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Solutions to Problem Set 1 (Due Wednesday, 9/13)

1. Acompetitive market has n buyers, each one with the same downward-sloping demand function $d(p)$. Thus, the market demand function is $nd(p)$. The market supply function is $S(p)$, an upward-sloping function. Assume the functions d and S are both differentiable. Let $p^*(n)$ and $x^*(n)$ be the equilibrium price and quantity given a fixed n. Determine their comparative static properties. What further assumptions do you need to make?

***Soln*:** We make two more assumptions.[1] The first is that at the particular n we are interested in, an equilibrium price exists, i.e., there is a price $p^*(n)$ that equates demand and supply:

$$nd(p^*(n)) - S(p^*(n)) = 0. \tag{1}$$

Second, we assume that the derivatives of the given functions are not zero at $p^*(n)$:[2]

$$d^{*\prime}(p^*(n)) \neq 0 \text{ and } S^{*\prime}(p^*(n)) \neq 0.$$

This, together with the fact that d is downward sloping and S is upward sloping, implies $d^{*\prime}(p^*(n)) < 0$ and $S^{*\prime}(p^*(n)) > 0$. For later use, let's note that we now have

$$nd^{*\prime}(p^*(n)) - S^{*\prime}(p^*(n)) < 0. \tag{2}$$

The question asks us to find the signs of $p^{*\prime}(n)$ and $x^{*\prime}(n)$. We find $p^{*\prime}(n)$ by using the chain rule to differentiate the identity shown in (1) with respect to n :

$$d(p^*(n)) + \left[nd'(p^*(n)) - S'(p^*(n))\right] p^{*\prime}(n) = 0.$$

Since (2) implies the term in square brackets is not zero, we can divide by it to find

$$p^{*\prime}(n) = \frac{-d(p^*(n))}{nd'(p^*(n)) - S'(p^*(n))}.$$

Because the denominator is negative by (2), and $d(p^*(n)) \geq 0$ (this is an implicit assumption too – it should have been made explicit), we have determined the comparative statics properties of the equilibrium price:

$$p^{*\prime}(n) > 0 \text{ if } d(p^*(n)) > 0, \text{ and } p^{*\prime}(n) = 0 \text{ if } d(p^*(n)) = 0.$$

Lastly, since the quantity of the good sold is $x^*(n) = S(p^*(n)$, we have

$$x^{*\prime}(n) = S'(p^*(n))p^{*\prime}(n),$$

and so

$$x^{*\prime}(n) > 0 \text{ if } d(p^*(n)) > 0, \text{ and } p^{*\prime}(n) = 0 \text{ if } d(p^*(n)) = 0.$$

From (1) we see that $x^*(n) > 0$ iff $d(p^*(n)) > 0$, so we can write the comparative statics properties more transparently as

$$x^*(n) > 0 \;\Rightarrow\; p^{*\prime}(n), x^{*\prime}(n) > 0,$$
$$x^*(n) = 0 \;\Rightarrow\; p^{*\prime}(n), x^{*\prime}(n) = 0.$$

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[1] Actually, we should also make a third assumption, that the functions d and S are continuously differentiable. This allows us to use the implicit function theorem.

[2] We are told that $S(p)$ is an upward-sloping and differentiable function, but this does not imply that its derivative is everywhere positive. For example, the function $S(p) = 1 + (p-1)^3$ is strictly increasing and differentiable, but has a zero derivative at $p = 1$. Similarly, the properties of d that are stated in the problem do not imply its derivative is negative everywhere.

JR1.2 ***Soln***

(a) This asks you to prove that the set $\succsim$ is a subset of itself, which is silly – every set is a subset of itself.

(b) Let $(x, y) \in \sim$. We must show that $(x, y) \in \succsim$. This is trivial: the definition of $x \sim y$ alone implies $x \succsim y$, i.e., that $(x, y) \in \succsim$.

(c) To prove $\succ \cup \sim = \succsim$, we have to show (i) $\succ \cup \sim \subset \succsim$ and (ii) $\succsim \subset \succ \cup \sim$.

Proof of (i): Let $(x, y) \in \succ \cup \sim$. Then, either $x \succ y$ or $x \sim y$.
Case $x \succ y$: Then we have $x \succsim y$ and *not* $y \succsim x$. $x \succsim y$ implies $(x, y) \in \succsim$.
Case $x \sim y$: Then we have $x \succsim y$ and $y \succsim x$. $x \succsim y$ implies $(x, y) \in \succsim$.
Therefore, $\succ \cup \sim \subset \succsim$.

