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Suggested Solutions to Problem Set 7
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1. A monopoly has cost function $c(q) = 6q$. There is only one price-taking (!) consumer in the market for this good. Her utility function is $u(q) + x$, where x is the amount of income she spends on all other goods – view x as a single good with price 1. Assume also that $u(0) = 0$, and that the consumer's income, y, is large enough that for any relevant price p of q, her Marshallian demand for good x, $x(p, 1, y)$, is positive. (In particular, assume $y > 32$.) Her demand function for q is given by $D(p) = 10 - p$ for $p < 10$, and by $D(p) = 0$ for $p \geq 10$.

   (a) Find the function u.

   Soln: The FOC for the consumer's UMP are that $\lambda^* \geq 0$ exists such that

$$\begin{aligned} u'(q^*) - \lambda^* p \leq 0, &\text{ with equality if } q^* > 0; \\ 1 - \lambda^* \leq 0, &\text{ with equality if } x^* > 0; \\ pq^* + x^* \leq y, &\text{ with equality if } \lambda^* > 0. \end{aligned}$$

   Let $p < 10$. Then we know $q^* = 10 - p > 0$, and we've been told that $x^* > 0$. It follows that the first inequality in each line of the FOCs is an equality. So, from the first one we have $u'(q^*) = p$, or rather, since $p = 10 - q^*$,

$$u'(q^*) = 10 - q^*. \tag{1}$$

   Now, for any $q \in [0, 10]$, by repeating this argument for the price $\hat{p} = 10 - q$ we obtain this same expression for q.[1] Thus, (1) is an identity in q on $[0, 10]$, a differential equation we can integrate directly. Integrating from 0 to any $q \in [0, 10]$ yields

$$u(q) = 10q - \tfrac{1}{2}q^2,$$

   since $u(0) = 0$. This is the answer for $q \in [0, 10]$. For $q > 10$ the nature of u is not pinned down – it simply needs to not increase so rapidly in q that the consumer will want to purchase a positive amount of q for some $p > 10$. The simplest u is constant for $q > 10$, say $u(q) = u(10)$. This u is

$$u(q) = \begin{cases} 10q - \frac{1}{2}q^2 & \text{for } 0 \leq q \leq 10 \\ 50 & \text{for } q > 10 \end{cases}.$$

   ■

   (b) A two-part tariff consists of a fixed fee f the consumer must pay in order to purchase the good q, and a price p that must be paid per unit purchased. So, if she purchases an amount $q > 0$, she pays $pq + f$. Find the firm's profit-maximizing two-part tariff, (f, p).

   Soln: The basic idea is that the firm sets f so that the consumer is just willing to buy her optimal amount at that price, $q(p) = 10 - p$, as opposed to not buying at all – this makes the firm's total revenue as large as possible. Given this strategy for choosing f, the firm's profit is the total surplus (consumer

[1] In other words, $u'(q)$ is the inverse demand function $P(q)$ on $[0, 10]$.

surplus minus cost), and so it wants to set p to maximize total surplus, i.e., set p so that the consumer's choice of q maximizes the total surplus $u(q) - c(q)$. This price, of course, is $p = MC = 6$. (There are nice pictures to go along with this informal argument.)

Formally, for any $q \geq 0$ let $b(q)$ be the most the consumer would be willing to pay to get the amount q rather than have none of this good at all. It is given by the equation

$$u(q) + y - b(q) = y \;\Rightarrow\; b(q) = u(q).$$

(We ignore for now the constraint $b(q) \leq y$.) Assuming the consumer can actually afford to pay this amount, i.e., assuming $u(q) \leq y$, the maximal profit the firm can get if it sells q to the consumer is $u(q) - 6q$. In this case the firm would set q to satisfy the FOC, $u'(q) = 6$, which here yields $q^* = 4$.

