Cross products and Translation

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Cross product

Definition Let $\vec{u} = (u_1, u_2, u_3)$ and $v_1, \vec{v_2}, v_3$ in $\mathbb{R}^3$. The *cross product* of $\vec{u}$ and $\vec{v}$ is the vector

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$$\begin{aligned} \vec{u} \times \vec{v} &= (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1) \\ &= (2 \times (-3) - 1 \times 0, 1 \times 2 - 1 \times (-3), 1 \times 0 - 2 \times 2) \\ &= (-6, 5, -4). \end{aligned}$$

For memorising

A (2×2)-matrix is given by the following

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By the notation of determinant, the cross product of $\vec{u} \times \vec{v}$ is given by

$$\left(\det \begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix}, \ -\det \begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix}, \ \det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \right)$$

where $\begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix}$, $\begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix}$ and $\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$ are obtained by deleting the first, second and third columns of

$$\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}.$$

Theorem Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^3$. Let c be a scalar. Then

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

$$\begin{cases} \vec{u} \times (\vec{v} + \vec{w}) & = \vec{u} \times \vec{v} + \vec{u} \times \vec{w} \\ (\vec{u} + \vec{v}) \times \vec{w} & = \vec{u} \times \vec{w} + \vec{u} \times \vec{w} \end{cases} \quad \text{(distributive law)}$$

$$c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v}) \quad \text{(scalars factor out)}$$

$$\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$$

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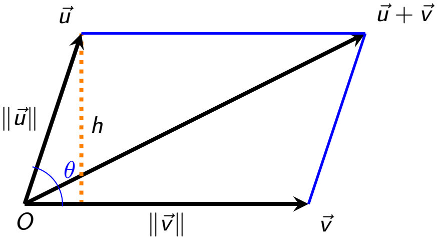
$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \, sin\theta = \sqrt{\|\vec{u}\|^2 \, \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2}$$

where θ is the angle determined by $\vec{u}$ and $\vec{v}$.

Theorem The area of the parallelogram determined by $\vec{u}$ and $\vec{v}$ is given by $\|\vec{u} \times \vec{v}\|$.



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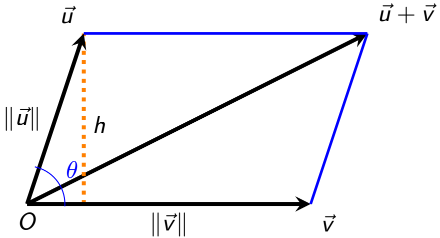
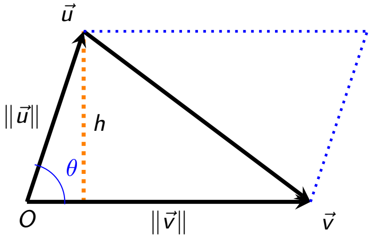


Example Consider the vectors $\vec{u} = (1, 2, 3)$ and $\vec{v} = (2, 1, 0)$. Find the area of the parallelogram determined by these vectors.

Theorem The area of the triangle determined by $\vec{u}$ and $\vec{v}$ is given by $\frac{1}{2}\,\|\vec{u} \times \vec{v}\|$.



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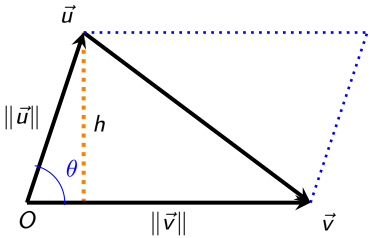

Example Find the area of the triangle OAB, where O is the origin, A is the point $(2, -3, 1)$ and B is the point $(4, 6, 2)$.

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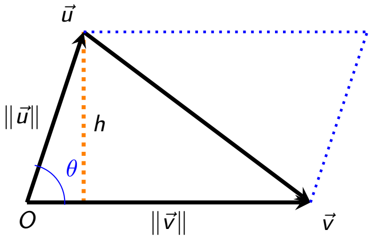


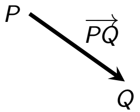
Example Find the area of the triangle OAB, where O is the origin, A is the point $(2, -3, 1)$ and B is the point $(4, 6, 2)$.

How about the area of the triangle $O'AB$, where O' is the point $(1, 4, -2)$ and A, B are the same as above?

Translation

Directed line segment $\overrightarrow{PQ}$

For points P and Q in $\mathbb{R}^2$ or $\mathbb{R}^3$, we denote the directed line segment from P to Q by $\overrightarrow{PQ}$.



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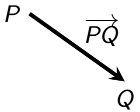
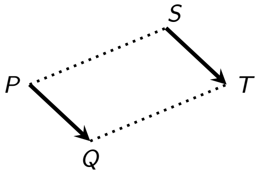


Definition Two directed line segments $\overrightarrow{PQ}$ and $\overrightarrow{ST}$ are *equivalent* if they have the same direction and length.



Theorem Let $\overrightarrow{PQ}$ be a directed line segment from P to Q, where P and Q are two distinct points in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $\overrightarrow{PQ}$ is equivalent to the vector $\vec{u} - \vec{v}$, where $\vec{u} = \overrightarrow{OQ}$ and $\vec{v} = \overrightarrow{OP}$ and O denotes the origin.

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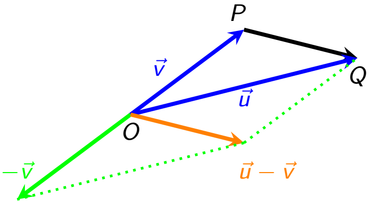

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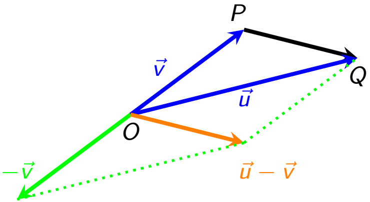

Remark The process of replacing $\overrightarrow{PQ}$ by the vector $\vec{u} - \vec{v}$ is called *translating P to the origin*.

Definition The process of replacing a directed line segment $\overrightarrow{PQ}$ with the equivalent vector is called *translating* $\overrightarrow{PQ}$ to the origin. Similarly, the process of replacing the vector $\vec{v}$ with an equivalent directed line segment which starts at some point P is called translating $\vec{v}$ to P.

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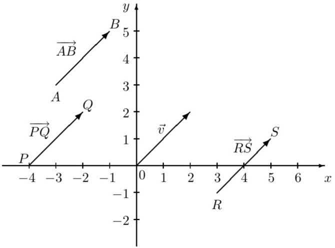
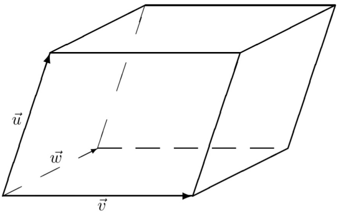


*Pic is from the online note.

Example

Find the area of the triangle $O'AB$, where O' is the point $(1, 4, -2)$, A is the point $(2, -3, 1)$ and B is the point $(4, 6, 2)$.

Definition The parallelepiped determined by vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ is the 6-faced solid whose faces are the parallelograms determined by $\vec{u}$ and $\vec{v}$, by $\vec{u}$ and $\vec{w}$, and by $\vec{v}$ and $\vec{w}$.



Theorem The volume of the parallelepiped determined by the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ is given by

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Example Find the volume of the parallelepiped determined by the directed line segments $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ for the points $A(1,0,1)$, $B(2,1,1)$, $C(2,2,1)$ and $D(1,1,2)$.