Determinant

The determinant is only defined for square matrices.

Preparation:

- deleting procedure
- a sign table

Deleting procedure

For an $m \times n$ matrix A

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

the submatrix A_{ij} is obtained by deleting the i-th row and j-th column.

For example, let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & -5 & 0 & 2 \\ 1 & 1 & 4 & 3 \\ -2 & 3 & -1 & 0 \end{bmatrix}.$$

Find A_{23}.

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The left index 3 of A_{23} indicates the third column of A

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & -5 & 0 & 2 \\ 1 & 1 & 4 & 3 \\ -2 & 3 & -1 & 0 \end{bmatrix}$$

Put together, we have a highlight row and a highlight column

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & -5 & 0 & 2 \\ 1 & 1 & 4 & 3 \\ -2 & 3 & -1 & 0 \end{bmatrix}$$

In our mind, we delete the hightlight row and the highlight column. The submatrix is the rest of horizontal and vertical numbers. Then

$$A_{23} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 3 \\ -2 & 3 & 0 \end{bmatrix}$$

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For a matrix, we can associate a sign table by filling the (i, j)-entry with a sign $(-1)^{i+j}$. Graphically, we start with $+$ at the $(1, 1)$-entry and alternate the sign when moving horizontally or vertically from one position to another.

For instance,

$$\begin{bmatrix} + & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

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$$\begin{bmatrix} + & - & & \\ - & & & \\ & & & \\ & & & \end{bmatrix}$$

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$$\begin{bmatrix} + & - & + & - \\ - & + & - & \\ + & - & & \\ - & & & \end{bmatrix}$$

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The following are $n = 3$ and $n = 5$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \text{ and } \begin{bmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{bmatrix}$$

Definition of Determinant

From now on, we consider square matrices only, unless stated otherwise.

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For a 1×1 matrix $A = [a]$, the determinant of $A = [a]$ is

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For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant of A is

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

The determinant of any $n \times n$ matrix ($n \geq 3$) is defined recursively.

Definition Let $n \geq 3$ and let $A = [a_{ij}]$ be an $n \times n$ matrix. The determinant of A is

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots + (-1)^{n+1} a_{1n} \det A_{1n}.$$

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Note 1: each term is a multiplication of three parts: a_{1j} (entries at the first row of A), the sign $(-1)^{1+j}$, and the determinant $\mathrm{det} A_{1j}$.

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Examples

(1) Compute $\det A$, where A is given by

$$A = \begin{bmatrix} 2 & 1 & -3 \\ -3 & -2 & 0 \\ 2 & 1 & 2 \end{bmatrix}$$

(2) Compute $\det A$, where A is given by

$$A = \begin{bmatrix} 2 & 1 & 3 & -1 \\ -3 & -2 & 0 & 2 \\ 2 & 1 & 2 & -1 \\ 1 & 0 & 1 & -2 \end{bmatrix}$$

Definition Let A be an $n \times n$ matrix and let A_{ij} denote the sub-matrix (of order $(n-1)$) obtained from A by deleting its i-th row and its j-th column.

(1) the *i, j-minor* of A is given by $\det A_{ij}$

(2) the *i, j-cofactor* of A is given by $(-1)^{i+j} \det A_{ij}$.

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Example Find the $(2, 1)$-minor and the $(2, 1)$-cofactor of

$$A = \begin{bmatrix} 2 & -2 & 1 & -1 \\ 1 & 3 & 3 & 0 \\ 10 & 1 & 0 & -1 \\ 4 & 3 & -2 & 1 \end{bmatrix}$$

Theorem Let A be a square matrix of order n. Let i be a fixed row number. Then

$$\begin{aligned}\det A =& (-1)^{i+1} a_{i1} \det A_{i1} + (-1)^{i+2} a_{i2} \det A_{i2} \\ & + \ldots + (-1)^{i+n} a_{in} A_{in}\end{aligned} \tag{1}$$

We call the equation (1) cofactor expansion along the i-th row (or simply expansion along the i-th row).

Theorem Let A be a square matrix of order n. Let j be a fixed column number. Then

$$\begin{aligned} \det A =& (-1)^{1+j} a_{1j} \det A_{1j} + (-1)^{2+j} a_{2j} \det A_{2j} \\ & + \ldots + (-1)^{n+j} a_{nj} \det A_{nj} \end{aligned} \tag{2}$$

We call the equation (2) cofactor expansion along the j-th column (or simply expansion along the j-th column).

Remark Equations (1) and (2) allow us to expand along a row or a column which contains as many zero as possible for an efficiency of calculation of determinant.

Example

Compute $\det A$, by expansion along the row or column with the maximum number of zeros.

$$(1)\quad A = \begin{bmatrix} -3 & 0 & 2 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix} \qquad (2)\quad A = \begin{bmatrix} 2 & 10 & -3 & 4 \\ 4 & 0 & 0 & 7 \\ 3 & -1 & 1 & 3 \\ 6 & 0 & 0 & 2 \end{bmatrix}$$