# Partial derivatives

**Defination:** given function $z = f(x, y)$

the first partial is the derivate of $x$

the second partial is the derivate of $y$

$$\frac{\partial z}{\partial x} \equiv \frac{\partial f}{\partial x} \equiv f_1(x, y) \equiv f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial z}{\partial y} \equiv \cdots \equiv f_2(x, y) \equiv f_y(x, y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

In the above definitions, $f_1(x, y)$ means the partial derivative of $f$ with the respect of the first assumment. i.e. $x$ and $f_2(x, y)$ is the partial derivative of $f$ with $y$ variable.

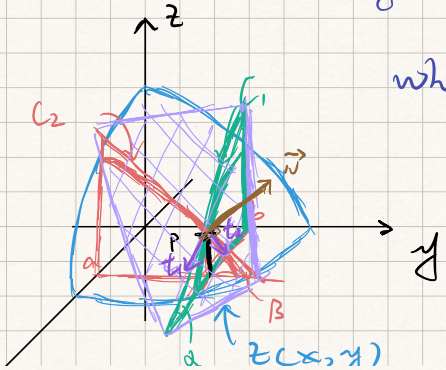
Rule of finding partial derivatives of $z = f(x, y)$ :

1. to obtain $f_x$ : consider $y$ as a constant and differentiate $x$
2. $f_y$ : $x$ $y$

e.g. 1 : $z = xy + x^2$ , find: 1. $f_x$ 2. $f_y$ at $(2, 0)$

$$f_x = y(x)' + (x^2)' \qquad f_y = x(y') + x^2(1')$$
$$= y + 2x \qquad\qquad = x$$
$$= 4 \text{ // Ans} \qquad\qquad = 2 \text{ // Ans}$$

## Geomethic meaning of partial derivatives




when $x$ varies only, curve $C_1$ generated

$y$ $C_2$

take tangent line $l_1$ at $p$, $\alpha$ generate

$l_2$ $\beta$

$f_x(x, y)| \quad = k_1 = \tan\alpha$

$f_x(x,y)|_{(a,b)}$

$$f_y(x,y)|_{(a,b)} = k_2 = \tan\beta$$

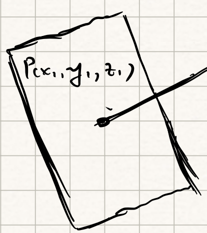
let $\vec{t}_1, \vec{t}_2$ be tangent vectors to $C_1$ and $C_2$ to the surface $z = f(x,y)$ at the point $(a, b, f(a,b))$. Then $\vec{t}_1 = (1, 0, f_x(a,b))$, $\vec{t}_2 = (0, 1, f_y(a,b))$

then the normal vector $\vec{n}$ to the surface $z = f(x,y)$ at point $P$ is: $\vec{n} = \vec{t}_1 \times \vec{t}_2$. $\Delta$!

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x(a,b) \\ 0 & 1 & f_y(a,b) \end{vmatrix} = \hat{i}\begin{vmatrix} 0 & f_x(a,b) \\ 1 & f_y(a,b) \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & f_x(a,b) \\ 0 & f_y(a,b) \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= -f_x(a,b)\,\hat{i} - f_y(a,b)\,\hat{j} + \hat{k}$$

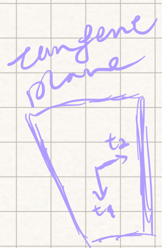
Recall Chapter 12:




then the equation of tangent surface is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

$\Downarrow$

$P(a, b, f(a,b))$ $\quad \vec{N} = -f_x(a,b)\hat{i} - f_y(a,b)\hat{j} + \hat{k}$




the equation of tangent plane of the surface is: $-f_x(a,b)(x-a) - f_y(a,b)(y-b) + (1)(z - f(a,b)) = 0$

$$\Rightarrow \boxed{z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)}$$

$\vec{n}$ the equation of normal line to the surface $z = f(x,y)$ at point $P(a, b, f(a,b))$

is: $\boxed{\dfrac{x-a}{-f_x(a,b)} = \dfrac{y-b}{-f_y(a,b)} = \dfrac{z - f(a,b)}{1}}$

e.g. 2. Find equation of tangent plane to $z = y\ln x$ at the point $(\underset{a}{1}, \underset{b}{4}, \underset{f(a,b)}{0})$

check: $f(a,b) = 4\ln 1 = 0$

$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$.

