Economics 681 **Steven A. Matthews**
Fall 2017 **University of Pennsylvania**

Lecture Outlines
Today's Date: December 3, 2017

The relevant portions of the textbook, *Advanced Microeconomic Theory* (3rd edition) by Jehle and Reny (JR), are denoted in the format chapter.section.subsection.

Lecture 1. Why theory and math?

1. Comparative statics example: $\hat{D}(p,t) = \hat{S}(p,t) \Rightarrow p^*(t), x^*(t) \Rightarrow$?
2. Math concepts: chain rule, implicit function theorem

Lecture 2. Preferences (JR1.1, 1.2.1)

1. Primitives: $x \in X = \mathbb{R}^n_+$, $\succsim$, $B \in \mathfrak{B} \subseteq 2^X$
2. Preference axioms
   (a) $\succsim$ is a *preference* relation, i.e., complete (A1) and transitive (A2)
   (b) $\succsim$ is a *continuous* relation (A3)
   (c) $\succsim$ satisfies *local non-satiation* (A4$'$) or *strict monotonicity* (A4)
   (d) $\succsim$ is a *convex* relation (A5$'$) or a *strictly convex* relation (A5)
3. Math concepts: binary relations, contour sets, indifference sets, continuity, convex sets

Lecture 3. Utility

1. Utility (JR1.2.2)
   (a) Definition of a function *representing* a binary relation. The *ordinal* properties of a function $u : X \to \mathbb{R}$.
   Theorem 1.2. *The preferences represented by a function $u : X \to \mathbb{R}$ are also repesented by $\hat{u} := u \circ f$, where f is any strictly increasing function from $u(X)$ to $\mathbb{R}$.*
   (b) **Debreu's Representation Theorem.** *A binary relation $\succsim$ on a connected set $X \subseteq \mathbb{R}^n$ is represented by a continuous $u : X \to \mathbb{R}$ iff it is complete, transitive, and continuous.*
   (c) Representation of strictly monotonic preferences
   Theorem 1.1. *A binary relation $\succsim$ on $\mathbb{R}^n_+$ is represented by a continuous $u : \mathbb{R}^n_+ \to \mathbb{R}$ if $\succsim$ is complete, transitive, continuous, and strictly monotonic.*

Lecture 4. Utility Properties and Maximization (JR1.2, 1.3)

1. No function represents lexicographic preferences on $\mathbb{R}^n_+$ (if $n \geq 2$)
2. Properties of u that correspond to those of $\succsim$
   Theorem 1.3. For a binary relation $\succsim$ on $\mathbb{R}^n_+$ represented by u,
   (a) u is strictly increasing iff $\succsim$ is strictly monotonic,
   (b) u is quasiconcave iff $\succsim$ is convex, and
   (c) u is strictly quasiconcave iff $\succsim$ is strictly convex.
3. Consumer's Utility Maximization Problem

$$\text{(UMP)} \quad \max_{x \in X} u(x) \text{ such that } p \cdot x \leq y$$

   (a) Non-Calculus Examples: (i) Leontief and

$$\text{(ii) } u(x_1, x_2) = \min\{3x_1, x_1 + x_2, 3x_2\}$$

4. Math: Definitions of quasiconcave and concave functions.

Lecture 5. Utility Properties and Maximization (con't) (JR1.2, 1.3)

1. Discuss the non-calculus example (ii) from Lecture 4.
2. Differentiable utility: We often focus on strictly quasiconcave C^2 utility functions that for which $\nabla u(\cdot) \gg 0$.
   (a) Formula for the marginal rate of substitution of good j for good i :

$$MRS_{ij}(x) = \frac{u_i(x)}{u_j(x)}$$

   (b) Quasiconcavity of C^1 and C^2 functions: picture "proofs" of
      i. A C^1 function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is quasiconcave iff for all x and y,

$$u(y) \geq u(x) \;\Rightarrow\; \nabla u(x) \cdot (y - x) \geq 0.$$

      ii. A C^2 function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is quasiconcave iff the Hessian matrix $\mathbf{H}(x) := [u_{ij}(x)]$ is constrained negative semi-definite: for all $x \in \mathbb{R}^n_+$ and $v \in \mathbb{R}^n$,

$$\nabla u(x) \cdot v = 0 \;\Rightarrow\; v^T H(x) v \leq 0.$$

3. First-order conditions (FOC) for the UMP given a C^2 function $u : \mathbb{R}^n_+ \to \mathbb{R}$ with $\nabla u(\cdot) \gg 0$
   (a) Necessary FOC: If $x^* \gg 0$ solves the UMP, then $\lambda^* \in \mathbb{R}_+$ exists s.t.

$$\nabla u(x^*) = \lambda p \text{ and } p \cdot x^* = y.$$

   (b) Sufficient FOC: If u is quasiconcave and $(x^*, \lambda) \geq 0$ satisfies these two equations, then x^* solves the UMP.
   (c) Example: $u(x_1, x_2) = \ln(x_1) + x_2$.
4. Math: Taylor expansions, geometry of gradients, Lagrange multipliers.

