1 Propositional Logic - Axioms and Inference Rules

Axioms

Axiom 1.1 [Commutativity]

$$\begin{aligned}
(p \wedge q) &= (q \wedge p) \\
(p \vee q) &= (q \vee p) \\
(p = q) &= (q = p)
\end{aligned}$$

Axiom 1.2 [Associativity]

$$\begin{aligned}
p \wedge (q \wedge r) &= (p \wedge q) \wedge r \\
p \vee (q \vee r) &= (p \vee q) \vee r
\end{aligned}$$

Axiom 1.3 [Distributivity]

$$\begin{aligned}
p \vee (q \wedge r) &= (p \vee q) \wedge (p \vee r) \\
p \wedge (q \vee r) &= (p \wedge q) \vee (p \wedge r)
\end{aligned}$$

Axiom 1.4 [De Morgan]

$$\begin{aligned}
\neg(p \wedge q) &= \neg p \vee \neg q \\
\neg(p \vee q) &= \neg p \wedge \neg q
\end{aligned}$$

Axiom 1.5 [Negation]

$$\neg\neg p = p$$

Axiom 1.6 [Excluded Middle]

$$p \vee \neg p = T$$

Axiom 1.7 [Contradiction]

$$p \wedge \neg p \quad = \quad F$$

Axiom 1.8 [Implication]

$$p \Rightarrow q \quad = \quad \neg p \vee q$$

Axiom 1.9 [Equality]

$$(p = q) \quad = \quad (p \Rightarrow q) \wedge (q \Rightarrow p)$$

Axiom 1.10 [or-simplification]

$$\begin{aligned} p \vee p &= p \\ p \vee T &= T \\ p \vee F &= p \\ p \vee (p \wedge q) &= p \end{aligned}$$

Axiom 1.11 [and-simplification]

$$\begin{aligned} p \wedge p &= p \\ p \wedge T &= p \\ p \wedge F &= F \\ p \wedge (p \vee q) &= p \end{aligned}$$

Axiom 1.12 [Identity]

$$p \quad = \quad p$$

Inference Rules

$$\frac{p_1 \ = \ p_2 \quad , \quad p_2 \ = \ p_3}{p_1 \ = \ p_3} \quad \text{Transitivity}$$

$$\frac{p_1 \ = \ p_2}{E(p_1) \ = \ E(p_2) \quad , \quad E(p_2) \ = \ E(p_1)} \quad \text{Substitution}$$

$$\frac{q_1 \ , \ q_2 \ , \ \ldots \ , \ q_n \ , \ q_1 \wedge q_2 \wedge \ldots \wedge q_n \Rightarrow (p_1 = p_2)}{E(p_1) \ = \ E(p_2) \quad , \quad E(p_2) \ = \ E(p_1)} \quad \text{Conditional Substitution}$$

2 Propositional Logic - Derived Theorems

Equivalence and Truth

Theorem 2.1 **[Associativity of $=$]**

$$((p = q) = r) \quad = \quad (p = (q = r))$$

Theorem 2.2 **[Identity of $=$]**

$$(T = p) \quad = \quad p$$

Theorem 2.3 **[Truth]**

$$T$$

Negation, Inequivalence, and False

Theorem 2.4 **[Definition of F]**

$$F \quad = \quad \neg T$$

Theorem 2.5 **[Distributivity of $\neg$ over $=$]**

$$\neg(p = q) \quad = \quad (\neg p = q)$$
$$(\neg p = q) \quad = \quad (p = \neg q)$$

Theorem 2.6 **[Negation of F]**

$$\neg F \quad = \quad T$$

Theorem 2.7 [Definition of $\neg$]

$$\begin{aligned} (\neg p \;=\; p) \quad &= \quad F \\ \neg p \quad &= \quad (p \;=\; F) \end{aligned}$$

Disjunction

Theorem 2.8 [Distributivity of $\vee$ over $=$]

$$\begin{aligned} (p \vee (q = r)) \quad &= \quad ((p \vee q) = (p \vee r)) \\ ((p \vee (q = r)) = (p \vee q)) \quad &= \quad (p \vee r) \end{aligned}$$

