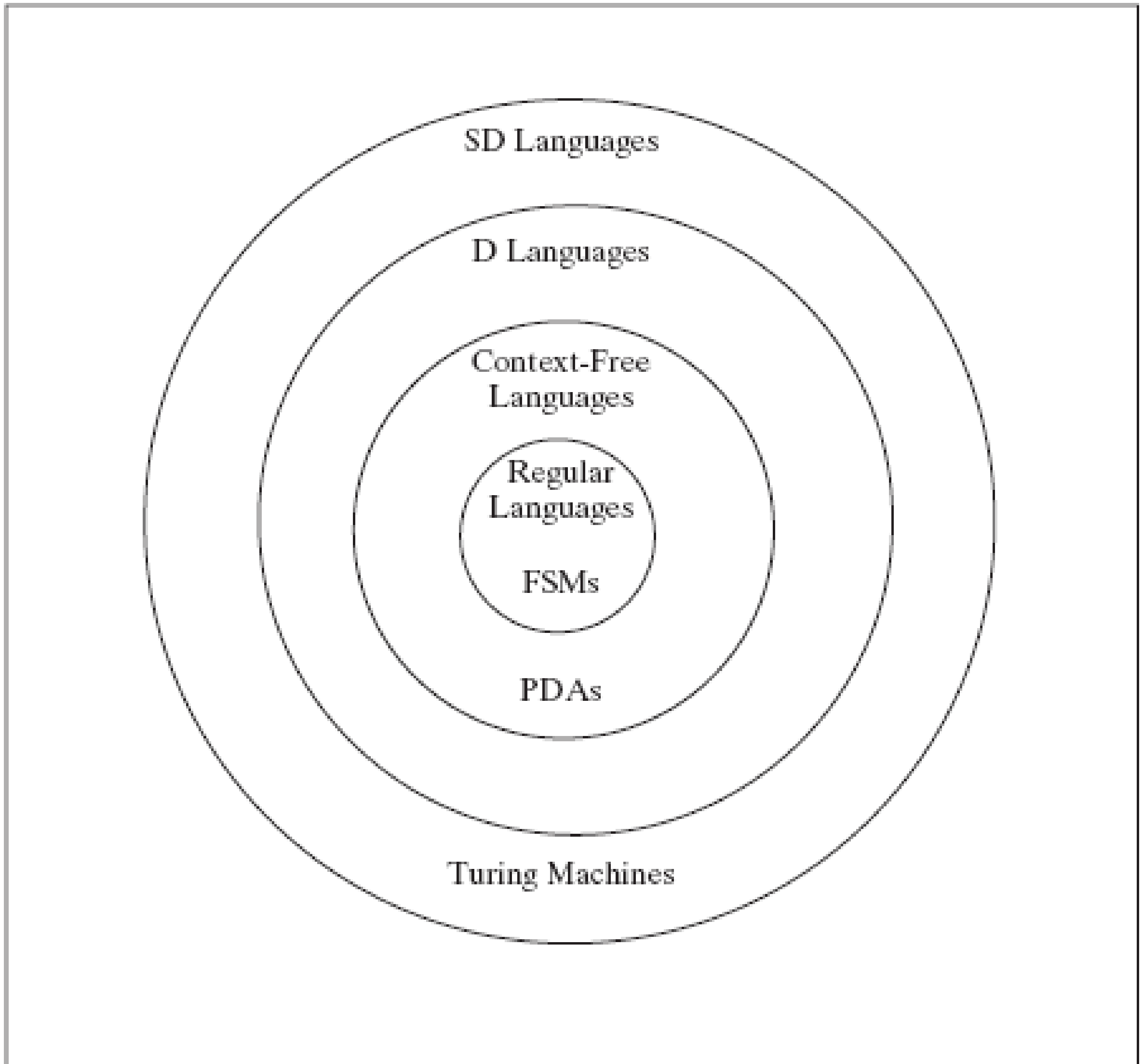
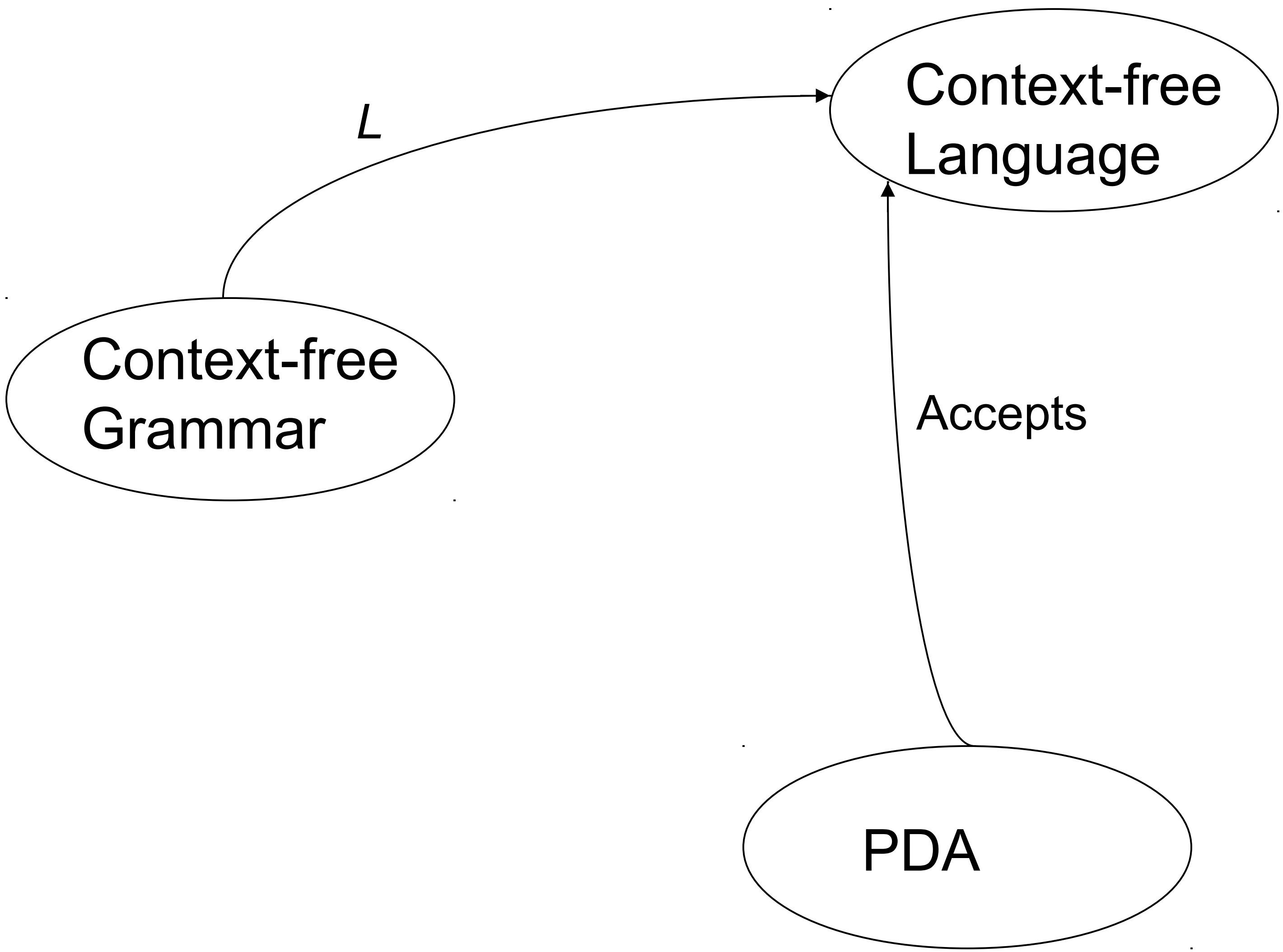
Context-Free Grammars

