Introduction to a General Theory of Elementary Propositions

Author(s): Emil L. Post

Source: *American Journal of Mathematics*, Jul., 1921, Vol. 43, No. 3 (Jul., 1921), pp. 163-185

Published by: The Johns Hopkins University Press

Stable URL: https://www.jstor.org/stable/2370324

INTRODUCTION TO A GENERAL THEORY OF ELEMENTARY PROPOSITIONS.

By Emil L. Post.

Introduction.

In the general theory of logic built up by Whitehead and Russell* to furnish a basis for all mathematics there is a certain subtheory† which is unique in its simplicity and precision; and though all other portions of the work have their roots in this subtheory, it itself is completely independent of them. Whereas the complete theory requires for the enunciation of its propositions real and apparent variables, which represent both individuals and propositional functions of different kinds, and as a result necessitates the introduction of the cumbersome theory of types, this subtheory uses only real variables, and these real variables represent but one kind of entity which the authors have chosen to call elementary propositions. The most general statements are formed by merely combining these variables by means of the two primitive propositional functions of propositions Negation and Disjunction; and the entire theory is concerned with the process of asserting those combinations which it regards as true propositions, employing for this purpose a few general rules which tell how to assert new combinations from old, and a certain number of primitive assertions from which to begin.

This theory in a somewhat different form has long been the subject matter of symbolic logic.‡ However, although it had reached a high state of development as a theory of classes, it had this incurable defect as a logic of propositions, that it used informally in its proofs the very propositions whose formal statements it tried to prove. This defect appears to be entirely overcome in the development of 'Principia.' But owing to the particular purpose the authors had in view they decided not to burden their work with more than was absolutely necessary for its achievement, and so gave up the generality of outlook which characterized symbolic logic.

It is with the recovery of this generality that the first portion of our paper deals. We here wish to emphasize that the theorems of this paper

* A. N. Whitehead and B. Russell, *Principia Mathematica*, Vol. 1, 1910; Vol. 2, 1912; Vol. 3, 1913. Camb. Univ. Press.

† *Ibid.*, Vol. 1, part 1, section A.

‡ See C. I. Lewis, "A Survey of Symbolic Logic," University of California Press, 1918. An extensive bibliography is given there.

are *about* the logic of propositions but are *not included* therein. More particularly, whereas the propositions of 'Principia' are *particular* assertions introduced for their interest and usefulness in later portions of the work, those of the present paper are about the set of *all* such possible assertions. Our most important theorem gives a uniform method for testing the truth of any proposition of the system; and by means of this theorem it becomes possible to exhibit certain general relations which exist between these propositions. These relations definitely show that the postulates of 'Principia' are capable of developing the complete system of the logic of propositions without ever introducing results extraneous to that system—a conclusion that could hardly have been arrived at by the particular processes used in that work.

Further development suggests itself in two directions. On the one hand this general procedure might be extended to other portions of 'Principia,' and we hope at some future time to present the beginning of such an attempt. On the other hand we might take cognizance of the fact that the system of 'Principia' is but one particular development of the theory—particular in the primitive functions it employs and in the postulates it imposes on those functions—and so might construct a general theory of such developments. This we have tried to do in the other portions of the paper. Our first generalization leads to systems which are essentially equivalent to that of 'Principia' and connects up with the work of Sheffer* and Nicod† in reducing the number of primitive functions and of primitive propositions respectively. The second generalization, on the other hand, while including the first also seems to introduce essentially new systems. One class of such systems, and we study these in detail, seems to have the same relation to ordinary logic that geometry in a space of an arbitrary number of dimensions has to the geometry of Euclid. Whether these "non-Aristotelian" logics and the general development which includes them will have a direct application we do not know; but we believe that inasmuch as the theory of elementary propositions is at the base of the complete system of 'Principia,' this broadened outlook upon the theory will serve to prepare us for a similar analysis of that complete system, and so ultimately of mathematics.

Finally a word must be said about the viewpoint that is adopted in this paper and the method that is used. We have consistently regarded the system of 'Principia' and the generalizations thereof as purely *formal de-*

* H. M. Sheffer, "A Set of Five Independent Postulates for Boolean Algebras, with Applications to Logical Constants," *Trans. Amer. Math. Soc.*, 14 (1913), pp. 481–88.

† J. G. P. Nicod, "A Reduction in the Number of the Primitive Propositions of Logic," *Proc. Camb. Phil. Soc.*, Vol. XIX, Jan., 1917.

*velopments,** and so have used whatever instruments of logic or mathematics we found useful for a study of these developments. The fact that one of the interpretations of the system of 'Principia' is part of the informal logic we have used in this study makes the full significance of this *interpretation,* at least with regard to proofs of consistency, uncertain, but it in no way affects the actual content of the paper which is in connection with the *formal systems.*

THE SYSTEM OF PRINCIPIA MATHEMATICA.

1. Description of the System.—Let $p, p_1, p_2, \cdots, q, q_1, q_2, \cdots, r, r_1, r_2, \cdots$ arbitrarily represent the variable elementary propositions mentioned in the introduction. Then by means of the two primitive functions $\sim p$ (read not p—the function of Negation) and $p \vee q$ (p or q—the function of Disjunction) with the aid of the primitive propositions

I. If p is an elementary proposition $\sim p$ is an elementary proposition,
If p and q are elementary propositions $p \vee q$ is an elementary proposition,

we combine these variables to form the various propositions or rather ambiguous values of propositional functions of the system. It is desirable in what follows to have before us the vision of the totality of these functions streaming out from the unmodified variable p through forms of ever growing complexity to form the infinite triangular array

$$
\begin{array}{c}
p \\
p \vee p, \qquad p_1 \vee p_2, \qquad \sim p \\
p \vee \sim p, \qquad \cdots, \qquad \sim p_1 \vee \sim p_2, \qquad \cdots, \qquad (p_1 \vee p_2) \vee (p_3 \vee p_4), \\
\sim (p_1 \vee p_2), \qquad \sim (p \vee p), \qquad \sim \sim p \\
\cdot \quad \cdot
\end{array}
$$

and to note and remember that this array of functions formed merely through combining p's by $\sim$'s and $\vee$'s constitutes the entire set of enunciations it is possible to make in the theory of elementary propositions of 'Principia.'

But the actual theory is concerned with the assertion of a certain subset of these functions. We denote the assertion of a function by writing $\vdash$ before it. Then the motive power for the resulting process of deduction is furnished by the two rules of operation:

---

* For a general statement of this viewpoint see C. I. Lewis, *Loc. Cit.,* Chapter VI, section III.