Theorem 2.9 [Distributivity of $\vee$ over $\vee$]

$$p \vee (q \vee r) = (p \vee q) \vee (p \vee r)$$

Conjunction

Theorem 2.10 [Mutual definition of $\wedge$ and $\vee$]

$$\begin{aligned} (p \wedge q) \quad &= \quad (p = (q = (p \vee q))) \\ (p \wedge q) \quad &= \quad ((p = q) = (p \vee q)) \\ ((p \wedge q) = p) \quad &= \quad (q = (p \vee q)) \\ ((p \wedge q) = (p = q)) \quad &= \quad (p \vee q) \\ (((p \wedge q) = p) = q) \quad &= \quad (p \vee q) \end{aligned}$$

Theorem 2.11 [Distributivity of $\wedge$ over $\wedge$]

$$p \wedge (q \wedge r) = (p \wedge q) \wedge (p \wedge r)$$

Theorem 2.12 **[Absorption]**

$$p \land (\neg p \lor q) = p \land q$$
$$p \lor (\neg p \land q) = p \lor q$$

Theorem 2.13 **[Distributivity of $\land$ over $=$]**

$$\begin{aligned}(p \land q) &= ((p \land \neg q) = \neg p)\\ ((p \land q) = (p \land \neg q)) &= \neg p\\ p \land (q = p) &= (p \land q)\end{aligned}$$

Theorem 2.14 **[Replacement]**

$$(p = q) \land (r = p) = (p = q) \land (r = q)$$

Theorem 2.15 **[Definition of $=$]**

$$(p = q) = (p \land q) \lor (\neg p \land \neg q)$$

Theorem 2.16 **[Exclusive or]**

$$\neg(p = q) = (\neg p \land q) \lor (p \land \neg q)$$

Implication

Theorem 2.17 **[Definition of Implication]**

$$\begin{aligned}(p \Rightarrow q) &= ((p \lor q) = q)\\ ((p \Rightarrow q) = (p \lor q)) &= q\\ (p \Rightarrow q) &= ((p \land q) = p)\\ ((p \Rightarrow q) = (p \land q)) &= p\end{aligned}$$

Theorem 2.18 **[Contrapositive]**

$$(p \Rightarrow q) \quad = \quad (\neg q \Rightarrow \neg p)$$

Theorem 2.19 **[Distributivity of $\Rightarrow$ over $=$]**

$$p \Rightarrow (q = r) \quad = \quad ((p \Rightarrow q) = (p \Rightarrow r))$$

Theorem 2.20 **[Shunting]**

$$p \wedge q \Rightarrow r \quad = \quad p \Rightarrow (q \Rightarrow r)$$

Theorem 2.21 **[Elimination/Introduction of $\Rightarrow$]**

$$\begin{array}{rcl}
p \wedge (p \Rightarrow q) & = & p \wedge q \\
p \wedge (q \Rightarrow p) & = & p \\
p \vee (p \Rightarrow q) & = & T \\
p \vee (q \Rightarrow p) & = & \neg q \vee p \\
(p \vee q) \Rightarrow (p \wedge q) & = & (p = q) \\
p \Rightarrow F & = & \neg p \\
F \Rightarrow p & = & T
\end{array}$$

Theorem 2.22 **[Right Zero of $\Rightarrow$]**

$$(p \Rightarrow T) \quad = \quad T$$

Theorem 2.23 **[Left Identity of $\Rightarrow$]**

$$(T \Rightarrow p) \quad = p$$

Theorem 2.24 **[Weakening/Strengthening]**

$$\begin{array}{rcl} p & \Rightarrow & p \lor q \\ p \land q & \Rightarrow & p \\ p \land q & \Rightarrow & p \lor q \\ p \lor (q \land r) & \Rightarrow & p \lor q \\ p \land q & \Rightarrow & p \land (q \lor r) \end{array}$$

