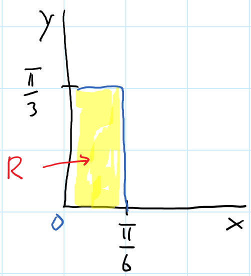
Q1. Evaluate $I = \iint_R x \sin(x+y)\, dA$ where $R = [0, \frac{\pi}{6}] \times [0, \frac{\pi}{3}]$.



$$I = \int_0^{\pi/6} \int_0^{\pi/3} x \sin(x+y)\, dy\, dx$$

$$= \int_0^{\pi/6} x \left[-\cos(x+y)\right]\Big|_{y=0}^{\pi/3} dx$$

$$= \int_0^{\pi/6} \left[-x\cos\left(x+\frac{\pi}{3}\right) + x\cos(x)\right] dx$$

$$I = -\int_0^{\pi/6} x\cos\left(x+\frac{\pi}{3}\right) dx + \int_0^{\pi/6} x\cos x\, dx$$

$$\underset{\text{IBPs}}{=} -\left[x\sin\left(x+\frac{\pi}{3}\right) - (1)\left(-\cos\left(x+\frac{\pi}{3}\right)\right)\right]\Big|_0^{\pi/6}$$

$$+ \left[x\sin x - (-\cos x)\right]\Big|_0^{\pi/6}$$

$$= \left(-x\sin\left(x+\frac{\pi}{3}\right) - \cos\left(x+\frac{\pi}{3}\right)\right)\Big|_0^{\pi/6} + \frac{\pi}{6}\underbrace{\sin\frac{\pi}{6}}_{=\frac{1}{2}} + \cos\frac{\pi}{6} - \cos 0$$

$$= -\frac{\pi}{6}\underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} - \underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} + \cos\left(\frac{\pi}{3}\right) + \frac{\pi}{6}\left(\frac{1}{2}\right) + \frac{\sqrt{3}}{2} - 1$$

$$= -\frac{\pi}{6} + \frac{1}{2} + \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

$$I = -\frac{\pi}{12} + \frac{\sqrt{3}}{2} - \frac{1}{2} \quad // \text{Ans.}$$

Q2. Sketch the region of integration and change the order of integration of

$$I = \int_1^2 \int_0^{\ln x} f(x,y)\, dy\, dx$$

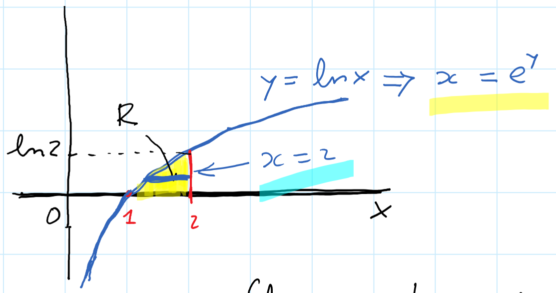
Solution

$$R = \{(x,y) \in \mathbb{R}^2 \mid 1 \le x \le 2,\ 0 \le y \le \ln x\}$$

$y_1 = 0$ and $y_2 = \ln x$

$y = \ln x \Rightarrow x = e^y$

$y_1 = 0$ and $y_2 = \ln x$



$$I = \int_0^{\ln 2} \int_{e^y}^{2} f(x,y)\, dx\, dy \quad // \text{Ans.}$$

Check: let $f(x,y) = 1$. Then

$$A(R) = \int_1^2 \int_0^{\ln x} dy\, dx = \int_1^2 \ln x\, dx = (x\ln x - x)\Big|_1^2$$

$$= 2\ln 2 - 2 - (\cancel{\ln 1}^{\,0} - 1)$$

$$= 2\ln 2 - 1 \quad // \text{Ans.}$$

Also
$$A(R) = \int_0^{\ln 2} \int_{e^y}^{2} dx\, dy = \int_0^{\ln 2} (2 - e^y)\, dy = (2y - e^y)\Big|_0^{\ln 2}$$

$$= 2\ln 2 - \underbrace{e^{\ln 2}}_{=2} - (0 - e^0)$$

$$= 2\ln 2 - 2 + 1$$

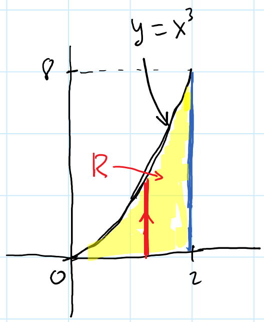
$$= 2\ln 2 - 1 \quad // \text{Ans.}$$

the same!

Q3. Evaluate $I = \int_0^8 \int_{\sqrt[3]{y}}^{2} e^{x^4}\, dx\, dy$ by reversing the order of integration.

