Economics 681 **Steven A. Matthews**
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Suggested Solutions to Problem Set 2
Today's Date: September 24, 2017

1. Show that if $u : \mathbb{R}^2_+ \to \mathbb{R}$ is a C^2 quasiconcave utility function, with $u_1 > 0$ and $u_2 > 0$, then its indifference curves slope down and exhibit (weakly) diminishing marginal rates of substitution. [Show that $g' < 0$ and $g'' \geq 0$, where the function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is defined, for a given $\bar{u}$, by $u(x_1, g(x_1)) = \bar{u}$.]

   Soln: The indifference curve $\{x : u(x) = \bar{u}\}$ is the graph of the function $g : \mathbb{R}_+ \to \mathbb{R}_+$ defined implicitly by $u(x_1, g(x_1)) = \bar{u}$. Differentiate this identity in x_1 to obtain
   $$u_1 + u_2 g'(x_1) = 0, \tag{1}$$
   where the argument of each function u_i is $(x_1, g(x_1))$. Thus, $g'(x_1) = -u_1/u_2$, which is indeed negative given that u_1 and u_2 are positive. So the indifference curve does slope down.

   Now differentiate again with respect to x_1 : differentiating (1) yields
   $$u_{11} + u_{12}g' + u_{21}g' + u_{22}(g')^2 + u_2 g'' = 0. \tag{2}$$
   Fix x_1, and let
   $$v := \begin{pmatrix} 1 \\ g' \end{pmatrix}, \quad H = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}.$$
   Note that (1) is $v \cdot \nabla u = 0$. Note too that
   $$u_{11} + u_{12}g' + u_{21}g' + u_{22}(g')^2 = v^T H v,$$
   and so (2) can be written as
   $$g'' = -\frac{v^T H v}{u_2}.$$
   We have $u_2 > 0$ and, since $v \cdot \nabla u = 0$ and u is a C^2 quasiconcave function, $v^T H v \leq 0$. Thus, $g'' \geq 0$, proving that the MRS diminishes (at least, does not increase) as one moves down the indifference curve. ■

2. Preferences are called *homothetic* if they satisfy the following property:
   $$x \succsim y \Rightarrow \alpha x \succsim \alpha y \quad \forall \alpha \geq 0$$
   Suppose $\succsim$ is a complete, transitive, strictly monotonic, continuous preference relation on $\mathbb{R}^L_+$. Show that $\succsim$ is homothetic if and only if there exists a utility representation u such that $u(\alpha x) = \alpha u(x)$ for all $\alpha \geq 0$.

   Soln: ($\Leftarrow$) Suppose $u(\cdot)$ represents $\succsim$ and is homogeneous of degree one. Let $x, y \in \mathbb{R}^L_+$ be such that $x \succsim y$ and let $\alpha \geq 0$. Then
   $$\begin{aligned} & u(x) \geq u(y) \quad \text{(since } u \text{ represents } \succeq) \\ \Rightarrow\ & \alpha u(x) \geq \alpha u(y) \text{ (since } \alpha \geq 0) \\ \Rightarrow\ & u(\alpha x) \geq u(\alpha y) \text{ (since } u \text{ is homogeneous of degree 1)} \\ \Rightarrow\ & \alpha x \succeq \alpha y \quad \text{(since } u \text{ represents } \succeq). \end{aligned}$$

So any preference relation represented by a utility function that is homogeneous of degree 1 is homothetic.

($\Rightarrow$) Suppose $\succsim$ is homothetic, and let u be the function representing $\succsim$ constructed in the proof of the Monotone Representation Theorem we sketched in class. Thus, for any $x \in \mathbb{R}^L_+$, $u(x)$ is the number such that

$$x \sim (u(x), \ldots, u(x)). \tag{3}$$

We show that this u is homogeneous of degree 1. Fixing x, applying the definition of homothetic preferences to (3) (twice, once in each direction) yields

$$\alpha x \sim (\alpha u(x), \ldots, \alpha u(x)).$$

We also know, by the definition of u, that

$$\alpha x \sim (u(\alpha x), \ldots, u(\alpha x)).$$

Transitivity now implies

$$(\alpha u(x), \ldots, \alpha u(x)) \sim (u(\alpha x), \ldots, u(\alpha x)),$$

and so by monotonicity we have $\alpha u(x) = u(\alpha x)$, as desired. ■

3. JR Exercise 1.29

Soln: This utility maximization problem is to choose $x = (x_1, x_2, \ldots)$ to solve the program

$$\max_{x \geq 0} \sum_{t=0}^{\infty} \beta^t \ln(x_t) \text{such that} \sum_{t=0}^{\infty} x_t \leq 1.$$