II. The assertion of a function involving a variable p produces the assertion of any function found from the given one by substituting for p any other variable q, or $\sim q$, or $(q \vee r)$.*

III. "$\vdash P$" and "$\vdash : \sim P . \vee . Q$" produce "$\vdash Q$."
These enable us to assert new functions from old, or rather in the form in which we have put them, generate new assertions from old. And the complete set of assertions is produced by applying II and III both to the following assertions which give us the start, and to all derived assertions that may result:

IV. $\vdash : \sim (p \vee p) . \vee . p, \quad \vdash : \sim [p \vee (q \vee r)] . \vee . q \vee (p \vee r),$

$\vdash : \sim q . \vee . p \vee q, \quad \vdash :. \sim (\sim q \vee r) . \vee : \sim (p \vee q) . \vee . p \vee r,$

$\vdash : \sim (p \vee q) . \vee . q \vee p.$

We here again point out what was emphasized in the introduction that this theory concerns itself exclusively with the production of particular assertions through the detailed use of the rules of operation upon the primitive assertions, and as a consequence the set of theorems of this portion of 'Principia' consists of the assertions of a certain number of particular functions of the above infinite set.†

2. Truth-Table Development.‡—Let us denote the truth-value of any proposition p by $+$ if it is true and by $-$ if it is false. This meaning of $+$ and $-$ is convenient to bear in mind as a guide to thought, but in the actual development that follows they are to be considered merely as symbols which we manipulate in a certain way. Then if we attach these two primitive truth-tables to $\sim$ and $\vee$

* This operation is not explicitly stated in 'Principia' but is pointed out to be necessary by B. Russell in his "Introduction to Mathematical Philosophy," London, 1919, p. 151. Its particular form was suggested to us by the first portion of the operation of "Substitution" given by Lewis, *loc. cit.*, p. 295. It will be noticed that the effect of II is to enable us to substitute any function of the system for a variable of an asserted function.

† We have consistently ignored the idea of definition in this description. We here rigorously follow the authors in saying that definition is a convenience but not a necessity and so need not be considered part of the theoretical development. And so although we too shall at times use its shorthand, we do not encumber our theoretical survey with it.

‡ Truth-values, truth-functions and our primitive truth-tables are described in 'Principia,' Vol. 1, p. 8 and p. 120, but the general notion of truth-table is not introduced. This notion is quite precise with Jevons and Venn (see Lewis, *loc. citus*, p. 74 and pp. 175 et seq. respectively) and has its foundation in the formula for the expansion of logical functions first given by Boole. (G. Boole, "An Investigation of the Laws of Thought," London, Walton, 1854, especially pp. 72–76.) For the relation to Schröder see the footnote to section 3.

p	$\sim p$
$+$	$-$
$-$	$+$

p, q	$p \vee q$
$+ +$	$+$
$+ -$	$+$
$- +$	$+$
$- -$	$-$

we have a means of calculating the truth-values of $\sim p$ and $p \vee q$ from those of their arguments. Now consider any function $f(p_1, p_2, \cdots p_n)$ in our system of functions, which we will designate by F. Then since f is built up of combinations of $\sim$'s and $\vee$'s, if we assign any particular set of truth-values to the p's, successive application of the above two primitive tables will enable us to calculate the corresponding truth-value of f. So corresponding to each of the 2^n possible truth-configurations of the p's a definite truth-value of f is determined. The relation thus effected we shall call the truth-table of f.

For example consider the function

$$\sim (\sim (\sim p \vee q) \vee \sim (\sim q \vee p))$$

which is the ultimate definition of the function $p \equiv q$ of Principia. We have when p is $+$ and q is $+$ the following truth-values of the successive components of the function and so finally of the function:

$$p : +, \quad \sim p : -, \quad \sim p \vee q : +, \quad \sim (\sim p \vee q) : -$$

$$q : +, \quad \sim q : -, \quad \sim q \vee p : +, \quad \sim (\sim q \vee p) : -$$

$$\sim (\sim p \vee q) \vee \sim (\sim q \vee p) : -, \quad \sim (\sim (\sim p \vee q) \vee \sim (\sim q \vee p)) : +$$

the successive truth-values being found by direct application of the primitive tables. In the same way the truth-values for $p +$, $q -$ etc. can be calculated and so we finally get the truth-table of $p \equiv q$, i.e.,

p, q	$p \equiv q$
$+ +$	$+$
$+ -$	$-$
$- +$	$-$
$- -$	$+$

It is needless to say that in actual work this amount of detail is quite unnecessary.

We shall call the number of variables which appear in a function the order of that function as well as that of its truth-table. It is evident that there are 2^{2^n} tables of the nth order. We now prove the

THEOREM. *To every truth-table of whatever order there corresponds at least one function of F which has it for its truth-table.*

For first corresponding to the four tables of the first order $\begin{smallmatrix}+\\-\end{smallmatrix}\big|\begin{smallmatrix}+\\-\end{smallmatrix}$, $\begin{smallmatrix}+\\-\end{smallmatrix}\big|\begin{smallmatrix}+\\+\end{smallmatrix}$, $\begin{smallmatrix}+\\-\end{smallmatrix}\big|\begin{smallmatrix}-\\-\end{smallmatrix}$, $\begin{smallmatrix}+\\-\end{smallmatrix}\big|\begin{smallmatrix}-\\+\end{smallmatrix}$ we have the functions $p \vee p$, $p \vee \sim p$, $\sim (p \vee \sim p)$, $\sim p$. Now assume there is a function for each mth order table. Then in any table of order $m + 1$ the configurations for which p_{m+1} is $+$ constitute an mth order table for which there is some function $f_1(p_1, p_2, \cdots p_m)$. Likewise corresponding to p_{m+1} $-$ we obtain $f_2(p_1, p_2, \cdots p_m)$. Let $p \,.\, q$ stand for $\sim (\sim p \vee \sim q)$ a function which has the truth-table

p, q	$p \cdot q$
$+ \; +$	$+$
$+ \; -$	$-$
$- \; +$	$-$
$- \; -$	$-$

Then it easily follows that the function

$$p_{m+1} \cdot f_1(p_1, p_2, \cdots p_m) \,.\, \vee \,.\, \sim p_{m+1} \cdot f_2(p_1, p_2, \cdots p_m)$$

has for its truth-table the given $m + 1$st order table.

The functions of F can then be classified according to their tables as follows: those which have all their truth-values $+$, all $-$, or some $+$ and some $-$. We shall call these functions respectively positive, negative, and mixed. This classification is of great importance in connection with the process of substitution which is so fundamental in the postulational development. We shall say that any function obtained from another by the process of substitution is contained in that function. We then have the

THEOREM. *Every function contained in a positive function is positive; every function contained in a negative function is negative; every mixed function contains at least one function for every possible truth-table.*

The first two results are immediate. In the third case note that any mixed function $f(p_1, p_2, \cdots p_n)$ has at least one configuration which yields $+$ and one which yields $-$. Let the truth-value of p_i in the positive configuration be denoted by t_i and in the negative by t'_ι, and construct a function $\phi_\iota(p)$ with the truth-table

p	$\phi_\iota(p)$
$+$	t_ι
$-$	t'_ι

.

Then $\psi(p) = f(\phi_1(p), \phi_2(p), \cdots \phi_n(p))$ will be $+$ when p is $+$ and $-$ when p is $-$. But by our first theorem there is at least one function $g(q_1, q_2, \cdots q_m)$ corresponding to any table of order m. Hence $\psi[g(q_1, q_2, \cdots q_m)]$ is a function contained in $f(p_1, p_2, \cdots p_n)$ corresponding to that table.

