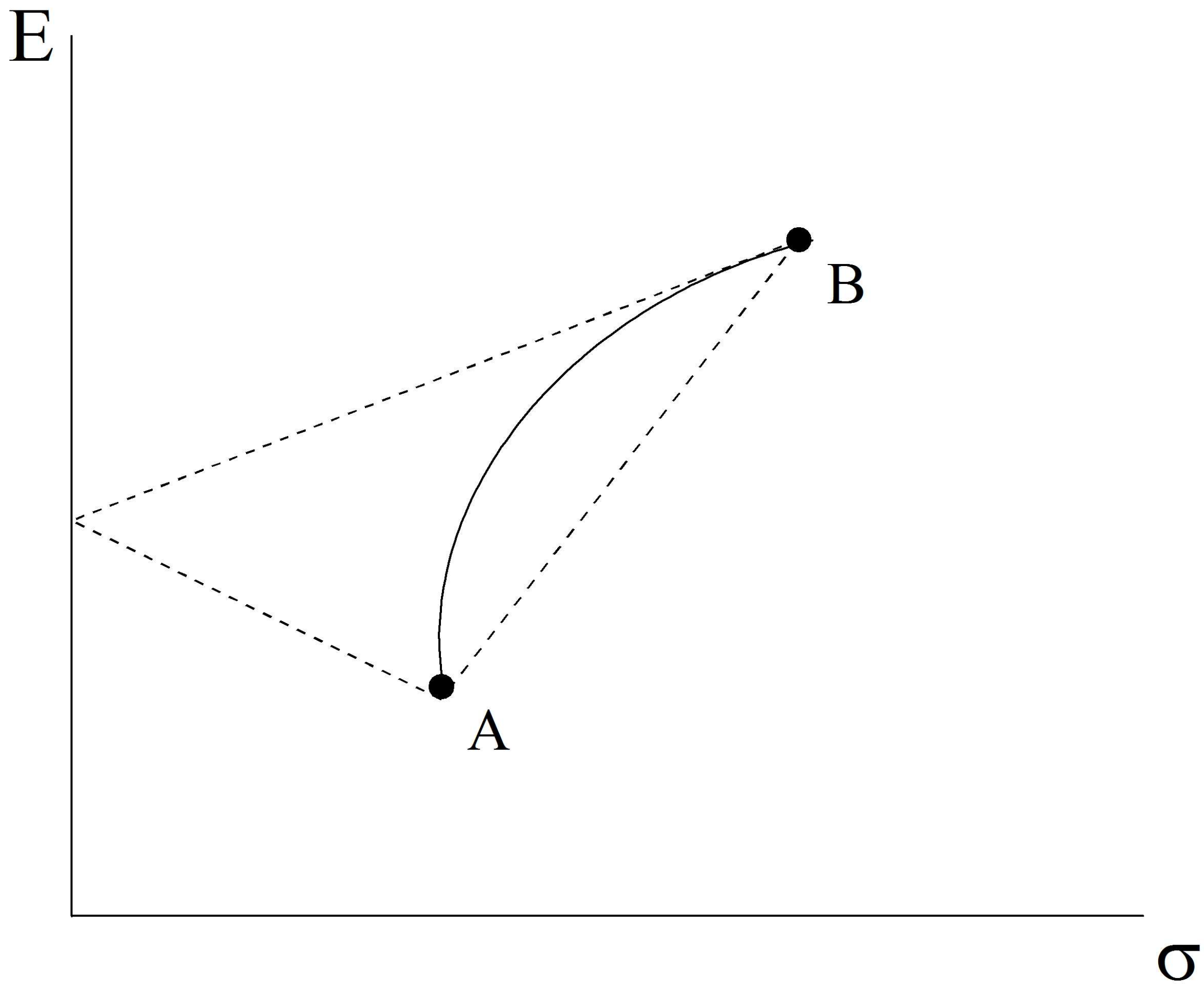
- The segment between point A and the riskless point is governed by the first case in (8), where w_A exceeds the hedge ratio w^*.
- The segment between the riskless point and point B is governed by the second case in (8), where w_A is less than w^*.

Two Risky Assets: General Case

- When the correlation lies between the extremes, that is $-1 < \rho_{AB} < 1$, then the two cases analyzed above provide boundaries for the more general case.
- Examine again the variance in the general case:

$$\sigma_P^2 = w_A^2\sigma_A^2 + (1 - w_A)^2\sigma_B^2 + 2w_A(1 - w_A)\sigma_A\sigma_B\rho_{AB} \tag{10}$$

