Economics 701A **Steven A. Matthews**
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Solutions to the Exam

100 points, 105 minutes. Closed books, notes, calculators.
Indicate your reasoning, using clearly written words as well as math.

DO JUST *FOUR* OF THE FIVE PROBLEMS.
(Only 1-4 will be graded if you do them all.)

1. (25 pts) Consider a differentiable demand function $x(p, m)$ generated by a strictly increasing $u : \mathbb{R}^n_+ \to \mathbb{R}$ that is homogeneous of degree 1.

   (a) (10 pts) Show that for any $p \in \mathbb{R}^n_{++}$ and $m > 0$, $x(p, m) = mx(p, 1)$.
   Soln: Fix (p, m), and let $\bar{x} = x(p, 1)$. Then $p \cdot \bar{x} \leq 1$. (We actually know $p \cdot \bar{x} = 1$, but don't need this.) Hence, $p \cdot m\bar{x} \leq m$, which shows that the bundle $m\bar{x}$ is feasible for the consumer problem at (p, m).
   Let x be another feasible bundle at (p, m). Then $p \cdot x \leq m$, and so $p \cdot \frac{x}{m} \leq 1$. This shows that $\frac{x}{m}$ is feasible at $(p, 1)$. Hence,

$$\begin{aligned} u(m\bar{x}) &= mu(\bar{x}) \text{ (as } u \text{ is homogeneous of degree 1)} \\ &\geq mu\left(\frac{x}{m}\right) \text{ (as } \bar{x} = x(p,1) \text{ and } \frac{x}{m} \text{ is feasible at } (p,1)) \\ &= u(x) \text{ (as } u \text{ is homogeneous of degree 1).} \end{aligned}$$

   So $m\bar{x}$ solves the consumer problem at (p, m), and hence $x(p, m) = m\bar{x}$. ■

   (b) (15 pts) Show that x satisfies the Law of Reciprocity.
   Soln: From (a) we have

$$x_j \frac{\partial x_i}{\partial m} = (m\bar{x}_j) \frac{\partial (m\bar{x}_i)}{\partial m} = m\bar{x}_j\bar{x}_i.$$

   Reversing the subscripts yields $x_i \frac{\partial x_j}{\partial m} = m\bar{x}_i\bar{x}_j$, and hence

$$x_j \frac{\partial x_i}{\partial m} = x_i \frac{\partial x_j}{\partial m}.$$

   The result now follows from the Slutsky equation and the fact that differentiable Hicksian demand satisfies the Law of Reciprocity:

$$\begin{aligned} \frac{\partial x_i}{\partial p_j} &= \frac{\partial h_i}{\partial p_j} - x_j \frac{\partial x_i}{\partial m} \\ &= \frac{\partial h_i}{\partial p_j} - x_i \frac{\partial x_j}{\partial m} \\ &= \frac{\partial h_j}{\partial p_i} - x_i \frac{\partial x_j}{\partial m} = \frac{\partial x_j}{\partial p_i}. \end{aligned}$$

■

2. (25 pts) An investor has wealth w and can choose to invest any amount $x \geq 0$ in a risky asset. The rate of return on the investment is r, so that the income the investor will consume is

$$y = w - x + (1+r)x = w + rx.$$

The consumer's Bernoulli utility function for income, $u : \mathbb{R} \to \mathbb{R}$, satisfies $u' > 0$ and $u'' < 0$.

Both the investor's wealth and the asset's rate of return are random at the time the investment decision is made. Denoting these random variables as $\tilde{w}$ and $\tilde{r}$, assume $\mathbb{E}\tilde{r} > 0$ and that they are perfectly and linearly correlated: $w_0, \beta \in \mathbb{R}$ exist such that

$$\tilde{w} = w_0 + \beta\tilde{r}.$$

Fix w_0, and assume that for any $\beta \in \mathbb{R}$, the investor has a finite optimal investment level. Denote it as $x^*(\beta)$.