COROLLARY. *Every mixed function contains at least one positive function and one negative function.*

3. The Fundamental Theorem.*—*A necessary and sufficient condition that a function of F be asserted as a result of the postulates II, III, IV is that all its truth-values be +.*

Note first that each of the primitive assertions of IV is a positive function. Furthermore from the assertion of positive functions we can only get positive functions. For the only method we have of producing new assertions from old is through the use of II and III. Now II can only produce positive functions since every function contained in a positive function is positive. As for III, if P is $+$ and Q is $-$, $\sim P \vee Q$ is $-$, so that so long as P is a positive function and $\sim P \vee Q$ is a positive function Q must be positive, so that III can only produce positive functions. Hence every asserted function is positive and we have proved the condition necessary.

In order to prove it also sufficient we give a method for deriving the assertion of any positive function. It will simplify the exposition to introduce the other two defined functions of 'Principia' besides $p \,.\, q$ (p and q) given above, viz.,

$$p \supset q \,.\!=\!.\, \sim p \vee q \quad Df\dagger; \qquad p \equiv q \,.\!=\!.\, p \supset q \,.\, q \supset p \quad Df$$

read "p implies q" and "p is equivalent to q" respectively, and having the tables

p, q	$p \supset q$
$+ +$	$+$
$+ -$	$-$
$- +$	$+$
$- -$	$+$

p, q	$p \equiv q$
$+ +$	$+$
$+ -$	$-$
$- +$	$-$
$- -$	$+$

* The method for testing propositions embodied in this theorem is essentially the same as that given by Schröder for the logical system he has developed. (Ernst Schröder, Vorlesungen über die Algebra Der Logik, Leipzig, Teubner; 2. Bd. 1. Abth, 1891; § 32.) But we believe the range of significance of the proof we have given to be quite different from that of the work of Schröder. For first, as has been emphasized by Lewis (*Loc. cit.*, Chap. IV), formal and informal logic are inextricably bound together in Schröder's development to an extent that prevents the system as a whole from being completely determined. As a result the necessity of the condition of the theorem, which evidently requires such a complete determination if it is to be proved, remains unproved. As for the sufficiency, parts E and C of our proof appear in the proof for the expression of functions given by Schröder. (1. Bd, 1890). Part A, however, seems not to have been given explicitly, while corresponding to part D are all the theoretical difficulties met with in passing from the theory of classes to that of propositions when the development is not strictly formal. Hence the sufficiency of the condition is only incompletely proved. The theorem as given by Schröder is therefore of only partial significance even in his own system; and when transplanted to the system of *Principia* requires independent proof. Finally we may mention that the applications we have made of the theorem depend for their significance on those parts of the proof which do not appear, and could not appear, in Schröder.

† III can now be written "$\vdash P$" and "$\vdash P \supset Q$" produce "$\vdash Q$."

It will be noticed that if we have "$\vdash f_1(p_1, \cdots p_n) \equiv f_2(p_1, \cdots p_n)$" this asserted equivalence must have a positive table by the first part of our theorem, and so f_1 and f_2 must have the same truth-values for the same configurations, i.e., they must have the same truth-table.

The proof is most conveniently given in four stages.

A. We prove the theorem $p \equiv q \,.\supset.\, f(p) \equiv f(q)$ where the function f may involve other arguments besides the one indicated and need not involve that. By means of this theorem we shall be able to replace a constituent of a given function by any equivalent function, and have the result equivalent to the given function.

It becomes necessary for the first time to introduce the notion of the rank of a function which we define inductively as follows: the unmodified variable p will be said to be of rank zero, the negative of a function of rank m will be of rank $m+1$; the logical sum of two functions the rank of one of which equals and the other does not exceed m will be of rank $m+1$. Each function of F then is of finite rank as well as of finite order.* Returning now to the theorem we notice that it is true for a function of rank zero since it reduces either to $p \equiv q \,.\supset.\, p \equiv q$ which follows from $p \supset p$† by II, or to $p \equiv q \,.\supset.\, r \equiv r$ which follows from $p \,\supset.\, q \supset p$, $r \equiv r$, III and II. Assume now that the theorem holds for functions of rank m and lower. Then it also holds for functions of rank $m+1$. For if f is of rank $m+1$ it can be written in the form $\sim f_1(p)$, or, $f_2(p) \vee f_3(p)$ where f_1, f_2 and f_3 are at most of rank m; and then the theorem follows by using $p \equiv q \,.\supset.\, \sim p \equiv \sim q$, $p \equiv q \,.\supset:.\, r \equiv s : \supset : p \vee r \,.\equiv.\, q \vee s$ along with $p \supset q : \supset : q \supset r \,.\supset.\, p \supset r$, III and II.

B. Consider now any function $f(p_1, p_2, \cdots p_n)$. Using $\sim (p \vee q) \equiv . \sim p \,.\, \sim q$ and $\sim \sim p \equiv p$ with the aid of the equivalence theorem of A and $p \equiv q : \supset : q \equiv r \,.\supset.\, p \equiv r$ we finally obtain $f(p_1, p_2, \cdots p_n)$ equivalent to a function $f'(p_1, p_2, \cdots p_n)$ which is expressed merely through combinations of p's and $\sim p$'s by $\cdot$'s and $\vee$'s.

C.‡ If we then apply the distributive law of logical multiplication to f', it will be reduced to an equivalent function consisting of successive logical sums of successive logical products of the p's and $\sim p$'s. If any of these products has neither p_n nor $\sim p_n$ as a factor we can introduce them through the propositions $p \vee \sim p$, and $p : \supset : q \,.\equiv.\, p \,.\, q$, whence $q : \equiv : (p \vee \sim p) \,.\, q : \equiv : p \cdot q \,.\vee.\, \sim p \,.\, q$. Now apply the commutative and associative laws

* But whereas the number of functions of given order is infinite those of given rank are finite.

† This as well as all other particular assertions that we use without an indication of proof appear in *Principia*, Vol. I, Part A.

‡ This portion of the proof is essentially that given by A. N. Whitehead in his "Universal Algebra," p. 46. Camb. Univ. Press, 1898.

of logical multiplication along with $p \,.\, p \,. \equiv .\, p$ so that each product has at most one p_i and one $\sim p_i$. Again using the distributive law for purposes of factorization along with the commutative and associative laws of addition we finally obtain f equivalent to

$$f_1(p_1, p_2, \cdots p_{n-1}) \,.\, p_n \,.\sim p_n \colon \vee \colon f_2(p_1, \cdots p_{n-1}) \,.\, p_n \,. \vee .\, f_3(p_1, \cdots p_{n-1}) \,.\sim p_n$$

where one or more of the terms and arguments may not appear.