The Lagrangian is

$$\mathcal{L}(x, \lambda) = \sum_{t=0}^{\infty} \beta^t \ln(x_t) + \lambda \left\{ 1 - \sum_{t=0}^{\infty} x_t \right\} \tag{4}$$

The necessary first order conditions (NFOCs) for $x^* = (x_1^*, x_2^*, \ldots)$ to be a solution are that $\lambda^* \geq 0$ exist such that

$$\beta^t \frac{1}{x_t^*} - \lambda^* \leq 0 \ (= 0 \text{ if } x_t^* > 0) \text{ for } t = 0, 1, \ldots, \tag{5}$$

$$1 - \sum_{t=0}^{\infty} x_t^* \geq 0 \ (= 0 \text{ if } \lambda^* > 0). \tag{6}$$

The utility function is quasiconcave, and so these NFOCs are also SFOCs. From (5) we see that each $x_t^* > 0$, and so $\lambda^* > 0$. It follows that all the inequalities in (5) and (6) are equalities. From (5), for $\forall t \geq 0$ we have

$$x_t^* = \frac{\beta^t}{\lambda^*},$$

and so from (6) we obtain

$$\lambda^* = \sum_{t=0}^{\infty} \beta^t = \frac{1}{1 - \beta}.$$

(The second equality is the number one formula to remember for summing a geometric series, and it is easy to derive.) Hence, the solution x^* is given by

$$x_t^* = \frac{\beta^t}{\lambda^*} = \beta^t(1-\beta) \text{ for } t = 0, 1, \ldots.$$

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4. Find the demand and indirect utility functions for these utility functions:

   (a) $u(x) = x_1 + x_2$

   Soln: The consumer problem is:

$$\max_{x_1, x_2 \geq 0} \; x_1 + x_2 \text{ such that } p_1 x_1 + p_2 x_2 \leq m.$$

   There are three cases to consider:

   i. $p_1 = p_2$. In this case the standard figure shows that any x on the budget line is a solution, since the budget line is also an indifference "curve." That is, any (x_1, x_2) satisfying $x_1 + x_2 = m/p_1$ is a solution.

   ii. $p_1 < p_2$. Intuitively, it should be clear that the consumer in this case should only buy the cheaper good, so the solution is $x^* = (m/p_1, 0)$. This is also clear from the standard figure, since the budget line is now flatter than the indifference curves. Here is a more formal proof. Let (x_1, x_2) satisfy $p_1 x_1 + p_2 x_2 \leq m$ with $x_2 > 0$. Then

$$\begin{aligned} u(x_1, x_2) &= x_1 + x_2 \\ &\leq \frac{m - p_2 x_2}{p_1} + x_2 \\ &= \frac{m}{p_1} + x_2 \left(1 - \frac{p_2}{p_1}\right) \\ &< \frac{m}{p_1}, \end{aligned}$$

   where the final inequality comes from $x_2 > 0$ and $p_1 < p_2$. Since x^* satisfies $u(x^*) = m/p_1$, this proves that x^* solves, uniquely, the consumer's problem.

   iii. $p_2 < p_1$. By the same argument as in (ii), $(0, m/p_2)$ is the unique solution.

