Economics 681 **Steven A. Matthews**
Fall 2017 **University of Pennsylvania**

Solutions for Midterm 1

75 minutes, 80 points. Closed books, notes, calculators.
Indicate your reasoning, using clearly written words as well as math.

1. (20 pts) Consider the utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ defined by

$$u(x) = \max\left\{x_1 + x_2, (x_1 + x_2)^2\right\}.$$

(a) (10 pts) Find a differentiable function on $\mathbb{R}^2_+$ which represents the same preferences as u.

Soln: Note that

$$u(x) = \begin{cases} x_1 + x_2 & \text{if } x_1 + x_2 \leq 1 \\ (x_1 + x_2)^2 & \text{if } x_1 + x_2 \geq 1 \end{cases}.$$

Thus, $u(x) \geq u(y)$ iff $x_1 + x_2 \geq y_1 + y_2$. So the differentiable function $\hat{u}(x) = x_1 + x_2$ for all $x \in \mathbb{R}^2_+$ represents the same preferences as does u. (The strictly increasing f for which $\hat{u} = f \circ u$ is

$$f(v) = \left\{ \begin{array}{ll} v & \text{if } v \leq 1 \\ \sqrt{v} & \text{if } v \geq 1 \end{array} . \right)$$

■

(b) (10 pts) Is u quasiconcave, concave, or neither? Prove your answer.

Soln: Quasiconcavity is a property of the upper level sets, namely, that they all be convex sets. In fact, any level set of u is the intersection of a halfplane with the nonnegative orthant $\mathbb{R}^2_+$, which is clearly a convex set. So yes, u is quasiconcave.

But u is not concave. Proof: Let $x^0 = (0, 0)$, $x^1 = (2, 0)$, and $\hat{x} = \frac{1}{2}\left(x^0 + x^1\right) = (1, 0)$. If u were to be concave, then $u(\hat{x}) \geq \frac{1}{2}\left(u(x^0) + u(x^1)\right)$. However, we have

$$u(x^0) = 0,\ u(x^1) = 4 \ \Rightarrow\ \frac{1}{2}\left(u(x^0) + u(x^1)\right) = 2,$$

whereas

$$u\left(\hat{x}\right) = 1 < 2.$$

So u is not concave. (Although $\hat{u}$ is.) ■

2. (20 pts) Suppose $u : \mathbb{R}^n_+ \to \mathbb{R}$ gives rise to a C^1 Hicksian demand function $x^h(p, U)$. Show that for any j,

$$\sum_{i=1}^{n} p_i \frac{\partial x_i^h}{\partial p_j} = 0.$$

Soln: We have $p \cdot x^h(p, U) = e(p, U)$. Differentiating this w.r.t. p_j (using the chain rule) yields

$$x_j^h + \sum_{i=1}^{n} p_i \frac{\partial x_i^h}{\partial p_j} = \frac{\partial e}{\partial p_j}.$$

Since $x_j^h = \frac{\partial e}{\partial p_j}$ by Shepard's lemma, we are left with

$$\sum_{i=1}^{n} p_i \frac{\partial x_i^h}{\partial p_j} = 0.$$

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Do ONLY one of the following two problems.
If you do both, only #3 will be graded.

3. (40 pts) Consider the utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ defined by

$$u(x) = \sqrt{x_1 + 1} + x_2,$$

which is a strictly quasiconcave function. Let $x^*(p, y)$ be the corresponding demand function. Assume $p_1 = 1$.

(a) (20 pts) Find a necessary and sufficient condition on p_2 for $x_1^* = 0$, another for $x_2^* = 0$, and yet another for $x^* \gg (0, 0)$. [Hint: Draw a figure, and then compare the MRS at each endpoint of the budget line to the slope of the budget line.]

Soln: The absolute value of the slope of an indifference curve at any x is

$$MRS(x) = \frac{u_1(x)}{u_2(x)} = u_1(x) = \frac{1}{2\sqrt{x_1 + 1}}.$$

The absolute value of the slope of the budget line is $p_1/p_2 = 1/p_2$. Note that

$$\frac{1}{p_2} \geq MRS\left(\left(0, \frac{y}{p_2}\right)\right) = \frac{1}{2} \;\Leftrightarrow\; p_2 \leq 2.$$

We therefore conclude that $p_2 \leq 2$ is a necessary and sufficient condition for $x_1^* = 0$ (draw the figure to see this clearly). We also have

$$\frac{1}{p_2} \leq MRS\left(\left(\frac{y}{p_1}, 0\right)\right) = \frac{1}{2\sqrt{y + 1}} \;\Leftrightarrow\; p_2 \geq 2\sqrt{y + 1}.$$

We conclude that $p_2 \geq 2\sqrt{y + 1}$ is a necessary and sufficient condition for $x_2^* = 0$. (Draw the figure!) Thus, a necessary and sufficient condition for $x^* \gg (0, 0)$ is that p_2 be between these two numbers,

$$2 < p_2 < 2\sqrt{y + 1}.$$

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(b) (20 pts) Find the demand for good 1, $x_1^*(1, p_2, y)$, for all $(p_2, y) \in \mathbb{R}^2_{++}$.

