# Black-Litterman: The Basic Steps

- Specify
  - **normal** portfolio
    - weights ($\bar{x}_i$'s) when views about expected returns are **neutral**
    - e.g. market-portfolio allocations (CAPM)
    - e.g. a long-run "policy" target portfolio
  - Risk measures:
    - volatilities, $\sigma_i$'s
    - correlations, $\rho_{i,j}$'s
  - Sharpe measure of the normal portfolio, $\bar{S}$
- Compute the neutral expected returns ($\bar{E}_i$'s) implied by the above, i.e., the expected returns that would make the normal portfolio optimal
- Input one or more **views** about departures from neutral expected returns

## Black-Litterman: The Basic Steps (cont.)

- Examples of views about expected returns
  - asset 2's expected return over the next month, $E_2$, is 10 basis points higher than its neutral expected return

$$E_2 = \bar{E}_2 + .001$$

  - over the next month, asset 2's expected return is 20 basis points higher than asset 1's expected return

$$E_2 - E_1 = .0020$$

- Produce a new set of revised expected returns, $E_i^*$'s, for all assets.
- Compute an optimal portfolio using the $E_i^*$'s—along with the $\sigma_i$'s and $\rho_{i,j}$'s—as inputs to the portfolio optimization.

# Computing Neutral Expected Returns

- Basic idea: use the conditions for portfolio optimality to find the expected returns for which the normal portfolio is the "tangent" portfolio—the portfolio with the maximum Sharpe ratio
- Portfolio optimality conditions: if portfolio $p$ combines $N$ assets to produce the maximum Sharpe ratio, then each asset $i$ of those $N$ assets must obey

$$E_i = r_f + (E_p - r_f)\left(\frac{\sigma_{ip}}{\sigma_p^2}\right) \tag{1}$$

- Can rewrite (1) as

$$E_i = r_f + S_p\frac{\sigma_{ip}}{\sigma_p}, \quad i = 1, \ldots, N. \tag{2}$$

  since the Sharpe ratio of $p$ is given by $S_p = (E_p - r_f)/\sigma_p$.

- To obtain neutral expected returns, assume that the normal portfolio is the tangent portfolio $p$.

# Computing Neutral Expected Returns (cont.)

- Use the assumed
  - volatilities ($\sigma_i$'s),
  - correlations ($\rho_{ij}$'s), and
  - weights in the normal portfolio ($\bar{x}_i$'s),
- to compute
  - $\sigma_p^2$, the variance of the normal portfolio,

$$\sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} \bar{x}_i \bar{x}_j (\underbrace{\rho_{ij} \sigma_i \sigma_j}_{\text{cov}(r_i, r_j)}) \tag{3}$$

  - $\sigma_{ip}$, the covariance of each risky asset with the normal portfolio,

$$\sigma_{ip} = \sum_{j=1}^{N} \bar{x}_j \rho_{ij} \sigma_i \sigma_j \tag{4}$$

  for $i = 1, \ldots, N$.

## Computing Neutral Expected Returns (cont.)

- Then, using the assumed Sharpe measure of the normal portfolio, $\bar{S}$, compute the right-hand side of (2).
- The results of this calculation are the neutral expected returns:

$$\bar{E}_i = r_f + \bar{S}\frac{\sigma_{ip}}{\sigma_p}, \quad i = 1, \ldots, N-1.$$

- Expected returns implied by a current portfolio can be assessed
  - Given a client's current portfolio, one can compute the expected returns that would make such a portfolio optimal
  - Apply same approach used to compute neutral expected returns from a normal portfolio
  - Can often reveal extreme expected returns implied by a client's current portfolio

## A Simple Example

- Assume two risky assets: large-cap stocks (CRSP deciles 1–2) and mid-cap stocks (CRSP deciles 3–5) with
  - volatilities (sample estimates)

$$\sigma_1 = 5.29\%$$
$$\sigma_2 = 6.68\%$$

  - correlation (sample estimate)

$$\rho_{12} = 0.9493$$

- Assume that cash returns .33% per month.
- Sample estimates of expected returns:
  - Suppose that, using sample averages, the expected returns would be

$$\hat{E}_1 = 0.928\%$$
$$\hat{E}_2 = 1.132\%$$

  - If those estimates are used as expected returns for optimization, the weights in the resulting tangent portfolio are rather extreme: -0.125 and 1.125.

## A Simple Example (cont.)

- In the Black-Litterman approach, those sample averages, and the implied extreme weights, are not used.
- Instead, the approach begins with a normal portfolio with
  - weights assumed to be

$$\bar{x}_1 = 0.80 \qquad \bar{x}_2 = 0.20$$

  - and a Sharpe measure assumed to be 0.12 (often a sample estimate for that portfolio)
- The normal portfolio then implies neutral expected returns given by

$$\bar{E}_1 = 0.966\%$$

$$\bar{E}_2 = 1.111\%.$$

- If used as inputs in an optimization, the neutral expected returns simply give back the normal portfolio as the optimum.

