THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

MATHEMATICS 1600B MIDTERM EXAMINATION — SOLUTIONS
7 February 2019 7:00–8:30 PM

1. For each of the following statements, circle **T** if the statement is always true and **F** if it can be false. Give a one-sentence justification for your answer.

   (a) (2 marks) If two lines are parallel, then they do not intersect.

   Solution: False. Two *distinct* parallel lines do not intersect, but a line is parallel to itself and we do not know the lines are distinct. (For example, $x+y=1$ and $2x+2y=2$ are parallel and intersect.)

   (b) (2 marks) If a homogeneous linear system over $\mathbb{R}$ has a non-zero solution, then it has infinitely many solutions.

   Solution: True. A homogeneous linear system always has the trivial solution where all variables are 0, so it either has exactly the trivial solution or infinitely many solutions. If there is a non-zero solution, then there is more than one solution, so there are infinitely many solutions.

   (c) (2 marks) The rank of a matrix is an integer > 0.

   Solution: False. The rank is the number of leading entries in an echelon form of the matrix so is an integer ≥ 0. However, the zero matrix has no leading entries, so has rank 0 not > 0.

   (d) (2 marks) If $\mathbf{u}, \mathbf{v} \in (\mathbb{Z}_{10})^{12}$ represent valid UPC codes, then so does $\mathbf{u} - \mathbf{v}$.

   Solution: True. The valid UPC codes correspond to the vectors $\mathbf{x} \in (\mathbb{Z}_{10})^{12}$ satisfying $\mathbf{x} \cdot \mathbf{c} = 0$ where $\mathbf{c}$ is the weight vector of the code. In particular, if $\mathbf{u} \cdot \mathbf{c} = 0$ and $\mathbf{v} \cdot \mathbf{c}$, then
   $$(\mathbf{u} - \mathbf{v}) \cdot \mathbf{c} = \mathbf{u} \cdot \mathbf{c} - \mathbf{v} \cdot \mathbf{c} = 0 - 0 = 0,$$
   so $\mathbf{u} - \mathbf{v}$ represents a valid code vector.

2. Let $\mathbf{v} = [\sqrt{2}, 0, -1]$ and $\mathbf{w} = [0, \sqrt{2}, 3]$.

(a) (2 marks) Find a non-zero vector $\mathbf{x}$ which is orthogonal to $\mathbf{v}$ and $\mathbf{w}$.

Solution: We must find $\mathbf{x}$ satisfying $\mathbf{v} \cdot \mathbf{x} = 0$ and $\mathbf{w} \cdot \mathbf{x} = 0$.

First Approach: Write $\mathbf{x} = [x, y, z]$ and

$$\mathbf{v} \cdot \mathbf{x} = \sqrt{2}x + 0y - z, \quad \mathbf{w} \cdot \mathbf{x} = 0x + \sqrt{2}y + 3z.$$

Thus we must solve the homogeneous system of equations

$$\begin{array}{rcrcrcl} \sqrt{2}x & & & - & z & = & 0 \\ & & \sqrt{2}y & + & 3z & = & 0. \end{array}$$

The first equation gives $z = \sqrt{2}x$ and the second gives $y = -\frac{3}{\sqrt{2}}z = -3x$, so

$$\mathbf{x} = [x, -3x, \sqrt{2}x] = [1, -3, \sqrt{2}]x$$

is the general solution, and $[1, -3, \sqrt{2}]$ is a specific solution.

Second Approach: Calculate $\mathbf{v} \times \mathbf{w}$. Compare 4a.

(b) (3 marks) Write $\mathbf{w}$ as a sum $\mathbf{w} = \mathbf{y} + \mathbf{z}$ of a vector $\mathbf{y}$ parallel to $\mathbf{v}$ and a vector $\mathbf{z}$ perpendicular to $\mathbf{v}$.

Solution: Let $\mathbf{y} = \text{proj}_{\mathbf{v}}(\mathbf{w})$ be the projection of $\mathbf{w}$ onto $\mathbf{v}$. By definition,

$$\mathbf{y} = \frac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \frac{0 + 0 - 3}{2 + 0 + 1}[\sqrt{2}, 0, -1] = [-\sqrt{2}, 0, 1]$$

and is parallel to $\mathbf{v}$. Moreover, the difference

$$\mathbf{z} = \mathbf{w} - \mathbf{y} = [\sqrt{2}, \sqrt{2}, 2]$$

is perpendicular to $\mathbf{v}$ (since $\mathbf{v} \cdot \mathbf{z} = 2 + 0 - 2 = 0$) and $\mathbf{w} = \mathbf{y} + \mathbf{z}$.

(c) (2 marks) Find a unit vector $\mathbf{u}$ pointing in the opposite direction as $\mathbf{v} + \mathbf{w}$.

