Economics 681 **Steven A. Matthews**
Fall 2017 **University of Pennsylvania**

Suggested Solutions to Problem Set 4
Today's Date: November 3, 2017

1. JR Exercise 2.19

   Soln: Suppose that for a gamble $g \in \mathcal{G}$ and $\alpha, \beta \in [0, 1]$,

$$(\alpha \circ a_1, (1-\alpha) \circ a_n) \sim g \sim (\beta \circ a_1, (1-\beta) \circ a_n).$$

   This and transitivity imply $(\alpha \circ a_1, (1-\alpha) \circ a_n) \sim (\beta \circ a_1, (1-\beta) \circ a_n)$. Hence,

$$(\alpha \circ a_1, (1-\alpha) \circ a_n) \succsim (\beta \circ a_1, (1-\beta) \circ a_n),$$

   and so Axiom G4 implies $\alpha \geq \beta$. We also have

$$(\beta \circ a_1, (1-\beta) \circ a_n) \succsim (\alpha \circ a_1, (1-\alpha) \circ a_n),$$

   and so Axiom G4 implies $\beta \geq \alpha$. Hence, $\alpha = \beta$. ∎

2. JR Exercise 2.24

   Soln: Let $y_n(x) = w_0 - \rho x$ and $y_l(x) = w_0 - \rho x - L + x$. The individual chooses her coverage x to maximize

$$\alpha u(y_l(x)) + (1-\alpha) u(y_n(x)).$$

   The solution, x^*, satisfies the FOC

$$\alpha u'(y_l(x^*))(1-\rho) - (1-\alpha) u'(y_n(x^*))\rho = 0.$$

   Rearranging terms yields

$$\frac{1-\alpha}{\alpha} \frac{u'(y_n(x^*))}{u'(y_l(x^*))} = \frac{1-\rho}{\rho} < \frac{1-\alpha}{\alpha},$$

   where the inequality follows from $\rho > \alpha$. Hence, $u'(y_n(x^*)) < u'(y_l(x^*))$. This and $u'' < 0$ imply $y_n(x^*) > y_l(x^*)$, or rather,

$$w_0 - \rho x^* > w_0 - \rho x^* - L + x^*.$$

   This simplifies to $x^* < L$; the individual less than fully insures when $\rho > \alpha$. ∎

3. JR Exercise 2.26, restricting attention to values of w and b satisfying $w < b$.

   Soln: For any $w < b$ we have

$$u'(w) = c(b-w)^{c-1} > 0 \;\Leftrightarrow\; c > 0.$$

   Given that c is positive, for any $w < b$ we have

$$u''(w) = -c(c-1)(b-w)^{c-2} < 0 \;\Leftrightarrow\; c > 1.$$

   Hence, u is strictly increasing and strictly concave on the interval $(-\infty, b)$ iff $c > 1$.

The Arrow-Pratt measure (coefficient) of absolute risk aversion is

$$R_a(w) = -\frac{u''(w)}{u'(w)} = -\frac{-c(c-1)(b-w)^{c-2}}{c(b-w)^{c-1}} = \frac{c-1}{b-w}.$$

Hence, $c > 1$ implies $u(w)$ displays increasing absolute risk aversion:

$$R_a'(w) = (c-1)(b-w)^{-2} > 0.$$

■

4. JR Exercise 2.27

   Soln: Note that $u(w)$ is defined only for $w > 0$. The corresponding measure of absolute risk aversion is

   $$R_a(w) = -\frac{u''(w)}{u'(w)} = -\frac{-\beta/w^2}{\beta/w} = \frac{1}{w},$$

   which indeed satisfies $R_a'(w) = -1/w^2 < 0$ for all $w > 0$. ■

5. JR Exercise 2.36

   Soln: For this problem to make sense, the two individuals must have the same initial wealth, say w. So the gambles of interest are of the form

   $$g_s = (s \circ (w+h), (1-s) \circ (w-h)),$$

   where s is the probability of winning. So $s \in S^i$ iff

   $$u_i(g_s) = s u_i(w+h) + (1-s) u_i(w-h) \geq u_i(w),$$

   or rather,

   $$s \geq \frac{u_i(w) - u_i(w-h)}{u_i(w+h) - u_i(w-h)} =: s_i.$$

   And similarly for j. We thus have $S^i = [s_i, 1]$ and $S^j = [s_j, 1]$.

   Now, note that since $u_j(g_{s_j}) = u_j(w)$, we have $c(g_{s_j}, u_j) = w$. By the strict version of Pratt's theorem, $c(g_{s_j}, u_i) < c(g_{s_j}, u_j)$ because $R_a^i(\cdot) > R_a^j(\cdot)$. It follows that $c(g_{s_j}, u_i) < w$, and so $u_i(g_{s_j}) < u(w)$. This shows that $s_j \notin S^i$, and so $s_j < s_i$. Consequently, $S^i \subset S^j$. ■

6. (Note, I've changed the \$400 to \$300.01 in this problem to make it more striking.) My vNM utility function is strictly increasing and satisfies $u(0) = 0$, $u(\$300) = \frac{1}{2}$, and $\lim_{w\to\infty} u(w) = 1$. Consider a gamble $g = (\frac{1}{2} \circ 0, \frac{1}{2} \circ x)$, where x is a prize in dollars. How large must x be in order for me to prefer this gamble to one in which I receive \$300.01 for sure?