## Incorporating Views about Expected Returns

- Suppose we expect that, over the next month, the portfolio of mid-cap stocks will outperform its neutral expected return by $q$:

$$E_2 = \bar{E}_2 + q,$$

where $q = .10\%$.

- The new set of revised expected returns is

$$E_1^* = 1.041\%$$

$$E_2^* = 1.211\%$$

- These are calculated as

$$E_1^* = \bar{E}_1 + \frac{\sigma_1}{\sigma_2}\rho_{12}(E_2^* - \bar{E}_2) \tag{5}$$

$$E_2^* = \bar{E}_2 + q \tag{6}$$

- Notice that both differ from the neutral expected returns, even though a view about asset 1 was not expressed.

# Incorporating Views about Expected Returns (cont.)

- The positive correlation between the assets implies
  - optimism about asset 2 should also extend somewhat to asset 1
  - the expected return on asset 1 should also be adjusted upward
- When the revised expected returns are used in an optimizer, the new weights in the tangent portfolio become

$$x_1^* = 0.725 \qquad x_2^* = 0.275$$

- The 10-basis-point optimistic view about asset 2 results in a 7.5% portfolio rebalancing.
- Suppose, instead, that the neutral expected returns had simply been revised, more naively as

$$\hat{E}_1 = \bar{E}_1 = 0.966\%$$
$$\hat{E}_2 = \bar{E}_2 + .10\% = 1.211\%$$

- The weights in the tangent portfolio would, in that case, be

$$\hat{x}_1 = -0.57 \qquad \hat{x}_2 = 1.57$$

## Uncertain Views

- Suppose one is less confident about specifying the amount by which the mid-cap stocks are expected to outperform their neutral expected return.
- For example, the uncertain view is

$$E_2 = \bar{E}_2 + q + \epsilon$$

  where again, say, $q = .10\%$, but $\epsilon$ is now a random term with zero mean.

- The variance of $\epsilon$, $\sigma_\epsilon^2$
  - quantifies uncertainty about the view
  - separate from $\sigma_2^2$, variance of the mid-cap portfolio's *unexpected* return.
- The extent to which this uncertain view receives weight depends on the magnitude of $\sigma_\epsilon^2$ relative to the uncertainty about the neutral expected return, $\bar{E}_2$.
- Uncertainty about $\bar{E}_2$ is denoted by the variance $\tau\sigma_2^2$, specified by choosing $\tau$.
  - Interpretation: uncertainty about $\bar{E}_2$ equivalent to that of a mean estimated using an average from a hypothetical sample of size $1/\tau$.

## Uncertain Views (cont.)

- The new set of revised expected returns is then computed as

$$E_1^* = \bar{E}_1 + \frac{\sigma_1}{\sigma_2}\rho_{12}(E_2^* - \bar{E}_2) \tag{7}$$

$$E_2^* = w\bar{E}_2 + (1-w)(\bar{E}_2 + q), \tag{8}$$

where

$$w = \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + \tau\sigma_2^2}.$$

- If, for example, $\sigma_\epsilon^2 = \tau\sigma_2^2$, then $w = 0.5$ and the revised expected returns are equal to

$$E_1^* = 1.004\%$$
$$E_2^* = 1.161\%$$

- The new weights in the tangent portfolio are

$$x_1^* = 0.761 \qquad x_2^* = 0.239.$$

# Relative Views

- Views can be expressed in relative terms
- One might have views about the expected return on one asset relative to that of another.
- No view about whether either asset will perform above or below neutral.
- Suppose one believes that, for a given value $q$, $E_2 - E_1 = q$.
- With no uncertainty about this *relative* view, the Black-Litterman revised expected returns are computed as

$$E_1^* = \bar{E}_1 + h[q - (\bar{E}_2 - \bar{E}_1)] \tag{9}$$

$$E_2^* = \bar{E}_2 + (1 + h)[q - (\bar{E}_2 - \bar{E}_1)], \tag{10}$$

where

$$h = \frac{\sigma_1(\rho_{12}\sigma_2 - \sigma_1)}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}}.$$

# Relative Views (cont.)

- For example, suppose we expect that, over the next month, mid-cap stocks will outperform large-cap stocks by 20 basis points:

$$q = E_2 - E_1 = .20\%,$$

- The revised expected returns then equal

$$E_1^* = 1.021\%$$

$$E_2^* = 1.221\%$$

- The new weights in the tangent portfolio are

$$x_1^* = 0.344 \qquad x_2^* = 0.656.$$

- Relative views can also be uncertain.
- Relative views are difficult to incorporate in other approaches

