The Tree ADT

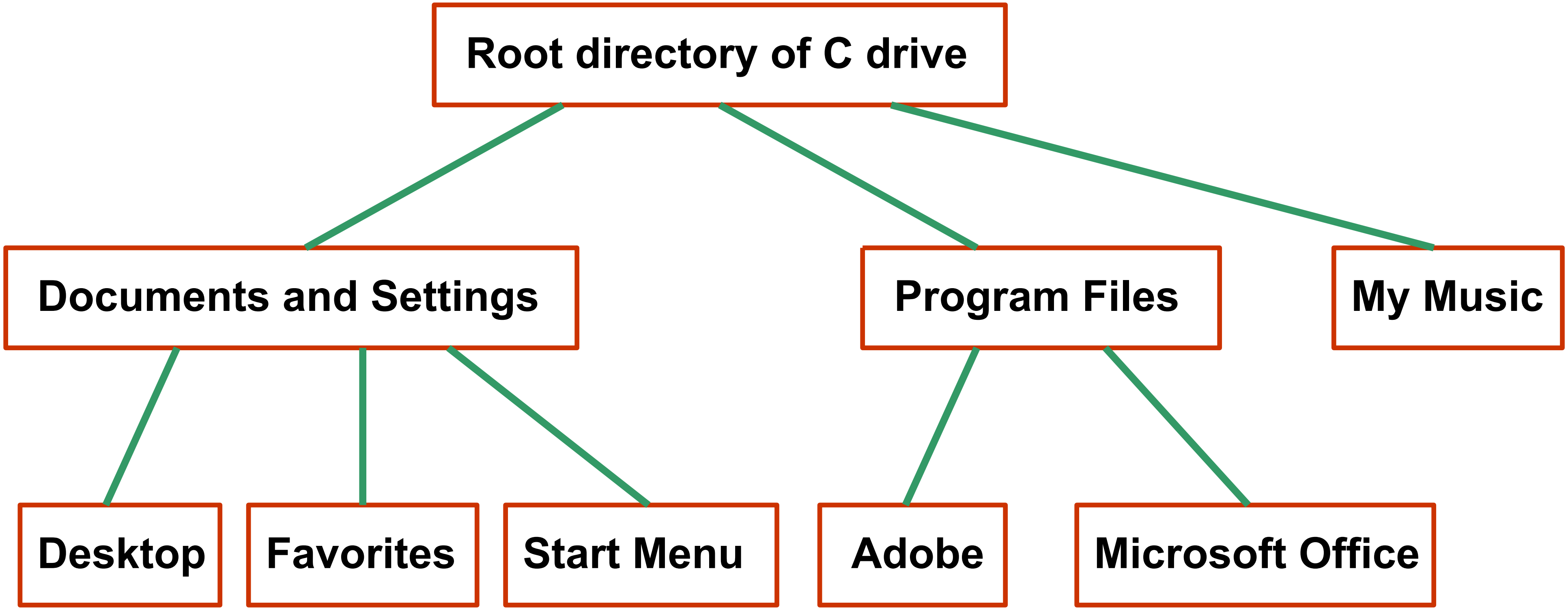
Objectives

- **Define trees as data structures**
- **Define the terms associated with trees**
- **Perform tree traversal algorithms**
- **Discuss a binary tree implementation**
- **Examine binary trees used for mathematical expressions**

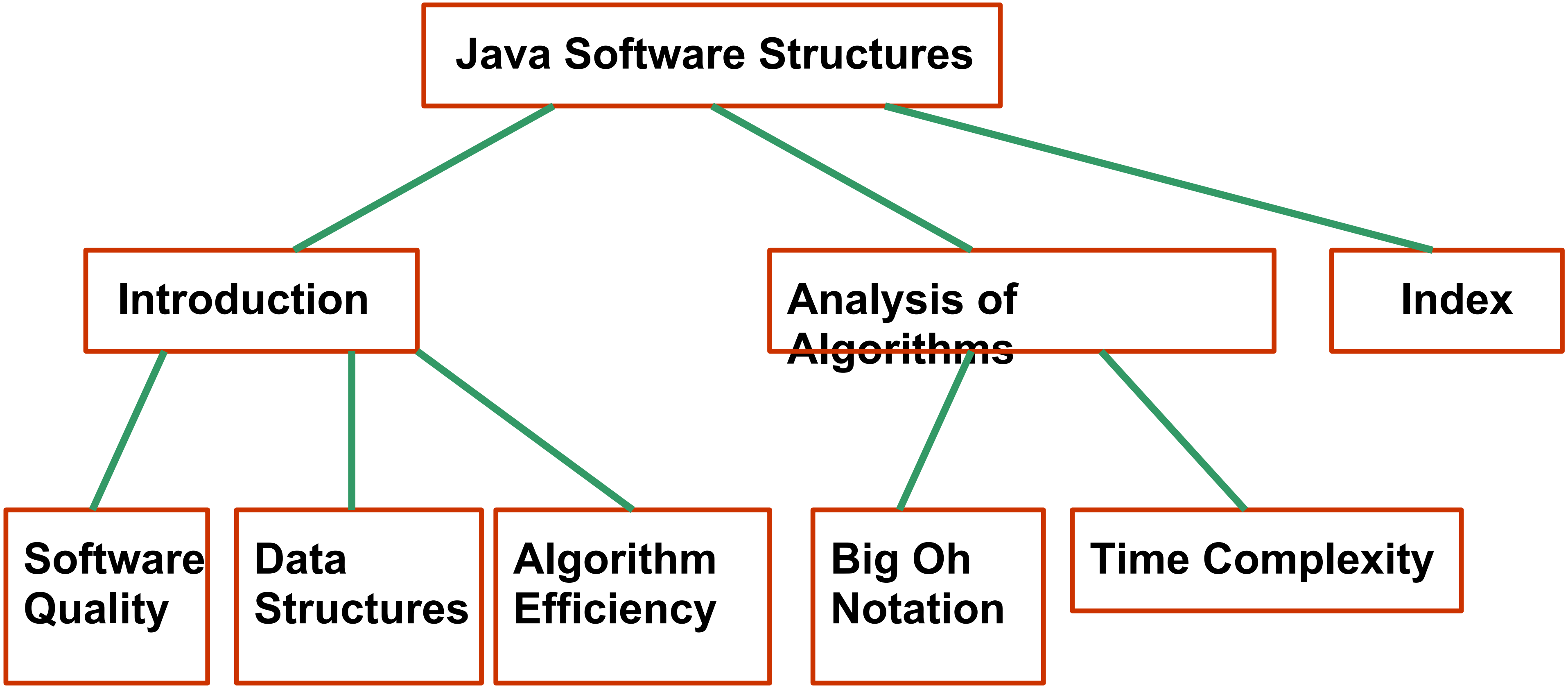
Trees

- A ***tree*** is a ***non-linear*** abstract data type that stores elements in a hierarchy.
- ***Examples*** in real life:
  - Family tree
  - Table of contents of a book
  - Class inheritance hierarchy in Java
  - Computer file system (folders and subfolders)
  - Decision trees

