Economics 701A **Steven A. Matthews**
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Suggested Solutions to the Quiz

20 points, 30 minutes. Closed books, notes, calculators.
Indicate your reasoning, using clearly written words as well as math.

1. Consider a C^1 strictly quasiconcave utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ that has positive partial derivatives everywhere. Suppose it gives rise to a differentiable demand function $x(p,m)$, and assume the indirect utility function $v(p,m)$ is homogeneous of degree 1 in m.

   (a) (2 pts) Define what it means for an arbitrary function $f : \mathbb{R}^k_+ \to \mathbb{R}$ to be homogeneous of degree 1.

   Soln: $f : \mathbb{R}^k_+ \to \mathbb{R}$ is *homogeneous of degree* 1 iff for all $x \in \mathbb{R}^k_+$ and $t > 0$, $f(tx) = tf(x)$. ∎

   (b) (5 pts) Prove that $x(p,m)$ is homogeneous of degree 1 in m.

   Soln: "Over-the-top" Proof. Since $v(p,m)$ is homogeneous of degree 1 in m, $v_m(p,m)$ is homogeneous of degree 0 in m, and $v_{p_i}(p,m)$ is homogeneous of degree 1 in m. Hence, from Roy's identity,

$$\begin{aligned} x_i(p,tm) &= -\frac{v_{p_i}(p,tm)}{v_m(p,tm)} \\ &= -\frac{tv_{p_i}(p,m)}{v_m(p,m)} = tx_i(p,m). \end{aligned}$$

   This proves $x(p,m)$ is homogeneous of degree 1 in m. ∎

   More Elementary (Better) Proof. For a continuous function of one variable, homogeneity of degree 1 is equivalent to linearity. Thus, a function $\bar{v}(p)$ exists such that $v(p,m) = \bar{v}(p)m$. Roy's identity now yields

$$x_i(p,m) = -\frac{v_{p_i}(p,m)}{v_m(p,m)} = \left(-\frac{\bar{v}_{p_i}(p)}{\bar{v}(p)}\right) m.$$

   This is linear in m, so $x(p,m)$ is homogeneous of degree 1 in m. ∎

   (c) (5 pts) Using (b), prove that for any $(p,m) \in \mathbb{R}^{n+1}_{++}$,

$$u(tx(p,m)) = tu(x(p,m)).$$

   Soln: We have

$$\begin{aligned} u(tx(p,m)) &= u(x(p,tm)) \text{ (as } x \text{ is homog deg 1 in } m) \\ &= v(p,tm) \text{ (by the def of } v) \\ &= tv(p,m) \text{ (as } v \text{ is homog deg 1 in } m) \\ &= tu(x(p,m)) \text{ (by the def of } v). \end{aligned}$$

   ∎

(d) (8 pts) Using (c) and the supporting hyperplane theorem, conclude that u is homogeneous of degree 1.

Soln: Fix $\bar{x} \in \mathbb{R}^n_{++}$ and $t > 0$. We shall prove $u(t\bar{x}) = tu(\bar{x})$. The continuity of u will then imply the same equality on the boundary of $\mathbb{R}^n_{++}$, so that u is indeed homogeneous of degree 1 on $\mathbb{R}^n_+$.

From (c) we know $u(tx(p,m)) = tu(x(p,m))$ for all $(p,m) \in \mathbb{R}^{n+1}_{++}$. Therefore, to show that $u(t\bar{x}) = tu(\bar{x})$, it suffices to prove the existence of $(p,m) \in \mathbb{R}^{n+1}_{++}$ such that $x(p,m) = \bar{x}$. In two dimensions this is graphically obvious – we find p by finding the line through $\bar{x}$ that is tangent to the upper contour set of $\bar{x}$, and then define m as $p \cdot \bar{x}$.

So, let $\bar{u} := u(\bar{x})$ and

$$U := \{y \in \mathbb{R}^n_+ : u(y) \geq \bar{u}\}.$$

Then U is closed because u is continuous, and it is convex because u is quasiconcave. Also, $\bar{x}$ is a boundary point of U because u is strictly increasing. The supporting hyperplane theorem thus tells us that $p \in \mathbb{R}^n$ exists such that $p \neq 0$ and

$$p \cdot \bar{x} \leq p \cdot y \text{ for all } y \in U. \tag{1}$$

In other words,

$$\bar{x} \in \arg\min_{y \geq 0} p \cdot y \text{ such that } u(y) \geq \bar{u}.$$

Hence, $\bar{x} = h(p,\bar{u})$ and $p \cdot \bar{x} = e(p,\bar{u})$. Define $m := e(p,\bar{u})$. Then $m = p \cdot \bar{x} > 0$ if $p \in \mathbb{R}^n_{++}$, and in this case we have $h(p,\bar{u}) = x(p,m)$, and hence the desired result, $\bar{x} = x(p,m)$.

It remains only to show that $p \in \mathbb{R}^n_{++}$. Suppose $p_j < 0$ for some j. Define y by $y_i = \bar{x}_i$ if $i \neq j$, and $y_j = \bar{x}_j + 1$. Since u is monotonic, $y \in U$. However, $p_j < 0$ implies $p \cdot y < p \cdot \bar{x}$. This contradiction of (1) implies $p \in \mathbb{R}^n_+$.

Now suppose $p_j = 0$ for some j. Defining y as above, we now have

$$p \cdot y = p \cdot \bar{x}.$$

The strict monotonicity of u implies $u(y) > \bar{u}$. Since $\bar{x} \in \mathbb{R}^n_{++}$, we have $y \in \mathbb{R}^n_{++}$. Thus, since u is continuous, $\varepsilon > 0$ exists such that, letting $y'_i = y_i - \varepsilon$ for all i, we have $y' \in \mathbb{R}^n_{++}$ and $u(y') > \bar{u}$. This implies that $y' \in U$. We obtain

$$p \cdot y' < p \cdot y$$

from $y' \ll y$, $p \in \mathbb{R}^n_+$, and $p \neq 0$. We conclude that $y' \in U$ and $p \cdot y' < p \cdot \bar{x}$. As this contradicts (1), we have proved $p \in \mathbb{R}^n_{++}$. ■