CHAPTER 6
Relations and Identity

This chapter brings quantificational logic up to full power by adding **identity statements** (like “a=b”) and **relational statements** (like “Lrj” for “Romeo loves Juliet”). We’ll end with definite descriptions.

6.1 Identity translations

Our rule 3 for forming quantificational wffs introduces “=” (“equals”):

> 3. The result of writing a small letter and then “=” and then a small letter is a wff.

This rule lets us construct wffs like these:

x=y	=	x equals y.
r=l	=	Romeo is the lover of Juliet.
~r=l	=	Romeo isn’t the lover of Juliet.

We negate an identity wff by writing “~” in front. Neither “r=1” nor “~r=1” use parentheses, since these aren’t needed to avoid ambiguity.

The simplest use of “=” is to translate an “is” that goes between singular terms. Recall the difference between general and singular terms:

Use capital letters for **general terms** (terms that *describe* or put in a *category*):	Use small letters for **singular terms** (terms that pick out a *specific* person or thing):
L=a lover C=charming R=drives a Rolls	l=the lover of Juliet c=this child r=Romeo
Use capitals for “a so and so,” adjectives, and verbs.	Use small letters for “the so and so,” “this so and so,” and proper names.

Compare these two forms:

Predication	*Identity*
Lr	r=1
Romeo is a lover.	Romeo is the lover of Juliet.

Use “=” for “is” if both sides are singular terms (and thus represented by small letters). The “is” of identity can be replaced with “is identical to” or “is the same entity as”—and can be reversed (so if x=y then y=x).

We can translate "other than," "besides," and "alone" using identity:

Someone *other than* Romeo is rich =Someone *besides* Romeo is rich	=	Someone who isn't Romeo is rich.
	=	For some x, x≠Romeo and x is rich.
	=	$(\exists x)(\sim x{=}r \cdot Rx)$

Romeo *alone* is rich	=	Romeo is rich and no one besides Romeo is rich.
	=	$(Rr \cdot \sim(\exists x)(\sim x{=}r \cdot Rx))$

We also can translate some numerical notions, for example:

At least two are rich	=	For some x and some y:x≠y, x is rich, and y is rich.
	=	$(\exists x)(\exists y)(\sim x{=}y \cdot (Rx \cdot Ry))$

The pair of quantifiers "$(\exists x)(\exists y)$" ("for some x and some y") doesn't say whether x and y are identical; so we need "~x=y" to say that they aren't.

Henceforth we'll often need more variable letters than just "x" to keep references straight. It doesn't matter what letters we use; these two are equivalent:

$(\exists x)Rx$	=	For some x, x is rich	=	At least one being is rich.
$(\exists y)Ry$	=	For some y, y is rich	=	At least one being is rich.

Here's how we translate "exactly one" and "exactly two":

Exactly one being is rich	=	For some x:x is rich and there's no y such that y≠x and y is rich.
	=	$(\exists x)(Rx \cdot \sim(\exists y)(\sim y{=}x \cdot Ry))$
Exactly two beings are rich	=	For some x and some y:x is rich and y is rich and x≠y and there's no z such that z≠x and z≠y and z is rich.
	=	$(\exists x)(\exists y)(((Rx \cdot Ry) \cdot \sim x{=}y) \cdot \sim(\exists z)((\sim z{=}x \cdot \sim z{=}y) \cdot Rz))$

So our notation can express "There are exactly n F's" for any specific whole number n.

We also can express addition. Here's an English paraphrase of "1+1=2" and the corresponding formula:

If exactly one being is F and exactly one being is G and nothing is F-and-G, then exactly two beings are F-or-G.	$((((\exists x)(Fx \cdot \sim(\exists y)(\sim y{=}x \cdot Fy)) \cdot (\exists x)(Gx \cdot \sim(\exists y)(\sim y{=}x \cdot Gy))) \cdot \sim(\exists x)(Fx \cdot Gx)) \supset (\exists x)(\exists y)(((Fx \vee Gx) \cdot (Fy \vee Gy)) \cdot (\sim x{=}y \cdot \sim(\exists z)((\sim z{=}x \cdot \sim z{=}y) \cdot (Fz \vee Gz)))))$

We could prove our "1+1=2" formula by assuming its denial and deriving a contradiction. While we won't do this, it's interesting that it could be done. In principle, we could prove "2+2=4" and "5+7=12"—and the additions on your income tax form. Some mean logic teachers assign such things for homework.