Chapter 11

Languages and Machines



Context-free Grammars, Languages, and PDAs



Grammars

A grammar G is a quadruple (V, Σ, R, S), where:

- V is the rule alphabet, which contains nonterminals and terminals,

- Σ (the set of terminals) is a subset of V,

- R is a finite set of rules of the form:

$$\mathbf{X \rightarrow Y, \quad X,Y \in V^*}$$

- $S \in V - \Sigma$ -- the start symbol

Context-Free Grammars

A context-free grammar G is a quadruple,
(V, Σ, R, S), where:

- V is the rule alphabet
  - Σ, a subset of V, the set of terminals
  - $V - \Sigma$, the set of nonterminals
- R, a finite subset of $(V - \Sigma) \times V^*$, the set of rules
- S, an element of $V - \Sigma$, the start symbol

Example:

$(\{S, \mathtt{a}, \mathtt{b}\}, \ \{\mathtt{a}, \mathtt{b}\}, \ \{S \rightarrow \mathtt{a}\ S\ \mathtt{b}, S \rightarrow \varepsilon\}, \ S)$

Derivations

$x \Rightarrow_G y$ iff $x = \alpha A \beta$

and $A \rightarrow \gamma$ is in R

$y = \alpha \gamma \beta$

$w_0 \Rightarrow_G w_1 \Rightarrow_G w_2 \Rightarrow_G \ldots \Rightarrow_G w_n$ is a derivation in G.

Let $\Rightarrow_G{}^*$ be the reflexive, transitive closure of $\Rightarrow_G$.

Then the language generated by G, denoted $L(G)$, is:

$$L(G) = \{w \in \Sigma^* : S \Rightarrow_G{}^* w\}.$$

An Example Derivation

Example:

Let G = $(\{S, \texttt{a}, \texttt{b}\}, \ \{\texttt{a}, \texttt{b}\}, \ \{S \rightarrow \texttt{a}\ S\ \texttt{b}, S \rightarrow \varepsilon\}, \ S)$

$S \Rightarrow \texttt{a}\ S\ \texttt{b} \Rightarrow \texttt{aa}\ S\ \texttt{bb} \Rightarrow \texttt{aaa}\ S\ \texttt{bbb} \Rightarrow \texttt{aaabbb}$

$S \Rightarrow^* \texttt{aaabbb}$

Definition of a Context-Free Grammar

A language *L* is ***context-free*** iff it is generated by some context-free grammar *G*.

Regular Grammars

In a regular grammar, all rules in R must:

- have a left hand side that is a single nonterminal
- have a right hand side that is:
  - ε, or
  - a single terminal, or
  - a single terminal followed by a single nonterminal.

Legal: $S \rightarrow \mathtt{a}$, $S \rightarrow \varepsilon$, and $T \rightarrow \mathtt{a}S$
Not legal: $S \rightarrow \mathtt{a}S\mathtt{a}$ and $\mathtt{a}S\mathtt{a} \rightarrow T$

The language defined by a grammar: all terminal strings that can be obtained starting from S and applying the rules

A^nB^n

$S \rightarrow \varepsilon$

$S \rightarrow \mathtt{a}S\mathtt{b}$

Balanced Parentheses

$S \rightarrow \varepsilon$
$S \rightarrow SS$
$S \rightarrow (S)$

Recursive Grammar Rules

- A rule is ***recursive*** iff it is $X \rightarrow w_1 Y w_2$, where:

  $Y \Rightarrow^* w_3 X w_4$ for some w_1, w_2, w_3, and w_4 in V^*.

- A grammar is recursive iff it contains at least one recursive rule.

- Examples: $S \rightarrow (S)$ $\quad\quad$ $S \rightarrow (T)$

  $T \rightarrow (S)$

Self-Embedding Grammar Rules

- A rule in a grammar G is ***self-embedding*** iff it is :
  $X \rightarrow w_1 Y w_2$, where $Y \Rightarrow^* w_3 X w_4$ and
  both $w_1 w_3$ and $w_4 w_2$ are in Σ^+.

- A grammar is self-embedding iff it contains at least one self-embedding rule.

- Example: $S \rightarrow \mathtt{a} S \mathtt{a}$ is self-embedding
  $S \rightarrow \mathtt{a} S$ is recursive but not self-embedding
  $S \rightarrow \mathtt{a} T$
  $T \rightarrow S \mathtt{a}$ is self-embedding

Where Context-Free Grammars Get Their Power

- If a grammar *G* is not self-embedding then *L*(*G*) is regular.
- If a language *L* has the property that every grammar that defines it is self-embedding, then *L* is not regular.

PalEven = $\{ww^R : w \in \{\mathtt{a}, \mathtt{b}\}^*\}$

$G = \{\{S, \mathtt{a}, \mathtt{b}\}, \{\mathtt{a}, \mathtt{b}\}, R, S\}$, where:

$$R = \{\ S \rightarrow \mathtt{a}S\mathtt{a}$$
$$S \rightarrow \mathtt{b}S\mathtt{b}$$
$$S \rightarrow \varepsilon\ \}.$$

Equal Numbers of a's and b's

Let $L = \{w \in \{\mathtt{a}, \mathtt{b}\}^*: \#_{\mathtt{a}}(w) = \#_{\mathtt{b}}(w)\}$.

$G = \{\{S, \mathtt{a}, \mathtt{b}\}, \{\mathtt{a}, \mathtt{b}\}, R, S\}$, where:

$$
\begin{aligned}
R = \{\ & S \rightarrow \mathtt{a}S\mathtt{b} \\
& S \rightarrow \mathtt{b}S\mathtt{a} \\
& S \rightarrow SS \\
& S \rightarrow \varepsilon\ \}.
\end{aligned}
$$

Arithmetic Expressions

$G = (V, \Sigma, R, E)$, where

$V = \{+, *, (,), \texttt{id}, E\}$,

$\Sigma = \{+, *, (,), \texttt{id}\}$,

$R = \{$

$E \rightarrow E + E$

$E \rightarrow E * E$

$E \rightarrow (E)$

$E \rightarrow \texttt{id}\ \}$

BNF

A notation for writing practical context-free grammars

- The symbol | should be read as “or”.