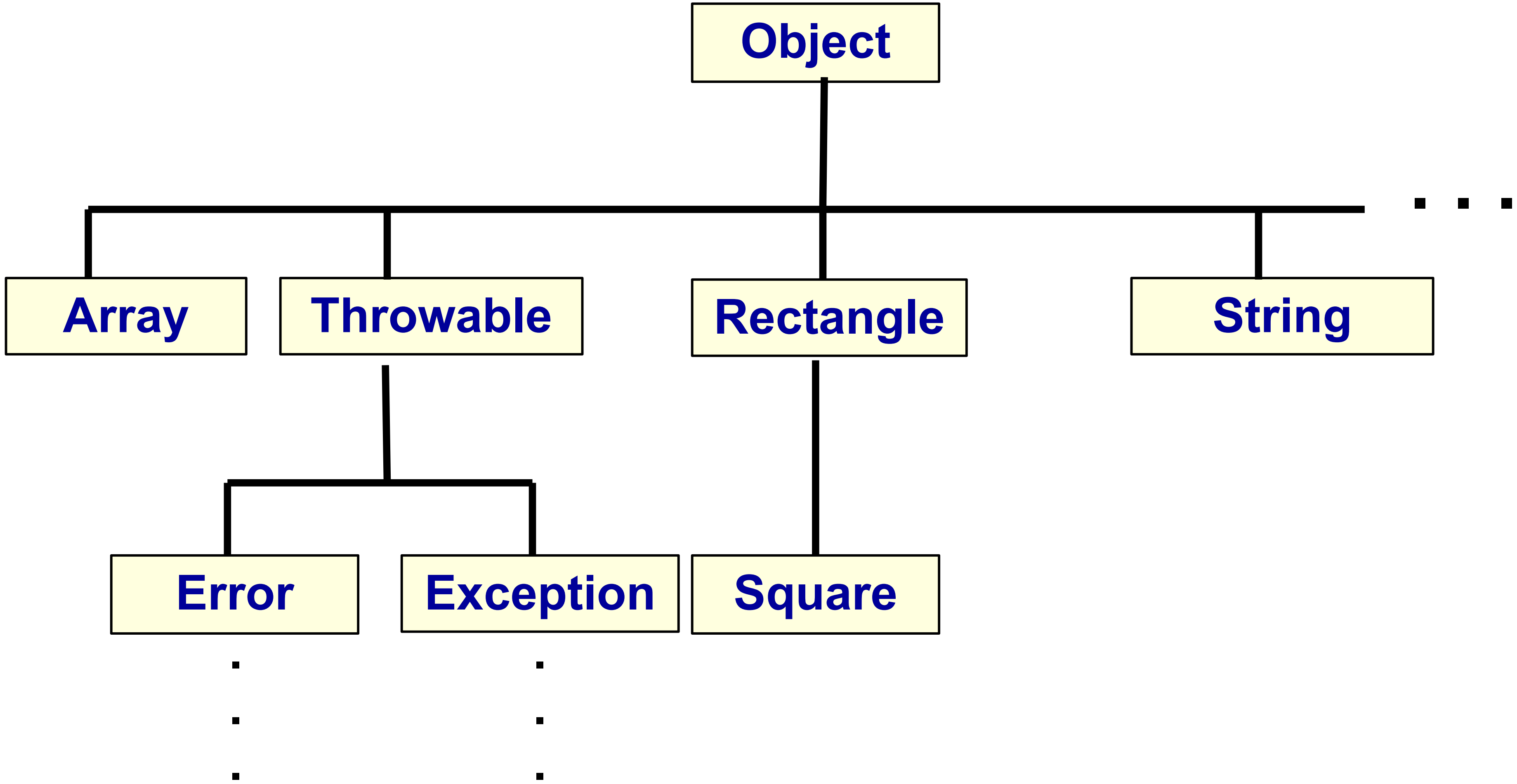
Example: Computer File System



Example: Table of Contents



Example: Java's Class Hierarchy



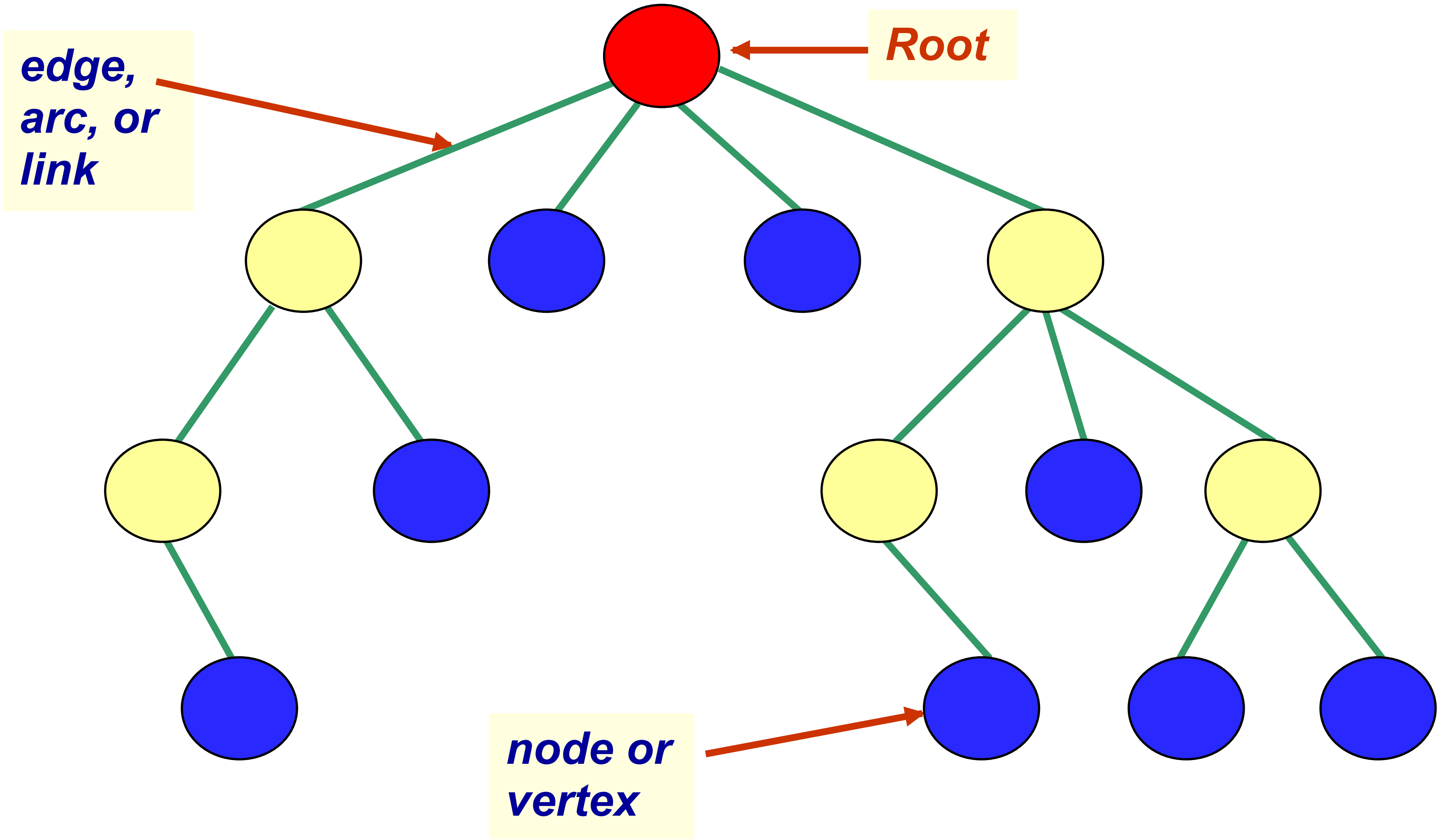
Tree Definition

- ***Tree:*** **a set of elements that either**
  - **it is empty**
  - **or, it has a distinguished element called the *root* and zero or more *trees* *(called subtrees of the root)***
- **What kind of definition is this?**
  - **What is the base case?**
  - **What is the recursive part?**

Tree Terminology

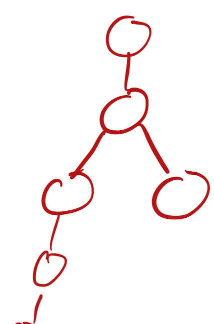
- ***Nodes***: the elements in the tree
- ***Edges***: connections between nodes
- ***Root***: *the* distinguished element that is the origin of the tree
  - There is only one root node in a tree
- ***Empty tree***: has no nodes and no edges

Tree Terminology



Tree Terminology

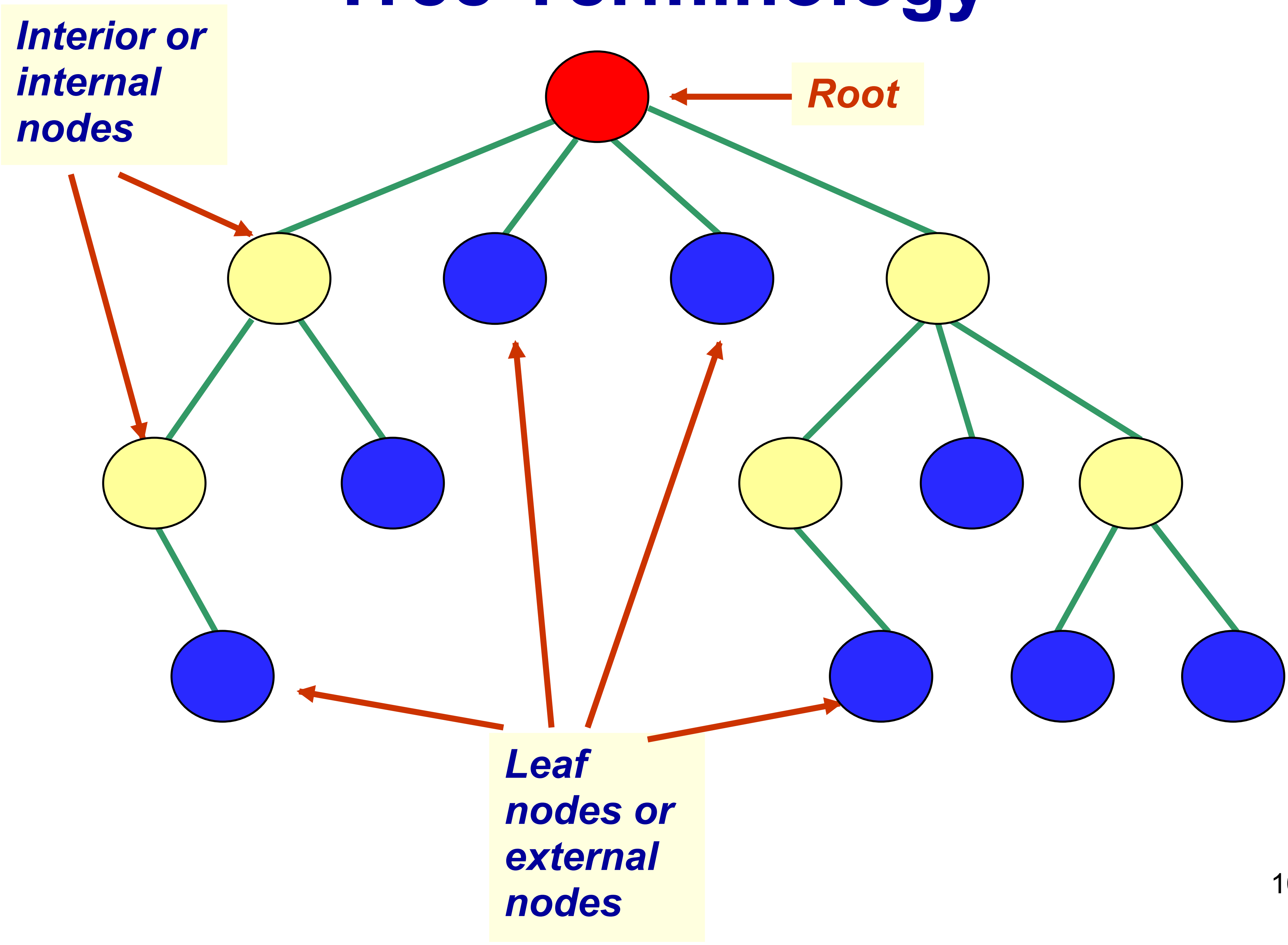
- ***Parent*** **or** ***predecessor*****: the node directly above another node in the hierarchy**
  - **A node can have only one parent**
- ***Child*** **or** ***successor*****: a node directly below another node in the hierarchy**
- ***Siblings*****: nodes that have the same parent**
- ***Ancestors*** **of a node: its parent, the parent of its parent, etc.**
- ***Descendants*** **of a node: its children, the children of its children, etc.**



Tree Terminology

- ***Leaf node***: a node without children
- ***Internal node***: a node that is not a leaf node

Tree Terminology



Discussion

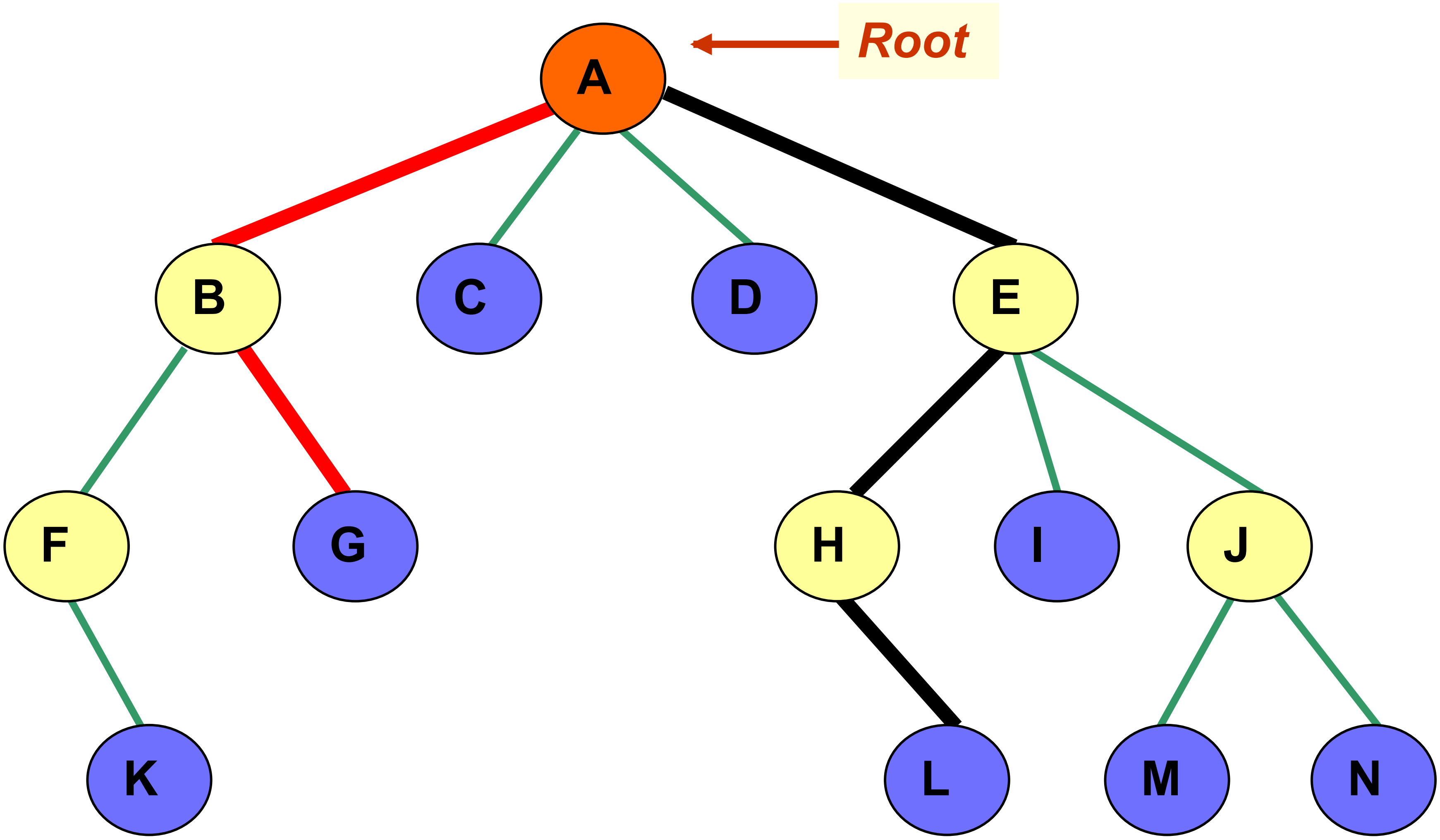
- Does a leaf node have any children? X
- Does the root node have a parent? X
- How many parents does every node (other than the root node) have?

