Math 2155, Fall 2021: Homework 10

Instructions for GradeScope:

You may handwrite your answers or type them. Only type solutions if you are able to type the appropriate symbols. If **typed**, use at least a 12pt font, with **one question per page**. You can then directly submit the pdf file to GradeScope.

If **handwritten**:

- Don't erase or cross out more than a word or two.
- Only write on one side of the page. Use dark, large, neat writing.
- If you have a real scanner, great! Most people will scan their solutions using their phone. Use one of the recommended apps: For iOS, use Scannable by Evernote or Genius Scan. For Android, use Genius Scan. Do *not* just take regular photos.
- When scanning, have good lighting and try to avoid shadows.
- It is best to have **one question per scan**. If the solution is short, **fold the page in half** and scan just the half it is on, so there isn't so much blank space. Or, you can scan and then crop the scan. You don't need to scan parts (a), (b), etc. separately or put them on separate pages.
- It works well to scan each question separately and produce one pdf file with one question per page. Most scanning apps will automatically combine your images into one pdf file.
- You must **check the quality** of your scans afterwards, and rescan if needed.

Make sure you are going to `http://gradescope.ca` not `http://gradescope.com`. You can access it through the course OWL site.

You can **resubmit** your work any number of times until the deadline.

See the GradeScope help website for lots of information: `https://help.gradescope.com/`
Select "Student Center" and then either "Scanning Work on a Mobile Device" or "Submitting an Assignment".

Instructions for writing solutions:

- Homework is graded both on **correctness** and on **presentation**.
- Your proofs should be written in **complete sentences**, starting with capitals and ending with periods. They should be in **paragraph form**, similar to the proofs in the textbook. For example, you shouldn't put each sentence on a new line. Paragraphs should be used to break the proof into logical chunks.
- Don't use informal abbreviations such as three dots for "therefore." Logical symbols can be used when they clarify things to the reader, but English words can often be more effective.
- Include all of the steps that are needed to logically justify every claim you make. Do not include unnecessary steps. Try to be **concise and complete**.
- Do not submit any **rough work**. Do your rough work on scratch paper, and only submit a clearly and neatly written answer. Do not cross out or erase more than a word or two. If you write each final solution on a new page, it's easy to start over on a fresh page when you've made a large error.
- You should do the work **on your own**. Read the course syllabus for the rules about scholastic offences, which include sharing solutions with others, uploading material to a website, viewing material of others or on a website (even if you don't use it), etc. The penalty for cheating on homework will be a grade of **0** on the homework set as well as a penalty of **negative 5%** on the overall course grade.
- We may not grade every question.

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Due **Friday, November 26 at 11:59pm**. You can **resubmit** your work any number of times until the deadline.

H10Q1: Prove: For all $n \in \mathbb{Z}$, if $n \equiv 2 \pmod 6$ or $n \equiv 4 \pmod 6$, then $n^2 \equiv 4 \pmod{12}$.

Solution: Let $n \in \mathbb{Z}$. Suppose that $n \equiv 2 \pmod 6$ or $n \equiv 4 \pmod 6$. We consider two cases.

Case 1: $n \equiv 2 \pmod 6$. Then $6 \mid (n-2)$, so there exists an integer k such that $n - 2 = 6k$. That is, $n = 6k + 2$. Therefore,

$$n^2 = 36k^2 + 24k + 4 = 12(3k^2 + 2k) + 4.$$

So $n^2 - 4 = 12(3k^2 + 2k)$ is divisible by 12, since $3k^2 + 2k$ is an integer. Therefore, $n^2 \equiv 4 \pmod{12}$.

Case 2: $n \equiv 4 \pmod 6$. Then, as above, $n = 6k + 4$ for some integer k. Then

$$n^2 = 36k^2 + 48k + 16 = 12(3k^2 + 4k + 1) + 4.$$

Therefore, $n^2 - 4$ is divisible by 12 and so $n^2 \equiv 4 \pmod{12}$.

By assumption, the cases are exhaustive, so we have proved the claim.

H10Q2: Consider the relation S on $\mathbb{R}$ defined by xSy if and only if $\exists a \in \mathbb{R}\,(a > 0 \wedge x = ay)$. Prove that S is an equivalence relation, and completely describe $\mathbb{R}/S$ and all of the equivalence classes.

Solution: Proof that S is an equivalence relation:

Reflexivity: Let $x \in \mathbb{R}$. Then xSx since $x = 1x$.

Symmetry: Let $x, y \in \mathbb{R}$ and assume xSy. There there is an $a > 0$ such that $x = ay$. Therefore, $y = \frac{1}{a}x$ (using that $a \neq 0$). Note that since $a > 0$, $\frac{1}{a} > 0$ as well. So ySx, as required.

Transitivity: Let $x, y, z \in \mathbb{R}$ be such that xSy and ySz. There there are $a > 0$ and $b > 0$ such that $x = ay$ and $y = bz$. Therefore, $x = (ab)z$. Since $a > 0$ and $b > 0$, we have $ab > 0$. So xSz.

Determining $\mathbb{R}/S$:

Note that if $x > 0$ and $y > 0$, then xSy, since $x = (x/y)y$ and $x/y > 0$. Similarly, if $x < 0$ and $y < 0$, then xSy, since $x = (x/y)y$ and again $x/y > 0$. However, 0 is only related to itself, since $a0 = 0$ for any a, and $ax \neq 0$ when $a > 0$ and $x \neq 0$. So there are three equivalence classes: $[1] = \mathbb{R}^+$, $[-1] = \mathbb{R}^-$, and $[0] = \{0\}$. And $\mathbb{R}/S = \{[-1], [0], [1]\}$.

H10Q3: Write down all functions from $A = \{1, 2\}$ to $B = \{3, 4\}$ as sets of ordered pairs. No need for explanation.

Solution: There are four functions corresponding to the four ways to replace the question marks in $\{(1, ?), (2, ?)\}$ with elements of B:

$f_1 = \{(1, 3), (2, 3)\}$

$f_2 = \{(1, 3), (2, 4)\}$

$f_3 = \{(1, 4), (2, 3)\}$

$f_4 = \{(1, 4), (2, 4)\}$

H10Q4: Let A be a set and let $\mathcal{F}$ be a partition of A. Consider the relation T from A to $\mathcal{F}$ defined by

$$aTB \text{ iff } a \in B,$$

for $a \in A$ and $B \in \mathcal{F}$. Give a careful proof that T is a function.

Solution: T is a function.

We must show that for each $a \in A$, there is a unique $B \in \mathcal{F}$ such that $a \in B$.

Let $a \in A$.

Existence: Since $\mathcal{F}$ is a partition of A, we know that $\bigcup_{B \in \mathcal{F}} B = A$. Since $a \in A$, we have $a \in \bigcup_{B \in \mathcal{F}} B$, so there exists a $B \in \mathcal{F}$ such that $a \in B$, as required.

Uniqueness: Suppose that there exist B and B' in $\mathcal{F}$ with $a \in B$ and $a \in B'$. Then $B \cap B' \neq \emptyset$. Since $\mathcal{F}$ is a partition, distinct elements are disjoint. So we must have $B = B'$, as required.

(This applies to A/R for any equivalence relation R on A. It produces the function sending $x \in A$ to $[x]_R$.)