  Example: $S \rightarrow \mathtt{a}S\mathtt{b} \mid \mathtt{b}S\mathtt{a} \mid SS \mid \varepsilon$

- Allow a nonterminal symbol to be any sequence of characters surrounded by angle brackets.

  Examples of nonterminals:

  <program>
  <variable>

BNF for a Java Fragment

```
<block> ::= {<stmt-list>} | {}
<stmt-list> ::= <stmt> | <stmt-list> <stmt>
<stmt> ::= <block> | while (<cond>) <stmt> |
           if (<cond>) <stmt> |
           do <stmt> while (<cond>); |
           <assignment-stmt>; |
           return | return <expression> |
           <method-invocation>;
```

HTML

```
<ul>
    <li>Item 1, which will include a sublist</li>
        <ul>
            <li>First item in sublist</li>
            <li>Second item in sublist</li>
        </ul>
    <li>Item 2</li>
</ul>
```

A grammar:

$$HTMLtext \rightarrow Element\ \ HTMLtext \mid \varepsilon$$

$Element \rightarrow UL \mid LI \mid \ldots$ (and other kinds of elements that are allowed in the body of an HTML document)

$$UL \rightarrow \texttt{
}\ HTMLtext\ \texttt{
}$$

$$LI \rightarrow \texttt{
}\ HTMLtext\ \texttt{
}$$

English

$S \rightarrow NP\ VP$

$NP \rightarrow$ `the` $Nominal$ | `a` $Nominal$ | $Nominal$ |
$ProperNoun$ | $NP\ PP$

$Nominal \rightarrow N$ | $Adjs\ N$

$N \rightarrow$ `cat` | `dogs` | `bear` | `girl` | `chocolate` | `rifle`

$ProperNoun \rightarrow$ `Chris` | `Fluffy`

$Adjs \rightarrow Adj\ Adjs$ | Adj

$Adj \rightarrow$ `young` | `older` | `smart`

$VP \rightarrow V$ | $V\ NP$ | $VP\ PP$

$V \rightarrow$ `like` | `likes` | `thinks` | `shots` | `smells`

$PP \rightarrow Prep\ NP$

$Prep \rightarrow$ `with`

Designing Context-Free Grammars

- Generate related regions together.

  A^nB^n

- Generate concatenated regions:

  $A \rightarrow BC$

- Generate outside in:

  $A \rightarrow \mathtt{a}A\mathtt{b}$

Concatenating Independent Languages

Let $L = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^m : n, m \geq 0\}$.

The $\mathtt{c}^m$ portion of any string in L is completely independent of the $\mathtt{a}^n\mathtt{b}^n$ portion, so we should generate the two portions separately and concatenate them together.

$G = (\{S, N, C, \mathtt{a}, \mathtt{b}, \mathtt{c}\}, \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}, R, S\}$ where:

$$
\begin{aligned}
R = \{\ & S \rightarrow NC \\
& N \rightarrow \mathtt{a}N\mathtt{b} \\
& N \rightarrow \varepsilon \\
& C \rightarrow \mathtt{c}C \\
& C \rightarrow \varepsilon\ \}.
\end{aligned}
$$

$L = \{ a^{n_1}b^{n_1}a^{n_2}b^{n_2}\dots a^{n_k}b^{n_k} : k \geq 0 \text{ and } \forall i\ (n_i \geq 0)\}$

Examples of strings in L: ε, abab, aabbaaabbbabab

Note that $L = \{a^nb^n : n \geq 0\}^*$.

$G = (\{S, M, a, b\}, \{a, b\}, R, S\}$ where:

$$R = \{ S \rightarrow MS$$
$$S \rightarrow \varepsilon$$
$$M \rightarrow aMb$$
$$M \rightarrow \varepsilon\}.$$

Unequal a's and b's

$L = \{\mathtt{a}^n\mathtt{b}^m : n \neq m\}$

$G = (V, \Sigma, R, S)$, where

$V = \{\mathtt{a}, \mathtt{b}, S, A, B\}$,

$\Sigma = \{\mathtt{a}, \mathtt{b}\}$,

$R =$

$S \rightarrow A$ /* more a's than b's

$S \rightarrow B$ /* more b's than a's

$A \rightarrow \mathtt{a}$ /* at least one extra a generated

$A \rightarrow \mathtt{a}A$

$A \rightarrow \mathtt{a}A\mathtt{b}$

$B \rightarrow \mathtt{b}$ /* at least one extra b generated

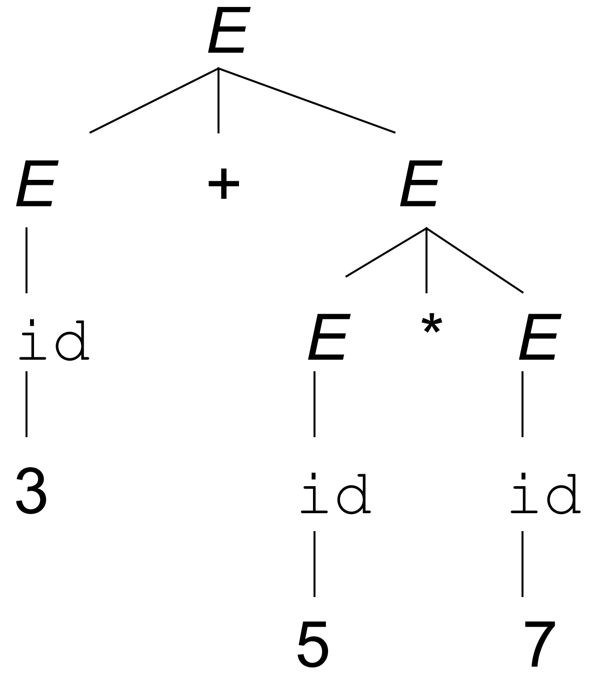
$B \rightarrow B\mathtt{b}$

$B \rightarrow \mathtt{a}B\mathtt{b}$

Structure

Context free languages:

We care about structure.



Derivations

To capture structure, we must capture the path we took through the grammar. **Derivations** do that.

Example:

$$S \rightarrow \varepsilon$$
$$S \rightarrow SS$$
$$S \rightarrow (S)$$

$$S \overset{1}{\Rightarrow} SS \overset{2}{\Rightarrow} (S)S \overset{3}{\Rightarrow} ((S))S \overset{4}{\Rightarrow} (())S \overset{5}{\Rightarrow} (())(S) \overset{6}{\Rightarrow} (())()$$
$$S \underset{1}{\Rightarrow} SS \underset{2}{\Rightarrow} (S)S \underset{3}{\Rightarrow} ((S))S \underset{5}{\Rightarrow} ((S))(S) \underset{4}{\Rightarrow} (())(S) \underset{6}{\Rightarrow} (())()$$

But the order of rule application doesn’t matter.