6.1a Exercise—LogiCola H (IM & IT)

Translate these English sentences into wffs.

Jim is the goalie and is a student.	(j=g·Sj)

1. Aristotle is a logician.
2. Aristotle is the greatest logician.
3. Aristotle isn't Plato.
4. Someone besides Aristotle is a logician.
5. There are at least two logicians.
6. Aristotle alone is a logician.
7. All logicians other than Aristotle are evil.
8. No one besides Aristotle is evil.
9. The philosopher is Aristotle.
10. There's exactly one logician.
11. There's exactly one evil logician.
12. Everyone besides Aristotle and Plato is evil.
13. If the thief is intelligent, then you aren't the thief.
14. Carol is my only sister.
15. Alice runs but isn't the fastest runner.
16. There's at most one king.
17. The king is bald.
18. There's exactly one king and he is bald.

6.2 Identity proofs

We need two new rules for identity. This self-identity (SI) rule holds regardless of what constant replaces "a":

Self-identity $\boxed{\rightarrow \quad a=a}$

This is an **axiom**—a basic assertion that isn't proved but can be used to prove other things. Rule SI says that we may assert a self-identity as a "derived step" anywhere in a proof, no matter what the earlier lines are. Adding "a=a" can be useful if this gives us a contradiction (since we already have "~a=a") or lets us apply an I-rule (since we already have "(a=a⊃Gb)").

The equals-may-substitute-for-equals (SE) rule is based on the idea that identicals are interchangeable: if a=b, then whatever is true of a also is true of b, and vice versa. This rule holds regardless of what constants replace "a" and "b" and what wffs replace "Fa" and "Fb"—provided that the two wffs are alike except that the constants are interchanged in one or more occurrences:

Substitute equals $\boxed{Fa,\ a=b \quad \rightarrow \quad Fb}$

Here's a simple identity proof:

I weigh 180 pounds.
My mind doesn't weigh 180 pounds.
∴ I'm not identical to my mind.

```
1      Wi      Valid
2      ~Wm
  [ ∴ ~i=m
3   ┌ asm: i=m
4   └ ∴ Wm    {from 1 and 3}
5   ∴ ~i=m    {from 3; 2 contradicts 4}
```

Line 4 follows by substituting equals; if i and m are identical, then whatever is true of one is true of the other.

Here's a simple invalid argument and its refutation:

The bankrobber wears size-twelve shoes.
You wear size-twelve shoes.
∴ You're the bankrobber.

```
1      Wb            Invalid   b, u
2      Wu
  [ ∴ u=b            Wb, Wu, ~u=b
3      asm: ~u=b
```

Since we can't infer anything (we can't do much with "~u=b"), we set up a possible world to refute the argument. This world contains two distinct persons, the bankrobber and you, each wearing size-twelve shoes. Since the premises are all true and conclusion false in this world, our argument is invalid.

Our next example involves *pluralism* and *monism:*

Pluralism	*Monism*
There's more than one being.	There's exactly one being.
(∃x)(∃y)~x=y	(∃x)(y)y=x
For some x and some y: x≠y.	For some x, every y is identical to x.

Here's a proof that pluralism entails the falsity of monism:

There's more than one being.
∴ It's false that there's exactly one being.

```
* 1     (∃x)(∃y)~x=y   Valid
     [ ∴ ~(∃x)(y)y=x
* 2   ┌ asm: (∃x)(y)y=x
* 3   │ ∴ (∃y)~a=y   {from 1}
  4   │ ∴ ~a=b   {from 3}
  5   │ ∴ (y)y=c   {from 2}
  6   │ ∴ a=c   {from 5}
  7   │ ∴ b=c   {from 5}
  8   └ ∴ a=b   {from 6 and 7}
  9   ∴ ~(∃x)(y)y=x   {from 2; 4 contradicts 8}
```

Lines 1 and 2 have back-to-back quantifiers. We can drop only quantifiers that are initial and hence outermost; so we have to drop the quantifiers one at a time, starting from the outside. After dropping quantifiers, we substitute equals to get line 8. Our "b=c" premise lets us take "a=c" and substitute "b" for the "c," thus getting "a=b."