D. Suppose now that the original function is positive; then this equivalent function will be positive. If in particular it be of first order, it can only be $p \vee \sim p$ or $p \,.\sim .\, p \vee .\, p \vee \sim p$. The first is an asserted function; likewise the second through $p \,.\supset .\, q \vee p$. Hence also $f(p)$ will be asserted through $p \equiv q \,.\supset .\, q \supset p$; and so every positive first order function is asserted. Assume now that this is true for all mth and lower ordered functions and let f be any positive $(m+1)$st order function. The reduced function being then positive, both f_2 and f_3 will be positive, and hence will be asserted. From the use of $p : \supset : q \,.\supset .\, p \equiv q$, $p \,.\, r \,. \vee .\, p \,.\sim r : \equiv : p \;(r \vee \sim r)$, $p : \supset : S \,.\supset .\, p \,.\, S$, and $p \,.\supset .\, q \vee p$, the reduced function will be asserted and so finally f. Hence every positive function can be asserted and so the proof is complete.

We thus see that given any function the theorem gives a direct method for testing whether that function can or cannot be asserted; and if the test shows that the function can be asserted the above proof will give us an actual *method for immediately writing down a formal derivation of its assertion by means of the postulates of Principia.*

Before we pass on to theorems about the system itself irrespective of truth-tables we give the following definitions which apply directly to the system: a true function is one that can be asserted as a result of the postulates, any other is false; a completely false function is a false function such that every function therein contained is false—otherwise we call it incompletely false. We then have the

COROLLARY. *The set of true, completely false, and incompletely false functions is identical with the set of positive, negative, and mixed functions respectively.*

4. Consequences of the Fundamental Theorem.—In the above development the truth-values $+$, $-$ were arbitrary symbols which were found related in certain suggestive ways through the fundamental theorem. We are now in a psoition to give direct definitions of these truth-values in terms of the postulational development. In fact we shall define $+$ to be the set of true functions, $-$ the set of completely false functions. The truth-value of a function will then exist when and only when it is true or completely false, and it will be defined as that class $(+, -)$ of which it is a member. The content of the fundamental theorem consists now of these two theorems:

1. The truth-value of $\sim p$ and $q \vee r$ exists whenever the truth-values of p, q and r exist, and depends only on those truth-values as given by the primitive tables. It therefore follows that the same is true of any function of F, and that the truth-table of such a function can be directly calculated from the primitive tables.

2. The fundamental theorem as stated, or else in the form: if f_1 and f_2 is any pair of positive and negative functions respectively, then a necessary and sufficient condition that a function $f(p_1, p_2, \cdots p_n)$ be asserted is that each of the 2^n contained functions found by substituting f_1 and f_2 for the p's is asserted. It will be noticed that theorem (1) tells us how to determine whether these latter are asserted.

We now pass on to several theorems about the system.

THEOREM. *It is possible to find 2^{2^n} functions of order n such that no two of them are equivalent and such that every other function of order n is equivalent to one of these.*

For we can find 2^{2^n} functions corresponding to the 2^{2^n} different tables of order n. The equivalence of any two of these will then not have a positive table and so will not be asserted. On the other hand any other nth order function will have the same table as one of the 2^{2^n} possible tables, and so the corresponding equivalence will be positive and hence asserted.

THEOREM. *An incompletely false function contains at least one function for each given function which is equivalent to that given function.*

COROLLARY. *An incompletely false function contains at least one true function and one completely false function.*

THEOREM. *The negative of a completely false function is true.*

For a completely false function has a negative truth-table, and so its negative will have a positive table and hence be asserted. It is worth noticing that although this theorem is immediate once we have the fundamental theorem it would be quite difficult without it.

COROLLARY. *Every function of F is either true, or its negative is true, or it contains both a true function and one whose negative is true.*

THEOREM. *The system of elementary propositions of 'Principia' is consistent.*

For if it were inconsistent we would have both a function and its negative asserted. But then both the function and its negative would have to have positive tables whereas if a function has a positive table its negative has a negative table.*

THEOREM. *Every function of the system can either be asserted by means of the postulates or else is inconsistent with them.*

* This argument requires merely the first part of the fundamental theorem which was proved quite simply.

For if a function be not asserted as a result of the postulates it will contain a function whose negative can be so asserted. If then we assert the original function, the contained function will be asserted so that we have asserted both a function and its negative, i.e., we have a contradiction.

COROLLARY. *A function is either asserted as a result of the postulates or else its assertion will bring about the assertion of every possible elementary proposition.*

For by the theorem we would obtain the assertion of both a function and its negative and so by $\sim p . \supset . p \supset q$ the assertion of the unmodified variable q. But q then represents any elementary proposition.

In conclusion let us note that while the fundamental theorem shows that the postulates bring about the assertion of those and only those theorems which should belong to the system, this last theorem enables us to say that they also automatically exclude the very possibility of any added assertions.

GENERALIZATION BY TRUTH-TABLES.

5. General Survey of the Systems Generated.—The system we have studied in the preceding sections is a particular system depending upon the two primitive functions $\sim p$ and $p \vee q$. Two modes of attack have presented themselves. On the one hand we have the original postulational method, on the other the truth-table development. In passing to a general study of systems of the kind discussed these two methods present themselves as instruments of generalization. We reserve the postulational generalization for the next portion of our paper and now take up the truth-table generalization.

To gain complete generality let us assume for our primitives μ arbitrary functions with an arbitrary number of arguments which we will designate by

$$f_1(p_1, p_2, \cdots p_{m_1}), f_2(p_1, p_2, \cdots p_{m_2}), \cdots f_\mu(p_1, p_2, \cdots p_{m_\mu})$$

and let us attach an arbitrary truth-table to each. By successive combinations of these functions with different or repeated arguments we generate the set of derived functions which as before we designate by F. Again each function of F will possess a truth-table in virtue of the tables of the primitive functions of which each is composed. Denote the set of truth-tables thus generated by T. Then whereas in the system of 'Principia' T consists of all possible truth-tables, this will not necessarily be the case here.

In another paper we completely determine all the possible systems T and show that there are 66 *systems that can be generated by tables of third and lower order, and* 8 *infinite families of systems that are generated by the introduction of fourth and higher ordered tables.*

If two systems have the same truth-tables the primitives of each can evidently be expressed in terms of those of the other so that truth-tables are preserved. We can then say that each system has a representation in the other and the two are equivalent. In particular *every truth-system has a representation in the system of Principia* while *every complete system,* i.e., having all possible truth-tables, *is equivalent to it.* In the aforementioned paper we also determine the ways in which a complete system may be generated, and it turns out that one table alone is sufficient to generate it, and it can be either of these two

$$\begin{array}{cc|c} + & + & - \\ + & - & + \\ - & + & + \\ - & - & + \end{array} \qquad \begin{array}{cc|c} + & + & - \\ + & - & - \\ - & + & - \\ - & - & + \end{array}$$

a result first given by Sheffer as stated in the introduction.

The truth-table development for complete systems is essentially the same as that given in section 2. It is easy to prove for all systems the

THEOREM. *Every function contained in a positive function is positive; every function contained in a negative function is negative; every mixed function contains a function for every table of the system.*

6. Postulates for a Complete System.—We now show how to construct a set of postulates for any complete system such that: *the set of asserted functions is identical with the set of positive functions, while the assertion of any other function brings about the assertion of every elementary proposition* a property which also characterized the system of 'Principia.'

