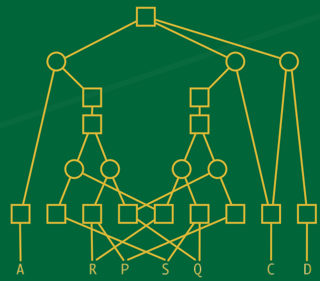
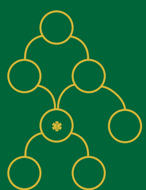
# Iconic Arithmetic




*simple*

*sensual*

*postsymbolic*



Volume II

## SYMBOLIC and POSTSYMBOLIC FORMAL FOUNDATIONS

William Bricken, Ph.D.

# Iconic Arithmetic

## Volume II

Any comments, corrections, refinements or
suggestions you may have will be greatly appreciated.

I'm available via email at
william@iconicmath.com

Thanks.


ISBN (soft-cover edition 2): 978-1-7324851-4-3
Unary Press, an imprint of
Unary Computers, Snohomish Washington USA

# Iconic Arithmetic

simple

sensual

postsymbolic

Volume II
SYMBOLIC and POSTSYMBOLIC
FORMAL FOUNDATIONS

William Bricken

in memoriam
Richard G. Shoup
George Spencer Brown

# Chapters Volume I

# Chapters Volume II

# Contents Volume II

# List of Figures

# Cast of Characters

## Visionaries

Aristotle
Gregory Bateson
John Horton Conway
Gottlob Frege
David Hilbert
Gottfried Leibniz
Giuseppe Peano
Charles Sanders Peirce
George Spencer Brown
Bertrand Russell & Alfred North Whitehead
Ludwig Wittgenstein
Stephen Wolfram

## Voices

Alain Badiou
Paul Bernays
Richard Dedekind
Stanislas Dehaene
Solomon Feferman
Louis Kauffman
Jaron Lanier
Brian Rotman

## Research and Perspective

Paul Benacerraf
Gregory Chaitin
Tobias Dantzig
John Derbyshire
Curtis Franks
Joseph Goguen
Mark Greaves

Verina Huber-Dyson
Stephen Kleene
Morris Kline
Donald Knuth
George Lakoff & Rafael Núñez
Rinus Plasmeijer & Marko van Eekelen
Hilary Putnam
Jean van Heijenoort
Francisco Varela
Fernando Zalamea

## Colleagues

Art Collings
Jack Engstrom
Thomas Etter
Fred Furtek
Robert Horn
Jeffrey James
David Keenan
Thomas McFarlane
Meredith Bricken Mills
Daniel Shapiro
Richard Shoup
William Winn

## Special Thanks to

Paul Allen
Colin Bricken
Ian Bricken
Julie Bricken
Doug Emory
Jeffrey James
Louis Kauffman
Ted Nelson
Daniel Shapiro
Richard Shoup
John Walker

| **Historical Figures** | ***circa*** | ***chapter*** |
|---|---|---|
| Pythagoras | 500 BCE | 21, 29 |
| Parmenides | 450 | 22, 29 |
| Plato | 400 | 16-17 |
| Aristotle | 350 | 17, 20, 22, 24, 26 |
| Euclid | 300 | 16, 20, 24, 26 |
| William of Ockham | 1320 CE | 26 |
| Viète | 1600 | 26 |
| Descartes | 1640 | 16, 26-27 |
| Leibniz | 1700 | 16-18, 21, 23, 26 |
| Euler | 1760 | 30 |
| Hume | 1760 | 22 |
| Lagrange | 1800 | 18 |
| Boole | 1860 | 28 |
| DeMorgan | 1860 | 16 |
| Cantor | 1880 | 21-22, 26-27 |
| Kronecker | 1880 | 24 |
| Dedekind | 1880 | 16, 19, 21-23, 27, 30 |
| Venn | 1880 | 24, 26 |
| Frege | 1900 | 16-17, 19-22, 24, 26 |
| Hilbert | 1900 | 16, 21, 23-27, 30 |
| Peano | 1900 | 16, 18-19, 21-24, 26-30 |
| Peirce | 1900 | 16, 22, 24, 26-30 |
| Poincaré | 1900 | 16, 27 |
| Brouwer | 1920 | 26 |
| Einstein | 1920 | 26 |
| Russell | 1920 | 21-22, 24, 26-28, 30 |
| Gödel | 1940 | 17, 19, 22-24, 26, 30 |
| Wittgenstein | 1940 | 16-17, 19, 24, 26-27 |

| **Quotes and Concepts** | ***chapter*** |
|---|---|
| Aristotle | 17, 20, open-22, 24, 26 |
| Alain Badiou | 17, 19, 22, 26-27, open-30 |
| Gregory Bateson | 19, 21, 30 |
| Paul Bernays | open-preface, 21, 23, 28 |
| Georg Cantor | 21-22, 26-27 |
| Richard Dedekind | 16, 19, 21-23, 27, 30 |
| Stanislas Dehaene | 19, 22, 26-27 |

## Supporting Quotes and Concepts *chapter*

| **Mentioned** | ***chapter*** |
|---|---|

## Editors and Translators

| (ed.) | *chapter* |
|---|---|
| W. Barfield | 19 |
| J. Barwise | 24 |
| J. Bendegem | 23 |
| J. Bransford | 26 |
| W. Brauer | 19 |
| W. Disney | 18 |
| M. Donovan | 26 |
| B. Engquist | 17 |
| B. Findell | 26 |
| T. Furness | 19 |
| C. Hartshorne | 29 |
| R. Hersh | 28 |
| C. Janvier | 26 |
| K. Ketner | 26 |
| J. Kilpatrick | 26 |
| D. Leivant | 19 |
| B. Magnusson | 26 |
| B. Meyer | 26 |
| J-F. Perrot | 26 |
| H. Putman | 26 |
| C. Rollinger | 19 |
| W. Schmid | 17 |
| J. Swalford | 26 |
| J. van Heijenoort | 17, 22, 28 |
| I. Wachsmuth | 19 |
| P. Weiss | 29 |

| (trans.) | *chapter* |
|---|---|
| W. Beman | 22, 23 |
| W. Ewald | 27 |
| R. Gaye | 22 |
| G. Halstead | 16 |
| R. Hardie | 22 |
| P. Mancosu | 23 |
| I. Mueller | 23 |
| L. Robinson | 26, 28 |
| D. Schlimm | 28 |

# Preface

*All fundamental questions can be settled*
*in a speci  cally mathematical way,*
*without having to rack one's brain about subtle logical dilemmas.*
*— Paul Bernays (1921)*

When this project began, it was envisioned as an application of boundary techniques to numeric arithmetic, to be followed by a report on my twenty years of work with boundary *logic*. I thought that arithmetic would be a friendlier and more familiar introduction to iconic thinking than would logic. As the chapters of *Iconic Arithmetic* accumulated, it became clear that there were three relatively separable areas: basic iconic arithmetic, historical grounding and the exotics of imaginary and infinite forms. Each of these areas has taken a separate volume to explore, primarily to make the technical case for iconic math both structurally and computationally, and to provide a thorough description of historical modes of thought. Several major themes have emerged:

**Comparative Theme:** The concepts of elementary mathematics have evolved over millenia based on what seemed to be good ideas at the time. However *math is designed* and different designs lead to different ways of thinking about mathematics, about structure and about the world. This theme grew out of the recent evolution of computer programming languages.

**Computational Theme:** If mathematics were to be measured by how it is used, then *almost all math is silicon computation*. With the introduction of metamathematical concepts such as effective procedures, algorithmic decidability, finite resources and ubiquitous computation the focus of formal thinking has shifted from valid proof to computational feasibility. Wolfram's *Mathematica* has demonstrated that symbol processing is no longer within the domain of humans. This perspective reflects my choice of profession as a computer scientist.

**Perceptual Theme:** Many aspects of math are *visual and experiential*. The utter dominance of symbolic forms of math narrows our thinking and disenfranchises the majority of folks who are exposed to elementary math education. This theme grow out of my study of Spencer Brown's *Laws of Form* and out of Francisco Varela's approach to the embodiment of concept, as well as dozens of years teaching mathematical ideas.

**Discovery Theme:** Iconic arithmetic incorporates some *fascinating design features*, such as the pervasiveness of *void*, the dual nature of boundaries, the structural freedom of multiple dimensions, the natural parallelism, three elegant yet comprehensive axioms, and the surprising new methods of transformation. These features are reflected in new ways to think about structure and about reality. This theme was inpired by the work of Louis Kauffman and is a result of several decades of implementing formal iconic software tools.

Perhaps needless to say, I also learn by writing.

I awoke one morning in 2015 to realize that my training in formal systems circa 1980 was, to say the least, antiquated. In the twenty-first century I was using a style of mathematical thinking from the early twentieth century, ignoring the fundamental evolution of mathematical perspective due to Grothendieck (algebraic geometry), Baez (n-categories), Chaitin (undecidability), Wolfram (universal computation) and others. I began to see that elementary logic and arithmetic are not determined, secure or natural. The iconic tools I had been working with for years have just as valid a claim to mathematical foundations as do the early explorations that led us today into set theory, Boolean logic and functional thinking. The bulk of the mathematical community still bases their formal thinking about numbers on Peano's axioms, while the modern evolution in mathematical thought appears to have taken place in the more rarefied atmosphere of the unification of advanced abstraction. Short of returning to school, I tried to leverage my antediluvian education to build what appear to be quite different systems of *elementary* mathematical thinking.

Volume I of this series describes two alternative ways to conceptualize numbers. **Ensemble arithmetic** mirrors the evolution of numeric math for thousands of years prior to the symbolic dominance that took hold less than two hundred years ago. **James algebra** embodies iconic structure to open new perspectives on elementary arithmetic by reformulating the "laws" of algebra. And there are plenty of surprises, especially the unexpected appearance of **infinite and imaginary forms**, both of which constitute the content of Volume III.

This Volume II is focused on **comparative axiomatics**, comparing James algebra to our current formal foundations for the arithmetic of numbers. Three simple structural James axioms ground iconic transformations throughout the three volumes. From a computational perspective nothing is remote, complicated or indirect, given the narrow focus on elementary math.

Here we explore the potential of a postsymbolic math that injects our current formal foundations with multiple heresies while still respecting the formality of mathematics itself. To justify the imposition of new iconic forms and transforms, I've described and compared the approaches of several technically different fields including numeric arithmetic and algebra, predicate logic, set theory, computational pattern-matching, educational methods and iconic boundary mathematics. Not only is there no agreement across fields about the basic structures of arithmetic, there is also no communality across tools and objectives. Iconic math adds yet more enrichment through diversity.

Volume II feels quite different than Volume I, with deeper, more historical questions at the foundations of the current philosophies of mathematics. This is a necessary volume to address the many technical details about the structures, assumptions and thought processes that we now expect grade school teachers and students to grasp intuitively.

In reading the text, backward reference to Chapters 1 through 15 refer to Volume I. All structural necessities are included in Figure 16-1 of Chapter 16. All references to online content have been verified as accessible during the latter half of 2018. The `iconicmath.com` website is the nexus for the content in these volumes and for forthcoming volumes focused more on computational logic.

The narrow side columns on each page hold handy illustrations and reminders in support of the text. In the case of formal transformation sequences, the rules being applied are listed line-by-line in the margins.

Aside from delimiting brackets that serve as unified iconic objects/operators, there are a few symbolic characters that have unusual roles.

I've used typographical delimiters, ( ), [ ], < > and others rather than Spencer Brown's spatial mark, ⏋, for easier typography and to make available several different representations for types of spatial containers.

A fixed width Monaco font identifies mathematical forms and functions, while the linguistic content is in Cochin.

The finger ☞ indicates a change in formal system, usually moving between iconic James forms and conventional string expressions.

The numeric unit represented by a round-bracket has two forms, ( ) and o.

The arbitrary James base is represented by #.

The quasi-token *void* is meant not to exist.

A *frame* is the James structure (A [B]) with A the **frame type** and B the **frame contents**.

The opening quote by Paul Bernays is a summary description of the goals of Hilbert's program to convert mathematics into a purely structural discipline. That indeed is the approach taken herein, with the fundamental difference that structure is illustrated by iconic images rather than expressed by typographical strings of symbols.

I do hope you too will enjoy this exploration of iconic form and postsymbolic thinking.

william bricken
Snohomish Washington, March 8, 2019

# Iconic Arithmetic

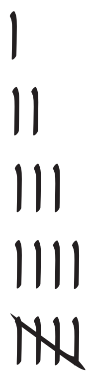
## Volume II

# Chapter 16

# Crossing

*We have allowed only one kind of relation...*
*a cross is said to contain what is on its inside and*
*not to contain what is not on its inside.*[1]
*— George Spencer Brown (1969)*

In Volume I **ensemble arithmetic** retraces the earliest recorded evidence that *homo sapiens* engaged in activities to answer the question "How many?". The system in use in antiquity was *tally arithmetic*: beads and knots and marks that record a one-to-one correspondence presumably with physical objects. As tallies accumulated they were gathered into groups. Putting tallies together *defines* addition. Grouping tallies defines multiplication. Hilbert, Frege, Peano and other founders of modern formal arithmetic at the turn of the twentieth century had tally arithmetic firmly in mind as the intuitively obvious foundation of numbers. Ensemble arithmetic extends this belief with the iconic techniques introduced by Charles Sanders Peirce in the 1890s and by George Spencer Brown in the 1960s.[2]



The second formal system in Volume I is **James algebra**, originally developed by Jeffrey James and myself in the early 1990s at the University of Washington, and inspired by the work of Louis Kauffman at the University of Illinois at Chicago. We migrated our approach to iconic representation and computation to Interval Research Corporation[3] during that decade, to implement under the direction of Richard Shoup several boundary mathematics software and hardware systems. The Natural Computing Project's mission was simple: If you could redesign computation from scratch, with no concern for backward compatibility, what would you do? These volumes are a relatively small yet foundational part of the answer.

*round* ( )
*square* [ ]
*angle* < >

Volume I explores the nature of a unit, how pattern variables support transformation, and the iconic path that connects accumulation with counting, addition, multiplication and exponentiation, the bread-and-butter of numeric operations. Two complementary boundaries define these operations. One more reflective boundary gives us all of the inverse operations. For completeness Volume I shows how polynomials, real numbers, fractions and bases can all be understood as patterns of containment. The three James boundaries unify the diversity of numeric expressions as patterns that make *no direct reference* to numeric concepts. Volume I then shows how these patterns are postsymbolic, their iconic form can be rendered in a wide variety of spatial and experiential dialects.

We'll now cross into Volume II, to compare James algebra to the foundational theories of formal arithmetic developed during the early twentieth century. How can a *new* formal structure, an **iconic structure**, come into being for something we know so well? How can arithmetic not be the arithmetic that we learned in grade school? History suggests that our understanding of arithmetic is undergoing continuous evolution. Formalization was a first rather than a last step in the definition of numeric structure.

## 16.1 James Algebra

Figure 16-1 is a summary of the structural basis of James algebra developed in Volume I. Transformation equations have names for application in either direction. All

## Axioms and Theorems of James Algebra

### *Ground Interpretations*

| | | | | | | |
|---|---|---|---|---|---|---|
| o = ( ) | ☞ | 1 | (o) | ☞ | # | |
| < > | ☞ | 0 | <o> | ☞ | −1 | |
| [ ] | ☞ | −∞ | <[ ]> | ☞ | ∞ | *(volume III)* |

### *Unit Definition*

| | | | |
|---|---|---|---|
| ( ) ≠ *void* | **existence** | | |
| ( ) ( ) ≠ ( ) | **unit accumulation** | | |
| [ ] [ ] ⇒ [ ] | **unification** | | |
| [ ]<[ ]> ⇒ *indeterminate* | **indeterminacy** | *(volume III)* | |

### *Pattern Axioms*

| | | |
|---|---|---|
| ([A]) = [(A)] = A | **inversion** | *enfold/clarify* |
| (A [B C]) = (A [B]) (A [C]) | **arrangement** | *collect/disperse* |
| A <A> = *void* | **reflection** | *create/cancel* |

### *Interpretative Axiom* *(volume III)*

| | |
|---|---|
| (<[ ]>) = <[ ]> = [<[ ]>] | **infinite interpretation** |

### *Theorems*

| | | |
|---|---|---|
| ( ) <( )> = *void* | **unit reflection** | *create/cancel* |
| ([ ]) = [( )] = *void* | **void inversion** | *enfold/clarify* |
| (A [ ]) = *void* | **dominion** | *emit/absorb* |
| A = ([A][o]) | **indication** | *unmark/mark* |
| A..$_N$..A = ([A][o..$_N$..o]) | **replication** | *replicate/tally* |
| <<A>> = A | **involution** | *wrap/unwrap* |
| <A><B> = <A B> | **separation** | *split/join* |
| <A <B>> = <A> B | **reaction** | *react/react* |
| (A [<B>] ) = <(A [B] )><br>(A <[<B>]>) = <(A <[B]>)> | **promotion** | *demote/promote* |

Figure 16-1: *Summary of definitions, axioms and theorems*

forms are represented in a minimal iconic language of typographical delimiters, with some exceptions explained more thoroughly in Volume I. Within James algebra

- The numeric unit has both a boundary form, `( )`, and a single character abbreviation, `o`.
- The Replication Theorem is generalized to `N` replications by the composite symbol $\cdot\cdot_{\mathtt{N}}\cdot\cdot\;\;\cdot$ `N` is finite, although there is no implication that `N` is a natural number.