We didn't bother to derive "c=c" from "(y)y=c" in line 5. From now on, it'll often be too tedious to drop universal quantifiers using *every* old constant. So we'll just derive instances likely to be useful for our proof or refutation.

Our substitute-equals rule seems to hold universally in arguments about matter or mathematics. But the rule can fail with mental phenomena. Consider this argument (where "Bx" stands for "Jones believes that x is on the penny"):

Jones believes that Lincoln is on the penny.	Bl
Lincoln is the first Republican US president.	l=r
∴ Jones believes that the first Republican US president is on the penny.	∴ Br

If Jones is unaware that Lincoln was the first Republican president, the premises could be true while the conclusion is false. So the argument is invalid. But yet we can derive the conclusion from the premises using our substitute-equals rule. So something is wrong here.

To avoid the problem, we'll disallow translating into quantificational logic any predicates or relations that violate the substitute-equals rule. So we won't let "Bx" stand for "Jones believes that x is on the penny." Statements about beliefs and other mental phenomena often violate this rule; so we have to be careful translating such statements into quantificational logic.[1]

So the mental seems to follow different logical patterns from the physical. Does this refute the materialist project of reducing the mental to the physical? Philosophers dispute this question.

6.2a Exercise—LogiCola IDC

Say whether each is valid (and give a proof) or invalid (and give a refutation).

a=b
∴ b=a

```
1     a=b    Valid
  [ ∴ b=a
2  ┌ asm: ~b=a
3  │ ∴ ~b=b    {from 1 and 2}
4  └ ∴ b=b    {self-identity, to contradict 3}
5  ∴ b=a    {from 2; 3 contradicts 4}
```

1. Fa
∴ ~(∃x)(Fx • ~x=a)

2. (a=b ⊃ ~(∃x)Fx)
∴ (Fa ⊃ ~Fb)

3. a=b
b=c
∴ a=c

4. ~a=b
c=b
∴ ~a=c

5. ~a=b
~c=b
∴ a=c

6. a=b
(x)(Fx ⊃ Gx)
~Ga
∴ ~Fb

7. a=b
∴ (Fa ≡ Fb)

8. Fa
∴ (x)(x=a ⊃ Fx)

9. ∴ (∃x)(y)y=x

10. ∴ (∃x)(∃y)~y=x

[1] Chapter 10 will develop special ways to symbolize belief formulas and will explicitly restrict the use of the substitute-equals rule with such formulas (see Section 10.2).

6.2b Exercise—LogiCola IDC

First appraise intuitively. Then translate into logic and say whether valid (and give a proof) or invalid (and give a refutation). You'll have to figure out what letters to use; be careful about deciding between small and capital letters.

1. Keith is my only nephew.
My only nephew knows more about BASIC than I do.
Keith is a ten-year-old.
∴ Some ten-year-olds know more about BASIC than I do. [I wrote this argument many years ago; now Keith is older and I have two nephews.]

2. Some are logicians.
Some aren't logicians.
∴ There's more than one being.

3. This chemical process is publicly observable.
This pain isn't publicly observable.
∴ This pain isn't identical to this chemical process. [This attacks the identity theory of the mind, which identifies mental events with chemical processes.]

4. The person who left a lighter is the murderer.
The person who left a lighter is a smoker.
No smokers are backpackers.
∴ The murderer isn't a backpacker.

5. The murderer isn't a backpacker.
You aren't a backpacker.
∴ You're the murderer.

6. If Speedy Jones looks back to the quarterback just before the hike, then Speedy Jones is the primary receiver.
The primary receiver is the receiver you should try to cover.
∴ If Speedy Jones looks back to the quarterback just before the hike, then Speedy Jones is the receiver you should try to cover.

