701A Prelim Questions & Solutions
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1. (June 2014) (25 pts) A strictly increasing utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ gives rise to a demand function $\mathbf{x}(\mathbf{p}, y) = (x_1(\mathbf{p}, y), x_2(\mathbf{p}, y))$. It is continuously differentiable in a neigborhood N of some $(\mathbf{p}^0, y^0) \gg \mathbf{0}$. Theory tells us much about the nature of such demand functions: use what it tells us to answer the following questions.

   (a) (3 pts) Write the definition of $\eta_i(\mathbf{p}, y)$, the income elasticity for good i.
   Soln: Dropping the understood arguments from η_i and x_i, we have

   $$\eta_i := \frac{y}{x_i}\frac{\partial x_i}{\partial y}.$$

   ■

   (b) (11 pts) Suppose the demand for good 1 takes the form

   $$x_1(\mathbf{p}, y) = \alpha_1(\mathbf{p})g_1(y)$$

   for all $(\mathbf{p}, y) \in N$. What is the most that this implies about $\eta_1(\mathbf{p}, y)$ on N?
   Soln: The given functional form immediately implies that η_1 does not depend on prices:

   $$\eta_1(\mathbf{p}, y) = \frac{y}{\alpha_1(\mathbf{p})g_1(y)}\left[\alpha_1(\mathbf{p})g_1'(y)\right] = \frac{yg_1'(y)}{g_1(y)}.$$

   But we can say more. Since $x_1(\mathbf{p}, y)$ is homogeneous of degree zero, Euler's law yields

   $$p_1\frac{\partial x_1}{\partial p_1} + p_2\frac{\partial x_1}{\partial p_2} + y\frac{\partial x_1}{\partial y} = 0.$$

   Divide this by x_1 and rearrange to obtain

   $$\eta_1(\mathbf{p}, y) = \frac{y}{x_1}\frac{\partial x_1}{\partial y} = -\left(\frac{p_1}{x_1}\frac{\partial x_1}{\partial p_1} + \frac{p_2}{x_1}\frac{\partial x_1}{\partial p_2}\right).$$

   Substitute into this from the given form of $x_1(\mathbf{p}, y)$ to obtain

   $$\eta_1(\mathbf{p}, y) = -\left(\frac{p_1}{\alpha_1(\mathbf{p})}\frac{\partial \alpha_1(\mathbf{p})}{\partial p_1} + \frac{p_2}{\alpha_1(\mathbf{p})}\frac{\partial \alpha_1(\mathbf{p})}{\partial p_2}\right).$$

   Hence, η_1 also does not depend on income. We conclude that a constant $e_1 \in \mathbb{R}$ exists such that for all $(\mathbf{p}, y) \in N$,

   $$\eta_1(\mathbf{p}, y) = e_1.$$

   ■

   (c) (11 pts) Now suppose in addition that in N, demand for good 2 takes the same form,

   $$x_2(\mathbf{p}, y) = \alpha_2(\mathbf{p})g_2(y),$$

and $\mathbf{x}(\mathbf{p}, y)$ satisfies the law of reciprocity:

$$\frac{\partial x_1}{\partial p_2} = \frac{\partial x_2}{\partial p_1}.$$

What can you now say about the two income elasticity functions on N?

Soln: By the same logic as in (b), we know that $\eta_2(\mathbf{p}, y)$ is also a constant. But we can now show this, and more, in another way.

Intermediate Result. $\eta_2(\mathbf{p}, y) = \eta_1(\mathbf{p}, y)$ for $(\mathbf{p}, y) \in N$.

Proof. There are two ways to show this. The first is to use $\frac{\partial x_1}{\partial p_2} = \frac{\partial x_2}{\partial p_1}$, the Slutsky equations,

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - x_j \frac{\partial x_i}{\partial y} \text{ for } i \neq j,$$

and the fact that Hicksian demand always satisfies the law of reciprocity, $\frac{\partial h_2}{\partial p_1} = \frac{\partial h_1}{\partial p_2}$, to obtain

$$x_1 \frac{\partial x_2}{\partial y} = x_2 \frac{\partial x_1}{\partial y}.$$

Dividing this by $x_1 x_2$ and multiplying by y yields

$$\eta_2(\mathbf{p}, y) = \frac{y}{x_2} \frac{\partial x_2}{\partial y} = \frac{y}{x_1} \frac{\partial x_1}{\partial y} = \eta_1(\mathbf{p}, y).$$

The second way to prove $\eta_2 = \eta_1$ is to use the given functional forms for x_1 and x_2. Using those forms, the equality $\frac{\partial x_1}{\partial p_2} = \frac{\partial x_2}{\partial p_1}$ becomes

$$\frac{\partial \alpha_1}{\partial p_2} g_1 = \frac{\partial \alpha_2}{\partial p_1} g_2.$$

Differentiating this equality with respect to y yields

$$\frac{\partial \alpha_1}{\partial p_2} g_1' = \frac{\partial \alpha_2}{\partial p_1} g_2'.$$

Dividing this equality by the preceding one yields $g_1'/g_1 = g_2'/g_2$, and hence

$$\eta_1(\mathbf{p}, y) = \frac{y g_1'}{g_1} = \frac{y g_2'}{g_2} = \eta_2(\mathbf{p}, y).$$

Final Result. $\eta_2(\mathbf{p}, y) = \eta_1(\mathbf{p}, y) = 1$ for $(\mathbf{p}, y) \in N$.

Proof. We know the demand function satisfies Walras' Law: $p_1 x_1 + p_2 x_2 = y$. Differentiate this with respect to y to obtain

$$\begin{aligned} p_1 \frac{\partial x_1}{\partial y} + p_2 \frac{\partial x_2}{\partial y} = 1 \;\Rightarrow\; & \frac{p_1 x_1}{y} \frac{y}{x_1} \frac{\partial x_1}{\partial y} + \frac{p_2 x_2}{y} \frac{y}{x_2} \frac{\partial x_2}{\partial y} = 1 \\ \Rightarrow\; & s_1 \eta_1 + s_2 \eta_2 = 1, \end{aligned}$$

where $s_i = p_i x_i / y$ is the share of income spent on good i. This last expression, since $\eta_1 = \eta_2$ (by the Intermediate Result) and $s_1 + s_2 = 1$ (by Walras' Law), implies $\eta_1 = \eta_2 = 1$ in the neighborhood N. ■

2. (June 2014) (25 pts) A competitive firm uses hops to make beer via a production function $f : \mathbb{R} \to \mathbb{R}$. The price of beer is $p > 0$ and the cost of new hops is $w > 0$. The firm has $x_0 > 0$ hops left over from last year, but an unknown amount of these old hops will either grow or spoil before production begins this year; the amount of them that will be usable is $x_0 + \theta\tilde{\varepsilon}$, where $\tilde{\varepsilon}$ is a nondegenerate random variable with mean zero, $\theta > 0$ is a parameter to allow easy comparative statics, and $\theta\tilde{\varepsilon} \in (-x_0, x_0)$. The firm will purchase more hops, $x \geq 0$, to maximize its expected profit,

$$\mathbb{E}\left\{pf(x_0 + \theta\tilde{\varepsilon} + x) - wx\right\}.$$

Assume f is smooth with derivatives $f' > 0$ and $f'' < 0$, and that the solution, $x^*(x_0, w, p, \theta)$, is positive. Make, if necessary, additional reasonable assumptions under which the signs of the partial derivatives,

$$x^*_{x_0},\ x^*_w,\ x^*_p,\ x^*_\theta,$$

can be determined, and find their signs under those assumptions.

