THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

MATHEMATICS 1600B FINAL EXAMINATION

April 6, 2020

INSTRUCTIONS:

1. This exam is available on April 6 starting at 7 PM[1].
2. You must upload your responses to questions no later than April 7 at 7 PM.
3. Show all your of your work and explain your answers fully:
   unjustified, irrelevant or illegible answers will receive little or no credit.
4. All vectors and equations involve real numbers.
5. You must use notes, our textbook, the online lectures, and other reference material.
6. You **may not** discuss this exam with anyone else.
7. You can write on whatever you want and upload scans of each answer separately.

[1] All times are in EASTERN DAYLIGHT TIME

1. For each of the following statements, circle **T** if the statement is always true and **F** if the statement can be false. Give a one sentence justification for your answer.

   (a) (2 marks) If $\mathbf{x}$ is an eigenvector of A and B, then $\mathbf{x}$ is an eigenvector of $A + B$.

   T **F**

   λ

   (b) (2 marks) If $\mathbf{x}, \mathbf{y}$ are eigenvectors of A, then $\mathbf{x} + \mathbf{y}$ is an eigenvector of A.

   T **F**

   (c) (2 marks) If A, B are invertible $n \times n$-matrices, then $((AB)^t)^{-1} = ((AB)^{-1})^t$.

   T **F**

   (d) (2 marks) If A is a 3×2 matrix and B is a 2×3 matrix, then AB is never invertible.

   3×3 **T** **F**

   (e) (2 marks) If A, B are 2×2 matrices, then $(A + B)^2 = A^2 + 2AB + B^2$.

   T **F**

2. (3 marks) For which values of k do the following vectors form a basis of $\mathbb{R}^3$?

$$\mathbf{v}_1 = \begin{bmatrix} k \\ 1 \\ 2 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 0 \\ 3 \\ 1-k \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Explain.

$av_1 + bv_2 = v_3$

$ak = 0$

$a + 3b = 1$

$2a + (1-k)b = 2.$

$\because$ form a basis of $\mathbb{R}^3$

$\therefore 1-k = 6$

$k = -5$

3. (3 marks) Let A, B be orthogonal matrices of the same size, and show that ABA is orthogonal.

$\because A, B$ are orthogonal matrices

$\therefore AA^T = I,\ BB^T = I$

$ABA \cdot (ABA)^T$

$= ABA \cdot A^T B^T A^T$

$= AB I B^T A^T$

$= A I^2 A^T = I^3 = I$

$\therefore ABA$ is orthogonal

4. (2 marks) Find a 3×3 orthogonal matrix A which is not diagonal.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

5. Find the determinants of the following matrices:

   (a) (3 marks) Let A be a 1000×1000 matrix with entries $a_{i,j} = j$, so

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & \dots & 1000 \\ 1 & 2 & 3 & 4 & \dots & 1000 \\ \dots & \dots & \dots & \dots & & \dots \\ 1 & 2 & 3 & 4 & \dots & 1000 \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & \dots & 1000 \\ 1 & 2 & 3 & 4 & \dots & 1000 \\ \dots & & & & & \\ 1 & 2 & 3 & 4 & \dots & 1000 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 2 & 3 & 4 & \dots & 1000 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

$\det A = 0$

   (b) (4 marks) Let A be a 1000×1000 matrix with entries $a_{i,j} = i + j$, so

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 & \dots & 1001 \\ 3 & 4 & 5 & 6 & \dots & 1002 \\ 4 & 5 & 6 & 7 & \dots & 1003 \\ \dots & \dots & \dots & \dots & & \dots \\ 1001 & 1002 & 1003 & 1004 & \dots & 2000 \end{bmatrix}.$$

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 & \dots & 1001 \\ 3 & 4 & 5 & 6 & \dots & 1002 \\ 4 & 5 & 6 & 7 & \dots & 1003 \\ \dots & & & & & \\ 1001 & 1002 & 1003 & 1004 & \dots & 2000 \end{bmatrix} \xrightarrow[\substack{\vdots \\ R_{1000} = R_{1000} - R_{999}}]{\substack{R_2 = R_2 - R_1 \\ R_3 = R_3 - R_2}} \begin{bmatrix} 2 & 3 & 4 & 5 & \dots & 1000 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

$\det A = 0$

Be sure to justify your answers.