Now, given (f, p), and assuming the consumer purchases, she purchases the amount q satisfying $u'(q) = p$, i.e., $q = 10 - p$. To ensure that she purchases $q^* = 4$, the firm sets price $p^* = 6$. Thus, to extract the maximal amount $b(q^*)$ from the consumer, we must have $p^*q^* + f = b(q^*)$, or rather, the optimal fixed fee is

$$\begin{aligned} f^* &= b(q^*) - p^*q^* \\ &= u(4) - (6)(4) \\ &= 32 - 24 \\ &= 8. \end{aligned}$$

This gives us the answer, so long as the constraint $u(q^*) \leq y$ is satisfied. Since $u(q^*) = 32$, this constraint is satisfied because $y \geq 32$. So the answer is $(f^*, p^*) = (8, 6)$. ■

2. Alice and Bob are the sole consumers in two competitive markets for goods 1 and 2. Their utility functions are

$$u^A(x_1^A, x_2^A) = \ln x_1^A + x_2^A \quad \text{and} \quad u^B(x_1^B, x_2^B) = \ln x_1^B + x_2^B,$$

respectively. Their endowments are $\omega^A = (2, 0)$ and $\omega^B = (1, 2)$.

(a) Find an equation for the part of the contract curve that is in the interior of the Edgeworth box, and graph the entire contract curve.

Soln: Equate the MRSs:

$$\frac{1}{x_1^A} = \frac{1}{x_1^B} \implies x_1^A = x_1^B.$$

Hence, since $x_1^A + x_1^B = \bar{\omega}_1 = 3$, the equation for the interior part of the contract curve is $x_1^A = x_1^B = \frac{3}{2}$.

The Edgeworth box is of width 3 and height 2. The contract curve starts at the origin in the lower left, goes right along the bottom boundary to $x_1^A = \frac{3}{2}$, then goes vertically through the box to the top of it, and then right along the top to the upper right corner. ■

(b) Find the Walrasian equilibrium allocation x and price vector $(p_1, 1)$ (good 2 is the numeraire).

Soln: The WE allocation is on the contract curve. By putting the endowment in the Edgeworth box, it's obvious that any equilibrium budget line through it will cut through the contract curve only in the interior of the box. Hence, the WE allocation satisfies

$$x_1^A = x_1^B = \frac{3}{2}.$$

As each consumer j sets her MRS $1/x_1^j$equal to the price ratio, $p_1/p_2 = p_1$, we have

$$p_1 = \frac{2}{3}.$$

The WE allocations of good 2 are now determined by calculating the value of their endowments and subtracting the cost of their good 1 purchases, and then dividing by $p_2 = 1$:

$$x_2^A = \left(\frac{2}{3}\right)(2) - \left(\frac{2}{3}\right)\left(\frac{3}{2}\right) = \frac{1}{3},$$

$$x_2^B = \left(\frac{2}{3}\right)(1) + (1)(2) - \left(\frac{2}{3}\right)\left(\frac{3}{2}\right) = \frac{5}{3}.$$

■

3. JR Exercise 5.11

(a) **Soln:** First note that we can assume u^1 takes the simpler form $u^1(x) = x_1 x_2$, as this is a monotonic transformation of $(x_1 x_2)^2$. For the same reason we can assume $u^2(x) = x_1(x_2)^2$.

We first find the interior Pareto efficient allocations, by equating their MRSs and using the resource constraints (which will obviously bind). Since

$$MRS^1 = \frac{x_2^1}{x_1^1},\ MRS^2 = \frac{x_2^2}{2x_1^2},$$

we have

$$\frac{x_2^1}{x_1^1} = \frac{x_2^2}{2x_1^2} = \frac{10 - x_2^1}{2(21 - x_1^1)}$$
$$\Rightarrow\ x_2^1 = \frac{10x_1^1}{42 - x_1^1}.$$

So the graph of this function in the Edgeworth box is the set of interior Pareto efficient allocations. Note that it goes from the lower left corner to the upper right corner. Hence, the only non-interior efficient allocations are those two corners. The entire set of efficient allocations is thus

$$PE = \left\{ ((x_1^1, x_2^1), (21 - x_1^1, 10 - x_2^1)) \mid 0 \le x_1^1 \le 21,\ x_2^1 = \frac{10x_1^1}{42 - x_1^1} \right\}.$$

