700 Prelim Questions
University of Pennsylvania
Steven A. Matthews

1. (Spring 2012) (20 pts) A strictly increasing utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ gives rise to a demand function $\mathbf{x}(\mathbf{p}, y) = (x_1(\mathbf{p}, y), \ldots, x_n(\mathbf{p}, y))$ defined on $\mathbb{R}^{n+1}_{++}$. Assume it and any other functions you use to answer this question are twice continuously differentiable.

   (a) (8 pts) State all the properties this demand function must necessarily satisfy.

   (b) (12 pts) For each property you listed in (a), sketch a proof of why it must be satisfied.

2. (Spring 2012) (20 pts) A consumer lives for two periods. In period 2 she will purchase a commodity bundle $x = (x_1, x_2)$ to maximize her utility $u(x) = x_1^\alpha x_2^\alpha$ subject to her budget constraint $p_1x_1 + p_2x_2 \le y$. These prices are fixed positive constants, known even in period 1.

   In period 1 the consumer invests in a risky asset that returns $(1+\tilde{r})z$ in period 2 if she invests an amount z. Her wealth is $w > 0$, and she is restricted to choosing $z \in [0, w]$. Her income in period 2 when she invests z will thus be the random variable $\tilde{y} = w + \tilde{r}z$. The asset has a positive expected return: $\mathbb{E}\tilde{r} > 0$. She chooses z to maximize the expected utility she will ultimately obtain in period 2.

   For each $\alpha > 0$, determine whether the consumer's optimal investment in the asset, $z^*(p, w)$, is decreasing, constant, or increasing in w.

3. (Fall 2012) A competitive firm uses two inputs, x_1 and x_2, to make one output, y. Consider the following possible cost function, where the exponents a and b are positive constants:

$$c(y, \mathbf{w}) = \left(\tfrac{1}{2}w_1^a + \tfrac{1}{2}w_2^b + \sqrt{w_1 w_2}\right) y.$$

   (a) (5 pts) For what values of a and b is c truly a cost function? Prove your answer.

   For the remaining questions, assume a and b satisfy the restrictions you just identified.

   (b) (5 pts) Find the firm's conditional factor demand functions, $\hat{x}_1(y, \mathbf{w})$ and $\hat{x}_2(y, \mathbf{w})$.

   (c) (5 pts) Find the firm's supply function, $y^*(p, \mathbf{w})$.

   (d) (5 pts) Find a production function f for which c is the corresponding cost function.

4. (Fall 2012) Mr. A has a complete and transitive preference ordering $\succeq_A$ over monetary lotteries $\tilde{x}$ that is monotone in the following sense: $\delta_x \succ_A \delta_y$ for all $x > y$, where δ_x and δ_y are the degenerate lotteries that put probability one on the amounts x and y, respectively.

   (a) (1 pt) Define what it means for Mr. A to be strictly risk averse.

   (b) (2 pts) Define Mr. A's certainty equivalent c_A for a given non-degenerate lottery $\tilde{x}$.

   (c) (3 pts) Assume Mr. A is strictly risk averse. For the $\tilde{x}$ and c_A from (b), what can you say about the relationship between c_A and $\mathbb{E}\tilde{x}$? Prove your answer.

   Now assume Mr. A has a Bernoulli utility function, u_A. Ms. B also has a Bernoulli utility function, u_B. Both functions are twice differentiable, with $u'_A(x) > 0$ and $u'_B(x) > 0$ for all $x \in \mathbb{R}$.

(d) (1 pts) Define their Arrow-Pratt coefficients of absolute risk aversion, $R_i(x)$ for $i = A, B$.

(e) (13 pts) Assume Mr. A is more risk averse than Ms. B, in the sense that $R_A(x) > R_B(x)$ for all $x \in \mathbb{R}$. Letting $\tilde{x}$ be a non-degenerate lottery, and c_A and c_B be their certainty equivalents for it, prove that $c_A < c_B$.

5. (June 2013) A competitive firm uses two inputs to produce one output according to a strictly increasing production function, $y = f(z_1, z_2)$. The input prices, w_1 and w_2, are constant in this problem, and hence we simplify notation by not writing them as arguments of functions.

   In the "long-run," the firm chooses z_1 and z_2 to maximize profit. Assume this gives rise to C^2 input demand and supply functions, $z_1^L(p)$, $z_2^L(p)$, and $y^L(p)$, defined on $\mathbb{R}_{++}$.

   In the "short-run," the firm chooses only z_2 to maximize profit, because the first input is fixed at some level $\bar{z}_1 > 0$. Assume this gives rise to a C^2 supply function, $y^S(\cdot, \bar{z}_1)$, defined on $\mathbb{R}_{++}$.Suppose $\bar{p} > 0$ is an output price such that $z_1^L(\bar{p}) = \bar{z}_1$. Show that at $\bar{p}$, the long-run and short-run supply functions specify the same output. Show also that at $\bar{p}$, the price elasticity of supply is larger, at least weakly, for the long-run supply curve than it is for the short-run supply curve.

6. (June 2013) An investor can invest any fracton $\alpha \in [0, 1]$ of her initial wealth $w > 0$ in a risky asset. Her random wealth when she invests αw is

$$\tilde{y} = (1 + \tilde{r}\alpha)w,$$

   where $\tilde{r}$ is the return of the asset. Assume the random variable $\tilde{r}$ is not degenerate, and $\mathbb{E}\tilde{r} > 0$. The investor's Bernoulli utility u satisfies $u' > 0$ and $u'' < 0$. Let $\alpha^*(w)$ be the investor's expected utility maximizing α. Assume that for any w of interest in this problem, $\alpha^*(w) < 1$. (In words, the asset is not so surely profitable that the investor would choose to invest all her wealth in it.)

   Kenneth Arrow claimed, as an empirical matter, that as an investor becomes wealthier, she invests a smaller proportion of her wealth in risky assets. Give a sufficient condition for this to be true, in this problem, in terms of the function that measures the investor's *relative risk aversion:*

$$R(y) := -\frac{yu''(y)}{u'(y)}.$$

7. (August 2013) (20 pts) A consumer has wealth w that she must consume over two periods. In period 2 she has a random income shock, $\theta\tilde{y}$, where $\theta > 0$, $\mathbb{E}\tilde{y} = 0$, and $\mathbb{E}\tilde{y}^2 > 0$. Her expected utility when she chooses to save an amount x is

$$u(w - x) + \mathbb{E}v(x + \theta\tilde{y}).$$

   She can save any amount, i.e., x can be any real number. The functions u and v are C^3, with strictly positive first derivatives and strictly negative second derivatives. Let $x^*(w, \theta)$ denote her optimal savings function.

   (a) (5 pts) Does x^* increase or decrease in w, or can it do either depending on the utility functions? Prove your answer.

   (b) (15 pts) Show that x^* strictly increases in θ if v exhibits nonincreasing absolute risk aversion (NIARA).

8. (August 2013) (20 pts) Let $Y \subseteq \mathbb{R}^N$ denote a production set and $\pi : \mathbb{R}^N_+ \to \mathbb{R}$ a profit function.

   (a) (5 pts) Suppose you are given the set Y. Show how the associated profit function π can be derived.

   (b) (15 pts) Suppose instead that you are given the function π, and told that it is the profit function asociated with a closed and convex Y that satisfies free disposal. Show how Y can be derived, and prove that your method works.