Theorem 2.25 **[Modus Ponens]**

$$p \land (p \Rightarrow q) \quad \Rightarrow \quad q$$

Theorem 2.26 **[Proof by Cases]**

$$\begin{array}{rcl} (p \Rightarrow r) \land (q \Rightarrow r) & = & (p \lor q \Rightarrow r) \\ (p \Rightarrow r) \land (\neg p \Rightarrow r) & = & r \end{array}$$

Theorem 2.27 **[Mutual Implication]**

$$(p \Rightarrow q) \land (q \Rightarrow p) \quad = \quad (p = q)$$

Theorem 2.28 **[Antisymmetry]**

$$(p \Rightarrow q) \land (q \Rightarrow p) \quad \Rightarrow \quad (p = q)$$

Theorem 2.29 **[Transitivity]**

$$\begin{array}{rcl} (p \Rightarrow q) \land (q \Rightarrow r) & \Rightarrow & (p \Rightarrow r) \\ (p = q) \land (q \Rightarrow r) & \Rightarrow & (p \Rightarrow r) \\ (p \Rightarrow q) \land (q = r) & \Rightarrow & (p \Rightarrow r) \end{array}$$

Theorem 2.30 **[Monotonicity of $\lor$]**

$$(p \Rightarrow q) \quad \Rightarrow \quad (p \lor r \Rightarrow q \lor r)$$

Theorem 2.31 **[Monotonicity of $\wedge$]**

$$(p \Rightarrow q) \quad \Rightarrow \quad (p \wedge r \Rightarrow q \wedge r)$$

Substitution

Theorem 2.32 **[Leibniz]**

$$(e = f) \quad \Rightarrow \quad (E(e) = E(f))$$

Theorem 2.33 **[Substitution]**

$$\begin{array}{rcl}
(e = f) \wedge E(e) & = & (e = f) \wedge E(f) \\
(e = f) \Rightarrow E(e) & = & (e = f) \Rightarrow E(f) \\
q \wedge (e = f) \Rightarrow E(e) & = & q \wedge (e = f) \Rightarrow E(f)
\end{array}$$

Theorem 2.34 **[Replace by T]**

$$\begin{array}{rcl}
p \wedge E(p) & = & p \wedge E(T) \\
p \Rightarrow E(p) & = & p \Rightarrow E(T) \\
q \wedge p \Rightarrow E(p) & = & q \wedge p \Rightarrow E(T)
\end{array}$$

Theorem 2.35 **[Replace by F]**

$$\begin{array}{rcl}
p \vee E(p) & = & p \vee E(F) \\
E(p) \Rightarrow p & = & E(F) \Rightarrow p \\
E(p) \Rightarrow p \vee q & = & E(F) \Rightarrow p \vee q
\end{array}$$

Theorem 2.36 **[Shannon]**

$$E(p) \quad = \quad (p \wedge E(T)) \vee (\neg p \wedge E(F))$$

3 Propositional Logic - Examples and Exercises

4 Predicate Logic - Axioms

Axiom 4.1 **[Definition of $\exists$]**

$$(m \geq n) \quad \Rightarrow \quad \begin{pmatrix} \exists i \ : \ m \leq i < n \ : \ p_i \\ = \\ F \end{pmatrix}$$

$$(m < n) \quad \Rightarrow \quad \begin{pmatrix} \exists i \ : \ m \leq i < n \ : \ p_i \\ = \\ \begin{pmatrix} \exists i \ : \ m \leq i < n-1 \ : \ p_i \\ \vee \\ p_{n-1} \end{pmatrix} \end{pmatrix}$$

Axiom 4.2 **[Definition of $\forall$]**

$$(m \geq n) \quad \Rightarrow \quad \begin{pmatrix} \forall i \ : \ m \leq i < n \ : \ p_i \\ = \\ T \end{pmatrix}$$

$$(m < n) \quad \Rightarrow \quad \begin{pmatrix} \forall i \ : \ m \leq i < n \ : \ p_i \\ = \\ \begin{pmatrix} \forall i \ : \ m \leq i < n-1 \ : \ p_i \\ \wedge \\ p_{n-1} \end{pmatrix} \end{pmatrix}$$

