Variance of a Portfolio Return

- For a portfolio p made up of N securities, the rate of return is

$$r_p = \sum_{i=1}^{N} w_i r_i, \tag{1}$$

where w_i is the weight in asset i ($\sum_{i=1}^{N} w_i = 1$).

[Notation: $\sum_{i=1}^{N} x_i = x_1 + x_2 + \cdots + x_{N-1} + x_N$]

- The variance of portfolio p's return is given by

$$\sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}, \tag{2}$$

where $\sigma_{ij} = \text{cov}\{r_i, r_j\}$.

- The $\sum\sum$ notation in (2) can be understood easily in terms of arrays.

Variance and Arrays

- In one array, place cross products of weights:

$$\begin{matrix} w_1^2 & w_1 w_2 & \cdots & w_1 w_N \\ w_2 w_1 & w_2^2 & \cdots & w_2 w_n \\ \vdots & \vdots & \ddots & \vdots \\ w_N w_1 & w_N w_2 & \cdots & w_N^2 \end{matrix}$$

- A second array contains covariances (the “covariance matrix”):

$$\begin{matrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_{NN} \end{matrix}$$

- Now multiply the elements in each array:

$$\begin{matrix} w_1^2 \sigma_{11} & w_1 w_2 \sigma_{12} & \cdots & w_1 w_N \sigma_{1N} \\ w_2 w_1 \sigma_{21} & w_2^2 \sigma_{22} & \cdots & w_2 w_n \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ w_N w_1 \sigma_{N1} & w_N w_2 \sigma_{N2} & \cdots & w_N^2 \sigma_{NN} \end{matrix}$$

- The sum of all N^2 elements is the portfolio’s variance.

Components of Variance

- The N diagonal entries contain the contributions of the variances of the individual securities (note $\sigma_{ii} = \text{cov}\{r_i, r_i\} = \text{var}\{r_i\} = \sigma_i^2$).
- The $N(N-1)$ off-diagonal entries contain the contributions of the pairwise covariances among the individual securities.
- Row i represents security i's contribution to total portfolio variance.
  - security i's own variance (the diagonal entry)
  - security i's covariances with other securities ($N-1$ off-diagonal entries)
- Portfolio variance with these contributions separated:

$$\sigma_p^2 = \underbrace{\sum_{i=1}^{N} w_i^2 \sigma_i^2}_{N \text{ terms}} + \underbrace{\sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} w_i w_j \sigma_{ij}}_{N(N-1) \text{ terms}} \tag{3}$$

- If w_i's have comparable magnitudes across securities $\Rightarrow$ N variance terms become unimportant relative to $N(N-1)$ covariance terms

Limits of Diversification in a Simplified Setting

- If we simplify to the case where $w_i = \frac{1}{N},\ i = 1, \ldots, N$, then the portfolio variance can be rewritten as

$$
\begin{aligned}
\sigma_p^2 &= \sum_{i=1}^{N} \left(\frac{1}{N}\right)^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} \left(\frac{1}{N}\right)^2 \sigma_{ij} \\
&= \frac{1}{N} \left[\frac{1}{N} \sum_{i=1}^{N} \sigma_i^2 \right] + \frac{N-1}{N} \left[\frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} \sigma_{ij} \right] \\
&= \frac{1}{N} \overline{\sigma_i^2} \ + \ \frac{N-1}{N} \overline{\sigma_{ij}} \qquad (4)
\end{aligned}
$$

  where a bar over a quantity denotes a simple arithmetic average.

- Then, as N grows large,

$$\sigma_p^2 \to \overline{\sigma_{ij}} \qquad (5)$$

Simplifying Further

- Assume that the securities are equal in volatility and pairwise covariances:
  - $\sigma_i = \sigma_j$ for all i and j
  - $\sigma_{ij} = \sigma_{jk}$ for all $i \neq j$ and $j \neq k$
- We can then remove the bars in (4), since those averages are now taken over identical items:

$$\sigma_p^2 = \frac{1}{N}\sigma_i^2 + \frac{N-1}{N}\sigma_{ij} \tag{6}$$

- Dividing through by σ_i^2 gives

$$\begin{aligned}\frac{\sigma_p^2}{\sigma_i^2} &= \frac{1}{N} + \frac{N-1}{N}\frac{\sigma_{ij}}{\sigma_i^2} \\ &= \frac{1}{N} + \frac{N-1}{N}\rho_{ij}\end{aligned} \tag{7}$$