1 only

Height of a Tree

- A *path* is a sequence of edges leading from one node to another
- *Length of a path*: number of edges on the path
- *Height of a (non-empty) tree*: length of the longest path from the root to a leaf
  - What is the height of a tree that has only a root node? 0 => no path
  - By convention, the *height of an empty tree* is -1

Tree Terminology



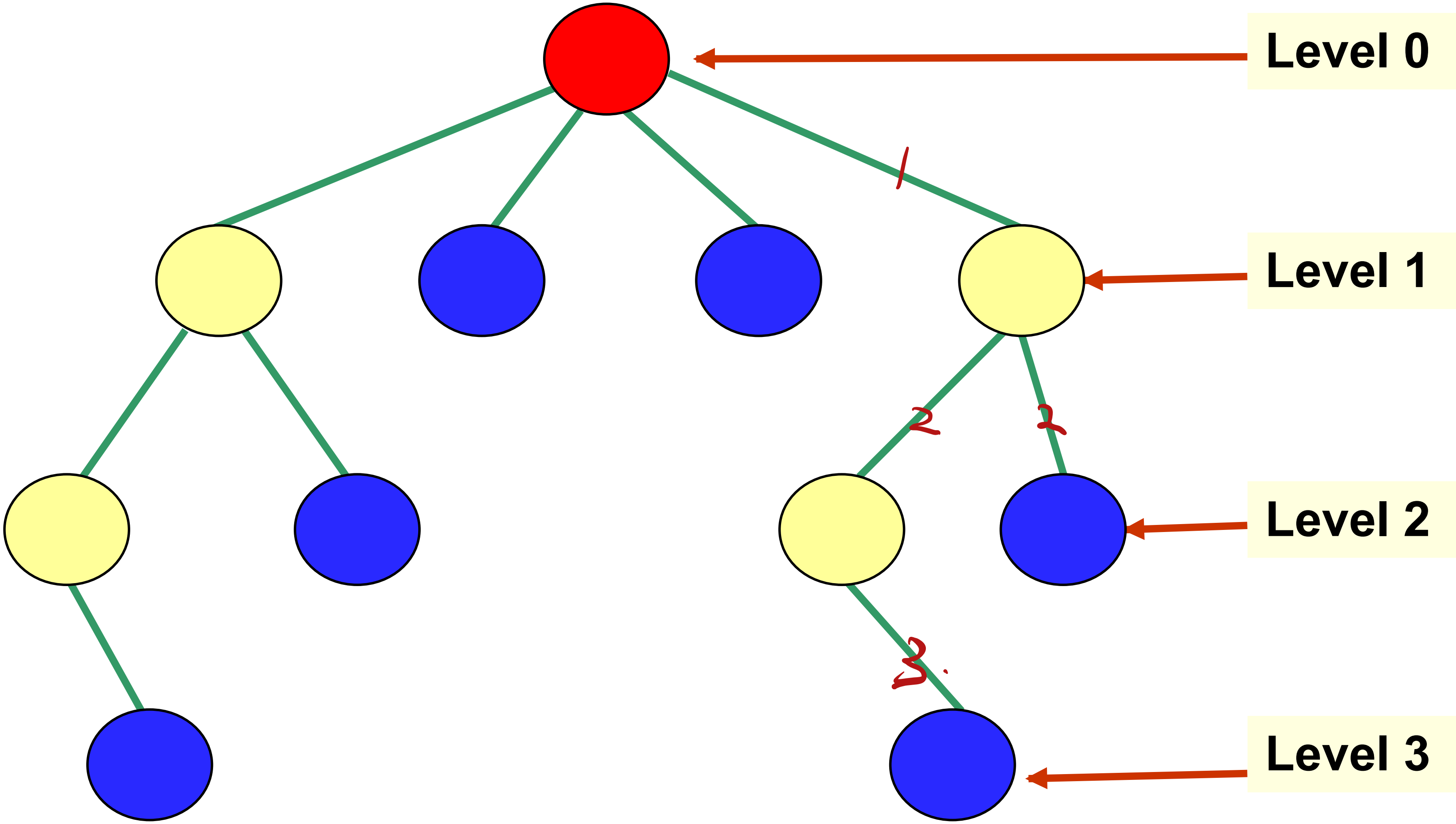
Height = 3

Level of a Node

- ***Level of a node:*** **number of *edges* between root and the node**
- **It can be defined *recursively*:**
  - **Level of root node is *0***
  - **Level of a node that is not the root node is *level of its parent + 1***
- ***Question: What is the height of a tree in terms of levels?***

the max level of all nodes.

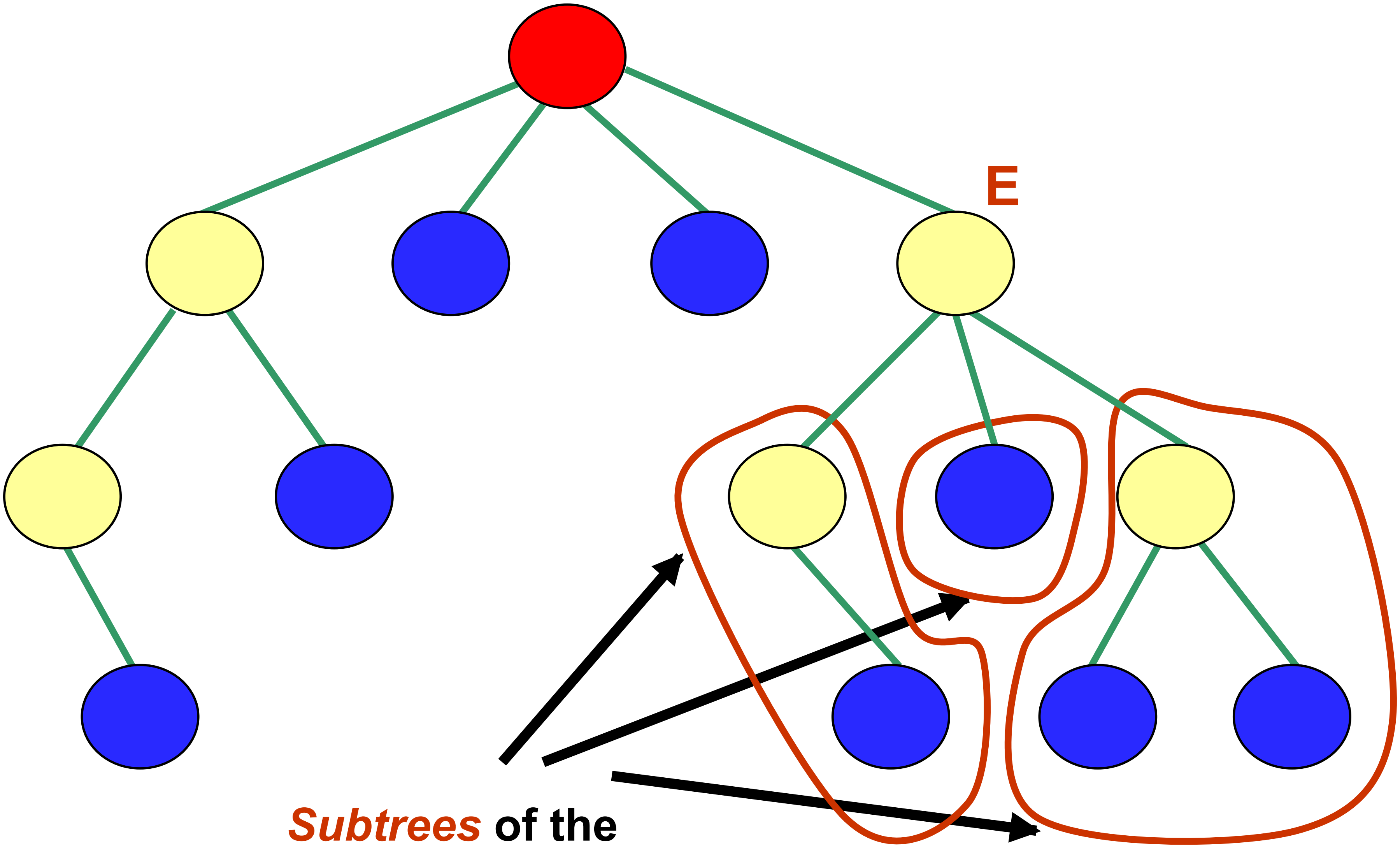
Level of a Node



Subtrees

- ***Subtree*** **of a node: consists of a child node and all its descendants**
  - **A subtree is itself a tree**
  - **A node may have many subtrees**

Subtrees



Subtrees **of the node labeled E**

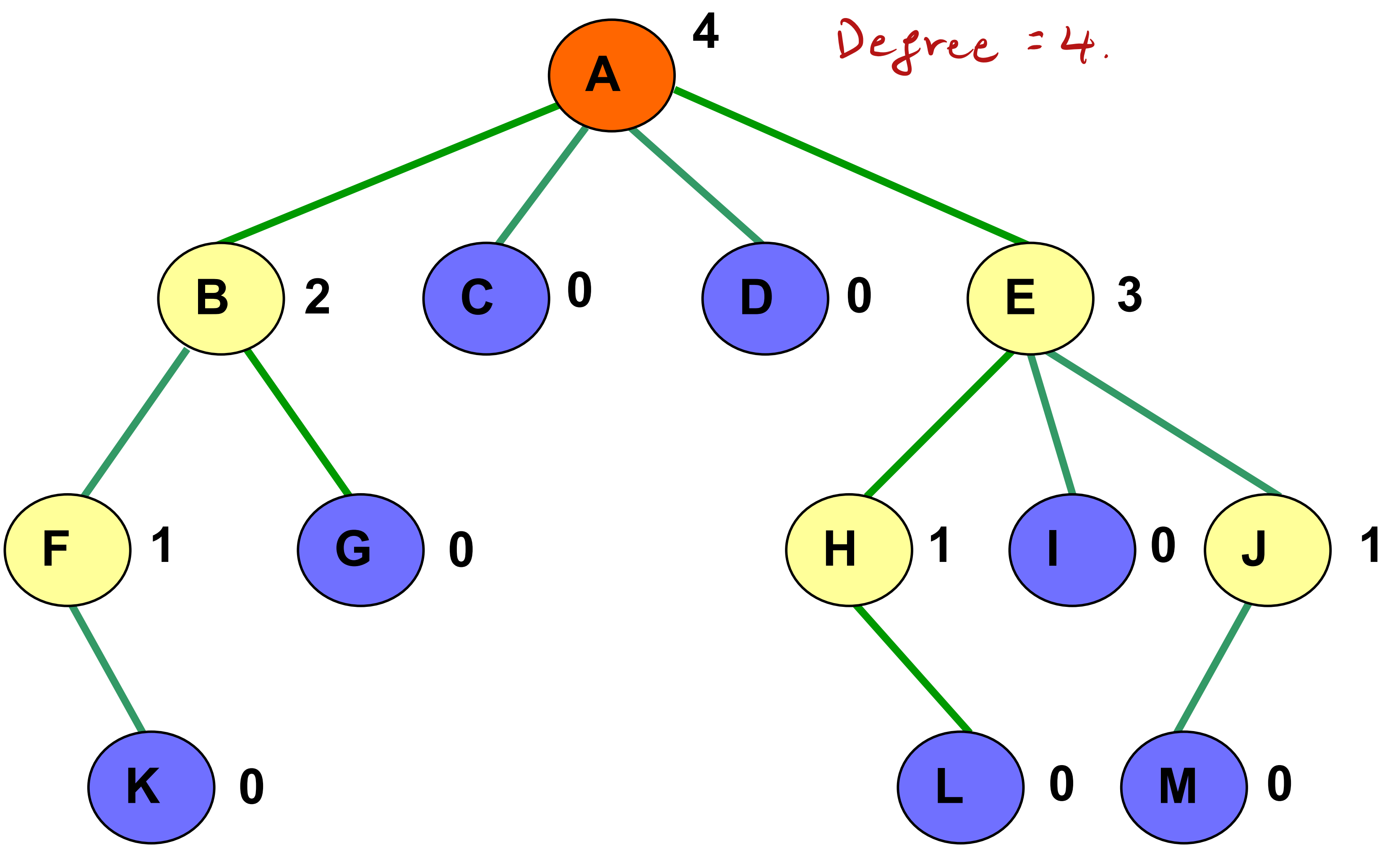
More Tree Terminology

- ***Degree*** **or** ***arity*** **of a** ***node*****: the number of children it has**

- ***Degree*** **or** ***arity*** **of a** ***tree*****: the** ***maximum*** **of the degrees of the tree's nodes**

