Slope of a parametric curve (Sec 10.2)

Slope of a tangent at a point (x, y) is defined as

$$\frac{dy}{dx}\bigg|_{x=x_0} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta t\to 0}\frac{\Delta y/\Delta t}{\Delta x/\Delta t} = \frac{\lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t}}{\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}\bigg|_{t=t_0}.$$

where $x(t_0) = x_0$.

参数 方程.

e.g. 1 Parametric equation of the circle $x^2+y^2=4$ are

$x = 2\cos t \quad y = 2\sin t.$

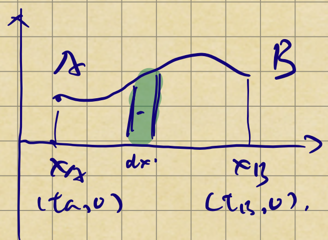
Find the slope of the tan to the curve at the point $(\sqrt{2}, \sqrt{2})$.

At $x=\sqrt{2}$, $\sqrt{2} = 2\cos t \quad t = \frac{\pi}{4}$

$y = \sqrt{2}$, $\sqrt{2} = 2\sin t \quad t = \frac{\pi}{4}$.

$$\frac{dy}{dx}\bigg|_{(\sqrt{2},\sqrt{2})} = \frac{\overset{(y')}{\frac{dy}{dt}}}{\underset{(x')}{\frac{dx}{dt}}}\Bigg|_{t=\frac{\pi}{4}} = \frac{2\cos t}{-2\sin t}\bigg|_{t=\frac{\pi}{4}} = -1.$$

Area of a parametric curve.



$dA = y \cdot dx$ where $x = f(t)$, $y = g(t)$, $t \in [t_A, t_B]$.

$\Downarrow \qquad \Downarrow$

$dx = f'(t)dt \quad dy = g'(t)dt.$

$$dA = |g(t)\cdot f'(t)dt|$$

$$A = \int dA = \int_{t_A}^{t_B} |g(t)f'(t)|\, dt.$$

e.g. $x^2+y^2=a^2$.

$x = a\cos t \qquad dx = -a\sin t\, dt$

$y = a\sin t$

$t \in [0, 2\pi]$.

$$A = \int_0 |a\sin t \cdot (-a\sin t)|\, dt$$

$$= a^2 \int_0^{2\pi} \sin^2 t \, dt.$$

$$= a^2/2 \int_0^{2\pi} (1-\cos 2t)\, dt$$

$$= a^2/2 \left[t - \tfrac{1}{2}\sin 2t\right]\Big|_0^{2\pi}$$

$$= a^2\pi$$

e.g. 2 $x = a\cos t$ $dx = -a\sin t\, dt$.

$y = b\sin t$.

$t \in [0, 2\pi]$.

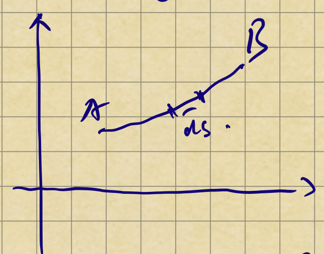
$$A = \int_0^{2\pi} |b\sin t \cdot (-a\sin t)|\, dt.$$

$$= |ab|/2 \int_0^{2\pi} (1-\cos 2t)\, dt.$$

$$= ab/2 \left[t - \tfrac{1}{2}\sin 2t\right]\Big|_0^{2\pi}$$

$$= ab/2 \cdot 2t = ab\pi.$$

Arc length of a parametric equation (Sec. 10.2).



$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$L = \int ds = \int_{t_a}^{t_b} \sqrt{(f'(t))^2 + (g'(t))^2}\, dt.$$

$x = f(t) \Rightarrow dx = f'(t)dt$.

$y = g(t) \Rightarrow dx = g'(t)dt$.

e.g. 3 $x = a\cos t$

$y = a\sin t$

$t \in [0, 2\pi)$

$$C = \int_0^{2\pi} \sqrt{(a^2\sin^2 t) + (a^2\cos^2 t)}\quad dt$$

$$= at\Big|_0^{2\pi}$$

$$= 2\pi a.$$

e.g. 4. $x = a\cos t$ $C = \int_0^{2\pi} \sqrt{(a^2\sin^2 t) + (b^2\cos^2 t)}\, dt.$

$y = b\sin t$.

$$= 4\int_0^{\pi/2} \sqrt{(a^2\sin^2 t) + (b^2\cos^2 t)}\, dt.$$

$$= 4\int_0^{\pi/2} \sqrt{(a^2\sin^2 t) + b^2(1-\sin^2 t)}\, dt$$

$$= 4b\int_0^{\pi/2} \sqrt{1 + \left(\frac{a^2-b^2}{b^2}\right)\sin^2 t}\; dt.$$

$$= 4b\int_0^{\pi/4} \sqrt{1 + e^2 \sin^2 t}\; dt.$$

How to obtain $\frac{d^2y}{dx^2}$ by using paramatric equation of a curve C defined by $\begin{cases} x = f(t) \\ y = g(t). \end{cases}$?

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\frac{dt}{dx}$$

$$= \frac{1}{\frac{dx}{dt}} \frac{d}{dt}\left(\frac{dy/dt}{dx/dt}\right)$$

$$= \frac{1}{dx/dt} \cdot \frac{\frac{d^2y}{dx^2}\cdot\frac{dx}{dt} - \frac{dy}{dt}\cdot\frac{d^2y}{dt^2}}{(dx/dt)^2}$$

$$\frac{d^2y}{dx} = \frac{\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{dy}{dt}}{(dx/dt)^3} = \frac{g''(t)f'(t) - f''(t)g'(t)}{(f'(t))^3}.$$

$\uparrow$

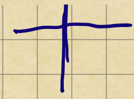
determine the concavity of a curve

sketch.

e.g. $\begin{cases} f(t) = t^3 - 3t \\ g(t) = t^2 \end{cases}$ $t \in [-2, 2]$.

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} = \frac{2t}{3t^2 - 3}$$

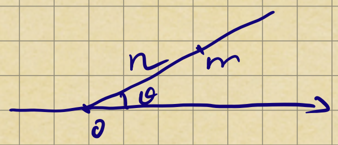
dy $\uparrow$

for horizontal tangents $\Rightarrow \frac{dy}{dt}=0 \quad t=0$ 

vertical $\frac{dx}{dt}=0 \quad t=\pm 1$.

极坐标.

Polar Coordinat (Sec 10.3).



Consider: an x-axis with a reference point O

then any point m in the plane A can be determined by

two variables: angle θ, $\theta \in [0, \pi)$ length L ($L \in [0, +\infty)$.

e.g. $(1, 0)$, $(2, \pi/2)$, $(3, \pi/6)$.