7. Judy isn't the world's best cook.
The world's best cook lives in Detroit.
∴ Judy doesn't live in Detroit.

8. Patricia lives in North Dakota.
Blondie lives in North Dakota.
∴ At least two people live in North Dakota.

9. Your grade is the average of your tests.
The average of your tests is B.
∴ Your grade is B.

10. Either you knew where the money was, or the thief knew where it was.
You didn't know where the money was.

∴ You aren't the thief.

11. The man of Suzy's dreams is either rich or handsome.
    You aren't rich.
    ∴ If you're handsome, then you're the man of Suzy's dreams.

12. If someone confesses, then someone goes to jail.
    I confess.
    I don't go to jail.
    ∴ Someone besides me goes to jail.

13. David stole money.
    The nastiest person at the party stole money.
    David isn't the nastiest person at the party.
    ∴ At least two people stole money. [See problem 4 of Section 2.3b.]

14. No one besides Carol and the detective had a key.
    Someone who had a key stole money.
    ∴ Either Carol or the detective stole money.

15. Exactly one person lives in North Dakota.
    Paul lives in North Dakota.
    Paul is a farmer.
    ∴ Everyone who lives in North Dakota is a farmer.

16. The wildcard team with the best record goes to the playoffs.
    Cleveland isn't the wildcard team with the best record.
    ∴ Cleveland doesn't go to the playoffs.

6.3 Relational translations

Our last rule for forming quantificational wffs introduces relations:

> 4. The result of writing a capital letter and then two or more small letters is a wff.

Lrj	=	Romeo loves Juliet.
Gxyz	=	x gave y to z.

Translating relational sentences into logic can be difficult, since there are few rules to help us. We mostly have to study examples and catch the patterns.

Here are further examples without quantifiers:

Juliet loves Romeo	=	Ljr
Juliet loves herself	=	Juliet loves Juliet.
	=	Ljj

Juliet loves Romeo and Antonio	=	Juliet loves Romeo and Juliet loves Antonio.
	=	(Ljr-Lja)

These use single quantifiers:

Everyone loves Juliet	=	For all x, x loves Juliet.
	=	(x)Lxj
Someone loves Juliet	=	For some x, x loves Juliet.
=Juliet is loved	=	(∃x)Lxj
Everyone is loved by Juliet	=	For all x, Juliet loves x.
	=	(x)Ljx

These are similar, but more difficult:

All Italians love Juliet	=	For all x, if x is Italian then x loves Juliet.
	=	(x)(Ix⊃Lxj)
Some Italians love Juliet	=	For some x, x is Italian and x loves Juliet.
	=	(∃x)(Ix · Lxj)

If the English has a quantifier just after "loves," put the quantifier first:

"Juliet loves everyone (some-one, no one)"	means	"For all (some, no) x, Juliet loves x."

Juliet loves everyone	=	For all x, Juliet loves x.
	=	(x)Ljx
Juliet loves someone	=	For some x, Juliet loves x.
=Juliet is a lover	=	(∃x)Ljx
Juliet loves no one	=	It is not the case that, for some x, Juliet loves x.
	=	~(∃x)Ljx

These are similar, but more difficult:

Juliet loves every Italian	=	For all x, if x is Italian then Juliet loves x.
	=	(x)(Ix⊃Ljx)
Juliet loves some Italian	=	For some x, x is Italian and Juliet loves x.
	=	(∃x)(Ix · Ljx)
Juliet loves no Italian	=	It is not the case that, for some x, x is Italian and Juliet loves x.
	=	~(∃x)(Ix · Ljx)

Here are sentences with two quantifiers:

Everyone loves everyone	=	For all x and for all y, x loves y.
	=	(x)(y)Lxy
Someone loves someone	=	For some x and for some y, x loves y.
	=	(∃x)(∃y)Lxy

In the second case, the “someone” who loves may or may not be the same “someone” who is loved. Compare these two:

Some love themselves	=	For some x, x loves x.
	=	(∃x)Lxx
Some love others	=	For some x and some y, x≠y and x loves y.
	=	(∃x)(∃y)(~x=y • Lxy)

Study carefully this next pair—which differs only in the order of the quantifiers:

Everyone loves someone.
=Everyone loves at least one person.
= (x)(∃y)Lxy
=For all x there’s some y, such that x loves y.

