Puzzle: $S = \{A \mid A \notin A\}$. $S \in S$? $S \notin S$?

$A \in S$ iff $A \notin A$

$S \in S$ iff $S \notin S$.

Russell's paradox, 1901.

Zermelo, 1908: only allow $\{x \in B \mid P(x)\}$
in some set.

Ex 3.4.5: $A \cap (B \setminus C) = (A \cap B) \setminus C$.

Given | Goal
$\forall x\,(x \in A \cap (B \setminus C) \leftrightarrow x \in (A \cap B) \setminus C)$

1st Given: $x \in A \cap (B \setminus C)$
$x \in A \wedge x \in (B \setminus C)$
$x \in A \wedge (x \in B \wedge x \notin C)$
$x \in A \wedge x \in B \wedge x \notin C$

1st Goal: $x \in (A \cap B) \setminus C$
$x \in A \wedge x \in B \wedge x \notin C$.

($\rightarrow$) Proof:
Let x be arbitrary. We have to show that $A \cap (B \setminus C)$ iff $x \in (A \cap B) \setminus C$. Both of them are equivalent to $x \in A \wedge x \in B \wedge x \notin C$ by associativity. So they are equivalent to each other. So $A \cap (B \setminus C) = (A \cap B) \setminus C$. $\square$

§3.5. Proofs involving $\vee$

Strategy: To use the given of $P \vee Q$, break the proof into two parts: 1) Case 1: suppose P is true
2) Case 2: Q

Form:

Case 1: Assume P. [Proof]
Case 2: Assume Q.
Since $P \vee Q$, the cases are exhaustive, so we have proof our goals.

Ex 2: $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$

Given | Goal
$\forall x\,(x \in (A \cup B) \setminus C \rightarrow x \in A \cup (B \setminus C))$.

$x \in (A \cup B) \setminus C$ | $x \in A \cup (B \setminus C)$

↳ $x \notin C$

↳ $x \in A \vee x \in B$.

Case 1: $x \in A$ → $x \in A$

2: $x \in B$ → $x \in B \setminus C$.

Proof: Let $x \in (A \cup B) \setminus C$. Then $x \in A \cup B$ and $x \notin C$.

Case 1: $x \in A$. Then clearly $x \in A \cup (B \setminus C)$.

Case 2: $x \in B$. Since $x \notin C$, then $x \in B \setminus C$. So $x \in A \cup (B \setminus C)$.

Since $x \in A \cup B$. These cover all cases, so $x \in (A \cup B) \setminus C$.

Since x is arbitrary, $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$ □.

Strategy: To prove a goal of form $P \vee Q$. Break into cases and prove P or Q in each cases

Ex 3.5.3. For any integer x. the remainder when $x^2/4$ is 0 or 1

Given	Goal
$x \in \mathbb{Z}$	$x^2 \% 4 = 0 \,\|\, x^2 \% 4 = 1$

x	x^2	$x^2 \% 4$
0	0	0
1	1	1
2	4	0
3	9	1
4	16	0
5	25	1
6	36	0
7	49	1
8	64	0
9	81	1
10	100	0

Cases x is even or odd.

Proof: Suppose $x \in \mathbb{Z}$,

Case 1: x is even. then there exist $k \in \mathbb{Z}$ that $2k = x$. so $x^2 = 4k^2$. Since $k \in \mathbb{Z}$, so the remainder of $4k^2/4$ is 0.

Case 2: x is odd. then there exist $k \in \mathbb{Z}$ that $x = 2k+1$. so $x^2 = 4k^2 + 4k + 1 = 4(k^2+k) + 1$. Since $k \in \mathbb{Z}$, the remainder of $(4(k^2+k)+1)/4$ is 1.

These cases are exhaustive, so in either cases we have proof our goal □.

Strategy: To prove a goal of the form $P \vee Q$, add $\neg P$
to given and try to proof Q.

Case: P ✓

Case: $\neg P \to Q \to P \vee Q$

Ex: $\forall x \in \mathbb{R}$, if $x^2 \geq x$ then $x \leq 0$ or $x \geq 1$.

Proof: Let $x \in \mathbb{R}$ assume $x^2 \geq x$. If $x \leq 0$. then we're done. So assume $x > 0$. Dividing both side by x giving $x \geq 1$. So either $x < 0$ or $x \geq 1$ $\square$.