Lecture 6. Solving the UMP, Properties of Demand and Indirect Utility (JR1.4)

1. Using Kuhn-Tucker to solve the UMP, focus on example $u(x) = \ln(x_1) + x_2$.
2. Properties of Demand
   (a) The set of solutions of the UMP is a convex set if u is quasiconcave. It is a singleton if u is strictly quasiconcave, in which case the solution is called a demand function, $x(p, y)$, mapping $\mathbb{R}^n_{++} \times \mathbb{R}_+$ to $\mathbb{R}^n_+$.
   (b) A demand function satisfies Walras' Law: $p \cdot x(p, y) = y$ for all (p, y), if u is a strictly increasing function. (Actually, WL holds whenever the preferences satisfy local nonsation, Axiom $4'$.)
   (c) A demand function is homogeneous (of degree 0).
3. The Indirect Utility Function
   (a) $v(p, y)$ is the value of the UMP program
   (b) Properties of $v(p, y)$ (JR Thm. 1.6)
      i. continuous (given that u is continuous)
      ii. homogeneous (of degree 0)
      iii. strictly increasing in y (given Walras' Law)
      iv. increasing in each p_i
      v. quasiconvex
      vi. The demand for good i can be obtained from v:

$$\text{Roy's Identity:} \quad x_i(p, y) = -\frac{v_{p_i}(p, y)}{v_y(p, y)}$$

      (given that v is differentiable and $v_y \neq 0$).

Lecture 7. Envelope Theorems, Roy's Identity, and Expenditure Functions (ET Handout, JR1.4, JR page 604)

1. Envelope Theorems, ET 1 and 2
2. Roy's Identity, derivation and interpretation
3. Cobb-Douglas Example: $u(x_1, x_2) = x_1^a x_2^b$, for $0 < a, b$ and $a + b = 1$.
4. Expenditure Function
   (a) $e(p, U) := \min_{x \geq 0} p \cdot x$ s.t. $u(x) \geq U$
   If the solution of this program is unique, call it the Hicksian demand function, $x^h(p, U)$
   (b) Properties of $e(p, U)$
   (JR Thm. 1.7): If $u : \mathbb{R}^n_+ \to \mathbb{R}^n$ is continuous and strictly increasing, then
      i. $e(p, U) = 0$ for all $U \leq u(0)$ (easy)
      ii. $e(p, U)$ is a continuous function
      iii. $e(p, U)$ *strictly* increases in U on $\mathcal{U} := [u(0), \sup_{x \geq 0} u(x))$ (the "strictly" part is not as easy)
      iv. $e(p, U)$ increases in p (easy)
      v. $e(p, U)$ is homogeneous of degree 1 in p (easy)

Lecture 8. Concavity, Expenditure Functions, and Duality (JR1.4, 1.5)

1. Concave Functions (Math)
   (a) **Def.** $f : \mathbb{R}^n_+ \to \mathbb{R}$ is *concave* iff the set below the graph of f,
   $$\left\{(x,y) \in \mathbb{R}^{n+1}_+ : y \le f(x)\right\},$$
   is a convex set. Equivalently, f is concave iff for all $x^0, x^1 \in \mathbb{R}^n_+$ and $t \in [0,1]$,
   $$f(x^t) \ge tf(x^1) + (1-t)f(x^0),$$
   where $x^t = tx^1 + (1-t)x^0$.
   (b) **Prop.** A C^1 function $f : \mathbb{R}^n_+ \to \mathbb{R}$ is concave iff for all x^0 and x^1,
   $$f(x^1) \le f(x^0) + \nabla f(x^0) \cdot (x^1 - x^0),$$
   i.e., iff the graph of f lies below all its tangent hyperplanes.
   (c) **Prop.** A C^2 function $f : \mathbb{R}^n_+ \to \mathbb{R}$ is concave iff the Hessian matrix $\mathbf{H}(x) := \left[f_{ij}(x)\right]$ is negative semi-definite (NSD) for all $x \in \mathbb{R}^n_+$, i.e., satisfies $v^T H(x) v \le 0$ for all $v \in \mathbb{R}^n$.
   (d) **Prop.** If a $n \times n$ matrix $A = \left[a_{ij}\right]$ is negative semi-definite, then $a_{ii} \le 0$ for all $i = 1, \ldots, n$. If A is negative definite ($v^T A v < 0$ for all nonzero $v \in \mathbb{R}^n$), then $a_{ii} < 0$ for all $i = 1, \ldots, n$.
2. Envelope Properties of the Expenditure Function
   JR Thm. 1.7. If $u : \mathbb{R}^n_+ \to \mathbb{R}$ is continuous and strictly increasing, then
   (a) $e(p, U)$ is concave in p, and
   (b) if $u(\cdot)$ is also strictly quasiconcave, then e is differentiable in p, and by ET 1,
   $$\text{Shepard's Identity:} \quad x_i^h(p,U) = e_{p_i}(p,U)$$
3. Duality Relationships
   If $u(\cdot)$ continuous and strictly increasing, then
   (a) **JR Thm. 1.8.** For any p, the functions $e(p, \cdot)$ and $v(p, \cdot)$ are inverses of each other:
      i. $e(p, v(p, y)) = y$
      ii. $v(p, e(p, U)) = U$
   (b) **JR Thm. 1.9.** If $u(\cdot)$ is also strictly quasiconcave, the Marshallian and Hicksian demand functions satisfy
      i. $x(p, y) = x^h(p, v(p, y))$
      ii. $x^h(p, U) = x(p, e(p, U))$
4. Cobb-Douglas example (con't)
5. If the Hicksian demand functions are C^1, they satisfy the Laws of Demand ($\partial x_i^h / \partial p_i \le 0$) and Reciprocity ($\partial x_i^h / \partial p_j = \partial x_j^h / \partial p_i$). The latter implies that if good i is a net substitute (complement) for good j, then good j is a net substitute (complement) for good i.

