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Chapter 1

Laws of Form: A Survey of Ideas

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1. Introduction

The purpose of this paper is to explore the idea of a distinction, using G. Spencer-Brown's work "Laws of Form" (Spencer-Brown, 1969) as a pivot, a reference and a place from which to make excursions into both simplicity and complexity. Spencer-Brown's language is an instrument of great delicacy. Throughout, we will refer to the subject of this essay as laws of form.

The Spencer-Brown mark $\urcorner$ is a sign that can represent any sign, and so begins in both universal and particular modes. The mark is seen to make a distinction in the space in which it is written, and so can be seen, through this distinction, to refer to itself. By starting with the idea of distinction, we find, in the mark, the first sign and the beginning of all possible signs. The mark stands for an observer inseparable from that which is observed.

This paper is meant to be an introduction to laws of form and to the way many topics emanate from its beginnings. Here is a description of the contents of the paper. Section 2 is a concise description of the formalism of laws of form and how it is based on a single sign that stands for and makes a distinction. Section 3 discusses the structure of a sign, how it may come about that a sign can stand for itself and how the Spencer-Brown mark can emerge. Section 4 is an introduction to the notational meaning of laws of form.

Section 5 continues with the structure of the Primary Arithmetic and its algebra. Section 6 discusses logic and syllogism. Section 7 discusses self-reference, re-entry and the notion of a sign for itself.

Section 8 discusses the beginning of mathematics and natural numbers, the construction of the integers in the form and continues into Peano arithmetic, the sieve of Eratosthenes, the infinity of primes in the eyes of Euclid and Euler and a quick movement into the Riemann Zeta function. It is striking that Euler's insight into the relationship of the primes and the Zeta function is based in the key re-entry form of elementary algebra:

$$\begin{aligned} u &= 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots \\ &= 1 + x(1 + x + x^2 + x^3 + x^4 + \cdots) \\ &= 1 + xu. \end{aligned}$$

Section 9 discusses continued fractions and an entry into real numbers and recursive processes. Section 10 looks at logic once again from the vantage of the real numbers.

Section 11 examines the entire range of finite expressions in laws of form and shows how they form a model for a fragment of Georg Cantor's transfinite ordinals. Laws of form notation is a perfect expression of this fragment of the ordinals, and goes beyond Boolean algebra to arithmetic and transfinite arithmetic. Here we give an exemplar of the overcoming of Goedelian incompleteness (of Peano arithmetic) by the use of the mark as an imaginary value.

Section 12 introduces ordered expressions in the mark so that a juxtaposition AB is distinct from a juxtaposition BA. We show how this structure is related to the structure of non-associative algebra and we show how to understand the fact that these forms are counted by the Catalan numbers $\mathrm{C}(n) = 1, 3, 5, 14, 42, \ldots$.

Letting S denote the formal summation of all such ordered expressions in Laws of Form (* is the unmarked state)

$$S = * + \overline{\;\;}| + \overline{\;\;}|\overline{\;\;}| + \overline{\overline{\;\;}|\;}| + \overline{\;\;}|\overline{\;\;}|\overline{\;\;}| + \overline{\overline{\;\;}|\;}|\overline{\;\;}| + \overline{\;\;}|\overline{\overline{\;\;}|\;}| + \overline{\overline{\;\;}|\overline{\;\;}|\;}| + \overline{\overline{\overline{\;\;}|\;}|\;}| + \cdots,$$

we derive the fundamental reentry equation

$$S = * + S\overline{S}|.$$

Furthermore, for E, the collection of all unordered expressions in the mark (these are the ones we are usually interested in for laws of form) we have the following remarkable self-referential formula:

$$\sum_{\alpha \in E} \alpha = \prod_{\overline{\alpha}| \in E} (1 - \overline{\alpha}|)^{-1}.$$

It is amazing that self-reference can encapsulate so much in so little space. This last relationship between summation and product is related to the corresponding Eulerian formula for the Zeta function that we have discussed in Section 8, and to the interlacing of the theory of partitions of integers with the enumeration of unordered expressions in the mark.

Section 13 shows how re-entry and oscillation leads to Fermion algebra, nilpotent operators and the Dirac equation. Here we see that the most elementary process of self-observation is related to the structure of relativistic quantum physics. Section 14 is a short briefing on the relationship of laws of form and fundamental physics.

Section 15 is a discussion of modulators in the synchronous mode, a complement to Chapter 11 of laws of form.

Section 16 introduces interactions of boundaries in the form of the Principle of Idemposition: Common boundaries cancel. We show how a calculus of the idemposition of boundaries reproduces the Laws of Calling and Crossing of laws of form, and we give a hint about the relationship of this calculus with the structure of colorings of maps and the four color theorem.

Section 17 is an epilogue, reflecting on the themes of this paper.

2. Finding Distinction

The system we describe is due to G. Spencer-Brown in his book "Laws of Form" (Spencer-Brown, 1969).

The sign $\overline{\;\;}|$ stands for the distinction that the sign is seen to make between its inside and its outside. Spencer-Brown calls $\overline{\;\;}|$ the mark, and allows it to refer to any given distinction, including itself. The inside of the mark is unmarked. The outside of the mark is marked (by the mark) (see Figs. 1 and 2).

The mark can be interpreted as an instruction to cross the boundary of a distinction. In that mode we have that $\overline{a}|$ denotes the value obtained by crossing from the state a. Thus $\overline{\overline{\;\;}|}|$ is unmarked since we have crossed from the marked state, and $\overline{\;\;}|$ is marked since we have crossed from the unmarked state. An extra mark in the space outside the mark is redundant since that space is already marked. Consequently, we may write $\overline{\;\;}|\overline{\;\;}| = \overline{\;\;}|$. Thus we have two basic replacement rules:

$$\text{Crossing: } \overline{\overline{\;\;}|}| = \quad ,$$

$$\text{Calling: } \overline{\;\;}|\overline{\;\;}| = \overline{\;\;}|.$$

A calculus arises from these equations. One can reduce or expand arbitrary expressions in the mark, for example,

$$\overline{\overline{\overline{\ }|\,\overline{\ }|}|\,\overline{\ }|}|\,\overline{\ }| = \overline{\overline{\overline{\ }|}|\,\overline{\ }|}|\,\overline{\ }| = \overline{\overline{\ }|}|\,\overline{\ }| = \overline{\ }|.$$

One can prove that the simplification of an expression is unique and one can consider the algebra that is related to this arithmetic.

In the algebra we have identities such as $AA = A$ for any expression A, and $\overline{\overline{A}|}| = A$ for any expression A. Remarkably, the algebra is quite nontrivial and leads to a new construction for Boolean algebra and new insights into the nature of logic.

An elementary structure of great significance appears from the equations

$$M = \overline{aN}|,$$
$$N = \overline{bM}|.$$

To see what happens here, let a and b be unmarked. Then we have

$$M = \overline{N}|,$$
$$N = \overline{M}|.$$

If $M = \overline{\ }|$ and $N = \overline{\overline{\ }|}|$, these values satisfy the equations and so the system is in a stable state. Similarly if $M = \overline{\overline{\ }|}|$ and $N = \overline{\ }|$, then the system is in a stable state. We see from this that M and N together form a memory. In a possible world of recursions, the memory can maintain a particular pair of values. In this way, the binding of structure across time emerges from the timeless eternity of forms.

Furthermore, if we were to change a or b to the marked state, we could influence the memory to change state. A momentary change in a and b can reset the memory. In this way, circular systems of equations can be made that correspond to circuitry at the base of computing, and the essential design of digital computers can be accomplished in the language of the mark and its algebra.

A key function that can be described in this algebra is the operation of exclusive or. I denote exclusive or of A and B by $A\#B$. It is expressed in the algebra of the mark as

$$A\#B = \overline{A\overline{B}|}|\,\overline{\overline{A}|\,B}|.$$

The reader will note that $A\#B$ is marked only when one of A or B is marked but not both. Thus $A\#B$ can indicate whether A and B

are distinct or indistinct. If $A = B$ then $A\#B$ is unmarked, but if A is not equal to B, then $A\#B$ is marked. It is this ability of the primary algebra to indicate distinctions that gives it the capability to model an act of distinction.

We can ask to reach deeper into the biological and physical world to find sources that underpin the emergence of distinctions. This will inevitably happen in the future development of our understanding.

3. From Mathematics to Laws of Form

Mathematics is often constructed by using the concept of a collection. A collection is a distinction of membership. For example the set of prime numbers connotes the distinction between composite and prime among the positive integers. At the level of sets themselves, the empty set, denoted by brackets containing nothing,

$$\{\quad\}$$

is a distinction between a void and the constructed empty container. The very sign for the empty set consists of two brackets (left and right) that together can be interpreted as a container for something that is placed between them. In the case of the empty set, nothing is placed between the brackets.

The brackets themselves are shaped as cusps. Each cusp can be seen as a process of bifurcation that gives rise to the distinction between its branches. The two cusps are mirror imaged, and it is this symmetry across an imaginary mirror between them that makes the possibility to see them together as one container. The brackets are two and yet they are one.

At this point (in an encounter with the empty set) we reach a semantic divide between the mode of speaking by mathematicians trained in logical formalism and a wider analysis of language.

To start with signs is to begin with something apparently definite and yet, as soon as the discussion begins, we find there are only signs. Thus what is in the mind of another person is also a sign, albeit a sign that is understood internally by that person. One can look and look for substance that may underlie the sign but the search always leads to more signs. In this expansion of signs related to signs, signs describing signs, the self becomes yet another sign standing in relation to all the signs that work at the nexus that the person represents. The sign of the self becomes a limit of all the signs that are the life of that self. The distinction of a person is a sign of distinction, a sign of self.

Spencer-Brown in "Laws of Form", makes a new start, beginning with the idea of distinction:

> "We take as given the idea of distinction and the idea of indication, and that we cannot make an indication without drawing a distinction. We take, therefore, the form of distinction for the form." (Spencer-Brown, 1969, p. 1).

In Chapter 12 in the last sentence of Laws of Form, Spencer-Brown writes "We see now that the first distinction, the mark, and the observer are not only interchangeable, but, in the form, identical." Here the mark is the first made sign or indication of a first distinction. The observer can be identified with the interpretant in so much as the interpretant (a term of C. S. Peirce (1976)) is an equivalent sign created in the mind of somebody, and must for its existence partake of the being of that somebody. At this nexus Spencer-Brown indicates the essential identity of sign, representamen and interpretant (C. S. Peirce, 1976). The three coalesce into the form that is the form of distinction.

The form of distinction becomes, in Spencer-Brown, a background for the entire play of signs. We take the form of distinction for the form. And in this, "the form" becomes a noun as elusive as it seems to be concrete, just as is the nature of the sign in Peirce. The form of a distinction drawn as a circle in the plane is a geometrical form. But what is the form of distinction?

Spencer-Brown's first paragraph is an amalgam of words that all stand for aspects of distinction: *definition, continence, boundary, separate, sides, point, draw a distinction, spaces, states, contents, side of the boundary, being distinct, indicated, motive, differ in value*. The paragraph is not a definition in the mathematical sense of definition: something in terms of previously defined things. There is no possibility to define distinction in terms of previous things that are not distinctions. The only possibility is to define distinction in terms of itself. We take the form of distinction for the form.

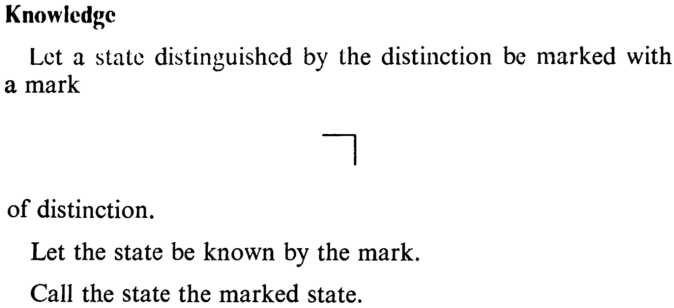
In the next few lines of Laws of Form. one finds the quote "If a content is of value, then a name can be taken to indicate this value.” Already we have faced the multiplicity of names for a distinction. Making an indication is a special act that cannot happen without the making of a distinction. Nevertheless, a *name* can be called forth. A name can be taken to indicate a value. A distinction can be performed. In this condensed place where there is only creation of distinction, boundary or the crossing of the boundary, the only distinction is at first the distinction between nothing (the unmarked) and the act of creation, and then arises a distinction between name and act.

We come to the creation of a name and find that this is the same as the creation of a distinction. They are one and the same. And yet a name can be separated from the distinction to which it refers. The name can be taken to be a new distinction that refers to the first distinction. Indeed we can imagine that the original distinction (for example a circle drawn in the plane) is seen to stand for, to indicate, itself. But in the act of recognizing this possibility that “it” could stand for “itself” we have made a distinction between “it” and “itself”. We have allowed a condensation by making the possibility of a separation. The name and the sign are born in that process.

The text tells its reader to “Draw a distinction” and to “Call it the first distinction”. This sweeps away any notion that first distinction is an absolute concept. The first distinction is the one that is under discussion. The form is the form of the first distinction. And so the form of the first chapter has shifted from the universal to the particular, and the form of distinction is the form of that first distinction. The form is inherent in any act of distinction. We find that at the point of intent “Let any mark, token, or sign be taken in any way with or with regard to the distinction as a signal. Call the use of any signal its intent”. Here is the entry of the word sign (signal) into Spencer-Brown’s consideration of distinction and form.

Finally, there enters upon the stage of distinction the mark that will be the pivot for the formalism of Laws of Form (see Fig. 1). Spencer-Brown does not say that this mark is the first sign, but sign it is and with it arrives the possibility to indicate the first distinction.

Spencer-Brown writes “Let a state distinguished by the distinction be marked with a mark $\urcorner$ of distinction.” The mark is written upon one side of the first distinction. We shall take the liberty of illustrating this in Fig. 2. The mark is chosen to make and to indicate



Knowledge

Let a state distinguished by the distinction be marked with a mark

of distinction.

Let the state be known by the mark.

Call the state the marked state.

Fig. 1. The Spencer-Brown mark.

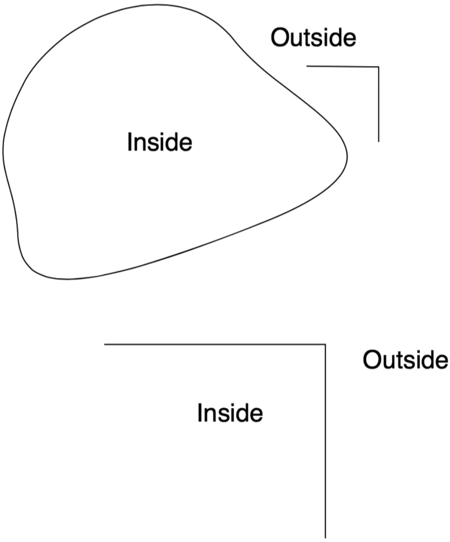


Fig. 2. The mark indicates the outside of the first distinction.

a distinction in its own form. The mark has (for the observer) an inside and an outside. Spencer-Brown says, quite explicitly "Let each token of the mark be seen to cleave the space into which it is copied. That is, let each token be a distinction in its own form." And before this, he gives permissions:

"Call the space cloven by any distinction, together with the entire content of the space, the form of the distinction. Call the form of the first distinction the form. . . . Let there be a form distinct from the form. Let the mark of distinction be copied out of the form into such another form. Call any such copy of the mark a token of the mark. Let any token of the mark be called as a name of the marked state. Let the name indicate the state."

Now the circle has closed. Each token of the mark is a sign and is a copy of the mark itself. Each mark or token of the mark is a distinction in its own form. There is a plethora of signs, marks and forms. They all indicate the marked side of the first distinction. Only one distinction is being discussed. As many marks as may be needed are available to signal this distinction. We embark upon, not just form, but formalism and the inception of calculation.

As in Fig. 2, the mark can indicate the outside of the first distinction, and we take the mark to make a distinction in the outer space of that first distinction. We see then that *we can take the mark itself as the first distinction.*

The mark is a sign for itself. It is a sign that makes a distinction and it is a sign that stands for the outer space of the distinction that the sign creates. All this is said in the inseparability of the sign and the observation of that sign.

The mark is a sign that makes a distinction in the plane within which it is drawn. In that plane there is a distinction between the (bounded) inside and the (unbounded) outside of the mark. The mark is chosen to refer to the outside of the distinction that it makes in the plane. The mark can be seen to refer (via referring to the outside of the distinction that it makes) to itself as the boundary of that distinction.

Thus we can write the Law of Calling in the form $\urcorner\urcorner = \urcorner$.

Each mark in the expression on the left is a sign or name for the outside of the distinction made by the other mark in the expression. Each mark is the name of the other mark. The calling of a name made again may be identified with the calling of the name. And so we have the equation as indicated above, condensing the two marks to a single mark.

We take the mark to indicate a crossing from the state indicated on its inside. The symbol $\overline{A\,}|$ denotes the state obtained by crossing from the state indicated by A. Hence $\overline{\urcorner\,}|$ indicates the state obtained by crossing from the marked state. Hence $\overline{\urcorner\,}|$ indicates the unmarked state. In equations, we have the Law of Crossing:

$$\overline{\urcorner\,}| = .$$

The value of a crossing made again is not the value of the crossing.

We have arrived at a self-referential nexus. The mark, first sign, refers to itself. The first sign is a name and it is identified with the action of crossing from the unmarked state (the state with no sign). We began with the idea of distinction.

The sign of the first distinction acts as the transformation and the boundary between the unmarked state (the state with no sign) and the marked state. The sign of the first distinction is a signal of the emergence of articulated form. The sign of the first distinction is, in the form, identical with the first distinction.

4. A Sign in Space

This section is an introduction to Laws of Form, to be read after one has appreciated the notion of the mark as a sign that itself makes a distinction in its plane space.

Let $\rceil$ be the Spencer-Brown mark.
Let there be a distinction, with Inside denoted I, and outside denoted O.
Regard the mark as an operator that takes inside to outside and outside to inside.

$$\text{Then}$$
$$\overline{I}\rceil = O,$$
$$\overline{O}\rceil = I.$$

Note that it follows that

$$\overline{\overline{I}\rceil}\rceil = \overline{O}\rceil = I,$$
$$\overline{\overline{O}\rceil}\rceil = \overline{I}\rceil = O.$$

For any state X, we have

$$\overline{\overline{X}\rceil}\rceil = X.$$

Introduce the unmarked state by letting the Inside be unmarked.

Then I =.

And so

$$\overline{\ }| = O,$$

$$\overline{O}| = .$$

Therefore the value of the outside is identified with the mark and

$$\overline{\overline{\ }|}| = .$$

The value of the outside is identified with the result of crossing from the unmarked inside.

$$\overline{\ }| = \overline{\ }|.$$

This equation can be read on the left as "cross from the inside" and it can be read on the right as "name of the outside".

Once the inside is unmarked, then the mark itself can be seen to be the first distinction.
The language of the mark is self-referential.
$\overline{\ }|\overline{\ }| = \overline{\ }|$ says that either mark names the other.

A sign can refer to another sign. The mark is seen as a sign and as a distinction between the inside of that sign and its outside. We take the mark, as sign, to refer to its own outside.

In the form, the mark and the observer are identical. In the form, a thing is identical to what it is not. The form we take to exist arises from framing nothing. We take the form of distinction for the form.