Soln: The FOC satisfied by $x^*(x_0, w, p, \theta)$ is

$$p\mathbb{E}f'(x_0 + \theta\tilde{\varepsilon} + x^*(x_0, w, p, \theta)) - w = 0.$$

Differentiating it "totally" with respect to each of the four parameters gives us our comparative statics results. Letting $\tilde{z} = x_0 + \theta\tilde{\varepsilon} + x^*$, we obtain the following:

(a) For x_0 :

$$p\mathbb{E}f''(\tilde{z})(1 + x^*_{x_0}) = 0 \quad \Rightarrow \quad x^*_{x_0} = -1 < 0.$$

(b) For w :

$$p\mathbb{E}f''(\tilde{z})x^*_w - 1 = 0 \quad \Rightarrow \quad x^*_w = \frac{1}{p\mathbb{E}f''(\tilde{z})} < 0.$$

(c) For p :

$$\mathbb{E}f'(\tilde{z}) + p\mathbb{E}f''(\tilde{z})x^*_p = 0 \quad \Rightarrow \quad x^*_p = \frac{-\mathbb{E}f'(\tilde{z})}{p\mathbb{E}f''(\tilde{z})} > 0.$$

(d) For θ :

$$p\mathbb{E}f''(\tilde{z})\left(\tilde{\varepsilon} + x^*_\theta\right) = 0 \quad \Rightarrow \quad x^*_\theta = \frac{\mathbb{E}f''(\tilde{z})\tilde{\varepsilon}}{-\mathbb{E}f''(\tilde{z})}.$$

So the sign of x^*_θ is the sign of $\mathbb{E}f''(\tilde{z})\tilde{\varepsilon}$. To sign it, we use the logic of the precautionary savings problem (Q4 of PS6 in 701A, 2013). Since $\mathbb{E}\tilde{\varepsilon} = 0$, we have

$$\begin{aligned}\mathbb{E}f''(\tilde{z})\tilde{\varepsilon} &= \mathbb{E}f''(x_0 + \theta\tilde{\varepsilon} + x^*)\tilde{\varepsilon} \\ &= \mathbb{E}\left[f''(x_0 + \theta\tilde{\varepsilon} + x^*) - f''(x_0 + x^*)\right]\tilde{\varepsilon}.\end{aligned}$$

As $\theta > 0$, this expression is positive (negative) if f'' is an increasing (decreasing) function. We conclude that we can sign x^*_θ under either assumption:

$$x^*_\theta \begin{cases} > 0 & \text{if } f''' > 0 \\ < 0 & \text{if } f''' < 0 \end{cases}.$$

■

3. (August 2014) (25 pts) There are two possible states of the world and one good, "money". It is commonly known that state s will occur with probability $\pi_s > 0$, for $s = 1, 2$. A state contingent allocation is a pair $(x_1, x_2) \in \mathbb{R}^2_+$. Consider a consumer who has a complete, transitive, and strongly monotonic ordering $\succsim$ over these allocations. Assume $\succsim$ is convex.

   (a) (10 pt) Prove or disprove: This consumer must be weakly risk averse.

   Soln: No.

   Proof. The following is a counterexample: Consider a function $U(x_1, x_2) := ax_1 + (1 - a)x_2$, with $a \in (0, 1)$ satisfying $a \neq \pi_1$. The $\succsim$ represented by U is clearly complete, transitive, strongly monotonic, and convex (the indifference curves are straight lines). The slope of the indifference lines is not equal to that of the iso-expected-value lines:

$$-\frac{a}{1-a} \neq -\frac{\pi_1}{\pi_2}.$$

   Hence, for any $(x_1, x_2) \in \mathbb{R}^2_{++}$, half of the iso-expected-value line containing $(\bar{x}, \bar{x})$, where $\bar{x}$ is the expected value $\pi_1 x_1 + \pi_2 x_2$, lies above the indifference line that contains $(\bar{x}, \bar{x})$. I.e., there are lots of risky allocations with expected value $\bar{x}$ that the consumer strictly prefers to the riskless allocation $(\bar{x}, \bar{x})$. So he is not weakly risk averse. ∎

   (b) (15 pts) Do the same as in (a), but under the assumption now that the consumer satisfies the expected utility hypothesis. Let u denote the consumer's Bernoulli utility function, and assume it is twice continuously differentiable, with $u' > 0$.

   Soln: Yes.

   Proof 1. We have that $\succsim$ is represented by

$$U(x_1, x_2) = \pi_1 u(x_1) + \pi_2 u(x_2).$$

   We show that u is concave, and hence the consumer is risk averse with respect to all monetary lotteries.

   Let $x > 0$. The indifference curve of U that contains (x, x) is the graph of the function $x_2(x_1)$ defined by the equation

$$\pi_1 u(x_1) + \pi_2 u(x_2(x_1)) = u(x).$$

   Differentiate this with respect to x_1 twice to obtain

$$\pi_1 u'(x_1) + \pi_2 u'(x_2(x_1))x_2'(x_1) = 0,$$

$$\pi_1 u''(x_1) + \pi_2 u''(x_2(x_1))x_2'(x_1)^2 + \pi_2 u'(x_2(x_1))x_2''(x_1) = 0.$$

   Replacing x_1 by x and noting that $x_2(x) = x$, these equalities become

$$u'(x)\left[\pi_1 + \pi_2 x_2'(x)\right] = 0, \tag{1}$$

$$u''(x)\left[\pi_1 + \pi_2 x_2'(x)^2\right] + \pi_2 u'(x)x_2''(x) = 0. \tag{2}$$

   From (1) we see that the MRS at (x, x) is $x_2'(x) = -\pi_1/\pi_2$. Use this in (2) to obtain

$$u''(x)\left[\frac{\pi_1}{\pi_2}\right] + \pi_2 u'(x)x_2''(x) = 0. \tag{3}$$

   As the indifference curves are convex, we have $x_2''(x) \geq 0$ and hence $\pi_2 u'(x)x_2''(x) \geq 0$. This and (3) imply $u''(x) \leq 0$. This is true of all $x > 0$, and so u is concave. ∎

Proof 2. We have that $\succsim$ is represented by

$$U(x_1, x_2) = \pi_1 u(x_1) + \pi_2 u(x_2).$$

We show u is concave. Assume not. Then $x > 0$ exists such that $u''(x) > 0$. So u is strictly convex in a neighborhood of x, implying that $\varepsilon^a < 0 < \varepsilon^b$ exist such that for $i = a, b$,

$$\pi_1 u\left(x - \frac{\varepsilon^i}{\pi_1}\right) + \pi_2 u\left(x + \frac{\varepsilon^i}{\pi_2}\right) > u(x).$$

Let $(x_1^i, x_2^i) = \left(x - \frac{1}{\pi_1}\varepsilon^i, x + \frac{1}{\pi_2}\varepsilon^i\right)$ for $i = a, b$. Then (x_1^a, x_2^a) and (x_1^b, x_2^b) are on the same iso-expected-value line as is (x, x), with (x, x) between them. Also, $U(x_1^a, x_2^a) > U(x, x)$ and $U(x_1^b, x_2^b) > U(x, x)$. This contradicts the convexity of $\succsim$. ■

4. (August 2014) (25 pts) Consider a society $N = \{1, \ldots, n\}$ and a finite set X of alternatives. Assume $n \geq 2$ and $\#X \geq 3$. Let $\mathfrak{R}$ be the set of complete and transitive binary relations on X. One alternative, $s \in X$, is the *status quo.* For each profile $\vec{R} \in \mathfrak{R}^n$, let G ("good") be the set of alternatives that are weakly Pareto preferred to s :

$$G = \{x \in X : xR_i s \ \forall i \in N\}.$$

(Note that $s \in G$.) Let B ("bad") be the complementary set, $B = X \backslash G$. For each $\vec{R} \in \mathfrak{R}^n$ define a binary relation $F(\vec{R})$ on X by

$$\begin{aligned} \forall x \in G, y \in B: &\quad xF(\vec{R})y \text{ and not } yF(\vec{R})x \\ \forall x, y \in G: &\quad xF(\vec{R})y \ \Leftrightarrow \ xR_n y \\ \forall x, y \in B: &\quad xF(\vec{R})y \ \Leftrightarrow \ xR_n y \end{aligned}$$

Answer the following questions, and prove your answers:

(a) (6 pts) Is F dictatorial?

(b) (6 pts) Does F satisfy Unanimity?

(c) (6 pts) Does F satisfy Independence of Irrelevant Alternatives?

(d) (7 pts) Is F an (Arrow) Social Welfare Function?

Soln: Given $\vec{R}$, let $R = F(\vec{R})$, and P and I be the strict and indifference relations derived from R.

