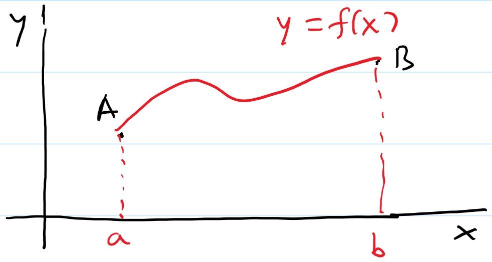
Lecture 12

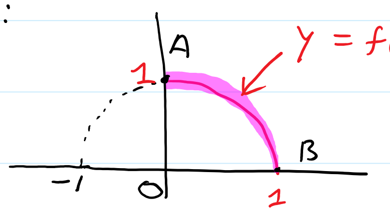
February 26, 2020 8:41 PM

Arc Length of a curve (sec 8.1)



Consider a function $y = f(x)$ defined on $[a, b]$. We want to compute the arc length of the arc $\overset{\frown}{AB}$.

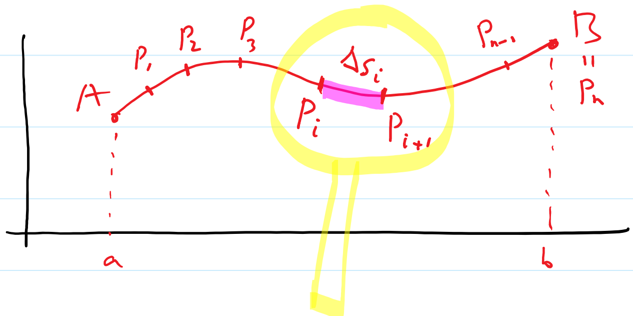
Ex1:

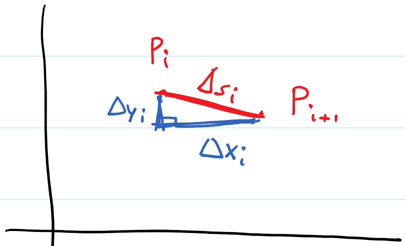


$y = f(x) = \sqrt{1-x^2}$ defined on $[0,1]$.

Then the arclength of the arc $\overset{\frown}{AB}$ is $\frac{1}{4}$ of the circumference of the circle of radius 1.

$$\overset{\frown}{AB} = L = \frac{1}{4}\left(2\pi(1)\right) = \frac{\pi}{2}$$





$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

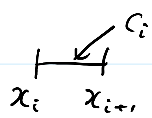
$$= \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\ \Delta x_i$$

$$L \approx \sum_{n=1}^{n} \Delta s_i$$

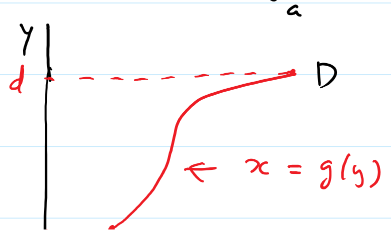
$$\approx \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)}\ \Delta x_i$$

$$\therefore\ L = \lim_{n\to\infty} \sqrt{1 + \underbrace{\left(\frac{\Delta y_i}{\Delta x_i}\right)^2}_{\downarrow\ \left(\frac{dy}{dx}\right)^2}}\ \Delta x_i$$

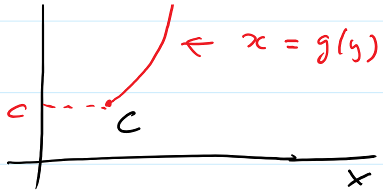
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum f(c_i)\,\Delta x_i$$



$$\therefore\ L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$



If the arc $\overset{\frown}{CD}$ has the equation $x = g(y)$ defined on $[c, d]$ then

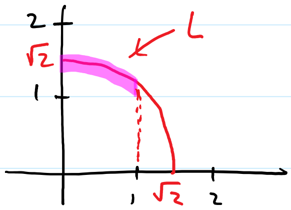


$x = g(y)$ defined on $[c, d]$ then

$$L = \widehat{CD} = \int_c^d \sqrt{1 + (g'(y))^2}\, dy$$

Ex 2: Use the arc length formula to obtain the arclength of the curve $y = f(x) = \sqrt{2 - x^2}$, $0 \le x \le 1$. Verify your answer by noting that the curve is a portion of the circle $x^2 + y^2 = 2$.