5. Finding Primary Arithmetic

We have one sign $\overline{\ }|$ and two laws or rules about that sign:

$$\text{The Law of Calling: } \overline{\ }| = \overline{\ }|\overline{\ }|,$$

$$\text{The Law of Crossing: } \overline{\overline{\ }|}| = .$$

At this stage in the development of the sign, these laws are statements about naming and about the crossing of the boundary of an initial distinction. The initial distinction can be the distinction made by the sign itself.

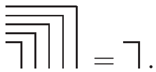
There is another sign. It is the equals sign, =.

With that sign we enter mathematics.

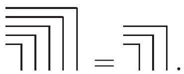
With the equals sign, we formalize condensations of reference and meaning.

It is implicit that we may write expressions such as

and wonder what this nest of signs can mean. We have already defined the meaning of this new sign as a five-fold act of crossing-from the previous state, starting from the unmarked state. And being able to count, we know that this means that we will arrive at the marked state after such a process. Thus we have

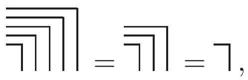


An infinity of possible equalities of concatenations of signs has opened up before us and since we know how to count, we can evaluate them all and find either the marked state or the unmarked state as an equivalent to each one. Do we need to know how to count to accomplish this task? We do not need to know how to count. We can apply the laws of calling and crossing where we find them. An empty cross with a cross over it can be regarded as an instance of the law of crossing:



The two innermost marks in the left hand nest of marks are an instance of the law of crossing and we can erase them, forming the right hand side with only three marks.

Doing this once more, we find



and the marked value of the nest has been uncovered without the need for counting.

Here is another way. Let u and m stand for the unmarked and marked states respectively. Agree that $\overline{u|}$ has the marked state as its outside value and write $\overline{u}|m$ to indicate this state of affairs. Agree that $\overline{m}|$ has the unmarked state as its outside value and let $\overline{m}|u$ indicate this state of affairs. Then we can evaluate the nest of marks by marking it with u and m.

$$\overline{\overline{\overline{\overline{\overline{u}|m}|u}|m}|u}|m.$$

Similarly, $\overline{\ }|\overline{\ }|\overline{\ }|\overline{\ }|\overline{\ }| = \overline{\ }|$ by repeated application of the law of calling.

$$\overline{\overline{\overline{\overline{\ }|}|}|\,\overline{\ }|}| = \overline{\overline{\ }|\,\overline{\ }|}| = \overline{\overline{\ }|}| = \ .$$

Here we combine uses of the laws of calling and crossing when they are available. We see that there is an arithmetic of expressions written in the mark and the equals sign has taken on the crucial role of connecting expressions that indicate the same value.

Oh! You want to know the meaning of $\overline{\overline{\overline{\overline{\ }|}|}|\,\overline{\ }|}|$! It is a multiple action. Think of putting an unmarked signal u at the deepest spaces in the expression and marking it with u and m as we did before.

$$\overline{\overline{\overline{\overline{u}|m}|m}|m\,\overline{u}|m}|u.$$

Note that in the space one crossing away from the outside there are two m's. We take the rules that $mm = m, uu = u, mu = um = m$. Then any expression can be seen as indicating a multiple process of crossing and recrossing from the unmarked state of the first distinction. The signals interact with one another and produce the value of the expression as either marked or unmarked. The result is the same as that obtained by using the laws of calling and crossing on the expression. Here is the simplest arithmetic generated by a sign that makes a distinction. Spencer-Brown calls this the primary arithmetic or the calculus of indications.

6. Finding Logic

The primary arithmetic is a two valued system. Every expression is either marked or unmarked. Remarkably, there is a translation to the two-valued logic of True (T) and False (F). Let $a \vee b$ denote

"a or b" (inclusive or – a or b or both a and b), $a \wedge b$ denote "a and b". Let $\neg a$ denote "not a" and let $a \rightarrow b$ denote "a implies b". Recall that in symbolic two-valued logic one takes the equivalence $a \rightarrow b = (\neg a) \vee b$. Now note that if we write algebraically about the primary arithmetic with the variables standing for either the marked or unmarked states, *then ab is marked exactly when a is marked or b is marked.* This suggests the we take the interpretation T for the marked state and F for the unmarked state. Lets write $T = \overline{\ \ }\rceil$ and $F =$. Then $\overline{T}\rceil = F$ and $\overline{F}\rceil = T$. Thus we can interpret $\overline{a}\rceil$ as $\neg a$. And then we have

$$a \rightarrow b = (\neg a) \vee b = \overline{a}\rceil\, b$$

so that implication in logic becomes the operation $\overline{a}\rceil\, b$ in the algebra of the primary arithmetic. It is then easy to see that *and* is expressed by the formula

$$a \wedge b = \overline{\overline{a}\rceil\, \overline{b}\rceil}\rceil$$

since the formula on the right is marked exactly when both a and b are marked.

In this way basic logic rests on the primary arithmetic and can be seen as a patterning of its operations and processes. I hope to have convinced the reader that this is a satisfactory entry into logic starting with the notions of sign and distinction. One can explore a great deal from this basis and I will stop here with only a hint of what may come.

One aspect of logic that comes forth at once is the role of paradox. Consider the Liar Paradox in the form $L = \neg L$. Rewriting into primary algebra, we find

$$L = \overline{L}\rceil\,.$$

Since the mark makes a distinction between its inside and its outside, this equation suggests that L must itself have a sign that indicates a form that re-enters its own indicational space. L must have a sign as shown below.

In crossing from the state inside the reentering mark, we arrive again at the inside. The inside is the outside and the outside is the inside. The sign connotes a distinction that controverts itself and yet it is still a sign in the constellation of all signs and it still distinguishes itself in its own form. Nothing is left but the time of circulation in the oscillation of inside and outside, and beyond this state of time we have returned to void.

Another aspect of logic is the articulation of the use of the words "all" and "there exists" in relation to "not", "and" and "or". By starting with the interpretation of $\overline{a|}\,b$ as "all a are b", we can proceed to interpret a collection of related expressions. Here is a partial list.

$$\overline{a|}\,b \text{ — "all } a \text{ are } b\text{"},$$

$$\overline{\overline{a|}\,b|} \text{ — "not all } a \text{ are } b\text{"} = \text{"some } a \text{ are not } b\text{"},$$

$$\overline{\overline{a|}\,\overline{b|}|} \text{ — "not all } a \text{ are not } b\text{"} = \text{"some } a \text{ are } b\text{"},$$

$$ab = \overline{\overline{a|}|}\,b \text{ — "all not } a \text{ are } b\text{"} = \text{"all not } b \text{ are } a\text{"},$$

$$\overline{ab|} \text{ — "not all not } a \text{ are } b\text{"} = \text{"some not } a \text{ are not } b\text{"}.$$

The reader will find it rewarding to work through these interpretations and to validate the consistency of this interpretation. With that in hand, it is possible to see the formal structure of syllogisms using laws of form. We refer the reader to Spencer-Brown's book (Laws of Form, 1969, Appendix 2) for more details. The basic syllogism is classically called BARBARA and is of the form "All a are b. All b are c. Therefore all a are c." We could write this in laws of form notation by stating that (All a are b.) and (All b are c.) implies (All a are c.). This becomes the statement

$$\overline{\overline{\overline{\overline{a|}\,b|}\,\overline{\overline{b|}\,c|}|}|}\,\overline{a|}\,c = \overline{\overline{a|}\,b|}\,\overline{\overline{b|}\,c|}\,\overline{a|}\,c.$$

We use the rules $\overline{\overline{x|}|} = x$ and $xy = yx$ in handling these expressions. *It is necessary at this juncture to not use the calculus of indications, since we are dealing with multiplicities of distinctions.* Note that BARBARA has now become

$$\overline{\overline{a|}\,b|}\,\overline{\overline{b|}\,c|}\,\overline{a|}\,c,$$

which we interpret as

$$\overline{P|}\,\overline{Q|}\,R,$$

where P and Q are premises of the syllogism and R is the conclusion. We are now in a position to derive new syllogisms from BARBARA. For example, we can rewrite

$$\overline{P|}\,\overline{Q|}\,R = \overline{P|}\,\overline{\overline{R|}|}\,\overline{Q|}\,.$$

This is the syllogism with premises (P) and (the negation of R). The conclusion is (the negation of Q). Apply this to BARBARA.

$$\overline{\overline{a|}\,b|}\,\overline{\overline{b|}\,c|}\,\overline{a|}\,c = \overline{\overline{a|}\,b|}\,\overline{\overline{\overline{a|}\,c|}|}\,\overline{\overline{b|}\,c|}\,.$$

The premises are now

$$\overline{a|}\,b \text{ — “all } a \text{ are } b\text{”},$$

$$\overline{\overline{a|}\,c|} \text{ — “Some } a \text{ are not } c,\text{”}$$

and the conclusion is

$$\overline{\overline{b|}\,c|} \text{ — “Some } b \text{ are not } c.\text{”}$$

This is a correct syllogism. Spencer-Brown remarks in Appendix 2 to that, in this way, BARBARA gives rise to all 24 syllogisms using “some” and “all” as well as “not”, “and” and “or”. The mark and expressions using that mark have taken a logical and linguistic role that reflects properties of our logic, speech and rhetoric. This aspect of the relationship of laws of form with logic and language deserves further study. In particular we see that the language of interpretations such as

$$\overline{ab|} = \text{“some not } a \text{ are not } b\text{”}$$

is here consistently mixed with the meta-language of implication, and or. We are aware of the fact that ordinary language can allow language and meta-language to interpenetrate. This fragment of interpenetration in elementary logic is implicit in the relationship of the universal and the particular that is always present in mathematics and logic.

Remark. Note that in this discussion of syllogism, we have assumed that "Some a are b." entails the existence of a's that are b's. In classical syllogistic it is often assumed that the collections under discussion are not empty. Under those assumptions there are other syllogisms than the ones coming from BARBARA. For example, the classical BARBARI is:

All a are b. All b are c. Therefore: Some a are c.

If a is empty then this syllogism is invalid, but if we assume that a is not empty, then it is correct, and we can rearrange it to the form:

All a are b. No a are c. Therefore: Some b are not c.

The rearrangement remains valid under the assumption that a is not empty. With this in mind, we can repeat the Spencer-Brown analysis for the remaining collections of classical syllogisms. I am indebted to Armahedi Mazar for making this significant observation. A complete treatment of this point of view is under preparation.

Paradox Resolved

It well known that taking an element J with $J = \overline{J\,}|$ and including it in the primary algebra with no change in conventions (including substitution) leads to the collapse of the marked and unmarked states. This occurs as follows:

We use the fact that $\overline{pp\,}| = \overline{\ \ }|$ for any p, that $pp = p$ for any p and that $\overline{\overline{p\,}|\,}| = p$ for any p in the Primary Algebra. Then

$$\overline{\ \ }| = J\overline{J\,}| = JJ = J$$

Hence

$$J = \overline{\ \ }|.$$

Hence

$$J = \overline{J\,}| = \overline{\overline{\ \ }|\,}|.$$

$$\text{Therefore}\ \overline{\ \ }| = \overline{\overline{\ \ }|\,}|.$$

Since one does not want the system to collapse, a change is required. The simplest and most profound change is the *Flagg Resolution* (Flagg and Kauffman, 2019):

In a given formula containing J, if J is changed to $\overline{J}\rceil$ *for any appearance of J, it must be changed to* $\overline{J}\rceil$ *for all appearances of J.*

If we follow the Flagg Resolution, then we cannot write $J\overline{J}\rceil = JJ$, but we can write $J\overline{J}\rceil = \overline{J}\rceil\,\overline{\overline{J}\rceil}\rceil = \overline{J}\rceil J$, and this produces no contradiction. The Flagg Resolution preserves the relationship between J and $\overline{J}\rceil$ that produces $J\overline{J}\rceil = \overline{\ \ }\rceil$ without sacrificing the statement that $J = \overline{J}\rceil$.

Note that the Flagg Resolution is not explicitly temporal but it is textually spatial, in the sense that all appearances of J in the text must be handled at once if a change is to be made. In this sense the spatiality of this resolution is non-local and in a process of transformation of the text, demands simultaneity. An example of this weaving of time and space is seen in the discrete waveform interpretations. We may take a waveform w as

$$w : \ldots abab \ldots = [a, b] \text{ and } \overline{w}\rceil : \ldots \overline{a}\rceil\,\overline{b}\rceil\,\overline{a}\rceil\,\overline{b}\rceil \ldots = [\overline{a}\rceil, \overline{b}\rceil].$$

Then

$$w\overline{w}\rceil = [a\overline{a}\rceil, b\overline{b}\rceil] = [\overline{\ \ }\rceil, \overline{\ \ }\rceil] = \overline{\ \ }\rceil.$$

But consider the case where a and b are alternately marked and unmarked. Then we have

$$i : \ldots \overline{\ \ }\rceil\,\overline{\overline{\ \ }\rceil}\rceil\,\overline{\ \ }\rceil\,\overline{\overline{\ \ }\rceil}\rceil\,\overline{\ \ }\rceil\,\overline{\overline{\ \ }\rceil}\rceil \ldots = [\overline{\ \ }\rceil, \overline{\overline{\ \ }\rceil}\rceil],$$

$$\overline{i}\rceil : \ldots \overline{\overline{\ \ }\rceil}\rceil\,\overline{\ \ }\rceil\,\overline{\overline{\ \ }\rceil}\rceil\,\overline{\ \ }\rceil\,\overline{\overline{\ \ }\rceil}\rceil\,\overline{\ \ }\rceil \ldots = [\overline{\overline{\ \ }\rceil}\rceil, \overline{\ \ }\rceil].$$

But we can also state that $\overline{i}\rceil = i$ in the sense that these two waveforms are indistinguishable from one another when they are seen individually. Nevertheless we have $i\ \overline{i}\rceil = \overline{\ \ }\rceil$ when they are taken together. The (phase shifted) relationship between them is held when they appear together in space or as processes, in simultaneity. Thus the Flagg Resolution stands prior to the emergence of space and time in these interpretations.

The Flagg Resolution can be applied to any and all elements J such that $J = \overline{J}|$, and thus allows us to incorporate apparently paradoxical elements into the Primary Algebra. The Resolution occurs at the level of the structure of the text and so can be regarded as precursor to both spatial and temporal resolutions of paradox. It is beneath a general theory of types and only makes a type distinction for the particular elements that are paradoxical. The reader may wish to think about how the Resolution will apply to entities such as the Russell Set or to the Barber who shaves everyone who does not shave himself. We leave such questions to explorations for the reader.

Other Logics

Along with finding the calculus of indications as a precursor to Boolean algebra, the stance of taking distinction as a starting point leads to other logics. Such logics have often been constructed to resolve paradox without changing any rules of substitution as we did with the Flagg Resolution above.

For example, one can take pairs of elements in the form (a, b) where a and b are elements in the calculus of indications. Regard $(\urcorner, \urcorner) = \urcorner$ and $(\overline{\urcorner}|, \overline{\urcorner}|) = \overline{\urcorner}|$, or in abbreviated form $(,) =$. The elements $I = (\urcorner, \overline{\urcorner}|)$ and $J = (\overline{\urcorner}|, \urcorner)$ are new elements and we combine pairs term wise as in $(a,b)(c,d) = (ac, bd)$. Thus, we have $IJ = (\urcorner\overline{\urcorner}|, \overline{\urcorner}|\urcorner) = (\urcorner, \urcorner) = \urcorner$. To make a different structure, we define crossing by the equation $\overline{(a,b)}| = (\overline{b}|, \overline{a}|)$. Notice that this new form of crossing is a combination of standard crossing and the permutation of the elements of the ordered pair. We then have $\overline{I}| = I$ and $\overline{J}| = J$. Thus this new arithmetic has two 'paradoxical' values. I have called this the *Waveform Arithmetic* (Kauffman, 1978b, 1980). The logic that corresponds to this arithmetic has four values, marked, unmarked, I and J. The self-referential values I and J can be interpreted as possibly true and possibly false in a four-valued logic. There is a lot of structure in the Waveform Arithmetic (WA) and its associated (DeMorgan) algebra. And there are other possibilities. A condensation of the WA gives a three-valued logic and a direct correspondence with the three valued logic of Lukasiewcz (Kauffman, 1987a).

Another possibility arises in the work of Art Collings (Collings, 2017). Collings constructed an arithmetic with a mark of order four. That is, Collings mark applied to itself four times is the unmarked state; Collings mark applied to itself twice is the usual mark. Collings and Kauffman (in these proceedings) show that his arithmetic can be modeled with ordered pairs (Kauffman, 2017) where we define $\lceil\overline{(a,b)\;} = (\overline{b}\rceil, a)$. The left-mark $\ulcorner$ is the Collings operator of order four. The reader will find it easy to check that $\lceil\overline{\ulcorner} = \urcorner$ and other properties of this system that we call BF arithmetic. This definition of the left-mark is in direct analogy with Sir William Rowan Hamilton's definition of the square root of minus one. The reader will recall that i, the square root of negative unity, can be defined as an operator on pairs of real numbers via the formula $i(a,b) = (-b,a)$. It is very interesting to think about BF in relation to Spencer-Brown's suggestions about imaginary values in logic and mathematics.

In Fig. 3, the geometry of the square root of minus one, juxtaposed with a diagrammatic geometry for the square root of negation is illustrated. The formal structure of the square root of negation is an extension of laws of form that is so close to the construction of the complex numbers, that we feel that it will be the source of significant insight into the nature of imaginary values.

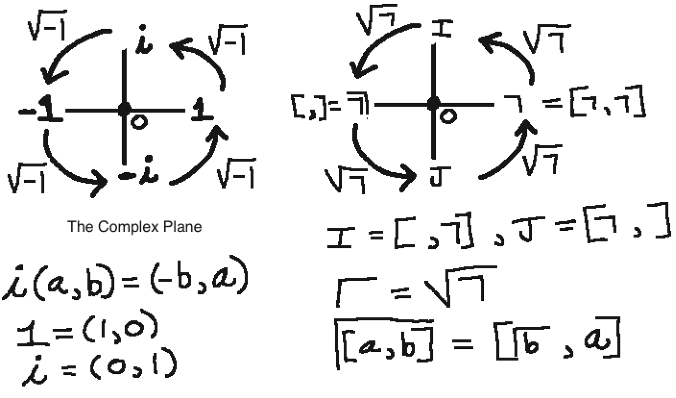


Fig. 3. Square root of minus one and square root of negation.

Another variation is to introduce a new arithmetical value @ that is *almost marked* in the sense that

$$\overline{@}\rceil =$$

$$@@ = @$$

$$\urcorner @ = \urcorner$$

with the usual laws of calling an crossing included. It is understood that @ and the mark have distinct values. Call this the shaded arithmetic. Note that in the shaded arithmetic we have $\overline{@}\rceil @ = @$ and so it is the case that Spencer-Brown's Integration ($\overline{a}\rceil a = \urcorner$ for all a) is not satisfied in this arithmetic. Furthermore, $\overline{\overline{\overline{@}\rceil}\rceil}\rceil = \overline{\urcorner}\rceil = \overline{@}\rceil$ while $\overline{\overline{@}\rceil}\rceil = \urcorner$ is not equal to @. See Kauffman (2017) for a discussion of this algebra and its relationship with Heyting algebras.