(a) **No.** Suppose $d \in N$ is a dictator. Let $\vec{R}$ and $x \neq s$ be such that $xP_d s$ and $sP_i x$ for some $i \neq d$. Then $x \in B$. Since $s \in G$, we have sPx, contradiction.

(b) **Yes.** Suppose $\vec{R}$, x, and y satisfy $xP_i y$ for all i. Then, since $xP_n y$, the only way xPy could not be true is if $x \in B$ and $y \in G$. But if $x \in B$, then $sP_j x$ for some j, and so by the transitivity of R_j we have $sP_j y$, which implies $y \in B$. So xPy.

(c) **No.** Let x and y be distinct alternatives, neither one equal to s. Let $\vec{R}$ and $\vec{R}'$ be preference profiles for which these three alternatives are at the top of each preference, and satisfy

R_1	$R_2 \cdots R_n$
y	x
x	y
s	s

R'_1	$R'_2 \cdots R'_n$
y	x
s	y
x	s

Given $\vec{R}$ we have $x, y \in G$, and so xPy since $xP_n y$. However, given $\vec{R}'$ we have $y \in G$ and $x \in B$, and so yPx. This violates IIA, since for all i, R_i and R'_i are the same on $\{x, y\}$.

(d) **Yes.** Let $\vec{R} \in \mathfrak{R}^n$ and $R = F(\vec{R})$. We must show R is complete and transitive. It is obviously complete. To show it is transitive, suppose xRy and yRz. If $x \in B$, then xRy implies $y \in B$, and so yRz implies $z \in B$, and thus xRz because R_n is transitive and equal to R on $\{x, y, z\}$. Similarly, xRz if $z \in G$. The only other case is $x \in G$ and $z \in B$, and then xPz by the definition of F. Thus, R is transitive.

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5. (June 2015) (35 pts) Axel is a newsboy. He can choose whether or not to buy a fixed amount of newpapers to resell. If he buys none, his profit will be 0. If he buys the fixed amount, his profit will depend on how many consumers come to his newsstand. This amount is a random variable D given by

$$D = \begin{cases} 0 & \text{with prob } p \\ 50 & \text{with prob } 1-p \end{cases},$$

where $p \in (0,1)$. If Axel buys the newspapers, his profit will be

$$\pi = \begin{cases} -15 & \text{if } D = 0 \\ 35 & \text{if } D = 50 \end{cases}.$$

Axel's Bernoulli utility function for money is u, which is C^2, strictly increasing, and concave.

Barb also owns a newsstand. She faces the exact same supply and demand environment as Axel. The only difference is that she has a different utility function, v, which is also C^2, strictly increasing, and concave. Lastly, *Barb is strictly more risk averse than Axel.*

(a) (5 pts) Show that $p_A \in (0,1)$ exists such that Axel's optimal decision is to buy the newspapers if and only if $p < p_A$ (he is indifferent in the knife-edge case $p = p_A$).

Soln: Axel will purchase the newspapers iff

$$pu(-15) + (1-p)u(35) > u(0),$$

which rearranges to

$$p < \frac{u(35) - u(0)}{u(35) - u(-15)} =: p_A.$$

■

(b) (10 pts) Letting p_B be the corresponding critical probability for Barb, prove which is larger, p_A or p_B.

Soln: $p_A > p_B$: the more risk averse agent requires a lower probability of a loss in order to be willing to take the risk.

Proof. (This proof, and that in (c) below, do not require Axel and Barb to satisfy the EUH, though it does require that p_A and p_B exist.) By definition, Barb is strictly more risk averse than Axel iff whenever she weakly prefers a nondegenerate lottery F to a degenerate lottery δ_x, Axel strictly prefers F to δ_x. The nondegenerate lotteries in this problem can be denoted as triples, (p, x, y), where $p = \Pr(x)$ and $\Pr(y) = 1 - p$. By its definition, p_B satisfies

$$(p_B, -15, 35) \sim_B \delta_0.$$

Hence,

$$(p_B, -15, 35) \succ_A \delta_0.$$

Since $\delta_0 \sim_A (p_A, -15, 35)$, transitivity implies

$$(p_B, -15, 35) \succ_A (p_A, -15, 35).$$

As Axel's preference for the lotteries $(p, -15, 35)$ decrease in p, this proves $p_B < p_A$. ■

Now suppose Axel, before deciding whether to buy the newspapers, is able to purchase perfect information about what his demand will be, i.e., Axel can learn whether $D = 0$ or $D = 50$. Obviously, if he acquires this information he will buy the newspapers if and only if he learns $D = 50$. Let I_A denote the maximum amount he is willing to pay for this information (I_A is the "value of information" to Axel). Let I_B be the corresponding amount for Barb.

(c) (10 pts) Assuming $p > \max\{p_A, p_B\}$, prove which is larger, I_A or I_B.

Soln: $I_A > I_B$: in this case, perhaps surprisingly, the more risk averse agent values information less.

Proof. In this case neither Axel nor Barb will buy the newspapers unless they acquire the information. Their values for information, I_A and I_B, are therefore determined by

$$(p, -I_A, 35 - I_A) \sim_A \delta_0, \tag{4}$$

$$(p, -I_B, 35 - I_B) \sim_B \delta_0. \tag{5}$$

As Barb is strictly more risk averse than Axel, (5) implies

$$(p, -I_B, 35 - I_B) \succ_A \delta_0.$$

In turn, this and (4) yield, by transitivity, Hence, as h is increasing,

$$(p, -I_B, 35 - I_B) \succ_A (p, -I_A, 35 - I_A).$$

As Axel's preferences for the lotteries $(p, -I, 35 - I)$ decrease in I, this proves $I_B < I_A$.
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(d) (10 pts) (Not used) Assuming $p < \min\{p_A, p_B\}$, prove which is larger, I_A or I_B.

Soln: $I_A < I_B$: this case confirms our casual intuition that a more risk averse agent should value information more.

Proof. In this case both Axel and Barb will buy the newspapers if they do not acquire the information. Hence, I_A and I_B are determined by

$$pu(-I_A) + (1-p)u(35 - I_A) = pu(-15) + (1-p)u(35), \tag{6}$$

$$pv(-I_B) + (1-p)v(35 - I_B) = pv(-15) + (1-p)v(35). \tag{7}$$

Now, note that the sign of $I_A - I_B$ is the same as the sign of Δ, where

$$\begin{aligned}\Delta &:= [pv(-I_B) + (1-p)v(35 - I_B)] - [pv(-I_A) + (1-p)v(35 - I_A)] \\ &= [pv(-15) + (1-p)v(35)] - [pv(-I_A) + (1-p)v(35 - I_A)],\end{aligned}$$

using (7). By replacing v by $h \circ u$ and rearranging, we obtain

$$\Delta = p\,[h(u(-15)) - h(u(-I_A))] + (1-p)\,[h(u(35)) - h(u(35 - I_A))].$$

Since h is strictly concave and differentiable, we have

$$h(u(-15)) - h(u(-I_A)) < h'(u(-I_A))\,[u(-15)) - u(-I_A)].$$

Similarly,

$$\begin{aligned}h(u(35)) - h(u(35 - I_A)) &< h'(u(35 - I_A))\,[u(35)) - u(35 - I_A)] \\ &< h'(u(-I_A))\,[u(35)) - u(35 - I_A)],\end{aligned}$$

where the second inequality follows from the concavity of h and $I_A > 0$. Putting the last three displayed expressions together yields

$$\begin{aligned}\Delta &< ph'(u(-I_A))\,[u(-15) - u(-I_A)] + (1-p)h'(u(-I_A))\,[u(35) - u(35 - I_A)] \\ &= h'(u(-I_A))\,\{[pu(-15) + (1-p)u(35)] - [pu(-I_A) + (1-p)u(35 - I_A)]\} \\ &= h'(u(-I_A))\,\{0\} \\ &= 0,\end{aligned}$$

using (6). This proves $I_A < I_B$, since Δ and $I_A - I_B$ have the same sign. ■

6. (June 2015) (25 pts) The Superior Coffee Shop (Starbucks?) sells a card that entitles its owner to a 10% discount on (tall) cups of coffee for a year. Denote such cups of coffee for Ms. Consumer as good 1, i.e., let x_1 denote the number of such coffees she will consume in a year. All other goods are represented as good 2. Ms. Consumer has a strictly increasing utility function for $x = (x_1, x_2)$ that gives rise to the Hicksian demand functions

$$h_i(p, u) = \frac{p_j}{p_i} u \text{ for } i, j = 1, 2 \text{ and } j \neq i.$$

Let B ("Buy price") be the maximum price Ms. Consumer would pay for this discount card. Let S ("Sell price") be the minimum price for which she would be willing to sell the card if she were to already own it. Let $p^0 = (p_1^0, p_2^0)$ denote the prices without the discount card, and $(p_1^1, p_2^1) = (.9p_1^0, p_2^0)$ the prices with the card.

