Regular Language Closure Properties

COMPSCI 3331

Closure Properties: Outline

- Closure Properties of Regular Languages.
- Union, Intersection, Complementation, Concatenation, Kleene Closure, Reversal.
- Non-Regular Languages: Pumping Lemma.

What is a closure property?

- A class of languages is closed under the operation $\diamond$ if you can apply $\diamond$ to languages in the language class and always get another language from the same class.
- e.g., "regular languages are closed under union."
- Use any representation of regular languages to prove closure properties: regular expressions, DFAs, (ε-) NFAs.

Why closure properties?

- Closure properties hold for all languages in a language class.
- If a language class is closed under an operation $\diamond$, it is useful to know that: use that fact later to build languages.
- If a language class is not closed under an operation, it is also useful to know: can't use that operation to build languages.
- Useful to compare language classes... do they have different closure properties?

Closure Properties

Are the regular languages closed under...

- union?
- concatenation?
- Kleene closure?
- complement? $\overline{L} = \Sigma^* - L$
- intersection? $L_1 \cap L_2$
- Other operations?

Easy Closure Properties

Theorem. The regular languages are closed under union, concatenation and Kleene closure.

- Easy because of regular expressions.
- What about intersection?
- What about complement?

Not Every Language is Regular

- There are languages that are not regular.
- Intuitively: any language that needs **unbounded** memory to accept words is not regular.

Non-Regular Languages: Example

$$L = \{a^n b^n \ : \ n \geq 0\}.$$

- We have to know if we have seen k occurrences of a for each $k \geq 0$.
- Argue that $L = \{a^n b^n \ : \ n \geq 0\}$ is not regular **by contradiction**.

Pumping Lemma for Regular Languages

Lemma. Let $L \subseteq \Sigma^*$ be a regular language. There exists a constant $n \geq 0$ (depending on L) such that for all $z \in L$ with $|z| \geq n$, we can write $z = uvw$ for words $u, v, w \in \Sigma^*$ such that

- $|uv| \leq n$;
- $v \neq \varepsilon$; and
- $uv^i w \in L$ for all $i \geq 0$.

Pumping Lemma for Regular Languages

- How do we prove the pumping lemma?

How to use the Pumping Lemma

Use the **contrapositive**:

IF the conditions of PL are **not** satisfied,
THEN L is **not** regular.

The pumping lemma **cannot** be used to prove that a language L **is** regular.

Contrapositive of Pumping Lemma

Let L be a language such that **for all** $n \geq 0$, **there exists** a word $z \in L$ with $|z| \geq n$ such that **for all** ways of writing $z = uvw$ (with $|uv| \leq n$ and $v \neq \varepsilon$), **there exists** an $i \geq 0$ such that

$$uv^i w \notin L$$

THEN L is not a regular language.

- **for all**: you have no control, cannot make any assumptions.
- **there exists**: you have control, pick something that makes things easy for you.

Pumping Lemma Example

- Show $L = \{a^n b^n \ : \ n \geq 0\}$ is not regular.

Using the Pumping Lemma Well

- Pick a word $z \in L$ such that for any n, $|z| \geq n$; i.e., z has to depend on n in some way.
- Consider all the ways to decompose your chosen word z into $z = uvw$ with $|uv| \leq n$ and $v \neq \varepsilon$.
- Pick an i which helps you out: $i = 0, 2$ are your best bets.