- As N grows large,

$$\frac{\sigma_p^2}{\sigma_i^2} \to \rho_{ij} \tag{8}$$

Example

- With $\rho_{ij} = 0.30$,

N	σ_p^2/σ_i^2
1	1.00
2	0.65
5	0.44
15	0.35
30	0.32
∞	0.30

- Diversification effect occurs pretty quickly (at roughly a $1/N$ rate)
- For a given degree of correlation among its securities, a diversified portfolio's variance still reflects the typical variance of its component securities.
- Rearranging (8),

$$\sigma_p^2 \to \rho_{ij}\sigma_i^2$$

Characteristic Lines and Portfolios

- Begin with a characteristic-line regression for each of N assets in a portfolio

$$R_{i,t} = \alpha_i + \beta_i R_{m,t} + \epsilon_{i,t}, \quad i = 1, \ldots, N \tag{9}$$

- Multiply both sides by the portfolio's weight in asset i, w_i
- Sum the resulting equations across the N assets

$$\sum_{i=1}^{N} w_i R_{i,t} = \sum_{i=1}^{N} w_i \left(\alpha_i + \beta_i R_{m,t} + \epsilon_{i,t}\right)$$

- Gives characteristic-line regression for the portfolio:

$$R_{p,t} = \alpha_p + \beta_p R_{m,t} + \epsilon_{p,t}, \tag{10}$$

where

$$\alpha_p = \sum_{i=1}^{N} w_i \alpha_i, \quad \beta_p = \sum_{i=1}^{N} w_i \beta_i, \tag{11}$$

$$\epsilon_{p,t} = \sum_{i=1}^{N} w_i \epsilon_{i,t}. \tag{12}$$

Diversification and Non-Market Risk

- Using the characteristic-line regression to decompose the portfolio's variance:

$$\text{var}(R_{p,t}) = \beta_p^2 \text{var}(R_{m,t}) + \text{var}(\epsilon_{p,t}) \tag{13}$$

- Same diversification analysis as earlier also applies to $\text{var}(\epsilon_{p,t})$.
- For example, if for all N securities in the portfolio,
  - $w_i = 1/N$
  - $\text{var}(\epsilon_{i,t}) = \sigma_\epsilon^2$
  - $\text{corr}\,(\epsilon_{i,t}, \epsilon_{j,t}) = \rho_\epsilon$ for all $i \neq j$

  then as N increases

$$\text{var}(\epsilon_{p,t}) \to \rho_\epsilon \sigma_\epsilon^2 \tag{14}$$

- The "single-index model" is the special case with $\rho_\epsilon = 0$. Then as N increases

$$\text{var}(\epsilon_{p,t}) \to 0 \tag{15}$$

Diversification and Portfolio Correlations

- General idea: diversification lowers non-market risk

$$\Downarrow$$

market exposure, common to portfolios, is then relatively more important

$$\Downarrow$$

correlations between portfolios become higher

- Using the characteristic-line regressions for each portfolio,

$$\begin{aligned} R_{p,t} &= \alpha_p + \beta_p R_{m,t} + \epsilon_{p,t} \\ R_{q,t} &= \alpha_q + \beta_q R_{m,t} + \epsilon_{q,t} \end{aligned}$$

we obtain

$$\text{cov}(R_{p,t}, R_{q,t}) = \beta_p \beta_q \text{var}(R_{m,t}) + \text{cov}(\epsilon_{p,t}, \epsilon_{q,t}) \tag{16}$$

[Uses rule for covariance: $\text{cov}(ax + by, cz) = ac\,\text{cov}(x, z) + bc\,\text{cov}(y, z)$]

Diversification and Portfolio Correlations (cont.)

- Simple example:
  - each portfolio composed of N securities, with none in common
  - $\beta_i = \beta$ for all i
  - var$(\epsilon_{i,t}) = \sigma_\epsilon^2$ for all i
  - corr $(\epsilon_{i,t}, \epsilon_{j,t}) = \rho_\epsilon$ for all $i \neq j$
- It can then be shown
$$\text{cov}(\epsilon_{p,t}, \epsilon_{q,t}) = \sigma_\epsilon^2 \rho_\epsilon \tag{17}$$
which is constant across N. [proof omitted, applies same covariance rule]
- Thus, as N increases, cov$(R_{p,t}, R_{q,t})$ remains constant.
- Since cov$(R_{p,t}, R_{q,t})$ is the numerator in the correlation,
$$\text{corr}\,(R_{p,t}, R_{q,t}) = \frac{\text{cov}(R_{p,t}, R_{q,t})}{\sqrt{\text{var}(R_{p,t})\text{var}(R_{q,t})}}, \tag{18}$$
diversification lowers the variances in the denominator and thus increases the correlation.