Solution: We must scale $\mathbf{v} + \mathbf{w} = [\sqrt{2}, \sqrt{2}, 2]$ by *minus* its length

$$\|\mathbf{v} + \mathbf{w}\| = \sqrt{(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w})} = \sqrt{2 + 2 + 4} = \sqrt{8}$$

to ensure it points in the *opposite* direction. Thus

$$\mathbf{u} = -\frac{1}{\|\mathbf{v} + \mathbf{w}\|}(\mathbf{v} + \mathbf{w}) = -\frac{1}{\sqrt{8}}[\sqrt{2}, \sqrt{2}, 2] = [-1/2, -1/2, -1/\sqrt{2}]$$

is the desired vector.

3. Consider the point $P = (1, -1)$ and vector $\mathbf{v} = [1, 3]$ in the Cartesian plane $\mathbb{R}^2$.

(a) (2 marks) Give an equation for the line ℓ_1 through P and parallel to $\mathbf{v}$.

Solution: The vector $\mathbf{n} = [-3, 1]$ is perpendicular to $\mathbf{v}$ and not $\mathbf{0}$, so it is a normal vector of ℓ_1. This means ℓ_1 has an equation of the form $-3x + y = c$ for a suitable c. We find $c = -3 - 1 = -4$ by substituting $(x, y) = P = (1, -1)$ into $-3x + y = c$, hence ℓ_1 is $y = 3x - 4$.

(b) (2 marks) Give an equation for the line ℓ_2 through P and perpendicular to ℓ_1.

Solution: Now $\mathbf{v} = [1, 3]$ is the desired normal vector, so ℓ_2 has the form $x + 3y = d$ for a suitable d. If we take $(x, y) = P = (1, -1)$, then we find $d = 1 - 3 = -2$ and ℓ_2 is $y = -\frac{1}{3}x - \frac{2}{3}$.

(c) (2 marks) Determine all points in the intersection $\ell_1 \cap \ell_2$ of the lines. Explain.

Solution: By definition, the lines ℓ_1 and ℓ_2 are not parallel since they are perpendicular, hence they intersect in exactly one point. Moreover, they both contain $P = (1, -1)$ by definition, so P is the only point in the intersection.

4. Let $\mathcal{P}$ be the plane in $\mathbb{R}^3$ given by the parametric equations

$$\begin{array}{rcrcrcr} x & = & 1 & + & s & & \\ y & = & 2 & & & + & t \\ z & = & 3 & - & s & - & t \end{array}$$

(a) (2 marks) Find a normal vector $\mathbf{n}$ to the plane $\mathcal{P}$.

Solution: Let $\mathbf{u} = [1, 0, -1]$ and $\mathbf{v} = [0, 1, -1]$ so that $\mathcal{P}$ is given by the parametric equation

$$[x, y, z] = [1, 2, 3] + s\mathbf{u} + t\mathbf{v}.$$

We must find a vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. Since $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^3$ and not parallel, the cross product

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = [0 + 1, 0 - (-1), 1 - 0] = [1, 1, 1]$$

satisfies $\mathbf{n} \cdot \mathbf{u} = 1 + 0 - 1 = 0$ and $\mathbf{n} \cdot \mathbf{v} = 0 + 1 - 1 = 0$ as desired.

(b) (3 marks) Compute the distance from $\mathcal{P}$ to the origin.

Solution: If $\mathbf{w} = [1, 2, 3]$ is the vector from the origin to the point $(1, 2, 3)$, then it suffices to compute the length of

$$\operatorname{proj}_{\mathbf{n}}(\mathbf{w}) = \frac{\mathbf{n} \cdot \mathbf{w}}{\mathbf{n} \cdot \mathbf{n}}\,\mathbf{n} = \frac{1+2+3}{1+1+1}\,[1, 1, 1] = [2, 2, 2].$$

Therefore

$$\left\|\operatorname{proj}_{\mathbf{n}}(\mathbf{w})\right\| = \sqrt{4+4+4} = 2\sqrt{3}$$

is the distance from $\mathcal{P}$ to the origin.

(c) (2 marks) Find a general equation for the plane $\mathcal{Q}$ through the origin and parallel to $\mathcal{P}$.

Solution: The equation of any plane parallel to $\mathcal{P}$ is of the form $ax + by + cz = d$ for suitable a, b, c, d. We can take $[a, b, c] = \mathbf{n} = [1, 1, 1]$, and then substituting $[x, y, z] = [0, 0, 0]$ gives $d = 0 + 0 + 0 = 0$. Thus $x + y + z = 0$ is the desired equation.

5. Consider the following 3×2 system of linear equations over $\mathbb{R}$:

$$\begin{array}{rcrcl} 2x & + & y & = & 1 \\ -3x & + & y & = & 1 \\ 4x & - & y & = & k \end{array}$$

Here k is a real constant.

(a) (1 mark) Write down the augmented matrix of this linear system.

Solution:

$$\left(\begin{array}{rr|r} 2 & 1 & 1 \\ -3 & 1 & 1 \\ 4 & -1 & k \end{array}\right)$$

(b) (3 marks) Compute the reduced row-echelon form of the augmented matrix and indicate all elementary row operations that you are performing.