Solution

$$R = \{(x,y) \mid 0 \le y \le 8,\ \sqrt[3]{y} \le x \le 2\}$$



$x_1 = \sqrt[3]{y}$, $x_2 = 2$

$\Updownarrow$

$y = x^3$

$$I = \int_0^2 \int_0^{x^3} e^{x^4}\, dy\, dx$$

$$= \int_0^2 e^{x^4} x^3\, dx$$

$\boxed{u = x^4}$

$du = 4x^3\, dx$

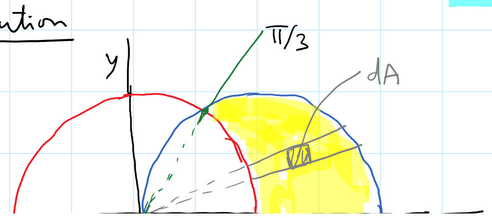
$x^3\, dx = \frac{du}{4}$

$$= \int_0^{16} e^u \left(\frac{du}{4}\right)$$

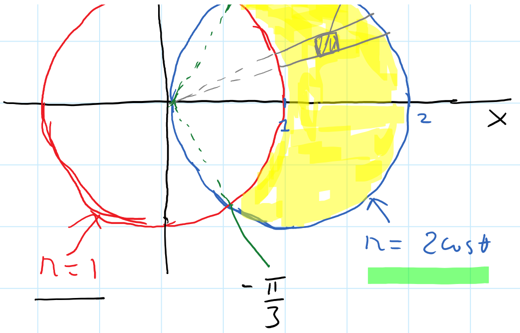
$$= \frac{1}{4} e^u \Big|_0^{16} = \frac{1}{4}(e^{16} - 1) \quad // \text{Ans.}$$

Q4. Find the area of the region inside the circle $(x-1)^2 + y^2 = 1$ and outside the circle $x^2 + y^2 = 1$.

Solution



In polar coordinates, the circle $(x-1)^2 + y^2 = 1$ has polar equation as



circle $(x-1)^2+y^2=1$ has polar equation as

$$r = 2\cos\theta$$

and the circle $x^2+y^2=1$ has its polar equation as

$$r=1$$

Next, we find the points of intersection

$$2\cos\theta = 1$$

$$\therefore \cos\theta = \frac{1}{2}$$

$$\theta = \pm\frac{\pi}{3}$$

$$A(R) = \iint_R dA$$

$$= \int_{-\frac{\pi}{3}}^{\pi/3}\int_1^{2\cos\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/3}\int_1^{2\cos\theta} r\,dr\,d\theta$$

symmetry! We only need to compute the portion above the x-axis then multiply the result by 2

$$= \cancel{2}\int_0^{\pi/3}\left(\frac{r^2}{\cancel{2}}\right)\Big|_{r=1}^{2\cos\theta} d\theta$$

$$= \int_0^{\pi/3}(4\cos^2\theta - 1^2)\,d\theta \qquad \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$= \int_0^{\pi/3}\left(2(1+\cos 2\theta) - 1\right)d\theta$$

$$= \int_0^{\pi/3}(1+2\cos 2\theta)\,d\theta$$

$$= \left[\theta + \sin 2\theta\right]\Big|_0^{\pi/3}$$

$$= \left[\frac{\pi}{3} + \sin\frac{2\pi}{3} - 0\right]$$

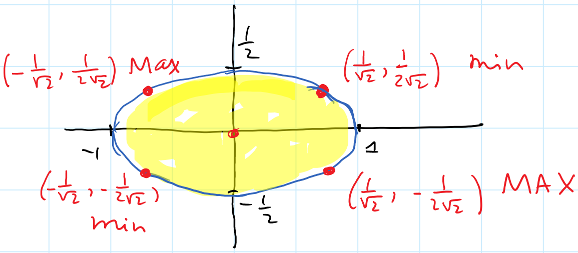
$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2} \quad //\text{Ans.}$$

Q5. Find extreme values of $f(x,y) = e^{-xy}$ on $x^2+4y^2 \leq 1$.

Solution

The boundary of the disk is the ellipse

$$x^2+4y^2 = 1$$



$$f_x = -y e^{-xy} = 0 \Rightarrow y = 0$$
$$f_y = -x e^{-xy} = 0 \Rightarrow x = 0$$

$\therefore$ $(0,0)$ is CP

$$f_{xx} = y^2 e^{-xy}$$
$$f_{xy} = -e^{-xy} - y(-x) e^{-xy} = -e^{-xy} + xy e^{-xy}$$
$$f_{yy} = x^2 e^{-xy}$$

$$H(x,y) = \begin{bmatrix} y^2 e^{-xy} & xy e^{-xy} - e^{-xy} \\ xy e^{-xy} - e^{-xy} & x^2 e^{-xy} \end{bmatrix}$$

At $(0,0)$, $H(0,0) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$$\det(H(0,0)) = (0)^2 - (-1)^2 = -1 < 0$$

$\therefore$ $(0,0)$ is a saddle point.