Axiom 4.3 [Range Split]

$$(m_1 \le m_2 \le m_3) \quad \Rightarrow \quad \left(\begin{array}{c} \left(\begin{array}{c} \forall i \ : \ m_1 \le i < m_2 \ : \ p_i \\ \wedge \\ \forall i \ : \ m_2 \le i < m_3 \ : \ p_i \end{array} \right) \\ = \\ \forall i \ : \ m_1 \le i < m_3 \ : \ p_i \end{array} \right)$$

$$(m_1 \le m_2) \ \wedge \ (n_1 \ge n_2) \quad \Rightarrow \quad \left(\begin{array}{c} \left(\begin{array}{c} \forall i \ : \ m_1 \le i < n_1 \ : \ p_i \\ \wedge \\ \forall i \ : \ m_2 \le i < n_2 \ : \ p_i \end{array} \right) \\ = \\ \forall i \ : \ m_1 \le i < n_1 \ : \ p_i \end{array} \right)$$

$$(m_1 \le m_2 \le m_3) \quad \Rightarrow \quad \left(\begin{array}{c} \left(\begin{array}{c} \exists i \ : \ m_1 \le i < m_2 \ : \ p_i \\ \vee \\ \exists i \ : \ m_2 \le i < m_3 \ : \ p_i \end{array} \right) \\ = \\ \exists i \ : \ m_1 \le i < m_3 \ : \ p_i \end{array} \right)$$

$$(m_1 \le m_2) \ \wedge \ (n_1 \ge n_2) \quad \Rightarrow \quad \left(\begin{array}{c} \left(\begin{array}{c} \exists i \ : \ m_1 \le i < n_1 \ : \ p_i \\ \vee \\ \exists i \ : \ m_2 \le i < n_2 \ : \ p_i \end{array} \right) \\ = \\ \exists i \ : \ m_1 \le i < n_1 \ : \ p_i \end{array} \right)$$

Axiom 4.4 [Interchange of Dummies]

$$\begin{array}{c} \forall i \ : \ m_1 \le i < n_1 \ : \ (\forall j \ : \ m_2 \le j < n_2 \ : \ p_{i,j}) \\ = \\ \forall j \ : \ m_2 \le j < n_2 \ : \ (\forall i \ : \ m_1 \le i < n_1 \ : \ p_{i,j}) \end{array}$$

$$\begin{array}{c} \exists i \ : \ m_1 \le i < n_1 \ : \ (\exists j \ : \ m_2 \le j < n_2 \ : \ p_{i,j}) \\ = \\ \exists j \ : \ m_2 \le j < n_2 \ : \ (\exists i \ : \ m_1 \le i < n_1 \ : \ p_{i,j}) \end{array}$$

Axiom 4.5 [Dummy Renaming]

$$\forall i \;:\; m \le i < n \;:\; p_i \quad = \quad \forall j \;:\; m \le j < n \;:\; p_j$$

Axiom 4.6 [Distributivity of $\vee$ over $\forall$]

$$(p \;\vee\; (\forall i \;:\; m \le i < n \;:\; q_i)) \quad = \quad \forall i \;:\; m \le i < n \;:\; (p \;\vee\; q_i)$$

Axiom 4.7 [Distributivity of $\wedge$ over $\forall$]

$$(m < n) \quad \Rightarrow \quad \left(\begin{array}{c} p \;\wedge\; (\forall i \;:\; m \le i < n \;:\; q_i) \\ = \\ \forall i \;:\; m \le i < n \;:\; p \;\wedge\; q_i \end{array} \right)$$

$$\left(\begin{array}{c} \forall i \;:\; m \le i < n \;:\; p_i \\ \wedge \\ \forall i \;:\; m \le i < n \;:\; q_i \end{array} \right) \quad = \quad \forall i \;:\; m \le i < n \;:\; (p_i \;\wedge\; q_i)$$