There’s someone that everyone loves.
=There’s some one specific person that everyone loves.
= (∃y)(x)Lxy
=There’s some y such that, for all x, x loves y.

In the first case, we might love *different* people. In the second, we love the *same* person; perhaps we all love God. These pairs emphasize the difference:

Everyone loves someone	≠	There’s someone that everyone loves.
Everyone lives in some house	≠	There’s some house where everyone lives.
Everyone makes some error	≠	There’s some error that everyone makes.

The sentences on the right make the stronger claim.

With back-to-back quantifiers, the order doesn’t matter if both quantifiers are of the same type; but the order matters if the quantifiers are mixed:

(x)(y) = (y)(x)   (∃x)(∃y) = (∃y)(∃x)   (x)(∃y) ≠ (∃y)(x)

Also, it doesn’t matter what variable letters we use, so long as the reference pattern is the same. These three are equivalent:

(x)(∃y)Lyx   (y)(∃x)Lxy   (z)(∃x)Lxz

Each has a universal, then an existential, then "L," then the variable used in the existential, and finally the variable used in the universal.

Many relations have special properties, such as reflexivity or symmetry. Here are examples:

"Is identical to" is *reflexive.* (Identity is a relation but uses a special symbol.)
= Everything is identical to itself.
= (x)x=x

"Taller than" is *irreflexive.*
= Nothing is taller than itself.
= (x)~Txx

"Being a relative of" is *symmetrical.*
= In all cases, if x is a relative of y, then y is a relative of x.
= (x)(y)(Rxy⊃Ryx)

"Being a parent of" is *asymmetrical.*
= In all cases, if x is a parent of y then y isn't a parent of x.
= (x)(y)(Pxy⊃~Pyx)

"Being taller than" is *transitive.*
= In all cases, if x is taller than y and y is taller than z, then x is taller than z.
= (x)(y)(z)((Txy·Tyz)⊃Txz)

"Being a foot taller than" is *intransitive.*
= In all cases, if x is a foot taller than y and y is a foot taller than z, then x isn't a foot taller than z.
= (x)(y)(z)((Txy·Tyz)⊃~Txz)

Love fits none of these six categories. Love is neither reflexive nor irreflexive: sometimes people love themselves and sometimes they don't. Love is neither symmetrical nor asymmetrical: if x loves y, then sometimes y loves x in return and sometimes not. Love is neither transitive nor intransitive: if x loves y and y loves z, then sometimes x loves z and sometimes not.

These examples are difficult:

Every Italian loves someone.
= For all x, if x is Italian then there's some y such that x loves y.
= (x)(Ix ⊃ (∃y)Lxy)

Everyone loves some Italian.
= For all x there's some y such that y is Italian and x loves y.
= (x)(∃y)(Iy · Lxy)

Everyone loves a lover.
= For all x, if x loves someone then everyone loves x.
= (x)((∃y)Lxy ⊃ (z)Lzx)

Juliet loves everyone besides herself.
= For all x, if x≠Juliet then Juliet loves x. =
(x)(~x=j⊃Ljx)

Romeo loves all and only those who don't love themselves.
For all x, Romeo loves x if and only if x doesn't love x.
(x)(Lrx≡~Lxx)

Study these and the other examples carefully, focusing on how to paraphrase the English sentences.

I have no tidy rules for translating relational sentences into formulas. But you might find these steps helpful if you get confused:

- Put a different variable letter after each quantifier word in English, and rephrase according to how logic expresses things.
- Replace each quantifier word in English by a symbolic quantifier, and keep rephrasing, until the whole sentence is translated.

Here are three examples:

Every Italian loves some Italian.
=Every x who is Italian loves some y who is Italian.
= (x)(x is Italian ⊃ (∃y)(y is Italian • x loves y))
= (x)(Ix ⊃ (∃y)(Iy • Lxy))

There's an unloved lover.
=For some x, no one loves x and x loves someone.
= (∃x)(no y loves x • x loves some z)
= (∃x)(~(∃y)Lyx • (∃z)Lxz)

Some Italian besides Romeo loves Juliet.
=Some x who is Italian and not Romeo loves Juliet.
= (∃x)(x is Italian • x isn't Romeo • x loves Juliet)
= (∃x)((Ix • ~x=r) • Lxj)

Paraphrase the English bit by bit, following the quantificational idiom.