   Soln: We have

   $$u(g) = \tfrac{1}{2}u(0) + \tfrac{1}{2}u(x) = \tfrac{1}{2}u(x).$$

   In order for you to prefer g to receiving \$300.01 for sure, x must satisfy $\frac{1}{2}u(x) > u(300.01)$. But this would imply

   $$u(x) > 2u(300.01) > 2u(300) > 1.$$

   There is no such $x < \infty$, since $\lim_{w\to\infty} u(w) = 1$. So no matter how large x is, you will prefer receiving \$300.01 for sure over the gamble g! ■

7. A consumer may invest in a risky asset that has a random *gross* return $\tilde{r}$, with $\mathbb{E}\tilde{r} = 1$. Her expected utility when she invests x in the asset is

$$\mathbb{E}u(w + \tilde{r}x - x).$$

Show, without using calculus, that if the consumer is risk averse, she will not invest in the asset.

Soln: By Jensen's inequality:

$$\mathbb{E}u(w + \tilde{r}x - x) \leq u(\mathbb{E}(w + \tilde{r}x - x)) = u(w).$$

However, at $x = 0$ we obtain

$$\mathbb{E}u(w + \tilde{r}x - x) = u(w).$$

This proves that $x = 0$ maximizes $\mathbb{E}u(w + \tilde{r}x - x)$ on $\mathbb{R}$. ■

8. When the consumer has non-random wealth w, define his risk premium, $\pi(w)$, for a gamble $\tilde{x}$ by

$$\mathbb{E}u(\tilde{x} + w) = u\left(\mathbb{E}\tilde{x} + w - \pi(w)\right).$$

Thus, the consumer is willing to pay at most $\pi(w)$ to exchange the gamble for its expected value $\mathbb{E}\tilde{x}$. Assume u is C^2, with $u' > 0$ and $u'' < 0$. Show that if u exhibits DARA, then the risk premium decreases in wealth.

Soln: For any w, define a utility function u_w by $u_w(x) := u(x + w)$. Now fix w and $w' > w$. Then by DARA, u_w is strictly more risk averse than $u_{w'}$. Note that the definition of $\pi(w)$ can be written as

$$\mathbb{E}u_w(\tilde{x}) = u_w\left(\mathbb{E}\tilde{x} - \pi(w)\right),$$

or rather, $E\tilde{x} - \pi(w)$ is equal to the certainty equivalent $c(\tilde{x}, u_w)$. By the same observation we have $E\tilde{x} - \pi(w') = c(\tilde{x}, u_{w'})$. Since u_w is strictly more risk averse than $u_{w'}$, Pratt's Theorem implies $c(\tilde{x}, u_w) < c(\tilde{x}, u_{w'})$, or rather,

$$E\tilde{x} - \pi(w) < E\tilde{x} - \pi(w').$$

This yields our conclusion, $\pi(w) > \pi(w')$. ■

9. A consumer has wealth w that she must consume over two periods. The only way to transfer wealth to or from period 2 is through buying or selling a risky asset with returns $\theta\tilde{r}$, where $\theta > 0$, $\mathbb{E}\tilde{r} = 0$, and $\mathbb{E}\tilde{r}^2 > 0$. Her expected utility when she chooses to save an amount x is

$$u(w - x) + \mathbb{E}v(\theta\tilde{r}x).$$

Suppose x can be any real number, and that u and v are C^2 with strictly positive first derivatives and strictly negative second derivatives. Let $x^* = x^*(w, \theta)$ be her optimal savings function.

(a) Does x^* increase or decrease in w, or can it do either?

Soln: The first order condition is

$$u'(w-x) = \mathbb{E}\theta\tilde{r}v'(\theta\tilde{r}x).$$

This must hold at the optimum. Differentiating the first-order condition with respect to w yields

$$x'(w) = \frac{u''(w-x)}{u''(w-x) + \mathbb{E}\theta^2\tilde{r}^2 v''(\theta\tilde{r}x)}.$$

The numerator and denominator are both negative, so $x'(w) > 0$. ■

(b) Is x^* always positive, always negative or neither?

Soln: x^* is always negative. At $x = 0$, the derivative of the objective function is

$$\mathbb{E}\theta\tilde{r}v'(0) - u'(w) = -u'(w) < 0.$$

Since the objective function is strictly concave in x, the critical point x^* must be negative. ■

(c) Sign the derivative x^*_θ.

Soln: $x^*_\theta > 0$. To see why, differentiate the first-order condition with respect to θ to obtain

$$x^*_\theta = -\frac{\mathbb{E}\tilde{r}v'(\tilde{y}) + \ \mathbb{E}\theta\tilde{r}^2 x v''(\tilde{y})}{u''(w-x) + \mathbb{E}\theta^2\tilde{r}^2 v''(\theta\tilde{r}x)} =: -\frac{N}{D}.$$

where $\tilde{y} = \theta\tilde{r}x$. Observe that $D < 0$.So the sign of x^*_θ is the same as that of N. The second term of N is $\mathbb{E}\theta\tilde{r}^2 x v''(\tilde{y})$, which is positive because it is the expectation of a positive function (as $x < 0$ and $v'' < 0$). The first term of N can be written as

$$\begin{aligned} \mathbb{E}\tilde{r}v'(\tilde{y}) &= \mathbb{E}\tilde{r}\left[v'(\tilde{y}) - v'(0)\right] \\ &= \mathbb{E}\tilde{r}\left[v'(\theta\tilde{r}x) - v'(0)\right]. \end{aligned}$$

Since $\theta x < 0$ and $v'' < 0$, we see that $r\left[v'(\theta r x) - v'(0)\right] > 0$ for all $r \neq 0$. We conclude that $N > 0$, and so $x^*_\theta > 0$. ■