Degree

Degree of a ***tree***: the ***maximum*** degree of its nodes



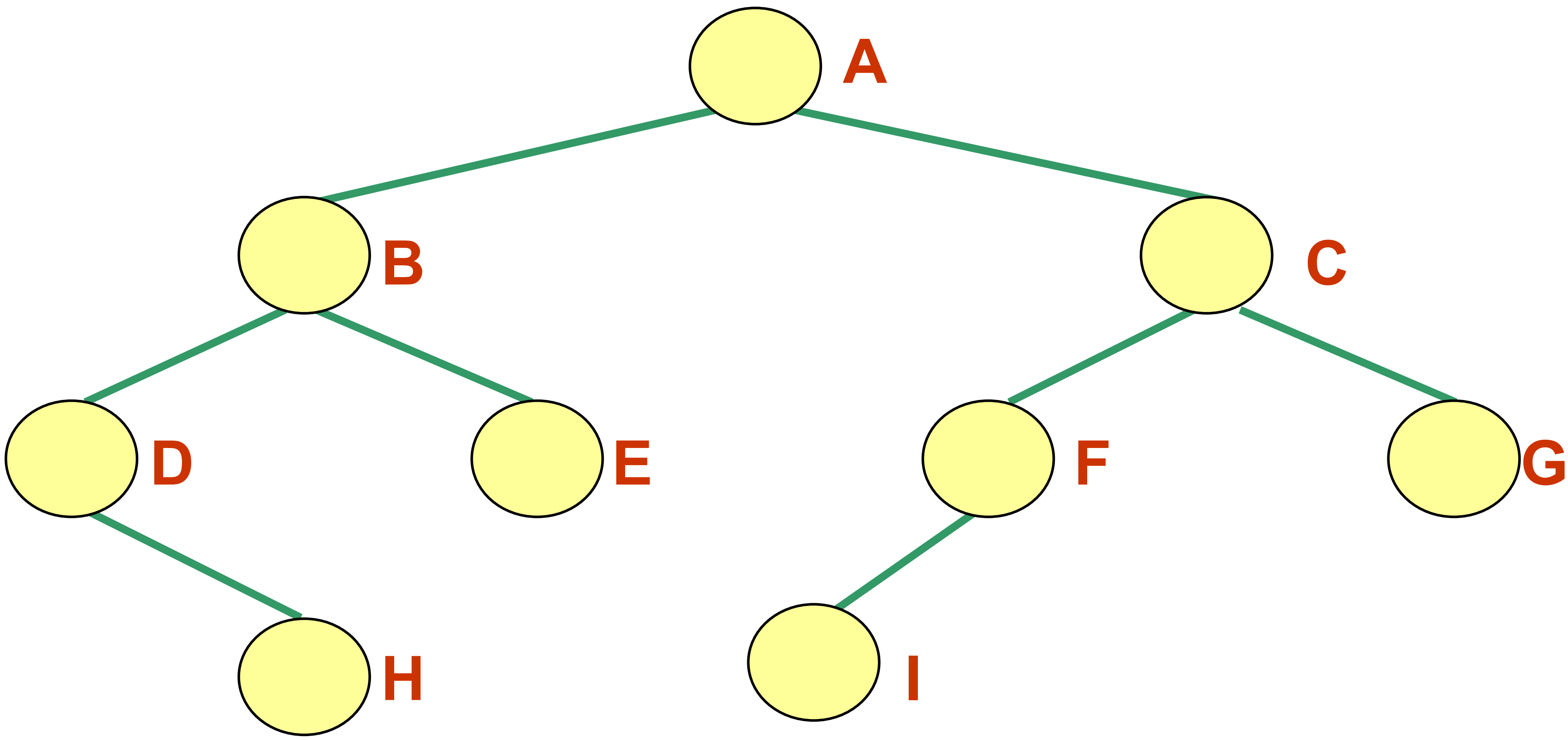
Binary Trees

- ***General tree***: a tree each of whose nodes may have any number of children
- ***n-ary tree***: a tree each of whose nodes may have no more than ***n*** children
- ***Binary tree***: a tree each of whose nodes may have no more than ***2*** children
  - ***i.e.*** a binary tree is a tree with ***degree (arity) 2***
  - The children (if present) are called the ***left child*** and ***right child***

Binary Trees

- ***Recursive definition*** **of a** ***binary tree*****: it is**
  - **The empty tree**
  - **Or, a tree which has a root whose left and right subtrees are binary trees**

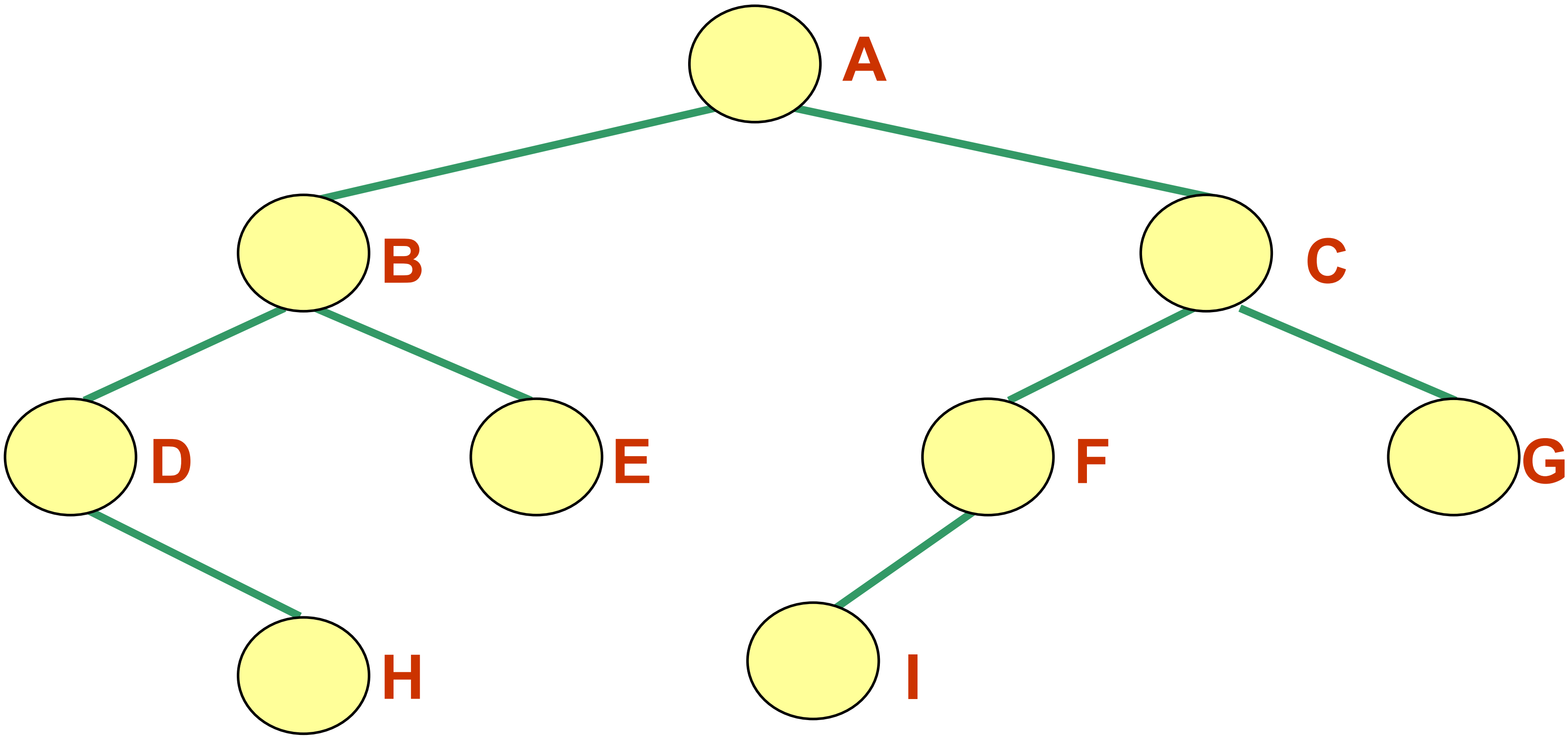
Binary Tree



Tree Traversals

- A *traversal* of a tree requires that each node of the tree be *visited* once
    - *Example*: a typical reason to traverse a tree is to display the data stored at each node of the tree
- Standard traversal orderings:
    - *preorder*
    - *inorder*
    - *postorder*
    - *level-order*

Traversals



We'll trace the different traversals using this tree; recursive calls, returns, and "visits" will be numbered in the order they occur

Preorder Traversal

- Start at the root
- Visit each node, followed by its children; we will choose to visit left child before right

- *Recursive algorithm* for preorder traversal:
  - If tree is not empty,
    - 1 Visit root node of tree
    - 2 Perform preorder traversal of its left subtree
    - 3 Perform preorder traversal of its right subtree

```
pT (tree) {
  tree.root.getValue();
  tree.pT(tree.leftSub()).
  tree.pT(tree.rightSub()).
}.
```

  - *What is the base case?* 1
  - *What is the recursive part?* 2 3,

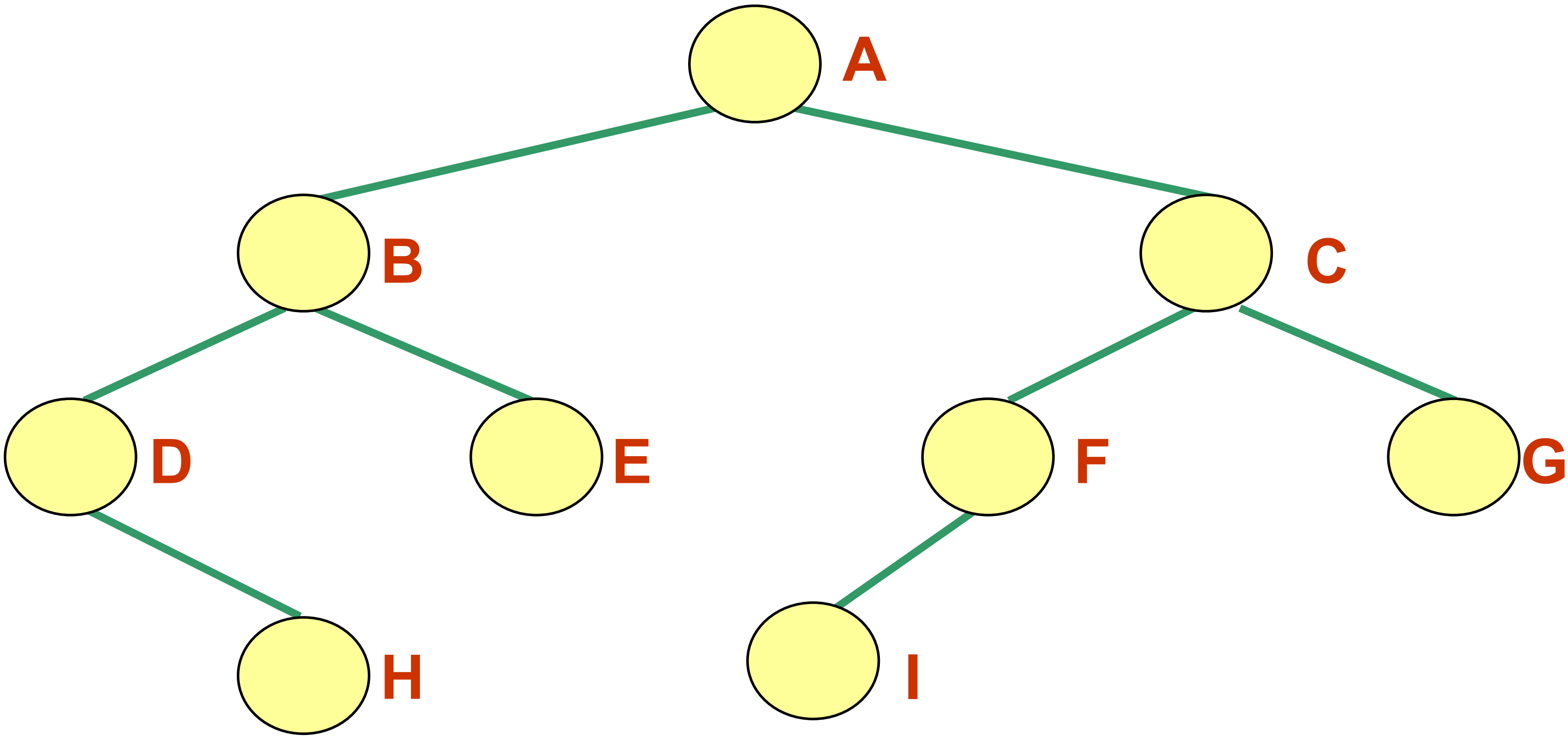
Preorder Traversal

```
public void preorder (BinaryTreeNode<T> r) {
    if (r != null) {
        visit(r);
        preorder (r.getLeftChild());
        preorder (r.getRightChild());
    }
}
```

