§ 4
Relations

§ 4.1
Order Pairs & Cartion Produts

$R^2 = R \times R = \{x, y \mid x, y \in R\}$.

$(x_1, y_1) = (x_2, y_2)$ iff $x_1 = x_2$, $y_1 = y_2$.

e.g. $(3,1) \neq (1,3)$.

everything would be in the order pairs, not only numbers.

Def 4.1.1.

Let A, B be sets, the Cartesion Product:

$A \times B = \{(a_1, b_1), (a_2, b_1) \cdots (a_n, b_1), (a_1, b_2) \cdots (a_n, b_1)\}$.

$= \{(a, b) \mid a \in A, b \in B\}$.

Ex $A = \{R, G\}$

$B = \{1, 2\}$.

$A \times B = \{(R,1), (R,2), (G,1), (G,2)\}$.

Note. if A, B are finite, then $|A \times B| = |A| \cdot |B|$

Truth sets.

Let $P(x,y)$ that depends on x and on y, $x \in A$ and $y \in B$, so for (a,b) in set $A \times B$, $P(a,b)$ make sense. The truth set of P is $\{(a,b) \in A \times B \mid P(a,b)\}$. It is a subset of $A \times B$.

Ex. $A \in Z$, $B \in Z$, let $D(x,y) = x \mid y$

the truth set of D is

$T = \{(x,y) \in Z^2 \mid D(x,y)\}$.

in this case, $(3,6) \in T$, $(6,3) \notin T$.

Ex 2: $A = B = R$, let $P(x,y)$: $y = 2x - 3$. the truth set $T = \{(x,y) \in R^2 \mid y = 2x - 3\}$.

=> then, you could get a graph.

Ex 3: $A = \{$uwo courses$\}$

$B = \{$uwo teachers$\}$

$Q =$ "x teaches Y".

$T = \{(x,y) \in A \times B \mid Q(x,y)\}$.

and the pair (Math 2155, Ran) $\in T$.

Facts: if T is a truth table of a statement $P(x,y)$ for $x\in A, y\in B$, then

$T \subseteq A\times B$.

$(x,y)\in T$ iff $P(x,y)$.

if $P(x,y)$ is a tautology, then $P(x,y) = A\times B$,

else contradiction $= \phi$

if S is the truth set of $Q(x,y)$, $x\in A, y\in B$, then the truth set of $T(x,y) \vee Q(x,y)$ is $T\cup S$.

similarly $\wedge$ $T\cap S$.

Thm 4.1.3. Let A, B, C, D are sets, then

1. $A\times(B\cap C) = (A\times B)\cap(A\times C)$
2. $A\times(B\cup C) = (A\times B)\cup(A\times C)$
3. $(A\cap D)\times(B\cap C) = (A\times B)\cap(C\times D)$
4. $(A\cup D)\times(B\cup C) \supseteq (A\times B)\cup(C\times D)$.
5. $A\times\phi = \phi = \phi\times A$.

Proof 2: let $P\in A\times(B\cup C)$, then $P=(a,y)$, $(a\in A)\wedge(y\in B\cup C)$

then $(a\in A)\wedge(y\in B \vee y\in C) \Rightarrow (a\in A\wedge y\in B)\vee$.

$\Rightarrow$ if $P\in B$, then $P\in A\times B \subseteq A\times(B\cup C)$

$P\in C$, then $P\in A\times C \subseteq A\times(B\cup C)$.

The other side: $P=(A\times B)\cup(A\times C)$:

if $P=A\times B$, then $P=(a,b)$ form $a\in A$ and $b\in B$,

so $b\in B\cup C$, so $P\in A\times(B\cup C)$.

similarly, let $P=A\times C$ ......

Proof of 5: suppose $p\in A\times\phi$, $p=(a,b)$, $a\in A$, $b\in\phi$, which is a contradiction, so p does not exist and $a\times\phi$ has no element, so $A\times\phi=\phi$.

similarly, $\phi\times A$

A counter example for 4 not be an equal sign:

if one of A, D and one of B, C is empty,

$A\times D=\phi$, $B\times C=\phi$, $(A\times D)\cup(B\times C)=\phi$.

$(A\cup D)\times(B\cup C) = A\cup C / A\cup B / D\cup B / D\cup C$

NOTE: There's **NO** commutative law for Cartesion Product.