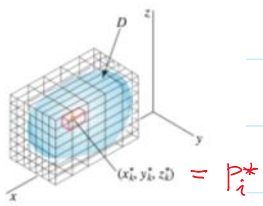
Lecture 16

October 16, 2020 11:11 AM



Consider an object D in 3-d space. Let $F(x,y,z)$ be the mass density of D. We want to compute the mass of D.

Dividing D into n- subregions ΔD_i's $(i=1,\dots,n)$ of volume $\Delta V_i = \Delta x \Delta y \Delta z$.

Let P_i^* be an arbitrary point in ΔD_i then the corresponding mass Δm_i of ΔD_i is

$$\Delta m_i = F(P_i^*)\,\Delta V_i = F(x_i^*, y_i^*, z_i^*)\,\Delta V_i$$

then form the Riemann sum

$$\sum_{i=1}^{n} \Delta m_i = \sum_{i=1}^{n} F(x_i^*, y_i^*, z_i^*)\,\Delta V_i$$

which is an approximation to the mass m of D. Let $n \to \infty$ in such a way that every ΔD_i is shrinking to a point, then if the above sum has a limit (which is the mass of D), then this limit is called a triple integral of F over D. In notation, we write

$$\iiint_D F(x,y,z)\,dV = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*, y_i^*, z_i^*)\,\Delta V_i \qquad (1)$$

In particular, if D is a rectangular box (rectangular prism) is defined by $a \le x \le b$, $c \le y \le d$, $e \le z \le f$ then Fubini's theorem says

$$\iiint_D F(x,y,z)\, dV = \int_a^b \int_c^d \int_e^f F(x,y,z)\, dz\, dy\, dx \qquad (2)$$

In addition, if F is seperable, ie,

$$F(x,y,z) = g(x)\, h(y)\, k(z)$$

then (2) becomes

$$\iiint_D g(x)\, h(y)\, k(z)\, dV = \left(\int_a^b g(x)\, dx\right)\left(\int_c^d h(y)\, dy\right)\left(\int_e^f k(z)\, dz\right) \qquad (3)$$

Ex 1: Evaluate $I = \iiint_D xyz^2\, dV$ where D is defined by

$0 \leq x \leq 1, \; -1 \leq y \leq 2, \; 0 \leq z \leq 3.$

Solution:

$$F(x,y,z) = xyz^2 = g(x)\, h(y)\, k(z)$$

where $g(x) = x$, $h(y) = y$, $k(z) = z^2$

Applying (3),

$$\iiint_D xyz^2\, dV = \left(\int_0^1 x\, dx\right)\left(\int_{-1}^2 y\, dy\right)\left(\int_0^3 z^2\, dz\right)$$

$$= \left(\frac{x^2}{2}\right)\Big|_0^1 \left(\frac{y^2}{2}\right)\Big|_{-1}^2 \left(\frac{z^3}{3}\right)\Big|_0^3$$

$$= \frac{1}{(2)(2)(3)}(1^2 - 0)\left((2)^2 - (-1)^2\right)(3^3 - 0)$$

$$= \frac{1}{(2)(2)\cancel{(3)}}(1)\cancel{(3)}(3^3) = \frac{27}{4} \quad // \text{Ans.}$$

The general case where D is NOT a rectangular box.

If a region D is defined by

$a \leq x \leq b, \; y_1(x) \leq y \leq y_2(x), \; z_1(x,y) \leq z \leq z_2(x,y)$

then Fubini's theorem states

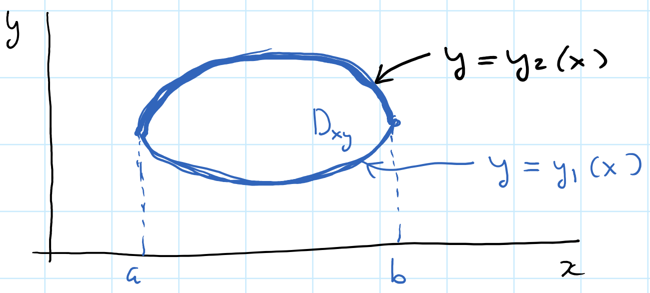
$$\iiint_D F(x,y,z)\, dV = \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} F(x,y,z)\, dz\, dy\, dx \qquad (\ast)$$