Let $\sim' p$ and $p \vee' q$ be functions in the given complete system with the tables of $\sim$ and $\vee$. Out of $\sim'$ and $\vee'$ we then construct $p \supset' q$ and $p \equiv' q$ as $p \supset q$ and $p \equiv q$ are found from $\sim$ and $\vee$, and also $f_1'(p_1, \cdots p_{m_1})$, $\cdots, f_\mu'(p_1, \cdots p_{m_\mu})$ with the same tables as $f_1(p, \cdots p_{m_1}), \cdots, f_\mu(p_1, \cdots p_{m_\mu})$. This is possible since $\sim$ and $\vee$, and so $\sim'$ and $\vee'$ can generate a complete system. All the functions $\sim'$, $\vee'$, $\supset'$, $\equiv'$, f_1', $\cdots$ f_μ' are ultimately expressed in terms of the f's and so belong to the system. Construct now the following set of postulates:

I. If $p_1, \cdots p_{m_1}$ are elementary propositions, $f_1(p_1, \cdots p_{m_1})$ is.

. . . .

If $p_1, \cdots p_{m_\mu}$ are elementary propositions, $f_\mu(p_1, \cdots p_{m_\mu})$ is.

II. The assertion of a function involving a variable p produces the assertion of any function found from the given one by substituting for p any other variable q, or $f_1(q_1, \cdots q_{m_1}), \cdots$ or $f_\mu(q_1, \cdots q_{m_\mu})$.

III. "$\vdash P$" and "$\vdash P \supset' Q$" produces "$\vdash Q$."

IV. (1) $\vdash : p \vee' p . \supset' p$ (a) $\vdash . f_1(p_1, p_2, \cdots p_{m_1}) \equiv' f_1'(p_1, p_2, \cdots p_{m_1})$,

. . . .

(5) $\vdash \cdots$ (u) $\vdash . f_\mu(p_1, p_2, \cdots p_{m_\mu}) \equiv' f_\mu'(p_1, p_2, \cdots p_{m_\mu})$.

where (1)–(5) are the assertions of IV in sec. 1 with $\sim'$ and $\vee'$ in place of $\sim$ and $\vee$.

That all asserted functions are positive can be verified as in the proof of sec. 4. As for the converse, note that III and IV (1)–(5) being of the same form as III and IV of sec. 4 will yield the assertion of all positive functions expressed in terms of $\sim'$ and $\vee'$. By the use of (a)–(u) every function can be shown to be equivalent ($\equiv'$) to some function expressed by $\sim'$ and $\vee'$ and so every positive function will be asserted. In the same way the assertion of any non-positive function will bring about the assertion of a non-positive function in $\sim'$ and $\vee'$ alone, and so of any proposition.

We thus see that complete systems are equivalent to the system of 'Principia' not only in the truth table development but also postulationally. As other systems are in a sense degenerate forms of complete systems we can conclude that no new logical systems are introduced.

7. Application to Nicod's Postulate Set.—Although, as in most existence theorems, the above set of postulates may not be the simplest in any one case, it can be used to advantage in showing that a given set has the same property as it possesses. For this purpose we show directly that all asserted functions are positive, and then that by means of the given postulates (a) each of our formal postulates may be derived (b) that the results derivable by our informal postulates can also be derived by the given ones.*

As an example we consider the set of postulates given by Nicod for the theory of elementary propositions in terms of the single primitive function of Sheffer's which Nicod denotes by $p|q$ and is termed incompatibility by Russell.† It is the first of the two functions given in section 5 as generating a complete system. Nicod gives the definitions

$$\sim p \,.=.\, p|p \quad Df, \qquad p \vee q \,.=.\, p/p|q/q \quad Df$$

which we take to be our $\sim' p$ and $p \vee' q$ respectively. His $p \supset q \,.=.\, p|q/q Df$ however is not our $p \supset' q$ which is $\sim' p \vee' q$. The primary distinction of his system is that he uses but one formal primitive proposition.

In carrying out the proof suggested we merely note that by means of his informal proposition "$\vdash P$" and "$\vdash P|R/Q$" produce "$\vdash Q$" we get the effect of "$\vdash P$" and "$\vdash P|Q/Q$" i.e., "$\vdash P \supset Q$" produce "$\vdash Q$" when $R = Q$. Since he has $p \supset' q \,.\supset.\, p \supset q$ we thus get the effect of "$\vdash P$"

* That the informal postulates of a system must be proved effectively replaced by others in another system is a precaution rarely taken in discussions of equivalence or dependence of logical systems. Such a discussion is unnecessary in ordinary mathematical systems since their distinctive postulates are all formal, the informal ones being those of a common logic. But in comparing logical systems, which usually do contain different informal postulates, such a discussion is fundamental.

† B. Russell, *loc. cit.*, chap. XIV.

and "$\vdash P \supset' Q$" produce "$\vdash Q$" our III. Likewise each function IV is proved with however $\supset$ in place of $\supset'$. But by means of $p \supset q . \supset . p \supset' q$ this too is remedied. We then easily complete the proof of the

THEOREM. *If in Nicod's system we give to $p|q$ the table*

p, q	$p\|q$
$+ +$	$-$
$+ -$	$+$
$- +$	$+$
$- -$	$+$

then the set of asserted functions is identical with the resulting set of positive functions; and the assertion of any other function would bring about the assertion of every elementary proposition.

GENERALIZATION BY POSTULATION.

8. The Generalized Set of Postulates.—As in the truth-table development we assume arbitrary primitive functions of propositions

$$f_1(p_1, p_2, \cdots p_{m_1}), \cdots, f_\mu(p_1, p_2, \cdots p_{m_\mu});$$

but in place of the arbitrary associated truth-tables we have a set of postulates of the following form. We have tried to preserve all the informal properties of the postulates of 'Principia' (and of sec. 5) but generalize the formal properties completely.

I. (As in sec. 5.)

II. (As in sec. 5.)

$$\begin{array}{llcl}
\text{III.} & \text{“}\vdash g_{11}(P_1, P_2, \cdots P_{k_1})\text{”} & \cdots & \text{“}\vdash g_{\kappa 1}(P_1, P_2, \cdots P_{k_\kappa})\text{”} \\
 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdots & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
 & \text{“}\vdash g_{1\kappa_1}(P, P_2 \cdots P_{k_1})\text{”} & \cdots & \text{“}\vdash g_{\kappa\kappa_\kappa}(P_1, P_1, \cdots P_{k_\kappa})\text{”} \\
 & \text{produce} & \cdots & \text{produce} \\
 & \text{“}\vdash g_1(P_1, P_2, \cdots P_{k_1})\text{”} & \cdots & \text{“}\vdash g_\kappa(P_1, P_2, \cdots P_{k_\kappa})\text{”}
\end{array}$$

where the P's are any combinations of f's including the special case of the unmodified variable, while the g's are particular combinations of this kind which need not have all the indicated arguments.

$$\begin{array}{ll}
\text{IV.} & \vdash h_1(p_1, p_2, \cdots p_{l_1}) \\
 & \vdash h_2(p_1, p_2, \cdots p_{l_2}) \\
 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
 & \vdash h_\lambda(p_1, p_2, \cdots p_{l_\lambda})
\end{array}$$

where the h's are particular combinations of the f's.