(a) (15 pts) Find the ratio B/S in terms of $u^0 = v(p^0, m)$ and $u^1 = v(p^1, m)$. Which is larger, B or S?

Soln: The Buy price, MWG's "compensating variation," is equal to the area between the prices, p_1^1 and p_1^0 to the left of the Hicksian demand curve $h_1(p_1, p_2^0, u^0)$:

$$B = \int_{p_1^1}^{p_1^0} \left(p_2^0/p_1\right)^{1/2} u^0 dp_1 = u^0 I,$$

where $I = \int_{p_1^1}^{p_1^0} \left(p_2^0/p_1\right)^{1/2} dp_1$. The Sell price, MWG's "equivalent variation," is equal to the area between the same prices to the left of the Hicksian demand curve $h_1(p_1, p_2^0, u^1)$, and so $S = u^1 I$. Hence,

$$\frac{B}{S} = \frac{u^0}{u^1}.$$

Since $p^0 > p^1$, and the utility function is strictly increasing, $u^0 < u^1$. Hence, $B < S$. ■

(b) (10 pts) Professor Behavior proclaims that for any consumer, owning a good makes the consumer attached to it, so that he/she will not sell it except for a higher price than he/she would have been willing to pay for it before, i.e., that $B < S$. (This is the so-called "endowment effect".) Propose an experiment that can provide evidence to distinguish Professor Behavior's hypothesis from the prediction of neoclassical consumer theory (as learned in Econ 701). Explain your reasoning.

Soln: The endowment effect hypothesis predicts $B < S$. So an experiment that can distinguish the two hypotheses must specify an environment in which the neoclassical theory predicts $B > S$. This requires a negative income effect (unlike in (a)), i.e., good 1 must be an inferior good. If good 1 is inferior, then

$$\frac{\partial}{\partial u} h_1(p, u) = \frac{\partial}{\partial u} x_1(p, e(p, u)) = \frac{\partial x_1}{\partial m} \frac{\partial e}{\partial u} < 0$$

because $\frac{\partial e}{\partial u} > 0$ and $\frac{\partial x_1}{\partial m} < 0$. Hence, since $u^1 > u^0$, in this case

$$B = \int_{p_1^1}^{p_1^0} h_1(p_1, p_2^0, u^0) dp_1 > \int_{p_1^1}^{p_1^0} h_1(p_1, p_2^0, u^1) dp_1 = S.$$

Thus, an experiment that might distinguish the two theories is to sell the discount card for an inferior good, say for a cheap bad coffee (Maxwell House?) instead of a normal good like Starbucks (?) coffee. If the consumer states that $B < S$ when the good is inferior, then classical theory can be rejected in favor of the endowment effect hypothesis. (If one is worried that the consumer would not simple "state" his true B and S, one could observe the price at which the consumer drops out of an ascending-price auction used to sell the card to obtain B, and when he owns the card already, observe the price at which he drops out of a descending-price auction used to buy the card to obtain S.) ■

7. (August 2015) (25 pts) Each day a worker consumes leisure, ℓ, and income, x, and has a strictly increasing utility function $u(\ell, x)$. Leisure is measured in hours. She works for the remaining $L = 24 - \ell$ hours, at wage w. The income she consumes must thus satisfy $x \leq wL$. She has differentiable demand functions.

   (a) (15 pts) Prove that if leisure is an inferior good for this consumer, then her labor supply curve must be upward sloping, i.e., $\widehat{L}'(w) \geq 0$.

   Soln: Letting the price of "income" be one, the budget constraint is $x \leq wL = 24w - w\ell$, or rather, $x + w\ell \leq 24w$. The consumer problem is

$$\max_{\ell, x} u(\ell, x) \text{ subject to } w\ell + x \leq 24w.$$

   Letting $m := 24w$ (the consumer's wealth), this problem is solved by the consumer's demands, $\ell^*(w, m)$ and $x^*(w, m)$. The Slutsky equation for $\partial \ell^* / \partial w$ is

$$\frac{\partial \ell^*}{\partial w} = \frac{\partial \ell^h}{\partial w} - \ell^* \frac{\partial \ell^*}{\partial m},$$

   where $\ell^h = \ell^h(w, v(w, m))$ is the Hicksian demand for leisure. Substituting $24w$ for m gives us her demand for leasure as a function of w: $\hat{\ell}(w) := l^*(w, 24w)$. Differentiating this equality with respect to w yields

$$\hat{\ell}'(w) = \frac{\partial \ell^*}{\partial w} + 24 \frac{\partial \ell^*}{\partial m}.$$

   Substituting into this the Slutsky expression for $\frac{\partial \ell^*}{\partial w}$ yields

$$\hat{\ell}'(w) = \frac{\partial \ell^h}{\partial w} + (24 - \ell^*) \frac{\partial \ell^*}{\partial m}.$$

   The first term on the right side of this expression is nonpositive (Hicksian demands satisfy the Law of Demand). The second term is also nonpositive: **by definition, the statement that leisure is an inferior good means** $\partial \ell^* / \partial m \leq 0$. Hence, the substitution and income effects work in the same direction, and $\hat{\ell}'(w) \leq 0$. Since the consumer's labor supply function is $\widehat{L}(w) := 24 - \hat{\ell}(w)$, we have $\widehat{L}'(w) = -\hat{\ell}'(w) \geq 0$. ■

   (b) (10 pts) Prove that if leisure is a normal good for this consumer, then her labor supply curve might slope down. In particular, show that $\widehat{L}'(w) < 0$ is possible even if her utility function is increasing and quasiconcave.

   Soln: This proof requires the construction of an example. Based on the logic of the proof of (a), we should look for a utility function for which the substitution effect is small. So let's try a Leontief utility function: $u(\ell, x) = \min(\ell, x)$. In this case utility maximization requires $x = \ell$, and the consumer problem reduces to

$$\max_{\ell} \ell \text{ subject to } (1 + w)\ell \leq 24w.$$

   The solution is $\hat{\ell}(w) = 24w/(1 + w)$, and so $\widehat{L}(w) = 24/(1 + w)$. As desired, the labor supply curve slopes down: $\widehat{L}'(w) < 0$. ■

8. (August 2015) (25 pts) In this problem you first prove a result comparing the riskiness of gambles, and then a result comparing the value of information to different decision makers.

   (a) (10 pts) Let $\hat{x} < x < y < \hat{y}$ and $p \in (0,1)$. Let g be the gamble yielding x with probability p and y with probability $1-p$. Similarly, let $\hat{g}$ be the gamble yielding $\hat{x}$ with probability p and $\hat{y}$ with probability $1-p$. Assume g and $\hat{g}$ have the same expected value. Rigorously prove the following.

   Lemma. Any risk averse expected utility maximizer strictly prefers g to $\hat{g}$. (Assume the EU maximizer has a strictly increasing, strictly concave C^1 utility function, $v : \mathbb{R} \to \mathbb{R}$.)