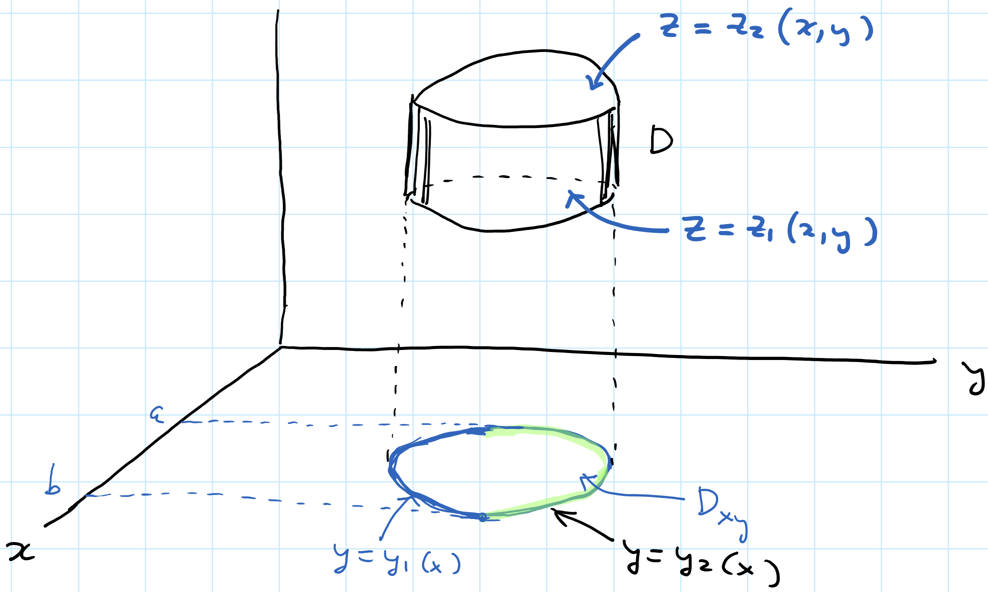
We can rewrite (★) as

$$\iiint_D F(x,y,z)\,dV = \iint_{D_{xy}} \left(\int_{z_1(x,y)}^{z_2(x,y)} F(x,y,z)\,dz \right) dA \qquad (A)$$

where D_{xy} is the projection of D onto the xy-plane. D_{xy} can be defined as

$$D_{xy} = \{ (x,y) \in \text{the xy-plane} \mid a \le x \le b,\ y_1(x) \le y \le y_2(x) \}$$



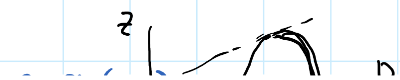


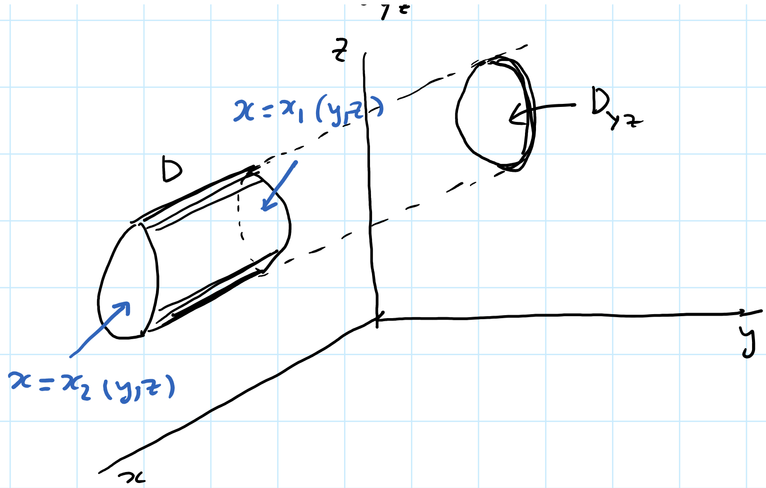
In this case, D is called a type I region.
In type II region, D is defined as

$$D = \{ (x,y,z) \mid (y,z) \in D_{yz},\ x_1(y,z) \le x \le x_2(y,z) \}$$

then Fubini's theorem states

$$\iiint_D F(x,y,z)\,dV = \iint_{D_{yz}} \left(\int_{x_1(y,z)}^{x_2(y,z)} F(x,y,z)\,dx \right) dA \qquad (B)$$





