Economics 700 **Steven A. Matthews**
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Suggested Solutions to the Exam

1. Let $\mathcal{A}$ be a set with at least three elements. Given $u : \mathcal{A} \to \mathbb{R}$, define a binary relation on $\mathcal{A}$ by

$$a \succeq b \;\Leftrightarrow\; u(a) \geq u(b) - 1.$$

Let $\succ$ and $\sim$ be the strict preference and indifference relations derived from $\succeq$ in the usual way. For each of these three relations, is it necessarily (a) complete? Is it necessarily (b) transitive? Prove your answers.

Soln: The definitions of $\succ$ and $\sim$ are:

$$\begin{aligned} a &\succ b \Leftrightarrow a \succeq b \text{ and not } b \succeq a, \\ a &\sim b \Leftrightarrow a \succeq b \text{ and } b \succeq a. \end{aligned}$$

(a) $\succeq$ is complete. To prove this, note that for any pair (a, b), we have $u(a) \geq u(b)$ or $u(b) \geq u(a)$. In the former case we have $u(a) \geq u(b) - 1$, and so $a \succeq b$. In the latter case we have $u(b) \geq u(a) - 1$, and so $b \succeq a$.

Neither $\succ$ nor $\sim$ need be complete. A pair (a, b) is not related by $\succ$ if $u(a) = u(b)$, as then $a \sim b$. A pair (a, b) is not related by $\sim$ if $u(a) < u(b) - 1$, as then not $a \succeq b$.

(b) $\succ$ is transitive. To prove this, suppose $a \succ b$ and $b \succ c$. The former implies $u(b) < u(a) - 1$, and the latter implies $u(c) < u(b) - 1$. Hence,

$$\begin{aligned} u(a) &> u(b) + 1 \\ &> [u(c) + 1] + 1 \\ &> u(c) - 1, \end{aligned}$$

and so $a \succeq c$. We also have

$$\begin{aligned} u(c) &< u(b) - 1 \\ &< [u(a) - 1] - 1 \\ &< u(a) - 1, \end{aligned}$$

and so not $c \succeq a$. Thus, as desired, $a \succ c$.

$\sim$ need not be transitive. A counterexample: $u(a) = 0$, $u(b) = 3/4$, and $u(c) = 3/2$. Then $a \sim b$ and $b \sim c$, but $c \succ a$.

$\succeq$ need not be transitive. In the previous counterexample, $a \succeq b$ and $b \succeq c$, but not $a \succeq c$.

■

2. A consumer has wealth $w > 0$ and a strictly concave Bernoulli utility function $u : \mathbb{R} \to \mathbb{R}$ for income. She has probability $\pi \in (0, 1)$ of having an accident, in which case she will suffer a monetary loss of $L > 0$. She can purchase insurance: there is a price p such that if she pays px dollars, the insurance company will pay her x dollars if she has an accident. Assume it is feasible for her to buy full insurance: $w > pL$.

(a) Suppose the price p is equal to the actuarially fair rate. Prove, *with no further assumptions*, that the consumer buys full insurance: $x^* = L$.

Soln: The actuarially fair price is $p = \pi$, as then the expected profit of the insurance company when it sells coverage x is $px - \pi x = 0$. Let $U(x)$ be her expected utility from purchasing the amount x at price $p = \pi$. For any $x \neq L$,

$$\begin{aligned} U(x) &= \pi u(w - L + x - \pi x) + (1 - \pi)u(w - \pi x) \\ &< u\left[\pi(w - L + x - \pi x) + (1 - \pi)(w - \pi x)\right] \\ &= u(w - \pi L), \end{aligned}$$

where the inequality follows from $x \neq L$, $\pi \in (0, 1)$, and the strict concavity of u (Jensen's inequality). For $x = L$ we have

$$\begin{aligned} U(L) &= \pi u(w - L + L - \pi L) + (1 - \pi)u(w - \pi L) \\ &= u(w - \pi L). \end{aligned}$$

Thus, $U(L) > u(x)$ for all $x \neq L$, proving that the consumer's optimal insurance choice is $x^* = L$. ■

(b) Now suppose p is strictly greater than the actuarially fair rate. Under what (reasonable) additional assumption will she now buy less than full insurance? Give an intuitive explanation of this result.

Soln: When $p > \pi$, then $x^* < L$ if u is differentiable, with $u' > 0$. Starting from a lottery with no uncertainty, which is the case when $x = L$, the differentiability of u means the consumer is approximately risk neutral, and so finds it optimal to change x in a direction that increases her expected income. This direction is to lower x if the price of insurance is greater than π. ■

(c) Prove the result you stated in (b).