$\therefore$ The extreme values of f must reside on the ellipse. We want to optimize $f(x,y) = e^{-xy}$ subject to the constraint $g(x,y) = x^2 + 4y^2 - 1 = 0$

Using Lagrange multiplier

$$\nabla f = \lambda \nabla g$$
$$\left(-y e^{-xy}, -x e^{-xy}\right) = \lambda (2x, 8y)$$
$$-y e^{-xy} = 2\lambda x \Rightarrow -2\lambda = \frac{y e^{-xy}}{x}$$
$$-x e^{-xy} = 8\lambda y \Rightarrow -2\lambda = \frac{x e^{-xy}}{4y}$$

$$\therefore \quad \frac{y \cancel{e^{-xy}}}{x} = \frac{x \cancel{e^{-xy}}}{4y} \Rightarrow 4y^2 = x^2 \Rightarrow x = \pm 2y$$

Subst these into the constraint $x^2 + 4y^2 = 1$, we obtain

$$8y^2 = 1 \Rightarrow y^2 = \frac{1}{8} \Rightarrow y = \pm \frac{1}{\sqrt{8}} = \pm \frac{1}{2\sqrt{2}}$$
$$x = \pm \frac{1}{\sqrt{2}}, \mp \frac{1}{\sqrt{2}}$$

$\therefore$ There are 4 points where f has extreme values

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right)$$

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = e^{-\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2\sqrt{2}}\right)} = e^{-\frac{1}{4}} \Rightarrow \text{min value}$$

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = e^{-\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2\sqrt{2}}\right)} = e^{-\frac{1}{4}} \Rightarrow \text{min value}$$

$$f\left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = e^{-\left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2\sqrt{2}}\right)} = e^{\frac{1}{4}} \Rightarrow \text{max value}$$

$$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right) = e^{-\left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{2\sqrt{2}}\right)} = e^{-\frac{1}{4}} \Rightarrow \text{min value}$$

$$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right) = e^{-\left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{2\sqrt{2}}\right)} = e^{\frac{1}{4}} \Rightarrow \text{max value}$$

Q6. The temperature at a point (x, y, z) is given by

$$T(x, y, z) = 200\, e^{-x^2 - 3y^2 - 9z^2}$$

where T is measured in °C and x, y, z in meters.

a) Find the rate of change of temperature at the point $P(2, -1, 2)$ in the direction toward the point $(3, -3, 3)$.

b) In which direction does the temperature increase fastest at P?

c) Find the maximum rate of change at P?

Solution

a)
$$\nabla T = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}$$

$$= 200\left[-2x\, e^{-x^2-3y^2-9z^2}, -6y\, e^{-x^2-3y^2-9z^2}, -18z\, e^{-x^2-3y^2-9z^2}\right]$$

$$\nabla T(2, -1, 2) = 200\left(-4e^{-43}, 6\, e^{-43}, -36 e^{-43}\right)$$

$$= 200\, e^{-43}\,(-4, 6, -36)$$

$$\vec{u} = (3, -3, 3) - (2, -1, 2)$$

$$= (3-2, -3-(-1), 3-2)$$

$$= (1, -2, 1)$$

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{(1, -2, 1)}{\sqrt{(1)^2 + (-2)^2 + (1)^2}} = \frac{(1, -2, 1)}{\sqrt{6}}$$

$$\therefore\ D_{\hat{u}}T(2, -1, 2) = \nabla T(2, -1, 2) \cdot \hat{u}$$

$$= 200\, e^{-43}\,(-4, 6, -36) \cdot \frac{(1, -2, 1)}{\sqrt{6}}$$

$$= \frac{200\, e^{-43}}{\sqrt{6}}\left[(-4)(1) + (6)(-2) + (-36)(1)\right]$$

$$= \frac{200\, e^{-43}}{\sqrt{6}}(-52)$$

$$= -\frac{10400}{\sqrt{6}}\, e^{-43} \quad °C/m \qquad //\text{Ans.}$$

b) In the direction of $\nabla T(2, -1, 2)$, ie, in the direction of $(-2, 3, -18)$, the rate of temperature increases fastest. //Ans.

c) max rate of change of T @ P $= \|\nabla T(2,-1,2)\|$

$$= 200\, e^{-43} \sqrt{(-4)^2 + (6)^2 + (-36)^2}$$

$$= 400\, e^{-43} \sqrt{337} \quad {}^\circ C/m \quad // \text{Ans.}$$

See you in tomorrow EXAM

Saturday, October 24, 2020

from 2:00 PM