Risky-Riskless Combination

- Consider a portfolio C that invests:
  - weight y in risky return r_P
  - weight $(1 - y)$ in riskless rate r_f

  Examples: $y = 0.5 \Leftrightarrow$ half in cash; $y = 2 \Leftrightarrow$ lever to 50% margin
- Realized return on C:

$$r_C = y r_P + (1 - y) r_f \tag{1}$$

- Risky return r_P
  - Expected return E_P
  - Volatility (standard deviation) σ_P
- Expected return on C:

$$E_C = y E_P + (1 - y) r_f, \tag{2}$$

- Volatility of return on C:

$$\sigma_C = |y| \sigma_P \tag{3}$$

Risk-Return Tradeoff

- Restrict $y > 0$ for now (i.e., not short P). Solving (3) for y gives

$$y = \frac{\sigma_C}{\sigma_P} \tag{4}$$

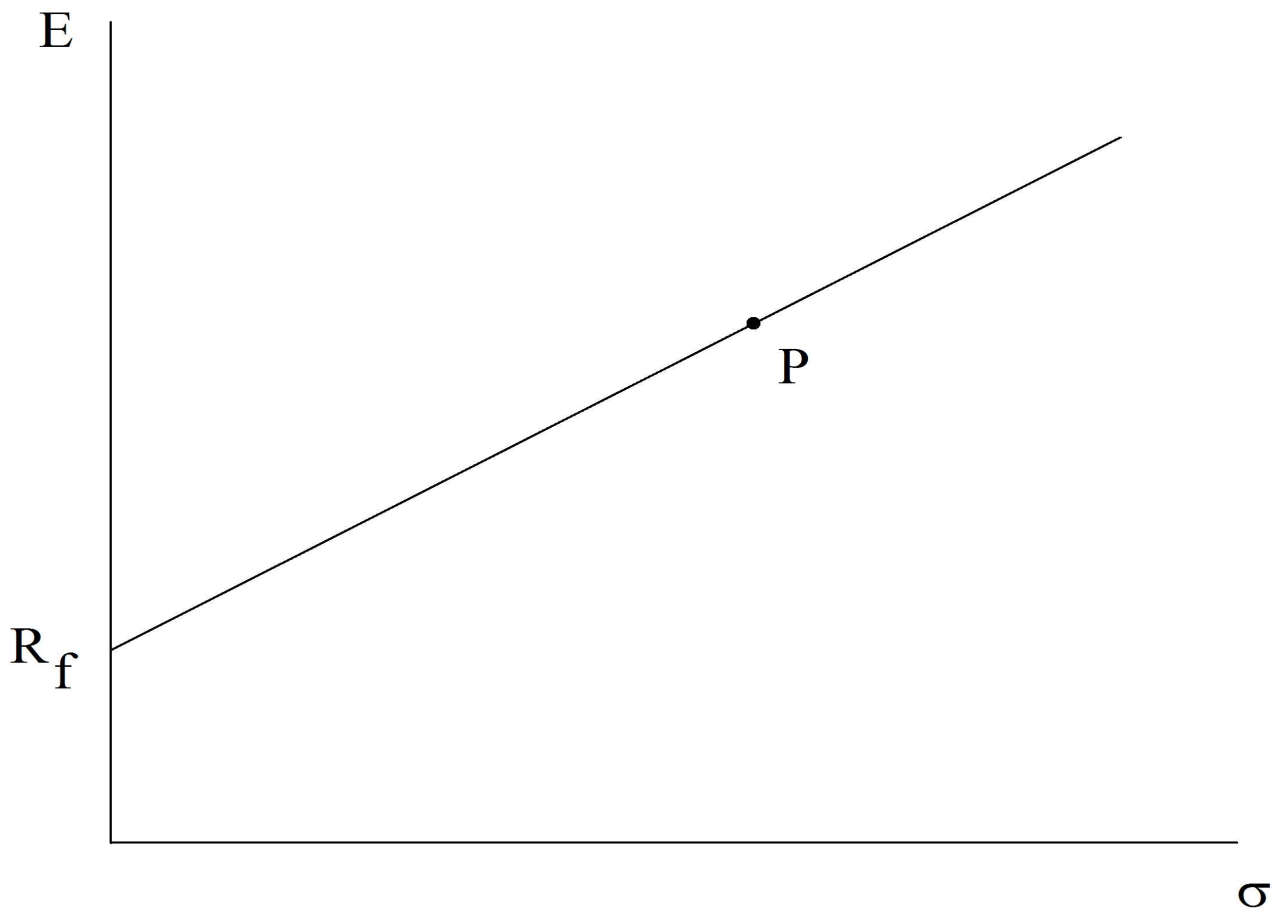
- Substituting for y in (2) gives

$$\begin{aligned} E_C &= \frac{\sigma_C}{\sigma_P} E_P + \left(1 - \frac{\sigma_C}{\sigma_P}\right) r_f \\ &= r_f + \left(\frac{E_P - r_f}{\sigma_P}\right) \sigma_C \end{aligned} \tag{5}$$

- Slope of this risk-return tradeoff is P's ***Sharpe Ratio***

$$S_P = \frac{E_P - r_f}{\sigma_P} \tag{6}$$

Capital Allocation Line



Example 1

- Riskless rate: 5%
- Portfolio P:

Expected Return:	11%	
Standard Deviation:	30%	
Sharpe ratio:	0.2	[= (.11 - .05)/.30]

- Portfolio C:

Portfolio P	$ 200
Borrowing	100
Amount invested	$ 100