6. For each of the following, give an example 2×2 real matrix A of the indicated kind:

(a) (3 marks) A is *not* diagonal and $\det(A - \lambda I) = -1 + \lambda^2$;

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $A - \lambda I = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$.

$\det(A - \lambda I) = ad - (a+d)\lambda + \lambda^2 - bc$

$ad - bc - (a+d)\lambda + \lambda^2 = 1 - \lambda^2$.

$ad - bc = -1$ $\quad a = 1 \quad d = -1$

$a + d = 0$ $\quad b = 1 \quad c = 2$.

$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

(b) (3 marks) A is *diagonalizable* with $\det(A - \lambda I) = (2 - \lambda)^2$;

$A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$. $A - \lambda I = \begin{bmatrix} a-\lambda & 0 \\ 0 & d-\lambda \end{bmatrix}$.

$\det(A - \lambda I) = ad - (a+d)\lambda + \lambda^2 = 4 - 4\lambda + \lambda^2$.

$\begin{cases} ad = 4 \\ a + d = 4. \end{cases}$

$a(4 - a) = 4$.

$a^2 - 4a + 4 = 0$

$a = 2 \quad \therefore d = 2$

$\therefore A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

(c) (3 marks) A is *non-diagonalizable* and $(2 - \lambda)^2$. $= \det(A - \lambda I)$.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad A - \lambda I = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$.

$4 - 4\lambda + \lambda^2 = ad - (a+d)\lambda + \lambda^2 - bc$

$ad - bc = 4$ $\quad a = 1$

$a + d = 4$. $\quad d = 3$

$b = -1$

$c = 4$.

$A = \begin{bmatrix} 1 & -1 \\ 4 & 3 \end{bmatrix}$.

In each case, you must justify how you know A has the indicated properties.

7. Suppose a matrix 4×4 real matrix A has eigenvalues $\lambda = -1, 1, 2, 3$, and let $B = A^2$.

(a) (2 marks) Is B diagonalizable? Explain.

$\because$ A is diagonalizable

$\therefore$ $A = PDP^{-1}$

$B = PD^2P^{-1}$

which $D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$, $D^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$.

$\therefore$ B is diagonalizable.

(b) (3 marks) Find the eigenvalues of B and their geometric multiplicities.

$\lambda_B = 1, 1, 4, 9$.

geo Mulipliciater for 1 : 2

geo Mulipliciater for 4 : 1

geo Mulipliciater for 9 : 1

(c) (2 marks) Find the characteristic polynomial of B.

$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$

characteristic polynomial: $(\lambda - 1)^2 (\lambda - 4)(\lambda - 9)$

8. Consider the 2×2 real matrices $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

(a) (2 marks) Show that A and D have the same eigenvalues.

$\det(\lambda I - A) = 0$

$\lambda I - A = \begin{bmatrix} \lambda-2 & -1 \\ 0 & \lambda-1 \end{bmatrix}$.

$(\lambda-2)(\lambda-1) = 0$

$\lambda_1 = 2$

$\lambda_2 = 1$

$\det(\lambda I - D) = 0$

$\lambda I - D = \begin{bmatrix} \lambda-2 & 0 \\ 0 & \lambda-1 \end{bmatrix}$

$(\lambda-2)(\lambda-1) = 0$

$\lambda_1 = 2$

$\lambda_2 = 1$.

(b) (4 marks) Find an eigenbasis of A.

$\lambda = 2$:

$A - 2I = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & -1 & 2 \end{array}\right]$

$\xrightarrow{R_2 = R_2 + R_1} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 2 \end{array}\right]$.

eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$\lambda = 1$:

$A - I = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 2 \end{array}\right]$.

(c) (3 marks) Find an invertible matrix P satisfying $P^{-1}AP = D$.

$P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

Prove: $P^{-1} = \frac{1}{\det P} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

$P^{-1}AP = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = D$.

Be sure to justify your answers.

9. Let W be the subspace of $\mathbb{R}^3$ spanned by the vectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

   (a) (3 marks) Find an orthogonal basis of W.

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$a_2 = a x_1 + b x_2 = \begin{bmatrix} a+3b \\ a+2b \\ a+b \end{bmatrix}.$$

$a_1 \, a_2 = 0$

$3a + 6b = 0.$

$a = -2 \quad b = 1.$

$\therefore$ orthogonal basis

$\therefore a_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$

   (b) (3 marks) Find an orthonormal basis of the orthogonal complement $W^{\perp}$.