Proof of (ii) Let $(x, y) \in \succsim$, i.e., $x \succsim y$. By completeness of $\succsim$, we have either $y \succsim x$ or *not* $y \succsim x$.
Case $y \succsim x$.: In this case we have $x \sim y$, that is, $(x, y) \in \sim$.
Case *not* $y \succsim x$: In this case we have $x \succ y$, that is, $(x, y) \in \succ$.
Therefore, $(x, y) \in \succ \cup \sim$.

(d) Suppose $(x, y) \in \succ$. Then, by the definition of $\succ$, $x \succsim y$ and not $y \succsim x$. Therefore, by the definition of $\sim$, not $x \sim y$, i.e., $(x, y) \notin \sim$. We conclude that $\succ \cap \sim = \emptyset$.

JR1.4 ***Soln***

Proof that $\succ$ is transitive: Suppose $\mathbf{x}^1 \succ \mathbf{x}^2$ and $\mathbf{x}^2 \succ \mathbf{x}^3$. Then, $\mathbf{x}^1 \succsim \mathbf{x}^2$ and $\mathbf{x}^2 \succsim \mathbf{x}^3$, and so the transitivity of $\succsim$ implies $\mathbf{x}^1 \succsim \mathbf{x}^3$. Now we want to show *not* $\mathbf{x}^3 \succsim \mathbf{x}^1$. Suppose instead that $\mathbf{x}^3 \succsim \mathbf{x}^1$. This and $\mathbf{x}^1 \succsim \mathbf{x}^2$ imply $\mathbf{x}^3 \succsim \mathbf{x}^2$ by transitivity, which contradicts $\mathbf{x}^2 \succ \mathbf{x}^3$. So it must be that *not* $\mathbf{x}^3 \succsim \mathbf{x}^1$. This and $\mathbf{x}^1 \succsim \mathbf{x}^3$ imply $\mathbf{x}^1 \succ \mathbf{x}^3$. We conclude that $\succ$ is transitive.

Proof that $\sim$ is transitive: Suppose $\mathbf{x}^1 \sim \mathbf{x}^2$ and $\mathbf{x}^2 \sim \mathbf{x}^3$. Then, $\mathbf{x}^1 \succsim \mathbf{x}^2$ and $\mathbf{x}^2 \succsim \mathbf{x}^1$, and $\mathbf{x}^2 \succsim \mathbf{x}^3$ and $\mathbf{x}^3 \succsim \mathbf{x}^2$. Hence, by transitivity, we have $\mathbf{x}^1 \succsim \mathbf{x}^3$ and $\mathbf{x}^3 \succsim \mathbf{x}^1$. Thus, $\mathbf{x}^1 \sim \mathbf{x}^3$. This proves that $\sim$ is transitive.

Proof that $\mathbf{x}^1 \sim \mathbf{x}^2$ and $\mathbf{x}^2 \succsim \mathbf{x}^3$ implies $\mathbf{x}^1 \succsim \mathbf{x}^3$: Suppose $\mathbf{x}^1 \sim \mathbf{x}^2$ and $\mathbf{x}^2 \succsim \mathbf{x}^3$. From the former we have $\mathbf{x}^1 \succsim \mathbf{x}^2$. From this and $\mathbf{x}^2 \succsim \mathbf{x}^3$, we have $\mathbf{x}^1 \succsim \mathbf{x}^3$ by transitivity.

JR1.7 ***Soln*:** We want to show that Axiom 5' implies that $\succsim (\mathbf{x}^0)$ is a convex set.

So, suppose $\mathbf{x}^1, \mathbf{x}^2 \in \succsim (\mathbf{x}^0)$. Without loss of generality (w.l.o.g.), we can assume $\mathbf{x}^1 \succsim \mathbf{x}^2$. By Axiom 5', $t\mathbf{x}^1 + (1-t)\mathbf{x}^2 \succsim \mathbf{x}^2$. Since $\mathbf{x}^2 \in \succsim (\mathbf{x}^0)$, we also have $\mathbf{x}^2 \succsim \mathbf{x}^0$. Hence, by transitivity, $t\mathbf{x}^1 + (1-t)\mathbf{x}^2 \succsim \mathbf{x}^0$, and so $t\mathbf{x}^1 + (1-t)\mathbf{x}^2 \in \succsim (\mathbf{x}^0)$. This proves that $\succsim (\mathbf{x}^0)$ is a convex set.

JR1.9 ***Soln*:** Note that the described preferences $\succsim$ are those represented by the Leontief function $u(x_1, x_2) = \min\{x_1, x_2\}$.