■

(b) **Soln:** The core here is the set of feasible allocations that are not blocked by any of the three possible coalitions, $\{1\}, \{2\}$, and $\{1,2\}$. Coalition $\{1\}$ blocks an allocation x iff $u^1(x^1) < u^1(e^1)$, i.e., iff $x_1^1 x_2^1 < 72$. Similarly, $\{2\}$ blocks x iff $x_1^2(x_2^2)^2 < (3)(6)^2 = 108$. The coalition $\{1,2\}$ blocks x iff x is not Pareto efficient. Thus, in using the answer to (a), we conclude that

$$\begin{aligned} Core \;&=\; \Big\{((x_1^1, x_2^1), (21 - x_1^1, 10 - x_2^1)) \mid 0 \le x_1^1 \le 21,\ x_2^1 = \frac{10x_1^1}{42 - x_1^1}, \\ &\qquad x_1^1 x_2^1 \ge 72,\ (21 - x_1^1)(10 - x_2^1)^2 \ge 108\Big\}. \end{aligned}$$

■

(c) **Soln:** Normalize the price of good 2 to be 1, and let p be the price of good 1. As both agents have Cobb-Douglas utility functions, consumer 1 will spend her wealth equally on each good and consumer 2 will spend 1/3 of his wealth on good 1 and 2/3 of it on good 2. Thus, the demand functions are

$$\begin{aligned} x^1(p,1) &= \left(\frac{18p+4}{2p}, 9p+2\right), \\ x^2(p,1) &= \left(\frac{3p+6}{3p}, 2p+4\right). \end{aligned}$$

This and market clearing in the market for good 1 give us $p:$

$$\frac{18p+4}{2p} + \frac{3p+6}{3p} = 21 \Rightarrow\ p = \frac{4}{11}.$$

(We need not check market clearing in the market for good 2, as we know, by Walras' law, that it clears because the good 1 market clears.) The Walrasian equilibrium is thus

$$(p,x) = \left(\left(\frac{4}{11}, 1\right), \left(\left(\frac{29}{2}, \frac{58}{11}\right), \left(\frac{13}{2}, \frac{52}{11}\right)\right)\right).$$

■

(d) **Soln:** The WEA satisfies market clearing, and the MRSs are equalized at the WEA (both equal to the price ratio), which implies it is Pareto efficient because both utility functions are quasiconcave.. It only remains to verify that both agents are at least weakly better off at the WEA than they are at the endowment:

$$u^1(x^1) = \left(\frac{29}{2}\right)\left(\frac{58}{11}\right) \approx 76, \quad u^1(e^1) = (18)(4) = 72 < 76,$$

$$u^2(x^2) = \left(\frac{13}{2}\right)\left(\frac{52}{11}\right)^2 \approx 145, \quad u^2(e^2) = (3)(6)^2 = 108 < 145.$$

■

4. JR Exercise 5.15

   (a) **Soln:** The Edgeworth box is a 100 by 100 square, with the endowment at the center. Consumer 1's indifference curves are vertical lines increasing to the right, and consumer 2's are horizontal straight lines increasing downwards in the box. ■

   (b) **Soln:** It is obvious from the picture that $(x^1, x^2) = ((100, 0), (0, 100))$ Pareto dominates any other point in the Edgeworth box, i.e., any move away from it will hurt both consumers, at least one of them strictly. It is thus Pareto efficient (it is the only PE allocation), and preferred by both consumers to the endowment. It is thus the unique core allocation. ■