- What is the standard deviation of Portfolio C's rate of return?

- What is the expected return on Portfolio C?

Total Return Swap

- Contract between two parties (generally privately negotiated) that specifies
  - "Notional" amount X dollars
  - Interest rate (e.g., LIBOR + spread)
  - Underlying asset (e.g., stock portfolio)
  - Time period (e.g., 1 year)
  - Payment dates (e.g., quarterly)
- No initial investment
- Contract parties "swap" interest rate r_f for risky return r_p
- One party (e.g., hedge fund or ETF)
  - receives Xr_p
  - pays Xr_f
  - profit/loss: $X(r_p - r_f)$
- Opposite for other party (e.g., large financial-services firm)

Swap Overlay

- Consider following investment of W dollars:

Position	Initial Investment	Profit
S&P Index Fund	yW	yWr_P
Cash	$(1-y)W$	$(1-y)Wr_f$
Swap notional position X	0	$X(r_P - r_f)$

- Overall rate of return, r'_C, is total profit divided by total initial investment:

$$\begin{aligned} r'_C &= \frac{yWr_P + (1-y)Wr_f + X(r_P - r_f)}{W} \\ &= y'r_P + (1-y')r_f, \qquad (7) \end{aligned}$$

where

$$y' = y + \frac{X}{W} \qquad (8)$$

- Swap overlay changes location on the capital allocation line for the S&P.
- Same as above using stock-index futures position of size X rather than swap