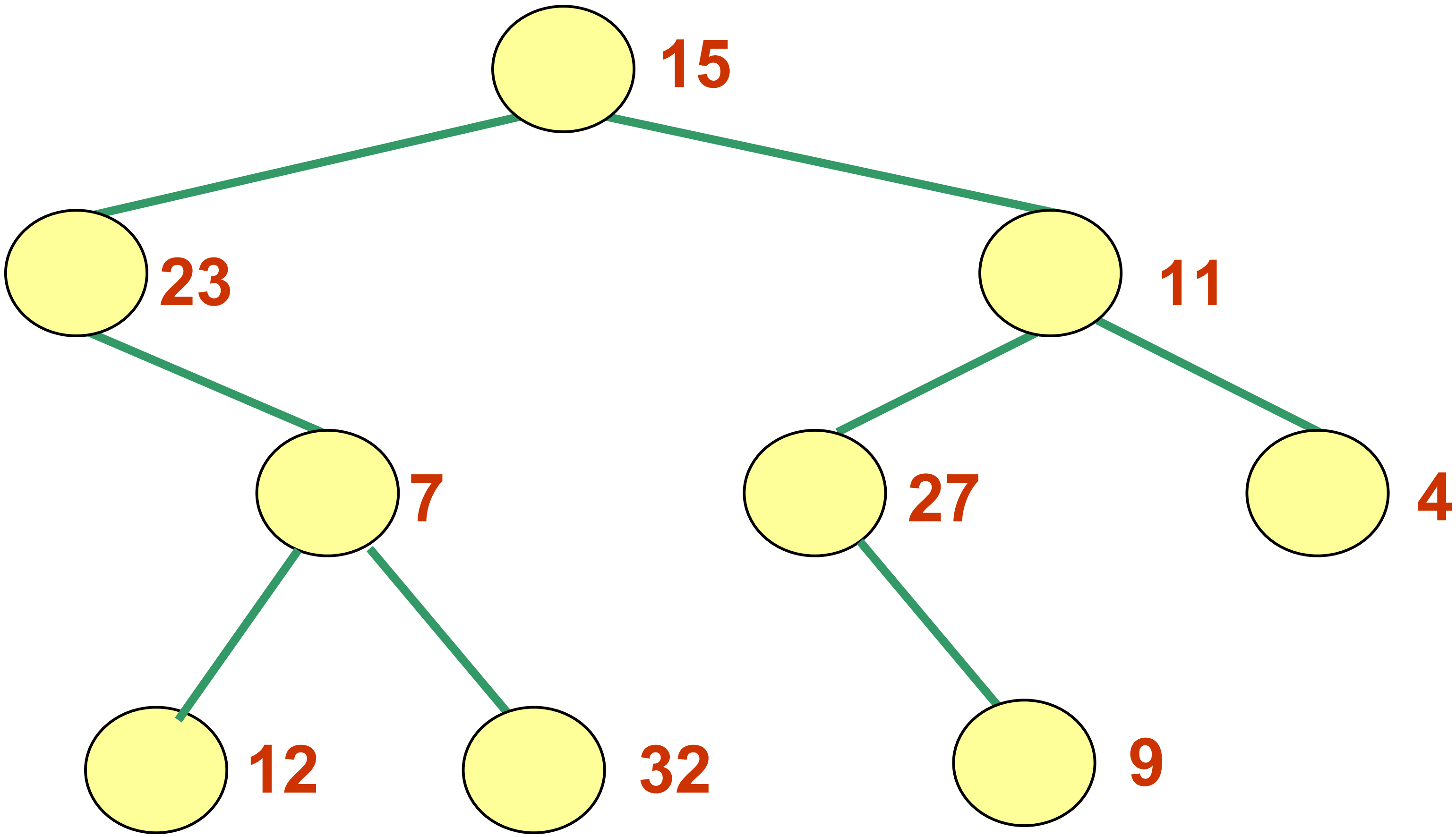
empty tree case.

top to bottom
left to right.

Preorder Traversal



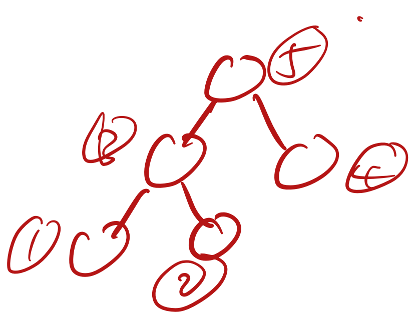
Preorder Traversal (Ex. 2)



Inorder Traversal

- Start at the root
- Visit the left child of each node, then the node, then any remaining nodes

- *Recursive algorithm* for inorder traversal
  - If tree is not empty,
    - Perform inorder traversal of left subtree of root
    - Visit root node of tree
    - Perform inorder traversal of its right subtree

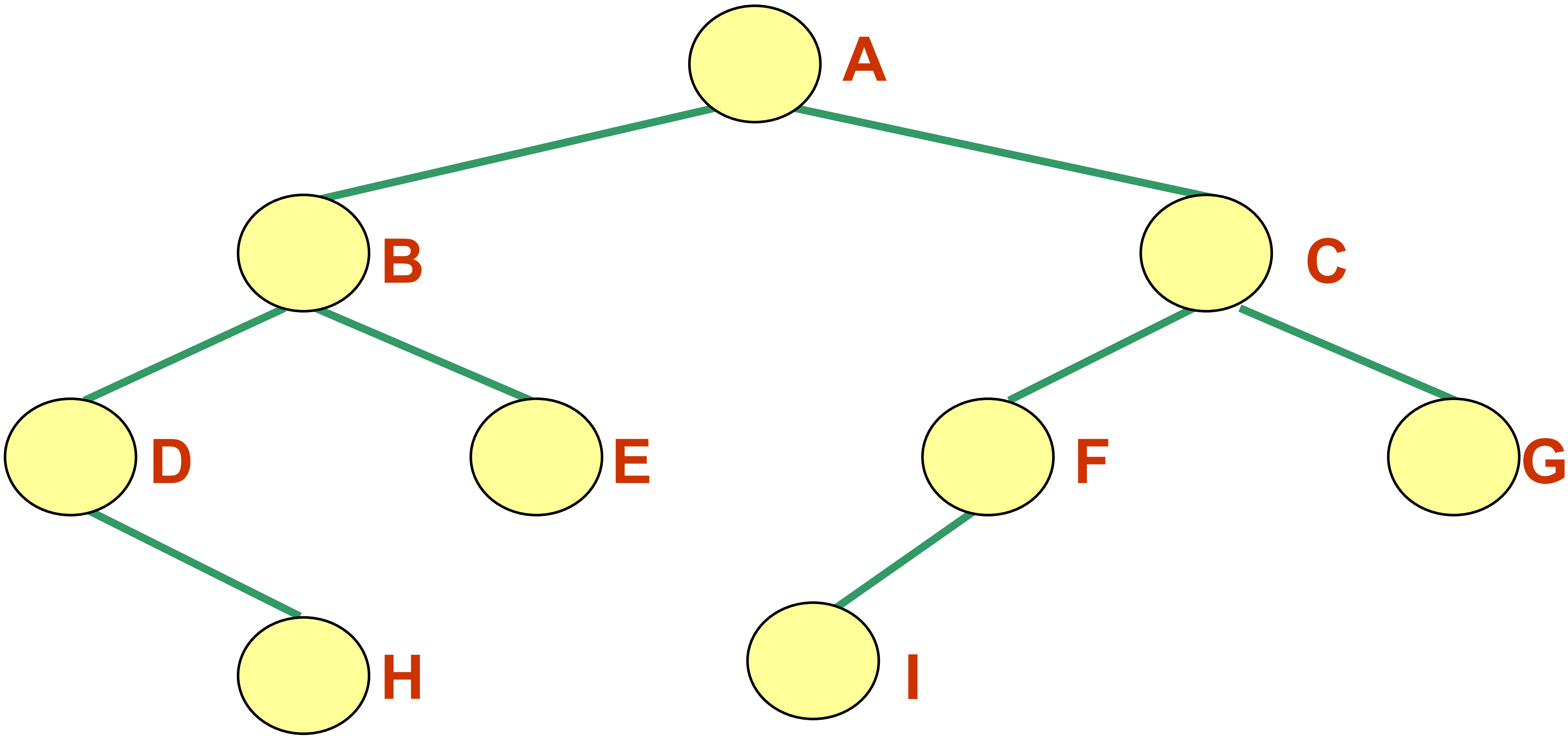


left to right, bottom to top.

Inorder Traversal

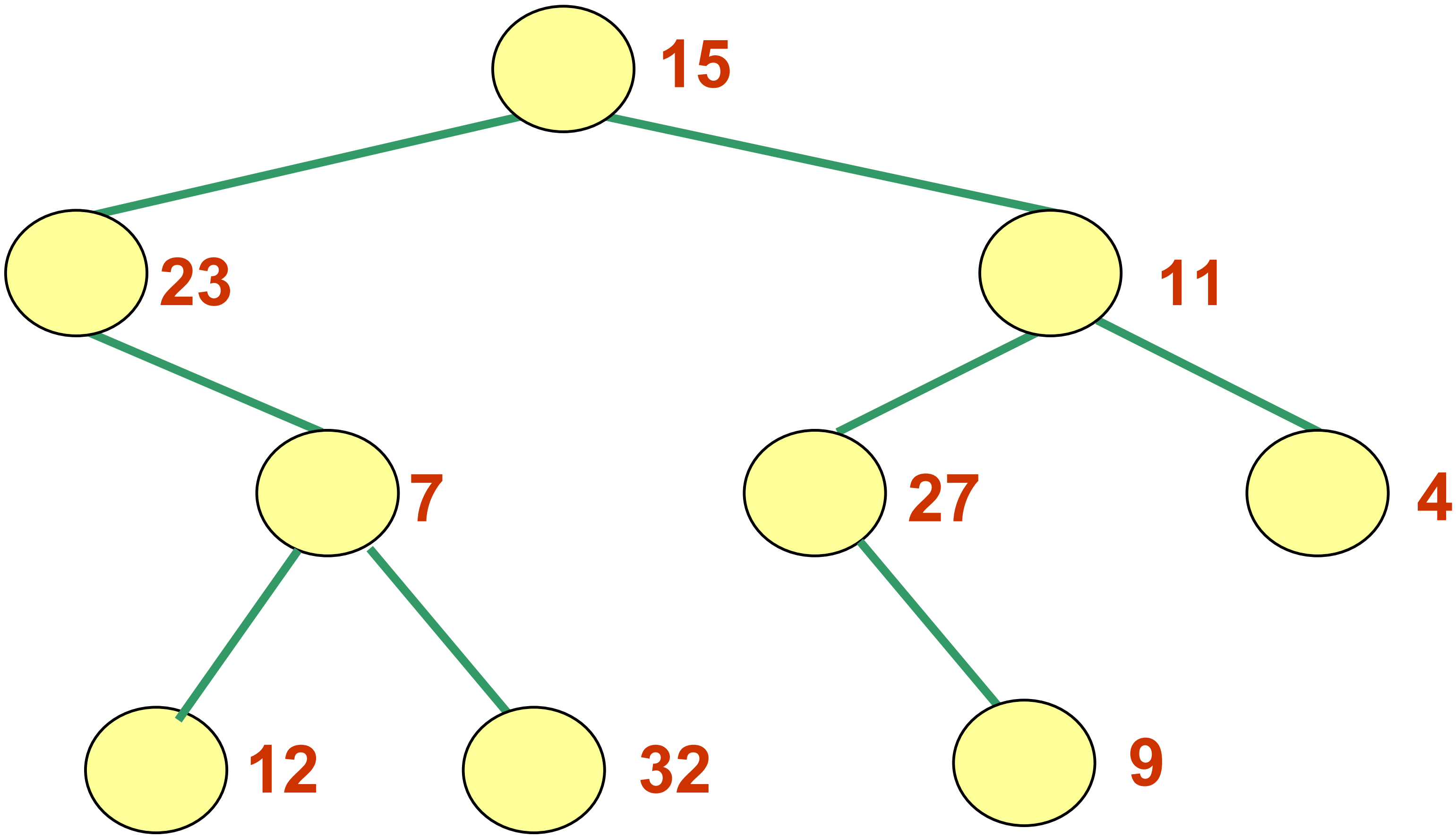
```
public void inorder (BinaryTreeNode<T> r) {
    if (r != null) {
        inorder (r.getLeftChild());
        visit(r);
        inorder (r.getRightChild());
    }
}
```

Inorder Traversal



HDEBIFGCA.