As metalanguage to straddle the chasm between iconic and symbolic languages,

- The **interpretation finger**, ☞, indicates when we have changed formal systems.
- The **arbitrary base** made specific by interpretation is symbolized by #.
- The **absence of a form** and the pervasive space underneath forms is brought to awareness by the virtual indicator *void*.

*accumulation*

● ● ≠ ●

While respecting the structural constraints of formal systems, common arithmetic and its algebra can be fully described within James algebra by the Accumulation Principle and three structural axioms that are analogous to the conventional operations of

| | |
|---|---|
| — additive inverse | **Reflection** |
| — functional inverse | **Inversion** |
| — distribution | **Arrangement** |

The Inversion and Arrangement axioms specify the interaction of round-brackets and square-brackets. They are sufficient for addition, multiplication and exponentiation. Reflection provides the definition of the angle-bracket, and is necessary to establish the Dominion theorem. Reflection provides the inverse operations, Dominion provides the behavior of conventional `0`. A Composition Principle governs construction and deconstruction of James forms and defines structural identity.

Numbers, and how they work, arise from ignoring differences within a vast panoply of structural uniqueness in three specific *iconic* ways.

— **No ambiguity:** Forms participate only in sameness or in difference.
— **Void-equivalence:** Two axioms identify structure that has no meaning.
— **Structural variety:** One axiom, Arrangement, generates a diversity of structures that look different but are not different.

A few structural theorems have been identified as useful, patterns that occur sufficiently often that we deem it convenient to provide them with a name. Thus far theorems have served only two purposes: to help to articulate how accumulation works, and to juggle around the structural location of angle-brackets. Two theorems, Indication and Replication, manage the generation and collection of replicated structure and help to explain the process of counting. The four theorems for angle-brackets are quite useful for transforming reflected forms, however only Promotion is essential. Arrangement and Promotion are the only patterns that rearrange containment structure. Promotion moves angle-brackets through inversion frames. Conversely, angle-brackets that cannot be promoted indicate sites of structural complexity. All other patterns create and delete structure. Form that can be arbitrarily deleted cannot impact fundamental structural variety, nor can its interpretation impact numeric value.

As indicated in the Figure 16-1, Volume III introduces one more unit definition and one more interpretative axiom. These are sufficient to organize the diversity of non-numeric forms, those forms that contain an empty square-bracket. Indeterminacy and Infinite Interpretation are postponed until infinite expressions and the exceptions they visit upon conventional arithmetic can be more thoroughly explored in Volume III.[4]

## Concept and Design

*a specific form*

(( )( ))

*an arbitrary form*

A

Iconic arithmetic addresses both arithmetic and algebra. Arithmetic deals with specifics, how empty boundaries work together. Algebra deals with generalities. An algebraic pattern-variable stands in place of any form. What web of interrelations between arbitrary forms do the three James axioms induce? Of course, we have the freedom to nominate whatever structural relations we are interested in. However there are many, shall we say, **meta-constraints** that not only define what we mean by a formal system but also limit the kinds of relationships that we may choose.

*iconic existence*
*is*
*physical existence*

A primary design decision in James algebra is to limit forms to the physical reality of containment, augmented by an ability to produce replicas of labels. We cast our anchor out onto the shore of constructibility in the physical world. This decision is both a preference and a discipline, based on a belief that school children will be able to understand what they can touch. This decision immediately separates our exploration from what many would consider to be mathematics.

*nothing*
*has no*
*structure*

The James axioms were not selected out-of-the-blue to define some forms that we would like to be equivalent. Rather the form of the axioms themselves comes with intention. A belief that guides the entire algebraic exploration is that *void* can have no relationships of any kind, contrary to symbolic concepts like zero, or the empty set, or logical FALSE. Our belief that *void has no structure* pays immediate dividends. We can simplify some structural consequences by ignoring them. **Void-equivalence** is a powerful ally in keeping it simple.

A second guiding belief is that *forms are unique* precisely because they are not other forms. Axioms create relatively small groups of forms that we can consider to be the same, and what is left outside of equality is *uniqueness*. It order to respect the uniqueness of forms we have elected to believe that *forms interact only with their container*, and with nothing else. When we put an orange into the basket with an apple, we choose to believe that the orange does not change the apple although it does change the basket by increasing its load.

*forms are independent*

To distinguish between numeric difference and conceptual uniqueness, we've incorporated the cognitive concept of **distinction** to join them both together. *A boundary is a distinction*, nothing more. It creates both inside and outside, two convenient concepts that allow us to localize differences. Axioms are choices to ignore differences, nothing more. By choosing to ignore just three kinds of difference (Involution, Arrangement and Reflection) we find ourselves able to identify within what is left that which our culture calls numbers.

## 16.2 Iconic Math

In this volume we return to the foundations of arithmetic developed over a century ago, to compare the metamathematical foundations of David Hilbert and his cohort to the perspectives and thinking induced by iconic form. We lurch into *postsymbolism* and find it necessary to abandon the comforts of set theory, logic and functions. And in the process we accomplish one of the primary goals of this work: a proof of principle, if you will, that our universally accepted way of thinking about numbers is *an option*, a temporal social decision not necessarily blessed with any absolute understanding of the nature of numeric thinking itself. Iconic arithmetic thus provides a path for numerics to follow logic into modernization from absolute Truth to relative truths.

A central idea that we are exploring is at the core of iconic notation: *a representation resembles what it means*. Boldly, we are reuniting human perception with the meaning of what we write down. The purely symbolic approach of predicate calculus cannot call upon the obvious:

*Look, ( ) exists. Look, it is empty.*

In symbolic notation, we must develop a non-intuitive language just to assert that we have elected to begin by drawing a distinction.

## Utter Simplicity

Although all results are formal, none of the three volumes on James algebra have a specifically *mathematical* style. There is little attempt to organize by definition, theorem and proof or to connect syntax to semantics. However, the philosophical commitment to remain **utterly simple** has resulted in two innovations. The first is that the use of *iconic representation* itself greatly simplifies the conceptual structure of numeric arithmetic and algebra. The second is that the consequent *iconic conceptual structure* greatly simplifies what would be considered to be algebraic computation and proof. The cost, which might not be surprising, is that iconic arithmetic throws us into unfamiliar territory. The classical styles of numeric calculation and the classical theorems of number theory are not directly motivated by the iconic foundation. Even the group theoretic foundations of modern algebra are abandoned. By looking in a pictorial direction we are able to see different facets of the mathematical enterprise. We are thus trespassing not only into Number, but also into the structure of elementary mathematics itself.

*iconic thinking*

## Mathematics of Mathematics

**Metamathematics** is about how mathematics works. What are the foundations of math? What are the essential principles? What can we believe? This entire volume is, in essence, an inquiry into the metamathematics of the arithmetic of numbers. The *computational approach* to mathematics is a variety of philosophy. The mathematical philosophies developed over one hundred years ago are being eclipsed by contemporary developments in both abstraction and computation. **Symbolic mathematics** now *means* mathematics performed by general purpose computational software such as *Mathematica*, not the kind of symbol juggling done by students in algebra class nor the symbolic metatheory envisioned by Hilbert. How should math be made accessible to humans, particularly to younger students? What is the *scope* of mathematical philosophy? Is it the historical thoughts of great minds, or perhaps very modern approaches that garner the Fields medal (the Nobel prize for mathematical accomplishment), or is it the foundational beliefs upon which mathematics rests, or possibly what practicing mathematicians actually do? Is there room in the philosophy of mathematics for what we teach to children? Is there sufficient tolerance to express formal concepts in multiple spatial-temporal dimensions?

## Aesthetics

With boundary math we are attempting to maintain the aesthetic values of formal systems. A benefit of iconic containment is that spatial boundaries permit relations to be exhibited as structure rather than abstracted as symbols or imagined as concepts. Another benefit is that iconic form provides sufficient compositional structure to support formal systems of great simplicity and wide-ranging power.

Scientific success is often associated with the aesthetic value of simplicity. Mathematics too values simplicity, in fact you might say that the *goal* of mathematics is to simplify. Poincaré said "Mathematics is the art of giving the same name to different things".[5] And certainly abstraction itself is a technique to render the complex simple.

*Henri Poincaré*
*1854–1912*

The philosophical infrastructure of abstraction, however, is not simple. For example, language cleaves our unified experience of wholeness into ambiguous and fabricated partitions that lack empirical existence.

*To describe is to make complex.*

We can, alternatively, perceive *images* as wholes. Experience is not as chunky as the words we use to describe it. Experience has no syntax.

Naming physical containers as a ground for semantics is only a convenience used to bridge the gap from mathematical abstraction to physical experience. We could just as easily name maps and territories, or a partial ordering structure, or terminating function calls in a software program. These linguistic anchors are a convenience; in the final analysis we must recognize Wittgenstein's main point, that *mathematics is devoid of reference and meaning*.

Iconic containment does not require the objects and operations that define numeric arithmetic. Everything instead is **patterns of containment**. Over the last two hundred years the primary candidate for an intuitive foundation for mathematics has been the arithmetic of whole numbers. James algebra can be *interpreted* as numeric arithmetic. The three James axioms themselves thus provide an alternative intuitive foundation, a simple visual approach to understanding that lacks only familiarity.

It is not a necessity that a new formal approach be instantly familiar, but it should be easy to follow and easy to learn. For that reason, the proofs (called demonstrations) of each new structural idea in these volumes have been overtly recorded with the names of transformations in the margins. Making the steps in each demonstration explicit accomplishes three important goals.

1) The demonstrations show that the three pattern-transformation axioms are indeed simple and powerful. Although we have appended a dozen theorems, these theorems themselves bundle only a

few proof steps. None are abstract or difficult. All are simple structural shortcuts.

2) There is a clear grounding in physicality: semantics consists only of putting labelled things into containers. The arithmetic of numbers is relegated to interpretation of containment structures.

3) Although it has evolved to be useful, our current number system is clumsy and difficult to use, as are the conceptual structures engendered by symbolic numbers. The demonstrations illustrate a conceptual approach, a way of thinking and a new method of organizing numeric structure built on one numeric concept, that of *accumulation*, and three transformation actions: arrangement, inversion and reflection.

## A New Perspective

We are exploring the James form to see if it sheds light upon the complexity of the arithmetic that our global culture has embraced. The goal is not to generate new mathematical theorems, this exploration is about foundations rather than elaborations. The intention is to explore a radically different conceptualization of arithmetic. This volume examines the relationships between iconic form and foundational mathematics circa 1900, well before Mac Lane, Grothendieck, Lawvere and others essentially refocused abstract mathematics to include the structure of transformation processes.

The symbolic perspective is that *a number is what it does.*[6] The iconic perspective is that a number is what it looks like. James algebra is a new method of exposing the behavior of arithmetic, not just by showing different conceptual pathways to understanding what we already know but by showing a different conceptual model of numeric behavior itself. Our modern understanding of unified abstraction (algebraic geometry, non-commutative algebras) is constructed on top of existing foundations, elaborations that have led to the evolving conviction that apparently different maths have the same deep structure, similar to Chomsky's notion of deep linguistic structure that allows the creation of thousands of indigenous languages[7] while providing a common cognitive infrastructure for all. Iconic math does not necessarily shed more light on the deep structure of mathematical conceptualization, rather it shifts to a *different sensory modality*, not linguistic but experiential, not only written and spoken but also seen and touched.

The concept of *equality* has also greatly expanded over the twentieth century due to the influence of **category theory**. Functions are maps between sets of objects. Category theory considers how these maps work. When maps are equivalent, they are *isomorphic*. Alternatively, within void-based thinking equality means non-existence.

To summarize, Figure 16-2 lists several of the changes in our conception of number that are embodied in James algebra. The figure includes a name and a short description for each concept, although it is the fusion of these conceptual fragments that forms a descriptive whole. The figure also includes a rough visual comparison of iconic form to symbolic expression. In the chapters that follow each of these new perspectives is compared to current ideas about number. The particular differences in the figure are characteristic of but certainly not definitive of iconic math in general.

## 16.3 Declaration of Independence

According to Plato's metaphysics, mathematics is the study of eternal and unchanging abstract Forms while science is an uncertain and changeable perspective about the world of mere becoming. Plato's view on the relative standing of mathematics and science is unambiguous: mathematics is the highest form of knowledge, science

| *concept* | *description* | *example iconic* ☞ *symbolic* |
|---|---|---|
| **iconic/symbolic** | pictures rather than words | `[]` ☞ *log* |
| **distinction** | difference rather than equality | ≠ ☞ = |
| **containment** | numeric forms are nested boundaries | `([2][3])` ☞ 2 x 3 |
| **object/process** | objects are processes and vice versa | `(2 3)` ☞ $n^2 \times n^3$ |
| **no zero** | eliminated in favor of pure absence | *void* ☞ 0 |
| **void-equivalent forms** | some forms are illusions without impact or meaning | `([2][ ])` ☞ 2 x 0 |
| **transformation** | patterns change by rule-based substitution | ⟦A B E⟧ ☞ B+C ⇒ A+C |
| **composition** | construction of forms replaces induction | A ⇒ A o ☞ n ⇒ n+1 |
| **unified inverse** | inverse functions are one boundary type | `(<[<3>]>)` ☞ 1 ÷ −3 |
| **arbitrary base** | forms are independent of a numeric base | `(2)` ☞ $n^2$ |
| **numeric and non-numeric** | forms mix numeric with non-numeric concepts | `([ ])` ☞ $n^{-\infty}$ |
| **no sets, no logic, no functions** | conventional foundations are not incorporated | A ☞ ¬(A = {A}) |
| **parallelism** | forms are independent and transform in parallel | 2 3 4 ☞ (2+3)+4 |
| **rigorously finite** | all forms and processes are bounded | $o.._n..o$ ☞ $1+.._n..+1$ |

Figure 16-2: *Comparison of concepts*

is mere opinion.[8] Math historian Morris Kline recounts how, as a consequence of Plato's opinion, "mathematics became the substance of scientific theories."[9] By the mid-nineteenth century, however, mathematics matured beyond its applications to the Earth, to re-inhabit Plato's purely abstract realm. Mathematical theories grew with no direct (or indirect) physical interpretation. Citing the rise of negative numbers, complex numbers, n-dimensional spaces and non-commutative algebras, Kline remarks that "mathematics was progressing beyond concepts suggested by experience."[10]

Since mathematics itself is **purely virtual** and does not necessarily connect to worldly objects or to concrete experiences, we can say that math is *pure abstract distinction*. Mathematician Michal Walicki: "Number is complete ability to ignore all differences in content."[11] Mathematics has been transformed from a overt mediation between description and reality to a covert contract between cognition and further cognition. This transition laid the groundwork for Hilbert's truly outrageous proposition that mathematics should be sufficient to justify itself without reliance upon any other discipline or grounding. This is in marked contrast to the perspective of working mathematician Verena Huber-Dyson: "The positive integers are mental constructs. They are tools shaped by the use they are intended for."[12]

## Symbolic Algebra

*polynomial* 347:
$(3 \times 10^2) +$
$(4 \times 10^1) +$
$(7 \times 10^0)$

The techniques of conventional algebra grew out of the study of polynomial expressions over seven centuries ago. Its techniques are particularly well adapted for polynomials, as is the now universal representation of digital numbers. Our interpretation of James algebra does not even have special containers that represent addition or multiplication. Its fundamental disconnection from polynomials provides new perspectives on the structure of our number system.

Algebraic equations rest upon an audacious extension of notation and language: we can choose an arbitrary symbol, in many cases $x$, to stand in place of both "everything" and what we do not know. **Literal symbolism**, in historian John Derbyshire's words "the systematic use of letters to stand for numbers"[13], is a relatively new mathematical invention introduced by Descartes (and others) in the mid-seventeenth century. Prior, mathematical problems were largely written in words as were mathematical expressions. The *idea* of abstraction itself came slowly. Before the innovations introduced by Descartes, mathematics was about concrete relations and concrete geometrical figures. Descartes' innovation is still in use today. It "inspired Leibniz's dream of a symbolism for all human thought"[14], what today has become **digital convergence**. Derbyshire observes,

> When we compare Descartes' mathematical demonstrations with the wordy expositions of earlier algebraists, we see that a good literal symbolism really does relieve the imagination [Leibniz's words], reducing complex high-level thought processes to some easily mastered manipulation of symbols.[15]

Throughout the next two centuries, until the invention of group theory, algebra was **universal arithmetic**, manipulation of relations between numbers using symbols. It took until the mid-nineteenth century for generalization to be introduced. At the time mathematician Augustus DeMorgan explained:

*Augustus DeMorgan*
*1806–1871*

> The formation of symbolic algebra itself is a separation of the essential conditions of operation from the non-essential: the rejection of all meaning over and above the *points of meaning* on which transformations depend.[16]

Here DeMorgan identifies, in modern terms, morphisms between numbers, geometry and trigonometry, noting that "many different sets of meanings may, when attached to the symbols, make the rules necessary consequences."[17] And with James algebra, we have done likewise. The axioms suffice to define an abstract formal system that stands alone. To *avoid* abstraction, for the purposes herein, iconic forms are mapped to numeric expressions. To provide meaning James algebra itself has a concrete mapping to nested physical containers. Thus symbolic abstraction can be replaced by material representation without losing the abstract power of an algebra. In the late nineteenth century founder Richard Dedekind pursued axiomatization of algebra, converting the objects of algebra into pure abstractions based on set theory. The James axiomatization is also pure abstraction, based on the theory of distinctions, for which containers are a concrete visualization.

