Determinant

The determinant is only defined for square matrices.

Preparation:

- deleting procedure
- a sign table

Deleting procedure

For an $m \times n$ matrix A

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

the submatrix A_{ij} is obtained by deleting the *i*-th row and *j*-th column.

For example, let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & -5 & 0 & 2 \\ 1 & 1 & 4 & 3 \\ -2 & 3 & -1 & 0 \end{bmatrix}.$$

Find A_{23} .

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The left index 3 of A_{23} indicates the third column of A

$$\left[\begin{array}{ccccc}
1 & 2 & 3 & 4 \\
3 & -5 & 0 & 2 \\
1 & 1 & 4 & 3 \\
-2 & 3 & -1 & 0
\end{array}\right]$$

Put together, we have a highlight row and a highlight column

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & -5 & 0 & 2 \\ 1 & 1 & 4 & 3 \\ -2 & 3 & -1 & 0 \end{bmatrix}$$

In our mind, we delete the highlight row and the highlight column. The submatrix is the rest of horizontal and vertical numbers. Then

$$A_{23} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 3 \\ -2 & 3 & 0 \end{bmatrix}$$

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Then

$$A_{23} = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 0 & 2 \\ -2 & -1 & 0 \end{bmatrix}$$

A sign (+,-) refers to the property of being positive or negative.

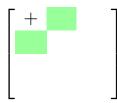
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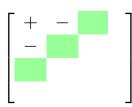
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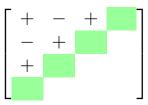
For a matrix, we can associate a sign table by filling the (i,j)-entry with a sign $(-1)^{i+j}$.

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For a matrix, we can associate a sign table by filling the (i,j)-entry with a sign $(-1)^{i+j}$. Graphically, we start with + at the (1,1)-entry and alternate the sign when moving horizontally or vertically from one position to another.







For instance, n = 4

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

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$$\begin{bmatrix}
+ & - & + & - \\
- & + & - & + \\
+ & - & + & - \\
- & + & - & +
\end{bmatrix}$$

The following are n = 3 and n = 5

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \text{ and } \begin{bmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{bmatrix}$$

Definition of Determinant

From now on, we consider square matrices only, unless stated otherwise.

Determinant

Definition

For a
$$1 \times 1$$
 matrix $A = [a]$, the determinant of $A = [a]$ is $\det A = a$.

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For a 1×1 matrix A = [a], the determinant of A = [a] is

$$\det A = a$$
.

For a 2 × 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant of A is

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

Definition Let $n \ge 3$ and let $A = [a_{ij}]$ be an $n \times n$ matrix. The determinant of A is

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Note 2: each A_{1j} is an $(n-1)\times(n-1)$ matrix. It means that to define a determinant of $n\times n$ matrix, we need to figure out the determinant of any $(n-1)\times(n-1)$ matrix. Recursively, we need the determinant of $(n-2)\times(n-2)$ until we know the determinant of 2×2 matrix.

Examples

(1) Compute $\det A$, where A is given by

$$A = \begin{bmatrix} 2 & 1 & -3 \\ -3 & -2 & 0 \\ 2 & 1 & 2 \end{bmatrix}$$

(2) Compute det A, where A is given by

$$A = \begin{bmatrix} 2 & 1 & 3 & -1 \\ -3 & -2 & 0 & 2 \\ 2 & 1 & 2 & -1 \\ 1 & 0 & 1 & -2 \end{bmatrix}$$

minor and cofactor

Definition Let A be an $n \times n$ matrix and let A_{ij} denote the submatrix (of order (n-1)) obtained from A by deleting its i-th row and its j-th column.

- (1) the i, j-minor of A is given by $\det A_{ij}$
- (2) the *i*, *j*-cofactor of A is given by $(-1)^{i+j} \det A_{ij}$.

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Example Find the (2,1)-minor and the (2,1)-cofactor of

$$A = \begin{bmatrix} 2 & -2 & 1 & -1 \\ 1 & 3 & 3 & 0 \\ 10 & 1 & 0 & -1 \\ 4 & 3 & -2 & 1 \end{bmatrix}$$

Theorem Let A be a square matrix of order n. Let i be a fixed row number. Then

$$\det A = (-1)^{i+1} a_{i1} \det A_{i1} + (-1)^{i+2} a_{i2} \det A_{i2} + \dots + (-1)^{i+n} a_{in} A_{in}$$

$$(1)$$

We call the equation (1) cofactor expansion along the i-th row (or simply expansion along the i-th row).

Theorem Let A be a square matrix of order n. Let j be a fixed column number. Then

$$\det A = (-1)^{1+j} a_{1j} \det A_{1j} + (-1)^{2+j} a_{2j} \det A_{2j} + \dots + (-1)^{n+j} a_{nj} \det A_{nj}$$
(2)

We call the equation (2) cofactor expansion along the j-th column (or simply expansion along the j-th column).

Remark Equations (1) and (2) allow us to expand along a row or a column which contains as many zero as possible for an efficiency of calculation of determinant.

Example

Compute $\det A$, by expansion along the row or column with the maximum number of zeros.