   Plugging these demands into the utility function in each case yields the indirect utility function:

$$v(p, m) = \min_i \frac{m}{p_i}.$$

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   (b) $u(x) = \ln x_1 + \ln x_2$

   Soln: The consumer problem is

$$\max_{x_1, x_2 \geq 0} \ln x_1 + \ln x_2 \text{ such that } p_1 x_1 + p_2 x_2 \leq m.$$

   The KT FOC $\mathcal{L}_{x_i} \leq 0$ cannot hold if $x_i = 0$, since $\partial \ln(x_i)/\partial x_i = \infty$ at $x_i = 0$. So the solution is interior, implying that the FOC are

$$\begin{aligned} \frac{1}{x_1} &= \lambda p_1, \\ \frac{1}{x_2} &= \lambda p_2. \end{aligned}$$

Divide these two inequalities to remove λ, and then rearrange to obtain

$$p_1 x_1 = p_2 x_2.$$

(There is economic content to this equality: the consumer spends equal amounts on the two goods.) Replacing $p_2 x_2$ by $p_1 x_1$ in the budget constraint yields $2p_1 x_1 = m$, or rather $x_1 = \frac{m}{2p_1}$. Similarly, $x_2 = \frac{m}{2p_2}$. Plugging these demands back into u gives us the indirect utility function:

$$v(p, m) = \ln \frac{m}{2p_1} + \ln \frac{m}{2p_2}.$$

∎

(c) $u(x) = e^{x_1 x_2}$

Soln: This is a strictly increasing transformation of the utility function from (b). Therefore, demand is the same as in (b). Plugging that demand function into the utility function gives us

$$v(p, m) = e^{\frac{m^2}{4p_1 p_2}}.$$

∎

(d) $u(x) = \sqrt{x_1} + x_2$

Soln: The consumer problem is:

$$\max_{x_1, x_2 \geq 0} \sqrt{x_1} + x_2 \text{ such that } p_1 x_1 + p_2 x_2 \leq m.$$

The KT FOC for x_1 and x_2 are

$$\begin{aligned} \frac{1}{2\sqrt{x_1}} &\leq \lambda p_1 \ (= \text{ if } x_1 > 0) \\ 1 &\leq \lambda p_2 \ (= \text{ if } x_2 > 0). \end{aligned}$$

The FOC for x_1 implies $x_1 > 0$, and so it holds as an equality. The FOC for x_2 implies $\lambda > 0$, and so the budget constraint binds (holds as an equality).

- Case 1: If $1 = p_2 \lambda$, we have

$$\frac{1}{2p_1\sqrt{x_1}} = \frac{1}{p_2}.$$

  Hence, $x_1 = p_2^2/(4p_1^2)$ and $x_2 = (m - p_1 x_1)/p_2 = m/p_2 - p_2/(4p_1)$. This (x_1, x_2) is a solution to the consumer's problem only when $x_2 \geq 0$, that is, $4p_1 m \geq p_2^2$.
- Case 2: If $1 < p_2 \lambda$, we have $x_2 = 0$. Then, from the budget equation we obtain $x_1 = m/p_1$. Note that the FOCs require

$$\frac{1}{p_2} < \lambda = \frac{1}{2p_1\sqrt{x_1}}.$$

  So, $(m/p_1, 0)$ is a solution only when $4p_1 m < p_2^2$.

We conclude that the demand function x is

$$x(p,m) = \begin{cases} \left(\frac{p_2^2}{4p_1^2}, \frac{m}{p_2} - \frac{p_2}{4p_1}\right) & \text{if } 4p_1 m \geq p_2^2 \\ \left(\frac{m}{p_1}, 0\right) & \text{if } 4p_1 m < p_2^2 \end{cases}.$$

The indirect utility function is

$$v(p,m) = \begin{cases} \frac{m}{p_2} + \frac{p_2}{4p_1} & \text{if } 4p_1 m \geq p_2^2 \\ \sqrt{\frac{m}{p_1}} & \text{if } 4p_1 m < p_2^2 \end{cases}.$$

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5. JR Exercise 1.47

(a) **Soln:** To avoid confusion between the function u and the its value u, let's use U for the latter. Hence, the expenditure function is

$$\begin{aligned} e(p,U) \quad : &= \min_{x \geq 0} p \cdot x \text{ s.t. } u(x) \geq U \\ &= \min_{x \geq 0} p \cdot x \text{ s.t. } \frac{1}{U} u(x) \geq 1 \\ &= \min_{x \geq 0} p \cdot x \text{ s.t. } u\left(\frac{1}{U} x\right) \geq 1 \text{ (homogeneity of degree 1)} \\ &= \min_{x \geq 0} p \cdot (Uz) \text{ s.t. } u(z) \geq 1 \left(\text{change variables from } x \text{ to } z := \frac{1}{U} x\right) \\ &= U \min_{z \geq 0} p \cdot z \text{ s.t. } u(z) \geq 1 \\ &= U e(p,1), \end{aligned}$$