   Soln: Proof. We must prove $\Delta < 0$, where

$$\begin{aligned}\Delta &:= [pv(\hat{x}) + (1-p)v(\hat{y})] - [pv(x) + (1-p)v(y)] \\ &= p\,[v(\hat{x}) - v(x)] + (1-p)\,[v(\hat{y}) - v(y)]\,. \end{aligned} \tag{8}$$

   Because v is a strictly concave C^1 function, we have

$$v(\hat{x}) - v(x) < v'(x)(\hat{x} - x) \tag{9}$$

   and

$$v(\hat{y}) - v(y) < v'(y)(\hat{y} - y).$$

   The second inequality, since (i) $\hat{y} - y > 0$ and (ii) $v'(y) < v'(x)$ because $y > x$, implies

$$v(\hat{y}) - v(y) < v'(x)(\hat{y} - y). \tag{10}$$

   >From (8)-(10) we obtain the desired result:

$$\begin{aligned}\Delta &< v'(x)\,\{p(\hat{x} - x) + (1-p)(\hat{y} - y)\} \\ &= v'(x)\,\{[p\hat{x} + (1-p)\hat{y}] - [px + (1-p)y]\} \\ &= v'(x) \cdot 0 \\ &= 0.\end{aligned}$$

   ■

   (b) (15 pts) Axel is a newsboy who must choose whether to open his newsstand today. His profit will be 0 if he keeps it closed. If he opens it, his profit will be the random variable

$$\pi = \begin{cases} -15 & \text{with probability } p \\ 35 & \text{with probability } 1-p \end{cases},$$

   where $p \in (0,1)$. Axel is risk neutral, and so maximizes expected profit.

   Before Axel decides whether to open his newsstand, he is able to purchase perfect information about whether his profit from opening will be $\pi = -15$ or $\pi = 35$. Let I_A be the maximum amount Axel is willing to pay for this information.

   Barb also owns a newsstand. She faces the same environment as Axel. Barb, however, is risk averse: she has a C^1 Bernoulli utility function for money, v, which is strictly increasing and strictly concave. She too can choose to purchase the perfect information about π. Let I_B be the maximum amount Barb is willing to pay for this information.

   Recall that $p_B \in (0,1)$ exists such that, if she does not acquire the information, Barb's optimal decision is to open her newsstand iff $p < p_B$. Assuming $p < p_B$, show that $I_B > I_A$.

(**Hint:** consider using the Lemma in (a), even if you were unable to prove it.)

Soln: Since $p < p_B$, Barb will open her newsstand if she does not acquire the information. Hence, I_B is given by

$$pv(-I_B) + (1-p)v(35 - I_B) = pv(-15) + (1-p)v(35). \tag{11}$$

Letting p_A be the corresponding critical probability for Axel, we know $p_A > p_B$. (This is easy to show; see the solution to Q1(b) on the June prelim.) So $p < p_A$, and hence I_A is given by

$$p(-I_A) + (1-p)(35 - I_A) = p(-15) + (1-p)(35).$$

This reduces to $I_A = 15p$ (the expected savings from acquiring the info).

Now, from (7) we see that $I_B > I_A$ iff

$$pv(-I_A) + (1-p)v(35 - I_A) > pv(-15) + (1-p)v(35). \tag{12}$$

The Lemma in (b) gives us this inequality. Let

$$\hat{x} = -15,\ x = -I_A,\ y = 35 - I_A,\ \hat{y} = 35,$$

and g and $\hat{g}$ be the gambles described in the Lemma. Then $\hat{x} < x$ because $I_A = 15p$, and $x < y < \hat{y}$ is obvious. Furthermore, the two gambles have the same expected value:

$$p\hat{x} + (1-p)\hat{y} = 35 - 50p,$$
$$px + (1-p)y = p(-15p) + (1-p)(35 - 15p) = 35 - 50p.$$

Hence, the Lemma tells us that Barb strictly prefers g to $\hat{g}$, which is (12). ■

9. (June 2016) (25 pts) In a two-good world, consider the function $e : \mathbb{R}^2_{++} \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$e(p, U) := U \min\left\{p_1,\ \frac{p_1 + p_2}{3},\ p_2\right\}.$$

(a) (5 pts) State four properties that an expenditure function must satisfy if it arises from a continuous monotonic utility function, and verify that the given e indeed satisfies them.

Soln: Such an expenditure function must be (1) continuous; (2) concave in p; (3) increasing in p and U (actually, strictly increasing in U); and (4) homogeneous of degree 1 in p.

The function $\min\left\{p_1,\ \frac{p_1+p_2}{3}, p_2\right\}$ is the minimum of three linear functions of p, and hence is continuous and concave in p. So the given e is the product of two continuous functions, and so continuous. It is obviously increasing in p, strictly increasing in U. It is homogeneous of degree 1 in p because

$$U \min\left\{tp_1,\ \frac{tp_1 + tp_2}{3},\ tp_2\right\} = tU \min\left\{p_1,\ \frac{p_1 + p_2}{3},\ p_2\right\}.$$

■

(b) (5 pts) Let $h(p, U)$ denote a Hicksian demand correspondence that gives rise to the expenditure function e. For each $(p, U) \in \mathbb{R}^2_{++} \times \mathbb{R}_+ \to \mathbb{R}_+$, find as many points in the set $h(p, U)$ as you can.

Soln: Letting $r = p_1/p_2$, we can write e as

$$e(p, U) = \begin{cases} p_1 U & \text{for } r \leq \frac{1}{2} \\ \frac{1}{3}(p_1 + p_2)U & \text{for } \frac{1}{2} \leq r \leq 2 \\ p_2 U & \text{for } 2 \leq r \end{cases}.$$

At any (p, U) at which e is differentiable with respect to p, the Hicksian demand correspondence is a singleton, given by the gradient of e with respect to p. From the above, we see that e is differentiable with respect to p at (p, U) iff $r \notin \{\frac{1}{2}, 2\}$. Hence, taking these derivatives yields

$$h(p, U) = \begin{cases} \{(U, 0)\} & \text{for } r < \frac{1}{2} \\ \{\frac{1}{3}(U, U)\} & \text{for } \frac{1}{2} < r < 2 \\ \{(0, U)\} & \text{for } 2 < r \end{cases}.$$

Because the Hicksian demand correspondence is uhc, we also know that

$$\begin{aligned} r = \tfrac{1}{2} \quad &\Rightarrow \quad h(p, U) \text{ contains } (U, 0) \text{ and } \tfrac{1}{3}(U, U), \\ r = 2 \quad &\Rightarrow \quad h(p, U) \text{ contains } \tfrac{1}{3}(U, U) \text{ and } (0, U). \end{aligned}$$

■

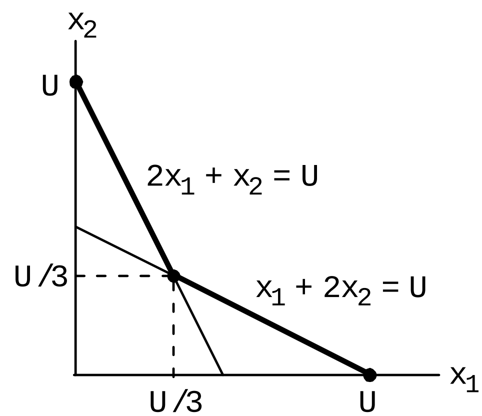
(c) (15 pts) For each property of a function listed below, state whether there is a utility function satisfying that property which gives rise to the expenditure function e, and sketch proofs of your answers.

i. (5 pts) strictly quasiconcave.

Soln: e does not arise from a strictly quasiconcave utility function.

Proof. The Hicksian demand correspondence arising from a strictly quasiconcave utility function is a function, i.e., its image never contains more than one point. But in (b) we found that any Hicksian demand correspondence that gives rise to e contains at least two points at any (p, U) for which $U > 0$ and $r \in \{\frac{1}{2}, 2\}$. ■

ii. (5 pts) quasiconcave.
Soln: e does arise from a quasiconcave utility function.
Proof. We find a quasiconcave u that gives rise to e. For a given $U > 0$, we know from (b) that the U indifference curve must contain the points $(0, U)$, $\frac{1}{3}(U, U)$, and $(U, 0)$, indicated by the three points below.

