Econ 701A
Lecture Slides 2
Consumer Choice and Preferences
MWG 2.A-E, 3.A-C

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Consumer Theory

- Consumer ($= DM$)
- Commodities (Goods) $\ell = 1, \ldots, L$
  - definition may include all physical properties, date, state, location, ...
  - quantities assumed measurable and observable
  - generally interpret as flows, with an implicit time period (quarts of milk per day or per week)
- Consumption Set X
  - contains all commodity bundles that can be conceivably consumed
  - reflects physical constraints (e.g., cannot consume negative amounts of food, discrete goods, survival needs, mutually exclusive goods, cannot work more than 24 hours per day)
  - reflects legal constraints (e.g., cannot work more than 16 hours a day, cannot own some types of gun)
  - Main example in the neoclassical theory: $X = \mathbb{R}^L_+$

Consumption Set Assumptions

- *Nonnegative*: $X \subseteq \mathbb{R}^L_+$
  - Important? No
- *Closed*: X is a closed set
  - Important? Not of economic importance
- *Convexity*: X is a convex set
  - Important? Yes, both technically and economically (discrete goods; bread in NY vs in DC at noon)

Competitive Budget Sets – Assumptions and Definitions

- Competitive Markets
  - all goods can be purchased in any amount at known prices,

$$p = (p_1, \ldots, p_L)$$

  - all prices are positive: $p \gg 0$ (Why this assumption?)
  - the consumer regards prices as fixed (price-taking assumption)

- Competitive (Walrasian) Budget Set
  - Given the price vector p and the consumer's income (wealth) $m \geq 0$, her *budget set* is

$$B(p, m) := \{x \in X : p \cdot x \leq m\}.$$

- **The consumer's choice problem:**
  - *Somehow choose a consumption bundle from* $B(p, m)$

Remarks on Budget Sets

- $B(p, m)$ is a convex polyhedron (intersection of a finite number of half spaces) that is closed and, provided $p \gg 0$, bounded (and hence compact and convex)

- Budget Line (Hyperplane)
  - It is a *level set* of the linear function $f(x) = p \cdot x$.

    Since $p = \nabla f(x)$, the price vector is perpendicular to the budget hyperplane, pointing in the direction in which expenditure increases most rapidly.
  - For $L = 2$, the slope of the budget line is $-\dfrac{p_1}{p_2}$

    This slope is the **real** price of good 1 in terms of good 2

    We sometimes assume $p_2 = 1$ and call good 2 "money" or "expenditure on goods $\ell > 1$"

Demand Correspondences

- The consumer's **(Walrasian) (ordinary) (Marshallian) demand correspondence** is a correspondence

$$x : \mathbb{R}^L_{++} \times \mathbb{R}_+ \rightrightarrows X$$

satisfying, for all (p, m),

$$x(p, m) \subseteq B(p, m).$$

If the set $x(p, m)$ is always a singleton, we call $x(\cdot, \cdot)$ a **demand function** and dispense with set notation,e.g., writing

$$x(p, m) = z$$

instead of

$$x(p, m) = \{z\}.$$

Choice-Based Demand Theory Overview

- The primitive of the choice-based approach to demand theory is just a demand correspondence $x(p, m)$. Chapter 2 of MWG is an introduction to this approach.
- The idea is to see what testable predictions can be made under some reasonable assumptions about the nature of $x(p, m)$.
- Three main assumptions are made: (i) homogeneity, (ii) Walras' law, and (iii) WARP. It turns out that most of the results of the traditional preference-based approach to demand theory can be derived from just these three assumptions.
- However, we will primarily consider homogeneity and Walras' law here. We thus skip most of MWG 2.F, which concerns WARP.

Homogeneity of Degree Zero

Is a demand correspondence a choice correspondence as we defined previously? That is, if we take $\mathfrak{B}$ to be the set of all budget sets $B(p, m)$, is $\langle \mathfrak{B}, x \rangle$ a choice structure?

