701A Prelim Questions
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1. (June 2014) (25 pts) A strictly increasing utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ gives rise to a demand function $\mathbf{x}(\mathbf{p}, y) = (x_1(\mathbf{p}, y), x_2(\mathbf{p}, y))$. It is continuously differentiable in a neigborhood N of some $(\mathbf{p}^0, y^0) \gg \mathbf{0}$. Theory tells us much about the nature of such demand functions: use what it tells us to answer the following questions.

   (a) (3 pts) Write the definition of $\eta_i(\mathbf{p}, y)$, the income elasticity for good i.

   (b) (11 pts) Suppose the demand for good 1 takes the form
   $$x_1(\mathbf{p}, y) = \alpha_1(\mathbf{p})g_1(y)$$
   for all $(\mathbf{p}, y) \in N$. What is the most that this implies about $\eta_1(\mathbf{p}, y)$ on N?

   (c) (11 pts) Now suppose in addition that in N, demand for good 2 takes the same form,
   $$x_2(\mathbf{p}, y) = \alpha_2(\mathbf{p})g_2(y),$$
   and $\mathbf{x}(\mathbf{p}, y)$ satisfies the law of reciprocity:
   $$\frac{\partial x_1}{\partial p_2} = \frac{\partial x_2}{\partial p_1}.$$
   What can you now say about the two income elasticity functions on N?

2. (June 2014) (25 pts) A competitive firm uses hops to make beer via a production function $f : \mathbb{R} \to \mathbb{R}$. The price of beer is $p > 0$ and the cost of new hops is $w > 0$. The firm has $x_0 > 0$ hops left over from last year, but an unknown amount of these old hops will either grow or spoil before production begins this year; the amount of them that will be usable is $x_0 + \theta\tilde{\varepsilon}$, where $\tilde{\varepsilon}$ is a nondegenerate random variable with mean zero, $\theta > 0$ is a parameter to allow easy comparative statics, and $\theta\tilde{\varepsilon} \in (-x_0, x_0)$. The firm will purchase more hops, $x \geq 0$, to maximize its expected profit,
   $$\mathbb{E}\left\{pf(x_0 + \theta\tilde{\varepsilon} + x) - wx\right\}.$$
   Assume f is smooth with derivatives $f' > 0$ and $f'' < 0$, and that the solution, $x^*(x_0, w, p, \theta)$, is positive. Make, if necessary, additional reasonable assumptions under which the signs of the partial derivatives,
   $$x^*_{x_0},\ x^*_w,\ x^*_p,\ x^*_\theta,$$
   can be determined, and find their signs under those assumptions.

3. (August 2014) (25 pts) There are two possible states of the world and one good, "money". It is commonly known that state s will occur with probability $\pi_s > 0$, for $s = 1, 2$. A state contingent allocation is a pair $(x_1, x_2) \in \mathbb{R}^2_+$. Consider a consumer who has a complete, transitive, and strongly monotonic ordering $\succsim$ over these allocations. Assume $\succsim$ is convex.

   (a) (10 pt) Prove or disprove: This consumer must be weakly risk averse.

(b) (15 pts) Do the same as in (a), but under the assumption now that the consumer satisfies the expected utility hypothesis. Let u denote the consumer's Bernoulli utility function, and assume it is twice continuously differentiable, with $u' > 0$.

4. (August 2014) (25 pts) Consider a society $N = \{1, \ldots, n\}$ and a finite set X of alternatives. Assume $n \geq 2$ and $\#X \geq 3$. Let $\mathfrak{R}$ be the set of complete and transitive binary relations on X. One alternative, $s \in X$, is the *status quo.* For each profile $\vec{R} \in \mathfrak{R}^n$, let G ("good") be the set of alternatives that are weakly Pareto preferred to s :

$$G = \{x \in X : xR_i s \ \forall i \in N\}.$$

(Note that $s \in G$.) Let B ("bad") be the complementary set, $B = X \backslash G$. For each $\vec{R} \in \mathfrak{R}^n$ define a binary relation $F(\vec{R})$ on X by

$$\begin{aligned} \forall x \in G, y \in B: &\quad xF(\vec{R})y \text{ and not } yF(\vec{R})x \\ \forall x, y \in G: &\quad xF(\vec{R})y \ \Leftrightarrow \ xR_n y \\ \forall x, y \in B: &\quad xF(\vec{R})y \ \Leftrightarrow \ xR_n y \end{aligned}$$

Answer the following questions, and prove your answers:

(a) (6 pts) Is F dictatorial?

