Properties of determinants (1)

Recap

Definition of determinant

Let $A = [a_{ij}]$ be a square matrix of order n.

The expansion along the i-th row is given by

$$\det A = (-1)^{i+1} a_{i1} \det A_{i1} + (-1)^{i+2} a_{i2} \det A_{i2} + \ldots + (-1)^{i+n} a_{in} \det A_{in}$$

The expansion along the j-th column is given by

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A note: Expand a row or a column of a square matrix A which contains the most zeros to find $\det A$.

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$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \text{ and } B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ ca_{i1} & ca_{i2} & \dots & ca_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

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Expansion along the i-th row of B, we have

$$\det B = (-1)^{i+1}(ca_{i1}) \det A_{i1} + \ldots + (-1)^{i+n}(ca_{in}) \det A_{in} = c \det A.$$

(1) Compute $\det A$ and $\det B$

$$A \underset{\frac{2}{3}}{=} \begin{array}{c} \begin{matrix} 4 & 0 & 24. \end{matrix} \\ \begin{bmatrix} 2 & 0 & 8 \\ 1 & -6 & 2 \\ 3 & 9 & 12 \end{bmatrix} \end{array} \quad \text{and} \quad B = \begin{array}{l} (\times 4) \\ \begin{bmatrix} \frac{1}{2} & -\frac{1}{6} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{4}{3} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix} \end{array}$$

(2) Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$ such that $\det A = -3$. Find $\det B$ and $\det C$, where

$$B = \begin{bmatrix} 3a & 9b & 3c \\ d & 3e & f \\ g & 3h & i \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 2g & 2h & 2k \end{bmatrix}.$$

$\det B = -9.$ $\cancel{\det C = -12.}$

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Example Find the determinant of $-3I$, where I is the identity matrix of order 17.

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Example Find the determinant of $-3I$, where I is the identity matrix of order 17. $\det(-3I) = (-3)^{17}$

Note that $\det(cA) \neq c \det A$ in general.

$\det I = 1.$

$\cancel{c \det}$ d

$\det -3I = -3^{17}.$

Theorem C Let A be a square matrix. If A has two identical rows (or two columns), then $\det A = 0$.

Theorem A and Theorem C imply the following statement.

Corollary If a square matrix A has a row that is a scalar multiple of another row (or a column is a scalar multiple of another column), then $\det A = 0$.

Examples

Compute det A

$$A = \begin{bmatrix} 2 & 1 & 3 & -4 \\ -1 & 0 & 1 & 2 \\ -3 & 2 & -1 & 6 \\ 4 & 1 & 4 & -8 \end{bmatrix}$$

Find det A.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 20 & 1 & -2 & 3 \\ 2 & 0 & -2 & 5 \\ 13 & -7 & 14 & -21 \end{bmatrix}$$

Find the determinant of the following matrices.

$$\begin{bmatrix} 0 & 2 & 4 \\ 1 & 13 & 26 \\ 0 & 4 & 16 \end{bmatrix}, \qquad \begin{bmatrix} 42 & 5 & 1 \\ 84 & 0 & 0 \\ 63 & 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 4 & 2 & 0 \\ 0 & 0 & 7 & 2 & 3 \\ -2 & 8 & 4 & 2 & 2 \\ 0 & 1 & 2 & 1 & 0 \end{bmatrix}$$

Find $\det A + (\det B)(\det C) + \det D$, where C is the identity matrix of order 6, and A, B, D are the following matrices

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 7 & 3 & 0 & 1 \\ 2 & 1 & 0 & 3 \\ 5 & 2 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 5 & -4 \\ 0 & -3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & -5 & 2 & -3 \\ -2 & 3 & 3 & -1 \\ -1 & 5 & -2 & 3 \\ 0 & -5 & 0 & 6 \end{bmatrix}$$