   (c) **Soln:** As the lower right corner is the only core allocation, it must be the WEA. So the price line must go through it and the endowment, which is the center of the square. As this line has slope -1, we conclude that WE prices satisfy $p_1 = p_2$. Thus, any WE must be of the form

$$(p, x) = ((p_1, p_2), ((100, 0), (0, 100)).$$

   To check algebraically that any such (p, x) is a WE, we need we need only check that the one with prices $\bar{p} = (1, 1)$ is a WE (homogeneity). Note that from the given preferences, their Marshallian demand functions are $x^1(\bar{p}) = (50 + 50, 0)$ and $x^2(\bar{p}) = (0, 50 + 50)$. Thus, the excess demand function at $\bar{p}$ is

$$z(\bar{p}) = x^1(\bar{p}) + x^2(\bar{p}) - \left(e^1 + e^2\right) = (100, 100) - (100, 100) = 90, 00.$$

   It follows that $(\bar{p}, ((100, 0), (0, 100))$ is a WE. ■

5. Jane will live T years. Her lifetime consumption bundle is $x = (x_1, ..., x_T)$, where $x_t = (x_{t1}, ..., x_{tn})$ and n is the number of goods. Her utility function for lifetime consumption is $u(x)$. Jane has inherited a computer company that can generate any profit stream $y = (y_1, ..., y_T)$ contained in some compact feasible set $Y \subset \mathbb{R}^T_+$. Each year Jane uses the profits y_t to purchase goods and to add or subtract from her savings account, which collects interest at the annual rate r. In year t her savings are $s_t = (1 + r)s_{t-1} + y_t - p_t \cdot x_t$ (negative savings indicate borrowing). (Set $s_0 = 0$.) Jane has *perfect foresight,* meaning that she knows what all future prices will be at the time she is born. She plans her entire life at the moment she is born, choosing x, y, and $s = (s_1, ..., s_T)$ subject to $y \in Y$ and the constraint that she not die in debt: $s_T \geq 0$.

   (a) Jane does not want to operate the company herself. Instead, she wants to tell a manager a rule for choosing y without revealing any private details about herself, such as her utility function. Can she do this and still maximize her lifetime utility? If so, what rule should she tell her manager? Prove your answer.

   Soln: This problem asks for the derivation of a famous separation theorem in a simple deterministic environment: given complete markets, a competitive firm should maximize its profit, which here is the discounted present value of the profit stream, regardless of the utility function(s) of the firm's owner(s).

Setting $s_0 \equiv 0$, Jane's overall problem when she controls the firm is

$$\text{(P)} \quad \max_{x \in \mathbb{R}^{nT}_+, s \in \mathbb{R}^T, y \in Y} u(x)$$
$$\text{s.t.} \quad s_t = (1+r)s_{t-1} + y_t - p_t \cdot x_t \quad \text{for } t = 1, \ldots, T$$
$$s_T \geq 0.$$

Divide the expression for s_t by $(1+r)^{t-1}$ to obtain

$$\frac{s_t}{(1+r)^{t-1}} = \frac{s_{t-1}}{(1+r)^{t-2}} + \frac{y_t - p_t \cdot x_t}{(1+r)^{t-1}}.$$

Sum these equalities over $t = 1, \ldots, T$, and note that $s_1, \ldots, s_{T-1}$ terms cancel out. The result, since $s_0 = 0$, is

$$\frac{s_T}{(1+r)^{T-1}} = \sum_{t=1}^{T} \frac{y_t - p_t \cdot x_t}{(1+r)^{t-1}}.$$

The constraint $s_T \geq 0$ is the same as the right hand side of this expression being nonnegative. We have thus eliminated the variables $s_1, \ldots, s_T$ and the T equations defining them, allowing us to rewrite (P) as

$$\text{(P}'\text{)} \quad \max_{x \in \mathbb{R}^{nT}_+, y \in Y} u(x)$$
$$\text{s.t.} \quad \sum_{t=1}^{T} \frac{p_t \cdot x_t}{(1+r)^{t-1}} \leq \text{DPV}(y),$$

where

$$\text{DPV}(y) := \sum_{t=1}^{T} \frac{y_t}{(1+r)^{t-1}}$$

is the discounted present value of the profit stream y. For a fixed y, (P$'$) is a standard consumer problem with the consumer's income being DPV(y). The problem obviously separates: it can be solved by first choosing $y \in Y$ to maximize DPV(y), and then choosing x to maximize $u(x)$ subject to the budget constraint shown in ((P$'$). Jane should thus tell the manager to choose $y \in Y$ to maximize DPV(y). ∎

(b) Let $n = 1$ and $T = 2$. Derive a Slutsky-Hicks expression for $\partial x_1^*/\partial r$ in terms of the first-year savings, s_1^*. Explain the intuition.