Lecture 9. Properties of Hicksian Demand (JR 1.5)

1. Hicksian Demand Satisfies the Generalized Law of Demand

   (a) **Thm.** For $x^0 = h(p^0, U)$ and $x^1 = h(p^1, U)$, we have $(p^1 - p^0) \cdot (x^1 - x^0) \leq 0$.
   Proof. For $i \neq j$, x^i is feasible for EMP(p^j). Hence, $p^j \cdot x^j \leq p^j \cdot x^i$. Add these two inequalities together to get the result.

2. **Thm.** If a Hicksian demand function is C^1 in prices, then for any (p, U) the matrix

$$S = \left[\frac{\partial x_i^h(p, U)}{\partial p_j} \right]$$

   is negative semidefinite (NSD) and symmetric.
   Proof. Differentiate the identity $x_i^h(p, U) = e_{p_i}(p, U)$ (Shepard's lemma) with respect to p_j to obtain

$$S = \left[\frac{\partial^2 e(p, U)}{\partial p_i \partial p_j} \right].$$

   Hence, since e is concave in p, S is NSD. Since x^h is *continuously* differentiable, e has continuous second partials, and we have $\frac{\partial^2 e(p,U)}{\partial p_i \partial p_j} = \frac{\partial^2 e(p,U)}{\partial p_j \partial p_i}$ by Young's theorem. So S is symmetric.

3. Slutsky Equations

   (a) For a given (p, y), let $U = v(p, y)$. Then the ij Slutsky equation is

$$\frac{\partial x_i(p, y)}{\partial p_j} = \frac{\partial x_i^h(p, U)}{\partial p_j} - x_j(p, y) \frac{\partial x_i(p, y)}{\partial y}.$$

   (b) **Derivation.** Differentiate the duality identity $x_i^h(p, U) = x_i(p, e(p, U))$ with respect to p_j, using the chain rule. Then apply Shepard's lemma to replace the $\partial e / \partial p_j$ term by x_j^h. Then let $y = e(p, U)$ and use the duality identity $x_j^h(p, v(p, y)) = x_j(p, y)$ to replace the x_j^h term by x_j. Rearrange the resulting equation to get the Slutsky equation shown.

   (c) The *Slutsky matrix* at (p, y) is

$$S(p, y) := \left[\frac{\partial x_i(p, y)}{\partial p_j} + x_j(p, y) \frac{\partial x_i(p, y)}{\partial y} \right],$$

   defined entirely from observable Marshallian demand. From the above,

$$S(p, y) = \left[\frac{\partial x_i^h(p, U)}{\partial p_j} \right],$$

   where $U = v(p, y)$, and so $S(p, y)$ is NSD and symmetric (if h is C^1 in prices).

   (d) Consequences

      i. The Marshallian demand for a normal good satisfies the Law of Demand.
      ii. A Giffen good must be an inferior good.
      iii. An inferior good is a Giffen good only if the demand for it is sufficiently large.

4. Figure: Income and Substitution Effects

Lecture 10. Marshallian Demand and Compensating Variation (JR 1.5, 2.2, 4.3.1)

1. Necessary Conditions a Demand Function Must Satisfy

   If $x : \mathbb{R}^{n+1}_{++} \to \mathbb{R}^n_+$ is a demand function arising from a continuous, strictly increasing and strictly quasiconcave $u : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, then it satisfies

   (a) Walras' law ("budget balancedness")
   (b) Symmetry and NSD of the Slutsky matrix $S(p, y)$
   (c) Homogeneity of degree 0
   (d) $S(p, y)p = 0$ (proved using (c) and Euler's theorem)

2. Sufficient Conditions for a Function to be a Demand Function

   (a) **JR Thm 2.5:** (a) & (b) $\Rightarrow$ (c) & (d)
   (Remark: JR's proof can be simplified by using the converse of Euler's theorem.)
   (b) **The Integrability Theorem (JR Thm 2.6)**: (a) & (b) are sufficient for an arbitrary C^1 function $x : \mathbb{R}^{n+1}_{++} \to \mathbb{R}^n_+$ to be the demand function arising from a continuous, strictly increasing and strictly quasiconcave $u : \mathbb{R}^n_+ \to \mathbb{R}^n_+$.