6.3a Exercise—LogiCola H (RM & RT)

Using these equivalences, translate these English sentences into wffs.

Cxy	=	x caused y	Ex	=	x is evil	g	=	God
Gxy	=	x is greater than y	a	=	Aristotle	w	=	the world

Aristotle caused nothing that is evil. ~(∃x)(Ex • Cax)

1. God caused the world.
2. The world caused God.
3. Nothing is greater than itself.
4. Aristotle is greater than anything else.
5. Aristotle is greater than some evil beings.
6. It is always true that if a first thing is evil and a second thing isn't evil then the first is greater than the second.
7. If God caused the world, then God is greater than the world.
8. It is not always true that if a first thing caused a second then the first is greater than the second.
9. Every entity is greater than some entity.
10. There's something than which nothing is greater.
11. God had no cause.
12. If God had no cause, then the world had no cause.
13. Nothing caused itself.
14. In all cases, if a first thing is greater than a second, then the second isn't greater than the first.
15. Everything is caused by something.
16. There's something that caused everything.
17. Something evil caused all evil things.
18. God caused everything besides himself.
19. In all cases, if a first thing caused a second and the second caused a third, then the first caused the third.
20. There's a first cause (there's some x that caused something but nothing caused x).

6.4 Relational proofs

In relational proofs, as before, we'll reverse squiggles, drop existentials (using new constants), and lastly drop universals. But now back-to-back quantifiers will be common (as in line 3 of this next proof); we'll drop such quantifiers one at a time, starting at the outside, since we can drop only an *initial* quantifier:

Romeo loves Juliet.
Juliet doesn't love Romeo.
∴ It's not always true that if a first person loves a second then the second loves the first.

```
    1      Lrj    Valid
    2      ~Ljr
         [ ∴ ~(x)(y)(Lxy ⊃ Lyx)
    3    ┌ asm: (x)(y)(Lxy ⊃ Lyx)
    4    │ ∴ (y)(Lry ⊃ Lyr)    {from 3}
*   5    │ ∴ (Lrj ⊃ Ljr)    {from 4}
    6    └ ∴ Ljr    {from 1 and 5}
    7    ∴ ~(x)(y)(Lxy ⊃ Lyx)    {from
               3; 2 contradicts 6}
```

Our older proof strategy would have us drop each initial universal quantifier twice, once using “r” and once using “j.” But now this would be tedious; so we instead derive only what is likely to be useful for our proof or refutation.

Here’s another relational proof:

There’s someone that everyone loves.
∴ Everyone loves someone.

```
*  1     (∃y)(x)Lxy    Valid
       [ ∴ (x)(∃y)Lxy
*  2   ┌ asm: ~(x)(∃y)Lxy
*  3   │ ∴ (∃x)~(∃y)Lxy   {from 2}
*  4   │ ∴ ~(∃y)Lay   {from 3}
   5   │ ∴ (y)~Lay   {from 4}
   6   │ ∴ (x)Lxb   {from 1}
   7   │ ∴ Lab   {from 6}
   8   └ ∴ ~Lab   {from 5}
   9   ∴ (x)(∃y)Lxy   {from 2; 7 contradicts 8}
```

This should be valid intuitively—since if there’s one specific person (God, for example) that everyone loves, then everyone loves at least one person.

Relational proofs raise interesting problems. With quantificational arguments that lack relations and identity:

1. there are mechanical strategies (like that of Section 5.2) that in every case will give a proof or refutation in a finite number of steps; and
2. a refutation never requires an infinite number of entities—and at most requires 2^n entities (where *n* is the number of distinct predicates).

Neither feature holds for relational arguments. Against 1, there’s no possible mechanical strategy that always will give us a proof or refutation of a relational argument. This result is called **Church’s theorem,** after Alonzo Church. As a result, working out relational arguments sometimes requires ingenuity and not just mechanical methods; the defect with our proof strategy, we’ll see, is that it can lead into endless loops. Against 2, refuting invalid relational arguments sometimes requires a possible world with an infinite number of entities.