Soln: Let $y^*(r) = (y_1^*(r), y_2^*(r))$ be a maximizer of DPV(y) on Y, and assume it is differentiable (which we need to do if we are to talk about Slutsky equations). For simplicity, multiply Jane's budget equation by $1 + r$ to get rid of the denominators. Given that her manager produces the profit stream $y^*(r)$, her problem can thus be written as

$$\max_{x_1, x_2 \geq 0} u(x_1, x_2) \quad \text{s.t.} \quad (1+r)p_1 x_1 + p_2 x_2 \leq m(r),$$

where

$$m(r) := (1+r)\text{DPV}(y^*(r)) = (1+r)y_1^*(r) + y_2^*(r). \tag{2}$$

This is a standard consumer problem, with prices $\hat{p}_1 := (1+r)p_1$ and p_2. Given an arbitary "income" m, let $x_1(\hat{p}_1, p_2, m)$ and $x_1^h(\hat{p}_1, p_2, U)$ be Jane's

Marshallian and Hicksian demands for period 1 consumption. We have the usual Slutsky equation for good 1:

$$\frac{\partial x_1}{\partial \hat{p}_1} = \frac{\partial x_1^h}{\partial \hat{p}_1} - x_1 \frac{\partial x_1}{\partial m}. \tag{3}$$

Now, the question refers to demands as a function of the interest rate, holding fixed the prices (p_1, p_2). The interest rate appears twice in the arguments of the Marshallian and Hicksian demands, in $\hat{p}_1 = (1+r)p_1$ and in $m(r)$ as defined in (2). Accordingly, define the functions

$$x_1^*(r) := x_1((1+r)p_1, p_2, m(r)), \tag{4}$$

$$\hat{x}_1^h(r) := x_1^h((1+r)p_1, p_2, U). \tag{5}$$

We must find a Slutsky-Hicks decomposition for $\partial x_1^*/\partial r$ in terms of the first-year savings, s_1^*.

Using the chain rule on (4), we obtain

$$\begin{aligned}\frac{\partial x_1^*}{\partial r} &= \frac{\partial x_1}{\partial \hat{p}_1} p_1 + \frac{\partial x_1}{\partial m} m'(r) \\ &= \frac{\partial x_1^h}{\partial \hat{p}_1} p_1 - p_1 x_1 \frac{\partial x_1}{\partial m} + \frac{\partial x_1}{\partial m} m'(r),\end{aligned}$$

where the second equality follows from the Slutsky equation (3). Using the chain rule on (5) yields

$$\frac{\partial \hat{x}_1^h}{\partial r} = \frac{\partial x_1^h}{\partial \hat{p}_1} p_1,$$

and so

$$\frac{\partial x_1^*}{\partial r} = \frac{\partial \hat{x}_1^h}{\partial r} - p_1 x_1 \frac{\partial x_1}{\partial m} + \frac{\partial x_1}{\partial m} m'(r). \tag{6}$$

Now, since $y^*(r)$ maximizes $y_1 + y_2/(1+r)$ on Y, it also maximizes $(1+r)y_1 + y_2$ on Y. Hence,

$$m(r) = \max_{y \in Y} (1+r)y_1 + y_2,$$

and so the Envelope Theorem tells us that $m'(r) = y_1^*(r)$. Substituting this into (6) yields

$$\begin{aligned}\frac{\partial x_1^*}{\partial r} &= \frac{\partial \hat{x}_1^h}{\partial r} - p_1 x_1 \frac{\partial x_1}{\partial m} + \frac{\partial x_1}{\partial m} y_1^*(r) \\ &= \frac{\partial \hat{x}_1^h}{\partial r} + \left[y_1^*(r) - p_1 x_1\right] \frac{\partial x_1}{\partial m} \\ &= \frac{\partial \hat{x}_1^h}{\partial r} + s_1^* \frac{\partial x_1}{\partial m}.\end{aligned}$$

This is the disired Slutsky-Hicks expression.