3. Compensating Variation

   (a) Def. $CV(p^0, p^1, y) := e(p^1, u^0) - m$, where $u^0 = v(p^0, y)$.
   (b) Interpretation 1. CV is the amount of money the consumer must be given ("compensated") after the price change from p^0 to p^1 to make her as well off as before the change. If the change in prices is good (bad) for the consumer, the CV is negative (positive).
   (c) Interpretation 2. The negative of CV, i.e. $-CV$, is the maximum amount the consumer would be willing to pay for the price change.
   (d) Shepard's lemma and the Fundamental Theorem of Calculus imply that $CV(p^0, p^1, y)$ is the integral of the Hicksian demand function $x^h(p, u^0)$ from p^0 to p^1. If the price of only one good, say good 1 changes, the integral is just with respect to one variable, p_1 :

$$CV(p^0, p^1, y) = \int_{p_1^0}^{p_1^1} x_1^h(p_1, \bar{p}_{-1}, u^0) dp_1.$$

   (e) Relationship to ΔCS, the change in Consumer Surplus.
   (f) Three figures.

4. Application to Commodity Taxes

   (a) Policy 0:a per-unit commodity tax on good 1
   Policy 1: a lump-sum income tax T equal to the tax revenue from Policy 0
   Result: the consumer prefers Policy 1.
   (b) Picture proof showing that $-CV > T$. Another proof by Revealed Preference.
   (c) Deadweight Loss. Figure.

Lecture 11. Decision Making under Uncertainty (JR 2.4)

1. Finish the Commodity Tax Application, Deadweight Loss from L10

2. Gambles, Preferences (JR 2.4.1)

   (a) Example: Decision to go to a concert

   (b) Can view the set G_S of simple gambles as the set Δ^{n-1} of all probability vectors on A, identifying its corners as the outcomes a_i

   (c) For any compound gamble $g \in G$ and outcome a_i, let $R_i(g) := \Pr(a_i|g)$. Thus,

   $$R(g) = (R_1(g) \circ a_1, \ldots, R_n(g) \circ a_n) \in G_S$$

   is the simple gamble induced by g, i.e., that puts the same probabilities on each a_i

3. Expected Utility Representation of Preferences (JR 2.4.2)

   (a) **Definition.** $\succsim$ on G has an *expected utility representation* if a there exists a function $u : G \to \mathbb{R}$ that represents $\succsim$ and satisfies the following *expected utility property*: for all $g \in G$,

   $$u(g) = \sum_{i=1}^{n} p_i u(a_i), \text{ where } p_i = R_i(g).$$

   We call a function u that has the expected utility property a *vNM utility function*, and if it represents $\succsim$ it is a *vNM utility function for* $\succsim$.

   (b) Preferences $\succsim$ satisfy the *Expected Utility Hypothesis (EUH)* if they have an EU representation.

Lecture 12. Decision Making under Uncertainty (JR 2.4) (con't)

1. If $\succsim$ satisfies the EUH, then either all of Δ^{n-1} is one indifference curve, or the indifference curve map is a collection of parallel affine sets (lines if $n = 3$). A strong property!

2. Allais Paradox

3. Cardinality of vNM Utility Functions

   (a) Ratios of vNM utility differences have meaning: any two vNM functions u and v for $\succsim$ must satisfy

   $$\frac{u_i - u_n}{u_1 - u_n} = \frac{v_i - v_n}{v_1 - v_n} \text{ (assuming } a_1 \succ a_n).$$

   (b) **JR Thm 2.8.** Suppose $\succsim$ on G satisfies $a_1 \succ a_n$. Then u and v are two vNM utility functions for $\succsim$ if and only if numbers $\alpha > 0$ and β exist such that

   $$v_i = \beta u_i + \alpha \text{ for all } i.$$

   I.e., v is a *positive affine transformation* of u.
   Proof.

4. The vNM Expected Utility Theorem

   (a) **JR Thm 2.7.** Any $\succsim$ on G satisfying Axioms G1-G6 satisfies the EUH. (The converse is also true if the assumption $a_1 \succ a_n$ is added – this direction is easy to show.)
   Proof Idea.

Lecture 13. Risk Aversion (JR 2.4.3)

1. Monetary Gambles

   (a) $A = \mathbb{R}$ is the set of all possible wealth levels

   (b) Simple gambles are those with a finite support: $g = (p_1 \circ w_1, \ldots, p_n \circ w_n)$ for some $n < \infty$

2. Utility

   (a) $u : G \to \mathbb{R}$ is a vNM utility function, where G is the set of all compound gambles.

   (b) Assume that on $A = \mathbb{R}$, u is C^2 and satisfies $u'(w) > 0$.

   (c) For $g = (p_1 \circ w_1, \ldots, p_n \circ w_n)$, the expected utility generated by g is

   $$u(g) = \sum_i p_i u(w_i) = E_g u(\tilde{w}),$$

   whereas the *expected value* of the gamble is its *mean*:

   $$Eg = \sum_i p_i w_i = E_g(\tilde{w}).$$

3. Define Risk Aversion

   (a) $\succsim$ on G is *risk averse* if $g \succsim Eg$ for all nondegenerate gambles g
   $\succsim$ on G is *risk neutral* if $g \sim Eg$ for all gambles g

   (b) u on G is *risk averse* if $u(Eg) > u(g)$ for all nondegenerate gambles g
   u on G is *risk neutral* if $u(g) = u(Eg)$ for all gambles g

4. **Theorem (Jensen Inequality).** A function $u : \mathbb{R}^n \to \mathbb{R}$ is concave iff for all simple gambles g on $\mathbb{R}^n$,

   $$u(g) > u(Eg).$$

   It is strictly concave iff this inequality is strict for all nondegenerate g.