We will not go further in these directions here. The main point is that by starting with a distinction, we start prior to particular choices of formal logic and so can see the development of these different aspects of logic from the root of the formation of distinctions.

7. Re-entry and a Sign for Itself

Charles Sanders Peirce came very close to inventing the mark $\urcorner$ in his "sign of illation" as shown in Fig. 4.

> [C. S. Peirce, *The New Elements of Mathematics*, edited by Carolyn Eisele, Volume IV—Mathematical Philosophy, Chapter VI—The Logical Algebra of Boole. pp. 106–115. Mouton Publishers, The Hague—Paris and Humanities Press, Atlantic Highlands, N.J.(1976).]

The Peirce sign of illation is used for logical implication and it is an amalgam of negation as the over-bar and logical or, written as a

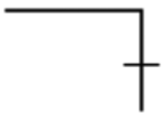


Fig. 4. Peirce's sign of illation.

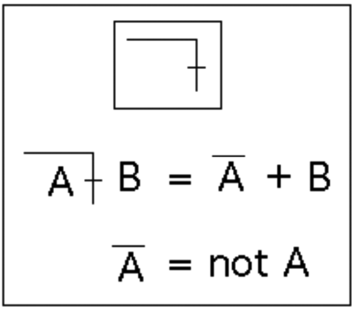


Fig. 5. The Peirce sign of illation.

$+$ sign on the left vertical part. See Fig. 5 for an illustration of this anatomy of the Peirce sign.

The mark $\urcorner$ goes further than Peirce since the unmarked state is allowed, and the operation of or is unmarked and indicated by juxtaposition. We still have the decomposition of $\overline{A}\rceil B$ as "Not(A) or B" once the mark is understood to operate as negation. The largest difference is semiotic, in that the mark can be taken as a universal sign and as a sign for itself. As such it has a conversational domain quite independent of Boolean logic. In this role, the mark can be seen as part of a wider context of distinction that informs and illuminates logic and mathematics.

Peirce spoke of a "Sign of Itself". Here is a key passage from his work.

"But in order that anything should be a Sign it must 'represent', as we say, something else called its Object, although the condition that a Sign must be other than its Object is perhaps arbitrary, since, if we insist upon it we must at least make an exception in the case of a Sign that is part of a Sign. Thus nothing prevents an actor who acts a character in a an historical drama from carrying as a theatrical 'property' the very relic that article is supposed merely to represent, such as the crucifix that Bulwer's Richelieu holds up with such an effort in his defiance. On a map of an island laid down upon the soil of that island there must, under all ordinary circumstances, be some position, some point, marked or not, that represents qua place on the map the very same point qua place on the island. . .

If a Sign is other than its Object, there must exist, either in thought or in expression, some explanation or argument or other

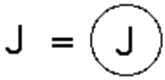


Fig. 6. Re-entrant equation.

context, showing how—upon what system or for what reason the Sign represents the Object or set of Objects that it does. Now the Sign and the explanation make up another Sign, and since the explanation will be a Sign, it will probably require an additional explanation, which taken together with the already enlarged Sign will make up a still larger Sign; and proceeding in the same way we shall, or should ultimately reach a Sign of itself, containing its own explanation and those of all its significant parts; and according to this explanation each such part has some other part as its Object."

> [C. S. Peirce, *Collected Papers—II, pp. 2.230–2.231*, edited by Charles Hartshorne and Paul Weiss, Harvard University Press, Cambridge (1933).]

There are extraordinary and topological ideas in this passage. There is an implicit reference to the notion of a fixed point so that a map and its image must have a coincidence. There is the notion that Sign and Explanation will undergo recursion until ultimately the Sign, the Explanation and the Object become One. We have begun with a sign $\rceil$ that is a sign for itself in the sense that it represents the distinction that is made by the sign in its coincidence with an Observer. And yet the recursion is always possible. Consider the re-entrant sign that was discussed in Section V. The re-entrant sign can be taken to be a solution to $J = \overline{J|}$ or, in a graphical mode, to be a solution to the re-embedding of J inside a circle as in Fig. 6.

The equation $J = \overline{J|}$ asserts the re-entry of J into its own indicational space, and it exhibits J as a part of itself. The equation is the explanation of the nature of J as re-entrant and can be taken as a description of the recursive process that generates an infinite nest of circles. It is J *as an equation* that yields J as a Sign of itself. If we wish to embody the equation in the Sign itself then we need to allow the Sign to indicate its own re-entry as we do in Figs. 7 and 8. This symbol does "contain its own explanation" in the sense that we

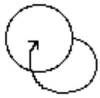


Fig. 7. A re-entrant form.

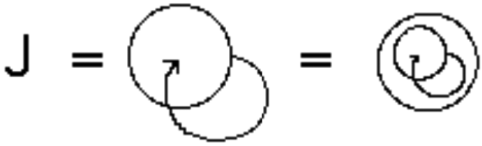


Fig. 8. Equation, indication and re-entry.

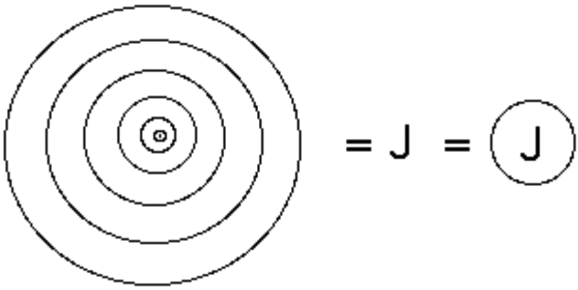


Fig. 9. Infinite regress and fixed point or eigenform.

interpret the arrow as an instruction to re-enter the form inside the circle.

> [See L. H. Kauffman (2001), The Mathematics of Charles Sanders Peirce, *Cybernetics Human Knowing*, 8(1–2): 79–110.]

In fact that re-entry occurs ad infinitum as indicated in Fig. 9, from which we see that the equational re-entry is recaptured from the self-standing form of Fig. 9.

We see that it is a matter of language that fuels the difference between a simple form that stands for itself such as the mark, and those re-entry forms that partake of infinite regress (as shown in Fig. 8). This infinite regress is a microcosm of the infinite regress of

Peirce and allows us to solve for J as an unending nest of marks.

$$J = \overline{\overline{\overline{\overline{\overline{\ldots}\rceil}\rceil}\rceil}\rceil}\rceil = \overline{J}\rceil.$$

It must be mentioned that the work of Church and Curry on the Lambda Calculus (see [8–10, 12–14]) gives another approach to re-entry.

The Lambda Calculus method goes as follows.

Let $GX = \overline{XX}\rceil$. Then $GG = \overline{GG}\rceil$ and so we can take $J = GG$ to obtain $J = \overline{J}\rceil$ without any infinite regress! How did this happen? At the level of sign and operation, G is a duplicating device. Given an X, G makes two copies of X and places them under a mark. The equals sign means that *GX is replaced by* $\overline{XX}\rceil$, and the operation of G is defined by this replacement. When we replace X by G in the equation, we have put G in the position to act on G. G does act and produces GG with a mark around it. But GG is now ready to act again and so GG moves into the temporal domain and instructs a recursion: $GG = \overline{GG}\rceil = \overline{\overline{GG}\rceil}\rceil = \overline{\overline{\overline{GG}\rceil}\rceil}\rceil = \overline{\overline{\overline{\overline{GG}\rceil}\rceil}\rceil}\rceil = \ldots$ The infinite regress of J has been replaced by the inherent temporality of GG. The Church-Curry idea of recursion is in fact an outgrowth of the Russell Paradox of the Set of all Sets that are not members of themselves. To see how this plays out in the realm of signs

let XY denote that Y is a member of X.

We define the Russell Set by the equation

$$RX = \sim XX.$$

As the reader sees immediately, R is now the duplicating Gremlin. We have shifted the interpretation of the mark to negation and we use ordered juxtaposition as membership. If $RX = \sim XX$, then $RR = \sim RR$. We have the self-denial of the Russell Set. This could just as well have been written $RX = \overline{XX}\rceil$ and $RR = \overline{RR}\rceil$.

It need not be a paradox. RR is a re-entrant form and can be taken on its own cybernetic ground. We have the option to view the Russell set temporally in the Church-Curry recursion. Then Russell oscillates in time between being a member of itself and not being a member of itself. The Russell pendulum avoids the Russell singularity.

Reflexive domains and self-reference

The method we have just indicated gives fixed points for any F by the same method: $Gx = F(xx)$, whence $GG = F(GG)$. A domain where this can be accomplished is called a *reflexive domain.* To see the mathematical issues, suppose we have a domain D so that there is a binary operation on the elements of D that is written ab for a and b in D. We do not assume any other axioms about this operation. It is not commutative or associative, but it is closed in the sense that given a and b in D then ab is in D. Now assume that for every algebraic expression $A(x)$ in one variable x (such as $A(x) = F(xx)$ where F is in A and $F(xx)$ denotes the binary operation between F and xx) there is an element A in D so that $Aa = A(a)$ for every a in A. Thus letting $[D, D]$ denote the collection of all such operations, we assume that $D = [D, D]$. *Elements of D are in one-to-one correspondence with operations of D.* We then call D a reflexive domain.

As the reader can see, we have the

Theorem. *Given F in a reflexive domain, there is an element J in the domain such that $FJ = J$.*

Proof. The proof is given above and we repeat it for emphasis: Let $Gx = F(xx)$. Then G is an element of D (by reflexivity) and so G can be applied to itself. Thus $GG = F(GG)$. So, taking $J = GG$, we have $FJ = F(J) = J$. $\square$

Set theoretically, it is complex to show that reflexive domains exist. From the point of view of language it is natural to think of such a domain as always under evolution, as in a computer language within which new algorithms can always be constructed. These algorithms can be applied to previously created algorithms and even to themselves.

Linguistically, we inhabit the reflexive domain of "ordinary language" where speech is always under evolution and what we say can be applied to itself and to other acts of speech. In the extraordinary reflexive domain of ordinary language there is always the possibility for reference and self-reference.

It is worth elaborating on this point. Let $A \to B$ (an arrow from A to B) denote a reference from A to B. We can say that "A names B". Now define the *shift* of a reference A $\to$ B to be a new reference denoted $\#A \to BA$. For example, if I am introduced to Albert then at the point of the introduction we have a reference

$$\text{Albert} \rightarrow \text{Person}.$$

This shifts to

$$\#\text{Albert} \rightarrow \text{PersonAlbert},$$

a representative of my cognitive state consisting in a superposition of the perception of Albert and his name. We could have used #Albert = "Albert" to indicate this, but I shall use #Albert to indicate the meta-name associated with Albert.

Take the shift operation to be a syntactic image of the continual shifting of names, references and symbols to the apparent objects of the reference. Then we can see a syntactic image of how self reference comes about.

Suppose that $M \rightarrow \#$ is a name (M) for the meta-naming operation #. Shift this reference. Then we have

$$M \rightarrow \#$$

shifts to

$$\#M \rightarrow \#M.$$

The meta name of the meta-naming operation refers to itself. We can contemplate the syntax of a possible definition of the self:

$$I = \#M.$$

I am the meta- name of the name of my meta-naming operation. Heinz von Forester (Von Forester (1981, 2003)) said it in a wider frame: "I am the observed relation between myself and observing myself."

And indeed the shift shows that "statements" can refer to themselves via the meta operator:

$$\text{Suppose that } M \rightarrow F\#, \text{ then } \#M \rightarrow F\#M.$$

And in this way, F refers to "itself" by referring to $\#M$ and $\#M$ refers to F. Of course we take it that FA can be interpreted as F referring to A.

This method of obtaining self-reference, now seen to be part and parcel of ordinary speech, was used by Kurt Goedel to show that in any mathematical system rich enough to formally handle arithmetic, indirectly self-referential statements can be constructed. Such statements can deny their own provability within the system. The incompleteness of formal systems arises from the inevitability of self-reference and fixed points in sufficiently rich languages.

8. Finding Mathematics — The Theory of Numbers

Up to this point we have not actually ventured across a boundary into numerical mathematics. The construction of a sign that can stand for any sign and is self-referential involves no counting, no calculation, no algebra and seemingly no arithmetic of any kind. It would appear that we have arrived at a pivot point where one could begin thinking about the growth of thought and language with no regard to the development of mathematics.

The reader may ask: Is this a mathematical subject? Can we start with non-circular definitions and make progress in the usual mathematical way? Lets recall how it is often done. For example, we wish to study the concept of a positive integer number. So I tell you that I shall represent integers. I represent them by collections of dots in a plane space (see Fig. 10).

Figure 10 includes many ideas and distinctions about numbers that we use in everyday life and in mathematics. We understand that two collections of dots are equivalent if they can be put into one-to-one correspondence with each other, and we understand that some other collection has three elements if the collection labeled “3”

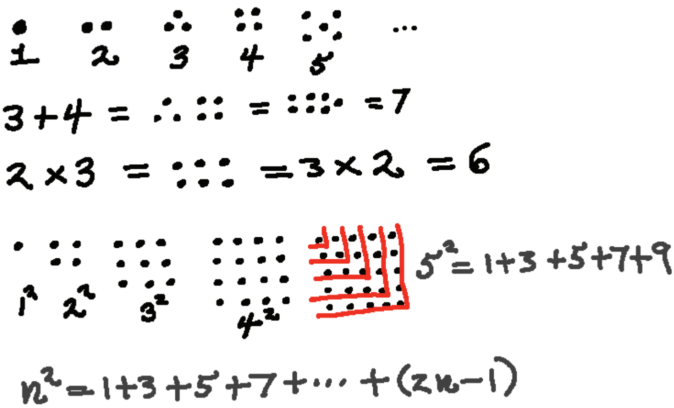


Fig. 10. Numbers and their distinctions.

in the figure can be put into one-to-one correspondence with those three dots. Another example is noting that I have "5" fingers on my right hand. This is determined by matching the five dots in the array above "5" with the fingers of my hand. We have all learned to perform these correspondences and distinctions since we were children.

In examining Fig. 10 and recalling what we know about arithmetic we can understand that natural numbers $(1, 2, 3, \ldots)$ are represented by collections of pebbles, by dots on a page, by notches on a stick or by knots on a rope. The history of this sort of representation goes back into the reaches of time.

The use of such "pebble numbers" is part of everyone's culture. We understand that five fingers on the hand are in one-to-one correspondence with the five pebbles that are labeled "five" or "5". The key to numbers is not just the pebbles but the idea/process of correspondence that allows us to declare that there are five fingers, five sides to a pentagon, five Platonic solids and so on. Each number becomes the pivot for all things that have such correspondence with it. Each number is a distinction of its own kind. We have learned a language of such distinctions and call it arithmetic.

It is in relation to arithmetic that we shall consider distinction itself. It is probably the case that no one could understand laws of form without knowing arithmetic in the above sense. But the system that Spencer-Brown created, the calculus of indications, a non-numerical arithmetic underlying Boolean algebra, this arithmetic can be understood on its own grounds and even taught to children. In fact, once one enters the domain of arithmetics of combinatorial interaction of distinctions then all manner of games and play that can arise among persons is seen to be related to the same domain that gives rise to the forms of mathematics.

Now return to Fig. 10 and note that it contains a pictorial proof that for any natural number n, $n^2 = 1 + 3 + 5 + \cdots + (2n - 1)$. That is, any square natural number of the form $n \times n$ is the sum of all of the odd numbers from 1 to $2n - 1$. To a person acquainted with the distinctions and conventions associated with pebble representations of natural numbers, the proof is transparent since the decomposition of the square into right angle bracket forms, each carrying an odd number of pebbles, is apparent. It is apparent that this pattern carries forward for a square of any size.

Number patterns and our notations for numbers evolve together. Thus we can, in the Arabic numeral system express numbers such as 10^(100^(1000)) that could never be expressed directly as a finite collection of pebbles on the beach. The sign we have just written still represents a distinction, but it is a distinction about distinctions. It is the power of the language of numbers and algebra that this forming of meta-statements can be accomplished seamlessly within the given language for number.

Before leaving this domain of number, consider the Fibonacci Sequence u{n} where

$$u\{n+1\} = u\{n\} + u\{n-1\} \text{ and } u\{1\} = 1, u\{2\} = 1.$$

We are all familiar with this inductive formula, and how it produces the series

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots.$$

Now, an experiment. Choose two consecutive Fibonacci numbers and add their squares. For example and $2^2 + 3^2 = 4 + 9 = 13$, and $5^2 + 8^2 = 25 + 64 = 144$.

You will find that *the sum of the squares of any two consecutive Fibonacci numbers is itself a Fibonacci number.* It is the appearance of patterns like this in fundamental mathematical systems that leads to the need for, and the desire for mathematical proofs, arguments for a phenomenon that are completely convincing. Perhaps you would like to find a proof of this statement about the squares of Fibonacci numbers.

Mathematics has symbolic beginnings and is woven into the structure of language. What signs are the least signs needed for number? We might take on the sign | for 1, the sign || for 2 and generally, the sign $n = |||\ldots|$ (with n vertical marks) for the integer n. In this mode we have $n+1 = n|$. And $n+m = nm$, the juxtaposition of the marks for n and the marks for m.

$$1 = |, 2 = ||, 3 = |||, 4 = ||||, 5 = ||||| \text{ and so on.}$$

$$3 + 2 = ||| + || = ||||| = 5.$$

Arithmetic can grow from elemental signs and indeed we can use the Spencer-Brown mark to represent numbers with zero as the unmarked state.

$$0 =$$

$$1 = \urcorner$$

$$2 = \urcorner\urcorner$$

$$3 = \urcorner\urcorner\urcorner$$

and so on. Note that in order to represent numbers in this way, we must rescind the Law of Calling so that multiplicities of marks stand for different numbers. With the Law of Calling removed, we are no longer working with only one distinction. Each new number is a distinction in its own form.

This is the beginning of arithmetic, the gateway into the depths and beauties of mathematics. This foundation for the theory of numbers will clarify the deep quests of number theory. One can begin by wondering about the prime numbers. Six is not prime. It is a product of 2 and 3. The row of six marks is two rows of three marks and it is three rows of two marks. It seems that numbers want their own distinctions. After conversing with six we see that six prefers to be seen as

$$\begin{array}{c} \urcorner\urcorner\urcorner \\ \urcorner\urcorner\urcorner \end{array}$$

or as

$$\begin{array}{c} \urcorner\urcorner \\ \urcorner\urcorner \\ \urcorner\urcorner \end{array}$$

but confinement to a single row is just not comfortable for a composite number. Let us find arithmetic anew by staying close to its origin in the origination of a sign.

If we wish to count, it will suffice to choose, from the vast collection of possible expressions, some that represent the numbers we wish to use in our counting. To this effect (without calling and without crossing) we shall now take the above representative forms for natural numbers, starting with zero as an empty form.

$$0 =$$
$$1 = \urcorner$$
$$2 = \urcorner\urcorner$$
$$3 = \urcorner\urcorner\urcorner$$
$$\vdots$$

Here is a sufficient collection of representative multiplicities, like marks on a tally stick. Yet, even here, the mark has a multiple role of container and counter. One way to obtain arithmetic operations is to allow the law of crossing and not allow the law of calling.