Derivations

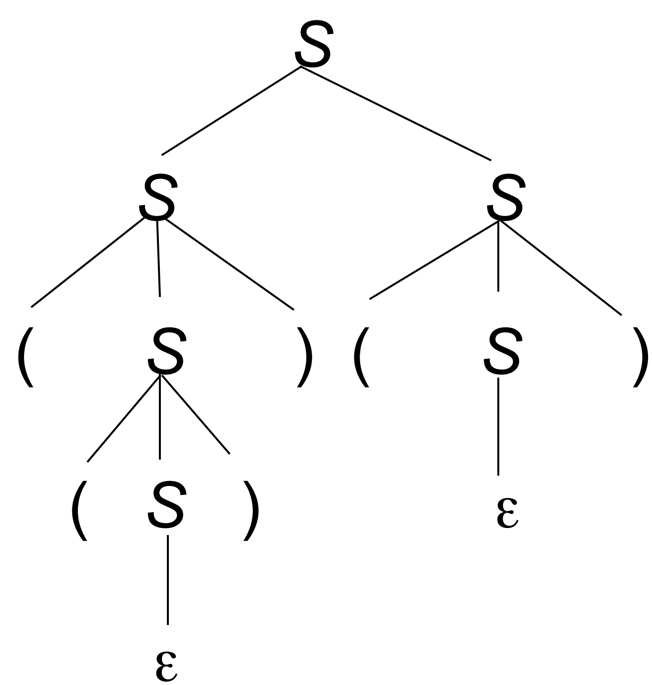
Parse trees capture essential structure:

1 2 3 4 5 6

$S \Rightarrow SS \Rightarrow (S)S \Rightarrow ((S))S \Rightarrow \quad (())S \quad \Rightarrow (())(S) \Rightarrow (())()$

$S \Rightarrow SS \Rightarrow (S)S \Rightarrow ((S))S \Rightarrow ((S))(S) \Rightarrow (())(S) \Rightarrow (())()$

1 2 3 5 4 6



Parse Trees

A **parse tree**, derived by a grammar $G = (V, \Sigma, R, S)$, is a rooted, ordered tree in which:

- Every leaf node is labeled with an element of $\Sigma \cup \{\varepsilon\}$,
- The root node is labeled S,
- Every other node is labeled with some element of:
  $V - \Sigma$, and
- If m is a nonleaf node labeled X and the children of m are labeled $x_1, x_2, \ldots, x_n$, then R contains the rule
  $X \rightarrow x_1, x_2, \ldots, x_n$.

Ambiguity

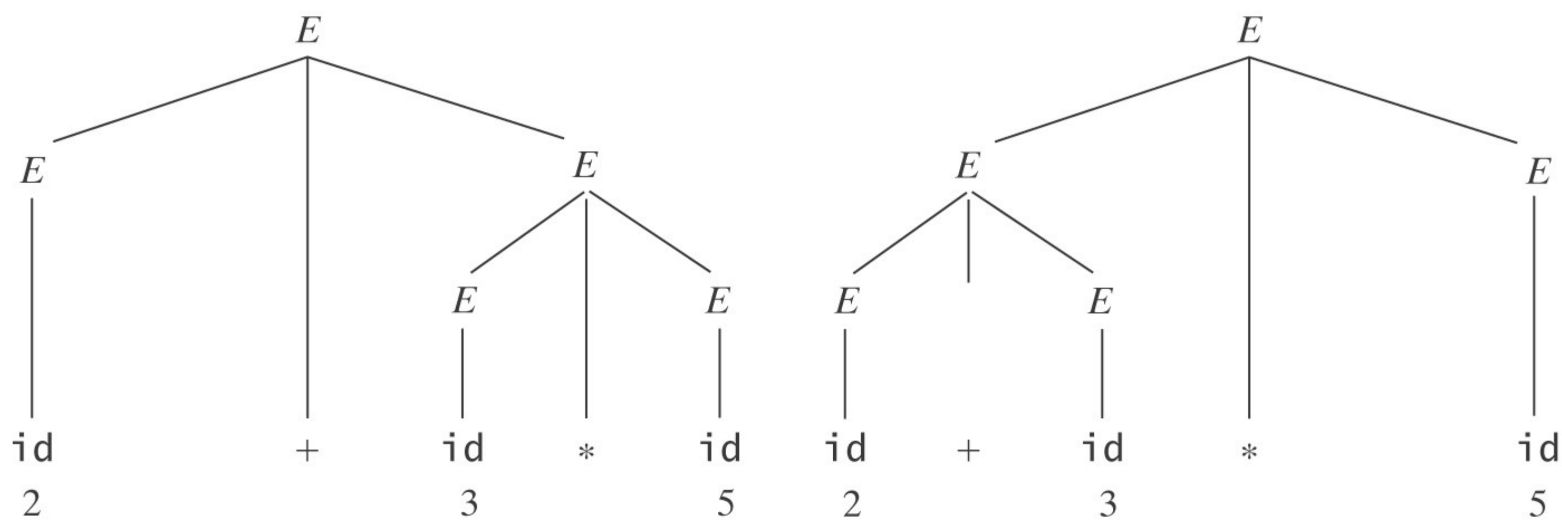
A grammar is ***ambiguous*** iff there is at least one string in $L(G)$ for which G produces more than one parse tree.

For most applications of context-free grammars, this is a problem.