(b) (6 pts) Does F satisfy Unanimity?

(c) (6 pts) Does F satisfy Independence of Irrelevant Alternatives?

(d) (7 pts) Is F an (Arrow) Social Welfare Function?

5. (June 2015) (35 pts) Axel is a newsboy. He can choose whether or not to buy a fixed amount of newpapers to resell. If he buys none, his profit will be 0. If he buys the fixed amount, his profit will depend on how many consumers come to his newsstand. This amount is a random variable D given by

$$D = \begin{cases} 0 & \text{with prob } p \\ 50 & \text{with prob } 1 - p \end{cases},$$

where $p \in (0, 1)$. If Axel buys the newspapers, his profit will be

$$\pi = \begin{cases} -15 & \text{if } D = 0 \\ 35 & \text{if } D = 50 \end{cases}.$$

Axel's Bernoulli utility function for money is u, which is C^2, strictly increasing, and concave.

Barb also owns a newsstand. She faces the exact same supply and demand environment as Axel. The only difference is that she has a different utility function, v, which is also C^2, strictly increasing, and concave. Lastly, *Barb is strictly more risk averse than Axel.*

(a) (5 pts) Show that $p_A \in (0, 1)$ exists such that Axel's optimal decision is to buy the newspapers if and only if $p < p_A$ (he is indifferent in the knife-edge case $p = p_A$).

(b) (10 pts) Letting p_B be the corresponding critical probability for Barb, prove which is larger, p_A or p_B.

Now suppose Axel, before deciding whether to buy the newspapers, is able to purchase perfect information about what his demand will be, i.e., Axel can learn whether $D = 0$ or $D = 50$. Obviously, if he acquires this information he will buy the newspapers if and only if he learns $D = 50$. Let I_A denote the maximum amount he is willing to pay for this information (I_A is the "value of information" to Axel). Let I_B be the corresponding amount for Barb.

(c) (10 pts) Assuming $p > \max\{p_A, p_B\}$, prove which is larger, I_A or I_B.

(d) (10 pts) (Not used) Assuming $p < \min\{p_A, p_B\}$, prove which is larger, I_A or I_B.

6. (June 2015) (25 pts) The Superior Coffee Shop (Starbucks?) sells a card that entitles its owner to a 10% discount on (tall) cups of coffee for a year. Denote such cups of coffee for Ms. Consumer as good 1, i.e., let x_1 denote the number of such coffees she will consume in a year. All other goods are represented as good 2. Ms. Consumer has a strictly increasing utility function for $x = (x_1, x_2)$ that gives rise to the Hicksian demand functions

$$h_i(p, u) = \frac{p_j}{p_i} u \text{ for } i, j = 1, 2 \text{ and } j \neq i.$$

Let B ("Buy price") be the maximum price Ms. Consumer would pay for this discount card. Let S ("Sell price") be the minimum price for which she would be willing to sell the card if she were to already own it. Let $p^0 = (p_1^0, p_2^0)$ denote the prices without the discount card, and $(p_1^1, p_2^1) = (.9p_1^0, p_2^0)$ the prices with the card.

(a) (15 pts) Find the ratio B/S in terms of $u^0 = v(p^0, m)$ and $u^1 = v(p^1, m)$. Which is larger, B or S?

(b) (10 pts) Professor Behavior proclaims that for any consumer, owning a good makes the consumer attached to it, so that he/she will not sell it except for a higher price than he/she would have been willing to pay for it before, i.e., that $B < S$. (This is the so-called "endowment effect".) Propose an experiment that can provide evidence to distinguish Professor Behavior's hypothesis from the prediction of neoclassical consumer theory (as learned in Econ 701). Explain your reasoning.

7. (August 2015) (25 pts) Each day a worker consumes leisure, ℓ, and income, x, and has a strictly increasing utility function $u(\ell, x)$. Leisure is measured in hours. She works for the remaining $L = 24 - \ell$ hours, at wage w. The income she consumes must thus satisfy $x \leq wL$. She has differentiable demand functions.

(a) (15 pts) Prove that if leisure is an inferior good for this consumer, then her labor supply curve must be upward sloping, i.e., $\widehat{L}'(w) \geq 0$.