5. Insurance Application

   (a) **Prop.** A risk averse person subject to a stochastic monetary loss who can buy insurance coveralge for it at an actuarily fair rate will buy full coverage.

   (b) **Three proofs.** (1) use Jensen's Inequality (no differentiability needed), (2) use first-order conditions, (3) use figure of contingent commodity space with budget line and indifference curves.

   (c) The EUH implies that the MRS of any indifference curve in the contingent commodity space where it crosses the 45degree line is equal to the ratio of the state probabilities.

6. Certainty Equivalent

   (a) Definition and figure as in JR Def 2.5

   (b) The CE of a gamble is the minimum amount its owner would be willing to sell it for, given his $\succsim$ (or u).

   (c) Notation: given u and g, let $c(g, u)$ be the corresponding CE for g

   (d) If u is risk averse, $c(g, u) < Eg$ for any nondegenerate g. Figure.

Lecture 14. Comparing Risk Aversion (JR 2.4.3)

1. **Definition.** $\succsim_a$ is *more risk averse than* $\succsim_b$ if for all $g \in G$ and $w \in \mathbb{R}$,

$$g \succsim_a w \;\Rightarrow\; g \succsim_b w.$$

2. Arrow-Pratt Measure (Coefficient) of Absolute Risk Aversion

   (a) $R_a(w) := u''(w)/u'(w)$

   (b) It is a measure of curvature

   (c) It is invariant to positive affine transformations of u

3. Risk Aversion Comparisons of of One Person to Another

   Pratt's Theorem. Given vNM utility functions u_a and u_b, let $\precsim_a$ and $\precsim_b$ be the preferences they represent. The following four properties are equivalent:

   (a) $\precsim_a$ is more risk averse than $\precsim_b$.

   (b) $u_a = h \circ u_b$ for some concave strictly increasing $h : \mathbb{R} \to \mathbb{R}$.

   (c) $R_a(w) \geq R_b(w)$ for all w.

   (d) $c(g, u_a) \leq c(g, u_b)$ for all g.

4. Risk Aversion Comparisons of One Person at Different Wealth Levels

   (a) DARA, CARA, and IARA

   (b) u is CARA iff $u(w) = (1 - e^{-\gamma w})/\gamma$ for some $\gamma \in \mathbb{R}$

   (c) Buy and sell prices for gambles

5. Application: Investment in a Risk Asset

6. Remarks on Ellsberg Paradox and Subjective Expected Utility

Lecture 15. Production (JR 3.2) (after doing 5 & 6 from L14)

1. Production Sets and Plans: $Y = \{y \in \mathbb{R}^m : \text{it is feasible to produce } y\}$
2. Production Functions of Single-Output Firms
   (a) $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ defined by

   $$f(x) := \max\{y \in \mathbb{R}_+ : (-x, y) \in Y\}$$

   (b) *Technological efficiency* holds if the firm produces $y = f(x)$ from inputs x
   (c) **Maintained Assumption 3.1.** f is continuous, strictly increasing, strictly quasiconcave, and satisfies $f(0) = 0$
   (d) Isoquant $Q(y)$
   (e) Cardinal properties of f are meaningful (in contrast to those of a consumer's u)
   (f) **Theorem 3.1.** Any f satisfying A3.1 is concave if it is homog. of degree $\alpha \in (0, 1]$.
   (g) Returns to Scale
   Definition 3.3. f exhibits *constant returns to scale* (CRS) if it is linearly homogeneous. It exhibits DRS (IRS) if

   $$f(tx) < (>) t f(x) \text{ for all } t > 1 \text{ and } x \geq 0.$$

   Proposition. If f is strictly concave (convex), then DRS (IRS).

Lecture 16. Cost Functions (JR 3.3) and Competitive Profit Maximization (JR 3.5)

1. Cost Function Definition, Rationale, and Ordinal properties

   (a) Define $c(w, y) := \min_{x \geq 0} w \cdot x$ such that $f(x) \geq y$.
   The solution of this problem is the firm's *conditional input demand function*, $x(w, y)$, and so $c(w, y) = w \cdot x(w, y)$.

   (b) Profit maximization $\Rightarrow$ cost minimization, since

   $$\begin{aligned} \max_y \max_x \{R(y) - w \cdot x \text{ st } f(x) \geq y\} &= \max_y \left\{R(y) - \left[\min_x w \cdot x \text{ st } f(x) \geq y\right]\right\} \\ &= \max_y R(y) - c(w, y). \end{aligned}$$

   (c) We've already studied this problem – it is the same as the consumer EMP. We thus know that the constraint binds, and the FOC for an interior solution boil down to the constraint and the tangency conditions $MRTS_{ij} = w_i/w_j$. Also, JR Theorems 3.2 & 3.3, and the Generalized Law of Demand for conditional input demand.

   (d) **Theorem 3.4.1.** If f is homothetic, then for all (w, y),

   $$c(w, y) = h(y)c(w, 1),$$

   where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing function satisfying $h(1) = 1$. Proved in JR and lecture.