The retention of I and II which are characteristic of the theory of elementary propositions is our justification for giving that name to the systems that may be generated by the above set of postulates. In what follows we give what we consider to be merely an introduction to the general theory.

9. Definition of Consistency and Related Concepts.—The prime requisite of a set of postulates is that it be consistent. Since the ordinary notion of consistency involves that of contradiction which again involves negation, and since this function does not appear in general as a primitive in the above system a new definition must be given.

Now an inconsistent system in the ordinary sense will involve the assertion of a pair of contradictory propositions which as we have seen will bring about the assertion of every elementary proposition through the assertion of the unmodified variable p. Conversely since p stands for any elementary proposition its assertion would yield the assertion of contradictory propositions and so render the system inconsistent. The two notions are thus equivalent in ordinary systems; and since one retains significance in the general case we are led to the

DEFINITION.—*A system will be said to be inconsistent if it yields the assertion of the unmodified variable p.*

In a consistent system we may then define a true function as one that can be asserted as a result of the postulates. Instead of defining a false function as one not true, we give the following

DEFINITION. *A false function is one such that if its assertion be added to the postulates the system is rendered inconsistent.*

We can then state that in the system of 'Principia' every function is true or false. This suggests the

DEFINITION. *If every function of a consistent system is true or false the system will be said to be closed.**

As a justification of this name we may note that the postulates of such a system automatically exclude the possibility of any added assertions—a state of affairs we believe to be highly desirable in the final form of a logical theory.

10. Properties of Consistent Systems.—In all that follows we assume that the system discussed is consistent. If it be inconsistent one could hardly say anything more about it.

We turn to a theorem which will give us most of the results of this section. But first we must state two lemmas which we do not further prove.

LEMMA 1.—If a given set of functions gives rise to some other function in accordance with II and III, and if these functions involve certain letters

* Had the name not been in use in a different connection we should have introduced the term categorical.

$r_1, r_2, \cdots r_i$ upon which no substitution is made in the process, then the same deductive process will be valid if we have given the original functions with an arbitrary substitution of the r's as described in II provided this substitution is also made throughout the process.

LEMMA 2.—The most general process of obtaining an assertion from a given set of assertions in accordance with II and III can be reduced to first asserting a number of functions in accordance with II, and then applying II and III in such a way that no substitutions are made on the arguments of those functions.

THEOREM. *Every false function contains a finite set of untrue first order functions $\phi_1(p), \phi_2(p), \cdots \phi_\nu(p)$ such that whenever p is replaced by an untrue function at least one of these functions remains untrue.*

By the definition of false functions there must be some deductive process whereby from the given false function and true functions we assert p. By lemma 2 we can replace this process by another where from the given false function and true functions we obtain certain contained functions from which without substitution of the arguments we obtain p. Now first by lemma 1 we can equate to p all the arguments thus appearing and still have a valid deductive process for obtaining p. Denote the resulting untrue functions which are contained in the original false function by $\phi_1(p)$, $\phi_2(p), \cdots \phi_\nu(p)$. Then secondly by lemma 1 we can replace p by any function ψ and still have a valid process which now consists in obtaining ψ from certain true functions and $\phi_1(\psi), \cdots \phi_\nu(\psi)$. If then each $\phi_\iota(\psi)$ were true, ψ, being obtained from true functions in accordance with II and III would be true. It follows that if ψ be untrue, some $\phi_\iota(\psi)$ must be untrue.

THEOREM. *Every false function contains an infinite number of untrue first order functions; and if the system has at least one false function of order greater than one, then each false function contains an infinite number of untrue functions of every order.*

By the above theorem the false function contains at least one untrue function $\phi_{\iota_1}(p)$. By the same theorem some $\phi_{\iota_2}\phi_{\iota_1}(p)$ must be untrue, etc., through $\phi_{\iota_j}, \phi_{\iota_{j-1}} \cdots \phi_{\iota_1}(p)$. These are all different being of different rank, and are all contained in the given function.

The last part of the theorem may then be proved by showing that by replacing equal by unequal variables in the infinity of functions thus gotten from the false function of order greater than one we get untrue functions of every order, and so by the above method an infinite number of every order in every false function.

We have immediately the

THEOREM. *A necessary and sufficient condition that a function of a closed system be true is that all contained first order functions be true.*

COROLLARY. *It is also necessary and sufficient that all those of rank greater than some finite integer ρ be true.*

In analogy with corresponding ideas in the system of 'Principia' define a completely untrue function as one in which all contained functions are untrue with a similar definition for completely false. We then have the interesting

THEOREM. *If a system has a completely untrue function, then every false function contains a completely untrue function.*

Every function contained in the completely untrue function makes at least one $\phi_\iota(p)$ of a false function untrue. If ψ is such a contained function which makes say $\phi_{\iota_1}(p)$ true, then ψ will be completely untrue, and all contained functions will make $\phi_{\iota_1}(p)$ true yet some remaining $\phi_\iota(p)$ untrue. By repeating this process we finally obtain a function ψ' such that all contained functions make each $\phi_\iota(p)$ of a set that remains untrue. Each such $\phi_\iota(\psi')$ will then be a completely untrue function in the given one.

COROLLARY. *If a closed system has a completely false function every false function contains a completely false function.*

If we call such a system completely closed we have the stronger

THEOREM. *In a completely closed system every false function $f(p_1, p_2, \cdots, p_n)$ contains a completely false function $f(\psi_1(p), \psi_2(p), \cdots, \psi_n(p))$ where each $\psi_\iota(p)$ is either true or completely false.*

By equating all variables to p in the function of the corollary we get such a completely false function where some ψ's may be incompletely false. These are then eliminated by successively substituting for p functions which make them true.

COROLLARY. *A necessary and sufficient condition that a function of a completely closed system be true is that all contained first order functions found by substituting true or completely false functions for the arguments be true.*

This property begins to approximate to the truth-table method. It leads us easily to the following criterion for a completely closed postulational system being a truth-system which we state without proof.

THEOREM. *A necessary and sufficient condition that a completely closed postulational system be a truth-system is that a true first order function remains true whenever we replace a true or completely false constituent function by any other true or completely false first order function respectively.**

* In making a more complete study of the postulational generalization it would be desirable to classify all the systems that may result more or less in the way in which we have classified truth-systems through the associated systems of truth-tables. In this connection we might define the order of a set of postulates as the largest number of premises used in deriving a conclusion in III, and the order of a system as the lowest order a set of postulates deriving it can have. It is then of interest to note that *whereas the set of postulates of the system of 'Principia' is of the second order, the system itself is of the first order.*

m-VALUED TRUTH-SYSTEMS.*

11. The Generalized ($\sim$, $\vee$) System.—We have seen that the truth-table generalization, at least with regard to complete systems, is included in the postulational development. We now show that the latter is more general by presenting a new class of systems, distinct from the two-valued systems of symbolic logic, which can be generated by a completely closed set of postulates.