*a generic frame*
`(A [B])`

*void inversion*
`(  [ ])`

*inversion*
`(  [B])`

*dominion*
`(A [ ])`

*indication*
`(A [o])`

*replication*
`(A [o...o])`

*promotion*
`(A [<B>])`

*arrangement*
`(A [B C])`

Arrangement provides an example of one of the recurrent organizing structures within James forms, the **inversion frame**. Frames are structural skeletons that themselves deserved names, not because they are generated often by transformations, but because they are associated with concepts that we consider frequently. Accumulation, for example, leads to frames associated with indication, with cardinality and with accumulation itself. The generic shape of an inversion frame is `(A [B])`, where `A` is the frame type and `B` is the frame contents. Arrangement permits frame contents to be collected or dispersed, both of which are useful for computation. Inversion and Dominion are degenerate types of inversion frames, within which respectively `A` or `B` are *void*. Indication, Replication and Promotion are also organized as inversion frames.

## Algebraic Dependencies

It is clear that a boundary system requires at least one axiom for each type of boundary. The number of useful theorems that derive from those axioms, however, can be very large. For example, the entire basis of plane geometry

*reaction*
*separation*
*indication* *involution*

~~~ **inversion** ~~~ **arrangement** ~~~ **reflection** ~~~

*replication* *dominion*
*promotion*

Figure 16-3: *Dependency of theorems upon axioms*

was developed from Euclid's five postulates. A theorem is an encapsulation of several transformation steps. A *useful* theorem summarizes sequences that occur often, usually across different applications of an axiom system.

The seven pattern-matching theorems included in Figure 16-3 are organized spatially by their dependency upon the axioms. The figure tracks which axioms are used in the proof of each theorem, thereby classifying each theorem by the type of boundaries that it, at its foundation, depends upon. This table is empirical. There may be other proof sequences that use different axioms or fewer axioms, the ultimate question of the most useful theorems of James algebra thus has an evolving answer. In Figure 16-3 three of the seven theorems are specialized extensions of Reflection, and one is an extension of Inversion. This limits the importance of these four theorems to that of convenience. The Replication theorem combines Indication with Arrangement of indications. It is at the heart of the process of identifying the cardinality of ensembles. But it is Dominion and Promotion that are fundamental to James transformations. Both are surprises since both facilitate and strengthen the Inversion axiom, yet neither requires Inversion for its proof. Dominion incorporates a *non-numeric form* (that is, [ ] is at the deepest level) but is derived directly from numeric axioms. It thus identifies a bridge between
~~~

numeric and non-numeric mathematical concepts. The power of Dominion is to render forms void-equivalent, in essence to convert numeric forms into a different non-numeric class, that of non-existence. The power of Promotion is to move angle-brackets outside of inversion frames and thus simplify the internal structure of a form.

## Arrangement

*arrangement*
`(A [B C]) =`
`(A [B])(A [C])`

Axioms that rearrange structure can be particularly expensive to use. There appears to be no good way to know which direction Arrangement may need to be applied (i.e. `disperse` or `collect`) to minimize a form. Heading simplification toward the canonical most dispersed form can run into intractable growth due to exponential generation of replicas. Heading toward a most deeply nested form runs into non-canonical pathways. What *deepest* means depends upon local choices about which forms to `collect` together. And there is a potentially exponential number of choices.

The refractory computational behavior of Arrangement is essential. Without such a rule, every transformation sequence would be *tractable*, computation would always be relatively trivial. Only systems that require a non-obvious choice of the direction of application for a transformation are powerful enough to represent useful systems such as simple arithmetic. Thus Arrangement is the primary source of computational and algorithmic complexity in James algebra.

## In Entirety

The axioms and definitions listed in Figure 16-1 are used throughout this volume. The new definitions and theorems not included in Figure 16-1 are presented in Figure 18-1 of Chapter 18. These new structural theorems are all related to the Principle of Composition, permitting the equal sign to be integrated into James forms.

## 16.4 Remarks

Boundary forms provide new structural insights. Most interesting is the substantial structural similarity between two central mathematical systems in our culture: logic and numbers. The distinction between *Arithmetic* and *Logic* is to permit or not permit accumulation. This difference can also be expressed by the behavior of distinction boundaries. Numeric boundaries are **impermeable**, that's why they accumulate. Logic boundaries are **semipermeable**, that's how asymmetric inference works.

Since the Additive Principle is both intuitive and the historical basis of numerics, it may be palatable to accept that addition is putting together and that zero is a symbolic artifact. The same freedoms are not available for logic, a discipline that has been associated over history with how our minds work. Western rationality is built upon the grounds of `TRUE` and `FALSE`. Duality is at the heart of our world view. However boundary logic treats *void* with respect, leaving us with only one logical ground. In boundary logic, `FALSE` does not exist. How can we reason without duality? How can logic be *unary* rather than binary? Peirce and Spencer Brown answer these questions through their iconic foundation for logic that confounds truth with existence.[18]

We begin this volume with a new perspective on equality, abandoning induction in favor of explicit construction of forms. We then explore the elementary structure of pure boundary math. The current conventional definitions of number and arithmetic, the nature of formalism, and its relation to computation are next. The final few chapters introduce postsymbolic thinking and describe how and why set theory, logic and functions are not particularly elegant ideas. The volume ends with the network version of James algebra.

## Endnotes

1. **opening quote:** G. Spencer Brown (1969) *Laws of Form* p.6-7. Online 8/18 at `http://www.manuelugarte.org/modulos/biblioteca/b/G-Spencer-Brown-Laws-of-Form.pdf`.

2. **and by George Spencer Brown in the 1960s:** During the 1980s and 90s I explored many representational and pedagogical varieties of iconic arithmetic, focusing especially on computational *parallelism*.

3. **our approach to iconic representation and computation to Interval Research Corporation:** Interval (IRC) was Paul Allen's research company located in Palo Alto California. IRC operated from 1993 to 2001 (thus the "interval") developing primarily new software tools and algorithms.

4. **can be more thoroughly explored in Volume III:** The center pieces of Volume III are the non-reducing composite units (<[ ]>), which is nominally called *divide-by-zero*, and [<( )>], called J. These forms provide a sufficient basis for the definition and exploration of infinite and imaginary numbers.

5. **the art of giving the same name to different things:** H. Poincaré (1908) *Science et Méthode* p.375. In G. Halstead (trans.) (1913) *The Foundations of Science*. Online 8/18 at `https://www.gutenberg.org/files/39713/39713-h/39713-h.htm`

6. **A number is what it does:** P. Lockhart (2017) *Arithmetic* p.182.

7. **the creation of thousands of indigenous languages:** There are estimated to be about 7000 different languages currently spoken by humans.

8. **mathematics is the highest form of knowledge; science is mere opinion:** P. Maddy (2008) How applied mathematics became pure. *The Review of Symbolic Logic*. **1**(1) p.16-41.

9. **mathematics became the substance of scienti c theories:** M. Kline (1972) *Mathematical Thought from Ancient to Modern Times* p.394.

10. **progressing beyond concepts suggested by experience:** Kline p.1030.

11. **Number is complete ability to ignore all differences in content:** M. Walicki (1995) The origin of mathematics. Online 8/18 at `https://www.ii.uib.no/~michal/phil/om/om.pdf`

12. **They are tools shaped by the use they are intended for:** V. Huber-Dyson (1998) On the nature of mathematical concepts: why and how do mathematicians jump to conclusions? Online 8/18 at `https://www.edge.org/conversation/verena_huber_dyson-on-the-nature-of-mathematical-concepts-why-and-how-do-mathematicians`

13. **the systematic use of letters to stand for numbers:** J. Derbyshire (2006) *Unknown Quantity* p.81.

14. **dream of a symbolism for all human thought:** Derbyshire p.94.

15. **easily mastered manipulation of symbols:** Derbyshire p.94.

16. **the points of meaning on which transformations depend:** A. DeMorgan (1849) *Trigonometry and Double Algebra* p.114. (Emphasis in original.) Online 8/18 at `https://archive.org/details/trigonometrydoub00demoiala/page/n8`

17. **make the rules necessary consequences:** DeMorgan p.93.

18. **iconic foundation for logic that confounds truth with existence:** On page xiv of *Laws of Form* Spencer Brown writes:

> It is possible to develop the primary algebra [his container-based rules of logic] to such an extent that it can be used as a restricted (or even as a full) algebra of numbers. There are several ways of doing this, the most convenient of which I have found is to limit condensation in the arithmetic, and thus to use a number of crosses in a given space to represent either the corresponding number or its image.

The result of limiting condensation is Accumulation. Spencer Brown's number and its image have been separately distinguished in James algebra as round-brackets and square-brackets.

*Chapter 26*

# Postsymbolism

*All of life is imbued with nonsymbolic communication.... A book is a book as an object prior to being a book that can be decoded as a bearer of symbols.*[1]
*— Jaron Lanier (1989)*

Mathematics has *schools of thought*. The traditional categories are **pure** mathematics and **applied** mathematics.[2] I'm from a third school of thought, computer science. **Computation** is mathematics without infinities, which excludes most of what a pure mathematician does. It is intensely practical, we are after all trying to get machines to do interesting things. James algebra is designed for a computer to implement.

*Alexander Grothendieck 1928–2014 Fields Medal 1966*

*Alan Turing 1912–1954*

Mathematical content and technique have changed so rapidly over the last fifty years that the concerns and contributions of Frege and Peano and Hilbert and Gödel seem antiquated at best. We have not even considered the revolution in contemporary mathematics initiated by Grothendieck (and many others) that has fundamentally changed what advanced math is. Nor have we integrated the algorithmic work initiated by Turing (and many others). This chapter continues in a relatively narrow vein, considering only one small addition to the structure of mathematics, a change that is already fully underway during the first two decades of the twenty-first century. **Postsymbolic math** is mathematics that recognizes the formal structure of non-textual forms.

What cognitive limitations are imposed by the choice of a particular style of representation? What are the costs of attempting to separate the content of a communication from the method of its transmission? Does formulating our criteria for clarity, proof and efficiency in terms of linear strings of symbols limit our capabilities for understanding the concepts that these strings supposedly represent? Could it be that what we consider to be rational and logical thought is burdened by approaches to communication that are excessively narrow?

A theme is that the representation of formal thought is a design choice, one that strongly interacts with both content and cognition. Reading the book is not isomorphic to seeing the movie.[3] Symbolic techniques support an ancient Greek belief that formal thinking is entirely cognitive, that sensation has no place within the rigor of mathematics. Iconic techniques support a postsymbolic perspective that **cognition is embodied**, that rigor arises from its biological and physiological context. A screenplay is expanded into a movie by enriching its sensual elements: sound, light, action! In reducing a movie to its readable text, sensuality retreats into the imagination, an enrichment of a different kind but one that weakens rigor by removing the validation of experience.

## 26.1 Contemporary Mathematics

Contemporary mathematics is not only lacking a foundation, it is adverse to the simplicity imposed by a foundation of any type. Logic and algebra and geometry have been woven into a tapestry that provides a rich diversity while also providing exciting glimpses of a richer unity. The time honored metaphor of a plethora of balconies overlooking a central town square, providing a diversity of perspectives on a singular center, has been inverted. Mathematics is now a grand hotel with a plethora of balconies each looking out in a different direction. The hotel's core, the elevator shaft, provides a way to reach each of the rooms, but only as a method of traversal, not as an organizational principle. Abstract mathematics has transcended the *concept* of a foundation, rendering Russell's and Brouwer's and Hilbert's dreams as ancient history.

Fernando Zalamea, in his philosophical study of contemporary mathematics, summarizes recent growth as

> a complex dialectic that delineates both the movement of concepts/objects (the functorial transit between the algebraic, the geometric, and the topological) and the relative invariants of form (cohomologies). At stake is a profound mathematical richness — a richness that vanishes and *collapses* if one restricts oneself to thinking in terms of elementary mathematics.[4]

Logic, set theory, algebra and geometry are each in themselves idealizations that apply false partitions within advanced mathematics, and critically, they also do not provide a ground for understanding. Zalamea sees the bulk of modern philosophy in mathematics as rehashing centuries old debates that themselves have been replaced by radical innovation.

## Shaking the Ground

We're not exploring in this chapter the specific axioms of James algebra so much as an iconic approach to understanding the classical mechanisms of mathematics. Two features enable boundary math to contribute to the vision of a new basis for math education:

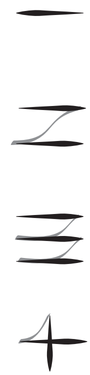
- Iconic form is fundamentally different than symbolic expression since it accommodates our physical senses.

- The available mapping of James forms to conventional expressions provides a metamathematical bridge.

In this and the following two chapters we will focus upon the *comparative* characteristics of different foundational axiom systems. And it is here that boundary math may appear to be most outrageous, partly because it violates so many of the presumed and established rules of how mathematical representation works, partly because such violations may appear to be arrogant or disrespectful or incompetent, and partly because comparing a new and different experimental approach to a culturally established pillar of wisdom will necessarily appear to be comparing a trickle to a river. Boundary math lacks the depth and diversity of a field that has been evolving over millennia. It both borrows and mirrors, while at the same time being audacious enough to suggest that for elementary mathematics to be complete, computation and experience are equally as important as abstraction. And incidentally, if Zalamea and his contemporaries are correct, we are but dancing on graves of the already departed.

## 26.2 Postsymbolic Math

Iconic form is **postsymbolic**. It enlists images and experiences as glue to repair the disconnection of words and textual symbols. The container is perhaps the smallest step away from words to icons. Certainly, apart from *emoticons,* containers as textual delimiters are the only iconic forms available on a conventional typewriter. Typographical characters do not support a meaning of their own.[5] They are building blocks for words but do not individually contribute to the meaning of a word. Only



the first few whole number digits have retained even a ghost of the image of their former selves. Parentheses in contrast overtly separate inside from outside both textually and visually.

The construction of mathematics (and our tacitly symbolic understanding of mathematics) from *strings of tokens* is somewhat a reaction to the separation of algebra from geometry in the nineteenth century. Math historian Israel Kleiner observes that "mathematics evolved for at least three millenia with hardly any symbols."[6] Iconic notation was widely used by Peirce, Frege, Venn and other founders of formal arithmetic at the turn of the twentieth century, but by the 1950s the currency of mathematical expression was typographical. I suspect the construction of the syntax/semantics barrier gave false security that all structural concepts could be stringified. The operation of juxtaposition (aka concatenation) created *de facto* sequences of symbols, while spatial ensembles were completely ignored. Here's Brian Rotman:

> Within the Platonist program, this alphabetic prejudice is given a literal manifestation: linear strings of symbols in the form of normalized sequences of variables and logical connectives drawn from a short, preset list determine the resting place for mathematical language in its purest, most rigorously grounded form.[7]

## Origins

Euclid's *Elements* was the primary mathematics textbook throughout Western history until the mid-eighteenth century. Although Euclid introduced the axiomatic method, his content was geometric structure without the inclusion of number as measurement. Geometry yielded to algebra in the 1800s, and algebra to logic at the end of that century, although only for a few decades. The enthusiastic exploration of logic led not only to monumental symbolic efforts

*Bertrand Russell*
*1872–1970*

*Alfred North Whitehead*
*1861–1947*

such as Whitehead and Russell's *Principia Mathematica*, but also to iconic techniques such as Venn's diagrams, Hasse's ordering diagrams, Frege's concept script and Peirce's existential graphs. The mutual heresy of these pioneers was to advocate concepts embedded in spatial arrangements rather than in strings of tokens. Ancient Greece scholar Reviel Netz observes:

> Mathematical diagrams may well have been the first diagrams. The diagram is not a representation of something else; it is the thing itself. It is not like a representation of a building, it is like a building, acted upon and constructed.[8]

## Peirce

*Charles S. Peirce*
*1839–1914*

Charles Sanders Peirce is recognized as America's greatest philosopher, having made foundational contributions to formal logic, semiotics and the entire panoply of philosophy (ethics, ontology, metaphysics, …). Peirce makes the case that *spatial visualization is the native vocabulary of rational thinking*. To Peirce, formal structure was a geometric not a textual property. Geometric properties can be observed directly. Therefore, the process of thought is directly observable in the structure and transformation of iconic forms of logic. This structure is obscured by textual expressions, since text cannot directly represent some essential concepts of iconic logic. Worse, text obscures the process of rational thinking by hiding its spatial structure behind essentially arbitrary tokens. Here is Peirce's commentary on his iconic logic:

> I dwell on these details which from our ordinary point of view appear unspeakably trifling, — not to say idiotic, — because they go to show that this syntax is truly diagrammatic, that is to say that its parts are really related to one another in forms of relation analogous to those of the assertions they represent, and that consequently in studying this