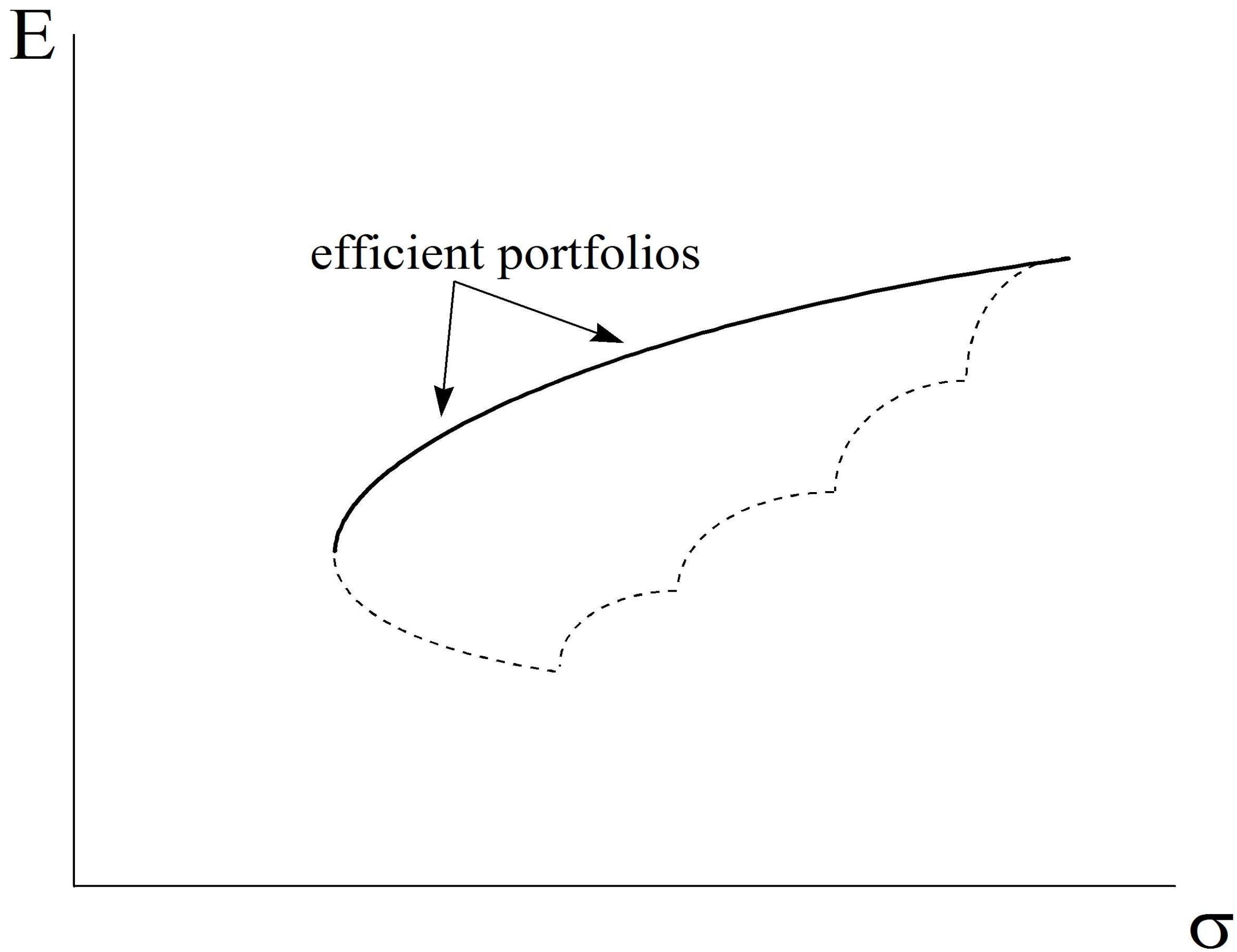
- Observe that, for any given value of w_A, the variance cannot be
  - greater than the variance when $\rho_{AB} = 1$,
  - less than the variance when $\rho_{AB} = -1$.
- Portfolio opportunities lie on a curve between the boundaries given by the two limiting cases:



E
B
A
σ

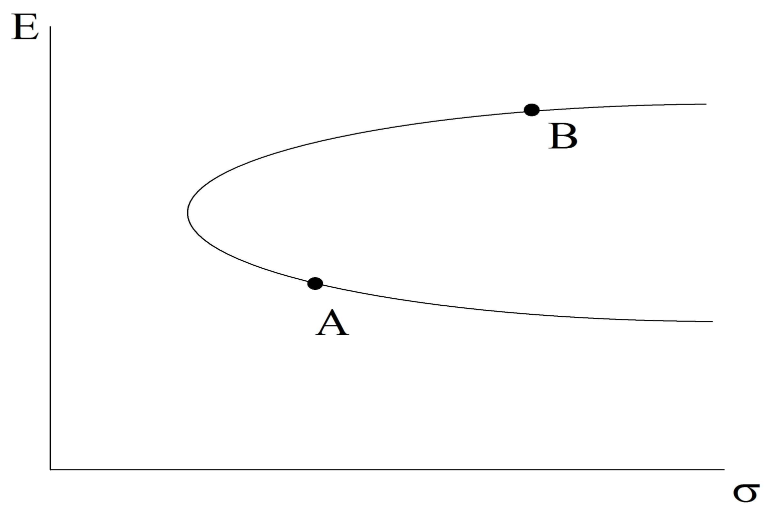
More than Two Risky Assets

- With only two risky assets, there are at most two different levels of expected return (E) for a given level of risk (σ), but with three or more risky assets, there can be many levels of E for a given σ.
- The upper boundary of any portfolio opportunity set represents the *efficient* portfolios, those offering the highest expected return for a given level of risk, and this collection of efficient portfolios is known as the *efficient set*.
- When shortselling of risky assets is prohibited, the opportunity set (and therefore the efficient set) has a right-hand boundary:

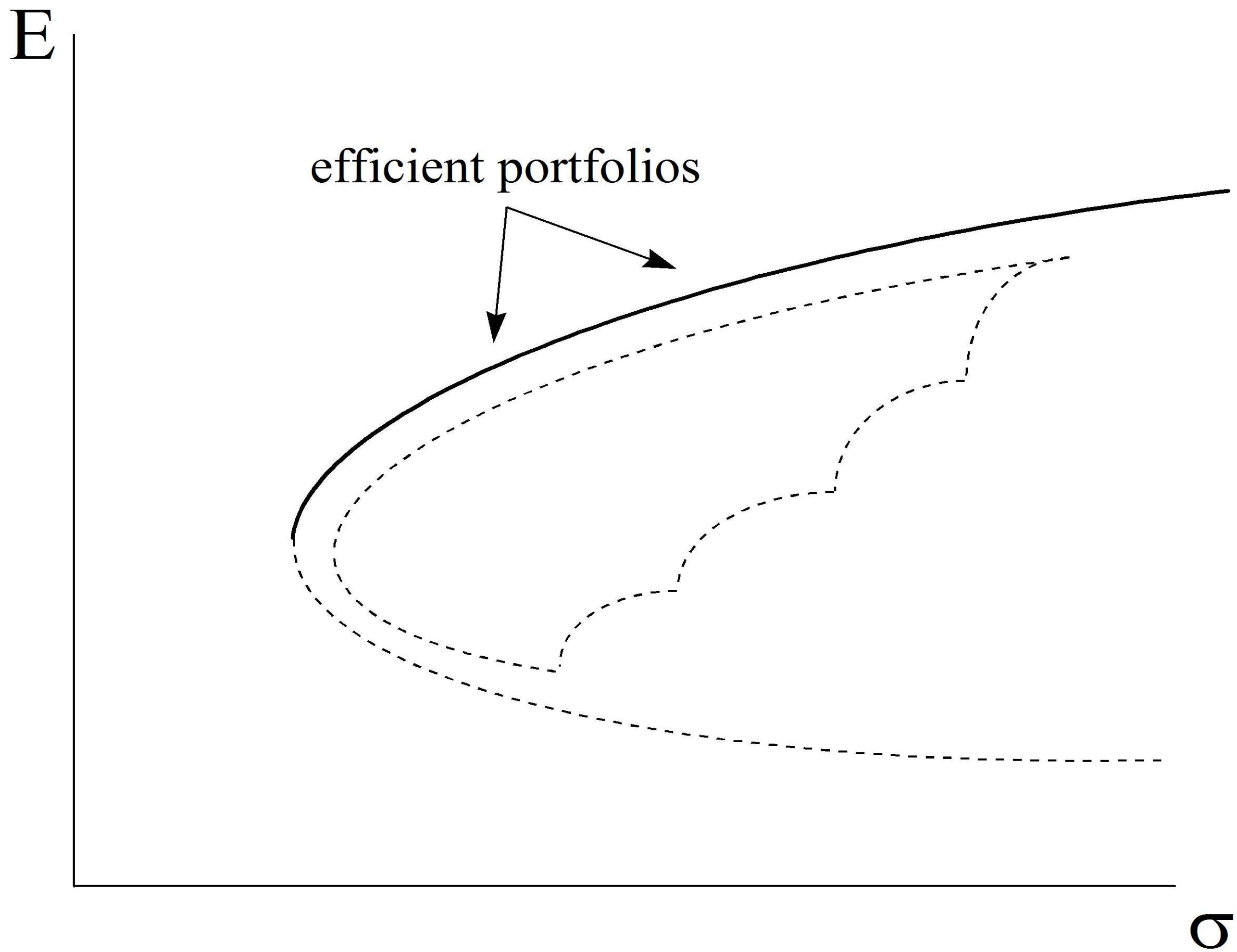

E
efficient portfolios
σ

Short Selling

- Short selling can be viewed as investing a *negative* amount in an asset: a positive return on the asset provides a loss to the short seller.
- A simplified representation of short selling is just to allow $w_A < 0$
- With two assets, the curve describing portfolio opportunities is simply extended beyond the endpoints:



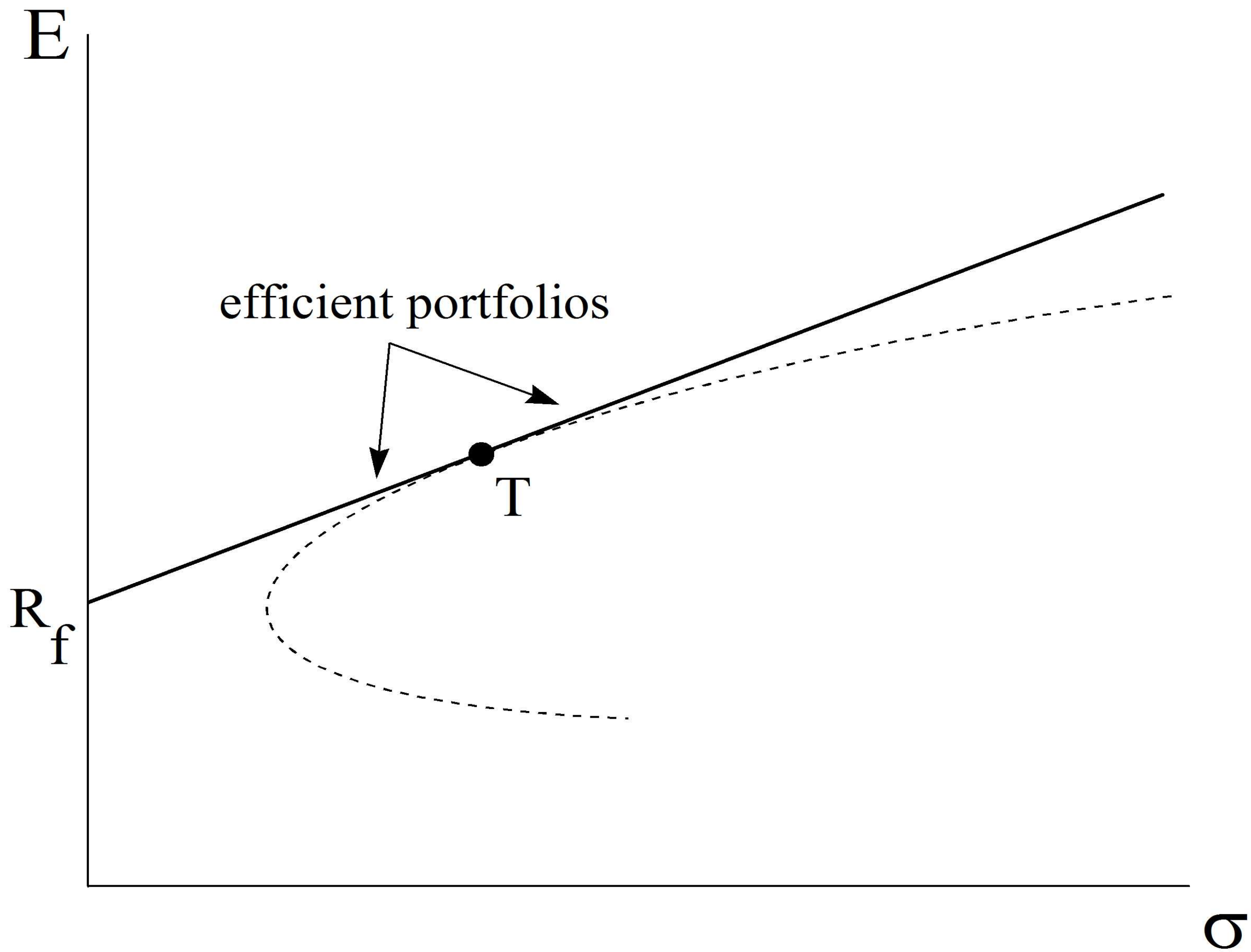
- With three or more assets, the opportunity set extends to the right and, in general, is also expanded a bit to the left:


E
efficient portfolios
σ

Multiple Risky Assets and a Riskless Asset

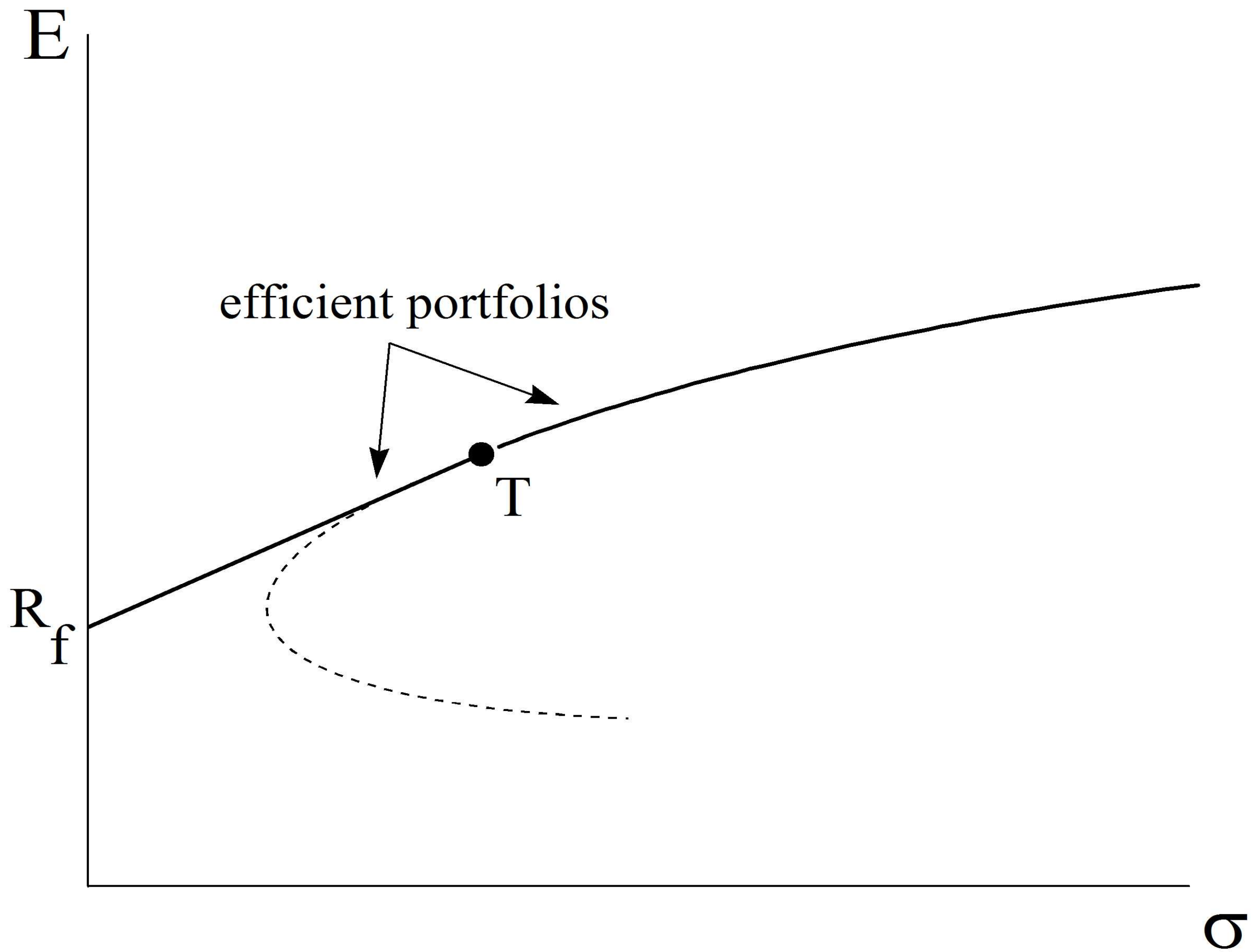
- With borrowing and lending at a common rate r_f, there is *only one* efficient portfolio composed solely of risky assets.
- The risky portfolio T, the "tangent" portfolio, is the risky component of all efficient portfolios.
- The risk-return tradeoff between expected return (E_C) and risk (σ_C) for efficient portfolios is then given by the capital allocation line constructed using portfolio T:

$$E_C = r_f + \left(\frac{E_T - r_f}{\sigma_T} \right) \sigma_C \tag{11}$$


E
efficient portfolios
T
R_f
σ

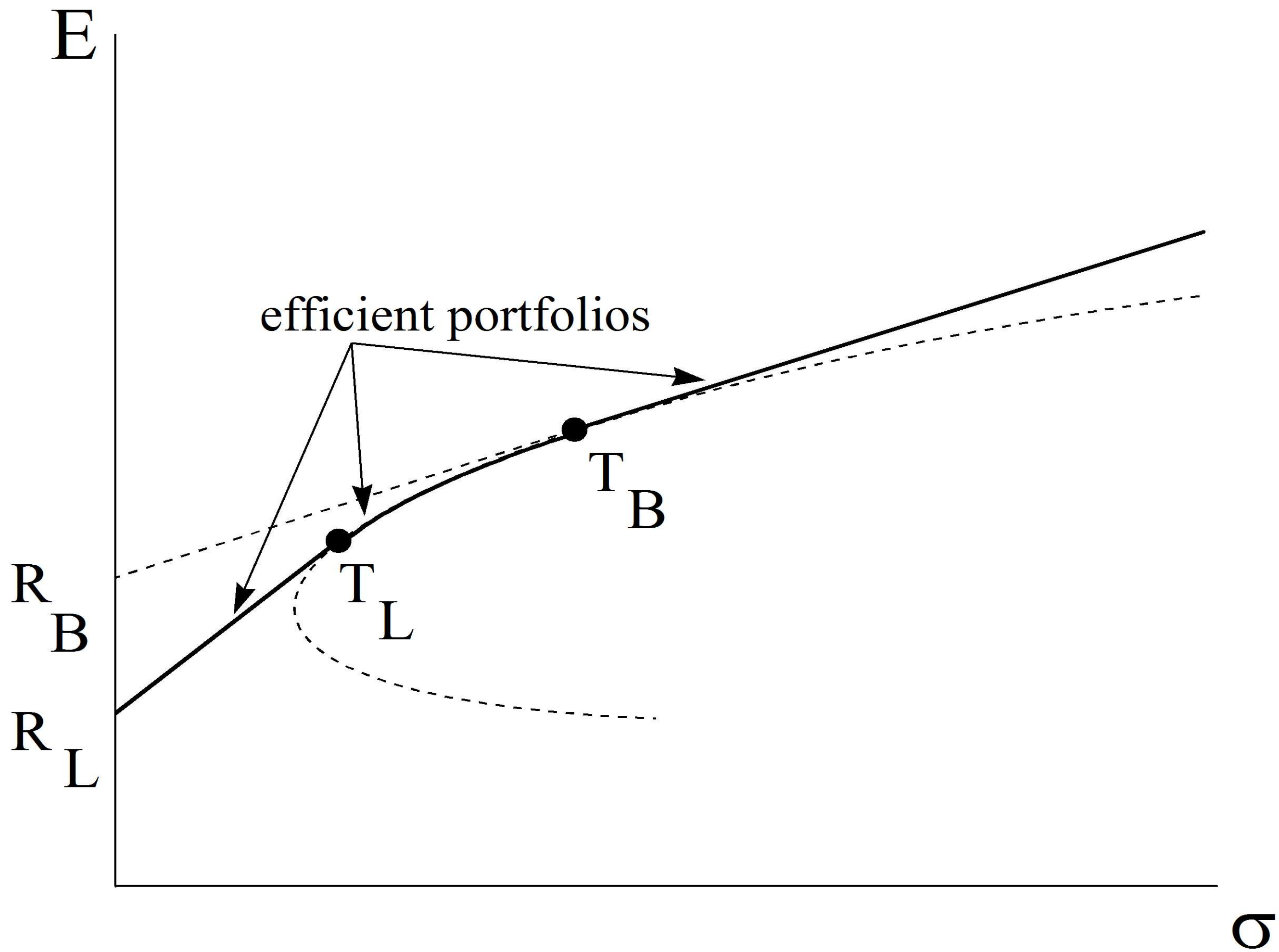
No Borrowing

- When only lending can be done at a riskless rate, then all efficient portfolios that involve lending will contain portfolio T as the risky-asset component.
  - For risk levels less than or equal to σ_T, equation (11) gives the risk-return tradeoff.
  - For risk levels higher than σ_T, the composition of the risky-asset portfolio changes as risk increases.


E
efficient portfolios
T
R_f
σ

Different Borrowing and Lending Rates

- When both riskless lending and borrowing are possible, but at different rates, then the efficient set consists of three parts:
  - Let T_L denote the tangent portfolio associated with the lending rate r_L. For all portfolios with risk less than or equal to that of portfolio T_L, the risk return tradeoff is linear, and all efficient portfolios combine T_L with lending.
  - Let T_B denote the tangent portfolio associated with the borrowing rate r_B. For all portfolios with risk greater than or equal to that of portfolio T_B, the risk return tradeoff is linear, and all efficient portfolios combine T_B with borrowing.
  - For levels of risk between those of portfolios T_L and T_B, efficient portfolios involve neither lending nor borrowing, and the optimal portfolio of risky assets changes for different levels of risk.


E
efficient portfolios
T_B
T_L
R_B
R_L
σ

Separating Investment Policy From Risk Preferences

- The efficient set can often be generated by a small number of portfolios.
- In general, a key message of modern portfolio theory is that there can be a fair amount of separation between the activities of portfolio-optimization and client-specific asset allocation.
- With a single riskless borrowing and lending rate
  - All efficient portfolios combining the optimal risky "tangent" portfolio with either lending or borrowing
  - Investment managers could disagree about the composition of the optimal risky portfolio.
  - No single manager would believe that more than one risky portfolio is necessary to satisfy the risk tolerances of a variety of clients.

Separating Investment Policy From Risk Preferences (cont.)

- With no riskless asset, or with borrowing that costs more or is restricted
  - The risky-asset (curved) boundary can be generated as combinations of any *two* portfolios on that boundary.
  - Managers could disagree about how to construct a pair of optimal risky portfolios.
  - No single manager would think it necessary to offer more than two funds to satisfy demands to hold efficient portfolios.
  - Example: Suppose there is no riskless asset. If all three of the portfolios below are efficient, then the composition of the third portfolio is determined by the information given.

	Portfolio 1	Portfolio 2	Portfolio 3
Large-Cap Stocks	$ 400,000	$ 400,000	
Small-Cap Stocks	200,000	1,200,000	
Government Bonds	400,000	400,000	
Total Value	1,000,000	2,000,000	1,000,000
Expected Return	12%	16%	14%

Asset Allocations

Mean-variance efficient?