(b) (10 pts) Prove that if leisure is a normal good for this consumer, then her labor supply curve might slope down. In particular, show that $\widehat{L}'(w) < 0$ is possible even if her utility function is increasing and quasiconcave.

8. (August 2015) (25 pts) In this problem you first prove a result comparing the riskiness of gambles, and then a result comparing the value of information to different decision makers.

(a) (10 pts) Let $\hat{x} < x < y < \hat{y}$ and $p \in (0, 1)$. Let g be the gamble yielding x with probability p and y with probability $1 - p$. Similarly, let $\hat{g}$ be the gamble yielding $\hat{x}$ with probability p and $\hat{y}$ with probability $1-p$. Assume g and $\hat{g}$ have the same expected value. Rigorously prove the following.

Lemma. Any risk averse expected utility maximizer strictly prefers g to $\hat{g}$. (Assume the EU maximizer has a strictly increasing, strictly concave C^1 utility function, $v : \mathbb{R} \to \mathbb{R}$.)

(b) (15 pts) Axel is a newsboy who must choose whether to open his newsstand today. His profit will be 0 if he keeps it closed. If he opens it, his profit will be the random variable

$$\pi = \begin{cases} -15 & \text{with probability } p \\ 35 & \text{with probability } 1 - p \end{cases},$$

where $p \in (0, 1)$. Axel is risk neutral, and so maximizes expected profit.

Before Axel decides whether to open his newsstand, he is able to purchase perfect information about whether his profit from opening will be $\pi = -15$ or $\pi = 35$. Let I_A be the maximum amount Axel is willing to pay for this information.

Barb also owns a newsstand. She faces the same environment as Axel. Barb, however, is risk averse: she has a C^1 Bernoulli utility function for money, v, which is strictly increasing and strictly concave. She too can choose to purchase the perfect information about π. Let I_B be the maximum amount Barb is willing to pay for this information.

Recall that $p_B \in (0,1)$ exists such that, if she does not acquire the information, Barb's optimal decision is to open her newsstand iff $p < p_B$. Assuming $p < p_B$, show that $I_B > I_A$.

(**Hint:** consider using the Lemma in (a), even if you were unable to prove it.)

9. (25 pts) In a two-good world, consider the function $e : \mathbb{R}^2_{++} \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$e(p, U) := U \min\left\{p_1,\ \frac{p_1 + p_2}{3},\ p_2\right\}.$$

   (a) (5 pts) State four properties that an expenditure function must satisfy if it arises from a continuous monotonic utility function, and verify that the given e indeed satisfies them.

   (b) (5 pts) Let $h(p, U)$ denote a Hicksian demand correspondence that gives rise to the expenditure function e. For each $(p, U) \in \mathbb{R}^2_{++} \times \mathbb{R}_+ \to \mathbb{R}_+$, find as many points in the set $h(p, U)$ as you can.

   (c) (15 pts) For each property of a function listed below, state whether there is a utility function satisfying that property which gives rise to the expenditure function e, and sketch proofs of your answers.

      i. (5 pts) strictly quasiconcave.

      ii. (5 pts) quasiconcave.

      iii. (5 pts) non-quasiconcave.

10. (25 pts) A consumer lives for two periods. In period 2 she will purchase a commodity bundle $x = (x_1, x_2)$ to maximize her utility $u(x)$ subject to her budget constraint $p \cdot x \le y$.

   (a) (5 pts) Let $v(y)$ be the consumers's indirect utility arising from $u(x)$ (we suppress the argument p since it is fixed in this question). Show that v is concave in y if u is concave in x.

In period 1 the consumer chooses an amount z to invest in a risky asset that returns $(1+\tilde{r})z$ in period 2. Her initial wealth is $w > 0$, and she is restricted to choosing $z \in [0, w]$. She uses her resulting income in period 2, $\tilde{y} = w + \tilde{r}z$, to purchase x at the prices p. She knows these prices in period 1, and chooses z in order to maximize the expected utility she will ultimately obtain in period 2. For each $w \ge 0$, let $z^*(w)$ be an optimal investment for this consumer.

Assume the random variable $\tilde{r}$ is continuously distributed on an interval $[\underline{r}, \bar{r}]$, with $\underline{r} > -1$ and $\mathbb{E}\tilde{r} > 0$.

   (b) (10 pts) Suppose v is C^2 and satisfies $v' > 0$, $v'' < 0$, and DARA ($A(y) := -v''(y)/v'(y)$ is a strictly decreasing function). Show that for any $w > 0$, if $z^*(w) < w$ holds, then $z^{*\prime}(w) > 0$.

