Math 2155, Fall 2021: Homework 11

Instructions for GradeScope:

You may handwrite your answers or type them. Only type solutions if you are able to type the appropriate symbols. If **typed**, use at least a 12pt font, with **one question per page**. You can then directly submit the pdf file to GradeScope.

If **handwritten**:

- Don't erase or cross out more than a word or two.
- Only write on one side of the page. Use dark, large, neat writing.
- If you have a real scanner, great! Most people will scan their solutions using their phone. Use one of the recommended apps: For iOS, use Scannable by Evernote or Genius Scan. For Android, use Genius Scan. Do *not* just take regular photos.
- When scanning, have good lighting and try to avoid shadows.
- It is best to have **one question per scan**. If the solution is short, **fold the page in half** and scan just the half it is on, so there isn't so much blank space. Or, you can scan and then crop the scan. You don't need to scan parts (a), (b), etc. separately or put them on separate pages.
- It works well to scan each question separately and produce one pdf file with one question per page. Most scanning apps will automatically combine your images into one pdf file.
- You must **check the quality** of your scans afterwards, and rescan if needed.

Make sure you are going to `http://gradescope.ca` not `http://gradescope.com`. You can access it through the course OWL site.

You can **resubmit** your work any number of times until the deadline.

See the GradeScope help website for lots of information: `https://help.gradescope.com/`
Select "Student Center" and then either "Scanning Work on a Mobile Device" or "Submitting an Assignment".

Instructions for writing solutions:

- Homework is graded both on **correctness** and on **presentation**.
- Your proofs should be written in **complete sentences**, starting with capitals and ending with periods. They should be in **paragraph form**, similar to the proofs in the textbook. For example, you shouldn't put each sentence on a new line. Paragraphs should be used to break the proof into logical chunks.
- Don't use informal abbreviations such as three dots for "therefore." Logical symbols can be used when they clarify things to the reader, but English words can often be more effective.
- Include all of the steps that are needed to logically justify every claim you make. Do not include unnecessary steps. Try to be **concise and complete**.
- Do not submit any **rough work**. Do your rough work on scratch paper, and only submit a clearly and neatly written answer. Do not cross out or erase more than a word or two. If you write each final solution on a new page, it's easy to start over on a fresh page when you've made a large error.
- You should do the work **on your own**. Read the course syllabus for the rules about scholastic offences, which include sharing solutions with others, uploading material to a website, viewing material of others or on a website (even if you don't use it), etc. The penalty for cheating on homework will be a grade of **0** on the homework set as well as a penalty of **negative 5%** on the overall course grade.
- We may not grade every question.

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Due **Wednesday, December 8 at 11:59pm**. Last one!

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H11Q1: Let $A = [0, \infty)$ and $B = [-2, \infty)$. Consider the function $f : A \to B$ defined by $f(x) = x^2 - 2$. Prove that f is one-to-one and onto and find $f^{-1} : B \to A$.

Hint: you might find it easiest to use Theorems 5.3.4 and 5.3.5 from the text, like the solution of Exercise 3 from Section 5.3 at the back of the book.

Solution: Consider the function $g : B \to A$ defined by $g(y) = \sqrt{y+2}$. Since $y \geq -2$, $\sqrt{y+2}$ is well-defined. Since $\sqrt{\ }$ denotes the positive square root, $\sqrt{y+2}$ is in A.

We first show that $g \circ f = i_A$. Let $x \in A$. Then

$$(g \circ f)(x) = g(f(x)) = g(x^2 - 2) = \sqrt{x^2 - 2 + 2} = \sqrt{x^2} = x,$$

where the last equality uses that $x \geq 0$. So $g \circ f = i_A$.

Next we show that $f \circ g = i_B$. Let $y \in B$. Then

$$(f \circ g)(y) = f(g(y)) = f(\sqrt{y+2}) = (\sqrt{y+2})^2 - 2 = y + 2 - 2 = y,$$

where the fourth step uses that $y + 2 \geq 0$. So $f \circ g = i_B$.

By Theorem 5.3.4, f is one-to-one and onto. By Theorem 5.3.5, $f^{-1} = g$. So $f^{-1}(y) = \sqrt{y+2}$. $\square$

Note: We did not explain where the formula for g came from. It was done in rough work.

H11Q2: Suppose that $g : A \to B$ and $h : B \to C$. Prove that if g is not onto and h is one-to-one, then $h \circ g$ is not onto.