Inorder Traversal (Ex. 2)



Postorder Traversal

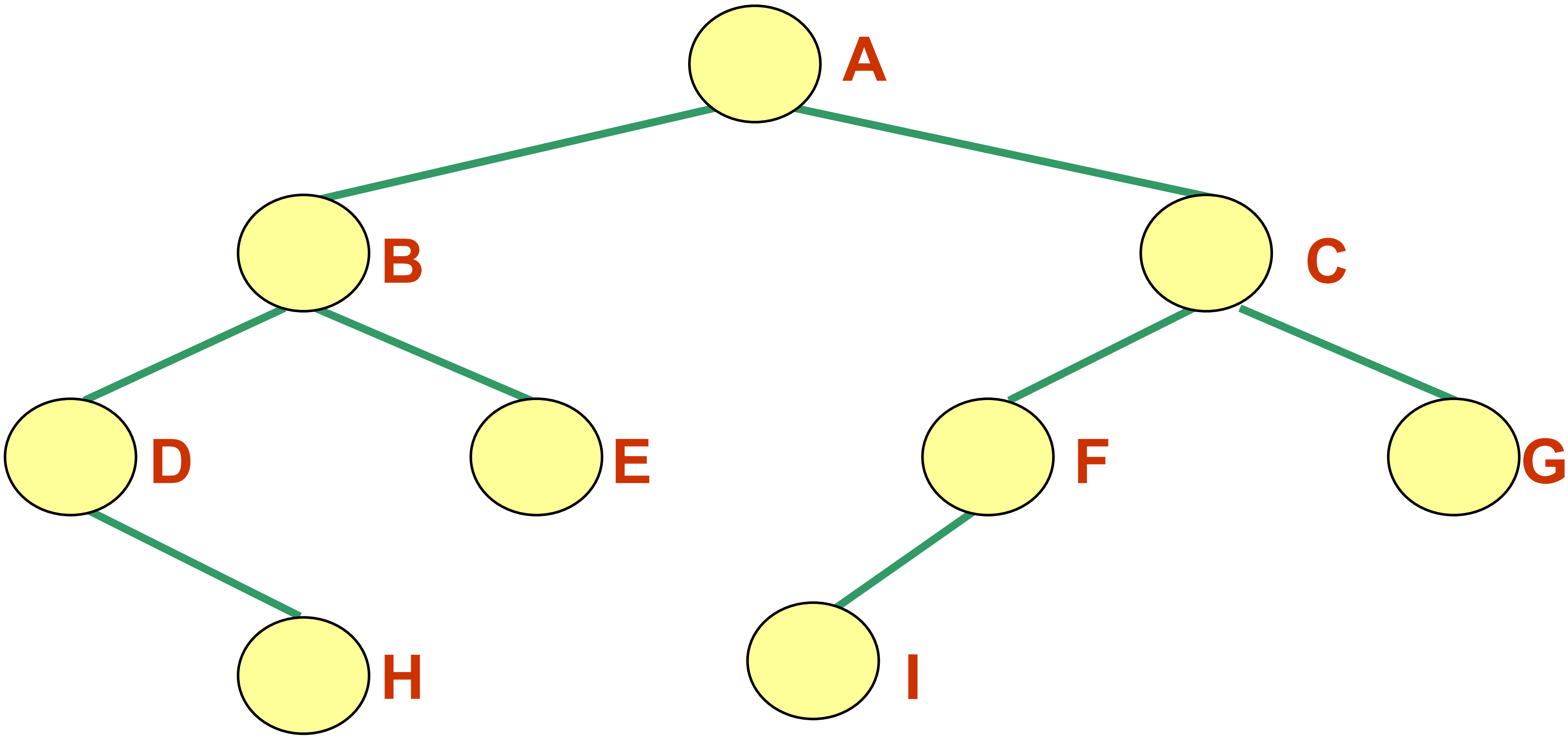
- Start at the root
- Visit the children of each node, then the node

- *Recursive algorithm* for postorder traversal
  - If tree is not empty,
    - Perform postorder traversal of left subtree of root
    - Perform postorder traversal of right subtree of root
    - Visit root node of tree

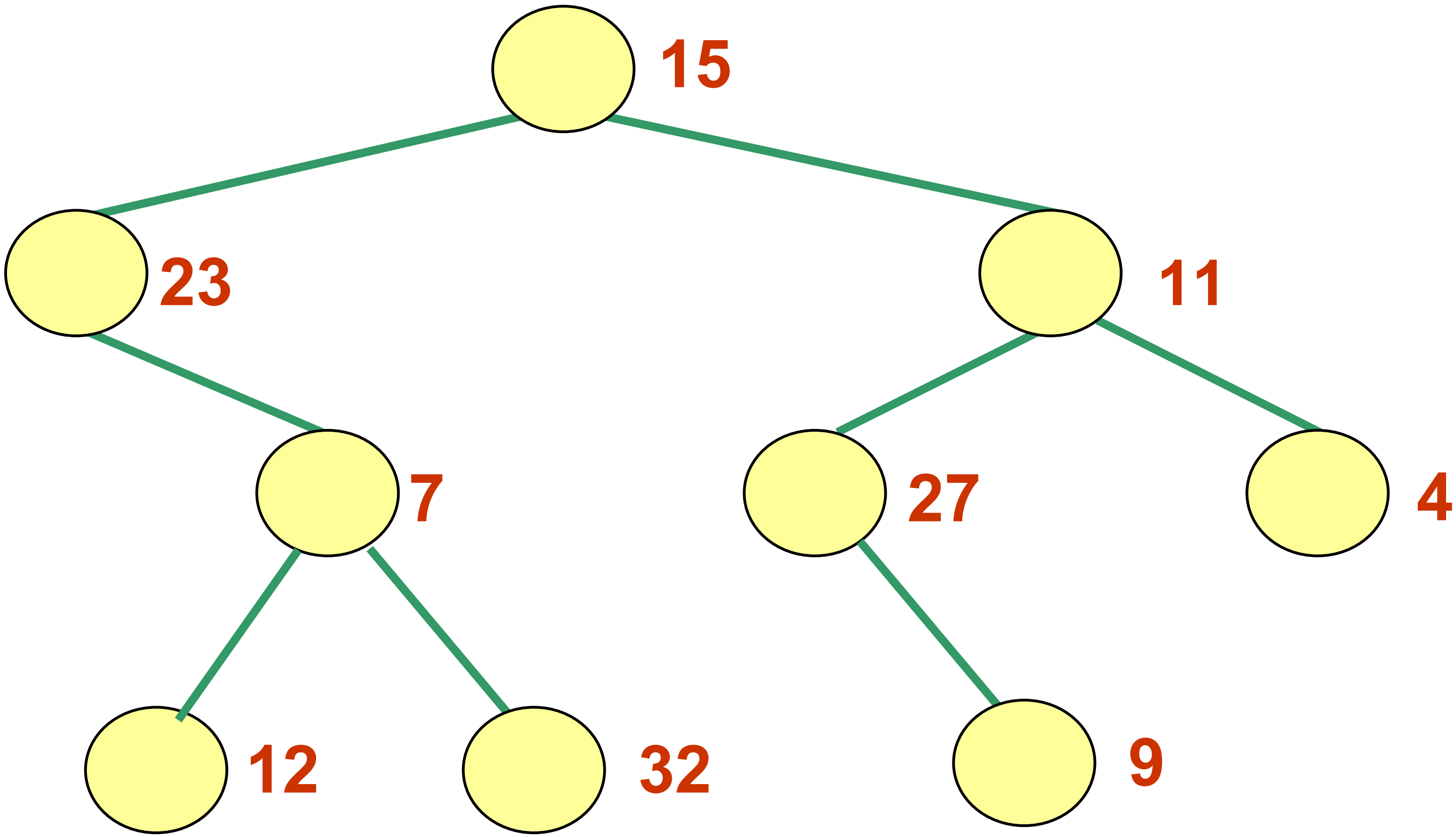
Postorder Traversal

```
public void postorder (BinaryTreeNode<T> r) {
    if (r != null) {
        postorder (r.getLeftChild());
        postorder (r.getRightChild());
        visit(r);
    }
}
```

Postorder Traversal



Postorder Traversal (Ex. 2)



Iterative Binary Tree Traversals

- In recursive tree traversals, the ***Java execution stack*** keeps track of where we are in the tree (by means of the activation records for each call)
- In ***iterative traversals***, the programmer needs to keep track!
  - An iterative traversal uses a ***container*** to store references to nodes not yet visited
  - Order of visiting will depend on the type of container being used (**stack**, **queue**, etc.)

Iterative Binary Tree Traversals

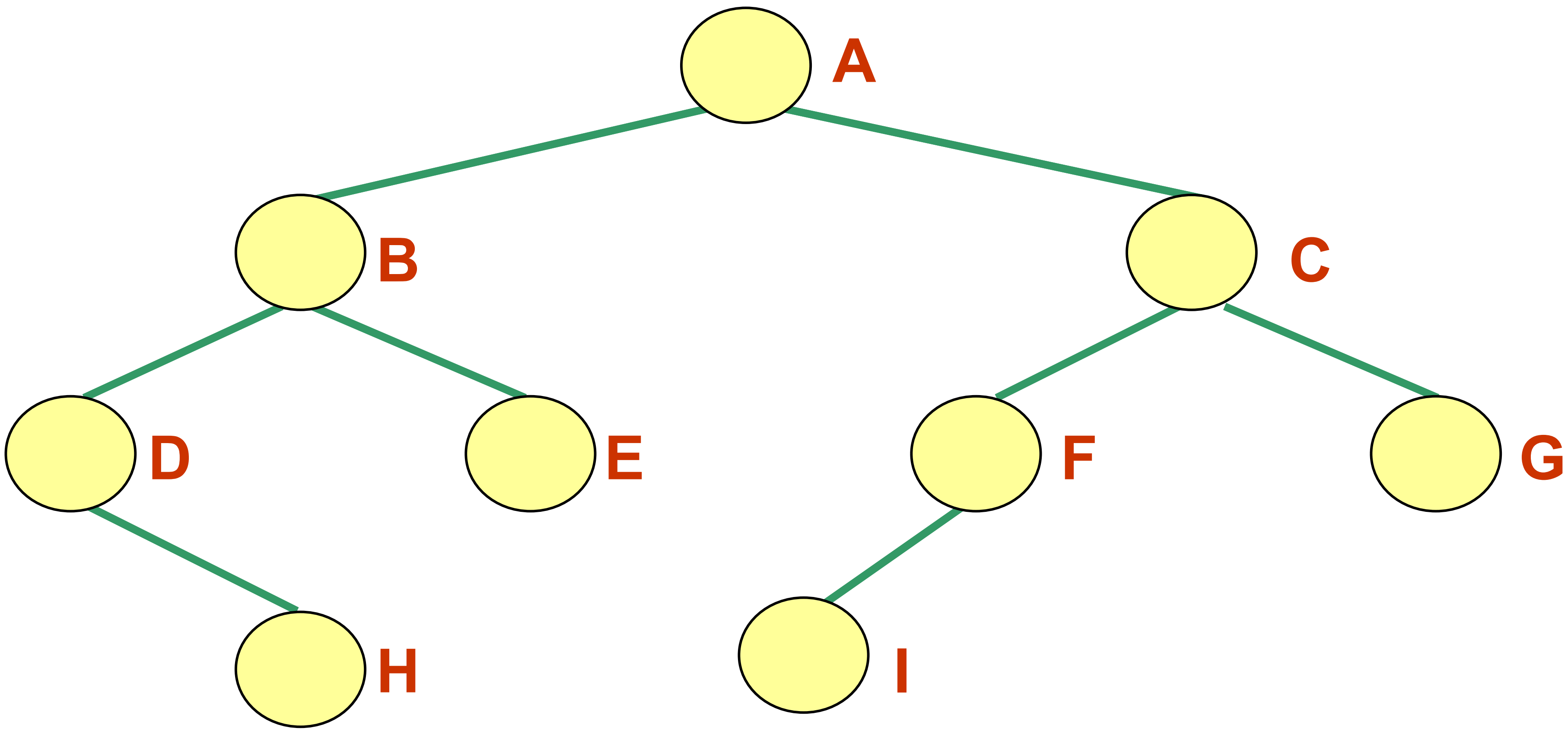
- **Container is a stack*:* if we push the *right* successor of a node before the *left* successor, we get *preorder* traversal**
- **Container is a queue*:* if we enqueue the *left* successor before the *right*, we get a *level order traversal***