We shall assume that $\overline{\overline{E}\rceil}\rceil = E$ for any expression E.
We shall define addition (+) by juxtaposition:

$$\urcorner\urcorner + \urcorner\urcorner\urcorner = \urcorner\urcorner\urcorner\urcorner\urcorner.$$

We have used the law of crossing, and this entails

$$0 = \overline{\urcorner}\rceil.$$

It entails that 0 can be taken to be the unmarked state.

Then we shall define multiplication (*) by the formula:

$$a^*b = \overline{\overline{a}\rceil\,\overline{b}\rceil}\rceil,$$

where it is understood that we shall resolve this expression by using transposition

$$a\overline{\overline{b}\rceil\,\overline{c}\rceil}\rceil = \overline{\overline{ab}\rceil\,\overline{ac}\rceil}\rceil$$

in the sense that

$$x^*3 = \overline{\overline{x}\rceil\,\overline{3}\rceil}\rceil = \overline{\overline{x}\rceil\,\overline{\urcorner\urcorner\urcorner}\rceil}\rceil$$
$$= \overline{\overline{\overline{x}\rceil}\rceil\,\overline{\overline{x}\rceil}\rceil\,\overline{\overline{x}\rceil}\rceil}\rceil = \overline{\overline{xxx}\rceil}\rceil = xxx = x + x + x.$$

In a multiplicative context we use the law of transposition:

$$a^*n = \overline{\overline{a}\rceil\,\overline{\urcorner\urcorner\urcorner\dots\urcorner}\rceil}\rceil = \overline{\overline{\overline{a}\rceil}\rceil\,\overline{\overline{a}\rceil}\rceil\,\overline{\overline{a}\rceil}\rceil\dots\overline{\overline{a}\rceil}\rceil}\rceil = aaa\dots a.$$

In this rule for multiplication, we take on an identity that does hold in primary arithmetic. Here it simply says that $a^{*}n$ is the addition of n copies of a. In the form, $\overline{a}|$ is inserted into each mark in the representative for $\overline{n}| = \overline{\urcorner\urcorner\urcorner\ldots\urcorner}|$.

If n = 0, then there is just one mark to insert and so we have $0^{*}a = 0$. Multiplication on the right by zero yields zero in the same way.

Let us call this structure *arithmetic* (as opposed to primary arithmetic). Here 1 is the multiplicative void and 0 is the additive void. *We have two voids in arithmetic, one for each of its fundamental operations.*

Clarification

In order to clarify the structure of this arithmetic it is important to distinguish the two voids: the multiplicative void whose value is 1 and the additive void whose value is 0. When we write an expression like $3 = \urcorner\urcorner\urcorner$, the outermost space is taken to be an additive space so that juxtaposition of forms corresponds to addition. If we were to write $3 = \overline{\urcorner\urcorner\urcorner}|$, then the outer space would be an multiplicative space, and the juxtaposition would be multiplication. *In an arbitrary expression every crossing of a mark switches the space from additive to multiplicative or from multiplicative to additive.* For example consider this expression.

$$\overline{\overline{\overline{\overline{\otimes}|\oplus}|\overline{\overline{\otimes}|\oplus}|\otimes}|\oplus}|\otimes.$$

We have labeled each space in the expression with its arithmetic type. We can then label the deepest spaces of the expression with the value of the corresponding arithmetical void. In this case the void is 1 since the deepest spaces are multiplicative.

We take as the rule for evaluation that *an arithmetical value passes through a mark unchanged, but is interpreted in the new space as additive or multiplicative according to the arithmetic type of that space.*

For example, in evaluating the expression above we have that each deepest space has value 1 and so we have

$$\overline{\overline{\overline{\overline{1}|}|\overline{\overline{1}|}|}|}|.$$

The two ones enter a multiplicative space and so are multiplied to a single one and this one survives to the outside. So the value of this expression is equal to one.

In this example,

$$\overline{\overline{\;}|\,\overline{\;}|\,\overline{\;}|}|_{\otimes}$$

an outer multiplicative space has three inner deepest multiplicative spaces. They each have value 1 and the 1's are each transmitted to the next space up. This next space is additive and the three 1's add to produce 3. The 3 is transmitted to the outer space and so $\overline{\overline{\;}|\,\overline{\;}|\,\overline{\;}|}|$ represents the value 3. But if the outer space of this same expression is taken to be additive, then the three deepest spaces are also additive voids and so have value 0. This 0 value is transmitted outward and the expression itself has value 0 with an additive outer space. Alternatively, the expression $\overline{\;}|\,\overline{\;}|\,\overline{\;}|$ has value 3 in an additive outer space, but is equal to 0 in a multiplicative outer space. We have the choice to designate the type of that outer space. Note that when a given expression is crossed, its value remains the same, but the space in which its value is used switches from additive to multiplicative or vice versa.

Negative Numbers

We now introduce negative numbers.

$$-1 = \overline{|\overline{\;}}|$$

$$-2 = \overline{|\overline{\;}\,|\overline{\;}}|$$

$$-3 = \overline{|\overline{\;}\,|\overline{\;}\,|\overline{\;}}|$$

$$\vdots$$

The negative (or reverse) mark $|\overline{\;}$ will interact with the positive mark $\overline{\;}|$ by a form of cancellation: $|\overline{\;}\,\overline{\;}| =$ and $\overline{\;}|\,|\overline{\;} =$.

And we shall also assume that $|\overline{|\overline{E}} = E$ and $\overline{\overline{E}|}| = |\overline{|\overline{E}}|$ for any extended expression E that uses both positive and negative marks.

In Depth

Returning to the evaluation from deepest spaces, we take as a rule of evaluation that *a value passes through a reversed mark unchanged* *except for its sign, which is changed from plus to minus, or from minus to plus.*

Thus we have that $\overline{x\,}|$ presents the value x to its outer space and $|\overline{\,x}$ presents the value $-x$ to its outer space.

$\overline{|\overline{\ }\;|\overline{\ }\;|\overline{\ }\,}|$ represents -3 in a multiplicative space because each deepest space contributes 1. These 1's are transmitted by reverse marks to -1's in an additive space, tallying to -3 and this -3 is then transmitted to the outer space.

Now consider $|\overline{\,\overline{\ }|\;\overline{\ }|\;\overline{\ }|}$. *This expression also represents -3 in the multiplicative space.* Each deepest space contributes 1 and these 1's transmit and add to produce 3. Then three crosses the reverse mark and becomes -3. In particular, both $\overline{|\overline{\ }\,}|$ and $|\overline{\,\overline{\ }|}$ represent -1 in the multiplicative space.

Note that in a multiplicative space a^*b is represented by juxtaposition, and

$$a + b = \overline{\overline{a\,}|\,\overline{b\,}|\,}|.$$

The reader can easily verify that $1 + (-1) = 0$.

$$1 + (-1) = \overline{\overline{\ }|\,}| + \overline{|\overline{\ }\,}| = \overline{\overline{\overline{\overline{\ }|\,}|\,}|\,\overline{\overline{|\overline{\ }\,}|\,}|\,}| = \overline{\overline{\ }|\,|\overline{\ }\,}| = \overline{\ }| = 0.$$

Now consider the product of negative unity with negative unity.

$$(-1)^*(-1) = \overline{|\overline{\ }\,}|\,\overline{|\overline{\ }\,}| = \overline{|\overline{\,\overline{|\overline{\ }\,}|\,}\,}| = \overline{\overline{|\overline{\,|\overline{\ }\,}\,}|\,}| = \overline{\overline{\ }|\,}| = 1.$$

We see that it will be so that $(-a)^*(-b) = a \times b$. The product of two negatives is a positive.

The Negation of Zero

In a multiplicative space, we have $0 = \overline{\ }|$ but also $0 = |\overline{\ }$ since 0 transmitted across a reversed mark becomes -0, and $-0 = 0$ in our evaluation arithmetic. Note that we also have $\overline{\ }|\,\overline{\ }| = \overline{\ }|$ in a multiplicative space, since $0 \times 0 = 0$. But these identities do not hold in an additive space and so we can say that $\overline{\overline{\ }|\,\overline{\ }|\,}|$ is 2, since the interior marks are in an additive space and there the law of calling does not hold. Equality is dependent upon context.

We have now constructed the integers

$$\ldots -3, -2, -1, 0, 1, 2, 3, \ldots$$

on the basis of the act of distinction and the given properties of a mark of distinction and a mark of negation. The arithmetical negation is of order two, just as is our crossing operation, but it is of a different kind than simple crossing. Arithmetical negation is a symmetry of the system of integers, exchanging the positive and the negative while pivoting on 0.

$$\sim \overline{\urcorner\urcorner\ldots\urcorner}\rceil = \overline{\ulcorner}\rceil\overline{\urcorner\urcorner\ldots\urcorner}\rceil = \overline{\ulcorner\ulcorner\ulcorner\ldots\ulcorner}\rceil.$$

$$\sim \urcorner = \overline{\ulcorner}\rceil\urcorner = \urcorner\overline{|\urcorner} = \overline{|\overline{\urcorner}|} = \ulcorner = \urcorner.$$

We have the fundamental arithmetical self-reference

$$\sim \urcorner = \urcorner.$$

The negation of zero is zero.

With this understanding, we can step back and see Boolean arithmetic and Boolean algebra in a new light.

Powers

How shall we represent a^b? A number a raised to the b-th power means a multiplied by itself b times. Thus $2^3 = 2{*}2{*}2$. Lets write this in an additive space. Then it reads $2^3 = \overline{\overline{2}\rceil\overline{2}\rceil\overline{2}\rceil}\rceil = 2.\overline{\urcorner\urcorner\urcorner}\rceil = 2.\overline{3}\rceil$ where we have used transposition in a notational way and indicated it by a dot. We can then write $a^b = a \cdot \overline{b}\rceil$ where a and b are numbers for an additive space. Zero in the additive space is the unmarked state and so we have

$$0^0 = \urcorner.$$

In this context, the void raised to the void power is the marked state. This can be taken to be an explanation of the ancient philosophical conundrum: How can nothing give rise to something?

Peano Arithmetic

There is a well-known axiomatization of the natural numbers due to Giuseppe Peano in 1889 (van Heiienoort, 1967). In Peano's approach to numbers there is posited a number 0 and a successor function S

that assigns to any given number n another number, its successor $S(n)$ sometimes denoted by $S(n) = n' = n + 1$. Then one takes axioms that say that equality of numbers is an equivalence relation, that $S(n)$ is never equal to 0, that $S(n) = S(m)$ if and only if $n = m$. Along with this one takes the Principle of Mathematical Induction*: If one has a proposition $P(n)$ about natural numbers n, and if $P(0)$ is true and if it is shown that the truth of $P(k)$ implies the truth of $P(S(k))$ for all k, then $P(n)$ is true for all natural numbers.*

It is of interest to see how we can formulate Peano arithmetic in terms of laws of form. To this end we start with just expressions in the primary arithmetic and only the algebraic law of crossing (reflection) so that it is given that $\overline{\overline{x|}\,|} = x$ for any expression x. Then let $0 = \overline{\;|}$, and define

$$S(x) = \overline{\overline{x|}\,\overline{\;|}\,|}$$

for any x. Note that we then obtain the crossed form of natural numbers discussed in this section.

$$0 = \overline{\;|},$$

$$S(0) = \overline{\overline{\overline{\;|}\,|}\,\overline{\;|}\,|} = \overline{\overline{\;|}\,|} = 1,$$

$$S(S(0)) = \overline{\overline{\overline{\overline{\;|}\,|}\,|}\,\overline{\;|}\,|} = \overline{\overline{\;|}\,\overline{\;|}\,|} = 2,$$

$$S(S(S(0))) = \overline{\overline{\overline{\overline{\;|}\,\overline{\;|}\,|}\,|}\,\overline{\;|}\,|} = \overline{\overline{\;|}\,\overline{\;|}\,\overline{\;|}\,|} = 3.$$

We can define addition directly by

$$x + y = \overline{\overline{x|}\,\overline{y|}\,|}$$

exactly as we have done before. We then define multiplication inductively via the formula

$$x^*Sy = \overline{\overline{x^*y|}\,\overline{x|}\,|}.$$

This is the same as saying that $x^*(y+1) = x^*y + x$, and we can then prove by induction that

$$x^*\overline{\overline{y|}\,\overline{z|}\,|} = \overline{\overline{x^*y|}\,\overline{x^*z|}\,|},$$

recovering our laws of form point of view about integral arithmetic.

The Division Algorithm, the Euclidean Algorithm and Prime Factorization

Given an two natural numbers n and m with $n < m$ we can write $m = q^*n + r$ where $0 \leq r < n$. If we think of m as a row of marks, this is a matter of subtracting copies of the row n as many times as possible and tallying the count and the remainder. Basic number theory develops naturally from numbers as rows of marks. Repeated application of this division algorithm leads to the Euclidean Algorithm that determines the greatest common divisor of two numbers, and shows that if d is the greatest common divisor of a and b then there are integers r and s so that $d = ra + sb$. Once one has this result one can prove that if a *prime number* p (a number, not equal to 1, with no divisors other then itself and 1) divides a product a^*b, the p must divide one of a or b. From this is obtained the uniqueness of the prime factorization of natural numbers.

Eratosthenes Sieve and the Riemann Zeta Function

Eratosthenes Sieve is the earliest method to enumerate the prime numbers and it can be seen to occupy a direct relation to our representation of numbers as rows of marks. Begin with a list of all the natural numbers from 1 onwards.

Here is everything that could be known about these numbers if only we could fathom it. In this form *the nth box contains n marks*. We can use the usual notation and the list becomes

$$1234567\ldots$$

and these are but the names of the boxes. We must attend to the marks within each box. Examine each box from left to right. *If the number in the box is a multiple of an earlier number, replace it with the name of that earlier number.* This is the Eratosthenes Sieve. What is left are the prime numbers, repeated at places where the box for that place is divisible by that prime. For the above we have

$$1232527\ldots$$

and continuing in this fashion we have

$$123456789[10][11][12][13][14][15][16][17][18][19][20][21][22][23]\ldots$$

$$123252723[2][11][2][13][2][3][2][17][2][19][2][3][2][23]\ldots$$

This gives us the beginning list of prime numbers

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots.$$

There is much more to say about number theory. We cannot resist pointing out how the study of numbers propels one upwards from the primary arithmetic of laws of form to the use of algebra and even infinite forms, self-reference and indeed imaginary values in the sense of the complex numbers. Euclid had proved in 500 BC that there are infinitely many primes. Euler found a new proof in 1744 that used these tools, *tools of re-entry forms in the guise of infinite series.*

To see this, begin with the algebraic series

$$\begin{aligned} u &= 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots \\ &= 1 + x(1 + x + x^2 + x^3 + x^4 + \cdots) \\ &= 1 + xu. \end{aligned}$$

We see that the series u re-enters its own indicational space, and is indeed defined in terms of itself by the equation $u = 1 + xu$. Numerically, we know that u is a convergent series when x has absolute value less than unity, and then we can solve the re-entry form for x and obtain

$$u = 1/(1 - x).$$

If we let $x = 1/p$ where p is a prime number, then we obtain

$$1 + 1/p + 1/p^2 + 1/p^3 + \cdots = 1/(1 - 1/p) = (1 - 1/p)^{-1}.$$

Letting s be a variable we can write

$$1 + 1/p^s + 1/p^{2s} + \cdots = (1 - 1/p^s)^{-1}$$

and, following Euler, we have the amazing formula where the product is taken over all prime numbers and the sum is taken over all natural numbers from 1 onwards

$$\prod_p (1 + 1/p^s + 1/p^{2s} + 1/p^{3s} + 1/p^{4s} + \cdots)$$

$$= \prod_p (1 - 1/p^s)^{-1} = \sum_n 1/n^s.$$

Euler's formula follows when we write out the product on the left-hand side of this equation to obtain the sum of all possible reciprocal products of prime powers. But a natural number n is given uniquely as just such a product of prime powers. The correspondence is perfect and only the finite product terms survive to the right-hand side. Euler then noted that when $s = 1$, the series of reciprocals of the natural numbers diverges. But if there were only finitely many prime numbers, then the product on the left-hand side would be a specific rational number. This contradiction shows that there must be infinitely many primes, and indeed Euler's formula led to remarkable new progress in number theory that continues to this day. Euler actually had many variants of this proof that there are infinitely many primes. For example, if $s = 2$ then Euler proved that the sum on the right-hand side, the sum of the reciprocals of all squares of natural numbers from 1 onwards, is equal to $\pi^2/6$. Furthermore, Euler could prove that $\pi^2/6$ is not a rational number. Thus we have the formula

$$\prod_p (1 - 1/p^2)^{-1} = \sum_n 1/n^2 = \pi^2/6$$

and this again proves that there are infinitely many primes since if there were finitely many primes, the above formula would tell us that $\pi^2/6$ is rational. Finally, we can state that

$$\prod_p (p^2/(p^2 - 1)) = \pi^2/6$$

and thus the primes in totality can conspire to produce geometry arising from the form alone.

We can trace a line that starts with the ideal of distinction and indication, a mark of distinction, numbers as rows of marks, patterns of numbers, and find ourselves in the most recondite areas of mathematics. There is no lower limit on the considerations of the nature of number.

The function $\xi(s) = \sum_n 1/n^s$ when $s = a + bi$ where i is the square root of negative unity, is called the Riemann Zeta Function. Riemann (1859) formulated Euler's function over the complex numbers. With Riemann, the problems relating the Zeta function and the prime numbers come to a head. Riemann conjectured that all the so-called non-trivial zeroes of the Zeta function occur for numbers of the form $s = 1/2 + it$ where t is a real number. This is a vertical line in the complex plane that intersects the real line in the number 1/2. See Kauffman, Flagg, and Sahoo (2021) for more information about Riemann's Conjecture, called the Riemann Hypothesis. The Riemann Hypothesis is the most famous unsolved mathematics problem in the world at the time of this writing. It is extraordinary that the symbol $s = 1/2 +$ it has so much in common with the mark at the semiotic level. 1/2 connotes a distinction. i connotes a ninety degree rotation. If we were in search of a sign to denote the idea of $1/2 + it$ we could do no better than the mark itself. And with this, our excursion into number theory comes full circle back to the emergence of a distinction.

9. Continued Fractions, Re-entry and Real Numbers

It is natural at this juncture to ask about rational numbers and real numbers. For this purpose, I shall add a new sign that can be taken to be a relative of the Spencer-Brown mark. Let $\boxed{\text{x}} = 1/x$ for any non-zero number x. This means, for example that $\boxed{\rceil\rceil\rceil}\,\rceil\rceil\rceil = \rceil$ is our formal statement for $(1/3)(3) = 1$.