(a) (5 pts) Give a verbal, "intuitive" (virtually no math!) argument for whether $x^*(\beta)$ is positive or zero when $\beta < 0$.

Soln: $x^*(\beta, w_0)$ is positive when $\beta < 0$. This is because the investor's wealth is negatively correlated to the asset's rate of return in this case. Hence, investing in the asset not only increases expected income, but it also decreases it's risk, both of which the investor desires. (Her income is not risky if she sets $x = -\beta$, so she sets it at least that high. In fact she sets it higher, as we see below.) ∎

(b) (20 pts) Find an expression for $x^*(\beta)$ in terms of β and $x^*(0)$, valid for all $\beta \in \mathbb{R}$.

Soln: $x^*(\beta)$ is the unique solution of the problem

$$\max_{x \geq 0} \mathbb{E}u(w_0 + (\beta + x)\tilde{r}).$$

(Unique because the objective function is strictly concave in x.) Change the choice variable from x to $z = \beta + x$. Then $x^*(\beta) = z^*(\beta) - \beta$, where $z^*(\beta)$ is the unique solution of

$$\text{(P)} \quad \max_{z \geq \beta} \mathbb{E}u(w_0 + z\tilde{r}).$$

Note that $z^*(0) = x^*(0)$. Note too that $x^*(0) > 0$, since $x^*(0) = 0$ would imply by Kuhn-Tucker that

$$\mathbb{E}\tilde{r}u'(w_0) \leq 0,$$

contrary to $\mathbb{E}\tilde{r} > 0$ and $u' > 0$. It follows that $x^*(0)$ satisfies

$$\mathbb{E}\tilde{r}u'(w_0 + x^*(0)\tilde{r}) = 0.$$

This and the concavity of $\mathbb{E}u(w_0 + z\tilde{r})$ implies that $x^*(0)$ is its global maximizer on $\mathbb{R}$. Hence, its maximizer on $[\beta, \infty)$ is

$$z^*(\beta) = \begin{cases} x^*(0) & \text{if } \beta \leq x^*(0) \\ \beta & \text{if } \beta > x^*(0) \end{cases},$$

and so

$$x^*(\beta) = \begin{cases} x^*(0) - \beta & \text{if } \beta \leq x^*(0) \\ 0 & \text{if } \beta > x^*(0) \end{cases} = \max(0, x^*(0) - \beta).$$

∎

3. (25 pts) Let $C = \{1, \ldots, N\}$ be a set of consequences, with $N \geq 3$. Let $\succeq$ be a binary relation on the set $\Delta(C)$ of simple lotteries.

(a) (5 pts) State the independence axiom.

Soln: $\succeq$ satisfies independence iff the following holds: for any simple lotteries L, L', and L'', and any number $a \in (0, 1]$,

$$L \succeq L' \Leftrightarrow aL + (1-a)L'' \succeq aL' + (1-a)L''.$$

■

(b) (10 pts) Show that if $\succeq$ satisfies independence, then $\succeq$ is convex.

Soln: Another assumption is needed for this part, namely, that $\succeq$ is transitive.

By definition, a preference relation is convex if its upper contour sets are convex. So, let L and L'' be lotteries in the upper contour set of a lottery L' : $L \succeq L'$ and $L'' \succeq L'$. Letting $a \in (0, 1)$, we must prove that

$$aL + (1-a)L'' \succeq L'.$$

From independence and $L \succeq L'$, we have

$$aL + (1-a)L'' \succeq aL' + (1-a)L''.$$

By independence and $L'' \succeq L'$, we have

$$aL' + (1-a)L'' \succeq aL' + (1-a)L' = L'.$$

Transitivity now yields $aL + (1-a)L'' \succeq L'$. ■

(c) (10 pts) Suppose $\succeq$ is represented by the median function m, where for $L = (p_1, \ldots, p_N)$,

$$m(L) := \min \left\{ c \in C : \sum_{k \leq c} p_k \geq .5 \right\}.$$

State and prove whether $\succeq$ satisfies independence.