Solution:

$$\left(\begin{array}{rr|r} 2 & 1 & 1 \\ -3 & 1 & 1 \\ 4 & -1 & k \end{array}\right) \sim \left(\begin{array}{rr|r} -1 & 2 & 2 \\ -3 & 1 & 1 \\ 4 & -1 & k \end{array}\right) \qquad R_1 \mapsto R_1 + R_2$$

$$\sim \left(\begin{array}{rr|r} -1 & 2 & 2 \\ 0 & -5 & -5 \\ 0 & 7 & 8+k \end{array}\right) \qquad R_2 \mapsto R_2 - 3R_1 \text{ and } R_3 \mapsto R_3 + 4R_1$$

$$\sim \left(\begin{array}{rr|r} 1 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & 7 & 8+k \end{array}\right) \qquad R_1 \mapsto -R_1 \text{ and } R_2 \mapsto -\frac{1}{5}R_2$$

$$\sim \left(\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1+k \end{array}\right) \qquad R_1 \mapsto R_1 + 2R_2 \text{ and } R_3 \mapsto R_3 - 7R_2$$

(c) (2 marks) For which values of k does the system have a solution? Explain.

Solution: The system is inconsistent if and only if $1 + k$ is a leading entry, that is, $1 + k \neq 0$, so $k = -1$ is the unique value where the system has a solution.

6. Consider the following system of linear equations over $\mathbb{Z}_5$:

$$\begin{array}{rcrcrcrcl} 2x & + & y & + & 2z & + & w & = & 3 \\ & & 3y & + & z & + & w & = & 3 \\ & & & & & & 4w & = & 2 \end{array}$$

(Read again the last instruction on the front page!)

(a) (2 marks) Show that $[x, y, z, w] = [0, 1, 2, 3]$ is a solution.

Solution: We must show that $[0, 1, 2, 3]$ satisfies all three equations. Indeed

$$2 \cdot 0 + 1 + 2 \cdot 2 + 3 = 0 + 1 + 4 + 3 = 8 \equiv 3 \bmod 5$$
$$3 \cdot 1 + 2 + 3 = 3 + 2 + 3 = 8 \equiv 3 \bmod 5$$
$$4 \cdot 3 = 12 \equiv 2 \bmod 5$$

as desired.

(b) (3 marks) Find all solutions.

Solution: Recall that the addition, subtraction, and multiplication tables for $\mathbb{Z}_5$ are given by:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

−	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

×	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

We form an augmented matrix and put the system in RREF:

$$\left(\begin{array}{cccc|c} 2 & 1 & 2 & 1 & 3 \\ & 3 & 1 & 1 & 3 \\ & & & 4 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 3 & 1 & 3 & 4 \\ & 1 & 2 & 2 & 1 \\ & & & 1 & 3 \end{array}\right) \qquad R_1 \mapsto 3R_1,\ R_2 \mapsto 2R_2,\ R_3 \mapsto 4R_3$$

$$\sim \left(\begin{array}{cccc|c} 1 & 3 & 1 & 0 & 0 \\ & 1 & 2 & 0 & 0 \\ & & & 1 & 3 \end{array}\right) \qquad R_1 \mapsto R_1 - 3R_3,\ R_2 \mapsto R_2 - 2R_3$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ & 1 & 2 & 0 & 0 \\ & & & 1 & 3 \end{array}\right) \qquad R_1 \mapsto R_1 - 3R_2$$

Therefore the solutions are given by

$$x = 0, \quad y = -2z = 3z, \quad z = \text{free}, \quad w = 3.$$

7. (3 marks) Do the following points line in a common plane? Explain.

$$P = (1,1,0), \quad Q = (1,0,1), \quad R = (0,1,1), \quad T = (2,2,2).$$

Solution: We consider the three vectors

$$\mathbf{u} = \overrightarrow{PQ} = (0,-1,1), \qquad \mathbf{v} = \overrightarrow{PR} = (-1,0,1), \qquad \mathbf{w} = \overrightarrow{PT} = (1,1,2).$$

We note that $\mathbf{u}$ and $\mathbf{v}$ are not parallel. Hence T lies in the plane spanned by P, Q and R if and only if $\mathbf{w}$ is orthogonal to a normal vector $\mathbf{n}$ of that plane. We take

$$\mathbf{n} = \mathbf{v} \times \mathbf{u} = (1,1,1),$$

so that

$$\mathbf{n} \cdot \mathbf{w} = 4 \neq 0.$$

This shows that the given points do not lie in a common plane.

8. (3 marks) Why are there no valid ISBN-10 codes with exactly one non-zero digit?

Solution: Assume that $\mathbf{x}$ is a code with only one non-zero digit, say at position k. Then $\mathbf{x} = a\,\mathbf{e}_k$ for some scalar $0 \neq a \in \mathbb{Z}_{11}$. The code is valid if and only if

$$0 = \mathbf{x} \cdot \mathbf{c} = a\,\mathbf{e}_k \cdot \mathbf{c} = a\,c_k$$

where c_k is the k-th component of $\mathbf{c}$. By the definition of $\mathbf{c}$, we have $c_k \neq 0$, hence the product with $a \neq 0$ is non-zero. In other words, the code $\mathbf{x}$ is not valid.