type II region

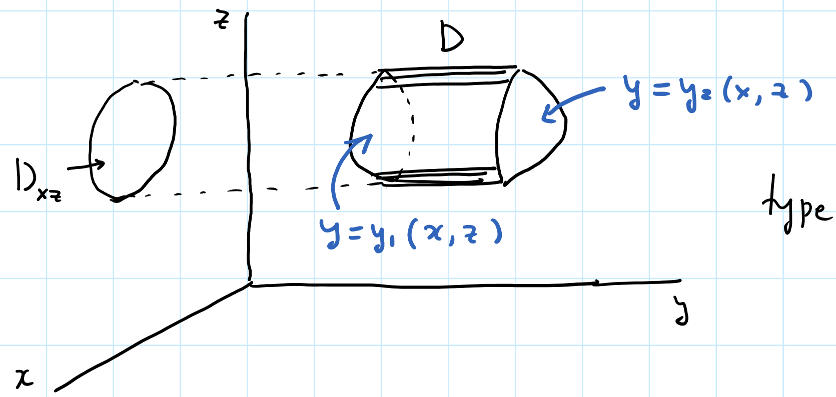
In type III region, D is defined by

$$D = \{ (x,y,z) \mid (x,z) \in D_{xz} \,,\ y_1(x,z) \le y \le y_2(x,z) \}$$

Then Fubini's theorem becomes

$$\iiint_D F(x,y,z)\, dV = \iint_{D_{xz}} \left(\int_{y_1(x,z)}^{y_2(x,z)} F(x,y,z)\, dy \right) dA \qquad (C)$$

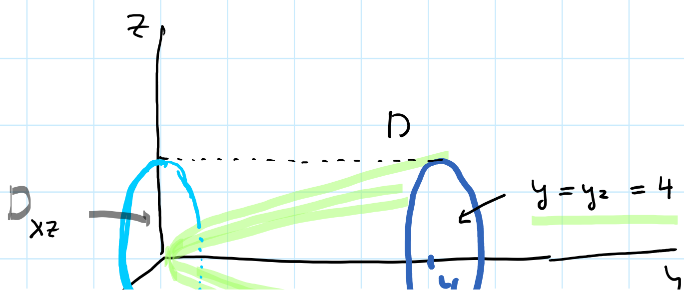
where D_{xz} is the projection of D onto the xz-plane.

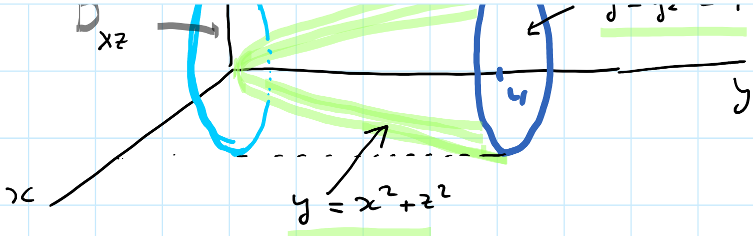


type III

<u>Ex1</u>: Evaluate $I = \iiint_D \sqrt{x^2+z^2}\, dV$ where D is the region bounded by the paraboloid $y = x^2+z^2$ and the plane $y = 4$.

<u>Solution</u>





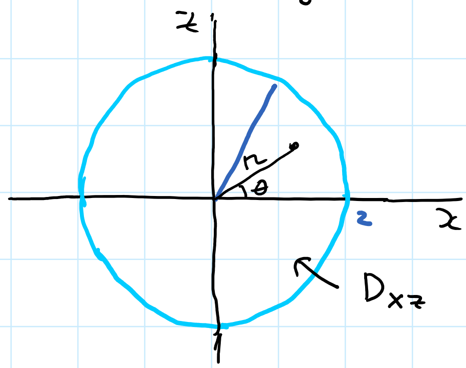
D is a type III region so we integrate wrt y first.

$$\iiint_D \sqrt{x^2+z^2}\, dV = \iint_{D_{xz}} \left(\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy \right) dA$$

$$= \iint_{D_{xz}} \sqrt{x^2+z^2}\, \left(4 - (x^2+z^2)\right) dA$$

Next, we want to determine D_{xz} which is the projection of the curve of intersection between $y = x^2+z^2$ and $y = 4$.

$$x^2 + z^2 = 4$$



$x = r\cos\theta$

$z = r\sin\theta$

$x^2+z^2 = r^2$

$$I = \int_0^{2\pi} \int_0^{2} \boxed{r\,(4 - r^2)\; r}\, dr\, d\theta$$

↑ polar coordinates

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{2} (4r^2 - r^4)\, dr\right)$$

$$= (2\pi)\left(\frac{4r^3}{3} - \frac{r^5}{5}\right)\Bigg|_0^2$$

$$= (2\pi)\left(\frac{4}{3}(2)^3 - \frac{(2)^5}{5}\right) = \frac{128\pi}{15} \quad // \text{Ans.}$$

See you on Monday!