Soln: In light of (a), we have three cases.

Case 1: $p_2 \leq 2$. In this case, from (a) we know

$$x_1^*((1, p_2, y) = 0.$$

Case 2: $p_2 \geq 2\sqrt{y+1}$. In this case, from (a) we know $x_2^*(1, p_2, y) = 0$. So by the budget equation (Walras' Law), we have

$$x_1^*(1, p_2, y) = y.$$

Case 3: $2 < p_2 < 2\sqrt{y+1}$. In this case, from (a) we know $x^*(1, p_2, y) \gg (0, 0)$. So the solution to the UMP is interior, and we have two equations and two unknowns to solve to find x^* :

$$MRS(x) = \frac{1}{2\sqrt{x_1+1}} = \frac{1}{p_2} \text{ and } x_1 + p_2 x_2 = y.$$

The demand for good 1 (obtainded from just the tangency condition) is thus

$$x_1^*(1, p_2, y) = \frac{p_2^2 - 4}{4}.$$

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4. (40 pts) A consumer in a two-good world has a strictly increasing utility function that gives rise to the indirect utility function

$$v(p, y) = \ln\left(\frac{p_2}{p_1}\right) + \frac{y}{p_2} - 1$$

at any (p, y) for which the consumer's demand satisfies $x(p, y) \gg (0, 0)$. At such (p, y), find the following functions:

(a) (10 pts) Marshallian demand,

Soln: We use Roy's Identity to obtain Marshallian demand. Note that

$$v_y = \frac{1}{p_2}, \quad v_{p_1} = \left(\frac{p_1}{p_2}\right)\left(-\frac{p_2}{p_1^2}\right) = -\frac{1}{p_1}, \text{ and}$$

$$v_{p_2} = \left(\frac{p_1}{p_2}\right)\left(\frac{1}{p_1}\right) - \frac{y}{p_2^2} = \frac{p_2 - y}{p_2^2}.$$

Hence,

$$x_1(p, y) = -\frac{v_{p_1}}{v_y} = \frac{p_2}{p_1},$$

$$x_2(p, y) = -\frac{v_{p_2}}{v_y} = \frac{y - p_2}{p_2}.$$

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(b) (10 pts) expenditure,

Soln: We invert v with respect to y to find e. So, letting $U = v(p, y)$, we solve the equation

$$U = \ln\left(\frac{p_2}{p_1}\right) + \frac{y}{p_2} - 1$$

for y, obtaining

$$y = p_2 U + p_2 - p_2 \ln\left(\frac{p_2}{p_1}\right).$$

As this y is $e(p, U)$, we have $e(p, U) = p_2 U + p_2 - p_2 \ln\left(\frac{p_2}{p_1}\right)$. ■

(c) (10 pts) Hicksian demand, and

Soln: We obtain the Hicksian demand from Shepard's lemma:

$$x_1^h(p, U) = e_{p_1} = -p_2\left(\frac{p_1}{p_2}\right)\left(-\frac{p_2}{p_1^2}\right) = \frac{p_2}{p_1},$$

$$\begin{aligned} x_2^h(p, U) &= e_{p_2} = U + 1 - \ln\left(\frac{p_2}{p_1}\right) - p_2\left(\frac{p_1}{p_2}\right)\left(\frac{1}{p_1}\right) \\ &= U - \ln\left(\frac{p_2}{p_1}\right). \end{aligned}$$

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(d) (10 pts) utility.

Soln: Fix x. To find $u(x)$, we want to solve the two equation system $x = x^h(p_1, p_2, U)$ for the three unknowns (p_1, p_2, U) in terms of x – the solution U^* will be the desired $u(x)$. We can solve the two-equation three-unknown system because the Hicksian demands are homogeneous of degree zero in prices. Thus, from (c) we have the two equations,

$$x_1 = \frac{p_2}{p_1}, \quad x_2 = U - \ln\left(\frac{p_2}{p_1}\right).$$

Solving for U in terms of x yields

$$U^* = \ln\left(\frac{p_2}{p_1}\right) + x_2 = \ln(x_1) + x_2.$$

We conclude that the utility function is $u(x) = \ln(x_1) + x_2$. ■