Now we can have rational numbers such as $p/q = p\boxed{\text{q}}$. To go on and construct real numbers, we need to choose a method to represent them. I choose here to use the fact that any positive real number has a unique expression as a continued fraction with positive integer terms in the form

$$R = [a_1, a_2, a_3, a_4, \ldots] = a_1 + 1/(1 + a_2/(1 + a_3/(1 + \cdots)))$$

$$= a_1 + \boxed{a_2 + \boxed{a_3 + \boxed{\cdots}}}.$$

The number of real numbers is indeed vast. Each real number corresponds to an infinite sequence of positive integers. Some real numbers

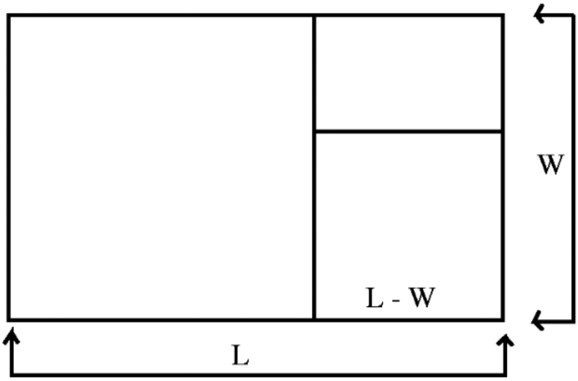


Fig. 11. Proportions of the Golden Rectangle.

become re-entry forms in this language. For example, we know that the Golden Ratio satisfies the equation

$$\phi = (1+\sqrt{5})/2 = [1,1,1,\ldots] = 1 + \boxed{1 + \boxed{1 + \boxed{1 + \ldots}}} = 1 + \boxed{\phi},$$

and thus, writing in an additive space, we have

$$\phi = \rceil\boxed{\phi} = \rceil\boxed{\rceil\boxed{\phi}} = \rceil\boxed{\rceil\boxed{\rceil\boxed{\phi}}}$$

and in the limit, this figure represents the Golden Ratio.

$$\phi = \rceil\boxed{\rceil\boxed{\rceil\boxed{\rceil\boxed{\rceil\boxed{\rceil\boxed{\rceil\boxed{\rceil\boxed{\ldots}}}}}}}}$$

At this point it is worth pointing out the relationship of the Golden Ratio to geometry. A rectangle is Golden if, upon excising a square from it, the resulting smaller rectangle has the same proportions as the original rectangle (see Fig. 11).

In this figure, we see a large rectangle with side L and width W. Excising a square (large square on the left) we see a smaller rectangle of side W and width $(L-W)$. To be Golden these lengths must satisfy the ratio

$$L/W = W/(L-W).$$

Note that $W/(L-W) = 1/((L/W)-1)$.

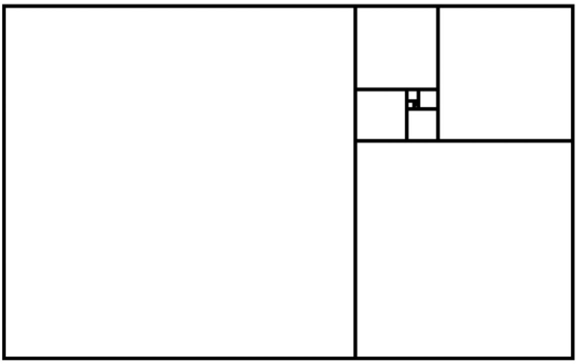


Fig. 12. The Golden Rectangle as a union of infinitely many squares.

Thus if $x = L/W$, then we require that

$$x = 1/(x - 1)$$

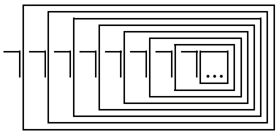
or $x^2 - x = 1$ which is the same as $x^\wedge 2 = x + 1$ which is the same as

$$x = 1 + 1/x.$$

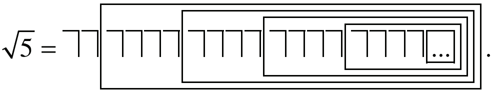
At this point we see that L/W is indeed the Golden Ratio as we have identified it above as $\phi = 1 + 1/\phi = (1 + \sqrt{5})/2$.

The infinite continued fraction for the Golden Ratio corresponds geometrically to the infinite decomposition of the Golden Rectangle into whirling squares of decreasing sizes as shown in Fig. 12.

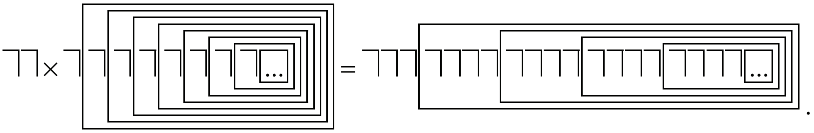
It is beautiful to compare the re-entry structure of the Golden Rectangle with the re-entry structure of its formal arithmetic precursor

$(1+\sqrt{5})/2 =$ 

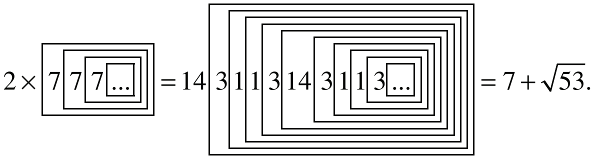
There are many relationships to explore in this realm of arithmetical re-entry forms. For example, we have the following direct formula for the square root of five.



Hence



Here is another example



In general, the periodic and hence simply re-entrant continued fractions correspond to solutions to quadratic equations. The other non-periodic or not eventually periodic continued fractions correspond to irrational numbers. The arithmetic of infinite continued fractions is a world of its own. It is natural to have laws of form arithmetic grow into the real numbers via continued fractions. Then we see the interior relationship of the non-numerical form, the calculus of indications and the beginnings of the deep relations of arithmetic with geometry.

10. Logic and Real Numbers

Given the field of real numbers there is a natural generalization of Boolean logic and the laws of form Arithmetic obtained by defining $\overline{x}\rceil = 1 - x$ for real numbers x. Note that $\overline{\overline{x}\rceil}\rceil = 1 - (1 - x) = x$ and $\overline{0}\rceil = 1$ and $\overline{1}\rceil = 0$. We can take 0 to stand for the unmarked state and 1 to stand for the marked state. Let + denote "or" and * denote "and". Then multiplication of real numbers give $00 = 0, 01 = 10 = 0,$

$11 = 1$ all in accord with the corresponding statements about the marked and unmarked states. Thus we take $x^\wedge y = xy$ (real number product) as our generalization of "and". To obtain or we use $a \vee b = \overline{\overline{a|}\,\overline{b|}\,|} = 1-(1-a)(1-b)$ by analogy with DeMorgan's Law.

Then we have $a \vee b = a + b - ab$ and indeed $1 \vee 1 = 1$ as desired for a generalization of Boolean algebra. Note that $a \vee b$ is not the usual addition of real numbers, but $a^\wedge b = ab$ is the usual multiplication of real numbers. In this extended logic with infinitely many values we have one solution to $\overline{x|} = x$, namely $x = 1/2$. The reader will also note that if we restrict the values to numbers between 0 and 1, then all operations preserve this property. Thus this arithmetic of distinctions can be compared with probability (since probabilities take values between 0 and 1). The logic we discuss here goes back to George Boole's original constructions of Boolean algebra and his applications of that algebra to probability. In this context the reader can verify that if x and y are between 0 and 1, then both $x \wedge y$ and $x \vee y$ are also between zero and 1. The calculus can be applied to probabilities just as Boole did.

Let $a^b = \overline{b|} \vee a = \overline{b|} + a - \overline{b|}\,a$. This is the analog of "b implies a" in this system. Vladimir Lefebvre uses it as a model for "a reflecting on b" where a and b are psychological subjects. It is interesting to examine some special cases of this formula:

$$0^b = \overline{b|} + 0 - \overline{b|}\,0 = \overline{b|}$$

$$0^b = \overline{b|} = 1 - b.$$

Here we see that there is a continuous spectrum from 0 implies 0 (which is True =1) and 1 implies 0 (which is False = 0).

$$a^a = \overline{a|} \vee a = \overline{a|} + a - \overline{a|}\,a = 1 - (1-a)a = 1 - a + a^2.$$

In this system, if self-reflection is 0, then a is equal to $(1+\sqrt{-3})/2$ or $(1-\sqrt{-3})/2$. These are imaginary (complex) sixth roots of unity. In this system, no one with a real self-reflection can have a fully false (zero) self evaluation.

This calculus has been called gamma algebra by Vladimir Lefebvre. Lefebvre has applied it to cybernetic reflexivity, ethical structures and game theory. Gamma algebra deviates significantly from the usual many valued logics in that its operators do not distribute over one another. Lefebvre points out that if

$$(1/2)^{x^x} = x$$

then $x = (-1 + \sqrt{5})/2 = .618\ldots$. This is the inverse of the Golden Ratio and is sometimes called the Golden Mean. A neutral observer reflecting on the self-reflection of x will have the value x when x is equal to the Golden Mean. As for the Golden Ratio, $(1 + \sqrt{5})/2 = 1.6.1828\ldots$, it is the solution to the equation

$$(1/2)^{x^x} = 0.$$

It is worth contemplating the possibilities inherent in this algebra of reflexion.

Another example of an arithmetical logic is obtained by defining the domain of values to be the real numbers with one infinite value added. Thus we use $R \cup \{\infty\} = R^*$ as the set of values. We define $\overline{x}\rceil = 1/x$ where $\overline{0}\rceil = \infty$ and $\overline{\infty}\rceil = 0$. We take $x \vee y = x + y$ with $0 + \infty = \infty + 0 = \infty$, $\infty + \infty = \infty$, $0 + 0 = 0$. Thus we can identify 0 as the unmarked state and ∞ as the marked state. We define $x \wedge y = \overline{\overline{x}\rceil + \overline{y}\rceil}\rceil = 1/((1/x)+(1/y))$. This extension of Boolean logic is closely related to the theory of electrical conductance with the rules for or and for and corresponding to the evaluations of series and parallel connections of conductors. The unmarked state corresponds to zero conductance while the marked state corresponds to infinite conductance. The reader can read more about this arithmetical logic in (Kauffman (1994a)) and about its relationships with topology. In this case the arithmetical logic fits directly with our construction of real numbers using reciprocation, addition and continued fractions. Note that in this system $+1$ and -1 are the paradoxical values, invariant under inversion.

11. Finding Ordinals and Imaginary Value

Remarkably, the structure of the laws of form expressions gives us a map of the transfinite ordinals. Let us explain. First recall that the transfinite ordinals of Georg Cantor (Cantor (1895)) are an extension of the natural numbers. We begin with the natural numbers $1, 2, 3, \ldots$

and then posit, as did Cantor, a new infinite number w that is greater than any natural number n. So now we have the ordered sequence

$$1, 2, \ldots; w.$$

And we can continue with

$$1, 23, \ldots; w, w+1, w+2, \ldots, w+w = 2w, \ldots,$$
$$3w, \ldots, w^2, \ldots, w^3, \ldots, w^w, w^{w+1} \ldots.$$

This sequence of transfinite ordinals never ends and is the beginning of the infinite structures that Cantor gave to the mathematical world.

We shall translate these ordinals into laws of form expressions as shown below.

$$1 = \overline{\ \ }|$$
$$2 = \overline{\ \ }|\,\overline{\ \ }|$$
$$3 = \overline{\ \ }|\,\overline{\ \ }|\,\overline{\ \ }|$$
$$\vdots$$

To get higher we shall notate that generally, $A + B = AB$ (juxtaposition) and

$$w^A = \overline{A}|.$$

This is the key point in our construction. An expression A that is crossed by a mark, is matched with the ordinal w raised to the A-th power.

Thus

$$w = \overline{\overline{\ \ }|}|,\ w^w = \overline{\overline{\overline{\ \ }|}|}|,\ w^{w^w} = \overline{\overline{\overline{\overline{\ \ }|}|}|}|,$$
$$w^w + w + 1 = \overline{\overline{\overline{\ \ }|}|}|\,\overline{\overline{\ \ }|}|\,\overline{\ \ }|,$$
$$w^{w^w+w+1} = \overline{\overline{\overline{\overline{\ \ }|}|}|\,\overline{\overline{\ \ }|}|\,\overline{\ \ }|}|.$$

It should be clear to the reader that the finite expressions in laws of form, taken only up to commutativity ($AB = BA$) each uniquely represent the tree-like polynomial expressions fragment of the transfinite ordinals! This means that we can order all the expressions in the mark so that it can be said of two expressions which is the greater with respect to this ordering. For example, we have that

$w^w + w + 1 =$  $< w^{w^w+w+1} =$ .

The simplest way to understand this ordering is to examine the expressions in w. We say that $w^w + w + 1 < w^{w^w+w+1}$ because this inequality is true for all natural numbers w that are sufficiently large.

A key property of the ordering that we have defined on the finite expressions in laws of form is that fact that *any descending sequence of expressions is finite.* This is called the *well-ordering* of the ordinal numbers.

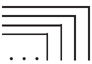
The reader should convince herself of this fact by thinking about some examples. The proof can be found in treatments of the Cantorian ordinal numbers. For example, consider a sequence of descending expressions starting with the expression

.

You might try

,

where that last row of marks is ten to the ten million marks long, but it is finite. And so the rest of any sequence you construct will have to be finite. This is the general phenomenon with the descending ordinal sequences. From the point of view of laws of form, there is this huge ordered structure of finite expressions right at the basis of the form.

Infinite expressions can be explored further. For example, let $J =$ .

Then

$$J = \overline{J}\rceil = w^J,$$

$$J = w^J.$$

This J is the important limit ordinal at the top of the hierarchy of the tree-like transfinite ordinals that we have used for the finite expressions.

$$J = w^{w^{w^{w^{w^{w^{\cdot^{\cdot^{\cdot}}}}}}}}$$

The Hydra Game

With the ordinal correspondence in place, we can translate a version of the Hercules and Hydra game of Kirby and Paris (Kirby, Paris (1982)) into laws of form expressions. This game illustrates a combinatorial fact that requires transfinite induction for its proof. The game illustrates a hole in our reasoning that can be filled by using the whole collection of expressions in their Cantorian ordering.

Take a finite expression such as $E =$ .

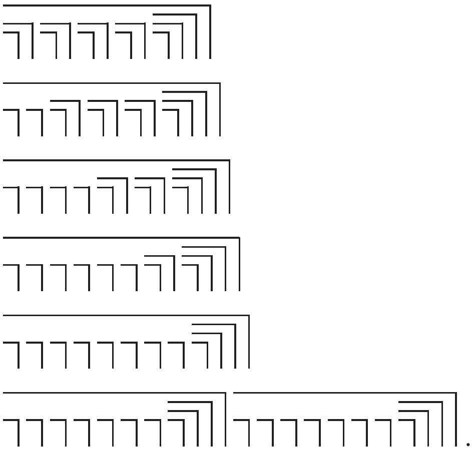
Choose an empty mark in the expression. Determine the first mark that encloses this mark, making a sub-expression. For example, E above has the sub-expression , with the left-most mark the one we have chosen. Now *remove the mark you have chosen and duplicate the resulting sub-expression to make a new expression E'.* Here the result is

$E' =$ .

(In the Kirby–Paris game, one can make any finite number of duplicates. We shall restrict to one duplicate.) *The object of the game is to eventually reduce the expression to nothing.* Note that by the rules above, a single empty mark can be erased. Here are a few more moves in the game starting with E, above:

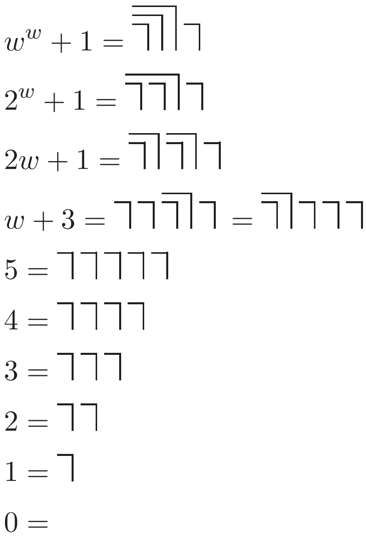
E = ,

E$'$ = 



We are sure that the reader would like to finish this game! It will take quite a few moves, but not too many.

Here is a shorter sequence. And in this shorter sequence we have labeled each expression with the corresponding ordinal. *The ordinals get smaller each time.* This always happens and it is the essence of the Kirby–Paris proof that *one can always win the Hydra game.* Any descending sequence of ordinals is finite, and so the game must end.



The most remarkable fact about the Hydra game is that, while we can prove that the game ends by using the transfinite ordinals (and hence by using the Cantorian ordering on the expressions in laws of form), there is no proof of this fact in Peano arithmetic (Kirby, Paris (1982)). Peano arithmetic is a highly restricted form of formal arithmetic where one is only allowed to use the simplest form of mathematical induction and the expressions in the language are basically of the form $|, ||, |||, \ldots$ for the numbers $1, 2, 3, \ldots$. In laws of form with its possibility of insertion of a given mark inside or outside of another mark, the expressions have a vastness that encompasses a good piece of Cantor's tranfinite ordinals. *Laws of Form notation is a perfect expression of this fragment of the ordinals, and goes beyond Boolean algebra to arithmetic and transfinite arithmetic. The Hydra Game is an exemplar of the overcoming of Goedelian incompleteness by the use of the mark as an imaginary value.*

12. Re-Entry, Parentheticals and the Enumeration of Expressions

In the last section we have considered the structure of the entire collection of expressions in Laws of Form. These can be written in terms of the mark itself or in terms of paired parentheses that represent the mark. In this section we will use both of these notations. Thus we take $\overline{\ \ }| = ()$ so that for more complex expressions, there is a corresponding parenthetical as in $\overline{\overline{\overline{\ \ }|}|\,\overline{\ \ }|}|\,\overline{\ \ }| = ((())())()$. When dealing with parentheses we do not consider that $()(())$ and $(())()$ are equal. By the same token, we can take it that $\overline{\overline{\ \ }|}|\,\overline{\ \ }|$ and $\overline{\ \ }|\,\overline{\overline{\ \ }|}|$ are unequal. In this way, we are arranging the expressions in relation to the linear order of a line. The resulting structures are of definite mathematical interest.

One reason that the ordered parentheticals are important is their relationship with non-associativity in products. Consider an algebraic product a^*b that is non-associative. You are familiar with many examples such as $a^*b = a - b$ where a and b are integers, or $a^*b = a \times b$ where a and b are vectors in three-dimensional space and the $a \times b$ denotes their vector cross product. We can use the mark to indicate a product by writing $a^*b = a\overline{b|}$. Then we have

$$(a^*b)^*c = a\overline{b|}\,\overline{c|},$$

$$a^*(b^*c) = a\overline{b\overline{c|}|},$$

and we have

$$((a^*b)^*c)^*d = a\overline{b|}\,\overline{c|}\,\overline{d|},$$

$$(a^*(b^*c))^*d = a\overline{b\overline{c|}|}\,\overline{d|},$$

$$a^*((b^*c)^*d) = a\overline{b\overline{c|}\,\overline{d|}|},$$

$$a^*(b^*(c^*d)) = a\overline{b\overline{c\overline{d|}|}|},$$

$$(a^*b)^*(c^*d) = a\overline{b|}\,\overline{c\overline{d|}|}.$$

As you can see, it is possible to write every possible associated product in this notation with no extra parentheses, just patterns of the mark. The single product for two variables corresponds to one mark. All ways of associating three variables corresponds to all ordered expressions in two marks. All ways of associating three variables corresponds to all ordered expressions in three marks. In general, all ways of associating $n+1$ variables corresponds to all ordered expressions in n marks.