2. Cost Function Properties Implied by Cardinal Properties of f

   (a) **Theorem 3.4.2.** If f is homogeneous of degree $\alpha > 0$, then

   $$c(w, y) = y^{1/\alpha} c(w, 1), \quad x(w, y) = y^{1/\alpha} x(w, 1).$$

   Hence, CRS $\Rightarrow$ constant MC and $\alpha < 1 \Rightarrow$ increasing MC.

   (b) **Proposition.** f strictly concave $\Rightarrow$ c strictly convex in y (increasing MC). Proved in lecture. Figures for case of convex-concave f.

3. Competitive Profit Maximization

   (a) A competitive firm takes output and input prices as "fixed". Its problem is thus

   $$\max_{y \geq 0} py - c(w, y).$$

   The value of this program is the *profit function*, $\pi(p, w)$, and the solution is the firm's *supply function*, $y^* = y(p, w)$. It satisfies the NFOC

   $$\begin{array}{ll} \text{(NFOC)} & p = c_y(w, y^*) \text{ (price equal to MC)}, \\ \text{(NSOC)} & c_{yy}(w, y^*) \geq 0 \text{ (MC increasing at } y^*). \end{array}$$

   Figures.

Lecture 17. Profit, Supply, and Demand Functions (JR 3.5)

1. Our single-output competitive firm's profit function can be written as

$$\pi(p, w) := \max_{x \geq 0} pf(x) - w \cdot x.$$

The solution of this program is the firm's input demand function, $x^* = x(p, w)$. The supply function is $y(p, w) = f(x(p, w))$. The NFOC for an interior solution is $p\nabla f(x^*) = w$. This is also the SFOC if f is concave.

Existence issue – figures.

2. Properties of $\pi(p, w)$: continuous; increasing in p and decreasing in w; homogeneous of degree one; and convex. Proofs by the usual arguments.

3. Properties of demand and supply, $y(p, w)$ and $x(p, w)$:

   (a) Homogeneous of degree zero

   (b) Generalized Law of Supply and Demand:

$$(p^1 - p^0)(y^1 - y^0) - (w^1 - w^0) \cdot (x^1 - x^0) \geq 0$$

   (c) Hotelling's lemma:

$$\frac{\partial \pi(p, w)}{\partial p} = y(p, w), \quad \frac{\partial \pi(p, w)}{\partial w_i} = -x_i(p, w)$$

   (d) At any (p, w), the matrix

$$\begin{bmatrix} \frac{\partial y}{\partial p} & \frac{\partial y}{\partial w_1} & \cdots & \frac{\partial y}{\partial w_n} \\ -\frac{\partial x_1}{\partial p} & -\frac{\partial x_1}{\partial w_1} & \cdots & -\frac{\partial x_1}{\partial w_n} \\ \vdots & \vdots & & \vdots \\ -\frac{\partial x_n}{\partial p} & -\frac{\partial x_n}{\partial w_1} & \cdots & -\frac{\partial x_n}{\partial w_n} \end{bmatrix}$$

   is PSD (Laws of Supply and Demand) and, if π is C^2, symmetric (Law of Reciprocity).

4. Short Run versus Long Run: Profit and Supply

$$\text{(LRP)} \quad \max_x pf(x) - w_1x_1 - w_2x_2 \quad \Rightarrow \quad x^L(p),\ y^L(p),\ \pi^L(p)$$

$$\text{(SRP)} \quad \max_x pf(x) - w_1x_1 - w_2x_2 \text{ s.t. } x_1 = \bar{x}_1 \quad \Rightarrow \quad x^S(p),\ y^S(p),\ \pi^S(p)$$

Proposition. If $\bar{x}_1 = x_1^L(\bar{p})$ at $\bar{p} > 0$, then $y^S(\bar{p}) = y^L(\bar{p})$ and $y^{S\prime}(\bar{p}) \leq y^{L\prime}(\bar{p})$.

Hence, this SR supply curve crosses the LR supply curve from below at $\bar{p}$, and so its price elasticity is lower. Starting from a LR profit-maximizing situation at $\bar{p}$, the firm's response to an increase in the output price will be larger in the LR than it is in the SR. Prove by Hotelling and the NSOC obtained by noting that

$$\bar{p} \in \arg\max_p \pi^S(p) - \pi^L(p).$$

Lecture 18. Short-Run Cost (JR 3.3) and Industry Equilibrium (JR 4.1)

1. Short-Run Cost Functions

   (a) Definition. Assuming $n = 2$ and input 1 is fixed in the short run (and surpressing the argument w) (figure, example):

   $$c^S(y, x_1) := \min_{x_2} w_1 x_1 + w_2 x_2 \quad \text{s.t. } f(x) \geq y.$$

   (b) Relationship to LR cost: First, note that $c^S(y, x_1) \geq c(y)$. Also,

   $$\begin{aligned} c(y) &= \min_{x} w_1 x_1 + w_2 x_2 \quad \text{s.t. } f(x) \geq y \\ &= \min_{x_1} \left\{ \min_{x_2} w_1 x_1 + w_2 x_2 \quad \text{s.t. } f(x) \geq y \right\} = \min_{x_1} c^S(y, x_1). \end{aligned}$$

   So $c(\cdot)$ is the lower envelope of all the SR cost functions, $\left\{ c^S(\cdot, x_1) : x_1 \geq 0 \right\}$.

