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THEORETICAL NEUROSCIENCE

TD5: RATE MODELS

All TD materials will be made available at `https://github.com/yfardella/Th_Neuro_TD_2025`.

This tutorial aims to study a very standard example of rate model: the Ring Model. This model was originally introduced for orientation coding in the visual cortex. More generally, this model could apply for any network encoding a circular one-dimensional variable. In order to understand the model's dynamics, we will use methods used in the previous tutorials to study nonlinear ODEs and introduce perturbation theory.

1 The ring model

We consider a network of neurons responsive to a stimulus θ which spans the range $[0, 2\pi]$. Each neuron has a preferred stimulus and the connection strength between two neurons respectively preferring θ_1 and θ_2 is proportional to $J_0 + J_1 \cos(\theta_1 - \theta_2)$. Additionally, each neuron receives an external input which depends on the neuron's preferred stimulus $h(\theta)$.

In the limit where the number of neurons is large and their preferred stimuli uniformly span the range $[0, 2\pi]$, we can write the neural activity as a continuous function $m(\theta, t)$. The input current to a neuron preferring θ can then be written as

$$I(\theta, t) = h(\theta) + \int_0^{2\pi} \frac{1}{2\pi} (J_0 + J_1 \cos(\theta - \theta')) m(\theta', t) d\theta'. \tag{1}$$

The activity then evolves according to

$$\frac{dm(\theta, t)}{dt} = -m(\theta, t) + f(I(\theta, t)), \text{ with } f(x > 0) = x \text{ and } f(x < 0) = 0. \tag{2}$$

1.1 Uniform state

We suppose that there is a constant, uniform and positive external current $h(\theta) = h_0$, which is sufficient for there to be positive, uniform network activity $m(\theta, t) = m_0$.

1. What is the current received by each neuron?
2. Deduce the equilibrium network activity m_0.
3. How does it depend on J_0 ?

1.2 Stability analysis through perturbations

We wish to study whether this uniform state is stable - we therefore consider small perturbations around it $m(\theta, t) = m_0 + \delta m(\theta, t)$. We wish to see how these evolve with time. To do so, we want to find a simple description of the dynamics. We introduce the order parameters

$$M(t) = \int_0^{2\pi} \frac{1}{2\pi} \delta m(\theta', t) d\theta', \tag{3}$$

$$C(t) = \int_0^{2\pi} \frac{1}{2\pi} \delta m(\theta', t) e^{i\theta'} d\theta'. \tag{4}$$

4. Give an interpretation of $M(t)$.

1.2.1 Uniform perturbation

We suppose that the perturbation is uniform

$$\delta m(\theta, t) = \epsilon.$$

5. Compute the values of $M(t)$ and $C(t)$.

1.2.2 Bumpy perturbation

We now suppose that the perturbation is a small bump centred around the angle ϕ

$$\delta m(\theta, t) = \epsilon \cos(\theta - \phi) = \epsilon \frac{e^{i(\theta-\phi)} + e^{-i(\theta-\phi)}}{2}.$$

6. Compute the values of $M(t)$ and $C(t)$.

7. Give an interpretation of $C(t)$.

Now that we have understood what $M(t)$ and $C(t)$ characterise, we will try to obtain a description of the dynamics in terms of the evolution of these two order parameters.

8. Linearise the dynamics of the activity around the equilibrium m_0 and express it as a function of $M(t)$ and $C(t)$.

9. Determine the differential equations governing the evolution of the order parameters.

10. Determine the conditions under which the uniform activity is stable. What happens when either these conditions is not met?

We consider $J_0 < 1$ and $J_1 < 2$. The network is submitted to an external input with weak modulation

$$h(\theta) = h_0 + \epsilon \cos(\theta), \text{ with } \epsilon \ll 1.$$

11. What is the profile of activity of the network induced by such an external input?

Suppose that the firing rate is given by

$$m(\theta, t) = m_0 + m_1 \cos(\theta).$$

12. Determine the conditions such that the network amplifies the input, i.e.

$$\frac{m_1}{m_0} > \frac{\epsilon}{h_0}.$$

2 Homogeneous Excitatory and Inhibitory Populations

We analyze a simple model in which all of the excitatory neurons are described by a single firing rate ν_E and all of the inhibitory neurons are described by a second firing rate ν_I. The equations describing the dynamics of the firing rates are

$$\tau_E \frac{d\nu_E}{dt}(t) = -\nu_E + [M_{EE}\nu_E + M_{IE}\nu_I - \gamma_E]_+ \,, \tag{5}$$

$$\tau_I \frac{d\nu_I}{dt}(t) = -\nu_I + [M_{EI}\nu_E + M_{II}\nu_I - \gamma_I]_+ \,, \tag{6}$$

where $[\cdots]_+$ represents the linear threshold function (or ReLU).

In this exercise, we consider the case where $M_{EE} = 1.25$, $M_{IE} = 1$, $M_{II} = 0$, $M_{EI} = -1$, $\gamma_E = -10$ Hz and $\gamma_I = 10$ Hz.

13. Draw the nullclines for ν_E and ν_I. What is the (qualitative) condition for the existence of a fixed point?
14. Draw the flow field of ν_E and ν_I above and below the nullclines (draw only the vertical and horizontal arrows).
15. Study the stability of the fixed point. What happens if the fixed point is not stable to perturbations?