Note how this plays out just in marks. There is one way to write one mark, two ways to write two marks and five ways to write three marks.

$$1{:}\,\overline{|}$$

$$2{:}\,\overline{|}\,\overline{|},\ \overline{\overline{|}|}$$

$$3{:}\,\overline{|}\,\overline{|}\,\overline{|},\ \overline{\overline{|}|}\,\overline{|},\ \overline{\overline{|}\,\overline{|}|},\ \overline{\overline{\overline{|}|}|},\ \overline{|}\,\overline{\overline{|}|}.$$

We can let $C(n)$ denote the number of ways to write ordered expressions in n marks. Then we have $C(1) = 1$, $C(2) = 2$, $C(3) = 5$, $C(4) = 14$, $C(5) = 42, \ldots$. The numbers $C(n)$ are called *Catalan*

Numbers and appear in many mathematical contexts. It turns out that the formula for $C(n)$ is

$$C(n) = (C_n^{2n})/(n+1),$$

where

$$C_r^m = m!/(r!(m-r)!)$$

is the choice coefficient: the number of ways to choose r things from m things.

Couldn't we make a direct count to find that $C(n)$ is given by the formula $C_n^{2n}/(n+1)$? How is choosing n things from $2n$ things involved? Well, there is a way. One way is to go back to individual parens. We have *left parens* (and *right parens*). Each parenthetical expression of type n has n left parens and n right parens. But we are enumerating parentheticals and these are not all the sequences of parens with n left and n right. For example,)(is not a parenthetical. It is not a legal parenthesis structure.

But look, you say, if we consider ALL sequences of parens with n left and n right there will be C_n^{2n} of these. After all, you simply choose the places in the sequence where the left parens appear, and that is a choice of n from $2n$. So, you say, all we have to do is to divide the collection of all paren sequences (with n left and n right) into $(n+1)$ classes such that each parenthetical is in a unique class. With the right notation this can be done. Onward into the next paragraph!

I shall introduce *right and left marks*.

$\urcorner$ is a right mark and it stands for ().

$\ulcorner$ is a left mark and it stands for)(.

Expressions involving left and right marks stand for parenthesis sequences. For example,

$$\urcorner\ulcorner\ = ())(, \text{ and } \ulcorner\ulcorner\urcorner\ \urcorner\urcorner\ \ulcorner\ =))()(((()))(.$$

The left- and right-mark representation is not a unique representation of parenthesis sequences. For example

$$\ulcorner\ulcorner\urcorner = ()()() = \urcorner\urcorner\urcorner.$$

It is an interesting puzzle to understand when two such forms represent the same parenthesis sequence. I leave this part as a project for the reader.

But now, *take a standard form in right marks. and for each right mark in it make a new left/right mark form by reversing that mark and all marks that contain it. The original form and all the ones you have made, make up* $(n+1)$ *left/right forms. Translate these into left right parenthesis sequences and you will find that you have achieved your goal: a division of all the left–right sequences into collections with each collection containing a unique parenthetical, a unique standard right form.*

The previous paragraph is an answer to the quest for a direct view of the count of parentheticals (equivalently of ordered mark expressions). I will not prove it, but lets look at some cases. You can enjoy finishing the proof yourself. Take the case $n = 1$. Then there is one standard mark $\urcorner$. Reversing it gives the left mark $\ulcorner$. Together, these two forms produce the sequences () and)(and these are all the sequences of length 2 with 1 left paren.

Next case is $n = 2$. The standard forms are $\urcorner\urcorner$ and $\overline{\urcorner\,}\!|$. The standard form $\urcorner\urcorner$ yields two reversals so that we have:

$$\urcorner\urcorner = ()()$$

$$\ulcorner\urcorner =)(()$$

$$\urcorner\ulcorner = ()(.$$

The second form also yields two reversals so that we have:

$$\overline{\urcorner\,}\!| = (())$$

$$|\!\overline{\,\urcorner} =)()($$

$$|\!\overline{\,\ulcorner} =))((.$$

Note that we have followed the rules in the italicized paragraph. The top of each list is the standard form and its corresponding parenthetical. Each other member of a list is obtained by flipping one mark and all the marks that contain it. You can see that the set of sequences we have obtained is the complete set of six sequences with two left parens and two right parens. Your next exercise is to work out all the lists for $n = 3$. To get you started here is one of the lists:

 = ((()))

 =)(())(

 =))()((

 =)))(((.

Remember, that when you flip a mark, you must flip all the marks that contain it. We have found a way to see the formula $C_n = C_n^{2n}/(n+1)$ directly in terms of counting sequences of parentheses. And this result depends upon a translation back and forth with the related category of left and right marks. *The result is seen by inventing new notation.* The new notation is really a new language for the mathematics of parentheticals and parenthesis sequences. By learning to speak this language we begin to gain insight into the structure of parentheses and what they can do.

There is much more to say here about the structure of parentheses. We end this section by showing how the collection of all ordered expressions in the mark has a simple re-entry description.

Consider the formal infinite summation S of all ordered expressions in the mark.

$$S =^{*} + \urcorner + \urcorner\urcorner + \overline{\urcorner}| + \urcorner\urcorner\urcorner + \overline{\urcorner}|\urcorner + \urcorner\overline{\urcorner}| + \overline{\urcorner\urcorner}| + \overline{\overline{\urcorner}|}| + \cdots .$$

Here * denotes the unmarked state.

It follows from the fact that every ordered expression has the unique form

$$E\overline{G}|,$$

where E and G are themselves ordered expressions, that

$$S =^{*} + S\overline{S}|,$$

where we take the conventions that

$$\overline{*}\rceil = \rceil$$

$$^{*}a = a^{*} = a$$

$$\overline{a+b}\rceil = \overline{a}\rceil + \overline{b}\rceil$$

for any expressions a and b.

It is very interesting to convince oneself of the truth of this self-nesting of the total structure of all ordered expressions. The main point is that every non-void term in S corresponds uniquely to a term of the form $E\overline{G}\rceil$, and this is a unique term in the product $S\overline{S}\rceil$ since E is a summand of S and G is a summand of S. Thus, we can see clearly that there is a precise one-to-one correspondence of the terms of S and the terms of $* + S\overline{S}\rceil$. It is a tautology, but a tautology that is worth exploring. The equation

$$S =^{*} + S\overline{S}\rceil$$

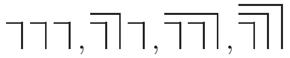
is the fundamental formal re-entry that generates all the expressions underlying the structure of Laws of Form. Remarkably, it is possible to use this re-entry equation to derive in another way the formula for $C(n)$. In the series for S, replace each mark by the algebraic variable x and replace * by 1. Then S becomes a series $F(x)$ with

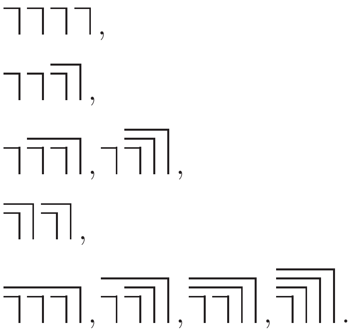
$$F(x) = 1 + C(1)x + C(2)x^2 + C(3)x^3 + \cdots .$$

But making this replacement in the re-entrant formula for S, yields

$$F(x) = 1 + xF(x)^2.$$

Thus the re-entrant equation becomes a quadratic equation, and we can solve it for $F(x)$, obtaining $F(x) = (1 - \sqrt{1-4x^2})/(2x)$. Then an application of Newton's series for the square root gives the formula for $C(n)$ in terms of choice coefficients that we have already seen.

Ah, but you say, this is all well and good, but I am really interested in the unordered expressions in Laws of Form where we have $\overline{\overline{\rceil}\rceil\;\rceil}\rceil =$ $\overline{\rceil\;\overline{\rceil}\rceil}\rceil$. How do you count those, and is there a re-entry formula for them? And I have to answer that I am glad you asked! Lets begin by considering how we would enumerate these forms. Suppose you want to enumerate all the unordered expressions with three marks. They are . There are four of them, as opposed to the five ordered forms. Lets do the same for four marks:



Note that we have divided the nine expressions into five groups corresponding to the partitions of 4:

$$1 + 1 + 1 + 1,$$
$$1 + 1 + 2,$$
$$1 + 3,$$
$$4.$$

For each partition expression we get as many "outer boxes" as there are terms in the partition. Thus $1 + 1 + 1 + 1$ has four boxes, while $2 + 2$ has two boxes. But note that a given number in the partition sum can have many expressions. Thus 4 connotes all expressions with 4 in a single box, and there are in fact four of these.

Inside a given box there are $(k-1)$ marks if that box contributes k to the partition. Thus the enumeration of unordered forms is layered in terms of the enumeration of partitions. The problem of enumerating partitions is a very vital area in the theory of numbers. Let A_n denote the number of unordered expressions in the mark that use n marks. Thus we have so far that $A_1 = 1$, $A_2 = 2$ and $A_3 = 4$. One more example should make the partition method clear. Consider the partitions of 4. They are

$$1+1+1+1$$
$$1+1+2$$
$$1+3$$
$$2+2$$
$$4$$

Corresponding to these, we have the following nine expressions in four marks:

$$\urcorner\urcorner\urcorner\urcorner$$
$$\urcorner\urcorner\overline{\urcorner}|$$
$$\urcorner\overline{\overline{\urcorner}|}|,\ \urcorner\overline{\urcorner\urcorner}|$$
$$\overline{\urcorner}|\overline{\urcorner}|$$
$$\overline{\urcorner\urcorner\urcorner}|,\ \overline{\urcorner\overline{\urcorner}|}|,\ \overline{\overline{\urcorner\urcorner}|}|,\ \overline{\overline{\overline{\urcorner}|}|}|.$$

In each case when we have a number k in a partition p of n, we must consider all the expressions in $(k-1)$ marks that can be placed inside a single mark. The single mark and the expression within it constitute an instance of that summand k in the partition. Thus for 4 as a partition of 4, we have all expressions in 3 marks under the roof of a single mark. In this way, we can recursively build up the values for the A_n in terms of an enumeration of the partitions of integers.

But we must look more closely. Let E denote all finite unordered expressions in the mark. Then $\alpha \in E \rightarrow \alpha = \overline{\beta_1}|^{k_1}\overline{\beta_2}|^{k_2}\overline{\beta_3}|^{k_3}\ldots\overline{\beta_n}|^{k_n}$ where $\beta_1, \beta_2, \beta_3, \ldots, \beta_n$ are distinct expressions in E. This is *a unique prime factorization* (*in the sense of juxtaposition of expressions*) *of* α. *Let* P *denote the set of prime elements of* E*, that is* P *is the set of elements of the form* $\overline{\beta}|$. *Then, just as in Euler's work relating the primes and the composites, we* have

$$\sum_{\alpha \in E} \alpha = \prod_{\overline{\beta}| \in P} (1 + \overline{\beta}| + \overline{\beta}|^2 + \overline{\beta}|^3 + \cdots)$$

and hence we have a fundamental self-reference of E:

$$\sum_{\alpha \in E} \alpha = \prod_{\overline{\alpha}\rceil \in E} (1 - \overline{\alpha}\rceil)^{-1}.$$

To appreciate the consequences of this formula, replace each mark in these expressions by an algebraic variable x. Then the equation becomes

$$1 + \sum_{n=1}^{\infty} A_n x^n = \prod_{n=0}^{\infty} (1 - x^{n+1})^{-A_n},$$

where A_n is the number of unordered expressions with n marks. This formula was derived by Cayley in 1857 (Cayley, 1857) for the equivalent problem of enumerating rooted trees (see also (Guy, 1973). Rewriting the formula as

$$1 + \sum_{n=1}^{\infty} A_n x^n = \prod_{n=0}^{\infty} (1 + x^{n+1} + x^{2(n+1)} + x^{3(n+1)} + \cdots)^{A_n}$$

and multiplying out and comparing terms, we see that it contains exactly the algorithm for recursively finding A_n in terms of the partitions of n as we have described above.

We have given two proofs of the method of recursion and partitions for the enumeration of expressions. One proof is by direct reasoning about the forms of these expressions. The other is, in the sense of laws of form, an example of using imaginary values. We point out a fundamental re-entering or self-referential property of the assembly of all expressions. From this emerges by diagrammatic formula magic, the very result we had found by careful reasoning. One could hope that all desired results would emerge in this way. It is a matter of finding the right point of view when such viewpoint can be found.

The reader may be interested to see that the series of values of A_n for $n = 1, 2, 3, \ldots$ goes as shown below:

$1, 2, 4, 9, 20, 48, 115, 286, 719, 1842, 4766, 12486, 32973, 87811, 235381,$
$634847, 1721159, 4688676, 12826228, 35221832, 97055181, 268282855,$
$743724984, 2067174645, 5759636510, 16083734329, 45007066269,$
$126186554308, 354426847597, \ldots$

Thus there are $354426847597 = 23 \times 15409862939$ distinct unordered expressions in 29 marks. Perhaps you would like to make a list of them.

But it is time to stop, with the inevitable affirmation that we shall return to this subject the next time 'round.

13. Discrimination and Fermions

Choose a circle in the plane.

How does it come about that this circle is seen as distinguishing its inside from its outside? The inside is a bounded space. The outside is an unbounded space. We, observers, make this extra distinction of bounded and unbounded. A simpler distinction could be just a vertical line segment, distinguishing left from right.

$$|$$

Here the vertical segment makes a difference and a boundary, but if we actually did not have the extra distinction of left and right, then the vertical mark would in fact not make a distinction at all.

A first distinction is eternal in the eternity of the moment. A first distinction can only become a distinction because the contents of the distinction must be seen to differ in value, and this difference is a further distinction beyond the first distinction.

The first distinction may at once disappear. And then it can reappear in a timeless oscillation. It is this oscillation that is the first thing that arises from no-thing.

How shall this primordial vibration be indicated? Let us use 0 for void or absence and 1 for presence.

$$\ldots 0101010101010101010101010101010101010101 \ldots$$

That is, we use 0 and 1 in the sense of Boolean arithmetic:

$$0^*0 = 0$$

$$0^*1 = 1^*0 = 0$$

$$1^*1 = 1$$

and

$$0 + 0 = 0$$
$$0 + 1 = 1 + 0 = 1$$
$$1 + 1 = 1.$$

The reader should recall the interpretations of Boolean arithmetic. We can take 0 to denote an empty universe and 1 to denote the whole universe. We take * to denote intersection and + to denote union. Then the rules for * are statements such as the intersection of emptiness with emptiness is empty. The intersection of the whole universe with itself is the whole universe. The rules for + are statements like the union of emptiness with emptiness is empty. The union of emptiness with the universe is the universe. The union of the universe with the universe is the universe. Boolean arithmetic also contains the complementation operation $'$ with $0' = 1$ and $1' = 0$. We understand Boolean arithmetic as an early step into complexity beyond the primary arithmetic of laws of form.

Primordial vibration is an oscillation between nothing, 0, and something, 1, where that something is all that there is in the primitive emergence. At this stage, while 1 may emerge, 1 is in fact not different from nothing, and 0 while it stands for nothing, is not nothing. Each is an attempt to become something, either by standing for nothing or by attempting to distinguish a something. Each attempt, in the first place, is not locked into existence. Each attempt falls back into the inchoate nothingness and loses its existence, only to re-emerge again and again.

The oscillation is not separate from its own observation. The oscillation is a movement of attention that each time joins with the that of which is attended and disappears only to reappear once again. Would you know it was a second time? Of course not. That would be an extra distinction. The world appears as its own observer, but being indistinct from itself, disappears.

What happens next, that we have the possibility to distinguish the oscillation itself? How do we leave the state of temporality and come to a discussion of a distinction apparently separate from us and yet in relation to us?

The Series . . . 010101010101. . . and the Fermion Algebra
Let's return to the series alternating between 0 and 1 with Boolean values.

$$\ldots 01010101010101 \ldots$$

Apply a temporal analysis to the series. We take as given two basic forms of observation/participation with the sequence, denoted by

$$p = (0, 1) \text{ and } q = (1, 0).$$

In the first case the participation is in rhythm from 0 => 1 and in the second case in rhythm from 1 => 0. We denote these by p and q, respectively. In q we have the sense of form dissolving to void. In p we have the sense of form arising from void. Here is the Boolean sense of form. We can regard 1 as a first form, dominant over void. And 0 as a notational representative for void.

In examining these waveforms, it is natural to have an operator that shifts from one wave form to the complementary waveform. To this end we introduce η such that $(a, b)\eta = \eta(b, a)$ and $\eta\eta = 1$. In this way we have $\eta(a, b)\eta = (b, a)$.

You can think of the flanking by η as an operator:

$$\{X\} = \eta X \eta.$$

Then $\{(a, b)\} = (b, a)$ performs the temporal shift of (a, b) to (b, a).

Since 1 is the identity operator, η is a new operator whose square is the identity:

$$\eta\eta = 1.$$

Think temporally, (a, b) denotes a process that shifts from a to b. $\eta(a, b)\eta$ denotes a process the shifts from b to a. We write $\eta(a, b)\eta = (b, a)$ and take $\eta\eta = 1$. From this it follows that $(a, b)\eta = \eta(b, a)$.

Shifting η across (a, b) converts it into (b, a). In this way we see that while (a, b) and (b, a) appear to be spatial forms in the sense that they interact with themselves and each other at their individual coordinate places, $(a, b)\eta$ has a built in temporality when we let it interact with another form or with itself.

For example, $(a, b)\eta(a, b)\eta = (a, b)\eta\eta(b, a) = (a, b)(b, a) = (ab, ba) = (ab, ab) = ab[1, 1] = ab$

while
$(a, b)(a, b) = (aa, bb) = (a, b)$ when a and b are either 0 or 1.

Take $p = (1, 0)$ and $q = (0, 1)$ as the two views of the sequence. Note that

$$pq = (1, 0)(0, 1) = (0, 0) = 0,$$

$$p + q = (1, 0) + (0, 1) = (1, 1) = 1.$$

We have

$$U = p\eta \text{ and } U^\dagger = q\eta$$

as the two basic temporal elements for the algebra of the 0-1 oscillation.

Note that

$$p\eta = (1, 0)\eta = \eta(0, 1) = \eta q,$$

$$q\eta = \eta p.$$

Hence $\eta p\eta = q$ and $\eta p\eta = q$, making p and q time shifts of each other.

Then

$$U^2 = p\eta p\eta = pq = 0,$$

and

$$(U^\dagger)^2 = q\eta q\eta = qp = 0.$$

Furthermore

$$UU^\dagger = p\eta q\eta = pp\eta\eta = pp = p,$$

$$U^\dagger U = q\eta p\eta = qq = q.$$

Hence

$$UU^\dagger + U^\dagger U = p + q = 1.$$

Thus we have shown that

$$U^2 = (U^\dagger)^2 = 0 \text{ and } UU^\dagger + U^\dagger U = 1.$$

These are the fundamental equations for the creation and annihilation operators for a Fermionic particle in quantum mechanics (Kauffman, 2016a). More than one Fermi particle cannot occupy the same space. This is the moral of the equations where the square of an operator is 0. Fermi particles can (irrespective of order) be created in pairs from the vacuum. This is the moral of the second equation where pairs $UU^\dagger$ or $U^\dagger U$ can appear from 1 (the *whole*, the vacuum).

Fermions embody special distinctions and it is remarkable that we find them emerging from the process algebra for a single distinction. In the next section we shall see how Fermions emerge from the Clifford algebra associated with the numerical plus/minus sequence.