Fund	Equities	Bonds	Short-Term/ Money markets
Fidelity Asset Manager 85%	85	15	0
Fidelity Asset Manager 70%	70	25	5
Fidelity Asset Manager 60%	60	35	5
Fidelity Asset Manager 50%	50	40	10
Fidelity Asset Manager 40%	40	45	15
Fidelity Asset Manager 30%	30	50	20
Fidelity Asset Manager 20%	20	50	30

Fund	Stocks	Bonds
Vanguard Life Strategy Income	20	80
Vanguard Life Strategy Conservative Growth	40	60
Vanguard Life Strategy Moderate Growth	60	40
Vanguard Life Strategy Growth	80	20

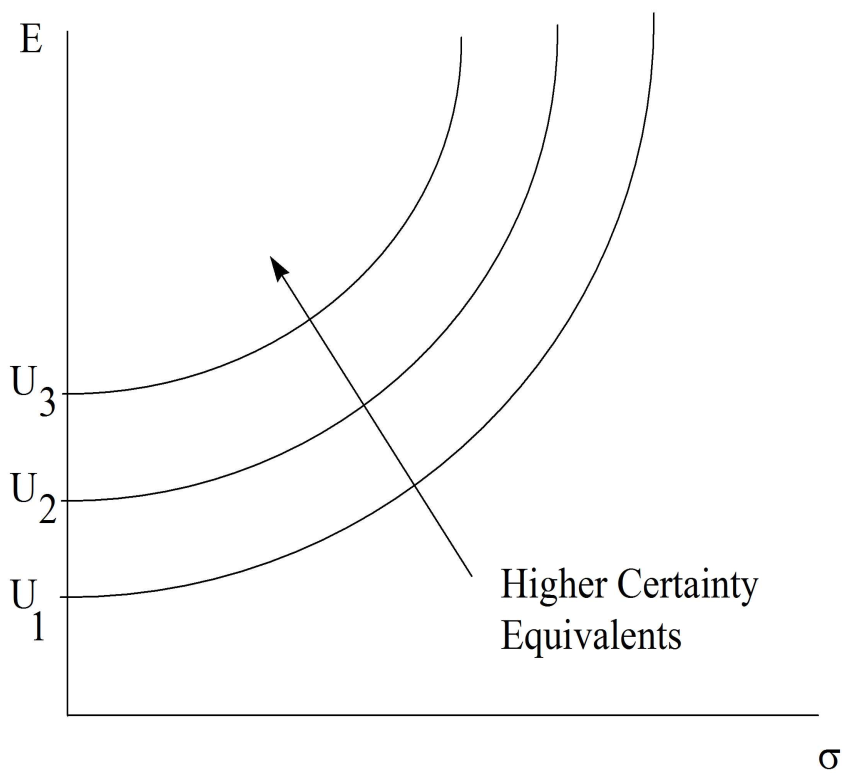
Risk Tolerance

- A tractable representation of an investor's required compensation for portfolio risk is the "utility" function
$$U = E - 0.5A\sigma^2, \tag{12}$$
where E is expected return and σ^2 is variance of return.
- Can interpret U as the investment's *certainty equivalent* rate of return: the investor regards
  - a risky investment with mean E and volatility σ
  - a riskless investment with rate of return U

  as equally desirable.
- The value of A is known as the investor's *coefficient of risk aversion*. The higher is A, the lower the tolerance for risk. In applications to asset allocation, A is generally taken to be somewhere between 1 and 10.

Risk Tolerance (cont.)

- The combinations of E and σ^2 that provide a given certainty equivalent produce an upward-sloping "indifference" curve when plotted with respect to mean (E) and standard deviation (σ).



