§ 3.7

More proofs Recall: $n \in \mathbb{Z}$. if n is a prime then it can't be written as $n = k \cdot l$ where both $k, l \in \mathbb{Z}$ and $1 < k, l < n$.

Thorem: Every $n > 1$ is either prime or product of primes.

Thorem 3.7.2. there are infinity many prime number.

Proof: Suppose there are finitely many primes. $p_1, p_2 \cdots p_n$.
let $m = p_1 p_2 \cdots p_n$. m is not divisible by p_1, since if $m = p_1 k$ from $k \in \mathbb{Z}$, then $1 = p_1 k - p_1 p_2 \cdots p_n = p_1(k - p_2 p_3 \cdots p_n)$, which is impossible since $p_1 > 1$. Similarly, m is divisible by any p_i.
Since $m > 1$, it is either prime or not prime:
Case 1: if $m > 1$, since $m > p_n$, it is a prime not in the list, but $p_1 \cdots p_n$ are list of all primes, so it is a contradiction.
Case 2: if m is a product of primes, then q be any prime factor of m, but q is not in the list, so it is a contradiction.

Thorem 3.7.3. for any $n \in \mathbb{Z}^*$, there's a sequence of n concecative position integers none of which are primes.

Proof: let $n \in \mathbb{Z}^*$, let $x = (n+1)! + 2$. We claims that $x, x+1, x+2 \cdots x+n-1$ are all not prime numbers. let $i \in [0, n-1]$,
$x + i = (n+1)! + 2 + i = 1 \times 2 \times 3 \times \cdots \times (n+1) + i + 2$
$= (i+2)(1 \times 2 \times 3 \cdots \times i \times (i+2) \times \cdots \times (n+1) + 1)$, which works since $2 \le i+2 \le n+1$. Since both factories are integers > 1, $x+i$ is not prime $\square$

§ 3.7.4 There is a unique $m \in \mathbb{R}$ such that
1) $\forall x \in \mathbb{R}$, $x^2 + 2x + 3 \geq m$.
2) if y has the same property 1), then $m \geq y$.

Note 1) m is a lower bound for $x^2 + 2x + 3$.
2) m is the greatest lower bound
$x^2 + 2x + 3 = (x-1)^2 + 2 \geq 2$.

Existence: $m = 2$

1) Given: $m = 2$ — Goal: $\forall x \; x^2 + 2x + 3 \geq 2$.
$\Rightarrow (x+1)^2 + 2 \geq 2$.

2) Given | Goal

$\forall x\,(x^2+2x+3) \geq y$ | $\forall y\,(\forall x\ x^2+2x+3 \geq y) \to m \geq y.$

$x=-1$ | $2 \geq y.$

$2 \geq y$

Uniqueness

Given | Goal

$\forall x\ x^2+2x+3 \geq m_1$ | $\forall m_1, m_2$ satisfy 1), 2)

$\forall x\ x^2+2x+3 \geq m_2$ | $\to m_1 = m_2.$

$\forall y\,(\forall x\ x^2+2x+3 \geq y \to m_1 = y)$

$\forall y\,(\forall x\ x^2+2x+3 \geq y \to m_2 = y).$

$y = m_1$

$m_2 = m_1.$