Solution



$$y = f(x) = \sqrt{2 - x^2}$$

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{2 - x^2}} = \frac{-x}{\sqrt{2 - x^2}}$$

$$\therefore\ L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 \sqrt{1 + \frac{x^2}{2 - x^2}}\, dx$$

$$= \int_0^1 \sqrt{\frac{2 - \cancel{x^2} + \cancel{x^2}}{2 - x^2}}\, dx$$

$$L = \int_0^1 \frac{\sqrt{2}}{\sqrt{2 - x^2}}\, dx = \int_0^{\pi/4} \frac{\sqrt{2}\ \sqrt{2}\cos\theta\, d\theta}{\sqrt{2 - 2\sin^2\theta}}\, d\theta$$

$x = \sqrt{2}\sin\theta \Rightarrow \sin\theta = \frac{x}{\sqrt{2}}$ $\qquad x = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

$dx = \sqrt{2}\cos\theta\, d\theta$ $\qquad x = 1 \Rightarrow \sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$$\therefore\ L = \int_0^{\pi/4} \frac{\sqrt{2}\ \cancel{\sqrt{2}}\ \cancel{\cos\theta}}{\cancel{\sqrt{2}}\ \cancel{\cos\theta}}\, d\theta = \sqrt{2}\int_0^{\pi/4} d\theta = \sqrt{2}\ (\theta)\Big|_0^{\pi/4} = \frac{\pi\sqrt{2}}{4}\ //\text{Ans.}$$

$$\int \frac{\sqrt{2}}{\sqrt{2 - x^2}}\, dx = \sqrt{2}\int d\theta = \sqrt{2}\,\theta + C = \sqrt{2}\sin^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

$$\theta = \sin^{-1}\left(\frac{x}{\sqrt{2}}\right)$$

$$\therefore\ \int_0^1 \frac{\sqrt{2}}{\sqrt{2 - x^2}}\, dx = \sqrt{2}\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\Big|_0^1$$

$$= \sqrt{2}\left(\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1}(0)\right)$$

$$= \sqrt{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi\sqrt{2}}{4}\ //\text{Ans.}$$

N.B.: If we compute the arc length L for f defined on $[0, \sqrt{2}]$

$$L = \int^{\sqrt{2}} \frac{\sqrt{2}}{\quad} \qquad \Big|^{\sqrt{2}}$$

Ex.1): If we compute the arc length L for f defined on $[0, \sqrt{2}]$

$$L = \int_0^{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{2-x^2}}\,dx = \sqrt{2}\,\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\Big|_0^{\sqrt{2}}$$

$$= \sqrt{2}\left[\underbrace{\sin^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)}_{\frac{\pi}{2}} - \cancel{\sin^{-1}(0)}^{\,0}\right] = \sqrt{2}\,\frac{\pi}{2}$$

check: the circumference of the circle is

$$C = 2\pi r = 2\pi(\sqrt{2}) = 2\pi\sqrt{2}$$

$$\therefore \quad L = \frac{1}{4}C = \frac{1}{4}(2\pi\sqrt{2}) = \frac{\pi\sqrt{2}}{2}$$

Ex2: Find the arc length of the curve

$$y = \frac{x^3}{3} + \frac{1}{4x}, \quad 1 \le x \le 2.$$

Solution:

$$y' = \frac{3x^2}{3} - \frac{1}{4x^2} = x^2 - \frac{1}{4x^2}$$

$$ds = \sqrt{1+y'^2}\,dx = \sqrt{1 + \left(x^2 - \frac{1}{4x^2}\right)^2}\,dx$$

$$\therefore \quad L = \int ds = \int_1^2 \sqrt{1 + \left(x^2 - \frac{1}{4x^2}\right)^2}\,dx$$

$$= \int_1^2 \sqrt{1 + \left(x^4 - \frac{1}{2} + \frac{1}{16x^4}\right)}\,dx$$

$$= \int_1^2 \sqrt{x^4 + \frac{1}{2} + \frac{1}{16x^4}}\,dx$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$\left(x^2 + \frac{1}{4x^2}\right)^2 = (x^2)^2 + 2(\cancel{x^2})\left(\frac{1}{4\cancel{x^2}}\right) + \left(\frac{1}{4x^2}\right)^2$$

$$= x^4 + \frac{1}{2} + \frac{1}{16x^4}$$

$$= \int_1^2 \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2}\,dx$$

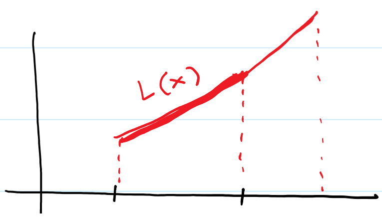
$$= \int_1^2 \left(x^2 + \frac{1}{4x^2}\right)dx$$