> syntax we may be assured that we are studying the real relations of the parts of the assertions and reasonings; which is by no means the case with the syntax of speech.[9]

The question is whether or not diagrams and images can convey the same formal information as strings. Venn developed his diagrams with set theory in mind; Peirce developed his existential graphs with logic in mind; Hasse developed his diagrams with partial orderings in mind. But with the development of metamathematics in the first half of the 20th century, all iconic techniques were ignored or rejected as insufficiently formal, which lead to a general disregard for the attempts by Venn, Peirce, Frege and others to incorporate the expressive power of spatial forms into formal mathematics. Peirce philosopher Randell Dipert observes:

*John Venn*
*1834–1923*

*Helmut Hasse*
*1898–1973*

> We should also give some hard thought to the difficult question of how much conceptual progress is made by symbolism and symbolic rigor alone....the recent history of logic has appeared to value any, and sometimes quite shallow and unenlightening, symbolisms and axiomatizations and tended to dismiss any non-symbolic historical account (for example those of Aristotle or Ockham) as so much empty verbiage.[10]

## Constructivism

Arend Heyting, student of the founder of the intuitionist school of logic L.E.J. Brouwer, contributed to turning constructivism into a formal system with rules slightly different than classical logic. Heyting's idea that all mathematical objects should be shown to exist, rather than just being inferred, essentially calls for algorithmic proof. Within *intuitionism* Truth gives away to Justification, allowing for context within logical expressions. In order to reach toward a mathematics built upon what is obvious

*L.E.J. Brouwer*
*1881–1966*

*Arend Heyting*
*1898–1980*

*double negation*
$\neg\neg A = A$

rather than obscure, **constructive logic** eliminates the Law of Double Negation, which is equivalent to eliminating one of Aristotle's grounding principles of logic, the Excluded Middle,

**excluded middle**

$$A \lor \neg A = \text{TRUE}$$

The *meaning* of an expression is no longer anchored to a truth-value, a first step toward iconic form in which meaning can be associated with physical circumstance. Heyting's perspective combines constructive existence with meaning accessible to intuition, leading the way for anchoring mathematics within diverse experience rather than within a binary evaluation. Heyting:

*A mathematical construction ought to be*
*so immediate to the mind and its result so clear*
*that it needs no foundation whatsoever.*[11]

For some, mathematical thought has always been post-symbolic. Here's Einstein:

> Words and language, whether written or spoken, do not seem to play any part in my thought processes. The psychological entities that serve as building blocks for my thought are certain signs or images, more or less clear, that I can reproduce and recombine at will.[12]

## Iconic Formality

*Saunders Mac Lane*
*1909–2005*

*Kurt Reidemeister*
*1893–1971*

*Stephen Wolfram*
*1959–*

*Richard Feynman*
*1918–1988*

Since the iconic approach incorporates types of structure that are simply not available within a string-based notation, spatial formalism is an asset rather than a liability. Today diagrammatic mathematics permeates modern formal systems. Premiere examples include Saunders Mac Lane's **category theory,** Reidemeister's **knot invariants,** Conway and Wolfram's **cellular automata**, and **Feynman diagrams**. The recent acceptance that human thought must have a physiological basis is essential to the recognition of iconic form as mathematical structure.

Wittgenstein emphasizes the relation between thinking and language. Other modern theorists have integrated the Platonic realm with the physical body, particularly

— Stanislas Dehaene (neurological substrate of numbers)
— John Horton Conway and Richard Guy (mathematical games),
— Benoit Mandelbrot (fractal geometry),
— George Lakoff and Rafael Núñez (embodiment of arithmetic), and
— Brian Rotman (mathematical communication and belief)

*Stanislas Dehaene 1965–*

*John H Conway 1937–*

*Benoit Mandelbrot 1924–2010*

*George Lakoff 1941–*

*Brian Rotman 1938–*

As well George Spencer Brown and Louis Kauffman are of special importance to the development of iconic mathematics and James algebra.

Although it appears as though James forms are symbolic in that they stand in place of putatively abstract concepts, the intention of an iconic system is that *representation looks like what it means*. This step is unique for several reasons.

— James forms have a physical as well as a representational manifestation.
— The physical manifestation can be read as concepts and be interpreted as numbers.
— The conceptual basis of James forms differs from that of numeric expressions.
— What James forms represent is not what numbers represent.
— Containment is not counting.

It is challenging to isolate one or two facets of boundary mathematics that deviate from conventional arithmetic. The change in perspective is *systemic* rather than local.

We have already highlighted many of these conceptual shifts.

- — Arithmetic is about physical experience. It is not abstract.
- — Formal rigor is not incompatible with direct experience.
- — Iconic forms provide their own meaning.
- — Reliance on strings of symbols limits the perspectives of mathematics.
- — Void-equivalence eliminates meaningless form.

*accumulation*

● ● ≠ ●

At the same time, boundary mathematics maintains rigorous formality. It is neither a psychological nor an educational technique. It is a collection of axiomatic systems that share common characteristics. *Formality* is an essential aspect of computational math. Only computational demonstration is sufficiently rigorous to meet the formalist criteria for mathematical veracity.

***Hilbert***

1 = |
2 = ||
3 = |||
n = |..$_n$..|

***Frege***

0 *exists*
1 = #0
2 = #1
3 = #2
n = #(n–1)

***Peano***

0 *exists*
1 = 0'
2 = 1'
3 = 2'
n = (n–1)'

***Zermelo***

0 = { }
1 = {0}
2 = {1}
3 = {2}
n = {n–1}

***vonNeumann***

0 = { }
1 = {0}
2 = {0,1}
3 = {0,1,2}
n = {0,...,n}

## Incremental Numbers

Each whole number is an accumulation of the numbers that precede it. Different formal models generally differ by the way in which accumulation is achieved. Hilbert describes numbers specifically as tally marks, as intuitive signs so obvious that they come prior to logic and prior to inferential definition. Frege defines numbers conceptually: 0 is the number of objects that are not identical to themselves; 1 is the number of things that are 0; 2 is the number of things that are either 0 or 1. Peano generates natural numbers by assuming both 1 and the successor operation, +1. The next number is the successor of the prior number. Ernst Zermelo, an originator of set theory, conceived of the natural numbers as the successive nesting of sets. Rather than incrementing, the successor function converts an existing number into a singleton set. VonNeumann's set theoretic successor constructs the union of the prior and the current number. The empty set `{ }` has zero members, while the set of the empty set `{{ }}`, has one member, and `{{ },{{ }}}`, the set of `0` and `1`, has two members. Frege and vonNeumann build numbers from the collection of all numbers before them. Peano and Zermelo do not. In all cases, each number has its own unique structure.

*Ernst Zermelo*
*1871–1953*

*John vonNeumann*
*1903–1957*

Paul Benacerraf observes that these definitions are mutually contradictory.[13] For Zermelo, the number `5` is a set with one member, for vonNeumann is it a set with five members, for Frege it is five conceptual differences, for Peano it is five unitary increments and for Hilbert it is five identical tallies. Boundary arithmetic identifies `1` as a distinction, `( )`. The emptiness inside serves as `0` but it is actually *nothing*. Then like Hilbert and like other tally systems, numbers grow in magnitude by the expedient of not permitting condensation.

The complexity of numbers themselves depends on the choice of foundational approach. Zermelo gives us the whole numbers; vonNeumann gives us the ordinal numbers. Cantor journeys all the way to actual (rather than potential) infinity. His definition of an infinite set is something that can be grasped as a whole. Modern theorists as such Badiou see the entirely of countable numbers as a structured whole. Numbers themselves are a unity. No single number makes sense without reference to all other numbers. You can tell a unity because it has a boundary rather than an unlimited expanse. This leads us to our current structural definition of a James number:

*A number is a reduced James form*
*that contains a round unit,* `( )`*,*
*and does not contain a square unit,* `[ ]`.

## Mathematics Education

The approach we are exploring is not intended to simplify or to modify the edifice of mathematics itself. We are not challenging professional mathematics, but we are challenging unprofessional mathematics education. The central question is whether or not there is a conceptually simpler approach to mathematical ideas that might benefit the 99.9% of people who do not engage in advanced mathematics, and the two-thirds of Americans who literally hate math.[14] As I tell my students, it is most likely that what they hate is *not* mathematics. Math is a tool, and it's hard to hate a hammer. What they hate is the experiences imposed upon them in the name of mathematics education. What they hate is the disrespect.

Until a student becomes a math major, in upper-division college, math teachers usually do not mention sets or logics or varieties of algebras. Yes, set theory and logical deduction and functional analysis are useful mathematical systems for students to know, but the fact is these mathematical skills are not taught in K-12 anyway. Some aspects of foundational thinking are embedded implicitly into math education but none are taught *as* math education. Making the ground upon which a learner stands implicit leads to confusion rather than to understanding. It's like asking a carpenter to built the walls of a house without having a cement slab to stand upon.

Even **group theory**, the conceptual mechanism underneath modern algebra, is not taught as high school algebra. Seeing symmetry is vital. Intuitive understanding of symmetric form is vital. But these conceptual skills are not in the K-12 mathematics curriculum. Group theory is an upper-division college course for math majors. Category theory, until very recently, was reserved for graduate school. Yet we still teach algebra with specifically selected group theoretic algebraic structures implicitly embedded as the *rules of algebra*. These rules provide a rather antiquated perspective on algebra. Useful in some circumstances but positively destructive to the growing understanding of young students. This is simply because these rules are taught as symbolic behavior, while students for the most part are learning about human behavior.

The Rules of Algebra (the familiar concepts of commutativity, associativity, zero, inverse and arity) pervade elementary and secondary mathematics education, perhaps due to a belief that these concepts somehow explain how arithmetic works. But prior to the introduction of symbolic forms, *preschool* mathematics emphasizes interactive manipulation, embodiment rather than abstraction of concepts. The tension between these two approaches "is a fundamental and unavoidable challenge for school mathematics."[15] The lesson of the Additive Principle is that the symbolic concepts that classify the arithmetic of numbers as an Abelian group are not the same concepts that have defined numbers throughout their evolutionary history.

Research in mathematics education recognizes the necessity of multiple modes of representation and multiple theoretic perspectives, placing mathematics learning in a pluralistic human context.[16] In contrast Hilbert's Program, the formal agenda of mathematics, removes from the operations of mathematics gross intuition, psychological necessity, physical interaction and concrete manipulation.[17] In support of Hilbert, here's mathematician Herbert Weyl: "We now come to a decisive step of mathematical abstraction: we forget about what the symbols stand for."[18]

*Hermann Weyl*
*1885–1955*

The symbolic model of arithmetic trades the visual and physical intuition that arises from direct experience for memorization of the rules of manipulation of structured strings of abstract tokens that explicitly divorce representation from meaning in order to protect rigor. The goals of advanced mathematics do not necessarily align with the needs of novice learners nor with the objectives of mathematics education.[19] Educator James Kaput is directly critical of the emphasis of form over content, and attempts to steer mathematics education toward representational

*James Kaput*
*1942–2005*

diversity.[20] The advent of computer graphics and web-based **virtual manipulatives**[21] has reinforced visual and manipulative techniques at all levels of math education, but arithmetic itself is still characterized by a single symbolic theory (algebraic group theory) to the exclusion of other conceptualizations of number.

## 26.3 Simplifying Foundations

Modern descriptions of natural numbers invariably enlist axioms that embed *sets* to identify domains and equivalence classes; predicate *logic* to provide connectives, quantification, and relations; and *functions* to enable transformations and invariants. What would numbers look like, indeed how would we be able to conceive of their structure without sets, logic and functions? What would a mathematical foundation that was calibrated, for example, for ease of understanding and teachability look like? Mathematician Alexandre Borovik states the case elegantly: "We cannot seriously discuss mathematical thinking without taking into account the limitations of our brain."[22]

The goal is to construct an iconic system that does not require the foundational axiomatic theories of sets, logic and functions. This does not mean that we abandon all mathematical concepts, just the dominant prepackaged component systems that currently serve as conventional foundations. We will keep the binary `contains` relation, the concept of equality, and the mechanism of pattern-matching and substitution constrained by pattern transformation rules. These features may be characterized as *algebraic*. In the following two chapters we will explicitly eliminate set membership, logical quantification, logical inference, induction and functional thinking. This makes James algebra challenging to talk about, so we will relax a bit and use conventional mathematical concepts in the metalanguage to converse *about* iconic concepts and to compare iconic structure to conventional structure. An experiential system though, in the final analysis, is characterized by actions rather than words.

A primary use of **set theory** is to provide standardized conceptual tools that can describe mathematical systems. The effort has been from its inception controversial. Apparently benign axioms lead to difficult to accept consequences.[23] Set theory itself is built upon the language of **predicate calculus**, aka first-order logic. A primary use of predicate calculus is to provide tools that can describe structural invariants across functions, relations, domains of objects and techniques of proof. The functions and relations of predicate calculus vary, depending upon the kind of mathematical structures we seek to describe. However quantification and logical inference are shared across almost all types of mathematical systems to describe domains and transformations.

The difficulty is, bluntly, that it might be imperative to quarantine these tools since many are toxic to the non-professional and most are represented by language that has grown to be outside of general human comprehension. It takes Russell and Whitehead, in *Principia Mathematica,* 345 pages to construct a sufficient groundwork of symbolic logic to introduce the concept One as "the class of all unit classes." This One requires on the order of 50,000 symbols.[24]

As symbolic form burgeons, newer mathematical approaches such as cellular automata and chaos theory are coming to the conclusion that complexity is just a whole lot of simplicity. In the 1920s, Frank Ramsey (and Bertrand Russell) made the case for simple mathematics. Here's Ramsey:

*Frank Ramsey*
*1903–1930*

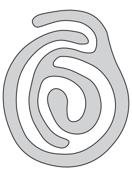
> So in saying that every thing in whose existence we have reason to believe is simple, I mean that there are no classes, complex properties or relations, or facts; and that the phrases which appear to stand for these things are incomplete symbols.[25]

**formal mathematics**

| ***established*** | ***boundary*** |
|---|---|
| propositions + connectives + truth + negation | forms + containment |
| relations + quantification | patterns + pattern-matching |
| specific domains + set theory | transformation rules |
| equality + substitution | equality + substitution |

Figure 26-1: *Deconstruction of formal mathematics*

For another contrast, we take as both axiomatic and obvious that a closed curve has an inside and an outside. This observation in mathematics is called the Jordan Curve Theorem: Every simple closed planar curve separates the plane into a bounded interior region and an unbounded exterior. Using the HOL automated theorem-prover, Thomas Hales completed the proof of the Jordan Curve Theorem, but at the cost of its simplicity.



> The formal proof of the Jordan curve theorem in HOL-Light consists of 138 definitions, 1381 lemmas, and over 44,000 proof steps spread over 59,000 lines of computer code. There are approximately 20 million primitive logical inferences in this proof.[26]

The iconic perspective is that the problem is *symbolic representation*, not the nature of closed curves.

## Mapping

The foundational theories of mathematics are designed to accommodate all mathematical ideas and notations. Naturally they have come into being, primarily in the twentieth century, through extensive study by the greatest mathematical minds of our time. Our goal here is far

| ***group theory*** | ☞ | ***James algebra*** |
|---|---|---|
| *commutativity* | independence of content | |
| *associativity* | independence of content | |
| *additive identity* | *void* | `n` |
| *multiplicative identity* | Indication | `([n] [o] )` |
| *additive inverse* | Reflection | `n < n >` |
| *multiplicative inverse* | | `([n]<[n]>)` |
| *distribution* | Arrangement | |

Figure 26-2: *Conventional algebraic axioms and James axioms*

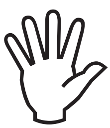
more modest: to design an adequate formal system that is simple enough to convey some basic ideas to naive minds. Thus we are pursuing a system that is visual and manipulable, that has only one binary relation, and that does not need the support of other foundational systems.

Figure 26-1 compares a hierarchical deconstruction of mathematical foundations to analogous components within the James system. Threaded throughout the objects of math are mappings between objects. The most common of these is the **function**, a relation between the inputs and outputs of a process or a transformation. The group theoretic structure of functions is compared to the mechanisms of James algebra in Figure 26-2. Functions themselves are objects of study in mathematics, leading to the concept of a **morphism**, or a structure preserving map between mathematical structures. **Structure preserving** means, roughly, that the objects and the transformations between two systems align. Recently the study of morphisms has led to the more abstract approach of **category theory**, in which the objects of study are mapping systems themselves. The premier example of a category is the *category of sets*, which includes collections of unique objects and the functions that relate them. Mathematician Barry Mazur:

> A category is a mathematical entity that, in the most succinct of languages, captures the essence of what a mathematical theory consists: objects of the theory, allowable transformations between these objects, and a composition law telling us how to compose two transformations when the range of the first transformation is the domain of the second.[27]

## Proof

There are two other great themes within mathematics: proof and interpretation. The simple perspective on **proof** is that it is following rules to get from one form to another. Since proof can be accomplished by different systems of transformation (logic, algebra, pattern-matching) deductive reasoning is not an essential component.