(c) (10 pts) Now assume $u(x) = u_1(x_1) + u_2(x_2)$, where each u_i is C^2 with $u_i' > 0$, $u_i'' < 0$, and $u_i'(0) = \infty$. Show that if u_1 and u_2 each satisfy DARA, then v satisfies DARA.

11. (25 pts) The inverse demand function for oil is given by a continuously differentiable function $P : \mathbb{R}_{++} \to \mathbb{R}_{++}$ satisfying $P' < 0$ and $P(x) \to \infty$ as $x \downarrow 0$. The price elasticity of the demand for oil is defined at any $x > 0$ as

$$e(x) := -\frac{P(x)}{P'(x)x}.$$

The total stock of oil below the ground is $0 < \bar{x} < \infty$. It is all owned by one oil company, which can extract it at zero cost. The firm's profit is zero if it sells no oil, and its profit is px if it sells an amount $x > 0$ at price p.

(a) (6 pts) Compare the competitive equilibrium (x^c, p^c) to the monopoly outcome (x^m, p^m) under (i) the assumption that $e(x) > 1$ for all $x \in (0, \bar{x}]$, and (ii) under the assumption that $e(\bar{x}) < 1$.

Now suppose there are two periods, $t = 1, 2$, and the firm discounts the second period at rate $r > 0$. The inverse demand function in period t is $P_t(x_t)$, which has the same properties as does the function P above. The firm's discounted payoff if it sells x_t in period t at price p_t is $p_1x_1 + (1 + r)^{-1}p_2x_2$, where (x_1, x_2) must satisfy $x_1 + x_2 \le \bar{x}$. Its profit in period t is 0 if $x_t = 0$.

(b) (6 pts) Suppose $(p_1^c, x_1^c, p_2^c, x_2^c)$ is a competitive equilibrium satisfying $x_1^c > 0$ and $x_2^c > 0$. Find a system of four equations this equilibrium must satisfy. Then compare x_1^c to x_2^c when $P_1(\cdot) = P_2(\cdot)$.

(c) (6 pts) Again allowing P_1 and P_2 to be different functions, assume now that for some $\underline{e} > 1$, the elasticities satisfy $e_t(x_t) > \underline{e}$ for all $x_t \in (0, \bar{x}]$ and $t = 1, 2$. Suppose $(p_1^m, x_1^m, p_2^m, x_2^m)$ is a monopoly outcome satisfying $x_1^m > 0$ and $x_2^m > 0$. Find a system of four equations this outcome must satisfy.

(d) (7 pts) Under the additional assumption that both P_1 and P_2 have constant elasticities, e_1 and e_2, satisfying $e_1 \ge e_2 > 1$, how does $(p_1^m, x_1^m, p_2^m, x_2^m)$ compare to $(p_1^c, x_1^c, p_2^c, x_2^c)$?

12. (25 pts) Consider a Bernoulli utility function $u : \mathbb{R} \to \mathbb{R}$ that has derivatives $u' > 0$ and $u'' < 0$, and exhibits DARA (decreasing absolute risk aversion). Prove each of the following.

(a) (10 pts) (Lemma) For any $k \in \mathbb{R}$ and any random gamble $\tilde{y}$,

$$\mathbb{E}u(\tilde{y}) = u(k) \;\Rightarrow\; \mathbb{E}u(\tilde{y} + a) > u(k + a) \;\forall a > 0.$$

Even if you were unable to prove the "Lemma" in (a), feel free to use it to prove (b)-(d) below.

(b) (5 pts) Let $\tilde{x}$ be a random gamble, and let $b(w)$ be the maximum price the agent is willing to pay for $\tilde{x}$ when her wealth is w. Then $b(w)$ increases in w.

(c) (5 pts) Let $\tilde{x}$ be a random gamble, and let $s(w)$ be the minimum price at which the agent is willing to sell $\tilde{x}$ when her wealth is w. Then $s(w)$ increases in w.

(d) (5 pts) Now let $\tilde{x}$ be a random gamble that is valuable at wealth w, in the sense that $\mathbb{E}u(w + \tilde{x}) > u(w)$. Then $s(w) > b(w)$, where $b(w)$ and $s(w)$ are defined in (b) and (c) from this $\tilde{x}$ and w.