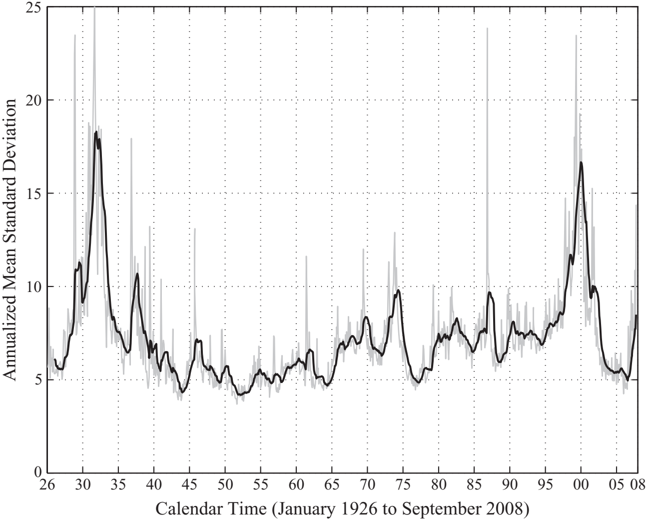

Figure 1
Idiosyncratic volatility from January 1926 to September 2008
This figure shows the annualized mean standard deviation (light line) and the 12-month backward moving average of this measure (dark line) for each month between January 1926 and September 2008. Idiosyncratic volatility is measured using daily returns following the CLMX volatility decomposition methodology.

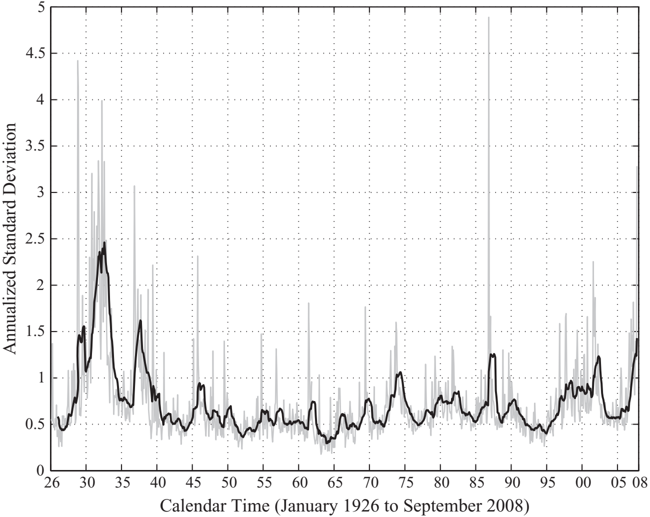

Figure 2
Market volatility from January 1926 to September 2008
This figure shows the annualized standard deviation (light line) of market returns and the 12-month backward moving average of this measure (dark line) for each month between January 1926 and September 2008. Market volatility is measured using daily market returns following the CLMX volatility decomposition methodology.

Estimating Volatility Using Return History

- Empirical regularities
  - volatility fluctuates over time
  - periods of high or low volatility persist for awhile
- Popular (Nobel-winning) approach ("GARCH") to predicting volatility

$$\text{var}(R_{i,t}) = a_0 + a_1 u_{i,t-1}^2 + a_2 \text{var}(R_{i,t-1})$$

  where

$$u_{i,t-1} = R_{i,t-1} - E(R_{i,t-1})$$

- Estimate a_0, a_1, and a_2 using historical data
- Typical values at a daily return frequency:
  - $a_0 \approx 0$
  - $a_1 = 0.08$
  - $a_2 = 0.91$

Estimating Volatility Using Option Prices

- Price of a put option or call option on an asset reflects
  1. asset's current price
  2. "strike" price at which option holder can buy (call) or sell (put) the asset
  3. expiration date of the option
  4. anticipated volatility (σ) of the asset's return until expiration
- Implied volatility: the σ consistent with items 1–3 and the option's price
- Since 1993, the CBOE has applied this approach in constructing the VIX
  - implied (annualized) volatility of S&P 500 index return over next 30 days
  - uses a number of put and call options having about 30 days to expiration
  - based on σ implied by the Black-Scholes pricing formula until 2003
  - now the implied (forward) price of a portfolio that replicates realized volatility
- Historical averages: VIX: 19%; realized volatility: 16%
- Upward bias in VIX — a "volatility premium" earned by option writers