An Arithmetic Expression Grammar

$$E \to E + E$$
$$E \to E * E$$
$$E \to (E)$$
$$E \to \texttt{id}$$

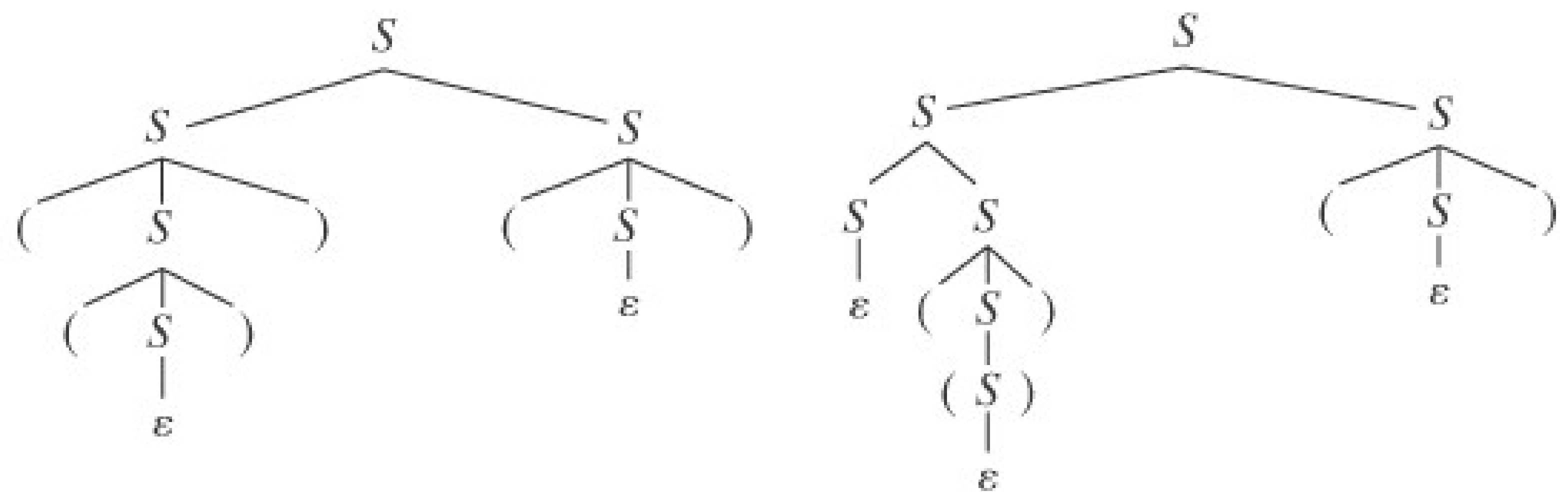
`id + id * id`



Even a Very Simple Grammar Can be Highly Ambiguous

$S \rightarrow \varepsilon$
$S \rightarrow SS$
$S \rightarrow (S)$

(())()



Inherent Ambiguity

Some languages have the property that every grammar for them is ambiguous. We call such languages ***inherently ambiguous***.

Example:

$L = \{a^n b^n c^m: n, m \geq 0\} \cup \{a^n b^m c^m: n, m \geq 0\}$.

It can be proved that L is inherently ambiguous.

We can generate $a^n b^n c^m$ and $a^n b^m c^m$ unambiguously but $a^n b^n c^n$ will be generated in two ways.

Inherent Ambiguity

Both of the following problems are undecidable:

- Given a context-free grammar G, is G ambiguous?
- Given a context-free language L, is L inherently ambiguous?

But We Can Often Reduce Ambiguity

We can get rid of:

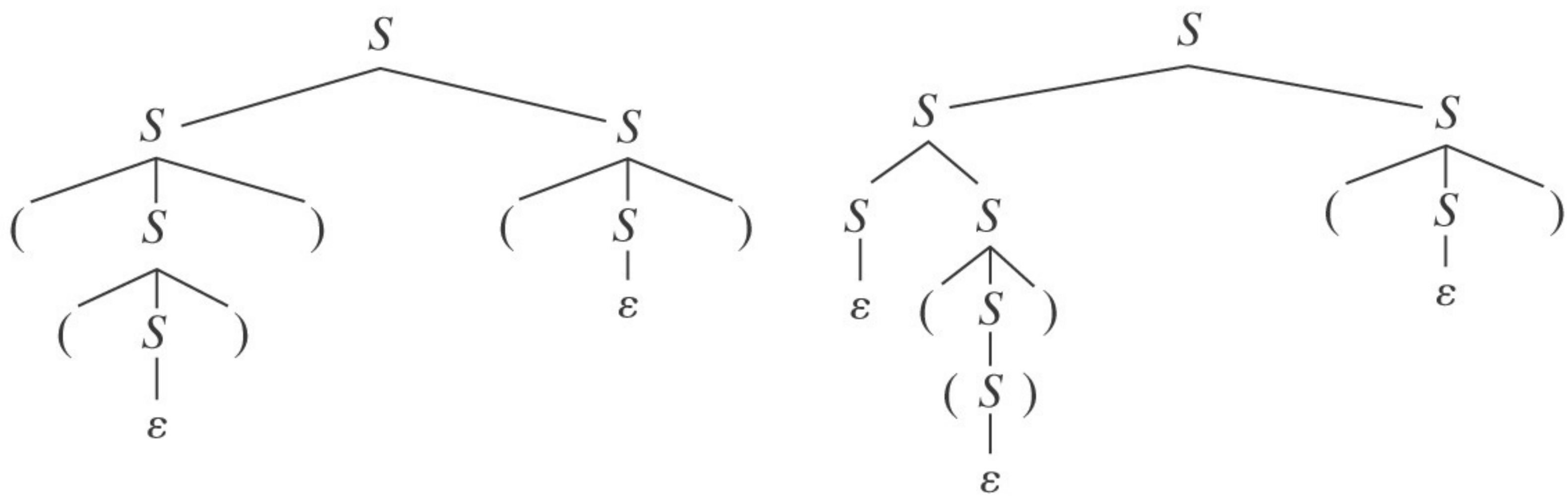
- ε rules like $S \rightarrow \varepsilon$,
- rules with symmetric right-hand sides, e.g.,

$$S \rightarrow SS$$
$$E \rightarrow E + E$$

- rule sets that lead to ambiguous attachment of optional postfixes.

A Highly Ambiguous Grammar

$S \rightarrow \varepsilon$
$S \rightarrow SS$
$S \rightarrow (S)$



Resolving the Ambiguity with a Different Grammar

The biggest problem is the ε rule.

A different grammar for the language of balanced parentheses:

$$
\begin{aligned}
S^* &\rightarrow \varepsilon \\
S^* &\rightarrow S \\
S &\rightarrow SS \\
S &\rightarrow (S) \\
S &\rightarrow ()
\end{aligned}
$$

Nullable Variables

A variable X is ***nullable*** iff either:

(1) there is a rule $X \to \varepsilon$, or

(2) there is a rule $X \to PQR\ldots$ and P, Q, R, …
are all nullable.

So compute N, the set of nullable variables, as follows:

1. Set N to the set of variables that satisfy (1).
2. Repeat until no change
   Add variables satisfying (2)

A General Technique for Getting Rid of ε-Rules

Definition: a rule is ***modifiable*** iff it is of the form:

$P \rightarrow \alpha R\beta$, for some nullable R, $P \neq \alpha\beta \neq \varepsilon$

removeEps(G: cfg) =

1. Let $G' = G$.
2. Find the set N of nullable variables in G'.
3. For each modifiable rule $P \rightarrow \alpha R\beta$ of G do
   Add the rule $P \rightarrow \alpha\beta$.
4. Delete from G' all rules of the form $X \rightarrow \varepsilon$.
5. Return G'.