Soln: We now have

$$U(x) = \pi u(w_a) + (1 - \pi)u(w_n),$$

where $w_a := w - L + x - px$ and $w_n := w - px$. We can assume $x^* > 0$, as $x^* = 0$ immediately implies $x^* < L$. We thus have the interior FOC:

$$U'(x^*) = \pi(1 - p)u'(w_a^*) - (1 - \pi)pu'(w_n^*) = 0.$$

From this, using $p > \pi$, we obtain

$$\frac{u'(w_a^*)}{u'(w_n^*)} = \left(\frac{1 - \pi}{\pi}\right)\left(\frac{p}{1 - p}\right) > 1.$$

Hence, $u'(w_a^*) > u'(w_n^*)$. The strict concavity of u now implies $w_a^* < w_n^*$, or rather, $x^* < L$. ■

3. Each day a worker consumes leisure, ℓ, and income, x, and has a strictly increasing utility function $u(\ell, x)$. Leisure is measured in hours. She works for the remaining $L = 24 - \ell$ hours, at wage w. The income she consumes must thus satisfy $x \leq wL$. She has differentiable demand functions.

   Prove or disprove: if leisure is a normal good for this consumer, then her labor supply curve must be upward sloping, i.e., $\widehat{L}'(w) \geq 0$.

   Soln: False, $\widehat{L}'(w) < 0$ is possible.

   Letting the price of "income" be one, the budget constraint is $x \leq wL = 24w - w\ell$, or rather, $x + w\ell \leq 24w$. The consumer problem is

$$\max_{\ell,x} u(\ell, x) \text{ subject to } x + w\ell \leq 24w.$$

   This is just the consumer problem with an endowment of leisure equal to 24. Recall the Slutsky equation derived for the consumer problem with endowments:

$$\frac{\partial \hat{x}_i}{\partial p_i} = \frac{\partial x_i^h}{\partial p_i} - (\hat{x}_i - \omega_i)\frac{\partial x_i^*}{\partial y}.$$

   Translating the symbols to those of this problem, we have $\ell = x_i$, $p_i = w$, and $\omega_i = 24$. Hence,

$$\hat{\ell}'(w) = \frac{\partial \ell^h}{\partial w} - (\hat{\ell} - 24)\frac{\partial \ell^*}{\partial y},$$

   where $\ell^* = \ell^*(w, y)$ evaluated at $y = 24w$, $\hat{\ell} = \hat{\ell}(w) = \ell^*(w, 24w)$, and $\ell^h = \ell^h(w, \bar{u})$ evaluated at $\bar{u} = v(w, 24w)$. The consumer's labor supply function is $\widehat{L}(w) = 24 - \ell^*(w, 24w)$, and hence

$$\hat{\ell}'(w) = \frac{\partial \ell^h}{\partial w} + \widehat{L}\frac{\partial \ell^*}{\partial y}.$$

   The first term on the right side of this expression is nonpositive (Hicksian demands satisfy the Law of Demand, i.e., own-substitution effects are negative). The second term is nonnegative: **by definition, the statement that leisure is a normal good means** $\partial \ell^* / \partial y \geq 0$. Hence, the substitution and income effects work in opposite directions here, and the overall sign depends on the functional forms and price. The same is true of $\widehat{L}'(w)$, since $\widehat{L}'(w) = -\hat{\ell}'(w)$.

   Of course, this is an explanation, not a proof. A real proof requires an example in which $\widehat{L}'(w) = -\hat{\ell}'(w) < 0$. But it tells us to look for an example in which the substitution effect is small. So let's try a Leontief utility function: $u(\ell, x) = \min(\ell, x)$. Then utility maximization requires $x = \ell$, and the consumer problem reduces to

$$\max_{\ell} \ell \text{ subject to } (1 + w)\ell \leq 24w.$$

   The solution is $\hat{\ell}(w) = 24w/(1 + w)$, and so $\widehat{L}(w) = 24/(1 + w)$. As desired, the labor supply curve slopes down: $\widehat{L}'(w) < 0$. ■

4. A competitive firm uses skilled and unskilled labor, L and ℓ, to produce one good. It has been observed that when the output price increases, the firm hires more skilled and fewer unskilled workers. Now the wage of the unskilled workers increases (they've unionized), but all other prices stay fixed. The firm has continuously differentiable demand and supply functions.