Soln: It does not satisfy independence. To show this, let $\bar{0}$ be the $N-3$ vector of all zeroes, and consider the following lotteries:

$$L = (0, 1, 0, \bar{0}),\ L' = (.6, 0, .4, \bar{0}),\ L'' = (0, 0, 1, \bar{0}).$$

Then $m(L) = 2$ and $m(L') = 1$, and so $L \succ L'$. However,

$$\begin{aligned} m(.8L + .2L'') &= m((0, .8, .2, \bar{0})) = 2, \text{ and} \\ m(.8L' + .2L'') &= m((.48, 0, .52, \bar{0})) = 3, \end{aligned}$$

which implies $.8L + .2L'' \prec .8L' + .2L''$. This contradicts independence.

[**Remark.** The preference relation m represents is easily shown to be convex. The converse of (b) is thus false.] ■

4. (25 pts) A competitive firm uses two inputs to produce one output according to a strictly concave and strictly increasing C^2 production function, $q = f(x, z)$. The input prices, $(w_x, w_z) \in \mathbb{R}^2_{++}$, are held fixed in this problem and so not written as arguments in functions. The firm has conditional demand functions $x^*(q)$ and $z^*(q)$, which give rise to a *long-run cost function* $c(q)$.

   In the short run, input z is fixed at $\bar{z}$, and only x is variable. The *short-run cost function* is

$$c_S(q, \bar{z}) = \min_{x \geq 0} w_x x + w_z \bar{z} \text{ such that } f(x, \bar{z}) \geq q.$$

   Suppose $\bar{z} = z^*(\bar{q})$ for some $\bar{q} > 0$.

   (a) (5 pts) For $q \neq \bar{q}$, how does $c(q)$ compare to $c_S(q, \bar{z})$? For $q = \bar{q}$?

   Soln: Since the short-run cost minimization problem is the same as the long-run problem but with one added constraint, $z = \bar{z}$,

$$c(q) \leq c_S(q, \bar{z}) \text{ for all } q \geq 0.$$

   For $q = \bar{q}$, the additional constraint $z = \bar{z}$ does not bind, since z is being held fixed at the level that would be cost minimizing anyway. Hence,

$$c(\bar{q}) = c_S(\bar{q}, \bar{z}).$$

   ■

   (b) (10 pts) Show that at $q = \bar{q}$, the short-run and long-run *marginal* cost curves cross, and the slope of the short-run curve is greater than that of the long-run curve.

   Soln: From (a), we see that

$$\bar{q} \in \arg\max_{q > 0} c(q) - c_S(q, \bar{z}).$$

   The necessary FOC and SOC are

$$\text{(NFOC)} \quad c'(\bar{q}) - c_S'(\bar{q}, \bar{z}) = 0,$$

$$\text{(NSOC)} \quad c''(\bar{q}) - c_S''(\bar{q}, \bar{z}) \leq 0,$$

   where c_S' and c_S'' denote the first and second derivatives with respect to q. The NFOC states that the short-run and long-run marginal cost curves cross at $\bar{q}$, and the NSOC states that the slope of the short-run curve is greater. ■

   (c) (10 pts) Let $q^*(p)$ and $q_S(p, \bar{z})$ denote the firm's long-run and short-run supply functions. Let $\bar{p}$ satisfy $q^*(\bar{p}) = \bar{q}$. Use the result of (b) to sketch an argument that at $\bar{p}$, the firm's price elasticity of supply is lower in the short run than it is in the long run.