An **interpretation** is a mapping between symbols and relations. Presumably, symbols are arbitrary, we can represent the concept of five by 5, by *cinq*, or by a hand with fingers spread open. Technically functions map symbols to other symbols, ones that we have perhaps a better understanding of. Mathematics seems to have a difficult time bridging the gap between the symbolic and the concrete. Indeed, we are using the word *interpretation* to mean a formal mapping between container forms and mathematical expressions composed of arithmetic symbols.

An entirely different use of interpretation is to assign *meaning* to symbols, where we are strict to maintain that meaning exists only in the physical world. The `plus` symbol might be interpreted to assert that we should put two things together, given that the things themselves are concrete rather than symbolic. 3 + 4 does not tell us to put together the two squiggles 3 and 4. It tells us to put together 3 and 4 *of something*. For elementary arithmetic that something is tallies, what we have been calling indications. Here we also have the option of converting container configurations to arrangements of physical containers, so that the meaning of `(( ))`, for example, is a physical container residing within another physical container. We might even say that when we see physical containers as mathematical relations we are interpreting the physical as something virtual.

The current postsymbolic challenge is to provide a proof of principle that it is possible to describe significant mathematical systems without symbols and without the tools that enforce a symbolic conceptualization. Our working example, James algebra of containers, can certainly be described symbolically, and we have done so by implying that the `contains` relationship is a *logical relation*. But which aspects of the concept of a *distinction* rely upon the infrastructure of sets and logic, and which parts are provided natively by the concept of containment? Can the relationship between a container and its contents be described without calling upon predicate calculus?

## 26.4 Doing without Symbols

Symbolic exposition dominates mathematics, especially foundational mathematics. Educators rationalize the frustration they visit upon students trying do math "in their heads" rather than with their eyes and fingers and bodies by claiming that symbolic math is both necessary and good for you. Presumably it helps the development of rational thinking. There is absolutely no evidence to support this. Most math teachers who express their distress publicly say that they love math, and that something is dreadfully wrong. The subtext of an exploration of iconic math is, of course, that one thing that is dreadfully wrong is *symbolic* math. Tristan Needham, in his ground-breaking textbook *Visual Complex Analysis* makes this appeal:

> When one opens a random modern mathematics text on a random subject, one is confronted by abstract symbolic reasoning that is divorced from one's sensory experience of the world....The present

> book openly challenges the current dominance of purely symbolic logical reasoning by using new, visually accessible arguments to explain the truths of elementary complex analysis.[28]

It is not difficult to find professional mathematicians and educators who are disturbed by the requirements that symbolic math has put on student learners. Here's math educator Norman Wildberger:

> I am confident that a view of mathematics as swimming ambiguously on a sea of potential Axiomatic systems strongly misrepresents the practical reality of the subject...at no point does one need to start invoking the existence of objects or procedures that we cannot see, specify, or implement.[29]

## Historical Context

Plus, +, the first widely used sign in arithmetic, originated in the West in the middle of the fourteen century. At that time algebra was written out in words. The words gradually morphed into shorthand symbols. From a book by Leibniz scholar Louis Couturat:

> The symbols now in use for the operations and relations of arithmetic mostly date from the sixteenth and seventeenth centuries; and these "constant" symbols together with the letters first used systematically by Viète (1540-1603) and Descartes (1596-1650), serve, by themselves, to express many propositions.[30]

Couturat emphasizes that Leibniz sought an iconic system of notation for his universal language, "by providing an ideography, in which the signs represent ideas and the relations between them directly (without the intermediary of words)."[31] Here's Leibniz:

> But it will be appropriate for the signs to be as natural as possible, e.g. for one, a point, for numbers, points.... The whole of the writing will therefore be made as if of geometrical gures and like pictures, just as the Egyptians once did and as the Chinese do today.[32]

Today, formal mathematics is expressed almost exclusively as strings of symbols. Successive transformation of strings generates the structure of proofs. The rules of string formation and transformation provide the syntax of mathematics. However, as Joseph Goguen observes, our actual notation for arithmetic is mixed.

> With Arabic numerals, the use of `1` for "one" is iconic (one stroke), but the others are symbolic; using the blank character for "zero" would be iconic.... When an operation like + is associative, it is usual to omit parentheses; thus we write `a+b+c` instead of `(a+b)+c` or `a+(b+c)`.... Dropping parentheses is iconic of the fact that it doesn't matter where they are; spread sheets also exploit this. Using `0` for the identity of addition is only symbolic, but using `1` for the identity of multiplication is indexical.[33]

Goguen concludes

> Perhaps mathematics could only get started through the iconic notation of its earliest achievements. And certainly hiding that iconicity is harmful to students trying to learn mathematics.[34]

## Semantics

If mathematics is to stand alone, without reliance of other more concrete disciplines, then the concept of semantics, of a tie to reality, is irrelevant. Mathematician Edward Nelson expresses the structuralist viewpoint,

> The role of syntax in mathematics is not to express semantic truths (because there are no semantic truths in mathematics to express). Mathematics is syntax, and syntax is mathematics itself.[35]

Nelson continues, "What is real in mathematics is simply formulas and proofs themselves, as strings of symbols."[36]

If we are to take this view seriously, then James algebra is *not mathematics*. But over the last few decades, diagrammatic mathematics has been widely accepted within many sub-disciplines of mathematics (as discussed in Chapter 1). The whole numbers are on a particularly firm iconic foundation. The numeral 3 is the label for Leibniz' conceptualization of *numbers as points*, forming the icon •••.

*Collatz function*

*If* `n` *is even,*
`f(n) = n/2`
*otherwise*
`f(n) = 3n + 1`

There are also severe foundational questions when one considers the definition and implementation of arbitrary symbolic specifications. Symbolic "rules" that appear to define sets or truth-valued logical expressions or functions can be arbitrarily incomprehensible, or undecidable or ambiguous. There are simple mappings such as the Collatz function that are believed to behave chaotically during iterated recursion. (There is no known proof that iteratively feeding the Collatz function back into itself always terminates at 1.) There are structural questions that are unanswerable, for example: What is the longest sequence of 7s in √2? And questions that seem answerable but may take longer than the age of the universe to compute. For example, find the next pair of twin primes greater than a `googolplex`. Many famous unproved theorems (e.g. the Riemann hypothesis) may not be provable. Well-formed algorithms may not terminate, or a well-formed result may be too large to record within the resources of the known universe. We are left having to acknowledge that whatever foundation we choose, whatever definition of all the numbers we adopt, we may still find ourselves in unknown territory.

But here is a bold suggestion: those ambiguities may be associated with our use of language rather than with our use of natural numbers. Penelope Maddy attributes the difficulty of learning numbers for toddlers to learning *number words*, acquisition of a linguistic rather than mathematical skill. She quotes cognitive scientist Paul Bloom:

> It is not that somehow children know that there is an infinity of numbers and infer that you can always produce a larger number word. Instead, they learn that one can always produce a larger number word and infer that there must therefore be an infinity of numbers.[37]

By admitting that mathematics is a human endeavor, we may not be able to avoid symbolic confusion. But perhaps some of these types of confusion can be avoided by iconic and behavioral communication.

## Embodiment

Postsymbolic math is embodied rather than abstract. There is a deeper alienation: symbolic representation denies that our eyes and our bodies and our thoughts are grounded in experience. Using symbolic tokens to convey structural ideas is neither intuitive nor natural. Symbols drastically increase cognitive load, we must memorize their meaning. String representation requires structural redundancy that is both technically inaccurate and cognitively misleading. We impose commutativity on addition to create symbolic addition but natural addition occurs in space as fusion, with no sequential first and second objects. And for most learners, symbolic systems engender insecurity since they ask for a mode of learning that has no biological or evolutionary basis.

Acceptance and memorization of concepts that have no possible basis in experience helps to undermine understanding. Young learners and their teachers appear to have little choice but to believe what they are told, that symbolic mathematics supports out-of-body experience, that it takes one to unreachable infinities, and that it denies common sense by expressing its ideas in generally incomprehensible strings of arcane symbols.

## 26.5 Remarks

In Chapter 27 and Chapter 28 we'll explore the rationale for avoiding the abstract formalism that now accompanies mathematical foundations. The mechanisms of set membership, logical conjunction and function composition are each equivalent to one another and to the act of physically putting things into containers. The binary connectives of logic and the arity-dependent composition of functions are to some extent not in place to clarify mathematics, but rather to limit operations to two arguments at a time. It is not that sequential thinking is in error, or that it was inappropriate to evolve through the various phases that lead to formalized thinking. It is just time to move on to newer models that include newer concepts such as parallel composition, pattern-driven transformation and iconic form. The next two chapters review the reasons to abstain from building the foundation of mathematics on logic, sets and functions. We'll then explore the implication of making mathematics more than strings of symbols.

## Endnotes

1. **opening quote:** J. Lanier (1989) Communication without symbols. *Whole Earth Review* 64 p.118-119. Online 8/18 at `http://www.jaronlanier.com/jaron%20whole%20earth%20review.pdf`

Not only is Lanier credited with coining the term *virtual reality*, his company VPL was the first to build a virtual reality system. Lanier's intention is to be able to explore direct non-mediated communication. To my knowledge in the cited article he is also the first person to use the term *postsymbolic communication*. In J. Lanier (2017) *Dawn of the New Everything* p.298 he writes:

> Consider that people have been innovating ways of connecting with each other since the dawn of the species. From spoken language tens of thousands of years ago, to written language thousands of years ago, to printed language hundreds of years ago, to photography, recording, cinema, computing, networking; then to virtual reality, and eventually to what I hoped my talk might provide a glimpse of: postsymbolic communication — and then on to what I could not imagine.

2. **traditional categories are pure mathematics and applied mathematics**: The old joke is that a pure mathematician is embarrassed if someone finds a use for his work, while an applied mathematician is embarrassed if no one finds a use.

3. **the book is not isomorphic to seeing the movie:** And a blank canvas is worth a thousand unspoken words.

4. **collapses if one restricts oneself to thinking in terms of elementary mathematics**: F. Zalamea (2009) *Synthetic Philosophy of Contemporary Mathematics* p.179.

5. **Typographical characters do not support a meaning of their own:** There are conventions that endow individual characters with meaning, such as adding an 's' to make a noun plural or replacing a '.' with '!' for emphasis.

6. **evolved for at least three millenia with hardly any symbols:** I. Kleiner (1991) Rigor and proof in mathematics: A historical perspective. *Mathematics Magazine* 64 p. 291-314. Online 8/18 at `https://www.maa.org/sites/default/files/pdf/upload_library/22/Allendoerfer/1992/0025570x.di021172.02p0031c.pdf`

7. **the resting place for mathematical language in its purest, most rigorously grounded form:** B. Rotman (2000) *Mathematics as Sign* p.55.

8. **it is like a building, acted upon and constructed:** R. Netz (1999) *The Shaping of Deduction in Greek Mathematics* p.60.

9. **which is by no means the case with the syntax of speech:** C. S. Peirce (1909) MS 514 "Existential graphs"

10. **(for example those of Aristotle or Ockham) as so much empty verbiage:** R. Dipert (1995) Peirce's underestimated place in the history of logic: A response to Quine. In K. Ketner (ed.) *Peirce and Contemporary Thought: Philosophical Inquiries* p.34.

11. **and its result so clear that it needs no foundation whatsoever:** A. Heyting (1971) Disputations. In P. Benacerraf & H. Putnam (1983) *Philosophy of Mathematics 2ed* p.70.

12. **signs or images, more or less clear, that I can reproduce and recombine at will:** Einstein, quoted by K. Devlin (2006) The Useful and Reliable Illusion of Reality in Mathematics. *Toward a New Epistemology of Mathematics Workshop*, GAP.6 Conference 2006. Online 8/18 at `https://web.stanford.edu/~kdevlin/Papers/Berlin06.pdf`

13. **these definitions are mutually contradictory:** P. Benacerraf (1965) What numbers could not be. In P. Benacerraf and H. Putnam (eds.) (1983) *Philosophy of Mathematics 2nd ed.* p.272-294.

14. **the two-thirds of Americans who literally hate math:** See, for example, M. Burns (1998) *Math: Facing an American Phobia*, and S. Tobias (1993) *Overcoming Math Anxiety*.

15. **a fundamental and unavoidable challenge for school mathematics:** J. Kilpatrick, J. Swalford & B. Findell (eds.) (2001) *Adding It Up: Helping Children Learn Mathematics* p.74.

16. **places mathematics learning in a pluralistic human context:** J. Greeno & R. Hall (1997) Practicing representation: learning with and about representational forms. *Phi Delta Kappan* 78 p.1-24. Online 4/18 at `http://www.pdkintl.org/kappan/kgreeno.htm`

17. **psychological necessity, physical interaction, and concrete manipulation:** M. Greaves (2002) *The Philosophical Status of Diagrams*.

18. **we forget about what the symbols stand for:** H. Weyl (1941) *The Mathematical Way of Thinking*.

19. **nor with the objectives of mathematics education:** M. Donovan & J. Bransford (eds.) (2005) *How Students Learn Mathematics in the Classroom*.

We do not need to understand how an electronic fuel injector works in order to drive a car. Knowledge of the electronic fuel injector is not even essential to understanding how a car works.

20. **steer mathematics education toward representational diversity:** J. Kaput (1987) Representation systems and mathematics. In C. Janvier (ed.) *Problems of Representation in the Teaching and Learning of Mathematics* p.19-26.

It is quite appropriate here to view representational diversity in a wider, cultural sense. Karen François comments:

> Traditional mathematics [curriculum] is strongly directed towards the performance of techniques and has little to do with the study of mathematics as a historical and cultural product nor with the underlying cultural values.

K. François (2007) The untouchable and frightening status of mathematics. In K. François & J. vanBendegem (eds.) (2007) *Philosophical Dimensions in Mathematics Education* p.14.

21. **computer graphics and web-based virtual manipulatives:** P. Moyer, J. Bolyard & M. Spikell (2002) What are virtual manipulatives? *Teaching Children Mathematics* **8**(6) p.373.

Funded primarily by the National Science Foundation, many of these interactive learning tools are available online. For example,

Utah State University (1999) National library of virtual manipulatives at `http://nlvm.usu.edu/en/nav/index.html`; and

D. Clements (1999) Concrete manipulatives, concrete ideas. *Contemporary Issues in Early Childhood* **1**(1) p.45-60. Online 8/18 at `http://www.gse.buffalo.edu/org/buildingblocks/Newsletters/Concrete_Yelland.htm`

22. **discuss mathematical thinking without taking into account the limitations of our brain:** A. Borovik (2007) *Mathematics under the Microscope: Notes on cognitive aspects of mathematical practice* p.vi. Online 8/18 at http://eprints.ma.man.ac.uk/844/1/covered/MIMS_ep2007_112.pdf

23. **benign axioms lead to difficult to accept consequences:** The Axiom of Choice, for example, facilitates the Banach-Tarski paradox, in which a single solid object can be disassembled into specific pieces and then reassembled into two of the same object with the same size.

24. **This One is on the order of 50,000 symbols:** A. Mathias heard this estimate and decided to verify the number. He deconstructed the structuralist definition of the number One from the Bourbaki school by identifying each symbol and expanding it to its most primitive definition. Going back to the very basics of symbolic definition, as if communicating in binary with a computer, he calculated that One would takeover 4.5 trillion symbols to define. A. Mathias (2002) A term length of 4,523,659,424,929. *Synthese* **133** p.75-86.

25. **phrases which appear to stand for these things are incomplete symbols:** F. Ramsey (1922) Truth and simplicity. *British Journal for the Philosophy of Science* (2007) 58 p.379-386.

26. **approximately 20 million primitive logical inferences in this proof:** T. Hales (2007) The Jordan curve theorem, formally and informally. *The Mathematical Association of America Monthly* 114 p.883. Online 8/18 at https://pdfs.semanticscholar.org/70ab/0431a8d59e1cd9147b54c5e99883a54190a1.pdf

27. **when the range of the first transformation is the domain of the second:** B. Mazur (2007) When is one thing equal to some other thing? Online 4/18 at http://www.math.harvard.edu/~mazur/preprints/when_is_one.pdf

28. **visually accessible arguments to explain the truths of elementary complex analysis:** T. Needham (1999) *Visual Complex Analysis* p.vii.

29. **the existence of objects or procedures that we cannot see, specify, or implement:** N. Wildberger (2005) Set theory: should you believe? Online 10/18 at http://web.maths.unsw.edu.au/~norman/views2.htm

30. **serve, by themselves, to express many propositions:** L. Couturat (1905) *The Algebra of Logic* L.G. Robinson (trans. 1914). From the Preface by P. Jourdain p.i.

31. **the relations between them directly (without the intermediary of words):** Couturat p.ii.

32. **just as the Egyptians once did and as the Chinese do today:** L. Couturat (1901) *The Logic of Leibniz* Ch 4. p.18. footnote 101 quoting Leibniz (§90; Phil., IV, 73; Math., V, 50). Online 8/18 (in French) at https://babel.hathitrust.org/cgi/pt?id=ien.35556036601318;view=1up;seq=1

33. **using 1 for the identity of multiplication is indexical:** J. Goguen (1993) On notation. In. B. Magnusson, B. Meyer & J-F. Perrot (eds.) *TOOLS 10: Technology of Object-oriented Languages and Systems* p.5-10.