Axiom 4.8 [Distributivity of $\wedge$ over $\exists$]

$$(p \;\wedge\; (\exists i \;:\; m \le i < n \;:\; q_i)) \quad = \quad \exists i \;:\; m \le i < n \;:\; (p \;\wedge\; q_i)$$

Axiom 4.9 [Distributivity of $\vee$ over $\exists$]

$$(m < n) \quad \Rightarrow \quad \left(\begin{array}{c} p \;\vee\; (\exists i \;:\; m \le i < n \;:\; q_i) \\ = \\ \exists i \;:\; m \le i < n \;:\; (p \;\vee\; q_i) \end{array} \right)$$

$$\left(\begin{array}{c} \exists i \;:\; m \le i < n \;:\; p_i \\ \vee \\ \exists i \;:\; m \le i < n \;:\; q_i \end{array} \right) \quad = \quad \exists i \;:\; m \le i < n \;:\; (p_i \;\vee\; q_i)$$

Axiom 4.10 [Universality of T]

$$\forall i \;:\; m \le i < n \;:\; T \quad = \quad T$$

Axiom 4.11 [Existence of F]

$$\exists i \; : \; m \leq i < n \; : \; F \quad = \quad F$$

Axiom 4.12 [Generalized De Morgan]

$$\begin{array}{rcl} \neg\,(\exists i \; : \; m \leq i < n \; : \; p_i) & = & \forall i \; : \; m \leq i < n \; : \; \neg p_i \\ \neg\,(\forall i \; : \; m \leq i < n \; : \; p_i) & = & \exists i \; : \; m \leq i < n \; : \; \neg p_i \end{array}$$

Axiom 4.13 [Trading]

$$\begin{array}{rcl} (m \leq i < n) \;\Rightarrow\; p_i & = & \forall i \; : \; m \leq i < n \; : \; p_i \\ (m \leq i < n) \;\wedge\; p_i & \Rightarrow & \exists i \; : \; m \leq i < n \; : \; p_i \end{array}$$

Axiom 4.14 [Definition of Numerical Quantification]

$$(m \geq n) \quad \Rightarrow \quad \left(\begin{array}{c} \mathcal{N} i \; : \; m \leq i < n \; : \; p_i \\ = \\ 0 \end{array} \right)$$

$$(m < n) \;\wedge\; \neg p_{n-1} \quad \Rightarrow \quad \left(\begin{array}{c} \mathcal{N} i \; : \; m \leq i < n \; : \; p_i \\ = \\ \mathcal{N} i \; : \; m \leq i < n-1 \; : \; p_i \end{array} \right)$$

$$(m < n) \;\wedge\; p_{n-1} \quad \Rightarrow \quad \left(\begin{array}{c} \mathcal{N} i \; : \; m \leq i < n \; : \; p_i \\ = \\ \left(\begin{array}{c} \mathcal{N} i \; : \; m \leq i < n-1 \; : \; p_i \\ + \\ 1 \end{array} \right) \end{array} \right)$$

Axiom 4.15 [Definition of Σ]

$$(m \geq n) \quad \Rightarrow \quad \left(\begin{array}{c} \Sigma i \ : \ m \leq i < n \ : \ e_i \\ = \\ 0 \end{array} \right)$$

$$(m < n) \quad \Rightarrow \quad \left(\begin{array}{c} \Sigma i \ : \ m \leq i < n \ : \ e_i \\ = \\ \left(\begin{array}{c} \Sigma i \ : \ m \leq i < n-1 \ : \ e_i \\ + \\ e_{n-1} \end{array} \right) \end{array} \right)$$

Axiom 4.16 [Definition of Π]

$$(m \geq n) \quad \Rightarrow \quad \left(\begin{array}{c} \Pi i \ : \ m \leq i < n \ : \ e_i \\ = \\ 1 \end{array} \right)$$

$$(m < n) \quad \Rightarrow \quad \left(\begin{array}{c} \Pi i \ : \ m \leq i < n \ : \ e_i \\ = \\ \left(\begin{array}{c} \Pi i \ : \ m \leq i < n-1 \ : \ e_i \\ * \\ e_{n-1} \end{array} \right) \end{array} \right)$$