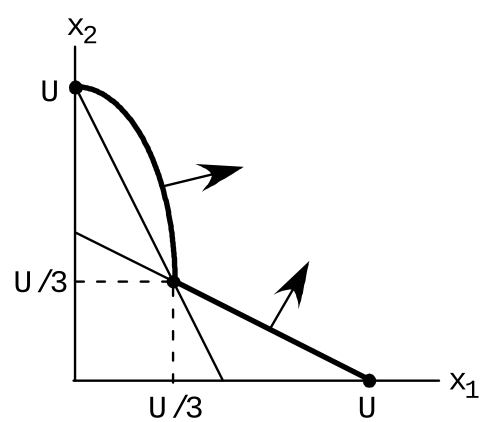


Since $(0, U)$ and $\frac{1}{3}(U, U)$ minimize the cost of getting utility U when $r = 2$, the U indifference curve must be weakly above the budget line given by $2x_1 + x_2 = U$. Thus, in order for u to be quasiconcave (and monotonic), the indifference curve must contain the line segment between $(0, U)$ and $\frac{1}{3}(U, U)$. By the same argument applied to the $x_1 + 2x_2 = U$ budget line, the indifference curve must contain the line segment between $\frac{1}{3}(U, U)$, and $(U, 0)$. The U indifference curve is thus the heavy kinked line, and the utility function is

$$u(x) = \begin{cases} x_1 + 2x_2 & \text{for } x_1 \geq x_2 \\ 2x_1 + x_2 & \text{for } x_1 \leq x_2 \end{cases} = \min\{x_1 + 2x_2, 2x_1 + x_2\}.$$

This u is obviously quasiconcave and gives rise to e. ■

iii. (5 pts) non-quasiconcave.
Soln: e does arise from a non-quasiconcave utility function.
Proof 1. Obtain a non-quasiconcave utility function $\hat{u}$ be deforming the indifference curves of the u given in (ii) by pushing them up on the set $\{x : 0 < x_1 < x_2\}$, so that a U indifference curve looks like the following:



(For example, let $\hat{u}(x) = u(x)$ for $x_1 \geq x_2$, and let $\hat{u}(x) = u(x) - x_1(1 - \frac{x_1}{x_2})$ for $x_1 < x_2$.) Such a $\hat{u}$ is clearly not quasiconcave, but its expenditure function is still the given e. (No point on the bowed part of an indifference curve, except its endpoints, ever minimizes the cost of getting to that indifference curve.) ■

Proof 2. The following discontinuous utility function gives rise to e and is clearly not quasiconcave (graph an indifference curve to see this): $u(x) = \max\{x_1, x_2\}$ if $x_1 \neq x_2$, and $u(x) = x_1$ if $x_1 = x_2$. ■

10. (June 2016) (25 pts) A consumer lives for two periods. In period 2 she will purchase a commodity bundle $x = (x_1, x_2)$ to maximize her utility $u(x)$ subject to her budget constraint $p \cdot x \leq y$.

   (a) (5 pts) Let $v(y)$ be the consumers's indirect utility arising from $u(x)$ (we suppress the argument p since it is fixed in this question). Show that v is concave in y if u is concave in x.

   Soln: For any $y \geq 0$ we have

$$\text{(P)} \quad v(y) := \max_x u(x) \text{ st } p \cdot x \leq y.$$

   Let $y = (1-\alpha)y_0 + \alpha y_1$, where y_0 and y_1 are two income levels and $\alpha \in [0, 1]$. Let x^i be a utility-maximizing bundle given income y_i. The bundles x^i satisfy the budget constraints $p \cdot x^i \leq y_i$. Hence, $\bar{x} := (1 - \alpha)x_0 + \alpha x_1$ satisfies the budget constraint in (P). Therefore,

$$\begin{aligned} v(y) &\geq u(\bar{x}) \\ &\geq (1 - \alpha)u(x^0) + \alpha u(x^1) \\ &= (1 - \alpha)v(y_0) + \alpha v(y_1), \end{aligned}$$

   where the second inequality follows from the concavity of u. This proves v is concave. ■

In period 1 the consumer chooses an amount z to invest in a risky asset that returns $(1 + \tilde{r})z$ in period 2. Her initial wealth is $w > 0$, and she is restricted to choosing $z \in [0, w]$. She uses her resulting income in period 2, $\tilde{y} = w + \tilde{r}z$, to purchase x at the prices p. She knows these prices in period 1, and chooses z in order to maximize the expected utility she will ultimately obtain in period 2. For each $w \geq 0$, let $z^*(w)$ be an optimal investment for this consumer.

Assume the random variable $\tilde{r}$ is continuously distributed on an interval $[\underline{r}, \bar{r}]$, with $\underline{r} > -1$ and $\mathbb{E}\tilde{r} > 0$.

   (b) (10 pts) Suppose v is C^2 and satisfies $v' > 0$, $v'' < 0$, and DARA ($A(y) := -v''(y)/v'(y)$ is a strictly decreasing function). Show that for any $w > 0$, if $z^*(w) < w$ holds, then $z^{*\prime}(w) > 0$.

   Soln: (This is based on 701A Lecture Slides 6.) Since v is strictly concave, the optimal investment is unique:

$$z^*(w) = \arg \max_{0 \leq z \leq w} \mathbb{E}v(w + \tilde{r}z).$$

   The derivative of the objective function is $\mathbb{E}v'(w+\tilde{r}z)\tilde{r}$. At $z = 0$, this derivative is equal to

$$\mathbb{E}v'(w)\tilde{r} = \mathbb{E}v'(w)(\mathbb{E}\tilde{r}) > 0,$$

   and so $z^*(w) \neq 0$. Thus, as $z^*(w) < w$, we know

$$\text{(FOC)} \quad \mathbb{E}v'(w + \tilde{r}z^*(w))\tilde{r} = 0.$$

   Differentiate this with respect to w to obtain

$$z^{*\prime}(w) = \frac{-\mathbb{E}v''(w + \tilde{r}z^*)\tilde{r}}{\mathbb{E}v''(w + \tilde{r}z^*)\tilde{r}^2} =: \frac{N}{D},$$

where $z^* = z^*(w)$. Note that $N = \mathbb{E}v'(w + \tilde{r}z^*)A(w + \tilde{r}z^*)\tilde{r}$. By DARA,

$$r > 0 \Rightarrow A(w + rz^*) < A(w) \Rightarrow A(w + rz^*)r < A(w)r,$$
$$r < 0 \Rightarrow A(w + rz^*) > A(w) \Rightarrow A(w + rz^*)r < A(w)r.$$

Thus,

$$N < \mathbb{E}v'(w + \tilde{r}z^*)A(w)\tilde{r} = A(w)\mathbb{E}v'(w + \tilde{r}z^*)\tilde{r} = 0,$$

by (FOC). Since $D < 0$, this proves $z^{*\prime}(w) > 0$ at any $w > 0$ satisfying $z^*(w) < w$.

Addendum. In fact, z^* is a strictly increasing function on $\mathbb{R}_+$, even in neighborhoods of points w for which $z^*(w) = w$.

Proof. (There may be a simpler one.) Let $w_1 > 0$. We must show that $z^*(w_1) > z^*(w)$ for all $w < w_1$. If $z^*(w_1) = w_1$ we are done, as in this case we have

$$z^*(w_1) = w_1 > w \geq z^*(w) \text{ for all } w < w_1.$$

We can now assume $z^*(w_1) < w_1$. Define

$$w_0 := \sup\{w \in [0, w_1] : z^*(w) = w\}.$$

This w_0 is well defined because the indicated set is nonempty (as $z^*(0) = 0$). By the maximum theorem z^* is continuous, and hence $w_0 < w_1$ and $z^*(w_0) = w_0$. For any $w \in (w_0, w_1)$ we have $z^*(w) < w$, and so $z^{*\prime}(w) > 0$ by the above proof. Hence, the mean value theorem implies that $z^*(w_1) > z^*(w)$ for all $w \in [w_0, w_1)$. Lastly, for $w < w_0$ we have

$$z^*(w_1) > z^*(w_0) = w_0 > w \geq z^*(w),$$

completing the proof. ∎

(c) (10 pts) Now assume $u(x) = u_1(x_1) + u_2(x_2)$, where each u_i is C^2 with $u_i' > 0$, $u_i'' < 0$, and $u_i'(0) = \infty$. Show that if u_1 and u_2 each satisfy DARA, then v satisfies DARA.