34. **hiding that iconicity is harmful to students trying to learn mathematics:** Goguen p.5-10.

35. **Mathematics is syntax, and syntax is mathematics itself:** E. Nelson (2002) Syntax and semantics. Presented to the International Conference: *Foundations and the Ontological Quest. Prospects for the New Millennium* p.5. Online 8/18 at https://web.math.princeton.edu/~nelson/papers/s.pdf

36. **mathematics is simply formulas and proofs themselves, as strings of symbols:** Nelson p.6.

37. **and infer that there must therefore be an infinity of numbers:** P. Maddy (2014) A second philosophy of arithmetic. *The Review of Symbolic Logic* **7**(2) p.234. Quoting P. Bloom (2000) *How Children Learn the Meanings of Words* p.238. Online 8/18 at http://www.socsci.uci.edu/~pjmaddy/bio/arithmetic%20in%20RSL.pdf

## Chapter 30

# Connection

*We live in an era of number's despotism:*
*thought yields to the law of denumerable multiplicities;*
*and yet...we have at our disposal no recent,*
*active idea of what number is.*[1]
*— Alain Badiou (2008)*

Volume I presents two iconic approaches to the representation of the formal structure of arithmetic. It also serves as an introduction to a different way of thinking about formality. This volume compares the iconic concepts embodied in James algebra to those of string-based numerics. As we will see in the next volume, James forms also include structures that behave like infinity and those that behave like imaginary numbers. These forms arise naturally out of the three James axioms and require no additional transformational mechanism, with the exception that we will need to add a fourth axiom to be able to reduce the many forms that act like infinity.

Postsymbolism introduces *representational freedom* without the loss of formality or expressibility. Iconic form permits many new families of representation that allow us to directly see and interact with the abstract concepts of logic and arithmetic. These new ways of thinking are intended to greatly simplify elementary mathematics. Our audience is grade school students.[2] However the iconic innovations are very unfamiliar, a barrier unlikely to be torn down due to the universal acceptance of textual expressions as vehicles for the formal concepts described in Chapter 24. Iconic formalism is making progress in calculus since many of the concepts within calculus have traditionally been associated with Cartesian graphs, surfaces and objects of rotation, visualizations of trajectory and flow, and other physical applications. Although *void* is identified as a valuable tool for solving equations in Chapters 17 and 18, I cannot over-emphasize the impact of void-equivalent forms. Within our culture *void* is abhorred by religion, Nature and typography. And yet it is this absent foundation that permits iconic form to flourish, as exemplified in Chapter 20.

## 30.1 Cognition

Our endeavor is guided by two fundamental perspectives about the nature of reality, that nothing is not something and that all things are unique unless we have elected explicitly to ignore differences. Our entire exploration centers around the nature of difference. And difference is not within the physical realm, it is purely mind stuff. Gregory Bateson delineates the essential characteristics of difference:[3]

— Difference is not material and cannot be localized.
— Difference cannot be placed in time.
— Difference is not a quantity. It is dimensionless and, for sense organs, digital.
— Information is news of difference. It is not energy.

We harness abstraction by making it concrete, only to discover that concrete form itself is a difference that our perceptions impose in support of abstract thought.

Containers are a physical envisionment of *cognitive distinctions*. A distinction constructs a difference. Since difference is not a physical quality, distinction is a purely cognitive act. Difference is the way that our senses construct order out of an essentially undifferentiated reality.[4]

Cognitive distinction constructs boundaries to create forms. The boundary can be as simple as a label, as categoric as an arbitrary property or as physical as a tree. Identifying a label or a property or an apparently discrete physical object creates a relation between the labeler and the labeled. The relation we have been calling *containment* is between object and environment, not between objects within a shared environment. Containers themselves cannot be separated from their inside and their outside. The One comes only as Three. It is our choice of perspective that defines what is inside and what is outside.

The concept we have been calling *distinction* is a recognition that it is difference that counts, and that what we consider to be the same is our unique personal choice. The mathematical language of axioms and theorems and transformations is a rigorously structured microcosm that allows us to practice with safety what is truly real. Postsymbolism seeks to explicitly enrich that microcosm.

An individual's world is the dynamic network of distinctions that person is constructing at the time. **Distinction network** *(dnet)* then is both the name of the graphic display of containment relations and the name of the cognitive construction that the containment relations represent. The primary difference is one of degree of interconnectivity. The dnets used herein to describe arithmetic are a tiny slice of a cognitive perspective. In a sense, mathematical philosophy is an exploration of different slices of our cognitive dnets, networks that have already be radically partitioned by the agreed upon distinctions that define mathematics. *There is a Platonic reality, but it is unique to each individual.* Although mathematical formalism seeks to greatly constrain that uniqueness, it is apparent that there is little agreement within mathematical philosophy about which constraints are appropriate. However, so long as mathematics embraces disembodiment it is isolating itself from the roots of its creation and thus embracing an unintelligible philosophy.

## 30.2 Last Century

From a formal perspective, what we believe to be numbers and arithmetic was established during the early twentieth century, as presented in Chapter 22. But as the opening quotes from John Bell in Chapter 23, from Bertrand Russell in Chapter 24, from Carl Sandburg in Chapter 25 and from Alain Badiou in this chapter testify, we are still very far from understanding what number and its arithmetic are.

*design choices depend upon notation*

In this volume, we have explored many of the founding concepts and strategies employed to define formal numerics. We have deconstructed equality, induction, set theory, logic, functional thinking and symbolic representation in an attempt to identify how and why iconic arithmetic is different. A minor objective in this effort is to dissuade conventional mathematics from its predilection of degrading iconic form to an isomorphism with linguistic expressions. It is a startlingly narrow perspective to insist that representation is independent of meaning, and worse, independent of formal thinking. One great learning over the last fifty years is that *concept* is **embodied**, it does not exist separate from *homo sapiens*. We have met our ideas and they are us. This is not to suggest that concept is anchored to physical manifestation. But we no longer need to project great thoughts outward to an unknown and unknowable Agent in the sky. Here is cybernetician Francisco Varela:

> The proper units of knowledge are primarily *concrete*, embodied, incorporated, lived.... The concrete is not a step toward something else; it is both where we are and how we get to where we will be.

The structure of mathematics is inseparable from the structure of our cognitive distinctions. Both are non-physical and both are irrevocably anchored to experience.

## 30.3 Computational Perspective

Chapter 25 declares that the perspective of James algebra is *computational,* specifically Hilbert's ideas that mathematics is both structural and operationally finite (Chapter 21). A more modern perspective, which is expressed in Chapter 27, is *ultrafinitism,* that the type of mathematics that is meaningful to a computer scientist is that which can be done using an algorithmic strategy with time constraints limited to the life of the universe. Another description of this position is **computational pragmatism**.

Chapter 23 introduces Primitive Recursive Arithmetic (PRA), a minimalist foundation that is supported by Friedman and Feferman as sufficient for pragmatic mathematics. In the late 1930s, prior to the domination of silicon computers, several different academic cultures converged in understanding that they each had been addressing the *same* formal concepts from different perspectives. This understanding is the **Church-Turing Thesis**. Kurt Gödel and Jacques Herbrand pioneered what later became PRA. Herbrand, following closely in Hilbert's footsteps, developed the equational formulation of PRA. Alonzo Church and Stephen Kleene developed the *lambda calculus,* a system based on substitution and abstraction. Alan Turing developed Turing machines, a formal specification of what a computer can do. And Emil Post demonstrated that string rewriting systems too were equivalent to PRA, lambda-calculus and Turing machines.

**Church-Turing Thesis**

*All reasonable formulations of the intuitive notion of effective computability are equivalent.*

This convergence then guided the development of software programming languages which provide formal specifications of what is computable.

The most recent advocacy of the computational perspective is Stephen Wolfram's fundamental theorem in 2002, which arose from his study of **cellular automata**, a vastly different computational approach with strong local parallelism and with simple accumulation rules determined by the state of adjacent neighbors within an unlimited discrete array. The distinction networks described in Chapter 29 are, by design, similar but with containment relations defining the meaning of adjacency between neighbors.

### Principle of Computational Equivalence

*Almost all processes that are not obviously simple can be viewed as computations of equivalent sophistication.*[6]

The Principle of Computational Equivalence widens the Church-Turing Thesis beyond computation to laws and processes of Nature across all varieties of machines and brains. Stephen Wolfram:

> No system can ever carry out explicit computations that are more sophisticated than those carried out by systems like cellular automata and Turing machines.[7]

The implication is that formal processes, those that we understand as embodied in computers and thoughts, are *universal*. The gauntlet thus thrown is that infinite mathematics, the kind used by almost all pure mathematicians, is not realizable. This certainly does not suggest that the concept of infinity is not valuable or worthwhile, in developing approximate models for example and in explaining how some models break down. But to be *useful*, for measurement in particular, infinite models need to be scaled to conform with what we have been describing as ultrafinitism. Specifically to qualify as science, mathematical theories should be bounded, local and determinate.[8] Areas of mathematics that do not qualify as computable include real numbers, infinity, ZFC set theory, existence proofs and *void*. Even comfortable transcendental concepts such as π and e do not qualify as exact numbers. In Volume III we will be able to identify the specific structures within James algebra that step outside of computability. Iconic mathematics can embrace both computable and non-computable structures and clearly identify which is which.

Modern study of the *non-computable* came hand-in-hand with the Church-Turing thesis. It is relatively easy to identify what a computer cannot do. The archetype is the **Halting Problem**: a computer cannot tell you when it will finish a computation. Obviously neither can another computer observing that computation. Both cellular automata and dynamic systems have exposed deterministic processes that are *immune to abstraction*. These **chaotic processes** are formal but cannot be simplified or predicted. The only way to know what will happen next is to carry out the process. And yet they are simple iterative algorithmic processes that fall well within PRA. These processes are reversible, include nothing random or probabilistic, and are not knowable but through experience. There are a plethora of examples for which a computation cannot tell us about upcoming results except by taking all of the steps to reach those results. Applied to human experience, Tor Norretranders observes

> There are no principal universal logical rules that tell us anything we did not already know. The Church-Turing thesis and Turing's halting problem tell us that we can learn nothing unless it is through experience.[9]

Bluntly, knowledge *must* be embodied. Iconic math merely attempts to provide a more consistent formal representation, one that aligns with the structure of cognition. In this volume, distinction networks have converged with whole numbers to provide what Charles Peirce identifies as *the form of formal thought*.

## 30.4 Beyond Arithmetic

As we will see in Volume III, we can call a container "infinity" and still manipulate it directly, even though we are associating a concrete object with a very abstract concept. This finesse has an unexpected consequence. We may find that what we think is abstract is actually within reach. The creations of our imagination may indeed be unreachable, but for entirely different reasons other than their abstraction. Infinity can be as concrete as One.

For whatever reasons, we have designed here a numeric arithmetic that accumulates while insisting that only numeric forms can accumulate. Infinity can exist but cannot accumulate, while accumulating *void* is an absurdity. Thus both [ ] and *void* are non-numeric. We have accepted two types of existence (round and square) and constrained their interaction with three axioms or beliefs. Given this belief structure, we rigorously limited our conceptual and structural tools to those permitted by the axioms. Well, and one additional tool, the idea of equality abstracted as the Composition Principle and implemented as pattern-matching and substitution. Some forms may look different but we construct *beliefs* that define them to be the same. Thus axioms guide us into complexity by expanding identity into equality. A *pattern-matcher* can provide search and recognition, but not construction. For that we need a *substitution-engine* that must perform exacting surgery on existent forms and thus create permitted structural variety without creating monster forms that change their equivalence class.

We have pursued two themes that do not negate our understanding of arithmetic but rather that attempt to expand its conceptualization. To embrace an unified reality, to see interaction, connectivity and feedback as essential to an understanding of numbers, *concurrency* as described in Chapter 19 is mandatory. Dedekind and Badiou envision our numbers as a unified whole, not as separate objects. To do so requires seeing them all at the same time, not visiting them one by one as encouraged by Peano's successor function and by today's fragmentary preschool educational practices described as "learning to count". Children need to learn first the cybernetic unity required for an appreciation not only of ecologies but also of social interaction. To combat prejudice we need to practice defining our world not by the properties of objects but by the network of connectivity that unifies object and environment.

The second theme is to reunite cognition with our bodies. What Chapter 26 calls postsymbolism is more than a visual and experiential approach to numbers, it is also an attempt to develop a deeper respect for the human being as constituted within a human body. The abstinence from sets and logic and functions described in Chapter 27 and Chapter 28 is based not only on coming to understand arithmetic better, it is also coming to understand social and cultural reality better by appreciating *uniqueness* rather than by collecting reality into sets defined by the properties of objects; by embracing *contradiction and context* rather than fabricating the dream that people should be rational; and by seeing feedback and evolution as the source of what we identify as structure rather than encouraging our senses to fracture cybernetic networks into input/output processes that distort both time and place.

## 30.5 Structure in Volume III

As is apparent the content of Volume III has been mentioned incidentally many times. One reason for this is that the content grew beyond the space available in this volume. Although not disconnected from the ongoing exploration, three major aspects of James arithmetic have been exiled into Volume III.

***AXIOMS***

| | |
|---|---|
| `[ ]  [ ]  ⇒  [ ]` | **unify** |
| `<[ ]><[ ]> ⇒ <[ ]>` | **unify II** |
| `[ ]  <[ ]> ⇒` *indeterminate* | **indeterminacy** |
| `(<[[ ]]>) ≠` *void* | **infinitesimal** |

***HYBRID AXIOM***

| | |
|---|---|
| `(<[ ]>) = <[ ]> = [<[ ]>]` | **infinite interpretation** |

***THEOREMS***

| | |
|---|---|
| `(A  [ ] ) = (  [ ]  ) =` *void* | **dominion** |
| `(A <[ ]>) = (<[ ]>)` | **dominion II** |
| `[ ]   = ([A][ [ ] ])` | **square replication** |
| `<[ ]> = ([A][<[ ]>]) ≠ ([<A>][[ ]])` | **square replication II** |

Figure 30-1: *Theorems of* `[ ]` *and* `<[ ]>` *(volume III)*

## Infinite Forms

The James form `(<[ ]>)` is stable and can be interpreted as `1/0`. Stable forms are grounds that are the constants of a formal system. Embedded within the James notation are forms of what might be called infinity, although that description relies upon our interpretation of these forms. It is a comfortable description because throughout history `1/0` has been associated with ∞. The empty square-bracket, `[ ]`, is a non-accumulating, non-numeric unit that lies within the deepest level of all existent non-numeric James forms. (Void-equivalent forms are also non-numeric and rely upon a completely different mechanism: they are non-numeric *and* non-existent.) We will explore the unifying influence that `[ ]` has upon many of the infinite, banned and exceptional expressions within symbolic arithmetic. From our new perspective, infinite and indeterminate expressions can be identified by specific James structures and integrated within numeric arithmetic to eliminate the confusion currently associated

| *name* | *derivative* | ☞ | *interpretation* |
|---|---|---|---|
| *constant* | dc = *void* | | $dc = 0$ |
| *variable* | dx = o | | $dx = 1$ |
| *power* | d(u) = ( u [du]) | | $d\#^{u} = \#^{u}/(\log_{\#}e)\ du$ |
| *logarithm* | d[u] = (<[u]>[du]) | | $d(\log_{\#} u) = (\log_{\#}e)/u\ du$ |
| *inverse* | d<u> = <du> | | $d(-u) = -du$ |
| *sum* | d〔u v〕 = 〔du dv〕 | | $d(u+v) = du + dv$ |

Figure 30-2: *Derivatives of James boundaries (volume III)*

with computational exceptions. Figure 30-1 shows the structural forms and transformations in Volume III that derive from forms containing [ ] after reduction. We will be able to explore not only infinite expressions but other exotics such as infinite powers, infinitesimals, indeterminate variables and logarithms with base 1, 0 and ∞.

## Differential Forms

Closely connected to infinite expressions are the topics of limit theory and the calculus of derivatives. Volume III takes our first excursion past algebra to explore elementary differential calculus. Figure 30-2 shows the structure of the derivatives of James boundaries. As is typical of each of our James explorations, some remarkable structural regularities emerge.

