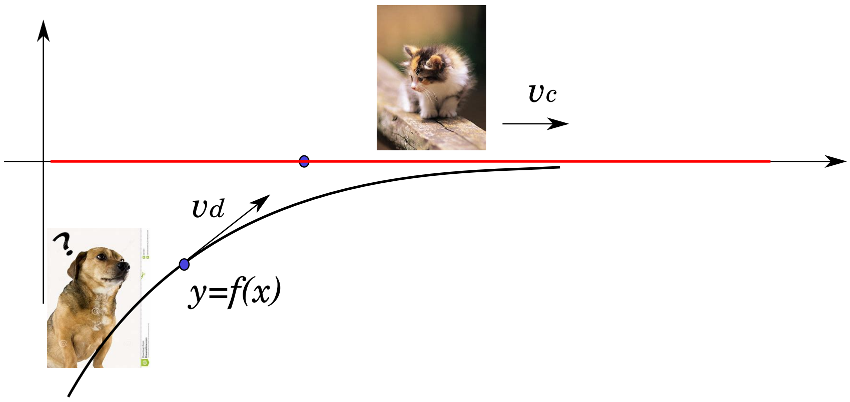
Problem description:

A cat runs along a straight line with a constant speed v_c. A Dog runs towards the cat with speed v_d.



Try to set up formulae to describe the curve that the dog runs.

Solution:

For mathematical simplicity, we invert coordinates x and y, ie. the cat runs along the y axis. Therefore, at time t, the positions of cat and dog are $(0, v_c t)$ and (x, y) respectively. Thus we have:

$$y' = \frac{y - v_c t}{x} \tag{1}$$

At point (x, y), the dog has moved $s = \int \sqrt{1 + (y')^2} dx = v_d t$. Thus:

$$t = \frac{1}{v_d} \int \sqrt{1 + (y')^2} dx \tag{2}$$

Replace t in eq.(1):

$$xy' = y - \frac{v_c}{v_d} \int \sqrt{1 + (y')^2} dx \tag{3}$$

Differentiating both side, we have:

$$xy'' = -\frac{v_c}{v_d} \sqrt{1 + (y')^2} \tag{4}$$

Let $u = y'$, then $y'' = \frac{du}{dx}$. Eq(4) becomes:

$$x \frac{du}{dx} = -\frac{v_c}{v_d} \sqrt{1 + u^2} \tag{5}$$

$$\frac{du}{\sqrt{1 + u^2}} = -\frac{v_c}{v_d} \frac{dx}{x} \tag{6}$$

$$\ln |u + \sqrt{1 + u^2}| = -\frac{v_c}{v_d} \ln |x| + C_1' \tag{7}$$

$$u + \sqrt{1 + u^2} = -C_1 x^{-v_c/v_d} \tag{8}$$

$$u=\frac{1-C_1^2x^{-2v_c/v_d}}{2C_1x^{-v_c/v_d}}=\frac{1}{2C_1}x^{v_c/v_d}-\frac{C_1}{2}x^{-v_c/v_d}=\frac{dy}{dx} \tag{9}$$

$$y=\begin{cases}\dfrac{1}{4C_1}x^2-\dfrac{C_1}{2}\ln|x|+C_2, v_c=v_d\\ \dfrac{1}{2C_1(v_c/v_d+1)}x^{v_c/v_d+1}+\dfrac{C_1}{2(v_c/v_d-1)}x^{1-v_c/v_d}+C_2, v_c\neq v_d\end{cases} \tag{10}$$

Assume initial conditions ($t=0$): cat at (0, 0), dog at (L, 0). At this moment, $y'=u=0$. Plugging $u=0, x=L$ into eq.(8), C_1 can be calculated:

$$C_1=-L^{v_c/v_d} \tag{11}$$

Then put $x=l, y=0$ into eq.(10) to calculate C_2:

$$C_2=\begin{cases}\dfrac{C_1}{2}\ln L-\dfrac{L^2}{4C_1}, v_c=v_d\\ -\dfrac{L^{v_c/v_d+1}}{2C_1(v_c/v_d+1)}-\dfrac{C_1L^{1-v_c/v_d}}{2(v_c/v_d-1)}, v_c\neq v_d\end{cases} \tag{12}$$

MORE EXERCISE: Try to draw the curve using MATLAB.
Assume v_c=5m/s, v_d=5m/s, L=50 m.