PRO SHARES ULTRA DOW 30 HOLDINGS, 9/1/2017

Description	Exposure Value	Market Value	Shares/Contracts
SPDR Dow Jones Industrial Average (DIA) SWAP Morgan Stanley	97,573,378		52,546
DJ Industrial Average SWAP Bank of America NA	89,938,816		4,090
DJ Industrial Average SWAP Citibank NA	62,364,079		2,836
DJ Industrial Average SWAP Credit Suisse International	62,328,732		2,835
DJ Industrial Average SWAP Societe Generale	46,415,380		2,111
DJ Industrial Average SWAP Goldman Sachs International	30,963,362		1,408
SPDR Dow Jones Industrial Average (DIA) SWAP Goldman Sachs	27,918,530		15,035
DJIA MINI 09/15/2017 (DMU7)	18,026,880		164
DJ Industrial Average SWAP UBS AG	9,185,944		418
BOEING CO		18,356,405	76,380
GOLDMAN SACHS GROUP INC		17,252,714	76,380
3M CO		15,547,913	76,380
UNITEDHEALTH GROUP INC		15,256,905	76,380
APPLE INC		12,530,139	76,380
MCDONALD'S CORP		12,206,288	76,380
HOME DEPOT INC		11,516,576	76,380
INTL BUSINESS MACHINES CORP		11,004,830	76,380
JOHNSON & JOHNSON		10,008,071	76,380
TRAVELERS COS INC/THE		9,157,962	76,380
CATERPILLAR INC		9,034,226	76,380
UNITED TECHNOLOGIES CORP		9,006,730	76,380
CHEVRON CORP		8,307,089	76,380
VISA INC		7,935,882	76,380
WALT DISNEY CO		7,752,570	76,380
PROCTER & GAMBLE CO		7,067,441	76,380
JPMORGAN CHASE & CO		7,004,046	76,380
AMERICAN EXPRESS CO		6,579,373	76,380
WAL-MART STORES INC		5,985,901	76,380
EXXON MOBIL CORP		5,848,417	76,380
MICROSOFT CORP		5,647,537	76,380
DOWDUPONT INC COMMON STOCK USD 0.01		5,131,208	76,380
MERCK & CO INC		4,875,335	76,380
NIKE INC		4,075,637	76,380
VERIZON COMMUNICATIONS INC		3,660,130	76,380
COCA-COLA CO/THE		3,496,676	76,380
INTEL CORP		2,680,174	76,380
PFIZER INC		2,593,865	76,380
CISCO SYSTEMS INC		2,467,074	76,380
GENERAL ELECTRIC CO		1,920,193	76,380
Net Other Assets / Cash		100,354,782	
TOTALS	444,715,102	344,262,091	

ProShares, Ultra Dow30

- Fund composition (9/1/2017):

	($mil.)
Fund value, W	344.3
Cash	100.3
Swap/futures position, X	444.7

- Fraction of W invested in Dow 30 stocks:

$$
\begin{aligned}
y &= \frac{344.3 - 100.3}{344.3} \\
&= 0.709
\end{aligned}
$$

- Exposure to Dow 30 stock return:

$$
\begin{aligned}
y' &= y + \frac{X}{W} \\
&= 0.709 + \frac{444.7}{344.3} \\
&= 0.709 + 1.292 \\
&= 2.00
\end{aligned}
$$

Exposure Changes and Rebalancing

- Let R denote the excess return on the risky asset $(r_P - r_f)$
- If constant y is desired, rebalancing is necessary (unless $y = 1$)
  - A positive R moves y toward 1
    - for $y < 1$, y rises
    - for $y > 1$, y falls
  - A negative R moves y away from 1
    - for $y < 1$, y falls
    - for $y > 1$, y rises
- Large reversals, which are more likely with high volatility
  - hurt a constant y strategy when $y > 1$, because rebalancing
    - raises y after positive R, worse than not raising if negative R follows
    - lowers y after negative R, worse than not lowering if positive R follows
  - help a constant y strategy when $y < 1$, because rebalancing
    - raises y after negative R, better than not raising if positive R follows
    - lowers y after positive R, better than not lowering if negative R follows

Change in Allocation Before Rebalancing

- Define

  y_t: allocation to risky asset at beginning of period $t+1$

  y_{t+1}: allocation to risky asset at end of period, *before* rebalancing

  W_t: value of investment at beginning of period

  W_{t+1}: value of investment at end of period

  $r_{p,t+1}$: return on risky asset in period $t+1$

  $r_{f,t+1}$: riskless rate in period $t+1$

- First note that the end-of-period investment value is

$$\begin{aligned} W_{t+1} &= W_t\left[1 + y_t r_{p,t+1} + (1 - y_t) r_{f,t+1}\right] \\ &= W_t\left[1 + r_{f,t+1} + y_t(r_{p,t+1} - r_{f,t+1})\right] \end{aligned}$$

- The end-of-period value of the risky portion is $W_t y_t (1 + r_{p,t+1})$
- Dividing the latter value by W_{t+1} gives y_{t+1}:

Change in Allocation Before Rebalancing (continued)

$$
\begin{aligned}
y_{t+1} &= \frac{W_t y_t (1 + r_{p,t+1})}{W_t \left[1 + r_{f,t+1} + y_t (r_{p,t+1} - r_{f,t+1})\right]} \\
&= y_t \left[\frac{1 + r_{f,t+1} + (r_{p,t+1} - r_{f,t+1})}{1 + r_{f,t+1} + y_t (r_{p,t+1} - r_{f,t+1})}\right] \\
&= y_t \left[\frac{1 + e^*_{t+1}}{1 + y_t e^*_{t+1}}\right], \quad e^*_{t+1} = \frac{r_{p,t+1} - r_{f,t+1}}{1 + r_{f,t+1}}
\end{aligned}
$$

- When $r_{p,t+1} > r_{f,t+1}$:

$$
\begin{aligned}
y_{t+1} &< y_t \text{ if } y_t > 1 \\
y_{t+1} &> y_t \text{ if } y_t < 1
\end{aligned}
$$

- When $r_{p,t+1} < r_{f,t+1}$:

$$
\begin{aligned}
y_{t+1} &> y_t \text{ if } y_t > 1 \\
y_{t+1} &< y_t \text{ if } y_t < 1
\end{aligned}
$$

Leveraged ETF's

- Leveraged exchange-traded funds (ETF's) rebalance each period (day) to maintain a constant y
- For a leveraged long ETF, $y > 1$

  $\Rightarrow$ fund adds to risky position after a period in which $r_{p,t+1} > r_{f,t+1}$

  $\Rightarrow$ fund subtracts from risky position after a period in which $r_{p,t+1} < r_{f,t+1}$
- Over N periods, ETF return $\neq$ *simple* (non-rebalanced) return
- ETF return:

$$r_{ETF} = \prod_{s=1}^{N} \left[1 + y r_{p,t+s} + (1-y) r_{f,t+s}\right] - 1$$

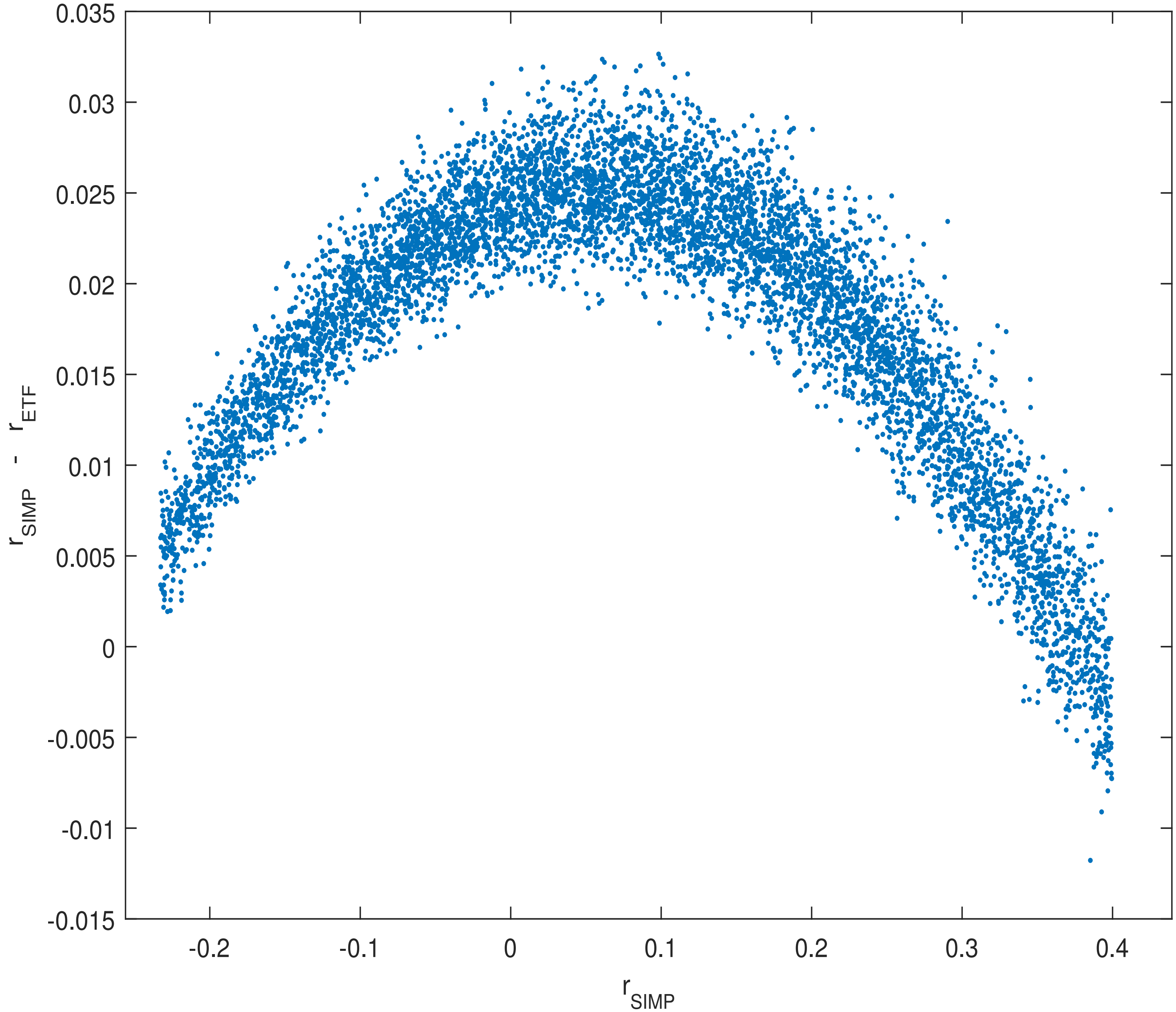
- Simple return:

$$r_{SIMP} = y \prod_{s=1}^{N} (1 + r_{p,t+s}) + (1-y) \prod_{s=1}^{N} (1 + r_{f,t+s}) - 1$$

($\prod_{s=1}^{N}$ denotes a product of the N quantities for $s = 1, \ldots, N$)

Simulation: ETF Return Versus Simple Return

- $E(r_{p,t}) = 0.08/250$
- $\sigma(r_{p,t+1}) = 0.01$
- $r_{p,t+1}$ normally distributed
- $r_{f,t+1} = 0.005/250$
- $N = 250$
- $y = 2$
- $10,000$ samples (years)
- Plots
  - $(r_{SIMP} - r_{ETF})$ vs. r_{SIMP}
  - r_{ETF} vs. r_{SIMP}
- When risky asset's value moves a lot in one direction, $r_{SIMP} < r_{ETF}$
- Otherwise, $r_{SIMP} > r_{ETF}$
- The leveraged ETF is "convex" relative to the simple (non-rebalanced) strategy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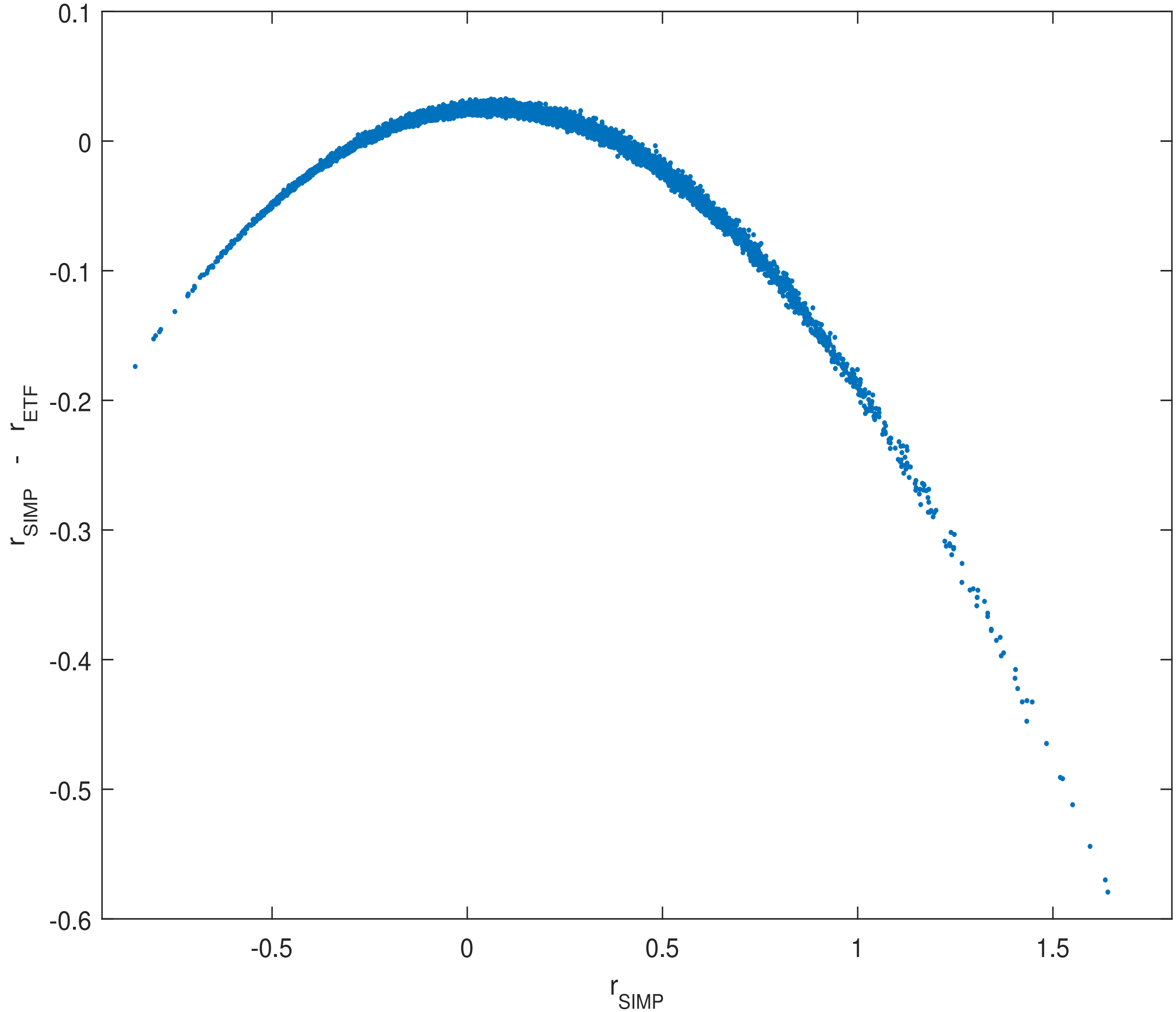
0.03
0.025
0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.2
-0.1
0
0.1
0.2
0.3
0.4
r_{SIMP} - r_{ETF}
r_{SIMP}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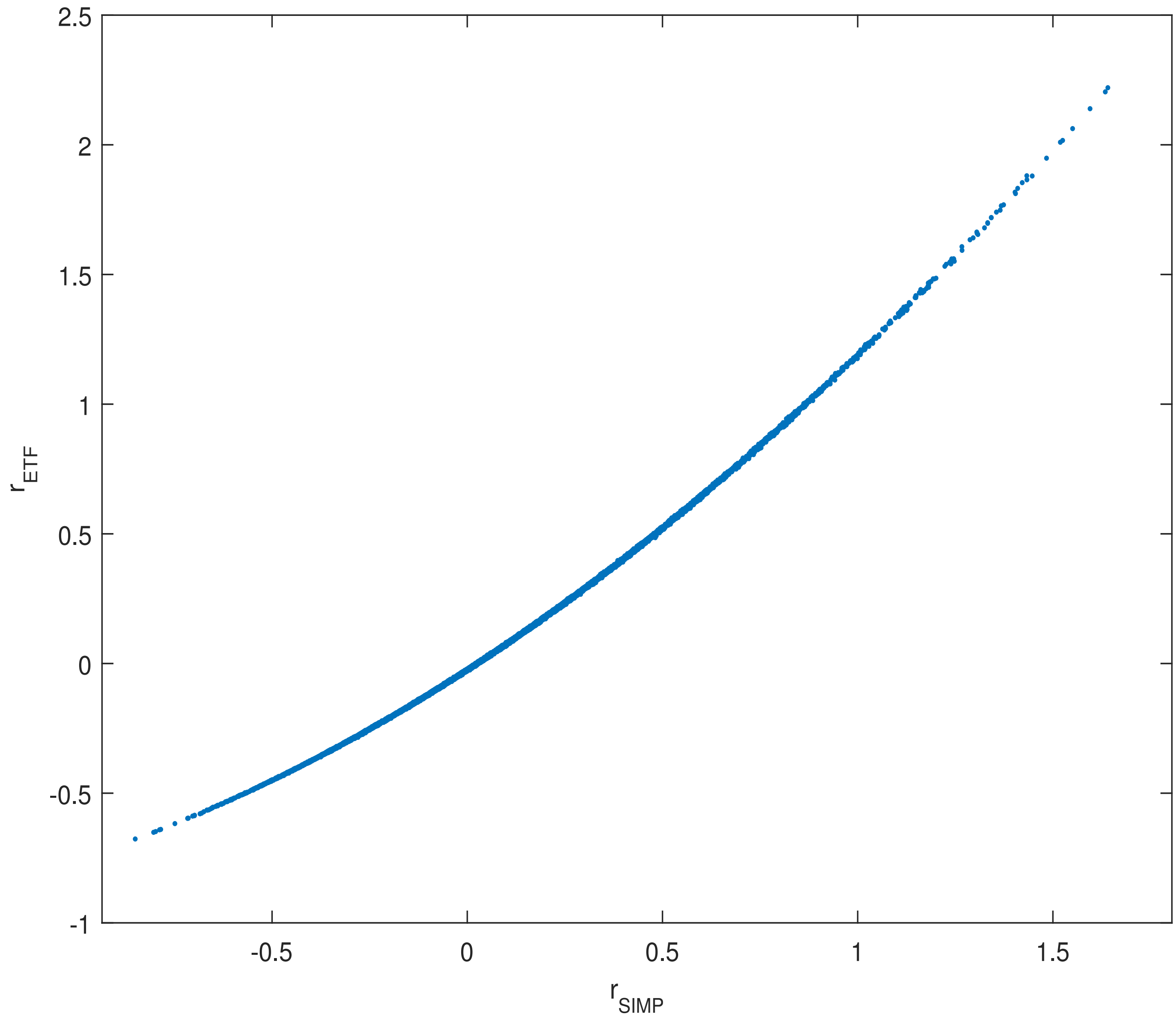
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.5
0
0.5
1
1.5
r_{SIMP}
r_{SIMP} - r_{ETF}