Proof that $\succsim$ satisfies Axiom 5′: Suppose $\mathbf{x}^1 \succsim \mathbf{x}^0$. Then

$$\min\{x_1^1, x_2^1\} \geq \min\{x_1^0, x_2^0\}. \tag{3}$$

Hence, for any $t \in [0, 1]$,

$$\begin{aligned}\min\{tx_1^1 + (1-t)x_1^0,\ tx_2^1 + (1-t)x_2^0\} &\geq \min\left\{\min\{x_1^1, x_1^0\},\ \min\{x_2^1, x_2^0\}\right\} \\ &= \min\{x_1^1, x_1^0, x_2^1, x_2^0\} \\ &= \min\{x_1^0, x_2^0\},\end{aligned}$$

where the last equality follows from (3). Hence, $t\mathbf{x}^1 + (1-t)\mathbf{x}^0 \succsim \mathbf{x}^0$, and so Axiom 5' holds.

Proof that $\succsim$ satisfies Axiom 4: If $\mathbf{x}^0 \geq \mathbf{x}^1$, then $\min\{x_1^0, x_2^0\} \geq \min\{x_1^1, x_2^1\}$, and so $\mathbf{x}^0 \succsim \mathbf{x}^1$. Similarly, if $\mathbf{x}^0 \gg \mathbf{x}^1$, then $\min\{x_1^0, x_2^0\} > \min\{x_1^1, x_2^1\}$, and so $\mathbf{x}^0 \succ \mathbf{x}^1$.

Proof that $\succsim$ does not satisfy Axiom 5: Let $\mathbf{x}^1 = (1, 2)$, and $\mathbf{x}^0 = (1, 1)$. Note that $\mathbf{x}^1 \neq \mathbf{x}^0$ and $\mathbf{x}^1 \succsim \mathbf{x}^0$. However, for any $t \in [0, 1]$,

$$\begin{aligned}\min\{tx_1^1 + (1-t)x_1^0,\ tx_2^1 + (1-t)x_2^0\} &= \min\{1, 2t + (1-t)\} \\ &= 1 \\ &= \min\{x_1^0, x_2^0\}.\end{aligned}$$

Hence, *not* $t\mathbf{x}^1 + (1-t)\mathbf{x}^0 \succ \mathbf{x}^0$.

2. For $n = 1, 2$, define $\succsim_n$ on $\mathbb{R}_+^n$ by $x \succsim y$ iff $x \geq y$. Determine whether $\succsim_1$ and $\succsim_2$ are complete, transitive, continuous, convex, and strictly convex.

Soln

Completeness: Since $\geq$ on $\mathbb{R}_+$ is complete, $\succsim_1$ is complete. But $\succsim_2$ is not complete because $\geq$ on $\mathbb{R}_+^2$ is not complete. (For example, $(1, 2) \not\geq (2, 1)$ and $(2, 1) \not\geq (1, 2)$.)

Transitivity: $\succsim_n$ for both $n = 1$ and $n = 2$ is transitive, because $\geq$ on $\mathbb{R}_+^n$ is transitive.

Continuity: $\succsim_1$ is continuous, since for all $x \in \mathbb{R}_+$, the contour sets

$$\succsim_1 (x) = [x, \infty) \text{ and } \precsim_1 (x) = (-\infty, x]$$

are closed. Similarly, $\succsim_2$ is continuous, since for all $x \in \mathbb{R}_+^2$, the contour sets

$$\succsim_2 (x) = [x_1, \infty) \times [x_2, \infty) \text{ and } \precsim_2 (x) = (-\infty, x_1] \times (-\infty, x_2]$$

are closed.

Convexity: $\succsim_1$ and $\succsim_2$ are both convex binary relations, since their upper contour sets are either intervals or the Cartesian product of intervals, and so are convex sets.

Strict Convexity: $\succsim_1$ is strictly convex. To see why, suppose $y, z \in \succsim_1 (x)$ and $y \neq z$. Then $y \geq x$ and $z \geq x$. Moreover, because $y \neq z$, at least one of them are strictly larger than x. Without loss of generality, assume $y > x$. This and $z \geq x$ imply that $ty + (1-t)z > x$ for all $t \in (0,1)$. Hence, $ty + (1-t)z \succ_1 x$ for all $t \in (0,1)$. This proves that $\succsim_1$ is strictly convex.

However, $\succsim_2$ is not strictly convex. To prove this, let $x = (1,1)$, $y = (2,1)$, and $z = (3,1)$. Then $y \succsim_2 x$, $z \succsim_2 x$ and $y \neq z$. But for any $t \in (0,1)$,

$$ty + (1-t)z = (3-t, 1)\,,$$

which is not strictly greater in each component than $x = (1,1)$. Hence, *not* $ty + (1-t)z \succ_2 x$.