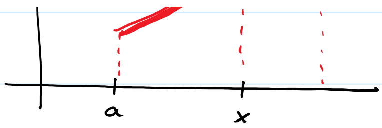
$$= \left(\frac{x^3}{3} - \frac{1}{4x}\right)\Big|_1^2$$

$$= \frac{8}{3} - \frac{1}{8} - \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= \frac{7}{3} + \frac{1}{8} = \frac{56+3}{24} = \frac{59}{24} \quad // \text{Ans.}$$

Arc Length function





Consider a curve whose equation is $y = f(x)$. Then the arc length $L(x)$ defined on $[a, x]$ is a function of x.

$$L(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

FTC Part I

$$\frac{d}{dx}\int_a^{x} f(t)\,dt = f(x)$$

Ex3: Find the arc length function of the curve

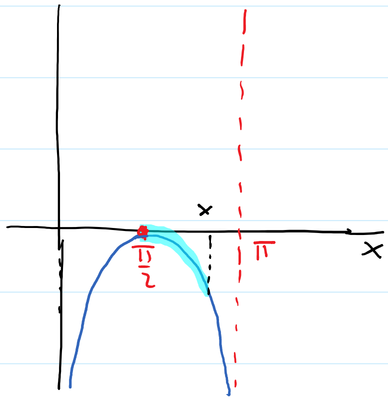
$y = f(x) = \ln(\sin(x))$, $0 < x < \pi$, with the starting point $\left(\frac{\pi}{2}, 0\right)$.

Solution

$$f(x) = \ln(\sin(x))$$

$x \to 0^+ \Rightarrow f(x) \to \lim_{x\to 0} \ln(\sin(x)) = -\infty \Rightarrow x = 0$ (the y-axis) is a vertical asymptote

$x \to \pi^- \Rightarrow f(x) \to \lim_{x\to\pi^-} \ln(\sin(x)) = -\infty \Rightarrow x = \pi$ is also a vertical asymptote



$x = \frac{\pi}{2} \Rightarrow \sin\left(\frac{\pi}{2}\right) = 1$

$f\left(\frac{\pi}{2}\right) = \ln\left(\sin\left(\frac{\pi}{2}\right)\right) = \ln 1 = 0$

$$f'(x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{\cos x}{\sin x}$$

$$\therefore L(x) = \int_{\frac{\pi}{2}}^{x} \sqrt{1 + (f'(t))^2}\, dt$$

$$\therefore L(x) = \int_{\pi/2}^{x} \sqrt{1 + \left(\frac{\cos t}{\sin t}\right)^2}\, dt$$

$$= \int_{\pi/2}^{x} \sqrt{\underbrace{1 + \cot^2(t)}_{\csc^2 t}}\, dt$$

$$= \int_{\pi/2}^{x} \csc t\, dt$$

$$= \ln|\csc t - \cot t|\Big|_{\pi/2}^{x}$$

$$= \ln|\csc x - \cot x| - \ln\left|\csc\frac{\pi}{2} - \cot\frac{\pi}{2}\right|$$

$$= \ln|\csc x - \cot x| - \ln|\cancel{1 - 0}|$$

$$= \ln|\csc x - \cot x| \quad // \text{Ans.}$$

$\int \csc t\, dt = -\csc t \cot t$?

$\frac{d}{dt}(\csc t) = -\csc t \cot t$

$= \ln|\csc t - \cot t|$

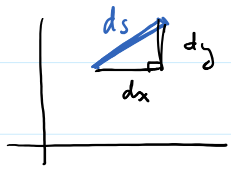
OR $= -\ln|\csc t + \cot t|$

$\csc\frac{\pi}{2} = \frac{1}{\sin\left(\frac{\pi}{2}\right)} = 1$

$\cot\frac{\pi}{2} = \frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}} = \frac{0}{1} = 0$

<u>N.B</u>: How to memorize the arc length formula

$y = f(x)$ defined on $[a,b]$



$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \sqrt{1 + (f'(x))^2}\, dx$$

$$\therefore\ L = \int ds = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$