Linear Algebra and Analytic Geometry II EXAMINATION PAPER

Reminder: According to the "Shantou University Rules for the Implementation of Bachelor's Degree Conferring Policy for Undergraduate Students", cheating in the exam will disqualify offenders for the award of the bachelor' s degree. Please take the exams with honesty and integrity.

Semester: _spring_ Year: _2021-2022_ SHANTOU UNIVERSITY

Department: Mathematics Class No: __________ Instructor: ____________

Student Name: __________ Student ID: __________ Grader/Instructor: __________

Section No.	One	Two	Three	Four	Five	Six	Seven	Eight	Nine	Ten	Total Score
Score											

Remark. (1) The score of each section is 10 points. (2) Please state the main procedure, otherwise you will get no point.

注意：(1) 每道题10分. (2) 请写出主要过程, 否则不得分.

Assume a right-handed rectangular coordinate system $[O; \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}]$ is used for problems 1–5.

1. Calculate the general equation and parameter equation of plane Π which passes through the point $P_1(1,1,1)$, parallels (平行) to the vector $(1,2,3)$, and is perpendicular (垂直) to plane $4x-2y+z=3$.

2. Let Π_1 and Π_2 be two planes:

$$\Pi_1 : x + 2y - 2z + 9 = 0; \qquad \Pi_2 : 2x + 4y - 4z + 7 = 0.$$

(1) Judge the position relation of Π_1 and Π_2.

(2) Let P' be the projection (投影) of $P(1, 2, -2)$ on Π_1, calculate the length (长度) of $\overrightarrow{OP'}$.

3. Judge the position relation of the following lines:

$$L_1 : \frac{x-3}{2} = \frac{y-1}{3} = \frac{z-2}{4}; \qquad L_2 : \begin{cases} x - 3y = 4, \\ 2y + z = 3. \end{cases}$$

4. Given two lines:

$$L_1 : \frac{x+2}{2} = \frac{y+1}{3} = \frac{z-2}{-1}; \quad L_2 : \frac{x}{1} = \frac{y-3}{-2} = \frac{z+1}{3}.$$

Calculate an equation for the common perpendicular line (公垂线) to L_1 and L_2.

5. Given two lines:

$$L_1 : \frac{x+1}{2} = \frac{y+1}{-1} = \frac{z-1}{1}; \quad L_2 : \frac{x-2}{1} = \frac{y-3}{1} = \frac{z+1}{2}.$$

(1) Find the angle between L_1 and L_2.
(2) Suppose the plane Π contains (包含) L_2 and parallels to L_1, find an equation of Π.
(3) Find the distance (距离) between L_1 and Π.

6. Suppose that $[v_1, v_2, v_3]$ is a basis of a linear space V.
   (i) Discuss whether $[v_1 - v_2,\ v_2 - v_3,\ v_1 + v_3]$ is a basis of V.
   (ii) Discuss whether $[v_1 - v_2,\ v_2 - v_3,\ v_1 + v_3,\ v_1 + v_2 + v_3]$ is a basis of V.

7. Given the follows subspaces of $\mathbb{R}^2$:
$W_1 = \text{span}\left[\begin{pmatrix}1\\1\end{pmatrix}\right]$, $W_2 = \text{span}\left[\begin{pmatrix}1\\-1\end{pmatrix}\right]$, $W_3 = \text{Span}\left[\begin{pmatrix}0\\1\end{pmatrix}\right]$, and $W_4 = \text{Span}\left[\begin{pmatrix}0\\-1\end{pmatrix}\right]$.
   (i) (4 points) What are the dimensions of $W_1 \cap W_2$ and $W_3 \cap W_4$?
   (ii) (6 points) What are the dimensions of $W_1 + W_2$, $W_3 + W_4$, and $W_1 + W_2 + W_3 + W_4$?

8. View the set of two-by-two matrices $M_2(\mathbb{R})$ as a four-dimensional vector space over $\mathbb{R}$. Consider the inner product

$$\psi\colon M_2(\mathbb{R}) \times M_2(\mathbb{R}) \longrightarrow \mathbb{R}$$

given by

$$(A, B) \longmapsto \mathrm{Tr}(AB^T).$$

Find the orthogonal complement (正交补) to the set

$$S = \left\{ P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, P_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}.$$

9. Let

$$V := \{ax + b \mid a, b \in \mathbb{R}\}.$$

Let $a, b, c, d, k \in \mathbb{R}$, and $\alpha = ax + b$ and $\beta = cx + d$, define the addition and scalar multiplication as follows:

$$\alpha + \beta := (a + c)x + (b + d), \qquad k\alpha := (ka)x + (kb).$$

Then V is a two-dimensional linear space over $\mathbb{R}$.

(1) Prove that $[e_1, e_2] := [1, x]$ is a basis of V.

(2) Define a bi-variate function (二元函数) on V:

$$(f, g) := \int_0^1 f(x)g(x)\mathrm{d}x, \quad f, g \in V.$$

Prove that (f, g) is an inner product on V.

(3) Apply the Gram-Schmidt process to $[e_1, e_2]$ to get an orthonormal basis (标准正交基) of V.

10. Let $\mathcal{F}$ be the set of lines in $\mathbb{R}^2$ passing through $(0,0)$:

$$\mathcal{F} = \{L \mid L \text{ is a 1-dimensional subspace of } \mathbb{R}^2\}.$$

Note that there is a unique line in $\mathcal{F}$ passing through $(0,1)$; we denote this unique line by L_∞. For each $a \in \mathbb{R}$, let L_a be the line given by

$$L_a := \{(x, ax) \mid x \in \mathbb{R}\}.$$

Clearly we have $\mathcal{F} = \{L_a\}_{a\in\mathbb{R}} \cup \{L_\infty\}$. Define the following addition and scalar product on the subset $\mathcal{F}_0 = \{L_a\}_{a\in\mathbb{R}}$: For every $a, b, \lambda \in \mathbb{R}$ we set

- $L_a + L_b := L_{a+b}$;
- $\lambda \cdot L_a := L_{\lambda a}$.

(i) Prove that $\mathcal{F}_0$ is a 1-dimensional space over $\mathbb{R}$ under the above operations.

(ii) Discuss whether there is a linear space structure on $\mathcal{F}$ making it contain $\mathcal{F}_0$ as a subspace under the above operations.