Soln: Proof 1. For an arbitrary y let $x = x(y)$ solve (P), and let $\lambda = \lambda(y)$ be the optimal value of the multiplier. Since each $u_i'(0) = \infty$, we know $x \gg 0$. The FOCs are thus equalities:

$$u_1'(x_1) = \lambda p_1,$$
$$u_2'(x_2) = \lambda p_2,$$
$$p \cdot x = y.$$

For future use, note that the the derivatives $x_i' = x_i'(y)$ are both positive. The easiest way to see this is to note that the first two FOCs imply

$$u_1''(x_1)x_1'/p_1 = \lambda' = u_2''(x_2)x_2'/p_2,$$

and so x_1' and x_2' have the same sign. The third FOC implies $p_1x_1' + p_2x_2' = 1$, so at least one of x_1' and x_2' is positive. So they are both positive.

We now use the FOCs to evaluate $A(y) = -v''(y)/v'(y)$. By the envelope theorem, the marginal utility of income is the multiplier: $v' = u_i'/p_i$, using the FOCs. To find $v'' = \lambda'$, differentiate all three FOCs with respect to y to obtain

$$\begin{bmatrix} u_1'' & 0 & -p_1 \\ 0 & u_2'' & -p_2 \\ p_1 & p_2 & 0 \end{bmatrix} \begin{pmatrix} x_1' \\ x_2' \\ \lambda' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Hence, by Cramer's rule,

$$\lambda' = \frac{\begin{vmatrix} u_1'' & 0 & 0 \\ 0 & u_2'' & 0 \\ p_1 & p_2 & 1 \end{vmatrix}}{\begin{vmatrix} u_1'' & 0 & -p_1 \\ 0 & u_2'' & -p_2 \\ p_1 & p_2 & 0 \end{vmatrix}} = \frac{u_1'' u_2''}{p_1^2 u_2'' + p_2^2 u_1''}.$$

Therefore,

$$A(y) = -\frac{v''}{v'} = \frac{-u_1'' u_2''}{\lambda p_1^2 u_2'' + \lambda p_2^2 u_1''}$$

$$= \frac{-1}{\left(\frac{u_1'}{p_1}\right) p_1^2 \frac{u_2''}{u_1'' u_2''} + \left(\frac{u_2'}{p_2}\right) p_2^2 \frac{u_1''}{u_1'' u_2''}} = \frac{1}{\frac{p_1}{A_1(y)} + \frac{p_2}{A_2(y)}},$$

where $A_i(y) := -u_i''(x_i(y))/u_i'(x_i(y))$. Since $x_i' > 0$ and u_i satisfies DARA, $A_i(y)$ is a decreasing function. This implies that

$$\frac{p_1}{A_1(y)} + \frac{p_2}{A_2(y)}$$

increases in y, and so $A(y)$ decreases in y. This shows that v satisfies DARA. ■

Proof 2. The FOCs and the envelope theorem imply

$$v'(y) = p_i^{-1} u_i'(x_i(y)),$$

and hence, $v''(y) = p_i^{-1} u_i''(x_i) x_i'$. Therefore,

$$A(y) = -\frac{p_i^{-1} u_i''(x_i) x_i'}{p_i^{-1} u_i'(x_i)} = A_i(y) x_i'(y),$$

where again $A_i(y) := -u_i''(x_i(y))/u_i'(x_i(y))$. Now, again, $A_i(y)$ is a decreasing function because $x_i' > 0$ and u_i satisfies DARA (the proof that $x_i' > 0$ is at the beginning of Proof 1 above). Since $p_1 x_1'(y) + p_2 x_2'(y) = 1$, it must be that $x_1'(y)$ or $x_2(y)$ is nonincreasing, say $x_1'(y)$ w.l.o.g.. The product $A_1(y) x_1'(y)$ therefore decreases in y, since $A_1(y)$ is positive and decreasing, and $x_1'(y)$ is positive and nonincreasing. Thus, $A(y)$ is decreasing, i.e., v satisfies DARA. ■

11. (August 2016) (25 pts) The inverse demand function for oil is given by a continuously differentiable function $P : \mathbb{R}_{++} \to \mathbb{R}_{++}$ satisfying $P' < 0$ and $P(x) \to \infty$ as $x \downarrow 0$. The price elasticity of the demand for oil is defined at any $x > 0$ as

$$e(x) := -\frac{P(x)}{P'(x)x}.$$

The total stock of oil below the ground is $0 < \bar{x} < \infty$. It is all owned by one oil company, which can extract it at zero cost. The firm's profit is zero if it sells no oil, and its profit is px if it sells an amount $x > 0$ at price p.

(a) (6 pts) Compare the competitive equilibrium (x^c, p^c) to the monopoly outcome (x^m, p^m) under (i) the assumption that $e(x) > 1$ for all $x \in (0, \bar{x}]$, and (ii) under the assumption that $e(\bar{x}) < 1$.

Soln: $(x^c, p^c) = (\bar{x}, P(\bar{x}))$. Proof: Given a price p, the consumer demands the x satisfying $P(x) = p$, and the firm supplies $\bar{x} = \arg\max_{0 \le x \le \bar{x}} px$. So supply = demand requires $x^c = \bar{x}$ and $p^c = P(\bar{x})$.

A monopoly chooses $x \in [0, \bar{x}]$ to maximize its revenue, which is given by $R(0) = 0$ and $R(x) = xP(x)$ for $x > 0$. Marginal revenue is

$$\begin{aligned} R'(x) &= P(x) + P'(x)x \\ &= \left(1 + \frac{P'(x)x}{P(x)}\right) P(x) = \left(1 - \frac{1}{e(x)}\right) P(x). \end{aligned}$$

Case (i) : $e(x) > 1$ for $x \in (0, \bar{x}]$. In this case $R'(x) > 0$ for all $x \in (0, \bar{x}]$. It follows that $R(\bar{x}) > R(x)$ for $x \in [0, \bar{x}]$. The monopoly outcome is thus $(x^m, p^m) = (\bar{x}, P(\bar{x}))$, the same as the competitive outcome.

Case (ii) : $e(\bar{x}) < 1$. In this case $R'(\bar{x}) < 0$, and so any maximizer of R on $[0, \bar{x}]$ is less than $\bar{x}$. Relative to the competitive firm, the monopoly produces less oil, $x^m < x^c$, at a higher price, $p^m = P(x^m) > p^c$. However, this statement is vacuous if the monopoly problem has no solution, which is possible. For example, R has no maximizer if $e(x) < 1$ for all $x \in (0, \bar{x}]$, as then R is a decreasing positive function on $(0, \bar{x}]$, with a discontinuity at 0 because $R(0) = 0$. ■

Now suppose there are two periods, $t = 1, 2$, and the firm discounts the second period at rate $r > 0$. The inverse demand function in period t is $P_t(x_t)$, which has the same properties as does the function P above. The firm's discounted payoff if it sells x_t in period t at price p_t is $p_1 x_1 + (1 + r)^{-1} p_2 x_2$, where (x_1, x_2) must satisfy $x_1 + x_2 \le \bar{x}$. Its profit in period t is 0 if $x_t = 0$.

(b) (6 pts) Suppose $(p_1^c, x_1^c, p_2^c, x_2^c)$ is a competitive equilibrium satisfying $x_1^c > 0$ and $x_2^c > 0$. Find a system of four equations this equilibrium must satisfy. Then compare x_1^c to x_2^c when $P_1(\cdot) = P_2(\cdot)$.

Soln: x_t^c must be what the consumers demand in period t at price p_t^c, and so

$$p_1^c = P_1(x_1^c), \quad p_2^c = P_2(x_2^c). \tag{13}$$

As these prices are positive, the price-taking firm must find it optimal to sell all its oil:

$$x_1^c + x_2^c = \bar{x}.$$

Thus, $x_1 = x_1^c$ maximizes $p_1^c x_1 + (1+r)^{-1} p_2^c(\bar{x} - x_1)$ on $[0, \bar{x}]$. Since $0 < x_1^c < \bar{x}$, the FOC is $p_1^c - (1+r)^{-1} p_2^c = 0$, which rearranges to the Hotelling no-arbitrage condition that the equilibrium price of oil increases over time at the rate of interest:

$$p_2^c = (1+r)p_1^c.$$

The equilibrium $(p_1^c, x_1^c, p_2^c, x_2^c)$ satisfies these four displayed equations, and they can in principle be solved to find the equilibrium.