Portfolio Choice on a Capital Allocation Line

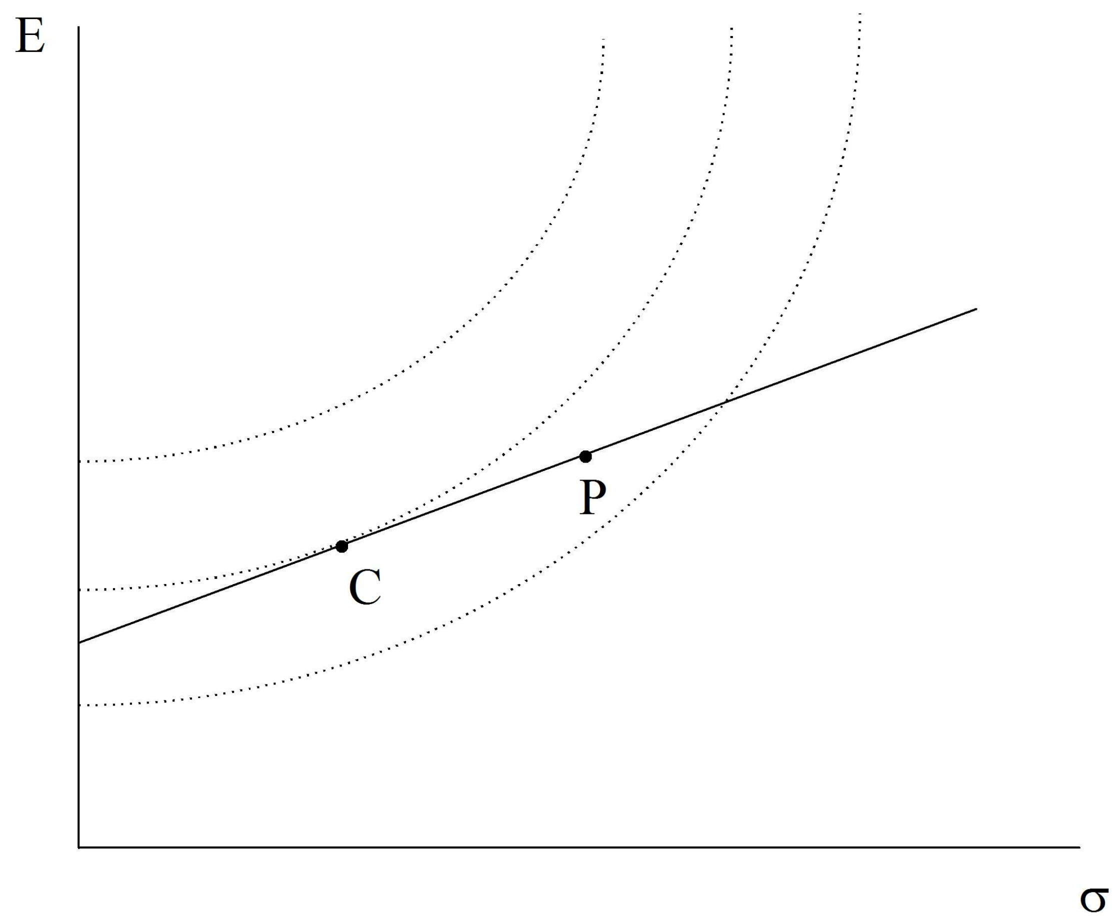
- If opportunities consist of asset P and cash, then

$$
\begin{aligned}
U &= E_C - 0.5A\sigma_C^2 \\
&= [yE_P + (1-y)r_f] - 0.5A[y^2\sigma_P^2] \qquad (13)
\end{aligned}
$$

- The optimal portfolio is given by the choice of y that maximizes U:

$$
y^* = \frac{1}{A}\left(\frac{E_P - r_f}{\sigma_P^2}\right) \qquad (14)
$$

- The optimal portfolio lies at the point where the capital-allocation line is tangent to the highest attainable indifference curve.


E
C
P
σ

Portfolio Choice on a Capital Allocation Line (cont.)

- Consider, for example, an investor with $A = 3$ who wishes to allocate \$200,000 between T-bills and an S&P index fund. Using Ibbotson Associates estimates,

$$\begin{aligned} y^* &= \frac{1}{3}\left(\frac{.130 - .038}{.203^2}\right) \\ &= \frac{1}{3}(2.23) \\ &= .74, \end{aligned} \tag{15}$$

so the optimal portfolio would place \$51,000 in T-bills and \$149,000 in the index fund.

Determining Risk Tolerance

- Evaluating an investor's risk tolerance is important but tricky.
- Common approaches: elicit an investor's responses to hypothetical:
  - lotteries
  - investment scenarios
- A lottery-based approach for determining A:
  - Let G denote 1% of your net worth.
  - Consider a coin-toss gamble that would have you win or loose G with equal probability.
  - Let B denote the amount you would have to be paid in order to induce you to take the gamble.
  - Compute
$$A = 200\frac{B}{G} \tag{16}$$
  - For example, if your ratio of B to G is 0.01, then your value of A is equal to 2.

Determining Risk Tolerance (cont.)

- Try it:

What is 1% of your net worth?	__________________(G)
How much would you have to be paid to bet that amount on a coin toss?	__________________(B)
Divide B by G and multiply by 200	__________________

Why does that work?

- Let net worth be W
- Gamble: invest W
- You are paid B as inducement to win or lose $G(= 0.01W)$ on a coin toss
  - expected return: $E = B/W$
  - return variance: $\sigma^2 = 0.5(0.01)^2 + 0.5(-0.01)^2 = (0.01)^2$
- Utility function: $U = E - (A/2)\sigma^2$
- What value of B makes your $U = 0$? (i.e., makes you indifferent to taking the gamble)

$$\begin{aligned}
0 &= E - \frac{A}{2}\sigma^2 \\
E &= \frac{A}{2}\sigma^2 \\
\frac{B}{W} &= \frac{A}{2}(0.01)^2 \\
\frac{B}{100G} &= \frac{A}{2}(0.01)^2 \\
A &= 200\frac{B}{G}
\end{aligned}$$