   State and prove what happens to (a) the firm's demand for unskilled workers, and (b) its supply of output.

   Soln:

   (a) The input demand functions of a competive firm satisfy the Law of Demand – see any text for a proof. (Easiest proof: revealed preference. Next easiest: use the convexity of the profit function and the envelope theorem.) Hence, the firm hires fewer unskilled workers when their wage rises.

   (b) The firm's supply of output increases.

   We prove this under the additional assumption that the firm's supply and demand functions are C^2 functions.

   Let $\ell(p, w)$ be the firm's demand for unskilled labor when its output price is p and the wage rate of unskilled workers is w, and let $y(p, w)$ be its labor supply function. (Supress the wage rate of skilled workers and all other factors, as they are held fixed here.) Let $\pi(p, w)$ be the corresponding profit function. We are told that $\ell_p(p, w) < 0$. Since the firm's supply and demand functions are C^2 functions, so is π. Hence, Young's Theorem tells us that π has symmetric cross partial derivatives. By the Envelope Theorem (called Hotelling's lemma in this application),

$$\pi_w(p, w) = -\ell(p, w), \quad \pi_p(p, w) = y(p, w).$$

   Hence,

$$y_w(p, w) = \pi_{wp}(p, w) = \pi_{pw}(p, w) = -\ell_p(p, w) > 0.$$

   The supply function indeed increases in w.

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5. A competitive firm has a continuous, strictly increasing and strictly concave production function, $f : \mathbb{R}^n_+ \to \mathbb{R}_+$, satisfying $f(\mathbf{0}) = 0$. It gives rise to a C^2 cost function $c(y, \mathbf{w})$. Holding $\mathbf{w} \gg \mathbf{0}$ fixed, derive the necessary properties of the function $c(\cdot, \mathbf{w})$ of output.

   Soln: Since $\mathbf{w}$ is held fixed, we simplify by writing $c(y)$. For a given y, $c(y)$ is defined by the problem

$$(\mathrm{P} - y) \quad c(y) = \min_{x \geq 0} w \cdot x \text{ subject to } f(x) \geq y.$$

   (a) $c(0) = 0$.
   This follows directly from $f(0) = 0$.

   (b) $c(y)$ is strictly increasing.
   To prove this, first note that the constriant of (P-y) binds at any solution x. For, $f(x) > y$ would imply $x > 0$, and this together with the continuity of f would imply the existence of $t < 1$ such that $f(tx) > y$, yielding the contradiction $c(y) = w \cdot x > w \cdot (tx) \geq c(y)$. Now, let $y > \hat{y} \geq 0$, and let x and $\hat{x}$ be solutions of (P-y) and $(\mathrm{P}-\hat{y})$, respectively. Then x is feasible for $(\mathrm{P}-\hat{y})$, and hence

$$c(\hat{y}) \leq w \cdot x = c(y).$$

   This inequality must actually be strict, since otherwise x would solve $(\mathrm{P}-\hat{y})$ and satisfy its constraint with slack. Thus, c is a strictly increasing function.

   (c) $c(y)$ is strictly convex.
   This follows from the strict concavity of f. To prove this, let $y \neq \hat{y}$, $a \in (0, 1)$, and $y_a := ay + (1 - a)\hat{y}$. Let x and $\hat{x}$ be solutions of (P-y) and (P-$\hat{y}$), respectively. Since $y \neq \hat{y}$, we have $x \neq \hat{x}$. The strict concavity of f now implies

$$\begin{aligned} f\left(ax + (1 - a)\hat{x}\right) &> af(x) + (1 - a)f(\hat{x}) \\ &= ay + (1 - a)\hat{y} \\ &= y_a. \end{aligned}$$

   Thus, $ax + (1 - a)\hat{x}$ satisfies the constraint of (P-y_a) with slack, and so does not solve the problem (as shown in (b)). Hence,

$$\begin{aligned} c(y_a) &< w \cdot (ax + (1 - a)\hat{x}) \\ &= a(w \cdot x) + (1 - a)(w \cdot \hat{x}) \\ &= ac(y) + (1 - a)c(\hat{y}). \end{aligned}$$

   This proves c is strictly convex.

   (d) $c'(y) > 0$ and $c''(y) \geq 0\ \forall y > 0$. This follows from (b) and (c), since c is C^2. (Furthermore, $c''(y) > 0$ for almost all $y > 0$.)

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