   (c) Envelope theorem result: letting $x_1(y) \in \arg\min_{x_1} c^S(y, x_1)$, from (b) we have $c(y) = c^S(y, x_1(y))$. Furthermore, by ET1 we have

   $$c'(y) = c^S_y(y, x_1(y)).$$

   (d) Fix $x_1 = \bar{x}_1$, and let $\bar{y}$ be the output satisfying $x_1(\bar{y}) = \bar{x}_1$. Then (b)-(c) imply

   $$\bar{y} \in \arg\max_{y} c(y) - c^S(y, x_1).$$

   The NSOC for this program is (figure)

   $$c''(\bar{y}) - c^S_{yy}(\bar{y}, x_1) \leq 0.$$

   Thus, SR supply curves are steeper than LR supply curves where they cross. (As we derived also in L17.) "SR supply is less elastic than LR supply."

2. Short-Run Cost Functions and Shut-Down Decisions

   $$\begin{aligned} c^S(y, x_1) &= w_1 x_1 + w_2 x_2(y, x_1) \\ &= F + c^V(y.x_1) = \text{ fixed cost + variable cost.} \end{aligned}$$

   Note that $py - F - c^V(y) > -F$ iff

   $$p > \frac{c^V(y)}{y} =: c^{AV}(y).$$

   So the firm's optimal decision is to shut down (choose $y = 0$) iff $p < \min_y c^{AV}(y)$.

   Geometry of AVC and MC curves: MC>AC implies the AC curve must be rising.

3. Short-Run Industry Equilibrium

   Fixed number J of firms, each with the same f and fixed x_1. Supply equal demand determines the SR equilibrium price, p^{SR}.

Lecture 19. Industry Equilibrium and Monopoly (JR 4)

1. Long-Run Industry Equilibrium

   (a) Basic result: Free entry and exit $\Rightarrow$ zero profit $\Rightarrow p^{LR} = \min_q c^A(q)$

   (b) So in the LR, a competitive industry acts *as though* the technology is CRS

   (c) Caveats: input prices must be constant; no learning-by-doing externalities

2. Monopoly

3. Cartels

Lecture 20. Oligopoly Models and Efficiency (JR 4)

1. Oligopoly

   (a) Cournot oligopoly

   (b) Bertrand oligopoly

2. Pareto Efficiency

   (a) **Definition.** Given an abstract collective choice environment $\langle N, X, (\precsim_i)_{i \in N} \rangle$, say that $x \in X$ *Pareto dominates (is Pareto superior to)* another possible choice $y \in X$ iff

   $$x \succsim_j y \text{ for all } j \in N, \text{ and } x \succ_j y \text{ for some } j \in N.$$

   A possible choice $x^* \in X$ is *Pareto efficient (Pareto optimal, efficient)* iff it is not Pareto dominated by any $x \in X$.

   (b) **Example.** $X = \{Barb, Karen, Nick\}$, candidates for President of Tiny USA. $N = \{$George, Rebecca, Mary$\}$, the citizens of Tiny USA. Which outcomes are PO in each of the following cases?

   Case I

$\precsim_G$	$\precsim_R$	$\precsim_M$
B	K	K
K	B	N
N	N	B

   Case II

$\precsim_G$	$\precsim_R$	$\precsim_M$
B	K	B
N	N	N
K	B	K

3. Evaluating the efficiency of industry equilibria

   (a) The relevant set of agents for evaluating the efficiency of an industry equilibrium consists of all consumers *and* firms.

   (b) Monopoly outcomes are inefficient because the monopoly output is too low. Proof based on price discrimination.

   (c) Same for Cournot equilibrium outcomes. Although, under fairly general conditions they converge to efficient outcomes as $J \to \infty$.

   (d) Bertrand equilibria are efficient so long as $J \geq 2$.

Lecture 21 (Post Midterm 2). Monopoly Loose Ends (JR 4); GE Introduction (JR 5.1&2)

1. Monopoly Loose Ends
   (a) Results regarding the price elasticity of demand
   (b) Why $p = mc(q)$ is necessary for efficiency: graphical & calculus proofs
2. Exchange Economies
   (a) Definitions, Edgeworth boxes
   (b) FOC satisfied by efficient (PO) allocations
      i. Pareto progams
      ii. **Prop.** Assuming the marginal utility functions u_k^i are all positive, any PO x^* satisfies (a) the resource constraints with equality, and (b) if $(x_\ell^{i*}, x_k^{j*}) \gg 0$ for distinct consumers i and j and distinct goods k and ℓ, then the tangency conditions
      $$MRS_{k\ell}^i(x^{i*}) = MRS_{k\ell}^j(x^{j*}).$$
      If instead $x_\ell^{i*} = 0$ or $x_k^{j*} = 0$, then $MRS_{k\ell}^i(x^{i*}) \geq MRS_{k\ell}^j(x^{j*})$.[1]
      iii. The converse of this Proposition holds if each u^i is quasiconcave.

