Economics 701A **Steven A. Matthews**
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Suggested Solutions to the Exam

80 points, 80 minutes. Closed books, notes, calculators.
Indicate your reasoning.
Use BOTH clearly written words and math.

1. (30 pts) A firm produces output q from inputs $z = (z_1, \ldots, z_n)$ using a strictly increasing C^2 production function f. Let $c(w, q)$ be the firm's cost function, $z(w, q)$ be its conditional factor demand function, and $D_w z(w, q) = [\partial z_i / \partial w_j]$ be the Jacobian matrix of $z(w, q)$ with respect to w. For each property below, state further assumptions that imply it is true, and prove your answer.

   (a) (10 pts) $c(w, q)$ is convex in q.
   Soln: True if f is concave.
   Proof. Given q^0, q^1, let z^0 and z^1 be the corresponding cost minimizing input bundles. Let $\lambda \in [0, 1]$, and define

   $$\begin{aligned} q^\lambda &= \lambda q^1 + (1-\lambda) q^1, \\ z^\lambda &= \lambda z^1 + (1-\lambda) z^1. \end{aligned}$$

   The concavity of f implies

   $$\begin{aligned} f(z^\lambda) &\geq \lambda f(z^1) + (1-\lambda) f(z^0) \\ &\geq \lambda q^1 + (1-\lambda) q^0 = q^\lambda. \end{aligned}$$

   Hence, z^λ is feasible for the cost minimization problem for producing output q^λ. It follows that c is convex in q since

   $$\begin{aligned} c(w, q) &\leq w \cdot z^\lambda \\ &= \lambda (w \cdot z^1) + (1-\lambda)(w \cdot z^0) \\ &= \lambda c(q^1) + (1-\lambda) c(q^0). \end{aligned}$$

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   (b) (10 pts) $c(w, q)$ is linear in q.
   Soln: True if f is homogeneous of degree one (constant returns to scale).
   Proof. (Drop the w here, as it is held constant). We have

   $$\begin{aligned} c(tq) &= \min_z w \cdot z \text{ such that } f(z) \geq tq \\ &= t \min_z w \cdot \frac{z}{t} \text{ such that } f(z) \geq tq \\ &= t \min_z w \cdot \frac{z}{t} \text{ such that } f(\frac{z}{t}) \geq q, \end{aligned}$$

   where the last equality holds because f is homogeneous of degree one. Letting $\hat{z} = z/t$, we reach our conclusion:

   $$\begin{aligned} c(tq) &= t \min_{\hat{z}} w \cdot \hat{z} \text{ such that } f(\hat{z}) \geq q \\ &= tc(q). \end{aligned}$$

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(c) (10 pts) $D_w z(w,q)w = 0$.

Soln: True at any (w,q) at which the derivatives $D_w z(w,q)$ exist.

Proof. The conditional factor demands are homogeneous of degree zero in w. Hence, by Euler's formula, for any i we have

$$\sum_j \frac{\partial z_i(w,q)}{\partial w_j} w_j = 0.$$

In matrix form, these n equations are $D_w z(w,q)w = 0$. ∎

2. (20 pts) Robinson Crusoe has an endowment $e \in \mathbb{R}_{++}$ of bananas that he can consume or use to make clothing. If he uses $x \in [0,e]$ bananas to make clothing, his utility will be $u(e-x, f(x))$, where $u : \mathbb{R}^2_+ \to \mathbb{R}$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$ are strictly increasing functions. Let

$$x^*(e) := \arg\max_{0 \le x \le e} u(e-x, f(x)).$$

(a) (10 pts) Show that $x^*(e)$ is convex if u is quasiconcave and f is concave.

Soln: Let $x^0, x^1 \in x^*(e)$, and $\bar{u} = u(e-x^0, f(x^0)) = u(e-x^1, f(x^1))$. Let $x^\lambda = \lambda x^1 + (1-\lambda)x^1$ for some $\lambda \in [0,1]$. Since f is concave, we have

$$f(x^\lambda) \ge \lambda f(x^1) + (1-\lambda) f(x^0).$$

Thus, as u is increasing in its second argument and is quasiconcave, we have

$$\begin{aligned} u(e-x^\lambda, f(x^\lambda)) &\ge u(e-x^\lambda, \lambda f(x^1) + (1-\lambda) f(x^0)) \\ &= u(\lambda(e-x^1) + (1-\lambda)(e-x^0),\ \lambda f(x^1) + (1-\lambda) f(x^0)) \\ &\ge \min\{u(e-x^1, f(x^1)),\ u(e-x^0, f(x^0))\} \\ &= \bar{u}. \end{aligned}$$

Since $\bar{u}$ is the maximal utility Robinson can obtain, x^λ must be a maximizer: $x^\lambda \in x^*(e)$. This proves $x^*(e)$ is a convex set. ∎

(b) (10 pts) Assume now that $x^*(e)$ is a singleton for any $e > 0$, u and f are C^2 functions, f and u are concave, and $u_{12} \ge 0$. Prove that $x^*(e)$ is a nondecreasing function, stating any further (minor) assumptions you need.