$f_x = \frac{y}{x} = 4$

$\Rightarrow z = 4(x-1) + 0(y-4) + 0$

$f_y = \ln x = 0$

$= 4x - 4$. // Ans

Like functions of one variable, we may define higher particial derivate. Given $z = f(x,y)$, the second derivate are

$$\frac{\partial^2 z}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \equiv \frac{\partial^2 f}{\partial x^2} \equiv f_{11}(x,y) \equiv f_{xx}(x,y)$$

$$\frac{\partial^2 z}{\partial y^2} \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \equiv \frac{\partial^2 f}{\partial y^2} \equiv f_{22}(x,y) \equiv f_{yy}(x,y).$$

$$\frac{\partial^2 z}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \equiv \frac{\partial^2 f}{\partial x \partial y} \equiv f_{21}(x,y) \equiv f_{yx}(x,y).$$

$$\frac{\partial^2 z}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \equiv \frac{\partial^2 f}{\partial y \partial x} \equiv f_{12}(x,y) \equiv f_{xy}(x,y).$$

mixed partial derivatives. of $f$

e.g. 3: Find the second derivates of $f(x,y) = x^3y^4 + x^4y^3$

$f_x = 3x^2y^4 + 4x^3y^3$

$f_{xx} = 6xy^4 + 12x^2y^3$

$f_{xy} = 12x^2y^3 + 12x^3y^2$.

$f_{yx} = 12x^2y^3 + 12x^3y^2$

$f_y = 4x^3y^3 + 3x^4y^2$

$f_{yy} = 12x^3y^2 + 6x^4y$

$\Rightarrow f_{xy} = f_{yx}$ :

Vairault's Theorimn:

If $f_x, f_y, f_{xy}, f_{yx}$ are continuous in a neighbhood of $(a,b)$

then $\boxed{f_{xy}(a,b) = f_{yx}(a,b)}$,

What is a differential equation (DE)?

A DE is an equation consisting of an unknown function and its derivates, there are two types of DE:

1. The unknown function is a function of one variable only then this DE is called an Ordinary Differential Equation (ODE)
2. The unknown function is a function of two or more independent variable, then this DE is called a Partial Differential Equation (PDE)

The following PDE are popular and useful in Science and Physics:

1. The Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad u_{xx} + u_{yy} = 0$$

where $u(x,y)$ is the unknown function.

2. The wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad u_{tt} = c^2 u_{xx}$$

3. The heat equation

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} \qquad u_t = D u_{x^2}$$

e.g. 4. Show that the function $u = e^{kx} \sin ky$ is a solution of Laplace's equation.

$$u_x = ke^{kx} \sin ky \qquad u_y = e^{kx} k \cos ky$$

$$u_{xx} = k^2 e^{kx} \sin ky \qquad u_{yy} = e^{kx} k^2 (-\sin ky)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u_{xx} + u_{yy} = 0$$

$$\Rightarrow k^2 e^{kx} \sin ky + e^{kx} k^2 (-\sin ky) = 0.$$

$\therefore u = e^{kx} \sin ky$ is a solution of Laplace's equation e.g.5. Show that the function $u(x,t) = \sin(x-at)$ is a solution of wave equation.

$u_x = \cos(x-at)$ $\quad$ $u_t = -a\cos(x-at)$

$u_{xx} = -\sin(x-at)$ $\quad$ $u_{tt} = -a^2\sin(x-at)$

$u_{tt} = a^2 u_{xx}$

$\therefore$ $u(x,t) = \sin(x-at)$ is a solution of wave equation.

$u(x,t) = \dfrac{e^{-x^2/4Dt}}{\sqrt{\pi \cdot 4Dt}}$ is a solution of the diffusion equation