A Fermionic Containment Calculus. We can make a *Fermionic containment calculus* by first defining two new operators on laws of form pairs. We define

$$\underline{\left|(a,b)\right.} = (b, \urcorner), \quad \textit{the left push},$$

and

$$\underline{\left.(a,b)\right|} = (\urcorner, a), \quad \textit{the right push}.$$

Note that for any (a,b), two pushes of the same type carry it to the marked state.

$$\underline{\left|\underline{\left|(a,b)\right.}\right.} = \urcorner,$$

$$\underline{\left.\underline{\left.(a,b)\right|}\right|} = \urcorner.$$

The left-and right-push operators do not commute.

$$\underline{\left|\underline{\left.(a,b)\right|}\right.} = (a, \urcorner),$$

$$\underline{\left.\underline{\left|(a,b)\right.}\right|} = (\urcorner, b).$$

but

$$\underline{\left|\underline{\left.(a,b)\right|}\right.} + \underline{\left.\underline{\left|(a,b)\right.}\right|} = (a,b),$$

where $x + y = \overline{\overline{x}| \; \overline{y}|}|$. We have

$$\underline{\left.\underline{\left.\;\;\right|}\right|} = \underline{\left|\underline{\left|\;\;\right.}\right.} = \urcorner$$

$$\underline{\left|\underline{\left.\;\;\right|}\right.} + \underline{\left.\underline{\left|\;\;\right.}\right|} =$$

and $\underline{\left.\overline{\;\;}\rfloor\right|} = \urcorner = \underline{\left|\overline{\;\;}\right|}$ while $\llcorner\!\overline{\;\;}| = \lrcorner$ and $\overline{\lrcorner|}| = \llcorner$.

Thus we have a non-commutative Fermionic calculus of containment operators. There is an algebra and an arithmetic to be explored, but we shall not do that in this paper. By taking $I = (\urcorner,)$ and $J = (, \urcorner)$, we have $\llcorner I = J$ and $J\lrcorner = I$, and we have the schematic containment calculus Fermion diagram as shown below.

$$\urcorner \xleftarrow[\llcorner]{} I \underset{\lrcorner}{\overset{\llcorner}{\rightleftarrows}} J \xrightarrow[\lrcorner]{} \urcorner.$$

The objects I and J are exchanged by the left and right pushes, but when a push is applied twice one ends up at the marked state, as indicated at the left and the right of the diagram. This diagram with arrows could be called the "Fermion Category", a small structure with arrows and compositions of arrows that embodies the essence of the Fermion operator algebra. See MacLane (1986) for a discussion of category theory.

Fermions and Standard Arithmetic

$UU = 0 = U^\dagger U^\dagger$ corresponds to the Pauli exclusion principle. Two identical Fermions cannot occupy the same place. Here the 0 denotes void and it is understood in the additive sense. Note that 0 is the unique self-referential value in standard arithmetic. $\sim 0 = 0$.

The other equation $UU^\dagger + U^\dagger U = 1$ can be interpreted as saying that the combination of particle and antiparticle can emerge from void where now void is the multiplicative void of 1. This corresponds to the unmarked state at the Boolean level.

If we allow standard arithmetic, then we can consider subtraction and we find that

$$\mathrm{e} = \mathrm{p} - \mathrm{q} = (0, 1) - (1, 0) = (-1, 1)$$

can be interpreted as as oscillation between -1 and $+1$. This is also an oscillation between absence and presence, but now in the language of standard arithmetic.

We have that $-\mathrm{e} = (1, -1)$, the complementary oscillation, and that

$$\eta \ \mathrm{e} \ \eta = \eta(\mathrm{p} - \mathrm{q})\eta = \eta \ \mathrm{p} \ \eta - \eta \ \mathrm{q} \ \eta = \mathrm{q} - \mathrm{p} = -\mathrm{e}.$$

Thus we have an algebra of e and η with $\mathrm{ee} = 1 = \eta\eta$ and $\mathrm{e}\eta = -\eta \ \mathrm{e}$. This is a *Clifford algebra* (See Kauffman (2016a)).

Note that if we write $\mathrm{i} = \mathrm{e}\eta$ then $\mathrm{ii} = \mathrm{e}\eta\mathrm{e}\eta = \mathrm{e}(-\mathrm{e}) = -1$. Thus, along with the emergence of the Clifford algebra, we have a construction of the square root of minus one that is essentially temporal. *The alternation of -1 and $+1$, made temporally sensitive, interacts with itself to produce negative unity.*

From Pythagoras to Einstein and Dirac

Now we can see a remarkable property of the Clifford Algebra we have just discussed. Suppose that we write a general element of this algebra in the form

$$U = \varepsilon A + \eta B + \varepsilon\eta C$$

where A, B and C are real numbers. Now compute the square of this element.

$$U^2 = (\varepsilon A + \eta B + \varepsilon\eta\, C)^2 =$$
$$(\varepsilon A)^2 + (\eta B)^2 + (\varepsilon\eta C)^2 + (\varepsilon\eta + \eta\varepsilon)AB + (\varepsilon\varepsilon\eta + \varepsilon\eta\varepsilon)AC$$
$$+(\eta\varepsilon\eta + \varepsilon\eta\eta)BC.$$

Just as $(\varepsilon\,\eta + \eta\,\varepsilon) = 0$, all the other cross terms vanish.

Since $\varepsilon^2 = \eta^2 = 1$ and $(\varepsilon\eta)^2 = -1$, we obtain

$$U^2 = A^2 + B^2 - C^2.$$

We have proved

Theorem. *If* $U = \varepsilon\, A + \eta\, B + \varepsilon\;\eta\; C$ *then* $U^2 = A^2 + B^2 - C^2$. *Hence* $U^2 = 0$ *if and only if* $C^2 = A^2 + B^2$.

This Theorem suggests that *the Pythagorean Theorem is involved with our algebra for the imaginary states of the reentering mark.* An operator with square zero is called a *nilpotent operator*.

We started with a nilpotent operator in the form of the mark: $\overline{\overline{}|}| = .$

We took a long journey in articulating the properties of the temporal behaviour of the reentering mark and arrived at a Clifford Algebra where the general element U of that algebra acts upon itself to produce zero, exactly when the parameters of that element U satisfy the Pythagorean Theorem!

In a philosophical mode we could reason that if U is the Universe then U can act upon itself to annihilate itself. In the algebraic operator mode of speaking, this is in the form of

$$UU = \text{Nothing}$$

or

$$\overline{\overline{\;\;}|}| = .$$

Thus, we have found a primordial algebraic expression $U = \varepsilon A + \eta B + \varepsilon\, \eta\, C$, where $C^2 = A^2 + B^2$, for *the fundamental vanisher*, the representative of universe that creates from nothing and takes back to nothing. $U^2 = 0$ when $C^2 = A^2 + B^2$.

In the last section we found operators U and $U^\dagger$, both of square zero and so that $UU^\dagger + U^\dagger U = 1$, corresponding to the algebra of a Fermion. In the present context, there is a good choice for $U^\dagger$ (and we shall explain below why this is a good choice). It is

$$U^\dagger = \varepsilon A + \eta B - \varepsilon\eta C$$

and you can easily check that $(U^\dagger)^2 = 0$. Note that we have the calculation

$$UU^\dagger + U^\dagger U = (U + U^\dagger)^2 = 4(\varepsilon A + \eta B)^2 = 4(A^2 + B^2) = 4C^2.$$

And so (up to a constant) U and $U^\dagger$ also satisfy the Fermion equations, and they have the variability afforded by solutions to the Pythagorean Equation $A^2 + B^2 = C^2$.

Now there is a relationship with physics that must be mentioned. First of all, there is the remarkable formula

$$E^2 = (pc)^2 + (mc^2)^2.$$

This is the Einstein formula for the energy of a particle with rest mass m that has a momentum p with respect to a given observer (Kauffman, 2016a). We usually think of the formula $E = mc^2$ for the relationship of mass and energy, but if the mass is moving past at momentum p, then there is a Pythagorean relationship of the total energy E and the energy mc^2 of the stationary particle. Thus we have the relativistic formula

$$E = \sqrt{(pc)^2 + (mc^2)^2}$$

making the energy the length of the hypotenuse of a right triangle whose sides are pc and mc^2.

Here, for the record is how to see the Einstein formula.

We have $E = m_0c^2$ where m_0 is the rest mass of the particle (Einstein's basic mass, energy relation). And we have $p = mv^2$ where v is the velocity of the particle and m is the relativistic mass with

$$m = m_0/\sqrt{1 - v^2/c^2}.$$

The formula then follows by squaring this (Pythagorean) mass equation.

$$\begin{aligned}
&m^2 = m_0^2/(1 - v^2/c^2)\\
&m^2(1 - v^2/c^2) = m_0^2\\
&m^2(c^2 - v^2) = m_0^2c^2\\
&(mc^2)^2 = (mv)^2c^2 + m_0^2c^4\\
&E^2 = p^2c^2 + m_0^2c^4.
\end{aligned}$$

What will happen if we combine our fundamental nilpotent U with this Pythagorean energy formula from special relativity? The remarkable answer is that we arrive at Dirac's relativistic equation for the electron.

To make the formalism easier, let us take $c = 1$. We use the convention that the speed of light is equal to 1. Then $E^2 = p^2 + m^2$ where m is the rest mass of the particle. With this we could write $U = m\varepsilon + p\eta + E\varepsilon\eta$ and obtain a physical nilpotent element. It follows at once from our theorem that $U^2 = 0$. So we have associated mass with ε, the representative of the on-going time series of the recursion. And we have associated momentum with the time shift operator, and energy with the time sensitive operator whose square is minus one.

We shall define the dual nilpotent $U^\dagger = m\varepsilon + p\eta - E\varepsilon\eta$. Here time goes backwards relative to U, and if U creates a particle, then $U^\dagger$ creates the corresponding anti-particle! See the discussion below.

We then find (as above) that

$$UU^\dagger + U^\dagger U = (U + U^\dagger)^2 = 4(m\varepsilon + p\eta)^2 = 4(p^2 + m^2) = 4E^2.$$

Thus, we have that

$$UU^\dagger + U^\dagger U = 4E^2$$

and

$$U^2 = U^{\dagger 2} = 0.$$

The first equation expresses that a creation of a particle and an antiparticle (in either order) can proceed from pure energy. The second two equations represent the Pauli Exclusion Principle that forbids the existence of identical particles at the same place. There is just no possibility for the production of two identical Fermi particles from pure energy.

Remark. Now recall that we have seen this algebra before in this section, where we found the Fermion algebra (with $E = 1/2$ in the above terms) coming directly from the re-entering mark. Here we took a longer road and used arithmetic rather than primary values of marked and unmarked, and we arrived at nilpotents in relation to the Pythagorean relationship $E^2 = p^2 + m^2$. And this relates to physics because this relationship holds for energy (E), momentum (p) and mass (m) in Special Relativity Theory. It is clear that even more thought is needed here about the roots of these relationships in the form and in the physics. From distinctions we find precursors to physics and the depths of this relationship is not yet fully apprehended.

We began this discussion with a tip of the hat to the Pauli Exclusion Principle, and we return to it in this way at the end. For the details about how this Fermion algebra is related to the Dirac equation, we recommend that the reader examine (Kauffman, 2016a) and the work of Peter Rowlands (2007) who discovered the nilpotent approach to the Dirac equation.

Dirac devised his equation so that it would correspond to the basic fact that we have $E^2 = p^2 + m^2$. He needed an operator form of the square root $E = \sqrt{p^2 + m^2}$. Dirac reasoned that if $\alpha^2 = \beta^2 = 1$ and $\alpha\beta + \beta\alpha = 0$, then $(\alpha p + \beta m)^2 = p^2 + m^2 = E^2$ and so he could identify the energy operator with $\alpha\hat{p} + \beta m$.

We can back-engineer the nilpotent Dirac equation from the requirement that

$$U\varphi = (m\varepsilon + p\eta + \varepsilon\eta E)e^{i(px - Et)}$$

should be a solution to the original Dirac equation. See the above references for the details. Now you can see why we took

$$U^{+} = m\varepsilon + p\eta - E\varepsilon\eta,$$

for this corresponds exactly to reversing time in the plane wave and so corresponds creating an antiparticle. The final expression of this Fermionic solution to the Dirac equation is quite remarkable, combining the algebraic form of U and the continuously varying exponential expression in space, time, momentum and energy. At this point, we have arrived at relativistic quantum mechanics by starting from laws of form. We shall say a good deal more elsewhere. The reader can examine Peter Rowland's work (Rowlands, 2007) and the papers of the author referenced above.

14. Laws of Form and Fundamental Physics

In Section 13, we discussed the construction of the Dirac operator, but we did not make it explicit. It will help in relating the physics to laws of form to point out that an essential component of quantum theory is the articulation of operators that correspond to each physical quantity. Thus to the energy E there is an energy operator $\hat{E}$. To the momentum there is a momentum operator $\hat{p}$. To the mass m there is a mass operator $\hat{m}$. And this situation of having operators and operands is very like the laws of form fundamental that every distinction can be seen as both a value and as an operator that corresponds to the act of crossing the distinction.

In the case of the physics the relationship between operators and operands occurs through a wavefunction such as $\psi = e^{i(px-Et)}$ and we will have that

$$\hat{E}\psi = E\psi,$$

$$\hat{p}\psi = p\psi,$$

$$\hat{m}\psi = \hat{m}\psi.$$

It is in relation to the wave function that the operators and operands become interchangeable. Dirac's original equation was then the statement that at the operator level

$$\hat{E} = a\hat{p} + b\hat{m},$$

where

$$a^2 = b^2 = 1,$$

$$ab + ba = 0.$$

We have discussed this in the last section.

The nilpotent Dirac operator is a variant of this relationship where we write $D = ab\hat{E} + b\hat{p} - a\hat{m}$ and we then have that

$$D\psi = (abE + bp - am)\psi$$

with the nilpotent $U = (abE + bp - am)$ so that

$$D\psi = U\psi \quad \text{and} \quad D(U\psi) = U^2\psi = 0.$$

Thus $U\psi$ is a solution to the Dirac equation. It is a Fermion.

The simplest precursor to the nilpotent Dirac operator is the laws of form operator

$$DX = \overline{X}\rceil.$$

Then

$$DDX = \overline{\overline{X}\rceil}\rceil = X$$

whence

$$D\rceil = $$

with the mark taken to void by the operator D.

At this level, we are speaking of a universe as it first emerges from one distinction and we are speaking of the distinction of an observer who finalizes a measurement in quantum mechanics. The final registration of an observation is identical with the creation of a universe in the moment of possibility becoming actuality.

The mark represents a primordial distinction and it also represents a primordial particle. As particle, the mark interacts with itself to annihilate itself (crossing) or to affirm/produce itself (calling). No particle could be more elementary than this. The mark is its own antiparticle, arising and returning to a void that should be compared with the physical vacuum.

We factor the mark typographically into left and right brackets to form a container { }. This factorization can be taken seriously in the typographical domain and we can define along with the container

$C = \{\}$ an extainer $E = \}\{$. There is an arithmetic of extainers and containers. Consider that

$$EE = \}\{\ \}\{\ = \}C\{$$

so that a container appears in the middle of the interaction of an extainer with itself. And indeed an extainer appears in the interaction of a container with itself

$$CC = \{\ \}\{\ \} = \{E\}.$$

The Dirac formalism of bras and kets has exactly this structure. In that formalism, the container is seen as a value while the extainer is seen as an operator. As a value the container can be moved about in the arithmetic, resulting in the equation

$$EE = \}C\{ = C\}\{ = CE.$$

Thus E becomes a projection operator with a basic equation that is a relative of the law of calling. See Kauffman, and Isaacson (2016a) for a discussion of the relationship of these structures with the recursion of distinctions. See Kauffman (2002) for the relationship of containers and extainers to the self-replication of DNA. See Kauffman (2012) for relationships of containers and extainers to Dirac bras and kets and to braids, knots and topology.

15. Modulators

In this section, we give a very brief introduction to temporal and recursive interpretations and applications of laws of form. See Kauffman (1978a, b, 1987b, 1994) Kauffman and Varela (1980). First consider the equation $J = \overline{J\,|}$. We have pointed out that it can indicate an imaginary logical value that is invariant under negation, and we have pointed out that it can be 'solved' by an infinite nest of marks as in $E = \overline{\overline{\overline{\ldots\,|}\,|}\,|}$. It is also the case that we can regard J as indicative of a temporal state in the form: At any given time, J is marked or unmarked, but at the next time J takes the opposite value. This can be modeled in analogy with a buzzer or feedback circuit by a circuit diagram as shown in Fig. 13. The recursion can be understood by writing $J' = \overline{J\,|}$ where J' is the value of J at the next time.

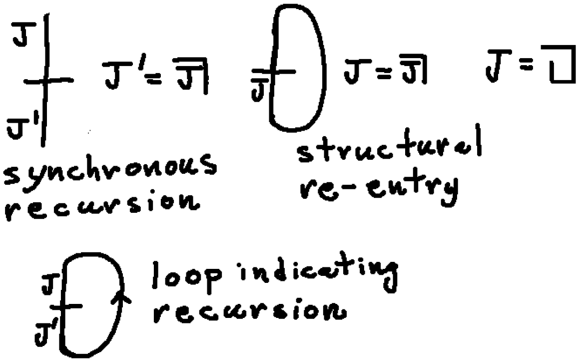


Fig. 13. Synchronous re-entering mark.

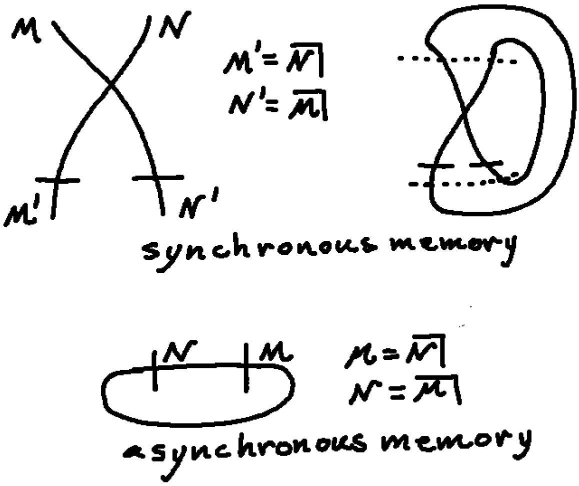


Fig. 14. Synchronous and asynchronous memory.

By the same token, we can consider $M = \overline{\overline{M|}\,|}$. We can turn this equation into a temporal process by writing $M' = \overline{\overline{M|}\,|}$. Here we see that $M' = \overline{\overline{M|}\,|} = M$ and so the value of M at the next time is its previous value. In this sense M is a *memory* element. As a circuit we write $M = \overline{N|}$, $N = \overline{M|}$ to instantiate the presence of the two marks in the formula for M (see Fig. 14). With this notion we can write the process as

$$M^{'} = \overline{N}\rceil,$$
$$N^{'} = \overline{M}\rceil.$$

At this point we realize that the pair $(M^{'}, N^{'})$ could be simultaneously updated from (M, N) or we could first change one of them and then change the other. When we speak of simultaneous updating, we call the process a *synchronous* process. If a process can be accomplished without simultaneity, it is called *asynchronous*.