2.5
2
1.5
1
0.5
0
-0.5
-1
-0.5
0
0.5
1
1.5
r_{ETF}
r_{SIMP}

Futures Accomplish the Same Thing as Swaps (derivation is optional)

- Underlying index portfolio P (e.g., S&P 500)
- Futures position
  - no initial outlay except margin (in most cases can be T-Bills).
  - profit on long (short) position equals (+)(-) change in the futures price
  - final futures "settlement" price at expiration set to price (index level) of P
- Consider, at time 0, a futures position that expires at time 1.
- Define

  S_0: current index level or "spot" price

  S_1: index level t time 1 (and futures settlement price) a

  F_0: current futures price

  D_1: dividends on current investment of S_0 in index portfolio (assume paid at time 1 but known at time 0)
- Profit from time 0 to time 1 when long futures: $S_1 - F_0$
- What is the futures price F_0?

- Consider two alternative investments with (i) same initial outlay and (ii) same uncertain component of future payoff (S_1)

	Investment	Cost at time 0	Payoff at time 1
1.	Buy index fund	S_0	$S_1 + D_1$
2.	Long futures	0	$S_1 - F_0$
	Invest \$$S_0$ in T-Bills	S_0	$S_0(1 + r_f)$

- No arbitrage $\Rightarrow$ equal payoffs on the investments:

$$S_1 + D_1 = S_1 - F_0 + S_0(1 + r_f)$$

so the futures price must be

$$F_0 = S_0(1 + r_f) - D_1$$

- Profit when long futures:

$$\begin{aligned} S_1 - F_0 &= S_1 - S_0(1 + r_f) + D_1 \\ &= S_0(1 + r_p) - S_0(1 + r_f) \\ &= S_0(r_p - r_f) \end{aligned}$$

- For futures position of size X dollars (in terms of current index value)

$$\text{profit} = X(r_p - r_f)$$