## Imaginary Forms

One of the most remarkable contributions of the study of James algebra is the stable form [<( )>], named J, which can be interpreted as $\log_{\#}{-1}$. J is structurally numeric, as can be seen by the innermost round-bracket. It cycles in value with a phase of 2. When two J forms accumulate, they return to *void*. The imaginary unit i is

### *THEOREMS*

| | |
|---|---|
| `J = [<o>]` | **definition of J** |
| `<A> = (J [A])` | **J-conversion** |
| `J J =` *void* | **J-void object** |
| `[<(J)>] =` *void* | **J-void process** |
| `([J][2]) =` *void* | **J-void tally** |
| `J = <J>` | **J-self-inverse** |
| `[<(A)>] = A J` | **J-transparency** |
| `A (J [A]) =` *void* | **J-occlusion** |
| `(J [J]) = J` | **J-self** |

### *COMPLEX NUMBERS*

| | | | |
|---|---|---|---|
| `i` | ☞ | `(J/2 [o])` | *form of* `i` |
| `π` | ☞ | `(J/2 [J])` | *form of* `π` |
| `a + bi` | ☞ | `a (J/2 [b])` | *form of complex numbers* |

Figure 30-3: `J` *patterns and transformations (volume III)*

phase-4 in exponential space. As a logarithm `J` shares this exponential space with `i`. In Volume III we'll show that `i` can be decomposed into `J`. That is, `i` is *not* the fundamental imaginary unit, it is one-half of `J`. The precise relation is

```
i  ☞  (J/2) = (( [J]<[2]>))
        J    =  ([[i]][2] )  = [i][i]
```

`J` is the *sum* of two logarithms of `i`.[10] To express this in exotic symbolism

$$\log_{\#} -1 = \log_{\#} i + \log_{\#} i$$

Thus `J` provides a third variety of accumulation.

| | |
|---|---|
| `( )( ) ≠ ( )` | *numeric accumulation* |
| `[ ][ ] ⇒ [ ]` | *unification* |
| `J  J  =` *void* | *cyclic accumulation* |

Figure 30-3 shows what to expect from the exploration of the structural relations of J. Since J is within numeric James algebra, there are no new axioms, just several new theorems that with familiarity provide an entirely different perspective on imaginary numbers. As an operator, J also provides a new perspective on the operations of arithmetic, since any form with an angle-bracket can be expressed as a J-form. For example,

$$
\begin{array}{lllll}
i^2 = -1 & ☞ & \langle o \rangle & & \\
 & & ([\langle o \rangle]) & & \texttt{enfold} \\
 & & (\;\; J \;\;) & ☞ \;\; \#^J & \texttt{substitute}
\end{array}
$$

It is $-1$ that is the mother of imaginary numbers, not $i$ and not J.

One final example. Euler's famous equation, $e^{i\pi} + 1 = 0$, is reputed to be mysterious yet we can derive it directly in a *base-free* form from $-1$ and the theorems in Figure 30-3. We first demonstrate that $J = i\pi$. In mixed notation,

$$
\begin{array}{llll}
J = ( & J \quad [J] \;) & & \texttt{J-self} \\
\;\;\;\;\; ( & J/2 \quad J/2\;[J] \;) & & \texttt{substitute} \\
\;\;\;\;\; ( & [(J/2\;[o])]\;[(J/2\;[J])]) & ☞ \;\; i\pi & \texttt{enfold}
\end{array}
$$

From above,

$$
\begin{array}{ll}
 & -1 = \#^J = \#^{i\pi} \\
 & \#^{i\pi} + 1 = 0 \\
\textit{let}\;\; \# = e & e^{i\pi} + 1 = 0
\end{array}
$$

## 30.6 Remarks

Spencer Brown's book *Laws of Form* is seminal for grounding the formal theory of distinctions. He presented an iconic form for logic, as did Peirce. Here we have presented a possible iconic form for numbers. We next step into the deep end of this pool of thought, to explore the non-numeric and imaginary James forms. Volume III provides the structure and interpretation of forms that redefine Accumulation to generate both imaginaries and indeterminacy. There is nothing more but to explore the consequences.

## Endnotes

1. **opening quote:** A. Badiou (2008) *Number and Numbers* p.1.

2. **Our audience is grade school students:** Several pilot projects are underway to introduce iconic thinking into the classroom. Past second grade, though, the utter dominance of textual form both in required curriculum and in the training of mathematics teachers makes traction nearly impossible. Even Euclidean geometry is being removed from the curriculum in the United States, both by suppressing visualization techniques and by converting geometry into exercises in symbolic logic proofs. Only a few experimental high school instructors who manage to mix "art" with "math" have succeeded in introducing symmetry, fractals, cellular automata, and information visualization into their math classrooms.

For pioneering work introducing Spencer Brown's *Laws of Form* and iconic algebra into the classroom, see

W. Bricken (1987) *Analyzing Errors in Elementary Mathematics*. Doctoral dissertation, Stanford School of Education.

W. Winn & W. Bricken (1992) Designing Virtual Worlds for Use in Mathematics Education: The Example of Experiential Algebra. *Educational Technology* **32**(12) p.12-19. Online 12/18 at `http://wbricken.com/pdfs/03words/03education/03iconic-math/07worlds-for-math.pdf`

W. Bricken (2007) *Presentation at WSMC07*. Online 12/18 at `http://wbricken.com/htmls/03words/0303ed/030305spacearith.html`

M. Klein & O. Pelz (2018) No Box Today. Online 12/18 at `https://www.noboxtoday.com/`

3. **delineates the essential characteristics of difference:** G. Bateson (1991) *A Sacred Unity* p.219. Online 8/18 at `https://monoskop.org/images/c/c3/Bateson_Gregory_Mind_and_Nature.pdf`

4. **our senses construct order out of an essentially undifferentiated reality:** When I look outside my window I see the yard covered in grass and an apparently different object, a tree. They are different because I have made cognitive distinctions to see them as different. They are the same when I make the distinction "where I live". They are also the same underground, out-of-sight where roots entwine to share water and nutrients. Tree and

grass belong to the same Plant kingdom, we differentiate them both from creatures that change location within the time frame of our perceptions. Trees and plants do travel by a different mechanism, that of spreading seeds. Reality is undifferentiated until we construct differences and then ignore them into similarities.

The cybernetic concept is **umwelt**, loosely the coupling between world and cognition that is the source of distinction. The distinctions we make are coupled to our physical capabilities. The dog in the yard is making fundamentally different distinctions based on his superb ability to smell. The honey bee differentiates flowers, not plants. The hummingbird is not bound by gravity. What *is* is what physiology and cognition construct as relevant differences.

5. **and how we get to where we will be:** F. Varela (1992) *Ethical Know-How: Action, Wisdom, and Cognition* p.7. (Emphasis in original.) Online 8/18 at `https://www.heartoftheart.org/wp-content/uploads/2017/08/Varela-F.-J.-1999-Ethical-know-how.-Action-wisdom-and-cognition-2119.pdf`

6. **can be viewed as computations of equivalent sophistication:** S. Wolfram (2002) *A New Kind of Science* p.716-717. Online 8/18 at `https://www.wolfram-science.com/nks/`

7. **than those carried out by systems like cellular automata and Turing machines:** Wolfram p.720.

8. **mathematical theories should be bounded, local, and determinate:** The ultrafinitist perspective would be that the use of infinite tools just identifies an immature science. In quantum mechanics, *renormalization* techniques have been developed to eliminate infinite "quantities" that arise during calculation. We do still have, though, *singularities* such as black holes that appear to step outside of boundedness. Again the ultrafinitist position would be that these are phenomena that we can describe only approximately, via an infinite model that is necessary due to a lack of complete understanding.

9. **we can learn nothing unless it is through experience:** T. Norretranders (1998) *The User Illusion* p.57.

10. **`J` is the sum of two logarithms of `i`:** This observation leads to an interesting math problem: Find the value of `x` in this equation. `x` is *not* zero.

```
x + x = 0
```

# Bibliography

*All entries in the bibliography are from the chapter endnotes.*


K. Appel & W. Haken (1977) Solution of the four color map problem. *Scientific American* **237**(4) p.108-121.

Aristotle (c. 350 BCE) *Metaphysics* 1022a. Online 8/18 at `http://classics.mit.edu/Aristotle/metaphysics.html`

_______ (c. 350 BCE) *Physics* Book IV. R. Hardie & R. Gaye (trans.) Online 11/18 at `http://classics.mit.edu/Aristotle/physics.html`

A. Badiou (1990) *Number and Numbers.*

J. Baez & J. Dolan (2001) From finite sets to Feynman diagrams. In B. Engquist & W. Schmid (eds.) *Mathematics Unlimited — 2001 and Beyond.*

G. Bateson (1972) Redundancy and Coding. *Steps to an Ecology of Mind* Online 8/18 at `http://shifter-magazine.com/wp-content/uploads/2015/11/gregory-bateson-steps-to-an-ecology-of-mind-1.pdf`

_______ (1979) *Mind and Nature: A necessary unity.* Online 8/18 at `https://monoskop.org/images/c/c3/Bateson_Gregory_Mind_and_Nature.pdf`

_______ (1991) *A Sacred Unity.*

J. Bell (1999) *The Art of the Intelligible: An elementary survey of mathematics in its conceptual development.*

P. Benacerraf (1965) What numbers could not be. In P. Benacerraf & H. Putnam (eds.) (1983) *Philosophy of Mathematics 2nd ed.* p.272-294.

P. Bernays (1922) On Hilbert's ideas for the foundation of arithmetic. P. Mancosu (trans.) Online 12/18 at `http://www.phil.cmu.edu/projects/bernays/Pdf/bernays02_2003-05-18.pdf`

_______ (1930) The philosophy of mathematics and Hilbert's proof theory. P. Mancosu & I. Mueller (trans.) Online 12/18 at `http://www.phil.cmu.edu/projects/bernays/Pdf/philmath.pdf`

_______ (1935) Hilbert's investigations into the foundations of arithmetic D. Schlimm (trans.) Online 12/18 at `http://www.phil.cmu.edu/projects/bernays/Pdf/untersuchungen.pdf`

P. Bloom (2000) *How Children Learn the Meanings of Words*.

A. Borovik (2007) *Mathematics under the Microscope: Notes on cognitive aspects of mathematical practice*. Online 8/18 at http://eprints.ma.man.ac.uk/844/1/covered/MIMS_ep2007_112.pdf

W. Bricken (1987) *Analyzing Errors in Elementary Mathematics*. Doctoral dissertation, Stanford University School of Education.

_______ (1995) Distinction networks. In I. Wachsmuth, C. Rollinger & W. Brauer (eds.) *KI-95: Advances in Artificial Intelligence* p.35-48. Online 8/18 at http://wbricken.com/pdfs/01bm/05arch/01dnets/04distinction-networks.pdf

_______ (2002) Generalized insertion. Online 11/18 at http://iconicmath.com/mypdfs/bl-general-insertion.020118.pdf

_______ (2007) *Presentation at WSCC07*. Online 12/18 at http://wbricken.com/htmls/03words/0303ed/030305spacearith.html

_______ (2008) Simplicity rather than knowledge *AI Magazine* **29**(2).

_______ (2017) Distinction is sufficient: Iconic and symbolic perspectives on *Laws of Form*. *Cybernetics and Human Knowing* **24**(3-4) p.29-74. Online 12/18 at http://iconicmath.com/new/distinctionissufficient/

W. Bricken & G. Coco (1995) VEOS: The virtual environment operating shell. In W. Barfield & T. Furness (eds.), *Virtual Environments and Advanced Interface Design* p.102-142. Online 8/18 at http://wbricken.com/pdfs/03words/02vr/02veos/01describe/04veos-proj-book.pdf

M. Burns (1998) *Math: Facing an American Phobia*.

S. Buss (1999) Bounded arithmetic, proof complexity and two papers by Parikh. Online 8/18 at https://www.math.ucsd.edu/~sbuss/ResearchWeb/parikh/paper.pdf

R. Casati & A. Varzi (1999) *Parts and Places*.

C. Cellucci (2002) "Introduction" to *Filosofia e mathematica*. In R. Hersh (ed.) (2006) *18 Unconventional Essays on the Nature of Mathematics*.

G. Chaitin (2004) *Meta Math!*

D. Clements (1999) Concrete manipulatives, concrete ideas. *Contemporary Issues in Early Childhood* **1**(1) p.45-60. Online 8/18 at http://www.gse.buffalo.edu/org/buildingblocks/Newsletters/Concrete_Yelland.htm

A. Collings (2017) The Brown-4 indicational calculus. *Cybernetics and Human Knowing* **24**(3-4) p.75-101.

A. Connes (2010) A view of mathematics. Online 11/18 at http://www.alain-connes.org/en/downloads.php

J. Conway (1976) *On Numbers and Games*.

J. Conway & R. Guy (1996) *The Book of Numbers*.

D. Corfield (2004) *Towards a Philosophy of Real Mathematics*.

L. Couturat (1901) *The Logic of Leibniz*. Online 8/18 (in French) at https://babel.hathitrust.org/cgi/pt?id=ien.35556036601318;view=1up;seq=1

_______ (1905) *The Algebra of Logic* L. Robinson (trans.) (1914) Online 8/18 at http://www.gutenberg.org/ebooks/10836

R. Cummings (1922) *The Girl in the Golden Atom*.

T. Danzig (1930) *Number: The language of science*. J. Mazur (ed.) (2005). Online 8/18 at http://www.engineering108.com/Data/Engineering/Maths/Number_the_language_of_science_by_Joseph-Mazur_and_Barry-Mazur.pdf

M. Davis, R. Sigal & E. Weyuker (1994) *Computability, Complexity, and Languages 2nd ed.*

R. Dedekind (1901) *Essays on the Theory of Numbers* W. Beman (trans.). Online 8/18 at https://www.gutenberg.org/files/21016/21016-pdf.pdf

S. Dehaene (2011) *The Number Sense: How the mind creates mathematics*.

_______ (2014) *Consciousness and the Brain*.

A. DeMorgan (1849) *Trigonometry and Double Algebra*. Online 8/18 at https://archive.org/details/trigonometrydoub00demoiala/page/n8

J. Derbyshire (2006) *Unknown Quantity*.

K. Devlin (2006) The useful and reliable illusion of reality in mathematics. *Toward a New Epistemology of Mathematics Workshop* GAP.6 Conference 2006. Online 8/18 at https://web.stanford.edu/~kdevlin/Papers/Berlin06.pdf

R. Dipert (1995) Peirce's underestimated place in the history of logic: A response to Quine. In K. Ketner (ed.) *Peirce and Contemporary Thought: Philosophical Inquiries*.

M. Donovan & J. Bransford (eds.) (2005) *How Students Learn Mathematics in the Classroom*.

A. Eddington (1927) *The Nature of the Physical World: The Gifford Lectures*. Online 8/18 at http://henry.pha.jhu.edu/Eddington.2008.pdf

*Encyclopedia Britannica, 1771 edition* Volume 2 "Logic".

T. Etter (2006) Three-place identity. Online 12/18 via the Wayback Machine: https://web.archive.org/web/20130510223843/http://www.boundaryinstitute.org/bi/articles/Three-place_Identity.pdf

Euclid (c. 300 BCE) *The Elements*. Online 12/18 at https://mathcs.clarku.edu/~djoyce/java/elements/toc.html

W. Ewald (1996) *From Kant to Hilbert: A source book in the foundations of mathematics* p.787-832.

S. Feferman (1992) Why a little bit goes a long way: Logical foundations of scientifically applicable mathematics. *Philosophy of Science Association* 1992 Vol. II p.442-455. Online 8/18 at https://math.stanford.edu/~feferman/papers/psa1992.pdf

_______ (1997) Does mathematics need new axioms? Online 4/18 at https://web.archive.org/web/20170911031521/http://math.stanford.edu/~feferman/papers.html

S. Feferman & G. Hellman (1995) Predicative foundations of arithmetic. *Journal of Philosophical Logic* p.1-17. Online 8/18 at http://math.stanford.edu/~feferman/papers/predarith.pdf

S. Feferman & T. Strahm (2010) Unfolding finitist arithmetic. *Review of Symbolic Logic* **3**(4) p.665-689. Online 8/18 at https://math.stanford.edu/~feferman/papers/UnfoldFA.pdf

K. François (2007) The untouchable and frightening status of mathematics. In K. François & J. van Bendegem (eds.) (2007) *Philosophical Dimensions in Mathematics Education.*

C. Franks (2009) *The Autonomy of Mathematical Knowledge*.

G. Frege (1884/1950) *The Foundations of Arithmetic*. Online 8/18 at http://www.naturalthinker.net/trl/texts/Frege,Gottlob/Frege,%20Gottlob%20-%20The%20Foundations%20of%20Arithmetic%20(1953)%202Ed_%207.0-2.5%20LotB.pdf

_______ (1903) *The Fundamental Laws of Arithmetic Vol. II.*

H. Friedman (2013) Concrete mathematical incompleteness. Online 8/18 at http://cage.ugent.be/programFriedman/slides/MathInc083113.pdf

J. Goguen (1988) What is unification? Online 8/18 at http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.16.9221

_______ (1993) On notation. In. B. Magnusson, B. Meyer & J-F. Perrot (eds.) *TOOLS 10: Technology of Object-oriented Languages and Systems* p.5-10.

R. Goodstein (1954) Logic-free formalizations of recursive arithmetic. *Mathematica Scandinavica* **2** p.247-261.

I. Grattan-Guinness (1997) *The Rainbow of Mathematics*.

M. Greaves (2002) *The Philosophical Status of Diagrams*.

J. Greeno & R. Hall (1997) Practicing representation: learning with and about representational forms. *Phi Delta Kappan* 78 p.1-24. Online 4/18 at `http://www.pdkintl.org/kappan/kgreeno.htm`

T. Hales (2007) The Jordan curve theorem, formally and informally. *The Mathematical Association of America Monthly* 114. Online 8/18 at `https://pdfs.semanticscholar.org/70ab/0431a8d59e1cd9147b54c5e99883a54190a1.pdf`

A. Heeffer (2007) Learning concepts through the history of mathematics. In K. François & J. van Bendegem (eds.) (2007) *Philosophical Dimensions in Mathematics Education*.