The simultaneous update can be expressed as a transformation:

$$(M^{'}, N^{'}) = S(M, N) = (\overline{N}\rceil, \overline{M}\rceil).$$

For convenience, let $0 = \rceil$ and $1 = \overline{\rceil}\rceil$. Then we have $S(0,1) = (0,1)$ and $S(1,0) = (1,0)$. From the point of view of memory, this means that the values of M and N remain constant in time, just as they should in a memory unit. However, if $M = 0$ and $N = 0$ then $S(M,N) = S(0,0) = (1,1)$ and similarly, $S(1,1) = (0,0)$. Thus, the synchronous memory oscillates if we start its sides M and N in the same state. In an asynchronous situation there are two transformations:

$M' = \overline{N}\rceil$ and $N' = \overline{M}\rceil$. If one of them happens before the other, we have a race. For example, if we start with $M = 0$ and $N = 0$ and apply the first equation $M' = \overline{N}\rceil$ then we have $M = 1$ and $N = 0$ at the very next time. From then on $M' = M$ and $N' = N$ and the memory is set in the state (1,0). If we had applied $N' = \overline{M}\rceil$ first, the memory would have landed in the state (0,1). Thus the asynchronous memory has different behaviour than its synchronous cousin. See Fig. 14 for the diagrammatics of this discussion.

Note that in Fig. 14 we see the synchronic transformation

$$(M', N') = S(M, N) = (\overline{N}\rceil, \overline{M}\rceil)$$

as a diagram that is a composition of crossed lines (permutation) and crossing operators on each line. In showing the memory as a feedback device we conceive this transformation feeding back both new values simultaneously. In conceiving the memory as an asynchronous device we think of the feedback as structural, demanding that the local transformations be satisfied at each node. The stable states of the memory are then labellings of this graph such that each local equation is satisfied.

Chapter 11 of Laws of Form introduces the notion of temporal forms and contains a number of asynchronous designs that are fundamental for the construction of counting in computers. An article by Spencer-Brown tells more (reference Modulators Article) and the author of the present paper has written a number of articles about these asynchronous modulators. Here we will consider synchronous structures in a short summary of ideas.

Consider the following transformation:

$$T(x,y) = (\overline{\overline{xy}|}|\,\overline{\overline{x}|\,y}|, \overline{y}|).$$

Think of this transformation as a device whose input is an variable y that oscillates between 0 and 1. We have $x' = \overline{\overline{xy}|}|\,\overline{\overline{x}|\,y}|$ for the next value of x. This means that if y = 0 then $x' = \overline{x}|$, but if $y = 1$, then $x' = x$.

Thus, as y oscillates between 0 and 1, x changes at half the rate of y. If we start x at 0 and y at 0 then the x values will be 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, ... as y oscillates in the pattern 0, 1, 0, 1, 0, 1, 0, 1, ... The iterations of T make a synchronous modulator. See Fig. 15 for a circuit illustration of the recursion of T, and for a depiction of T as a re-entry form. Is this the simplest synchronous modulator? The answer is no!

Consider $G(x,y) = (\overline{y}|, x)$. Then G has order four when iterated and so can be regarded as a modulator. Starting with (0,0) we have successive values of $G(x,y)$: $(0,0), (1,0), (1,1), (0,1), (0,0), \ldots$. Now look at the synchronous diagram of G. It looks like $J' = \overline{J}|$ wound around itself! (see Fig. 15). Indeed it is the winding and permuting inherent our definition of G that makes it have order 4. The pair of values move simultaneously down the lines, cross over, and one of them is flipped.

This ordering of events would be ignored in a corresponding asynchronous circuit and the modulator would not have order four in that realm. We can make many periodic structures in the synchronous realm that would disintegrate in the asynchronous realm. This is why the modulators in the Chapter 11 of Laws of Form are more complex than the synchronous examples. The asynchronous modulators will work no matter how you set them and how you assign time

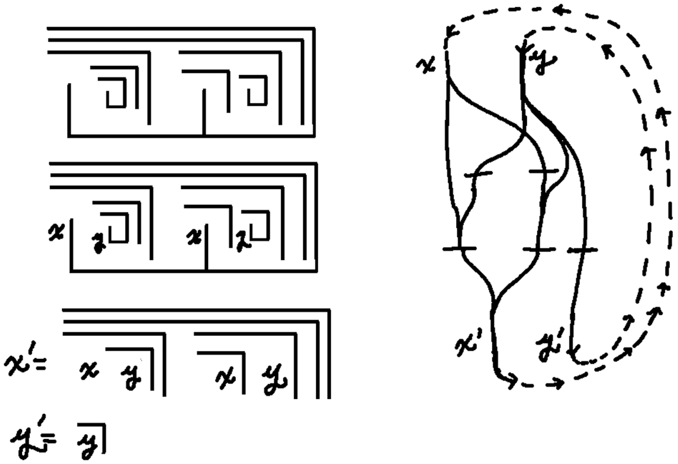


Fig. 15. A synchronous modulator.

delays to their parts.They operate independent of time and so can be seen as generators of time, structured in eternity.

We cannot resist pointing out that the synchronous clock of Fig. 16, is in fact our old friend the square root of minus one. Just transpose G to ordinary arithmetic and write it as

$$g(x, y) = (-y, x).$$

This is how i works on $(s, b) = a + ib$.

$$i(a, b) = i(a + ib) = ia - b = -b + ia = (-b, a).$$

Thus, we have come full circle and found the square root of minus one as the fundamental synchronous clock at the base of the dynamics of forms.

There is more to say about synchronicity and asynchronicity, but this is where we shall stop and return to the source.

16. Boundaries and the Calculus of Idemposition

We now discuss a calculus of boundaries. We shall consider curves in the plane and their interactions.

In Fig. 17, we illustrate two polygonal (i.e. made up from straight segments) closed curves interacting. We will use polygonal curves

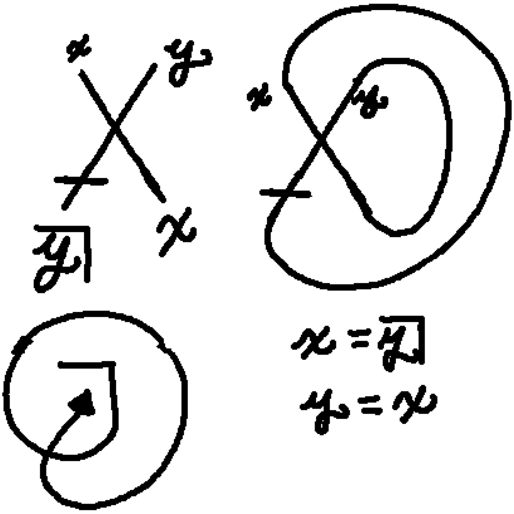


Fig. 16. A simplest synchronic modulator.

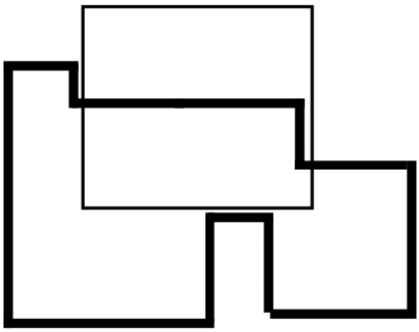


Fig. 17. An interaction of boundaries.

throughout the rest of this discussion. All polygonal curves will interact by sharing a segment of boundary in such a way that the two curves either cross over one another or not. In the case where they do not cross over, we call the interaction a *meeting*.

In a crossing, the two curves interact by sharing boundary, and each crosses over the other curve. In a meeting the two curves share boundary, but politely decline to cross. Meeting is analogous to a polite handshake or the sharing of common boundaries of two countries. Crossing is analogous to conversation, argument, or making

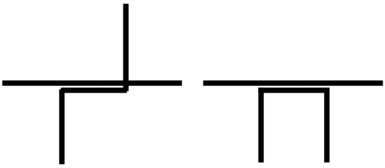


Fig. 18. Basic interactions.

love, where the distinctions that each party makes become part of the other party.

In Fig. 18, we illustrate an interaction between two curves A and B. The curve B is a simple rectangle, while A has a complex pattern of interaction with B that constitutes many crossings and meetings. That is one way to look at it, but of course if one traverses

B then from B's point of view there are many crossings and meetings with A. There is a relativity of interaction here, and the upshot is that while any single curve can be regarded as simple, two curves in interaction can create an arbitrary complexity.

The reader should note that while there are many metaphors linking these curve interactions to human interaction, it is certainly not necessary to take them literally, or to insist on such correspondences. Just as in Laws of Form where we considered the interactions of simple distinctions, these structures of interaction have a life of their own to be explored and examined both for their inner patterns and for their external relationships to our personal experience.

Analogous to the laws of crossing and calling in Laws of Form, we adopt the

Principle of Idemposition: **Common boundaries cancel.** *Idemposition is our term for the cancellation of superimposed* (*or common*) *boundaries.*

In Fig. 19, we illustrate the two basic local effects of idemposition at a crossing or at a meeting. Note the following two examples in Fig. 20.

Figure 20 shows that the familiar laws of calling and crossing are part of the more general curve arithmetic of idempositions. The law of calling corresponds to the meeting of two curves that results in their direct condensation to a single curve. The law of crossing

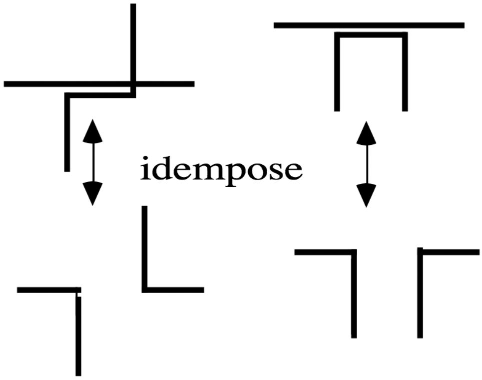


Fig. 19. Idemposition.

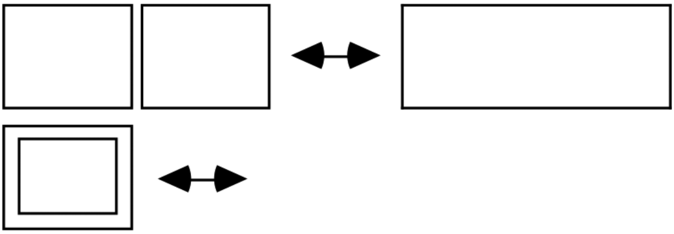


Fig. 20. Idemposition generates calling and crossing.

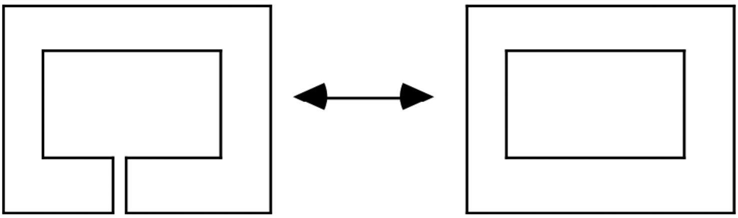


Fig. 21. A self-idemposition.

corresponds to a more total meeting of the two curves where they share boundaries completely and both disappear into void.

Of course, there are many more complex interactions possible. For example, consider Fig. 21.

Here a curve self-interacts and produces two curves.

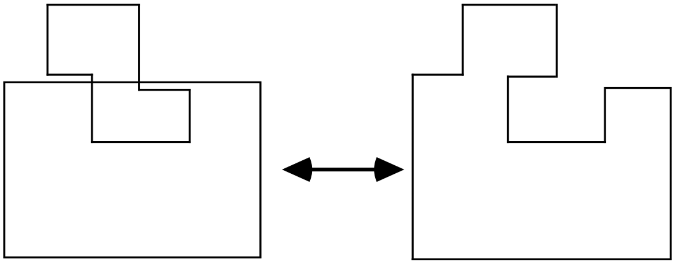


Fig. 22. From two to one.

An important first step in the calculus of idempositions is to be able to determine when the number of curves after the idemposition of two curves is even or odd. Here is one more example in Fig. 22.

In this example, two curves cross and idempose to form a single curve.

In fact it is possible to tell whether the number of curves in the idemposition of two curves is even or odd. What you do is take a walk along one of the curves and note how many times you see first a crossing on your left, how many times you see first a crossing on your right, and how many times you see a meeting. Let L be the number of left-first crossings, R the number of right-first crossings and M the number of meetings. Then form the number

$$M + (L - R)/2.$$

If this number is odd then there will be an odd number of curves after idemposition. If this number is even then there will be an even number of curves after idemposition. The reader should explore this statement and see how it works. This result is a first step in the calculus of idempositions. Working with it gives insight into how the calculus of indications can begin to become a calculus of interactions.

This calculus of boundaries is essential in Spencer-Brown's work on the coloring of maps. See Spencer-Brown (2015) for Spencer-Brown on this subject and see Kauffman (1985, 1990, 2005, 2016, 2020). Figure 23 gives a hint about the connection of idemposition calculus and the map theorem.

In Fig. 23 at the top left we see an expression in the idemposition calculus using two primary marks. These marks are *red* (solid color boundary) and *blue* (dotted boundary). When red and blue share a

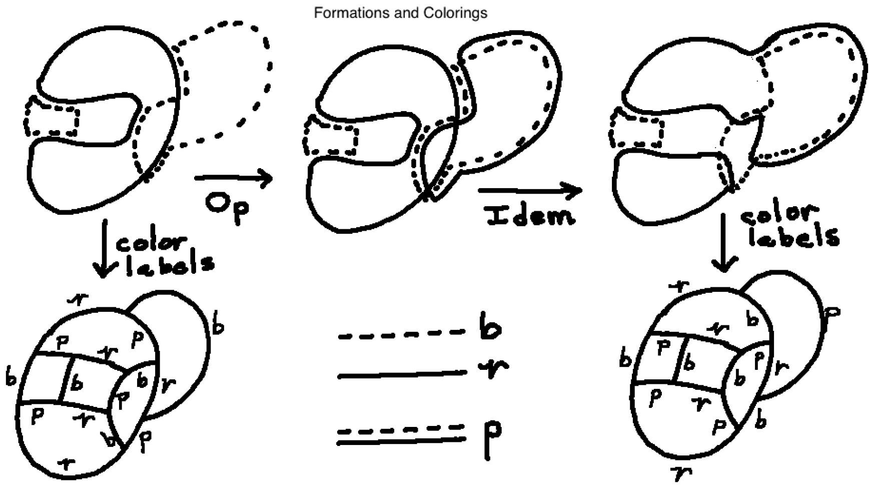


Fig. 23. Formations and map colorings.

boundary the combined color is (red/blue) *purple.* Any expression in the two primary marks (called a *formation*) can be seen as a coloring of the edges of a graph in the plane so that the graph has three edges at each vertex and these three edges have distinct colors (red, blue, purple). In the leftmost part of Fig. 23 we see the graph just after the arrow that says "color labels". The map coloring theorem (equivalent to the four color theorem) is equivalent to the statement that every trivalent plane graph that cannot be disconnected by removing an edge has the structure of a formation. In other words, all 1-connected planar trivalent graphs occur in the set of expressions involving two primary marks. The rest of the Fig. 23 illustrates how applying idemposition to individual colors (red can idempose red and blue can idempose blue) leads from expression to another and from one coloring to another. In this way, the idemposition calculus becomes a powerful tool in studying colorings and formations

17. Epilogue

In this paper, we have examined the use, development and dynamics of signs in relation to G. Spencer-Brown's Laws of Form. Consider the fragment in Fig. 24 from a Sumerian document, 26th century BC (https://en.wikipedia.org/wiki/Cuneiform).

Fig. 24. Sumerian Document, 26th Century B.C.

There in the document are a nest of left-shaped marks, and since they are nested, the distinction they each make in the plane was clearly part of their use. In modern typography a relative of the Spencer-Brown mark is the square root sign, a connected sign that can be nested and arranged for mathematical purposes.

$$\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{2}}}}}}$$

The language of Laws of Form was discovered, according to Spencer-Brown, in making a descent from Boolean algebra in which he found the notation of the mark, the role of the unmarked state and the double-carry of mark as name and mark as transformation. In Boolean algebra and in symbolic logic the negation sign connotes transformation and it does not stand for a value (True or False). In the calculus of indications, viewed from the stance of symbolic logic, the mark is a coalescence of the value True and the sign of negation.

This comes about because True is what is not False and the False is unmarked in Laws of Form. But this can not be said without confusion in symbolic logic since *there is no inside to the sign* $\sim$ *of negation.* In Laws of Form we can write $T = \overline{F}\rceil = \rceil$ and the mark as container (as parenthesis) makes it possible for it to take on its double role of value and operator.

Wittgenstein says (Wittgenstein (1922) — Tractatus [97] 4.0621)

"... the sign '$\sim$' corresponds to nothing in reality."

And he goes on to say (Wittgenstein (1922) — Tractatus 5.511)

"How can all-embracing logic which mirrors the world use such special catches and manipulations? Only because all these are connected into an infinitely fine network, the great mirror."

For Wittgenstein in the Tractatus, the negation sign is part of the mirror making it possible for thought to reflect reality through combinations of signs. These remarks of Wittgenstein are part of his early picture theory of the relationship of formalism and the world. In our view, the world and the formalism we use to represent the world are not separate. The observer and the mark are (in the form) identical.

This theme of formalism and the world is given a curious twist by an observation that the mark and its laws of calling and crossing can be regarded as the pattern of interactions of the most elementary of possible quantum particles, the Majorana Fermion (Kauffman, 2016a). A Majorana Fermion is a hypothetical particle that is its own anti-particle. It can interact with itself to either produce itself or to annihilate itself. In the mark we have these two modes of interaction as calling

$$\urcorner\,\urcorner = \urcorner_{\text{and crossing}} \overline{\urcorner\,}\!| = .$$

The curious nature of quantum mechanics is seen not in such simple interactions but in the logic of superposition and measurement. Measurement of a quantum state demands the coming into actuality of exactly one of a myriad of possibilities. Thus we may write

$$\urcorner * \urcorner = \urcorner\,\urcorner + \overline{\urcorner\,}\!|$$

to indicate that the quantum state $\urcorner * \urcorner$ of a self-interacting Majorana Fermion $\urcorner$ is a superposition of marked and unmarked states. Upon observation, one or the other (marked or unmarked) will be actual, but before observation, the state is neither marked nor is it unmarked.

We need a deeper step to enter into quantum sensibility. The equation for this interaction can be written in ordinary algebra as

$$PP = P + 1$$

where P stands for the Majorana Particle and 1 stands for the neutral state of pure radiation. Then we recognize a famous quadratic equation

$$P^2 = P + 1$$

with solution the Golden Ratio $(1 + \sqrt{5})/2$ and multifold relationships with the Fibonacci numbers. Indeed this is the legacy of the Majorana Fermion as Fibonacci particle, fundamental entity in the most idealistic and yet soon to be practical searches for quantum computing and understanding of particles as well-known as the electron. Each electron appears as an amalgam of two Majorana Fermions. The moving boundary of Sign and Space is changed from the time of Wittgenstein and we see insight of a different kind from now on.

When representation and explanation are insisted upon, then an infinite regress occurs due to the proliferation of signs that must indicate each stage of explanation. When this noise is reduced by the simple recognition of the presence of a distinction, then forms can stand alone and be recognized as being, in form, identical with their creators.

Along with the references quoted directly in the text, I have provided a selection of papers that are related to the themes of this essay. There is much to think about in this domain and we have only just begun.

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