Soln: Proof 1. To simplify notation, let $v(e,x) := u(e-x, f(x))$. Note that

$$v_{12} = -u_{11} + u_{12} f' \ge 0,$$

since the concavity of u implies $u_{11} \le 0$, f increasing implies $f' \ge 0$, and we've been told that $u_{12} \ge 0$. Now, for some $\bar{e}$ and $\hat{e} > \bar{e}$, let $\bar{x} = x^*(\bar{e})$ and $\hat{x} = x^*(\hat{e})$. We must show that $\hat{x} \ge \bar{x}$. This is obvious if the two are equal, so we can assume $\bar{x} \neq \hat{x}$. Then

$$\int_{\hat{x}}^{\bar{x}} \int_{\bar{e}}^{\hat{e}} v_{12}(e,x) de\, dx = [v(\hat{e},\bar{x}) - v(\hat{e},\hat{x})] - [v(\bar{e},\bar{x}) - v(\bar{e},\hat{x})] < 0,$$

since $\hat{x}$ is the unique maximizer of $v(\hat{e},\cdot)$ and $\bar{x}$ is the unique maximizer of $v(\bar{e},\cdot)$. Since $v_{12} \ge 0$ and $\hat{e} > \bar{e}$, the double integral would be nonnegative if $\hat{x} < \bar{x}$. Hence, $\hat{x} > \bar{x}$. ∎

Proof 2. This time we use the implicit function theorem to find $x^{*\prime}(e)$ at some e, making stronger derivative assumptions to make sure the troublesome denominator is not zero.

First additional assumption:

$$\text{(A1)} \quad 0 < x^*(e) < e.$$

The FOC thus holds with equality at $x^*(e) : -u_{11} + u_2 f' = 0$. Differentiate it with respect to e and solve for $x^{*\prime}(e)$:

$$x^{*\prime}(e) = \frac{u_{11} - u_{12} f'}{D}, \tag{1}$$

where

$$D = (1, -f') \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{pmatrix} 1 \\ -f' \end{pmatrix} + u_2 f''$$

and all functions are evaluated at $x = x^*(e)$. The concavity of u implies its Hessian is negative semidefinite, so the quadratic form term is nonpositive. The last term is nonpositive because $u_2 \geq 0$ and $f'' \leq 0$. To make sure D is not zero, we make a second additional assumption:

$$\text{(A2)} \quad [u_{ij}] \text{ is a negative definite matrix, or } u_2 > 0 \text{ and } f'' < 0.$$

Since (A2) implies $D < 0$, the IFT tells us x^* is differentiable at e and its derivative there is as shown in (1). Since $u_{11} \leq 0$, $u_{12} \geq 0$, and $f' \geq 0$, we conclude that $x^{*\prime}(e) \geq 0$. ■ ■

3. (30 pts) Let $\tilde{x} = a + \tilde{\varepsilon}$ be a gamble, where $a \in \mathbb{R}$ and $\tilde{\varepsilon}$ is a random variable with mean zero. A consumer has a C^2 Bernoulli utility function, $u : \mathbb{R} \to \mathbb{R}$, satisfying $u' > 0$ and $u'' \leq 0$. Her sale price for the gamble is the minimum amount she would sell the gamble for: it is the number $s(a)$ satisfying

$$u(s(a)) = \mathbb{E}u(a + \tilde{\varepsilon}).$$

(a) (10 pts) If u exhibits constant absolute risk aversion, what can you say about the derivative $s'(a)$? Prove your answer.

Soln: $s'(a) = 1$, i.e., $s(a) = a +$ *constant*.

Proof. Let A be the constant coefficient of absolute risk aversion. If $A = 0$, then we can normalize so $u(x) \equiv x$, and the definition of $s(a)$ implies $s(a) = a + \mathbb{E}(\tilde{\varepsilon}) = a$. So we can assume $A > 0$ (if $A < 0$, u would not be concave). We can now normalize u so that

$$u(x) = -e^{-Ax},$$

and so the definition of $s(a)$ implies

$$e^{-As(a)} = e^{-Aa}\mathbb{E}e^{-A\tilde{\varepsilon}} \;\Rightarrow\; e^{-A(s(a)-a)} = \mathbb{E}e^{-A\tilde{\varepsilon}}.$$

This implies that $s(a) - a$ does not depend on a, and so again $s'(a) = 1$. ■

(b) (20 pts) If u exhibits decreasing absolute risk aversion (DARA), what can you say about the derivative $s'(a)$? Prove your answer.

Soln: $s'(a) \geq 1$.

Proof 1. For any a, define a utility function u_a by $u_a(z) := u(a+z)$. DARA implies u_a is more risk averse than $u_{\hat{a}}$ if $a < \hat{a}$. The definition of $s(a)$ implies

$$u_a(s(a) - a) = u(s(a)) = \mathbb{E}u(a + \tilde{\varepsilon}) = \mathbb{E}u_a(\tilde{\varepsilon}).$$

Thus, $s(a) - a$ is the certainty equivalent of the risk $\tilde{\varepsilon}$ for the utility function u_a. Since u_a becomes less risk averse as a increases, this certainty equivalent increases in a. Hence, $s'(a) - 1 \geq 0$.

Proof 2. Differentiate the identity defining $s(a)$ to obtain.

$$s'(a) = \frac{\mathbb{E}u'(a + \tilde{\varepsilon})}{u'(s(a))}.$$

Now we observe that DARA implies u' is a convex function of u : letting $g := u' \circ u^{-1}$, we have $u'(z) = g(u(z))$ for any z, and at any $\bar{u}$,

$$g'(\bar{u}) = \frac{u''(u^{-1}(\bar{u}))}{u'(u^{-1}(\bar{u}))}$$

is increasing in $\bar{u}$ by DARA. Rewrite the expression for $s'(a)$ in terms of g :

$$s'(a) = \frac{\mathbb{E}g(u(a + \tilde{\varepsilon}))}{g(u(s(a)))}.$$

Since g is convex, Jensen's theorem and the definition of $s(a)$ yield

$$\mathbb{E}g(u(a + \tilde{\varepsilon})) \geq g(\mathbb{E}u(a + \tilde{\varepsilon})) = g(\mathbb{E}u(s(a))),$$

and so $s'(a) \geq 1$. ∎