H. Hellman (2006) *Great Feuds in Mathematics*.

A. Heyting (1971) Disputations. In P. Benacerraf & H. Putnam (1983) *Philosophy of Mathematics 2ed*.

V. Huber-Dyson (1998) On the nature of mathematical concepts: Why and how do mathematicians jump to conclusions? EDGE conversation 2/15/98. Online 8/18 at `https://www.edge.org/conversation/verena_huber_dyson-on-the-nature-of-mathematical-concepts-why-and-how-do-mathematicians`

D. Kahneman (2013) *Thinking, Fast and Slow*.

J. Kaput (1987) Representation systems and mathematics. In C. Janvier (ed.) *Problems of Representation in the Teaching and Learning of Mathematics* p.19-26.

L. Kauffman (1995) Arithmetic in the form. *Cybernetics and Systems: An International Journal* **26**(1) p.1-57. Online 8/18 at `http://homepages.math.uic.edu/~kauffman/ArithForm.pdf`

_______ (2017 in process) *Laws of Form — An Exploration in Mathematics and Foundations* (rough draft) Online 2/17 at `http://homepages.math.uic.edu/~kauffman/Laws.pdf`

L. Kauffman & F. Varela (1980) Form dynamics. *Journal of Social and Biological Structures* **3**. Online 8/18 at `http://homepages.math.uic.edu/~kauffman/FormDynamics.pdf`

C. Kieran (1981) Concepts associated with the equality symbol. *Educational Studies in Mathematics* **12** p.317-326. Online 8/18 at https://oak.ucc.nau.edu/smg224/401pdfs/algebrareadings/kieran1.pdf

J. Kilpatrick, J. Swalford & B. Findell (eds.) (2001) *Adding It Up: Helping children learn mathematics*.

S. Kleene (1952) *Introduction to Metamathematics*.

I. Kleiner (1991) Rigor and proof in mathematics: A historical perspective. *Mathematics Magazine* 64 p.291-314. Online 8/18 at https://www.maa.org/sites/default/files/pdf/upload_library/22/Allendoerfer/1992/0025570x.di021172.02p0031c.pdf

M. Klein & O. Pelz (2018) No Box Today. Online 12/18 at https://www.noboxtoday.com/

M. Kline (1972) *Mathematical Thought from Ancient to Modern Times*.

_______ (1980) *Mathematics: The loss of certainty*.

D. Knuth (1976) Mathematics and computer science: Coping with finiteness. *Science* **194**(4271) p.1235-1242.

E. Knuth et al (2006) Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education* **37**(4) p.297-312. Online 8/18 at http://www.sciacchitano.it/Spazio/Coping%20with%20Finiteness.pdf

A. Kornai (2003) Explicit finitism. *International Journal of Theoretical Physics* **42**(2) p.301-307.

G. Lakoff & R. Núñez (2000) *Where Mathematics Comes From*.

J. Lanier (1989) Communication without symbols. *Whole Earth Review* 64 p.118-119. Online 8/18 at http://www.jaronlanier.com/jaron%20whole%20earth%20review.pdf

_______ (2017) *Dawn of the New Everything*.

F. Lawvere & S. Schanuel (2009) *Conceptual Mathematics 2nd ed*.

J. Leach (2010) *Mathematics and Religion*.

M. Leng (2010) *Mathematics and Reality*.

T. Lenoir (2008) Machinic bodies, ghosts, and paraselves: confronting the singularity with Brian Rotman. Foreword to B. Rotman (2008) *Becoming Beside Ourselves*.

B. Lewin (2018) *Enthusiastic Mathematics*.

S. Lloyd (2002) Computational capacity of the universe. *Physical Review Letters* 88:237901.

P. Lockhart (2017) *Arithmetic*.

J. Łukasiewicz (1948) The shortest axiom of the implicational calculus of propositions. *Proceedings of the Royal Irish Academy* **52A**(3) p.25-33.

P. Maddy (1993) *Realism in Mathematics*.

_______ (2008) How applied mathematics became pure. *The Review of Symbolic Logic* **1**(1) p.16-41.

______ (2014) A second philosophy of arithmetic. *The Review of Symbolic Logic* **7**(2) p.222-249. Online 8/18 at http://www.socsci.uci.edu/~pjmaddy/bio/arithmetic%20in%20RSL.pdf

M. Mannucci & R. Cherubin (2008) Model theory of ultrafinitism I: Fuzzy initial segments of arithmetic. Online 12/18 at https://arxiv.org/pdf/cs/0611100v1.pdf

A. Mathias (2002) A term length of 4,523,659,424,929. *Synthese* 133 p.75-86.

B. Mazur (2007) When is one thing equal to some other thing? Online 4/18 at http://www.math.harvard.edu/~mazur/preprints/when_is_one.pdf

T. McFarlane (2007) Distinction and the foundation of arithmetic. Online 8/18 at http://www.integralscience.org/distinctionarithmetic.pdf

J. Misra (1994) Powerlist: A structure for parallel recursion. *ACM Transactions on Programming Languages and Systems* **16**(6) p.1737-1767.

R. Munroe xkcd *cartoon* "(". Online 8/18 at https://xkcd.com/859/

P. Moyer, J. Bolyard & M. Spikell (2002) What are virtual manipulatives? *Teaching Children Mathematics* **8**(6).

T. Needham (1999) *Visual Complex Analysis*.

E. Nelson (2015) *Elements*. Online 11/18 at https://arxiv.org/abs/1510.00369

______ (2002) Syntax and semantics. *Foundations and the Ontological Quest. Prospects for the New Millennium*. Online 8/18 at https://web.math.princeton.edu/~nelson/papers/s.pdf

T. Nelson (1974) *Computer Lib/Dream Machines*.

R. Netz (1999) *The Shaping of Deduction in Greek Mathematics*.

T. Norretranders (1998) *The User Illusion*.

J. Paris & L. Harrington (1977) A mathematical incompleteness in Peano arithmetic. In J. Barwise (ed.) (1982) *Handbook of Mathematical Logic*. Online 8/18 at `https://www.karlin.mff.cuni.cz/~krajicek/ph.pdf`

G. Peano (1889) The principles of arithmetic presented by a new method. In J. van Heijenoort (ed.) (1967) *From Frege to Gödel: A sourcebook in mathematical logic, 1879-1931*.

C. S. Peirce (1909) MS 514 Existential graphs.

______ (1931-35) *Collected Papers of Charles Sanders Peirce*. C. Hartshorne & P. Weiss (eds.).

R. Penrose (2004) *The Road to Reality*.

R. Plasmeijer & M. van Eekelen (1993) *Functional Programming and Parallel Graph Rewriting*.

H. Poincaré (1908) *Science et Méthode*. G. Halstead (trans.) (1913) *The Foundations of Science*. Online 8/18 at `https://www.gutenberg.org/files/39713/39713-h/39713-h.htm`

M. Potter (2004) *Set Theory and its Philosophy*.

S. Powell (2012) Equations and the equal sign in elementary mathematics textbooks. *Elementary School Journal* **112**(4) p.627-648.

F. Ramsey (1922) Truth and simplicity. *British Journal for the Philosophy of Science* (2007) **58** p.379-386.

A. Robinson (1974) *Non-Standard Analysis*.

R. Robinson (1950) An essentially undecidable axiom system. *Proceedings of the International Congress of Mathematics 1950* p.729-730.

V. Rodych (1999) Wittgenstein on irrationals and algorithmic decidability. *Synthese* 118 p.279-304.

______ Wittgenstein's philosophy of mathematics, *Stanford Encyclopedia of Philosophy*. Online 3/17 at `https://plato.stanford.edu/entries/wittgenstein-mathematics/`

B. Rotman (1993) *Ad Infinitum*.

______ (2000) *Mathematics as Sign: writing imagining counting*.

______ (2008) *Becoming Beside Ourselves: the alphabet, ghosts, and distributed human being*.

B. Russell (1918) *Mysticism and Logic and Other Essays*. Online 8/18 at `https://www.gutenberg.org/files/25447/25447-h/25447-h.htm`

_______ (1920) *Introduction to Mathematical Philosophy*. Online 4/18 at https://people.umass.edu/klement/imp/imp.html

C. Sandburg (1993) *Arithmetic*.

D. Schmandt-Besserat (1999) *The Evolution of Writing*. Online 4/18 at https://sites.utexas.edu/dsb/tokens/the-evolution-of-writing/

G. Spencer Brown (1969) *Laws of Form*. Online 8/18 at http://www.manuelugarte.org/modulos/biblioteca/b/G-Spencer-Brown-Laws-of-Form.pdf.

_______ (1969) *Laws of Form* Bohmeier Verlag Edition (2009) Appendix 4: *An algebra for the natural numbers* (1961).

R. Stansifer (1984) Presburger's article on integer arithmetic: remarks and translation. Technical Report 84-639. Dept. of Computer Science, Cornell University.

L. Stebbing (1931) *A Modern Introduction to Logic*. Online 8/18 at http://www.naturalthinker.net/trl/texts/Stebbing,LSusan/PostulationalSystems.html

W. Tait (1981) Finitism. In W. Tait (2005) *The Provenance of Pure Reason* p. 21-42.

A. Tarski (1953) Undecidable theories. *Studies in Logic and the Foundations of Mathematics*.

S. Tobias (1993) *Overcoming Math Anxiety*.

Utah State University (1999) National library of virtual manipulatives. Online 4/18 at http://nlvm.usu.edu/en/nav/index.html

J. van Heijenoort (ed.) (1967) *From Frege to Gödel: A sourcebook in mathematical logic, 1879-1931*.

F. Varela (1992) *Ethical Know-How: Action, Wisdom, and Cognition*. Online 8/18 at https://www.heartoftheart.org/wp-content/uploads/2017/08/Varela-F.-J.-1999-Ethical-know-how.-Action-wisdom-and-cognition-2119.pdf

U. Vishkin (2011) Using simple abstraction to reinvent computing for parallelism. *Communications of the ACM* **54**(1) p.75-85.

M. Walicki (1995) The origin of mathematics. Online 8/18 at https://www.ii.uib.no/~michal/phil/om/om.pdf

N. Weaver (2009) Is set theory indispensible? Online 8/18 at https://arxiv.org/abs/0905.1680

H. Weyl (1941) *The Mathematical Way of Thinking*.

J. Wheeler (1990) *Complexity, Entropy, and the Physics of Information*. Online 8/18 at https://archive.org/details/ComplexityEntropyAndThePhysicsOfInformation

A. Whitehead & B. Russell (1910) *Principia Mathematica*. Online 8/18 at https://archive.org/details/PrincipiaMathematicaVolumeI

N. Wildberger (2005) Set theory: should you believe? Online 10/18 at http://web.maths.unsw.edu.au/~norman/views2.htm

W. Winn & W. Bricken (1992) Designing virtual worlds for use in mathematics education: The example of experiential algebra. *Educational Technology* **32**(12) p.12-19. Online 12/18 at http://wbricken.com/pdfs/03words/03education/03iconic-math/07worlds-for-math.pdf

L. Wittgenstein (1921) *Tractatus Logico-Philosophicus*. Online 8/18 at https://www.gutenberg.org/files/5740/5740-pdf.pdf

_______ (1929-30) *Philosophical Remarks*.

S. Wolfram (2002) *A New Kind of Science*. Online 8/18 at https://www.wolframscience.com/nks/

_______ (2016) *Idea Makers*.

_______ (2017) *An Elementary Introduction to the Wolfram Language 2nd ed.*

F. Zalamea (2009) *Synthetic Philosophy of Contemporary Mathematics*.

# Index to the Index

The index is organized by concept and by content. Keywords are associated with each index category.

As a suggestion, decide first what kind of things you are looking for: *person*, *concept*, *pattern* or *iconic system*. Then look under the appropriate category for page and figure locations. Figures are listed by page number followed by the figure number in bold italics, for example: 124 ***5-2***. Boldface page numbers indicate primary definitions and content emphasis.

The five **Primary Reference Figures** summarize the structural forms of James algebra and have been isolated from the alphabetical Index as globally applicable.

# Index

## PEOPLE
**quotations in bold**

# Symbolic Concepts

**emphasis in bold**
**figure numbers follow their page number in *bold italic***

## MATHEMATICS AND LOGIC

## NUMBER AND ARITHMETIC

## COMPUTER SCIENCE

## EDUCATION AND OTHER



## ACRONYMS

# JAMES ALGEBRA

***Figure 16-1* summarizes the axioms and theorems of James algebra**

***Figure 16-2* summarizes the concepts of James Algebra**

***Figure 21-1* summarizes James Algebra for whole number arithmetic**

## CONTAINERS AND DELIMITERS

## PATTERNS AND PRINCIPLES

# VOLUMES

## Primary Reference Figures



## Typographical Delimiters

| *bracket* | *name* | *use* | *chapters* |
|---|---|---|---|
| | | ***James Algebra*** | |
| o, ( ) | `round` | numeric, exponential | all |
| [ ] | `square` | non-numeric, logarithmic | all |
| < > | `angle` | reflection, inverse | all |
| 〔 〕 | `shell` | void-equivalent outermost | 18, 20, 28 |
| { } | `curly` | generic boundary | 18, 24, 27, 29 |
| 〘 〙 | `double shell` | substitution operator | 16, 18, 23-**25**, 27, 28 |
| ⟨ ⟩ | `large angle` | logic, not numeric | 28, 29 |
| ⟦ ⟧ | `double square` | two-boundary system | 20 |
| | | ***Textual Mathematics*** | |
| ( ) | `parenthesis` | textual scoping | all |
| [ ] | `bracket` | function arguments | 19, 23 |
| { } | `brace` | set delimiter | 16-17, 20, 22, 26, 27 |
| ≪ ≫ | `double angle` | equivalence class | 17, 24 |
| | | ***Incidental*** | ***page*** |
| ⏋ | `cross, mark` | LoF distinction | xxvi |
| 《 》 | `large double angle` | two-boundary alternative | 124 |
| 〔 〕 | `shell` | semantic oscillation | 125-126 |
| “ ” | `quotation mark` | string expression | 191 |
| ⟦ ⟧ | `double bracket` | equality | 350 |

# Bumper Stickers

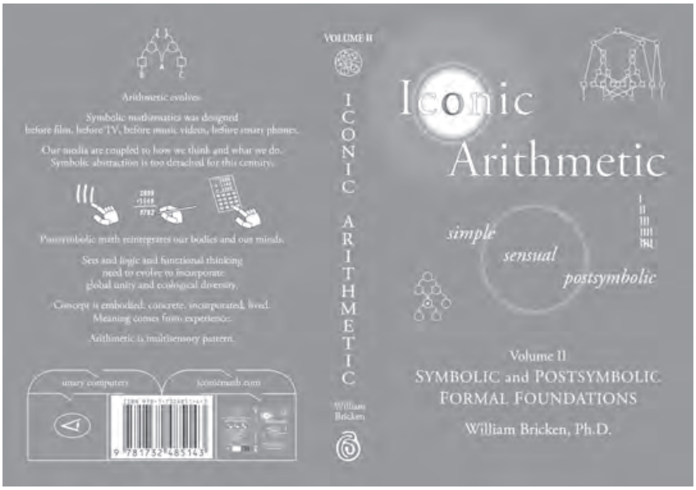
## COVER

page 357
*Arrangement axiom*
*dnet dialect*

page 140
*the number two*

page 251
*tangled*
*arrangement*

page 75
*wetware*
*software*
*hardware*




page 1
*tally*
*strokes*

Volume I page 324
*Arrangement axiom*
*room dialect*

page 282
*bounded space*

page 353
*structure sharing*
*dnet dialect*

## COVER WORDS

Arithmetic evolves.

Symbolic mathematics was designed
before film, before TV, before music videos, before smart phones.

Our media are coupled to how we think and what we do.
Symbolic abstraction is too detached for this century.

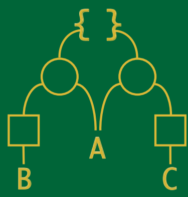
Postsymbolic math reintegrates our bodies and our minds.

Sets and logic and functional thinking
need to evolve to incorporate
global unity and ecological diversity.

Concept is embodied, concrete, incorporated, lived.
Meaning comes from experience.

Arithmetic is multisensory pattern.

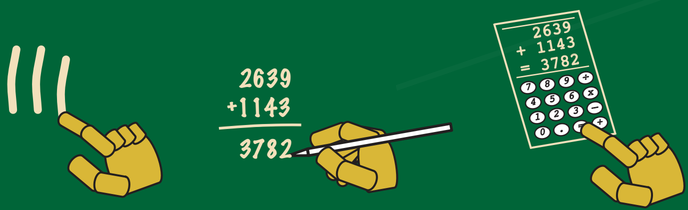
# Websites




Arithmetic evolves.

Symbolic mathematics was designed
before film, before TV, before music videos, before smart phones.

Our media are coupled to how we think and what we do.
Symbolic abstraction is too detached for this century.




Postsymbolic math reintegrates our bodies and our minds.

Sets and logic and functional thinking
need to evolve to incorporate
global unity and ecological diversity.

Concept is embodied, concrete, incorporated, lived.
Meaning comes from experience.

Arithmetic is multisensory pattern.

unary computers

iconicmath.com

ISBN 978-1-7324851-4-3
9 781732 485143