Level Order Traversal

- **Start at the root**
- **Visit the nodes at each level, from left to right**

- **Is there a recursive algorithm for a level order traversal?**

Level Order Traversal

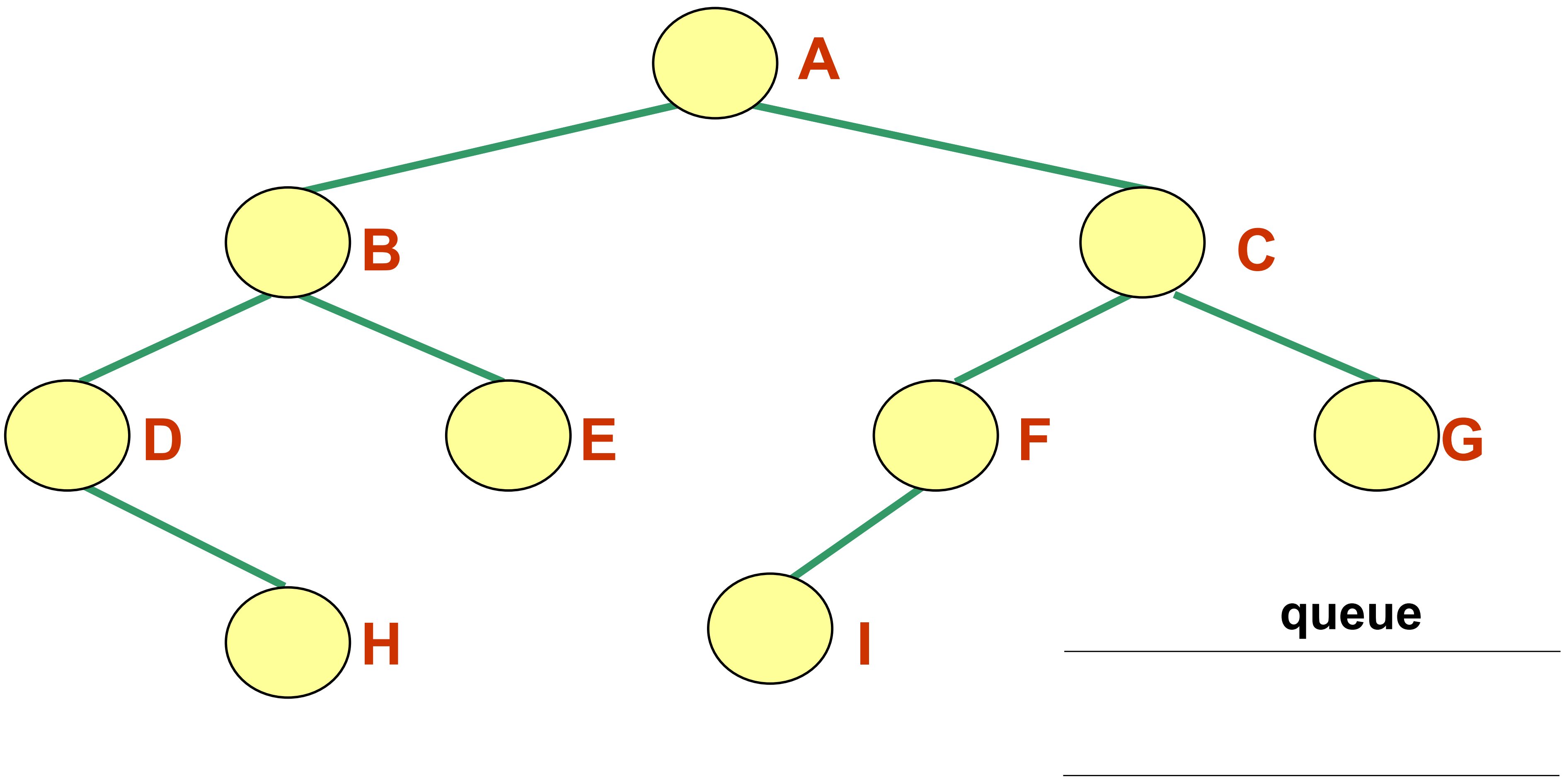


Level Order Traversal with a Queue

```
public void levelOrder (BinaryTreeNode<T> root) {
    if (root == null) return;
    LinkedQueue<T> Q = new LinkedQueue<T>();
    Q.enqueue(root);
    while (!Q.isEmpty()) {
        BinaryTreeNode<T> v = Q.dequeue();
        visit(v);
        if (v.leftChild() != null) Q.enqueue(v.leftChild());
        if (v.rightChild() != null)
Q.enqueue(v.rightChild());
    }
}
```

Level Order Traversal

- Start at the root
- Visit the nodes at each level, from left to right

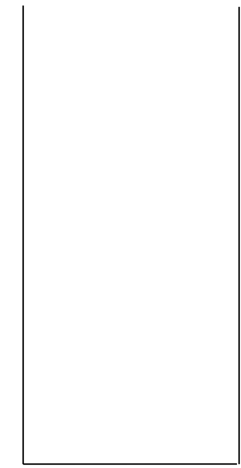


Preorder Traversal with a Stack

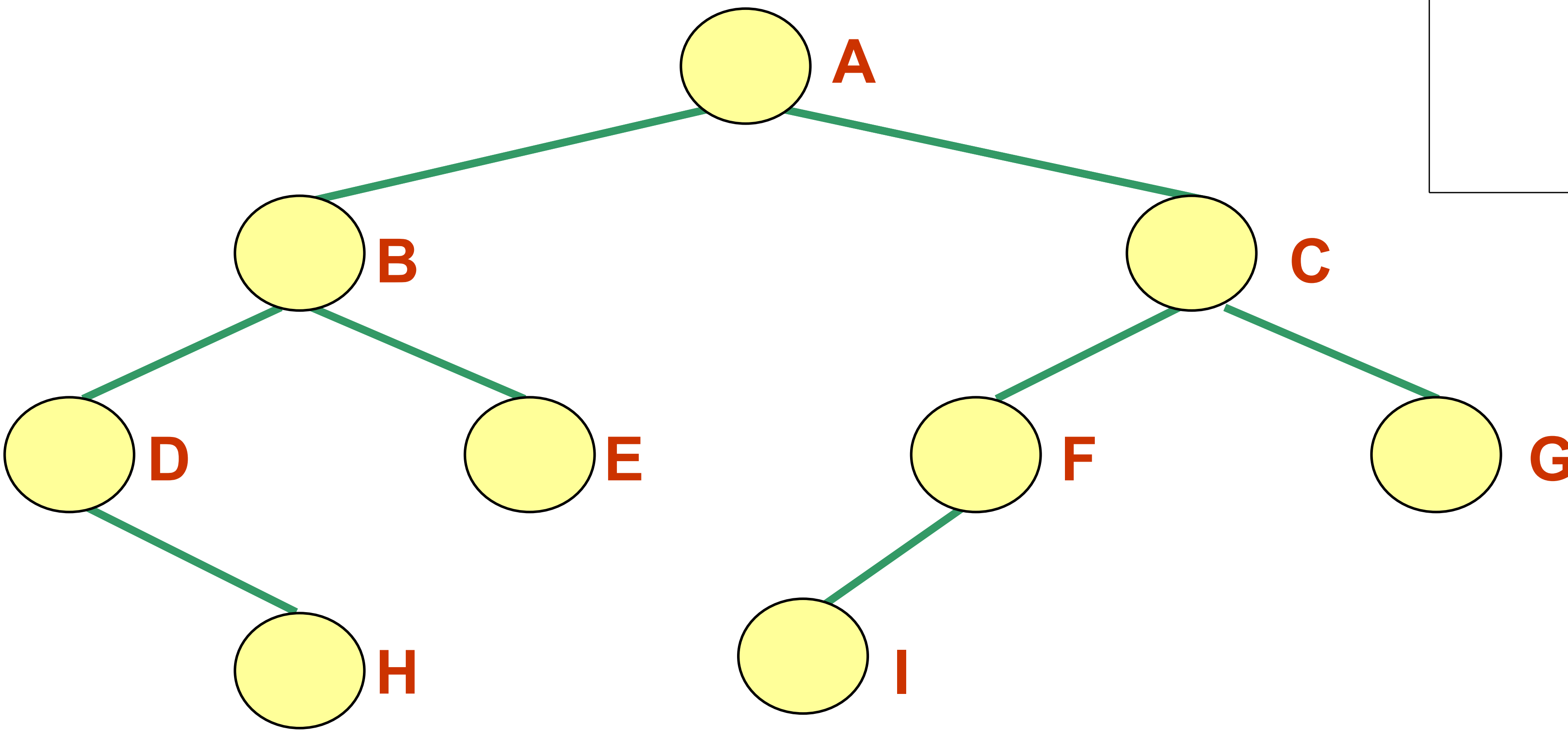
```
public void stackTraverse (BinaryTreeNode<T> root)
{
    if (root == null) return;
    LinkedStack<T> S = new LinkedStack<T>();
    S.push(root);
    while (!S.isEmpty()) {
        BinaryTreeNode<T> v = S.pop();
        visit(v);
        if (v.rightChild() != null) S.push(v.rightChild());
        if (v.leftChild() != null) S.push(v.leftChild());

    }
}
```

Stack Preorder Traversal

stack



- Visit a node (starting with root)
- Push right child if it exists
- Push left child if it exists



Traversal Analysis

- Consider a binary tree with n nodes
- How many recursive calls are there at most?
  - For each node, 2 recursive calls at most
  - So, 2*n recursive calls at most
- So, a traversal is O(n)

Operations on a Binary Tree

- What might we want to do with a binary tree?
  - Add an element (but where?)
  - Remove an element (but from where?)
  - Is the tree empty?
  - Get size of the tree (i.e. how many elements)
  - Traverse the tree (in preorder, inorder, postorder, level order)

Discussion

- **It is difficult to have a general *add* operation, until we know the purpose of the tree (we will discuss binary search trees later)**
  - **We could add "*randomly*": go either right or left, and add at the first available spot**

Discussion

- Similarly, where would a general ***remove*** operation remove from?
  - We could arbitrarily choose to remove, say, the leftmost leaf
  - If random choice, what would happen to the children and descendants of the element that was removed? What does the parent of the removed element now point to?
  - What if the removed element is the root?

Possible Binary Tree Operations

Operation	Description
getRoot	Returns a reference to the root of the tree
isEmpty	Determines whether the tree is empty
size	Determines the number of elements in the tree
find	Returns a reference to the specified target, if found
toString	Returns a string representation of tree’s contents
iteratorInOrder	Returns an iterator for an inorder traversal
iteratorPreOrder	Returns an iterator for a preorder traversal
iteratorPostOrder	Returns an iterator for a postorder traversal
iteratorLevelOrder	Returns an iterator for a levelorder traversal

Binary Tree ADT

```
package binaryTree;
import java.util.Iterator;
public interface BinaryTreeADT<T> {
   public T getRoot ();
   public boolean isEmpty();
   public int size();
   public T find (T targetElement) throws

ElementNotFoundException;
   public String toString();
   public Iterator<T> iteratorInOrder();
   public Iterator<T> iteratorPreOrder();
   public Iterator<T> iteratorPostOrder();
   public Iterator<T> iteratorLevelOrder();
}
```

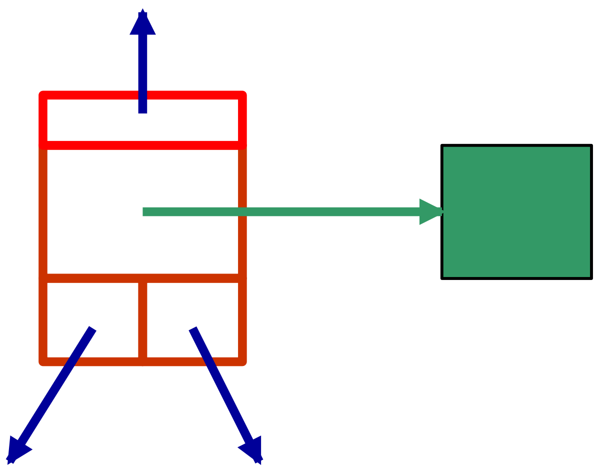
Linked Binary Tree Implementation

- To represent the binary tree, we will use a **linked** structure of nodes
  - **root**: reference to the node that is the **root of the tree**
  - **count**: keeps track of the number of nodes in the tree

- First, how will we represent a ***node of a binary tree***?