If P_1 and P_2 are the same function, then because it is a decreasing function and $p_2^c > p_1^c$, (13) implies $x_2^c < x_1^c$. Thus, more oil is consumed in the present than in the future, due to discounting and the no-arbitrage condition. ■

(c) (6 pts) Again allowing P_1 and P_2 to be different functions, assume now that for some $\underline{e} > 1$, the elasticities satisfy $e_t(x_t) > \underline{e}$ for all $x_t \in (0, \bar{x}]$ and $t = 1, 2$. Suppose $(p_1^m, x_1^m, p_2^m, x_2^m)$ is a monopoly outcome satisfying $x_1^m > 0$ and $x_2^m > 0$. Find a system of four equations this outcome must satisfy.

Soln: The firm's revenue function in period t is given by $R_t(x_t) = x_t P_t(x_t)$ for $x_t \in (0, \bar{x}]$, and $R_t(0) = 0$. The monopoly outputs maximize

$$R_1(x_1) + (1+r)^{-1} R_2(x_2)$$

subject to $x_1 + x_2 \leq \bar{x}$. Because both elasticities exceed 1, each R_t is an increasing function (see the solution to (a)). The constraint thus binds, and so $x_1 = x_1^m$ maximizes

$$R_1(x_1) + (1+r)^{-1} R_2(\bar{x} - x_1)$$

subject to $0 \leq x_1 \leq \bar{x}$. Since $x_1^m \in (0, \bar{x})$, the FOC is $R_1'(x_1^m) - (1+r)^{-1} R_2'(\bar{x} - x_1^m) = 0$. Substitute x_2^m for $\bar{x} - x_1^m$ and rearrange to obtain the monopoly no-arbitrage condition:

$$R_2'(x_2^m) = (1+r) R_1'(x_1^m). \tag{14}$$

The monopoly outcome $(p_1^m, x_1^m, p_2^m, x_2^m)$ satisfies this equation and $x_1^m + x_2^m = \bar{x}$, together with $p_1^m, = P_1(x_1^m)$ and $p_2^m = P_2(x_2^m)$. ■

(d) (7 pts) Under the additional assumption that both P_1 and P_2 have constant elasticities, e_1 and e_2, satisfying $e_1 \geq e_2 > 1$, how does $(p_1^m, x_1^m, p_2^m, x_2^m)$ compare to $(p_1^c, x_1^c, p_2^c, x_2^c)$?

Soln: Using the expression for R' shown in the solution to (a), we can now write the monopoly no-arbitrage condition (14) as

$$\left(1 - \frac{1}{e_2}\right) P_2(x_2^m) = (1+r)\left(1 - \frac{1}{e_1}\right) P_1(x_1^m),$$

and so

$$\frac{P_2(\bar{x} - x_1^m)}{(1+r)P_1(x_1^m)} = \frac{1 - \frac{1}{e_1}}{1 - \frac{1}{e_2}} \geq 1,$$

where the inequality follows from $e_1 \geq e_2 > 1$. The competitive no-arbitrage condition is

$$\frac{P_2(\bar{x} - x_1^c)}{(1+r)P_1(x_1^c)} = 1.$$

Hence,

$$\frac{P_2(\bar{x} - x_1^m)}{P_1(x_1^m)} \geq \frac{P_2(\bar{x} - x_1^c)}{P_1(x_1^c)},$$

with equality iff $e_1 = e_2$. This and $P_t' < 0$ imply $x_1^m \geq x_1^c$, and hence $x_2^m \leq x_2^c$. Therefore, if $e_1 > e_2$ then

$$p_1^m < p_1^c, \quad x_1^m > x_1^c, \quad p_2^m > p_2^c, \quad x_2^m < x_2^c,$$

and these are all equalities if $e_1 = e_2$. ■

12. (August 2016) (25 pts) Consider a Bernoulli utility function $u : \mathbb{R} \to \mathbb{R}$ that has derivatives $u' > 0$ and $u'' < 0$, and exhibits DARA (decreasing absolute risk aversion). Prove each of the following.

   (a) (10 pts) (Lemma) For any $k \in \mathbb{R}$ and any random gamble $\tilde{y}$,

   $$\mathbb{E}u(\tilde{y}) = u(k) \;\Rightarrow\; \mathbb{E}u(\tilde{y} + a) > u(k + a) \;\forall a > 0.$$

   Soln: Proof 1. Because $\mathbb{E}u(\tilde{y}) = u(k)$, "agent" u is indifferent between the random gamble $\tilde{y}$ and the deterministic gamble δ_k. Let u_a be the utility function defined by $u_a(z) = u(z+a)$. Then for $a > 0$, DARA implies that u_a is strictly more risk averse than u (essentially by Pratt's theorem). Hence, by definition of "strictly more risk averse," the indifference of u between $\tilde{y}$ and δ_k implies that u_a strictly prefers the former. Hence,

   $$\mathbb{E}u(\tilde{y} + a) = \mathbb{E}u_a(\tilde{y}) > u_a(k) = u(k + a).$$

   Proof 2. Let u_a be the utility function defined by $u_a(z) = u(z + a)$. Let $c(a)$ be the certainty equivalent $C(\tilde{y}, u_a)$, so that $\mathbb{E}u_a(\tilde{y}) = u_a(c(a))$. Note that $c(0) = k$. Since u_a exhibits DARA because u does, Pratt's Theorem implies $c(a)$ is an increasing function. Hence, for any $a > 0$ we have $c(a) > k$. Thus,

   $$\mathbb{E}u(\tilde{y} + a) = \mathbb{E}u_a(\tilde{y}) = u_a(c(a)) > u_a(k) = u(k + a).$$

   ■

Even if you were unable to prove the "Lemma" in (a), feel free to use it to prove (b)-(d) below.

   (b) (5 pts) Let $\tilde{x}$ be a random gamble, and let $b(w)$ be the maximum price the agent is willing to pay for $\tilde{x}$ when her wealth is w. Then $b(w)$ increases in w.

   Soln: Let $\hat{w} > w$. The buy price $b(w)$ is defined by

   $$\mathbb{E}u(w + \tilde{x} - b(w)) = u(w).$$

   By (a), adding $\hat{w} - w$ to the arguments on both sides of this equality yields

   $$\mathbb{E}u(\hat{w} + \tilde{x} - b(w)) > u(\hat{w}) = \mathbb{E}u(\hat{w} + \tilde{x} - b(\hat{w})),$$

   where the equality comes from the definition of $b(\hat{w})$. Hence, $b(w) < b(\hat{w})$. ■

   (c) (5 pts) Let $\tilde{x}$ be a random gamble, and let $s(w)$ be the minimum price at which the agent is willing to sell $\tilde{x}$ when her wealth is w. Then $s(w)$ increases in w.

   Soln: Let $\hat{w} > w$. The sell price $s(w)$ is defined by

   $$\mathbb{E}u(w + \tilde{x}) = u(w + s(w)).$$

   By (a), adding $\hat{w} - w$ to the arguments on both sides of this equality yields

   $$\mathbb{E}u(\hat{w} + \tilde{x}) > u(\hat{w} + s(w)).$$

   Thus, since $u(\hat{w} + s(\hat{w})) = \mathbb{E}u(\hat{w} + \tilde{x})$, we have

   $$u(\hat{w} + s(\hat{w})) > u(\hat{w} + s(w)),$$

   which implies $s(\hat{w}) > s(w)$.

   ■

(d) (5 pts) Now let $\tilde{x}$ be a random gamble that is valuable at wealth w, in the sense that $\mathbb{E}u(w + \tilde{x}) > u(w)$. Then $s(w) > b(w)$, where $b(w)$ and $s(w)$ are defined in (b) and (c) from this $\tilde{x}$ and w.

Soln: Again, the buy price $b(w)$ is defined by

$$\mathbb{E}u(w + \tilde{x} - b(w)) = u(w).$$

Because $\mathbb{E}u(w + \tilde{x}) > u(w)$, we have $b(w) > 0$. Therefore, by (a), adding $b(w)$ to the arguments on both sides of this equality yields

$$\mathbb{E}u(w + \tilde{x}) > u(w + b(w)).$$

Thus, since $u(w + s(w)) = \mathbb{E}u(w + \tilde{x})$, we have

$$u(w + s(w)) > u(w + b(w)),$$

which implies $s(w) > b(w)$. ■