Not necessarily! A demand correspondence is defined directly on price-income pairs (p, m), not on budget sets $B(p, m)$. We could have $x(p, m) \neq x(p', m')$, even if $B(p, m) = B(p', m')$.

This "money/price illusion" is ruled out by the following assumption.

Assumption. The demand correspondence $x(p, m)$ is **Homogeneous of Degree Zero**, i.e., for any $(p, m) \in \mathbb{R}^L_{++} \times \mathbb{R}_+$,

$$x(\alpha p, \alpha m) = x(p, m) \text{ for all } \alpha > 0.$$

Walras' Law (Adding Up)

The second assumption is that consumers do not waste money: all demanded commodity bundles are on the budget line.

Assumption. The demand correspondence $x(p, m)$ satisfies **Walras' Law**, i.e., for all $(p, m) \in \mathbb{R}^L_{++} \times \mathbb{R}_+$,

$$p \cdot x = m \text{ for each } x \in x(p, m).$$

- At this point, Homongeneity and Walras' Law are just properties our economic intuition tells us are plausible. When we get to the preference-based approach to consumer theory, we shall find fairly weak conditions under which they must hold.

Example

For what values of α and β does the following demand function for $L = 2$ goods satisfy Homogeneity and Walras' Law?

$$x_1(p, m) = \frac{\alpha p_2}{p_1 + p_2} \frac{m}{p_1}, \quad x_2(p, m) = \frac{\beta p_1}{p_1 + p_2} \frac{m}{p_2}$$

Answers

- $x(p, m)$ is homogeneous for any $(\alpha, \beta) \in \mathbb{R}^2$.
- It satisfies Walras' Law iff $\alpha = \beta = 1$.
  - **Proof.** Walras' Law holds at (p, m) iff

$$\begin{aligned} p \cdot x(p, m) = m \;&\Leftrightarrow\; \left[\frac{\alpha p_2 + \beta p_1}{p_1 + p_2}\right] m = m \\ &\Leftrightarrow\; (\beta - 1) p_1 = (1 - \alpha) p_2 . \end{aligned}$$

This can hold for all (p, m) iff $\alpha = \beta = 1$.

Demand Function Comparative Statics

- The comparative statics properties of a demand function $x(p, m)$ consist of how it changes when its arguments change.
- The most common comparative statics questions are about the signs of changes, e.g., is $\Delta x_\ell > 0$ if $\Delta m > 0$ and $\Delta p = 0$?
- We often assume $x(p, m)$ is differentiable. Then the questions are about the signs of the partial derivatives,

$$\frac{\partial x_\ell(p, m)}{\partial p_k} \text{ and } \frac{\partial x_\ell(p, m)}{\partial m}$$

Income Effects

- *Normal* $(\partial x_\ell / \partial m > 0)$ vs *inferior* $(\partial x_\ell / \partial m < 0)$ goods
  - these are local properties
  - we shall see when we consider preference-based demand, that even under standard assumptions goods can be inferior or normal
    - meals at a food truck, meals at the White Dog
- *Engel curves (functions)*
  - luxuries (convex) vs necessities (concave)
- *Income expansion paths* obtained as income increases

Price Effects

- A demand function $x_\ell(p, m)$ satisfies the **Law of Demand** iff $\partial x_\ell / \partial p_\ell < 0$.
- We shall see that preference-based demand, even under standard assumptions, may not satisfy the Law of Demand (in theory)
  - Good ℓ is called a *Giffen good* if its demand curve slopes up ($\partial x_\ell / \partial p_\ell > 0$)
    - potatoes in 1800s Ireland, rice in China
- Good ℓ is a *(gross) substitute* for good k iff $\partial x_\ell / \partial p_k > 0$
  Good ℓ is a *(gross) complement* for good k iff $\partial x_\ell / \partial p_k < 0$
- *Price expansion paths (offer curves)* obtained as one price changes

Comparative Statics Derived from Walras' Law

To see how comparative statics results can be derived rigorously without assuming specific functions, let us do so from Walras' law. Consider a differentiable demand function satisfying Walras' law, so that for all (p, m),

$$p \cdot x(p, m) = m.$$

As this is an **identity** in (p, m), the equality is maintained if we differentiate both sides. Differentiating with respect to p_j yields

$$x_j(p, m) + \sum_{i=1}^{L} p_i \frac{\partial x_i(p, m)}{\partial p_j} = 0.$$

Hence, assuming the consumer consumes good j, there must exist a good i that she consumes less of when the price of good j increases: for some good i, $\frac{\partial x_i}{\partial p_j} < 0$. (Which is obvious, given WL.)