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VIX Index (Chicago Board Options Exchange SPX Volatility Index)
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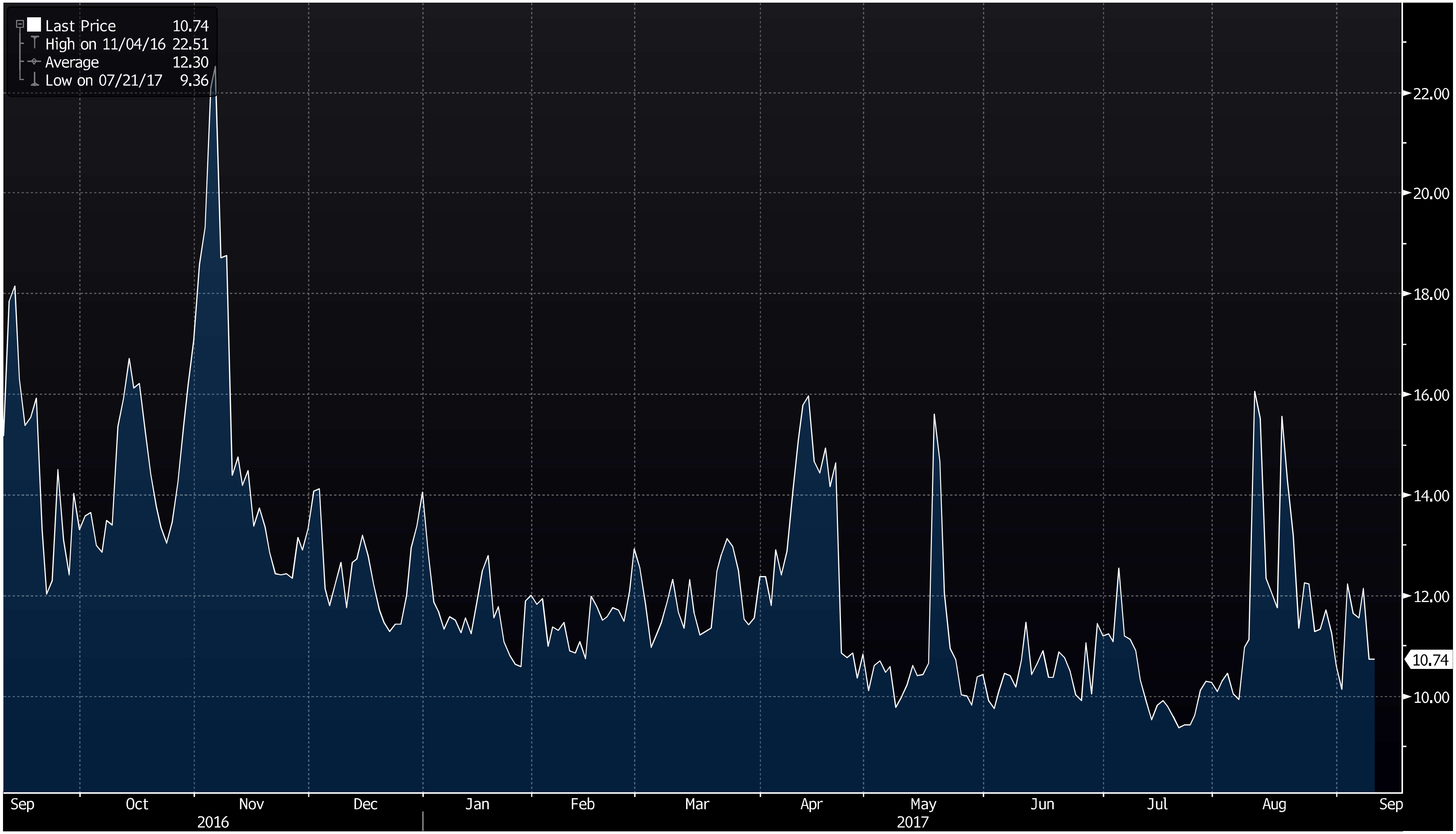

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