$L(G') = L(G) - \{\varepsilon\}$

An Example

$G = \{\{S, T, A, B, C, \mathtt{a}, \mathtt{b}, \mathtt{c}\}, \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}, R, S), R =$

$$\{\, S \rightarrow \mathtt{a}T\mathtt{a}$$
$$T \rightarrow ABC$$
$$A \rightarrow \mathtt{a}A \mid C$$
$$B \rightarrow B\mathtt{b} \mid C$$
$$C \rightarrow \mathtt{c} \mid \varepsilon \,\}$$

Nullable varibles = {A, B, C, T}

G´: add : remove:

$S \rightarrow \mathtt{aa}$	$T \rightarrow AC$	$C \rightarrow \varepsilon$
$T \rightarrow A$	$T \rightarrow BC$	
$T \rightarrow B$	$B \rightarrow \mathtt{b}$	
$T \rightarrow C$	$A \rightarrow \mathtt{a}$	
$T \rightarrow AB$		

What If $\varepsilon \in L$?

atmostoneEps(G: cfg) =

1. $G'' = removeEps(G)$.
2. If S_G is nullable then /* i. e., $\varepsilon \in L(G)$
   2.1 Create in G'' a new start symbol S^*.
   2.2 Add to $R_{G''}$ the two rules:

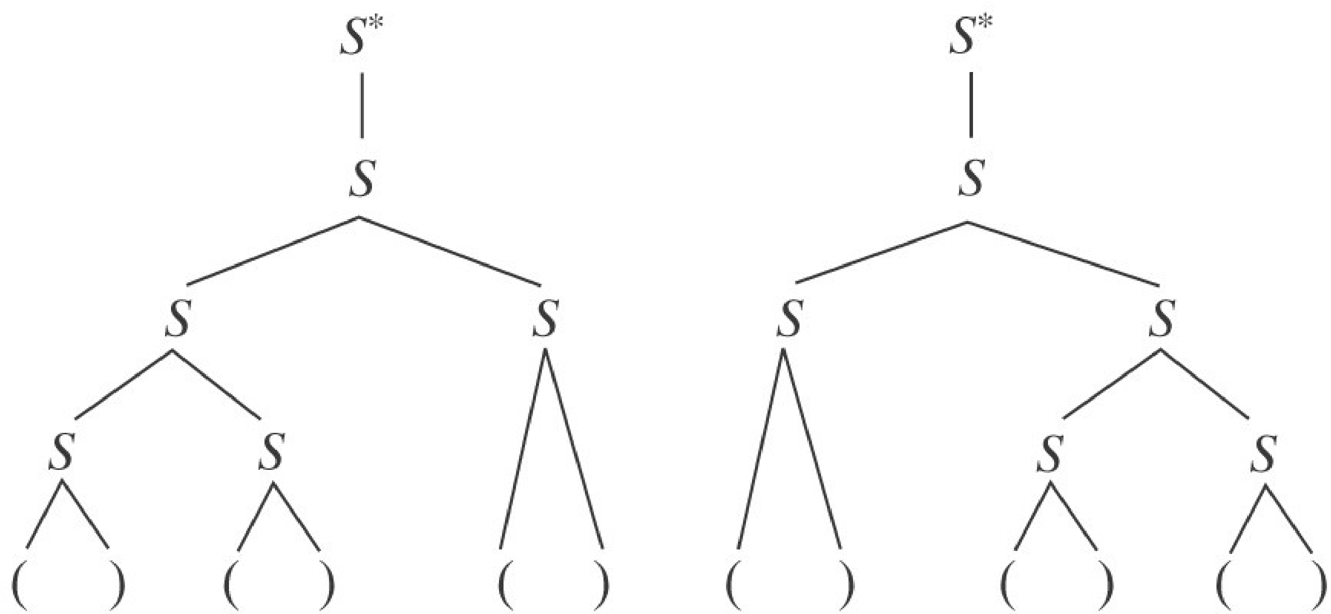
$$S^* \rightarrow \varepsilon$$
$$S^* \rightarrow S_G.$$

3. Return G''.

But There is Still Ambiguity

$S^* \rightarrow \varepsilon$
$S^* \rightarrow S$
$S \rightarrow SS$
$S \rightarrow (S)$
$S \rightarrow ()$

What about ()()() ?



Eliminating Symmetric Recursive Rules

$S^* \rightarrow \varepsilon$
$S^* \rightarrow S$
$S \rightarrow SS$
$S \rightarrow (S)$
$S \rightarrow ()$

Replace $S \rightarrow SS$ with one of:

$S \rightarrow SS_1$ /* force branching to the left
$S \rightarrow S_1S$ /* force branching to the right

So we get:

$S^* \rightarrow \varepsilon$
$S^* \rightarrow S$

$S \rightarrow SS_1$
$S \rightarrow S_1$
$S_1 \rightarrow (S)$
$S_1 \rightarrow ()$

Eliminating Symmetric Recursive Rules

So we get:

$S^* \rightarrow \varepsilon$

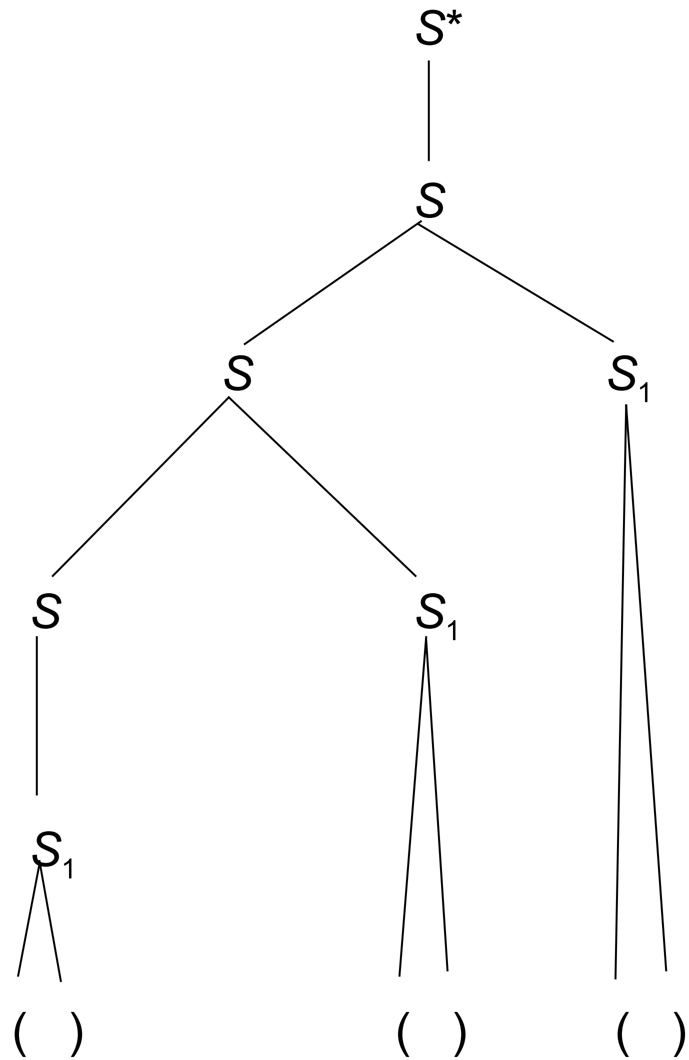
$S^* \rightarrow S$

$S \rightarrow SS_1$

$S \rightarrow S_1$

$S_1 \rightarrow (S)$

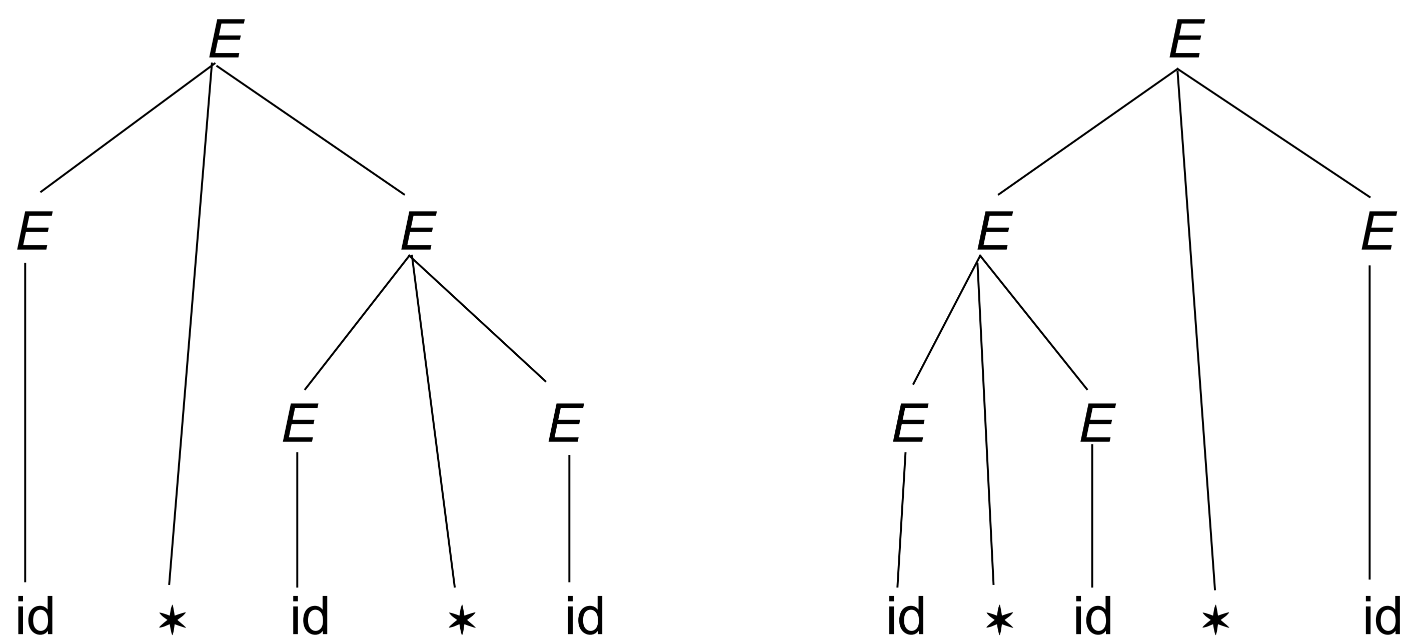
$S_1 \rightarrow ()$



Arithmetic Expressions

$$E \rightarrow E + E$$
$$E \rightarrow E \ast E$$
$$E \rightarrow (E)$$
$$E \rightarrow \texttt{id}$$

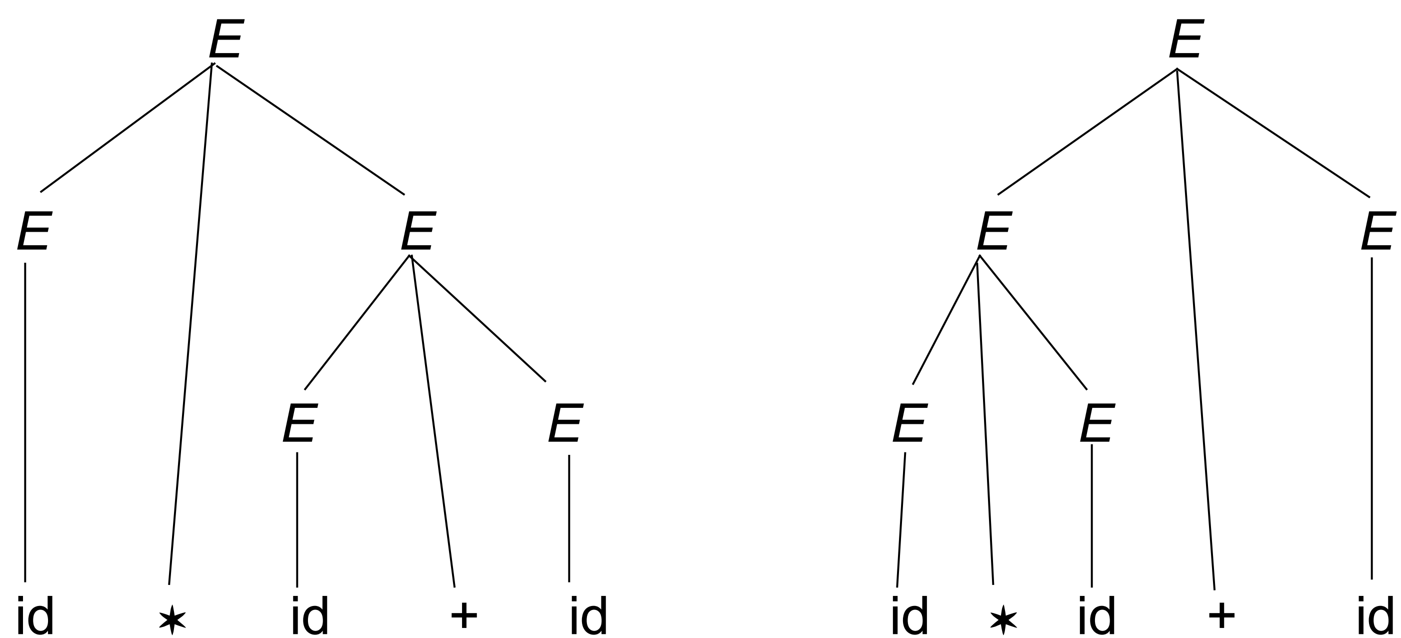
Problem 1: Associativity



Arithmetic Expressions

$$E \rightarrow E + E$$
$$E \rightarrow E \ast E$$
$$E \rightarrow (E)$$
$$E \rightarrow \texttt{id}$$