Discussion: Suppose first-period consumption is a normal good, so that $\partial x_1/\partial m > 0$. Then, if the consumer is a saver ($s_1^* > 0$) and the interest rate increases, there are two opposite effects. First, the substitution effect causes first-period consumption to decrease ($\partial x_1^h/\partial r < 0$), as it becomes more expensive relative to second-period consumption. But the income effect is positive because an increase in the interest rate is like an increase in income for a saver ($s_1^* \frac{\partial x_1}{\partial m} > 0$). Depending on which effect is stronger, the consumer may save more or less when r increases. ■

6. Consider an I-person, one-good, S-state exchange economy. Consumer $i = A, B$ has the utility function $U^i(x^i) = \sum_{s=1}^{S} \pi_s u_i(x_s^i)$, where $(\pi_1, \ldots, \pi_S) \gg 0$ is the probability vector reflecting their common beliefs about the states. The aggregate endowment of the good in state s is $\bar{\omega}_s$, and these endowments are ordered $\bar{\omega}_s > \bar{\omega}_{s+1}$ for all $s = 1, \ldots, S-1$. Assume the functions u_i have derivatives $u_i' > 0$ and $u_i'' < 0$.

   (a) Prove the *co-monotonicity property*: in any interior Pareto efficient allocation, each consumer obtains more of the good in states in which the aggregate endowment is greater.

   Soln: Let x be an interior Pareto efficient allocation. Suppose it does not satisfy co-monotonicity. Then $s < S$ and consumer i exist such that $x_s^i \leq x_{s+1}^i$. Since the resource constraints bind at an efficient allocation, and $\bar{\omega}_s > \bar{\omega}_{s+1}$, it follows that for some other consumer j, we have $x_s^j > x_{s+1}^j$. These inequalities and the strict concavity of u_i and u_j imply

$$\frac{u_i'(x_s^i)}{u_i'(x_{s+1}^i)} \geq 1 > \frac{u_j'(x_s^j)}{u_j'(x_{s+1}^j)}.$$

   However, because x is interior, its efficiency implies the tangency condition,

$$\frac{\pi_s u_i'(x_s^i)}{\pi_{s+1} u_i'(x_{s+1}^i)} = \frac{\pi_s u_j'(x_s^j)}{\pi_{s+1} u_j'(x_{s+1}^j)}.$$

   As the two displayed results cannot both hold, we have a contradiction. Hence, $x_s^i > x_{s+1}^i$ for all i and $s < S$, which is the co-monotonicity property. ■

   (b) For $I = S = 2$, what does this result imply about the location of the contract curve in the Edgeworth box? Graph it.

   Soln: The contract curve contains the lower left and upper right corner points, as usual. Otherwise it must lie strictly between the two 45° lines emanating from those two corners. ■

7. Two farmers must decide how many cattle to graze on the common. Each cow can be sold for p at the end of the season. Each farmer's cost of raising cows increases with the number of cows he raises, and also with the number of cows the other farmer raises because of the lower density of grass (food) on the common due to the grazing of the other farmer's cows. In particular, their cost functions are

$$C_1(q_1, q_2) = (2q_1 + q_2)\, q_1 \quad \text{and} \quad C_2(q_1, q_2) = (q_1 + 2q_2)\, q_2,$$

   where q_i is the number of cows raised by farmer i (we allow for fractional cows).

   (a) What is the Nash equilibrium (q_1^*, q_2^*) of this game?