as desired. ■

(b) **Soln:** From Theorem 1.8 in the textbook, we know that for a fixed p, the functions $e(p,\cdot)$ and $v(p,\cdot)$ are inverses of each other, so that $y = e(p, v(p,y))$. Thus, from (a) we have

$$y = v(p,y) e(p,1),$$

which can be solved trivially for $v(p,y)$:

$$v(p,y) = \frac{y}{e(p,1)}.$$

Thus, the marginal utility of income is $v_y(p,y) = 1/e(p,1)$, which indeed does not depend on y. ■

6. JR Exercise 1.54

(a) Derive the Marshallian demand functions.

Soln: The UMP is

$$\max_{x \geq 0} A \prod_{i=1}^{n} x_i^{\alpha_i} \text{such that} \sum_{i=1}^{n} p_i x_i \leq y.$$

The associated Lagrangian is

$$\mathcal{L}(x, \lambda) = A \prod_{i=1}^{n} x_i^{\alpha_i} + \lambda \left\{ y - \sum_{i=1}^{n} p_i x_i \right\}.$$

The KT FOC for x_j and λ are

$$A\alpha_j x_j^{\alpha_j - 1} \prod_{i \neq j} x_i^{\alpha_i} - \lambda p_j \leq 0 \ (= \text{ if } x_j > 0), \tag{7}$$

$$y - \sum_i p_i x_i \geq 0 \ (= \text{ if } \lambda > 0). \tag{8}$$

Assuming each $\alpha_j > 0$ (which the problem statement should have said), we have $\alpha_j - 1 < 0$. Thus, (7) implies that in any solution, each x_j is positive, (7) holds as as an equality (it binds), and λ is also positive. The latter implies that the budget equation (8) also binds.
Fix j, and divide (7) for an arbitriary i by (7) for j to obtain

$$\frac{\alpha_j x_i}{\alpha_i x_j} = \frac{p_j}{p_i}.$$

Hence, any x_i can be expressed in terms of x_j :

$$x_i = \frac{p_j \alpha_i}{p_i \alpha_j} x_j.$$

Substituting this into the budget constraint yields

$$\begin{aligned} y &= \sum_i p_i \frac{p_j \alpha_i}{p_i \alpha_j} x_j \\ &= \frac{p_j x_j}{\alpha_j} \left(\sum_i \alpha_i \right) = \frac{p_j x_j}{\alpha_j}. \end{aligned}$$

Solving this for x_i yields the optimal amount of good i, the consumer's Marshallian demand for it:

$$x_i^*(p, y) = \frac{\alpha_i y}{p_i}.$$

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(b) Derive the indirect utility function.

Soln: Plugging the Marshallian demand function from (a) into the utility function yields

$$\begin{aligned} v(p, y) &= A \prod_{i=1}^{n} \left(\frac{\alpha_i}{p_i} y \right)^{\alpha_i} \\ &= Ay \prod_{i=1}^{n} \left(\frac{\alpha_i}{p_i} \right)^{\alpha_i}. \end{aligned}$$

■

(c) Compute the expenditure function.

Soln: Use Theorem 1.8 in the textbook. For a fixed p and U,

$$\begin{aligned} U &= v(p, e(p,U)) \\ &= Ae(p,U) \prod_{i=1}^{n} \left(\frac{\alpha_i}{p_i}\right)^{\alpha_i}, \end{aligned}$$

and thus

$$e(p,U) = \frac{U}{A} \prod_{i=1}^{n} \left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}.$$

■

(d) Compute the Hicksian demands.

Soln: Apply Shepard's lemma to the expenditure function found in (c):

$$\begin{aligned} x_i^h(p,U) &= \frac{\partial e(p,U)}{\partial p_i} \\ &= \frac{U}{A} \left[\prod_{j \neq i} \left(\frac{p_j}{\alpha_j}\right)^{\alpha_j}\right] \cdot \left(\frac{p_i}{\alpha_i}\right)^{\alpha_i - 1}. \end{aligned}$$

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