   Soln: The FOCs for choosing q to maximize profit are the standard $p = MC$ equations:

$$p = c'(q^*(p)), \quad p = c_S'(q_S(p, \bar{z})).$$

   Differentiate with respect to p to obtain

$$q^{*\prime}(p) = \frac{1}{c''(q^*(p))}, \quad q_S'(p, \bar{z}) = \frac{1}{c_S''(q_S(p, \bar{z}))},$$

where $q_S'(p,\bar{z}) = \partial q_S/\partial p$. Since $q^*(\bar{p}) = q_S(\bar{p},\bar{z}) = \bar{q}$, from NSOC in (b) we obtain

$$q^{*\prime}(\bar{p}) = \frac{1}{c''(\bar{q})} \geq \frac{1}{c_S''(\bar{q})} = q_S'(\bar{p},\bar{z}).$$

This gives us the desired elasticity inequality:

$$q^{*\prime}(\bar{p})\frac{\bar{p}}{q^*(\bar{p})} = q^{*\prime}(\bar{p})\frac{\bar{p}}{\bar{q}} \geq q_S'(\bar{p},\bar{z})\frac{\bar{p}}{\bar{q}} = q_S'(\bar{p},\bar{z})\frac{\bar{p}}{q_S(\bar{p},\bar{z})}.$$

■

5. (25 pts) Consider a society $N = \{1,\ldots,n\}$ and a finite set X of alternatives. Assume $n \geq 2$ and $\#X \geq 3$. Let $\mathfrak{R}$ be the set of all complete and transitive binary relations on X. For each preference profile $\vec{R} \in \mathfrak{R}^n$, define a binary relation $F(\vec{R})$ on X by

$$\forall x,y \in X: \; xF(\vec{R})y \;\Leftrightarrow\; xR_1y \text{ or } xR_2y.$$

Answer the following questions, and prove your answers:

(a) (5 pts) Is F dictatorial?
(b) (5 pts) Does F satisfy Unanimity?
(c) (5 pts) Does F satisfy Independence of Irrelevant Alternatives?
(d) (5 pts) Is $C(B,\vec{R}) = \{x \in B : xF(\vec{R})y \;\forall y \in B\}$ nonempty for all $\vec{R} \in \mathfrak{R}^n$ and nonempty sets $B \subseteq X$?
(e) (5 pts) Is F an (Arrow) Social Welfare Function?

Soln: Given any profile $\vec{R}$, we let $R = F(\vec{R})$, and P and I be the strict and indifference relations derived from R (so xPy iff xRy and not yRx, and xIy iff xRy and yRx). Observe from the definition of F that for all $x,y \in X$ and $\vec{R} \in \mathfrak{R}^n$,

$$\begin{aligned} &xPy \text{ iff } xP_1y \text{ and } xP_2y, \\ &yPx \text{ iff } yP_1x \text{ and } yP_2x, \text{ and } xIy \text{ otherwise.} \end{aligned}$$

(a) **No.** Dictators dictate their strict preference: $d \in N$ is a *dictator* iff xP_dy implies xPy, for all $x,y \in X$. Neither person 1 nor 2 is a dictator, since a profile with xP_1y and yP_2x yields xIy. Nor is any $i > 2$ a dictator, since a profile with xP_iy and yR_1x yields yRx.
(b) **Yes.** Unanimity also refers to strict preference: F satisfies *Unanimity* iff xP_iy for all $i \in N$ implies xPy. If xP_iy for all $i \in N$, then xP_1y and xP_2y, and so indeed xPy.
(c) **Yes.** The definition of F directly implies that whether $xF(\vec{R})y$ holds depends only on the preferences (of persons 1 and 2) on $\{x,y\}$.
(d) **Yes.** For any $\vec{R}$, R_1 is complete and transitive. Hence, as B is nonempty and finite, $x \in B$ must exist such that xR_1y for all $y \in B$. Hence, xRy for all $y \in B$, implying $x \in C(B,\vec{R})$.
(e) **No.** An *Arrow SWF* on $\mathfrak{R}^n$ is a function from $\mathfrak{R}^n$ to $\mathfrak{R}$, i.e., it maps each $\vec{R} \in \mathfrak{R}$ into a complete and transitive binary relation on X. This F does not. Let x, y, and z be three distinct alternatives, and consider a profile satisfying

$$zP_1xP_1y, \quad yP_2zP_2x.$$

Then R is intransitive, since xIy, yIz, but zPx.

■