In these systems instead of the two truth-values $+$, $-$ we have m distinct "truth-values" $t_1, t_2, \cdots, t_m$ where m is any positive integer. A function of order n will now have m^n configurations in its truth-table, so that there will be m^{m^n} truth-tables of order n. Calling a system having all possible tables complete, we now show that the following two tables generate a complete system.

p	$\sim_m p$
t_1	t_2
t_2	t_3
$\cdots$	$\cdots$
t_m	t_1

p, q	$p \vee_m q$	
$t_1 t_1$	t_1	
$\cdots$	$\cdots$	
$t_{\iota_1} t_{j_1}$	t_{ι_1}	$\iota_1 \leq j_1$
$\cdots$	$\cdots$	$\iota_2 \geq j_2$
$t_{\iota_2} t_{j_2}$	t_{j_2}	
$\cdots$	$\cdots$	
$t_m t_n$	t_m	

We see that $\sim_m p$, the generalization of $\sim p$, permutes the truth-values cyclically, while $p \vee_m q$, the generalization of $p \vee q$ has the higher of the two truth-values.†

To construct a function for any first order table, of which there are m^m, note that

$$t_1(p) \,.=.\, p \vee \sim_m p \,.\vee_m \sim_m^2 p : \vee_m \cdots \sim_m^{m-1} p \quad Df,$$

where $\sim^2 p \,.=.\, \sim\sim p \quad Df$, etc., has all its truth values t_1. Then

$$\tau_{m_1}(p) \,.=.\, \sim_m^{m-1}(\sim_m^{m-1}(\sim_m t_1(p) \,.\vee_m.\, p) : \vee_m . \sim_m^{m_1} p) \quad Df$$

has all values t_m except the first which is t_{m_1}. Any first order table

p	$f(p)$
t_1	t_{m_1}
t_2	t_{m_2}
$\cdots$	$\cdots$
t_m	t_{m_m}

can then be constructed by the function

* See Lewis, *loc. cit.*, p. 222 for the term "Two-Valued Algebra."

† The higher truth-value has here the smaller subscript.

$$\tau_{m_1}(p) \,.\, \vee_m \,.\, \tau_{m_2}(\sim_m^{m-1} p) : \vee_m \,.\, \tau_{m_3}(\sim_m^{m-2} p) :. \vee_m \cdots \tau_{m_m}(\sim_m p).$$

Construct now a function for the table

p	$\overline{\sim}_m p$
t_1	t_m
t_2	t_{m-1}
$\cdots$	$\cdots$
t_m	t_1

and define $p \,._m q \,.=.\, \overline{\sim}_m(\overline{\sim}_m p \,.\, \vee_m \,.\, \overline{\sim}_m q)$ *Df* which is the generalization of $p \,.\, q$ and has the lower of the two truth values of its arguments. We can now construct a table all of whose values are t_m except for one configuration $t_{m_1}, t_{m_2}, \cdots, t_{m_n}$ when it is $t_{m_{m_1 m_2 \cdots m_n}} = t_\mu$ by the function

$$\tau_\mu(\sim_m^{m-m_1+1} p_1) \,._m \tau_\mu(\sim_m^{m-m_2+1} p_2) \,._m \cdots \tau_\mu(\sim_m^{m-m_n+1} p_n),$$

and so any table by constructing such a function for each configuration and then "summing up" by $\vee_m$.

12. Classification of Functions—the m dimensional Space Analogy.— The generalization of the classification of functions into positive, negative and mixed is afforded us by the following

THEOREM. *A function contains at least one function for every truth-table whose values are contained among the values of the given table.*

Let $t_{m_1} \cdots t_{m_\mu}$ be the truth-values that appear in the table of a given function $f(p_1, p_2, \cdots p_n)$. Then we can pick out μ configurations having these values respectively. Construct functions $\phi_\iota(p)$ such that when p has the vlaue t_{m_j} of one of these configurations, $\phi_\iota(p)$ have the value of p_i in that configuration. It is then easily seen that $f(\phi_1(p), \cdots, \phi_n(p))$ has the value t_{m_j} whenever p has the value t_{m_j}. If then $\psi(q_1, q_2, \cdots, q_l)$ have a table whose values are among the t_{m_j}'s, $f(\phi_1(\psi), \cdots, \phi_n(\psi))$ will be a function contained in the given function with that table.

We are thus led to a classification of functions by means of their truth-tables such that the set of tables of contained in a given function is the same for all functions in a given class. We then have m classes of functions where but one truth-value appears, $[m(m-1)]/2!$ with two truth-values, $\cdots$, $[m(m-1)\cdots(m-\mu+1)]/\mu!$ with μ truth-values, $\cdots$, one class with all m truth-values. We thus have $2^m - 1$ classes of functions which when $m = 2$ reduces to the three classes of positive, negative and mixed functions.

These formulæ suggest an analogy which, if well founded, is of great interest. For this purpose replace the set of functions having all of a given set of μ truth-values by all functions whose values are among these μ values. If then we compare the functions of our complete system to the points of a space of m dimensions,* the m classes of functions with but one truth-value would correspond to the m coordinate axes, the $[m(m-1)]/2!$ classes of functions with no more than two truth-values to the $[m(m-1)]/2!$ coordinate planes, etc., so that except for the absence of an origin all properties of determination and intersection within the coordinate configurations go over. If then we attach the name m-dimensional truth-space to our system, we observe the following difference, that whereas the highest dimensioned intuitional point space is three, the highest dimensioned intuitional proposition space is two. But just as we can interpret the higher dimensioned spaces of geometry intuitionally by using some other element than point, so we shall later interpret the higher dimensioned spaces of our logic by taking some other element than proposition.

13. Truth-Table Characteristics of Asserted Functions.—The following analysis presupposes that in constructing a set of postulates for the system we at least wish to impose the

CONDITION.—*If a function is asserted, all functions with the same truth-table will be asserted.*

It follows from the theorem of the preceding section that under the given condition, *if a function is asserted, every function of the truth-space it determines is asserted.*

We can now prove that *if the system is to be completely closed its asserted functions must constitute a single truth-space contained in the given truth space.* For if there were at least two such spaces, then a function having all their truth-values would be false, and so would contain a completely false function. This in turn would contain functions with but one truth-value; and these being therefore in one of the two given spaces would be true which contradicts their being in a completely false function.

No loss of generality ensues if we take the truth values of this contained truth-space of asserted functions to be $t_1, t_2, \cdots t_\mu$, where, to avoid degenerate cases $0 < \mu < m$. We now show that a completely closed set of postulates can be constructed for all such systems.