5 Predicate Logic - Derived Theorems

Theorem 5.1 **[Definition of $\exists$]**

$$(m \geq n) \quad \Rightarrow \quad \begin{pmatrix} \exists i \ : \ m < i \leq n \ : \ p_i \\ = \\ F \end{pmatrix}$$

$$(m < n) \quad \Rightarrow \quad \begin{pmatrix} \exists i \ : \ m < i \leq n \ : \ p_i \\ = \\ \begin{pmatrix} \exists i \ : \ m+1 < i \leq n \ : \ p_i \\ \vee \\ p_{m+1} \end{pmatrix} \end{pmatrix}$$

Theorem 5.2 **[Definition of $\forall$]**

$$(m \geq n) \quad \Rightarrow \quad \begin{pmatrix} \forall i \ : \ m < i \leq n \ : \ p_i \\ = \\ T \end{pmatrix}$$

$$(m < n) \quad \Rightarrow \quad \begin{pmatrix} \forall i \ : \ m < i \leq n \ : \ p_i \\ = \\ \begin{pmatrix} \forall i \ : \ m+1 < i \leq n \ : \ p_i \\ \wedge \\ p_{m+1} \end{pmatrix} \end{pmatrix}$$

Theorem 5.3 [Definition of Σ]

$$(m \geq n) \quad \Rightarrow \quad \left(\begin{array}{c} \Sigma i \ : \ m < i \leq n \ : \ e_i \\ = \\ 0 \end{array} \right)$$

$$(m < n) \quad \Rightarrow \quad \left(\begin{array}{c} \Sigma i \ : \ m < i \leq n \ : \ e_i \\ = \\ \left(\begin{array}{c} \Sigma i \ : \ m+1 < i \leq n \ : \ e_i \\ + \\ e_{m+1} \end{array} \right) \end{array} \right)$$

Theorem 5.4 [Definition of Π]

$$(m \geq n) \quad \Rightarrow \quad \left(\begin{array}{c} \Pi i \ : \ m < i \leq n \ : \ e_i \\ = \\ 1 \end{array} \right)$$

$$(m < n) \quad \Rightarrow \quad \left(\begin{array}{c} \Pi i \ : \ m < i \leq n \ : \ e_i \\ = \\ \left(\begin{array}{c} \Pi i \ : \ m+1 < i \leq n \ : \ e_i \\ * \\ e_{m+1} \end{array} \right) \end{array} \right)$$

6 Some Simple Laws of Arithmetic

Throughout this compendium, we assume the validity of all "simple" arithmetic rules. Examples of such rules are all *simplification rules*, e.g. =

$$
\begin{aligned}
2+3 &= 5 \\
x+x &= 2*x \\
x+y-y &= x \\
(x/3)*3 &= x \\
0*x &= 0 \\
1*x &= x \\
x*x &= x^2 \\
0^x &= 0 \\
1^x &= 1 \\
(2*x+10=20) &= (x=5) \\
(x+y<2*y) &= (x<y) \\
(x+y=x+z) &= (y=z) \\
x*(y+1)-(x+z) &= (x*y-z)
\end{aligned}
$$

Following is a collection of theorems that might be used. The list is not exhaustive but intend to show the level of complexity that you can specify theorems on.

Theorems on $<$ and $\leq$

$$
\begin{array}{rcl}
(x < y) & = & (y > x) \\
(x < y) & \Rightarrow & \neg(y = x) \wedge \neg(y < x) \\
(x < y) & \Rightarrow & (x \leq y) \\
(x < y) & = & (x \leq y) \wedge (x \neq y) \\
(x < y) \wedge (y \leq z) & \Rightarrow & (x < z) \\
(x \leq y) & = & \neg(x > y) \\
(x \leq x) & = & T \\
(x \leq y) \wedge (y \leq z) & \Rightarrow & (x \leq z) \\
(x \leq y) \wedge (y < z) & \Rightarrow & (x < z) \\
(x \leq y) \wedge \neg(x < y) & \Rightarrow & (x < y) \\
(x \leq y - 1) & = & (x < y) \\
(x \leq y) \vee (y \leq x) & = & T \\
(x \leq y) \vee (y < x) & = & T \\
(x \leq y) & = & (x < y) \vee (x = y)
\end{array}
$$

