

Recall: $f: A \rightarrow B$ is one-to-one (or injective) if
 $\neg \exists a_1 \in A \exists a_2 \in A (f(a_1) = f(a_2) \wedge a_2 \neq a_1)$
 Equality: $\forall a_1, a_2 \in A (f(a_1) = f(a_2) \rightarrow a_1 = a_2)$
 f is onto if: $\forall b \in B \exists a \in A (f(a) = b)$ (every element in range is taken)

Ex $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, $f = \{(1, 5), (2, 4), (3, 5)\}$.

* f is a function

is not injective: $f(1) = f(3)$.

not onto: $\nexists a \in A f(a) = 6$.

Ex 2 any A , $i_A: A \rightarrow A$ i.e. $i_A(a) = a$.

one-to-one + onto

Ex 3 $\mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$

injective \times onto \times

$f(1) = f(-1)$ $\nexists x \in \mathbb{R} f(x) = -1$.

Ex 4 $h: \mathbb{R} \rightarrow \mathbb{R}$ $h(x) = 2x + 3$.

injective \checkmark onto \checkmark

$g: \mathbb{Z} \rightarrow \mathbb{R}$ $g(x) = 2x + 3$

\checkmark

\times

$g(x) \in \mathbb{Z}$

Ex 5.1.4. $A = \mathbb{R} \setminus \{-1\}$, $f: A \rightarrow \mathbb{R}$, $f(a) = \frac{2a}{a+1}$

Proof: f is one-to-one but not onto.

Goal: $a_1, a_2 \in A$ $f(a_1) = f(a_2) \rightarrow a_1 = a_2$.

Proof: let $a_1, a_2 \in A$. Assume $f(a_1) = f(a_2)$. So $\frac{2a_1}{a_1+1} = \frac{2a_2}{a_2+1}$,
 therefore $2a_1a_2 + 2a_1 = 2a_1a_2 + 2a_2$, $a_1 = a_2$. \square

onto: roughwork. $b \in \mathbb{R}$. $f(a) = b$. $\frac{2a}{a+1} = b$.

$$2a = b(a+1)$$

$$a = \frac{b}{2-b} \text{ if } b \neq 2.$$

f is not onto: Assume $a \in A$ gives us $f(a) = 2$.

Then $\frac{2a}{a+1} = 2$, $0 = 2$, which is a contradiction,
 no such a exist. So it is not onto. \square

Theorem 5.2.5 Let $f: A \rightarrow B$, $g: B \rightarrow C$.

if f, g are one-to-one, then $g \circ f$ is one-to-one.
 onto onto.

Proof 1: Suppose f, g are one-to-one

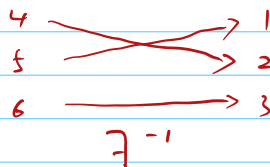
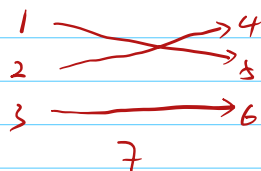
Let $a_1, a_2 \in A$. Assume $g \circ f(a_1) = g \circ f(a_2)$,

so $g(f(a_1)) = g(f(a_2))$. Since g is one-to-one, $f(a_1) = f(a_2)$.

Since f is one-to-one, $a_1 = a_2$. So $g \circ f$ is one-to-one. \square

2: Suppose f, g are onto. Let $c \in C$, $a \in A$. Since g is onto, $\exists b \in B$ that $g(b) = c$. Since f is onto, there exists $a \in A$ that $f(a) = b$, that is, $g(f(a)) = c$, so $g \circ f(a) = c$.

§ 5.3.
inverse of
functions.



f^{-1} is a function.

Thm 5.3.1 Suppose $f: A \rightarrow B$. if f is one-to-one and onto, then $f^{-1}: B \rightarrow A$ is also a function.

Proof: Assume f is one-to-one and onto, let $b \in B$.

Existence: Since f is onto, there is $a \in A$, $f(a) = b$.

so $(a, b) \in f$, $(b, a) \in f^{-1}$

Uniqueness: Assume $(b, a_1), (b, a_2) \in f^{-1}$, so $(a_1, b), (a_2, b)$ are in f . Since f is one-to-one, $a_1 = a_2$.