# Harvard Management Company (2010) - revisited

| | Expected Returns (%) | | | | Portfolio Weights (%)[a] | | | |
|---|---|---|---|---|---|---|---|---|
| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| | Values assumed by HMC | Neutral values implied by weights in (5)[b] | Changing only fixed-income views in (2) | Black-Litterman values given fixed-income views[c] | HMC Policy Portfolio - optimal under (2) | Optimal under (1) | Optimal under (3) | Optimal under (4) |
| Domestic Equity | 5.75 | 7.13 | 7.13 | 7.22 | 11 | 0 | 12 | 7 |
| Foreign Equity | 6.25 | 7.21 | 7.21 | 7.38 | 11 | 0 | 0 | 4 |
| Emerging Markets | 7.00 | 8.20 | 8.20 | 8.06 | 11 | 0 | 15 | 7 |
| Private Equity | 6.75 | 8.11 | 8.11 | 8.21 | 13 | 4 | 5 | 7 |
| Absolute Return | 5.00 | 4.57 | 4.57 | 4.99 | 16 | 24 | 0 | 10 |
| High-Yield | 4.75 | 5.32 | 5.32 | 5.76 | 2 | 0 | 0 | 1 |
| Commodities | 4.50 | 4.54 | 4.54 | 5.90 | 5 | 2 | 0 | 3 |
| Natural Resources | 5.00 | 2.08 | 2.08 | 2.14 | 9 | 43 | 7 | 8 |
| Real Estate | 6.00 | 4.05 | 4.05 | 4.62 | 9 | 17 | 0 | 6 |
| Domestic Bonds | 1.75 | 1.18 | 2.00 | 2.00 | 4 | 0 | 17 | 22 |
| Foreign Bonds | 2.25 | 1.48 | 2.25 | 2.25 | 2 | 8 | 27 | 16 |
| Inflation-Indexed | 2.25 | 1.50 | 2.25 | 2.25 | 5 | 41 | 48 | 36 |
| Cash | 1.00 | 1.17 | 1.00 | 1.00 | 2 | -40 | -31 | -27 |

[a] Weights are constrained within [0 100%] except for cash, which is constrained within [-50% 100%].

[b] The Sharpe ratio of the neutral policy portfolio is assumed to be 0.434, equal to the value implied by the policy weights and the HMC assumptions about asset-class expected returns, standard deviations, and correlations.

[c] The Black-Litterman expected returns assume no uncertainty about the fixed-income views.

Homework Problems

Consider an opportunity set consisting of cash, the S&P Composite Portfolio (asset 1), and a small-cap portfolio (asset 2). Assume that, in the normal portfolio (consistent with neutral views) the ratio of the amount invested in the S&P to the total amount invested in the S&P and small-cap is 0.90. Assume that the Sharpe ratio of the normal portfolio, under neutral views, is equal to 0.2. The volatilities of the two assets and the correlation between them are given by

$$\begin{aligned} \sigma_1 &= 0.06 \\ \sigma_2 &= 0.09 \\ \rho_{12} &= 0.8. \end{aligned}$$

Finally, assume that the current cash rate is 40 basis points (0.004) per month.

1. What are the implied neutral expected returns (per month) on the S&P and the small-cap portfolio?

Suppose you believe (with complete confidence) that the expected small-cap return over the next month exceeds its neutral expected return in your answer to part 1 by 50 basis points (0.50%).

2. Use the Black-Litterman approach to obtain the revised expected return on the S&P.
3. Given the revised expected returns, what are the weights in the optimal (tangent) combination of the S&P and the small-cap portfolio?
4. If you revise the expected return on the small-cap portfolio to exceed its neutral value by 30 basis points, but you do not revise the S&P expected return (keeping it at its neutral level), then what are the implied weights in the tangent portfolio?

Homework Answers

1. The covariance between the normal portfolio (with return $r_p$) and the S&P is equal to

$$\text{cov}\,(r_1, r_p) = 0.9(0.06)(0.06)(1) + 0.1(0.06)(0.09)(0.8) = 0.0037.$$

   Similarly, the covariance between the normal portfolio and the small-stock portfolio is equal to

$$\text{cov}\,(r_2, r_p) = 0.9(0.06)(0.09)(0.8) + 0.1(0.09)(0.09)(1) = 0.0047.$$

   The variance of $r_p$ is then

$$\text{var}(r_p) = 0.9\text{cov}\,(r_1, r_p) + 0.1\text{cov}\,(r_2, r_p) = 0.0038,$$

   so the standard deviation is

$$\sigma_p = 0.0614.$$

   Using equation (2) then gives the neutral expected return on the S&P as

$$E_1 = .0040 + 0.2\left(\frac{0.0037}{0.0614}\right) = 0.0160,$$

   and the the neutral expected return on the small-stock portfolio is

$$E_2 = .0040 + 0.2\left(\frac{0.0047}{0.0614}\right) = 0.0193.$$

2. The revised expected return on the small stocks is

$$E_2^* = 0.0193 + 0.0050 = 0.0243.$$

   Using equation (5) gives

$$E_1^* = 0.0160 + \frac{0.06}{0.09}(0.8)(0.0050) = 0.0186.$$

3. Using the revised Black-Litterman returns as inputs, along with the correlation and volatilities, gives a tangent portfolio of

$$x_1^* = 0.757 \quad x_2^* = 0.243.$$

(Run an optimizer using any positive value of risk aversion, with riskless lending and borrowing unrestricted, and then look at the relative weighting between the two risky assets.)

4. If only the small-stock expected return is revised, then the tangent-portfolio weights become

$$\hat{x}_1 = 0.300 \quad \hat{x}_2 = 0.700.$$