Linked Binary Tree Implementation

- A **binary tree node** will contain
  - a reference to a data element
  - references to its left and right children and parent



left and right children are binary tree nodes themselves

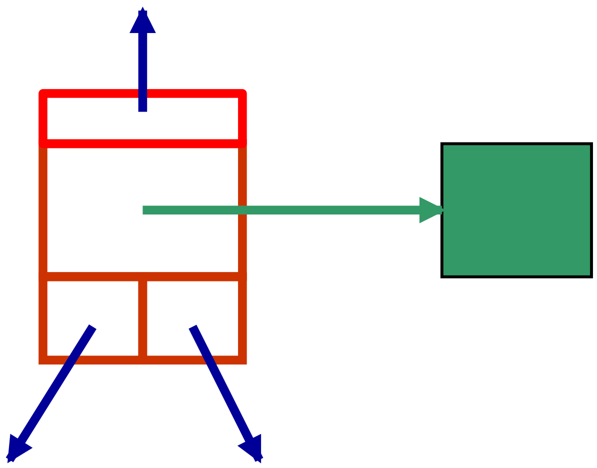
BinaryTreeNode class

- Represents a node in a binary tree
- Attributes:
  - **element**: reference to data element
  - **left**: reference to left child of the node
  - **right**: reference to right child of the node
  - **parent**: reference to the parent of the node

- See BinaryTreeNode.java
  - Note that the attributes here are **protected**
    - This means that they can be accessed directly from any class that is in the same package as BinaryTreeNode.java

A BinaryTreeNode Object

```
protected T element;
protected BinaryTreeNode<T> left, right,
   parent;
```



Note that either or both of the left and right references could be null

LinkedBinaryTree Class

- **Attributes:**

```
protected BinaryTreeNode<T> root;
protected int count;
```

- **The attributes are protected so that they can be accessed directly in any subclass of the LinkedBinaryTree class**
  - **We will be looking at a very useful kind of binary tree called a Binary Search Tree later**

LinkedBinaryTree Class

- **Constructors:**

```
//Creates empty binary tree
public LinkedBinaryTree() {
count = 0;
root = null;
}
//Creates binary tree with specified element as its root
public LinkedBinaryTree (T element) {
count = 1;
    root = new BinaryTreeNode<T> (element);
}
```

```
/* Returns a reference to the specified target element if it
   is found in this binary tree.
   Throws an ElementNotFoundException if not found. */

  public T find(T targetElement) throws

   ElementNotFoundException
  {
    BinaryTreeNode<T> current =
                                     findAgain( targetElement, root
   );
    if ( current == null )
       throw new ElementNotFoundException("binary
   tree");

    return (current.element);
  }
```

```
private BinaryTreeNode<T> findAgain(T targetElement,
                                     BinaryTreeNode<T> next)
{
   if (next == null)
      return null;
   if (next.element.equals(targetElement))
      return next;
   BinaryTreeNode<T> temp =
                     findAgain(targetElement, next.
  left);
   if (temp == null)
      temp = findAgain(targetElement, next.right);
   return temp;
}
```

```
/* Performs an inorder traversal on this binary
   tree by calling a recursive inorder method that
   starts with the root.
   Returns an inorder iterator over this binary tree
   */
  public Iterator<T> iteratorInOrder()
  {
    ArrayUnorderedList<T> tempList =
                          new
   ArrayUnorderedList<T>();

    inorder (root, tempList);
    return tempList.iterator();
  }
```

Discussion

- iteratorInOrder is returning an iterator object
  - It will perform the iteration in *inorder*
- But where is that iterator coming from?

```
return tempList.iterator();
```

- Let's now look at the helper method inorder …

```
/* Performs a recursive inorder traversal.
   Parameters are: the node to be used as the
   root for this traversal, the temporary list for
   use in this traversal */
  protected void inorder (BinaryTreeNode<T>
   node,
                        ArrayUnorderedList<T>
   tempList)
  {
    if (node != null)
    {
      inorder (node.left, tempList);
      tempList.addToRear(node.element);
      inorder (node.right, tempList);
    }
  }
```

Discussion

- Recall the *recursive algorithm* for inorder traversal:
  - If tree is not empty,
    - Perform inorder traversal of left subtree of root
    - Visit root node of tree
    - Perform inorder traversal of its right subtree
- That's exactly the order that is being implemented here!
    - What is "visiting" the root node here?

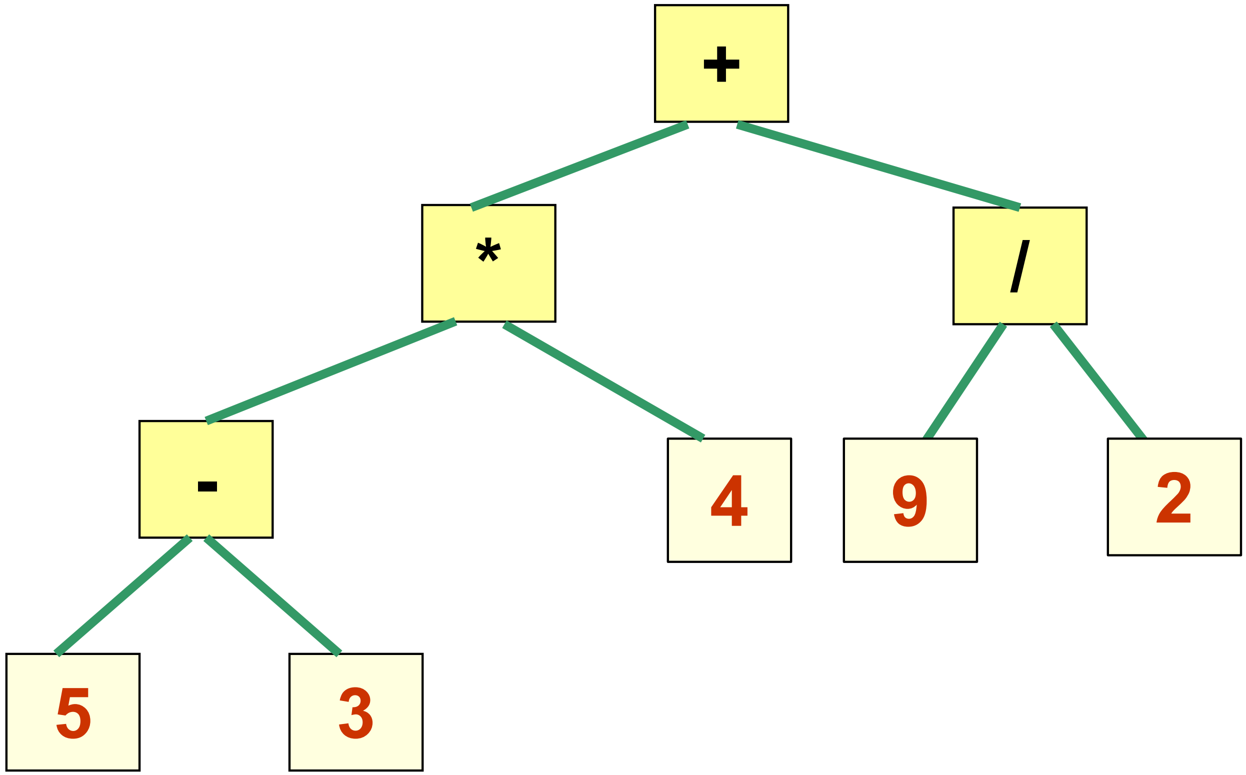
Discussion

- The data elements of the tree (i.e. items of type T) are being *temporarily* added to an unordered list, in *inorder* order
  - Why use an unordered list??
    - Why not? We already have this collection, with its iterator operation that we can use!

Using Binary Trees: Expression Trees

- **Programs that manipulate or evaluate arithmetic expressions can use binary trees to hold the expressions**
- **An *expression tree* represents an arithmetic expression such as (5 – 3) * 4 + 9 / 2**
  - **Root node and interior nodes contain *operations***
  - **Leaf nodes contain *operands***

Example: An Expression Tree

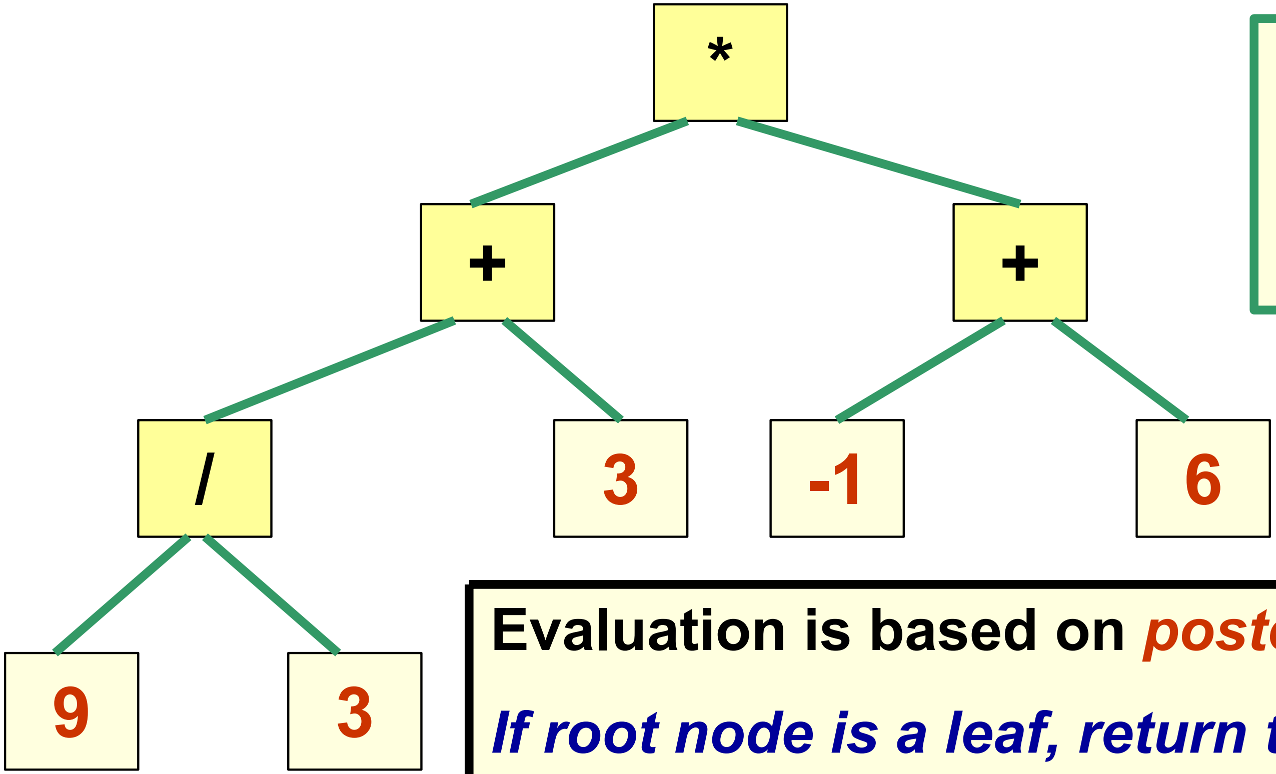


$$(5 - 3) * 4 + 9 / 2$$

Evaluating Expression Trees

- **We can use an expression tree to *evaluate an expression***
  - **We start the evaluation at the *bottom left***
  - **What kind of traversal is this?**

Evaluating an Expression Tree



This tree represents the expression

(9 / 3 + 3) * (-1 + 6)

Evaluation is based on *postorder* traversal:

If root node is a leaf, return the associated value.

Recursively evaluate expression in left subtree.

Recursively evaluate expression in right subtree.

Perform operation in root node on these two values, and return result.

Evaluating an Expression Tree