Theorems on properties about $+$ and $-$

$$
\begin{array}{rcl}
(x < y) & \Rightarrow & (x < y + 1) \\
(x < y + 1) & = & (x \leq y) \\
(x < y) & \Rightarrow & (z - y < z - x) \\
(0 < x) & = & (-x < 0) \\
(x - 1 < x) & = & T \\
(x \leq y - 1) & = & (x < y) \\
(x \leq y) & = & (x - 1 < y) \\
(x_1 < y_1) \wedge (x_2 < y_2) & \Rightarrow & (x_1 + x_2 < y_1 + y_2) \\
(x_1 \leq y_1) \wedge (x_2 \leq y_2) & \Rightarrow & (x_1 + x_2 \leq y_1 + y_2)
\end{array}
$$

Theorems on properties about $*$ and $/$

$$\begin{array}{rcl}
(0 < x) & = & (0 < 2 * x) \\
(0 < x) & = & (x < 2 * x) \\
(0 < x) & = & (x \div 2 < x) \\
(0 \leq x/2) & = & (0 \leq x) \\
(x = 0) & \Rightarrow & (x * y = 0) \\
2 * (x/2) & = & x
\end{array}$$

Theorems on equivalence relation

$$\begin{array}{rcl}
(x = x) & = & T \\
(x = y) & = & (x \leq y) \ \wedge \ (y \leq x)
\end{array}$$

Theorems about odd(n) and even(n)

$$\begin{array}{rcl}
\mathrm{odd}(x) & \Rightarrow & ((x-1) \div 2 \ = \ (x-1)/2) \\
\mathrm{even}(x) & \Rightarrow & (x \div 2 \ = \ x/2) \\
\mathrm{odd}(x + 2 * y) & = & \mathrm{odd}(x) \\
\mathrm{even}(x + 2 * y) & = & \mathrm{even}(x) \\
\mathrm{odd}(x) & = & \neg \mathrm{even}(x) \\
\mathrm{odd}(x) & \Rightarrow & ((x \geq 1) \ = \ (x \geq 0)) \\
\mathrm{odd}(x) \wedge (x = 0) & = & F
\end{array}$$

7 Predicate Logic - Examples and Exercises

8 Arrays - Axioms

8.1 Axioms

Axiom 8.1 [Assignment to Array Element]

$$((b;\ i : e)\,[j]\ =\ f) \quad = \quad \left(\begin{array}{c} (i = j)\ \Rightarrow\ (e\ =\ f) \\ \wedge \\ (i \neq j)\ \Rightarrow\ (b[j]\ =\ f) \end{array} \right)$$

Axiom 8.2 [Definition of Arithmetic Relations]

$$\begin{array}{rcl} (b[i:j]\ =\ x) & = & (\forall k\ :\ i \leq k < j+1\ :\ b[k]\ =\ x) \\ (b[i:j]\ <\ x) & = & (\forall k\ :\ i \leq k < j+1\ :\ b[k]\ <\ x) \\ (b[i:j]\ >\ x) & = & (\forall k\ :\ i \leq k < j+1\ :\ b[k]\ >\ x) \\ (b[i:j]\ \leq\ x) & = & (\forall k\ :\ i \leq k < j+1\ :\ b[k]\ \leq\ x) \\ (b[i:j]\ \geq\ x) & = & (\forall k\ :\ i \leq k < j+1\ :\ b[k]\ \geq\ x) \\ (b[i:j]\ \neq\ x) & = & (\forall k\ :\ i \leq k < j+1\ :\ b[k]\ \neq\ x) \\ x \in b[i:j] & = & (\exists k\ :\ i \leq k < j+1\ :\ x\ =\ b[k]) \end{array}$$