Exercise 4.3.23. Suppose A is a set and $F \subseteq \mathcal{P}(A)$. Let

$$R = \{(a, b) \in A \times A \mid \text{for every } X \subseteq A \setminus \{a, b\}, \text{if } X \cup \{a\} \in \mathcal{F} \text{ then } X \cup \{b\} \in \mathcal{F}\}.$$

Show that R is transitive.

Proof. Let $a, b, c \in A$ and assume aRb and bRc. To prove aRc, suppose that $X \subseteq A \setminus \{a, c\}$ and that $X \cup \{a\} \in \mathcal{F}$. We need to show that $X \cup \{c\} \in \mathcal{F}$.

Case 1: $b \notin X$. Then $X \subseteq A \setminus \{a, b\}$. So, since aRb, we have that $X \cup \{b\} \in \mathcal{F}$. Also, $X \subseteq A \setminus \{b, c\}$, so, since bRc, we have that $X \cup \{c\} \in \mathcal{F}$.

Case 2: $b \in X$. Let $X' = (X \cup \{a\}) \setminus \{b\}$ and $X'' = (X \cup \{c\}) \setminus \{b\}$. Since $X' \cup \{b\} = X \cup \{a\}$, we have that $X' \cup \{b\} \in \mathcal{F}$. Since bRc and $X' \subseteq A \setminus \{b, c\}$, it follows that $X' \cup \{c\} \in \mathcal{F}$. Now $X' \cup \{c\} = X'' \cup \{a\}$. Since aRb and $X'' \subseteq A \setminus \{a, b\}$, it follows that $X'' \cup \{b\} \in \mathcal{F}$. But $X'' \cup \{b\} = X \cup \{c\}$, so we also have $X \cup \{c\} \in \mathcal{F}$.

In either case, we have shown that $X \cup \{c\} \in \mathcal{F}$, as required. □