Solution: Suppose that g is not onto and h is one-to-one. To get a contradiction, assume that $h \circ g$ is onto.

Since g is not onto, there is a $b \in B$ such that for every $a \in A$, $g(a) \neq b$. Consider $h(b) \in C$. Since $h \circ g$ is onto, there is an $a \in A$ with $(h \circ g)(a) = h(b)$. That is, $h(g(a)) = h(b)$. Since h is one-to-one, we get that $g(a) = b$. But this contradicts the choice of b.

H11Q3: Prove that, for all $n \in \mathbb{N}$, $\displaystyle\sum_{k=0}^{n} (-1)^k\, k^2 = (-1)^n\, \frac{n(n+1)}{2}$.

Solution: We proceed by induction on n.

Base case: $n = 0$: The left hand side is $(-1)^0 0^2 = 0$, and the right hand side is $(-1)^0 0(0+1)/2 = 0$, so the claim is true.

Induction step: Let $n \in \mathbb{N}$ and assume

$$\sum_{k=0}^{n} (-1)^k\, k^2 = (-1)^n\, \frac{n(n+1)}{2}.$$

Then

$$\begin{aligned}
\sum_{k=0}^{n+1} (-1)^k k^2 &= \sum_{k=1}^{n} (-1)^k k^2 + (-1)^{n+1}(n+1)^2 \\
&= (-1)^n \frac{n(n+1)}{2} + (-1)^{n+1}(n+1)^2 \quad \text{(by induction hypothesis)} \\
&= (-1)^n \frac{(n+1)(n-2(n+1))}{2} \\
&= (-1)^n \frac{(n+1)(-n-2)}{2} \\
&= (-1)^{n+1} \frac{(n+1)(n+2)}{2}.
\end{aligned}$$

Thus, the equation holds with n replaced by $n+1$. $\square$

H11Q4: Consider the sequence c_n defined by $c_1 = 2$, $c_2 = 7$, and $c_n = c_{n-1} + 2c_{n-2}$ for $n \geq 3$. Prove that $c_n = 3(2^{n-1}) + (-1)^n$ for all $n \geq 1$.

Solution: We will prove this by strong induction.

Base cases: When $n = 1$, the left hand side is $c_1 = 2$, and the right hand side is $3(2^0) + (-1)^1 = 3 - 1 = 2$, so the equation holds. When $n = 2$, the left hand side is $c_2 = 7$ and the right hand side is $3(2^1) + (-1)^2 = 6 + 1 = 7$, so the equation holds.

Induction step: Let $n \geq 3$ be given and suppose that for $1 \leq k < n$ we have

$$c_k = 3(2^{k-1}) + (-1)^k.$$

Then

$$\begin{aligned}
c_n &= c_{n-1} + 2c_{n-2} && \text{(by recurrence for } c_n\text{)} \\
&= 3(2^{n-2}) + (-1)^{n-1} + 2(3(2^{n-3}) + (-1)^{n-2}) && \text{(by induction hypothesis)} \\
&= 3(2^{n-2} + 2(2^{n-3})) + (-1)^{n-1}(1-2) \\
&= 3(2^{n-1}) + (-1)^n,
\end{aligned}$$

as required. $\square$

H11Q5: Define a sequence $b_0, b_1, b_2, \ldots \in \mathbb{R}$ recursively by $b_0 = 0$ and $b_{n+1} = (b_n)^2 + \frac{1}{4}$. Prove that, for all $n \geq 1$, $0 < b_n < 1$.

Hint: It will be easier if you prove a *stronger* fact about the b_n's. Experiment!

Solution: We will prove the stronger claim that for all $n \geq 1$, $0 < b_n < 1/2$. We proceed by induction on n, starting at $n = 1$.

Base case: $n = 1$: $b_1 = 0^2 + 1/4 = 1/4$, and $0 < 1/4 < 1/2$.

Inductive step: Let $n \geq 1$ and assume that $0 < b_n < 1/2$. It is clear from the formula for b_{n+1} that $b_{n+1} > 0$, since $(b_n)^2 \geq 0$ and $1/4 > 0$.

Since $0 < b_n < 1/2$, we find by squaring that $0 < (b_n)^2 < 1/4$. It follows that
$b_{n+1} = (b_n)^2 + 1/4 < 1/4 + 1/4 = 1/2$, as required. $\square$