Note that this does not give us the law of demand: we do not know $j = i$, and so we cannot conclude that $\frac{\partial x_i}{\partial p_i} < 0$.

Now differentiate the Walras' law identity, $p \cdot x(p, m) = m$, with respect to income:

$$\sum_{i=1}^{L} p_i \frac{\partial x_i(p, m)}{\partial m} = 1.$$

An obvious conclusion: some good must be normal at (p, m), i.e., for some good i, $\frac{\partial x_i}{\partial m} > 0$. (Which is obvious, given WL.)

Preference-Based Demand Theory

This is the traditional approach to demand theory, covered in MWG Chapter 3. It is based on the assumption that for some $\succsim$, the demand correspondence is given by

$$x(p, m) = C^*(B(p, m), \succsim).$$

This assumption alone immediately implies homogeneity:

Theorem

Preference-based demand correspondences are homogeneous of degree zero in (p, m).

Indifference Curves and Contour Sets

Given a binary relation $\succsim$ on X and a bundle $x \in X$, it is useful to consider three subsets of X :

- The **indifference curve (or set)** containing x :

$$\{y \in X : y \sim x\}.$$

  Examples.

- The **upper** and **lower contour sets** of x :

$$\{y \in X : y \succsim x\} \text{ and } \{y \in X : x \succsim y\}.$$

Standard Assumptions about Consumer Preferences

(LNS) **local nonsatiation**: any open neighborhood of any $y \in X$ contains a bundle $x \in X$ such that $x \succ y$

(M) **monotonicity**: if $x \gg y$, then $x \succ y$

(SM) **strong monotonicity**: if $x \geq y$ and $x \neq y$, then $x \succ y$

Observation: SM $\Rightarrow$ M $\Rightarrow$ NS.

We have another immediate result:

Theorem

If $\succsim$ satisfies LNS, then $x(p, m) = C^*(B(p, m), \succsim)$ satisfies Walras' law.

Standard Assumptions on Consumer Preferences (con't)

(R) **rational** (complete and transitive)

(C) **continuous**, four equivalent (given completeness) definitions:

1. if $x \succ y$, then neighborhoods N_x and N_y exist such that $x' \succ y'$ for all $x' \in N_x$, $y' \in N_y$
2. the *graph* $\{(x, y) \in X^2 : x \succsim y\}$ of $\succsim$ is closed (i.e., $\succsim$ is a closed subset of X^2)
3. for any x, the upper and lower contour sets, $\{y \in X : y \succsim x\}$ and $\{y \in X : x \succsim y\}$, are closed
4. for any x, the strict upper and lower contour sets, $\{y \in X : y \succ x\}$ and $\{y \in X : x \succ y\}$, are open

Utility Representation of Preferences

Definition

A binary relation $\succsim$ on a set X is **represented** *by a function $U : X \to \mathbb{R}$ iff for all $x, y \in X$,*

$$x \succsim y \iff U(x) \geq U(y).$$

U is then said to be a utility function that represents $\succsim$.

- Two "obvious" results:
  1. Any representable $\succsim$ is rational.
  2. If U represents $\succsim$ and $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then $V = f \circ U$ also represents $\succsim$. The only important aspect of these "ordinal" utility functions are the preferences they represent.
- Utility functions are easier to work with than preferences. So, how restrictive is the assumption that preferences are representable?

