Lecture 29.

Recall: ex 5.2.4: $A = \mathbb{R} / \{-1\}$ $f: A \to \mathbb{R}$
$f(x) = \frac{2x}{x+1}$

Proof: f is one to one but not onto.
One to one: $\forall a_1, a_2 \in A$, $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.

Let $a_1, a_2 \in A$. assume $f(a_1) = f(a_2)$
$\Rightarrow 2a_1(a_2+1) = 2a_2(a_1+1)$.
$2a_1a_2 + 2a_1 = 2a_1a_2 + 2a_2$
$a_1 = a_2$.

not onto: try to proof it is onto.
$\forall b \in \mathbb{R}\ \exists a \in A\ f(a) = b$
Let $b \in \mathbb{R}$. assume $a \in A$ such that
$b = \frac{2a}{a+1}$
$b = (2-b)a$
$a = \frac{b}{2-b}$ $\Rightarrow 2 \notin \mathrm{Ran}(f)$
$\Rightarrow f$ is not onto because when $b=2$, $\sim \exists a \in A$, $f(a) = 2$.
Suppose such a exist:
$\frac{2a}{a+1} = 2$.
$2a = 2a+2$
$0 = 2$ $\Rightarrow$ never true, it is a contradiction.

5.2.5: Suppose $f: A \to B$ $g: B \to C$
1. if f and g are one to one, then $g \circ f$ is one to one.
2. same to onto.

Proof 1: suppose f, g are one to one.
Let $a_1, a_2 \in A$, assume $g \circ f(a_1) = g \circ f(a_2)$.
So $g(f(a_1)) = g(f(a_2))$
Since g is one to one, so $f(a_1) = f(a_2)$.
Since f is one to one, so $a_2 = a_1$
So $g \circ f$ is one to one.

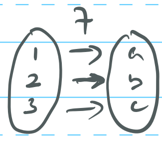
2: Suppose $g \circ f$ is onto.

Let $c \in C$. Since g is onto, there exist $b \in B$, $g(b) = c$.

Since f is onto, there exist $a \in A$, $f(a) = b$.

Then $g \circ f(a) = g(f(a)) = g(b) = c$

§ 5.3 Inverse of Functions



f^{-1} always exist, but not always a function

e.g. if two values has a same output.

uniqueness fail.

Theorem 5.3.1 Suppose $f: A \to B$ is one to one and onto,

then $f^{-1}: B \to A$.

proof: Let $b \in B$.

existence: Since f is onto, $\forall b \in B$, $\exists a \in A$ $f(a) = b$

That is, $(a, b) \in f$, so $(b, a) \in f^{-1}$

uniqueness: suppose $a_1, a_2 \in A$, $(b, a_1), (b, a_2) \in f^{-1}$.

Then, $(a_1, b), (a_2, b) \in f$,

So, $f(a_1) = b = f(a_2)$

Since f is one to one, $a_1 = a_2$ $\square$.

ex: $h: \mathbb{R} \to \mathbb{R}$, $h(x) = 2x + 3$. $h^{-1}(x) = (x-3)/2$.

Suppose $(y, x) \in h^{-1}$ iff $h(x) = y$

$h^{-1}(h(x)) = x$ $h(h^{-1}(y)) = y$

So $h^{-1}(y)$ is the unique x such that $h(x) = y$.

$\Rightarrow h^{-1}(x) = (x-3)/2$.

$h^{-1} \circ h = i_{\mathbb{R}}$ $h \circ h^{-1} = i_{\mathbb{R}}$ $\Uparrow$

Thm 5.3.2. Suppose $f: A \to B$ and $f^{-1}: B \to A$. Then $f \circ f^{-1} = i_A$, $f^{-1} \circ f = i_B$.

Thm 5.3.3: Suppose $f: A \to B$

1. if $\exists g: B \to A$ such that $g \circ f = i_A$, f is one to one
2. if $\exists g: B \to A$ such that $f \circ g = i_B$, then f is onto.

Proof 2: Let $b \in B$: Then $f(g(b)) = f \circ g(b) = i_B(b) = b$

Taking $a \in g(b)$ we have $f(a) = b$.

Since a is arbitrary, f is onto.

Thm 5.3.4: Let $f: A \to B$

5.3.1 $\hookrightarrow$ 1. f is one to one and onto

2. $f^{-1}: B \to A$.

5.3.2 $\hookrightarrow$ 3. $g: B \to A$ such that $f \circ g = i_B$ and $g \circ f = i_A$.

ex: $f: \mathbb{R} \to \mathbb{R}^+$ $f(x) = e^x$

Let $g: \mathbb{R}^+ \to \mathbb{R}$, $g(y) = \ln(y)$

Then $g(f(x)) = g(e^x) = x$

$f(g(y)) = e^{\ln y} = y$

so $f^{-1} = g$