[1] The convention here is $MRS_{k\ell}^i = u_k^i / u_\ell^i$.

Lecture 22. General Equilibrium in Exchange Economies (JR 5.1. 5.2)

1. The Core of an Exchange Economy

   (a) Edgeworth box example

   (b) General definition of blocking and core

   (c) An interpretation: Core = set of 'barter equilibrium' allocations (Coase theorem,)

      i. assuming costless coalition formation and well-defined and enforced property rights (frictionless bargaining) and

      ii. assuming costless full information about preferences and endowments,

      iii. do not need to assume preferences are selfish (i.e., there may be externalities)

2. Walrasian Equilibrium

   (a) **Definition.** A *Walrasian Equilibrium* (WE) is a pair (p^*, x^*) for which

      i. for each i, x^{i*} maximizes consumer i's utility at prices p^*, and

      ii. market clearing: $\sum_i x^{i*} = \bar{e} := \sum_i e^i$.

   (b) **Proposition.** If (p, x) is a WE, then so is $(\lambda p, x)$ for any $\lambda > 0$ (homogeneity in prices).

   (c) Consequently, we can wlog normalize prices if convenient. Usually either by assuming $p_n = 1$ (numeriaire), or by assuming $\sum_k p_k = 1$.

Lecture 23. General Equilibrium in Exchange Economies (con't) (JR 5.2)

1. Excess Demand

   (a) Consumer i's demand function in our exchange economy is $x^i(p, p \cdot e^i)$, where the function $x^i(p, y^i)$ is her Marshallian demand function. The *aggregate excess demand function* is

   $$z(p) = \sum_i x^i(p, p \cdot e^i) - \bar{e}.$$

   Thus, (p^*, x^*) is a WE iff $x^{i*} = x^i(p^*, p^* \cdot e^i)$ for each i, and $z(p^*) = 0$.

   (b) **Proposition.** $z(p)$ is homogeneous of degree zero and, if each u^i is locally non-satiated, satisfies Walras' Law:

   $$p \cdot z(p) = 0.$$

   (c) Consequently, a price vector $p \gg 0$ that clears $n-1$ markets must clear all n markets.

2. Example

3. **Existence Theorem.** A WE exists if, for all i, u^i is continuous, strongly increasing, and strictly quasiconcave, and $\bar{e} \gg 0$.

4. An Edgeworth box example of nonexistence when $e^i \gg 0$ does not hold for some i, and another for when u^i is not quasiconcave for some i.

5. **First Welfare Theorem (FWT)**. Every WE Allocation (WEA) is in the core (and hence Pareto optimal) if each u^i satisfies local non-satiation.

6. Edgeworth box example of a WEA x^* that is not Pareto optimal when some u^i is locally satiated at x^*.

Lecture 24. General Equilibrium (con't) (JR 5.2, 5.3, 5.4, and Handout)

1. Proof of the FWT.
2. **Second Welfare Theorem (SWT).** Under the same assumptions as in the Existence Theorem, if $x^*$is Pareto optimal, then a price vector $p^* \in \mathbb{R}^n_{++}$ and a "transfer of endowments," $\Delta e \in (\mathbb{R}^n)^I$ satisfying $\sum_i \Delta e^i = 0$, exists such that (p^*, x^*) is a WE of the economy with endowment $e + \Delta e$.
   (a) Proof.
   (b) Edgeworth box example of a PO x^* that is not a WEA for any transfers because some u^i is not quasiconcave.
3. Remarks on GE with Production
   (a) J firms, each with production set Y^j, $\Pi^j(p) = \max_{y \in Y^j} p \cdot y$.
   (b) Private ownership: consumer i's income is now

$$m^i(p) := p \cdot e^i + \sum_j \theta^{ij} \Pi^j(p),$$

   where θ^{ij} is the share of firm j owned by consumer i. (Hence, $\theta^{ij} \in [0,1]$ and $\sum_i \theta^{ij} = 1$ for all j.) Consumer i's demand function is $x^i(p) = x^i(p, m^i(p))$.
   - This is *why* the assumption that firms maximize profit makes sense in the competitive model!

   (c) Aggregate excess demand: $z(p) = \sum_i x^i(p) - \sum_j y^j(p) - \bar{e}$
   (d) Walrasian equilibrium: (p^*, x^*, y^*) such that $z(p^*) = 0$, $x^{i*} = x^i(p^*)$ for all consumers i, and $y^{j*} = y^j(p)$ for all firms j.
   (e) The existence and welfare theorems extend to production economies.
      i. The assumption that $\bar{e} \gg 0$ in the Existence Theorem and the SWT is replaced by the significantly weaker assumption that $y \in \sum_j Y^j$ exists such that $y + \bar{e} \gg 0$.
      ii. Proving existence of a WE is about the same as for an exchange economy, if one assumes each Y^j is bounded. But as JR discuss, assuming Y^j is bounded violates the spirit of decentralization. Boundedness can be replaced by alternative assumptions that rule out a firm making an arbitarily large amount of some good using a bounded amount of inputs.
4. Dates and States: Arrow-Debreu Economies (Handout)