Problem 2: Precedence

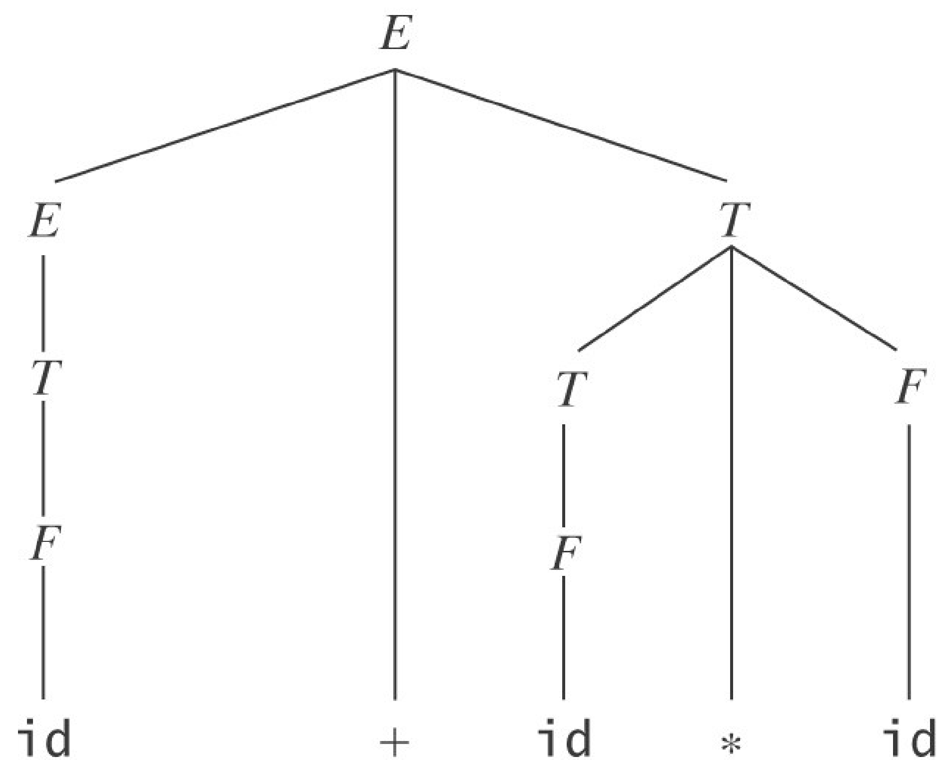


Arithmetic Expressions - A Better Way

$E \rightarrow E + T$
$E \rightarrow T$
$T \rightarrow T * F$
$T \rightarrow F$
$F \rightarrow (E)$
$F \rightarrow \text{id}$

Example:

`id + id * id`



Dangling "else"

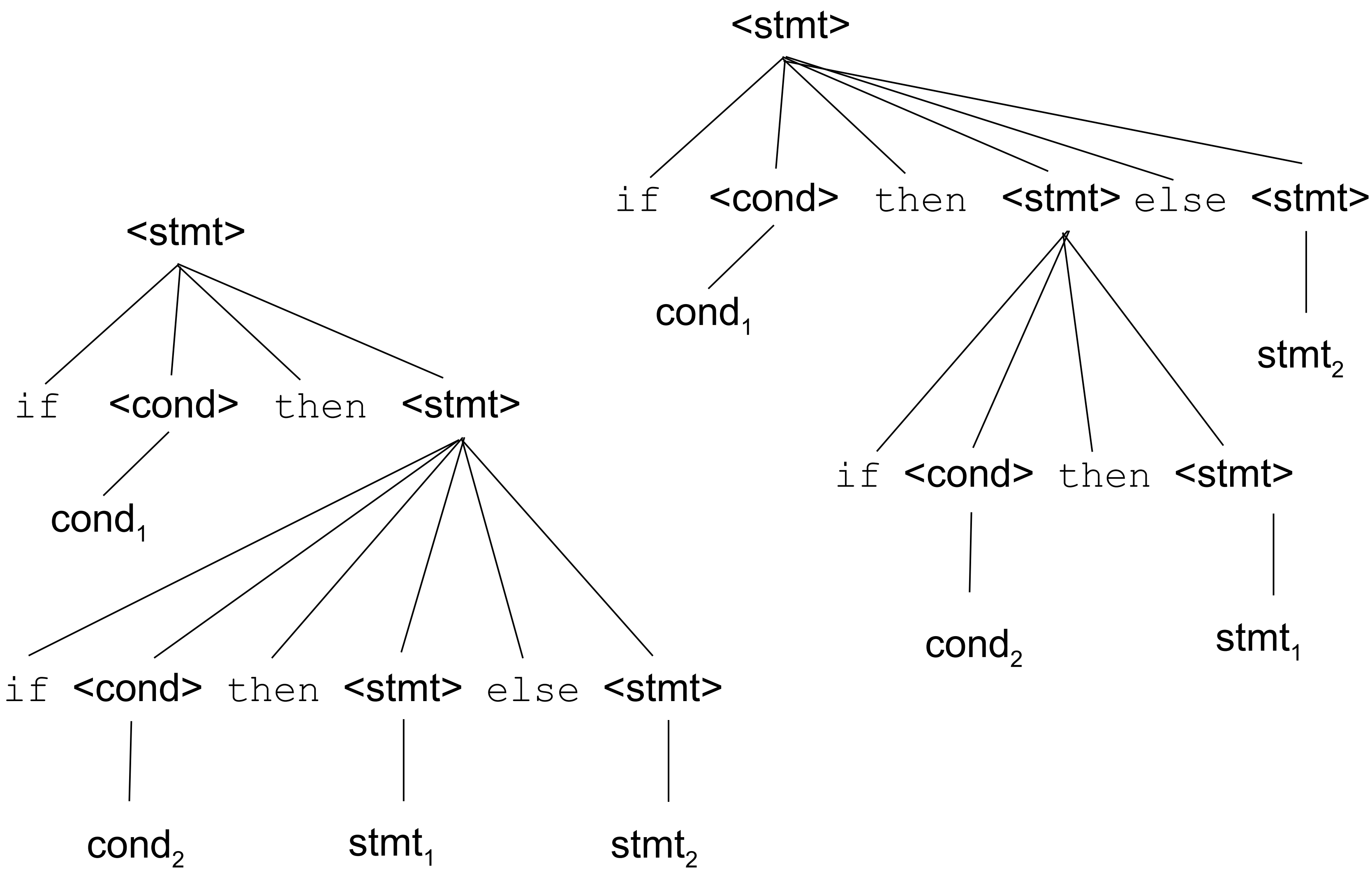
The dangling else problem:

<stmt> ::= `if` <cond> `then` <stmt>
<stmt> ::= `if` <cond> `then` <stmt> `else` <stmt>

Consider:

`if` cond_1 `then` `if` cond_2 `then` stmt_1 `else` stmt_2

Dangling “else” ambiguity



Dangling “else” solution

```
<stmt>          ::= <matched_if>
                 |  <unmatched_if>
<matched_if>    ::= if <cond> then <matched_if> else <matched_if>
                 |  <other_stmt>
<unmatched_if>  ::= if <cond> then <stmt>
                 |  if <cond> then <matched_if> else <unmatched_if>
```

Dangling “else” solution