Examples

- Cobb-Douglas
- Lexicographic
- The relation $\succsim$ on $\mathbb{R}$ represented by

$$u(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0. \end{cases}$$

- The $\succsim$ on $\mathbb{R}$ represented, for $\varepsilon > 0$, by

$$u(x) = \begin{cases} \varepsilon x - 1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ \varepsilon x + 1 & \text{for } x > 0. \end{cases}$$

Debreu's Representation Theorem

Theorem (Debreu)

For any a and $b > a$ in $\mathbb{R}$, a continuous rational preference relation $\succsim$ on a connected set $X \subseteq \mathbb{R}^n$ is representable by a continuous function $u : X \longrightarrow [a, b]$.

Observe the extra bit in the conclusion: not only is $\succsim$ representable, but it is representable by a *continuous (and bounded)* function.

Proving Representation Theorems

- Debreu's theorem proves that a representation exists under quite general conditions. It's proof, however, is mathematically technical and not very economically based.

  (See Rubinstein's text for a proof of most of the theorem, and Kreps' text for even more.)

- A weaker theorem has a more intuitive, economically-based proof:

Theorem (representation of monotone continuous rational preferences)

A monotone continuous rational preference relation $\succsim$ on $\mathbb{R}^L_+$ is representable by a continuous function u.

Proof of the Monotone Representaton Theorem

1. Construct the utility function.
   Fix $x \in \mathbb{R}^L_+$. Define $D := \{t\mathbf{e} : t \geq 0\}$, and two subsets of it:

$$B := D \cap \{y \in \mathbb{R}^L_+ : y \succsim x\}$$

$$W := D \cap \{y \in \mathbb{R}^L_+ : y \precsim x\}$$

- $B \neq \varnothing$ by monotonicity.
  $W \neq \varnothing$, since $0 \in W$ by monotonicity and continuity.
- B and W are both closed, by the continuity of $\succsim$.
- $\underline{t}$ and $\bar{t}$ exist s.t. $B = \{t\mathbf{e} : t \geq \underline{t}\}$ and $W = (t\mathbf{e} : t \leq \bar{t}\}$, by monotonicity.
- $\underline{t} \leq \bar{t}$ by completeness, and so $\underline{t} = \bar{t}$ by monotonicity.
- Define $u(x) := \bar{t}$.

Proof of the Monotone Representation Theorem (con't)

2. Show that u represents $\succsim$.

   - ($\Rightarrow$) Suppose $u(x) \geq u(y)$. Then

$$\begin{aligned} x &\sim (u(x), \ldots, u(x)) \text{ (by def of } u) \\ &\succsim (u(y), \ldots, u(y)) \text{ (by monotonicity)} \\ &\sim y \text{ (by def of } u). \end{aligned}$$

     Hence $x \succsim y$, by the transitivity of $\succsim$.

   - ($\Leftarrow$) Suppose $x \succsim y$. Then

$$(u(x), \ldots, u(x)) \sim x \succsim y \sim (u(y), \ldots, u(y)).$$

     Hence, by the transitivity of $\succsim$,

$$(u(x), \ldots, u(x)) \succsim (u(y), \ldots, u(y)).$$

     This and monotonicity yield $u(x) \geq u(y)$.

Proof of the Monotone Representation Theorem (con't con't)

3. Show that u is continuous.

   - Recall that u is continuous if for any $a \in \mathbb{R}$, the sets $u^{-1}((a,\infty))$ and $u^{-1}((-\infty,a))$ are open in the domain of u, $\mathbb{R}^L_+$.

   - The definition of u and the fact that it represents $\succsim$ imply that these sets are strict contour sets. For example:

$$\begin{aligned} u^{-1}((a,\infty)) &= \{x : u(x) > a\} \\ &= \{x : u(x) > u(a,\ldots,a)\} \\ &= \{x : x \succ (a,\ldots,a)\}. \end{aligned}$$

   Thus, the continuity of $\succsim$ implies $u^{-1}((a,\infty))$ is an open set. A similar argument shows that $u^{-1}((-\infty,a))$ is also open. ■