14. A Completely Closed Set of Postulates for the Systems.—I and II are determined directly as in the general case. To obtain III, construct a function $p \supset^\mu_m q$ whose table is given by the following: when the truth-value of p is that of q or lower, $p \supset^\mu_m q$ will have the value t_1, while if the truth-value of p is above that of q, then if the value of p is t_μ or higher, $p \supset^\mu_m q$ will have the value of q, while if it is below t_μ, say t_ν and that of q is $t_{\nu'}$, then the truth-value of $p \supset^\mu_m q$ will be $t_{\nu'-\nu+1}$. III will then be simply

* Or we might take the truth-table as element in which case the system is perhaps smoother than before.

$$\text{“}\vdash P\text{”}$$
$$\text{“}\vdash P \supset^{\mu}_{m} Q\text{”}$$
produce
$$\text{“}\vdash Q\text{”}$$

Now by generalizing each part A, B, C, D of the proof of the fundamental theorem of sec. 3 it can be shown that by the assertion of a finite number of functions with values from t_1 to t_μ all such can be obtained.* If then we assert these functions in IV we shall have every function in the μ-space asserted. Furthermore no others can be asserted for by the use of II and III we can only get functions with values from t_1 to t_μ by means of functions similarly restricted. This is obvious in II while in III if the value of P is from t_1 to t_μ while that of Q is below t_μ, then from the above definition of the table of $P \supset^{\mu}_{m} Q$ it would have the value of Q and so be below t_μ. But that contradicts the assumption that the premises had values from t_1 to t_μ.

This set of postulates will then give the proper set of true functions. Furthermore let us suppose that we assert a function with at least one value below t_μ. This will contain a function $\phi(p)$ with but one value, and that below t_μ. By II, $\phi(p)$ will be asserted. Furthermore since $\phi(p) \,.\supset^{\mu}_{m}.\, \phi(p) \supset^{\mu}_{m} \sim_m \phi(p)$ has its value t_1 it will be asserted, and so we obtain by III $\sim_m \phi(p)$. Repetition of this process will finally give us a function $\psi(p)$ with but one value t_m. But $\psi(p) \cdot \supset^{\mu}_{m} p$ is asserted having but one value t_1. We thus obtain the assertion of p. The system is therefore closed. And since all functions with values from $t_{\mu+1}$ to t_m are completely false, the system is completely closed.

15. Comparison of Systems.—As in the truth-table development we can generalize the systems by using arbitrary functions as primitives, and as was done there we can show how to generate a complete m-dimensioned system by one second order function, and how to give a completely closed set of postulates for all complete systems. The problem of determining all possible systems of m-dimensional truth-tables, however, is one we have not considered, though its solution would through considerable light on the ordinary problem.

We turn now to the following

DEFINITIONS. *A closed system S with primitives $f_1, f_2, \cdots f_n$ has a representation in a closed system S' with primitives $f'_1, f'_2, \cdots f'_{n'}$ if we can so replace the f's by functions in S' that a function in S will be true when and only when the correspondent in S' is true.*

Two systems are equivalent if each has a representation in the other.

Denote a complete m-dimensional truth-system with the asserted functions forming a truth-space of μ dimensions by ${}_{\mu}T_m$. We then have the

* Lack of space prevents us from giving the details.

THEOREM. *Two complete truth-systems* ${}_{\mu}T_m$ *and* ${}_{\mu'}T'_{m'}$ *are equivalent when and only when* $\mu = \mu'$ *and* $m = m'$.

The conditions are clearly sufficient since we can make truth-values correspond. To prove them necessary suppose $m > m'$. If we construct m^m functions of first order in T with different truth-tables then there will be two, $\phi_1(p)$, $\phi_2(p)$ whose correspondents $\phi'_1(p)$, $\phi'_2(p)$ have the same truth-tables since there are in T' only $m'^{m'}$ of first order. Let $\chi(p, q)$ have value t_1 when p and q have the same value and t_m otherwise. Then $\chi(\phi_1, \phi_1)$ is true; hence $\chi'(\phi'_1, \phi'_1)$ is. ϕ'_2 having the same table as ϕ'_1, $\chi'(\phi'_1, \phi'_2)$ is true, and hence $\chi(\phi_1, \phi_2)$ the correspondent. But that would make ϕ_1 have the same table as ϕ_2. Now suppose $\mu > \mu'$. If ϕ have all the values from t_1 to t_μ and no others there are μ^μ functions with values t_1 to t_μ of the form $\psi\phi(p)$. These will then be asserted and so the correspondents will be asserted and have values t'_1 to $t'_{\mu'}$. Since we can only have $\mu'^{\mu'}$ functions $\psi'\phi'(p)$ with different tables, we can find two of the μ^μ correspondents with the same table. The above contradiction then results as before.

For representation we have only found the

THEOREM. *To represent* ${}_{\mu}T_m$ in ${}_{\mu'}T'_{m'}$, *it is necessary to have* $\mu \leq \mu'$, $m \leq m'$; *it is sufficient to have* $\mu \leq \mu'$, $m - \mu \leq m' - \mu'$.

COROLLARY. *A necessary and sufficient condition that* ${}_{\mu}T_m$ *have a representation in* ${}_{\mu}T'_{m'}$, *is that* $m \leq m'$.

It is of interest to note as a result that the only complete truth-systems equivalent to the system of 'Principia' are ${}_1T_2$'s; and though it can be represented in every complete truth-system, only ${}_1T_2$'s can be represented in it. We have thus verified our statement that we obtain essentially new logical systems.

16. Interpretation of m-valued Truth-systems in Terms of Ordinary Logic.—Let the elementary proposition of the $(\sim_m, \vee_m)$ system be interpreted as an ordered set of $(m - 1)$ elementary propositions of ordinary logic $P = (p_1, p_2, \cdots p_{m-1})$ such that if one proposition is true all those that follow are true. P will be then be said to have the truth-value t_1 if all the p's are true, t_2 if all but one are true, etc. Also P will be said to be true if at most $(\mu - 1)p$'s are false.

If $P = (p_1, p_2, \cdots p_{m-1})$, $Q = (q_1, q_2, \cdots q_{m-1})$ we define

$$P \vee_m Q \,.=.\, (p_1 \vee q_1, p_2 \vee q_2, \cdots p_m \vee q_m) \quad Df$$

$$\sim_m P \,.=.\, (\sim(p_1 \vee p_2 \vee \cdots p_{m-1}), \sim(p_1 \vee \cdots p_{m-1}) \,.\vee.\, p_1 \,.\, p_2, \cdots, \sim(p_1 \vee \cdots p_{m-1}) \,.\vee.\, p_{m-2} \,.\, p_{m-1}) \quad Df$$

We easily justify these definitions by showing first that $P \vee_m Q$ and $\sim_m P$ are "elementary propositions" when P and Q are, and secondly that they have the proper truth tables. Thus in $P \vee_m Q$ the first $p_\iota \vee q_\iota$ to be true is the first for which either p or q is true; also all later terms will have p or q true and so will be true. $P \vee_m Q$ is therefore elementary and has the required table.

But in spite of this representation ${}_1T_2$ still appears to be the fundamental system since its truth-values correspond entirely to the significance of true and completely false, whereas in ${}_\mu T_m$, $m > 2$ either $\mu > 1$ or $m - \mu > 1$, and this equivalence no longer holds. We must however take into account the fact that our development has been given in the language of ${}_1T_2$ and for that very reason every other kind of system appears distorted. This suggests that *if* we translate the entire development into the language of any one ${}_\mu T_m$ by means of its interpretation, then it would be the formal system most in harmony with regard to the two developments.