   Soln: The problem for farmer 1 can be written as

$$\max_{q_1} \quad pq_1 - (2q_1 + q_2)q_1.$$

   Assuming an interior solution, the first order condition is

$$p - 4q_1 - q_2 = 0.$$

Similarly, the problem of farmer 2 can be written as

$$\max_{q_2} \quad pq_2 - (q_1 + 2q_2)q_2,$$

and its FOC (for an interior solution) is

$$p - q_1 - 4q_2 = 0.$$

Solving these two linear FOC equations in two unknowns yields for q_2 we obtain

$$q_1^* = q_2^* = \tfrac{1}{5}p.$$

■

(b) What outputs would maximize total profit?
Soln: Total profits are

$$p\,(q_1 + q_2) - (2q_1 + q_2)q_1 - (q_1 + 2q_2)q_2.$$

The first order conditions are

$$p - 4q_1 - 2q_2 = 0$$

and

$$p - 2q_1 - 4q_2 = 0.$$

Solving these gives, for each $i = 1, 2$,

$$q_i = \tfrac{1}{6}p < q_i^*.$$

The inequality illustrates the *Tragedy of the Commons.* ■

8. Each of two players, $i = 1, 2$, can contribute any amount $x_i \geq 0$ of a private good to the production of a public good y. Given a contribution vector $x = (x_1, x_2)$, the amount of public good produced is given by the production function, $y = 2\sqrt{x_1 + x_2}$. Player i's resulting utility is $u_i(y, x_i) = iy - x_i$.The players make their contributions simultaneously.

(a) Find the Nash equilibrium, $x^* = (x_1^*, x_2^*)$, of this game.
Soln: $x^* = (0, 4)$: only the player with the higher marginal valuation for the public good contributes.
To see this, note that a vector x^* is an equilibrium iff for each j, $x_j = x_j^*$ maximizes

$$2j\sqrt{x_j + x_i^*} - x_j,$$

which is player j's payoff given that the other player (player i) contributes x_i^*, subject to the nonnegativity constraint on contributions, $x_j \geq 0$. Letting $X^* = x_1^* + x_2^*$, the first-order condition for this maximization problem of player j is

$$\frac{j}{\sqrt{X^*}} - 1 \leq 0, \text{ with equality if } x_j^* > 0.$$

If $x_j^* > 0$, then $j/\sqrt{X^*} - 1 = 0$, and hence, $(j+1)/\sqrt{X^*} - 1 > 0$. As this would violate the first-order condition for player $j+1$, there had better not be a player $j+1$. Thus, $x_j^* > 0$ implies $j = 2$. That is, we have proved that any equilibrium satisfies $x_1^* = 0$. The first-order condition for player 2 now implies $2/\sqrt{x_2^*} - 1 = 0$, and so $x_2^* = 4$. ■

(b) Show that for any $a > 1$, there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$, players 1 and 2 will be made better off, relative to the equilibrium you just found, if they are forced to increase their contributions above x^* by ε and $a\varepsilon$, respectively.

Soln: Given any $\varepsilon \geq 0$, the players' utilities from contributing $x^* + (\varepsilon, a\varepsilon) = (\varepsilon, 4 + a\varepsilon)$ are given by

$$\hat{u}_1(\varepsilon) = 2\sqrt{4 + (1+a)\varepsilon} - \varepsilon,$$

$$\hat{u}_2(\varepsilon) = 4\sqrt{4 + (1+a)\varepsilon} - 4 - a\varepsilon.$$

Note that

$$\hat{u}_1'(0) = \tfrac{1}{2}(a-1), \quad \hat{u}_2'(0) = 1.$$

Both these derivatives are positive if $a > 1$. It follows that for any $a > 1$, $\bar{\varepsilon} > 0$ exists such that for all $\varepsilon \in (0, \bar{\varepsilon})$, $\hat{u}_i(\varepsilon) > \hat{u}_i(0) = u_i(x^*)$ for both $i = 1, 2$. ∎