Instructions lead into an endless loop if they command the same sequence of actions over and over, endlessly. I’ve written computer programs with endless loops by mistake. I put an endless loop into the glossary on purpose:

Endless loop See **loop, endless**.	**Loop, endless** See **endless loop**.

Our quantificational proof strategy can lead into such a loop. If you see this coming, quit the strategy and improvise your own refutation.

Trying to prove (“Not everything is identical to something”) leads into an endless loop:

Asm: Everything is identical to something.		[∴ ~(x)(∃y)x=y **Invalid**
∴ a is identical to something.	1	asm: (x)(∃y)x=y
∴ a is identical to b.	2	∴ (∃y)a=y {from 1}
∴ b is identical to something.	3	∴ a=b {from 2}
∴ b is identical to c.	4	∴ (∃y)b=y {from 1}
∴ c is identical to something….	5	∴ b=c {from 4}
	6	∴ (∃y)c=y {from 1} …

We drop the universal quantifier in 1, using a new constant "a" (since there are no old constants) to get 2; a step later, we get new constant "b." We drop the universal in 1 using "b" to get 4; a step later, we get new constant "c." And so on endlessly. To refute the argument, we can use a world with a single entity, a, that is identical to itself:

a $\boxed{\textbf{a=a}}$

In this world, everything is identical to at least one thing—and hence the conclusion is false. We have to think up this world for ourselves. The strategy doesn't provide it automatically; instead, it leads into an endless loop.

Wffs that begin with a universal/existential quantifier combination, like "(x)(∃y)," often lead into an endless loop. Here's another example:[1]

Everyone loves someone.	(x)(∃y)Lxy **Invalid**
∴ There's someone that everyone loves.	∴ (∃y)(x)Lxy

Here the premise by itself leads into an endless loop:

Everyone loves someone.	(x)(∃y)Lxy
∴ a loves someone.	∴ (∃y)Lay
∴ a loves b.	∴ Lab
∴ b loves someone.	∴ (∃y)Lby
∴ b loves c.	∴ Lbc
∴ c loves someone.…	∴ (∃y)Lcy …

[1] This example is like arguing "Everyone lives in some house, so there must be some (one) house that everyone lives in." Some great minds have committed this fallacy. Aristotle argued "Every agent acts for an end, so there must be some (one) end for which every agent acts." St Thomas Aquinas argued "If everything at some time fails to exist, then there must be some (one) time at which everything fails to exist." And John Locke argued "Everything is caused by something, so there must be some (one) thing that caused everything."

Again we must improvise, since our strategy doesn't automatically give us a proof or refutation. With some ingenuity, we can construct this possible world, with beings a and b, that makes the premise true and conclusion false:

a, b $\boxed{\begin{array}{l} Laa, \sim Lab \\ Lbb, \sim Lba \end{array}}$ (egoistic world)

Here all love themselves, and only themselves. This makes "Everyone loves someone" true but "There's someone that everyone loves" false. Here's another refutation:

a, b $\boxed{\begin{array}{l} Lab, \sim Laa \\ Lba, \sim Lbb \end{array}}$ (altruistic world)

Here all love the other but not themselves; this again makes the premise true and conclusion false.

We don't automatically get a refutation with invalid arguments that lead into an endless loop. Instead, we have to think out the refutation by ourselves. While there's no strategy that always works, I'd suggest that you:

1. try breaking out of the loop before introducing your third constant (often it suffices to use two beings, a and b; don't multiply entities unnecessarily),
2. begin your refutation with values you already have (maybe, for example, you already have "Lab" and "Laa"), and
3. experiment with adding other wffs to make the premises true and conclusion false (if you already have "Lab" and "Laa," then try adding "Lba" or "~Lba"—and "Lbb" or "~Lbb"—until you get a refutation).

We have to fiddle with the values until we find a refutation that works.

Refuting a relational argument sometimes requires a universe with an infinite number of entities. Here's an example:

In all cases, if x is greater than y and y is greater than z then x is greater than z.	$(x)(y)(z)((Gxy \cdot Gyz) \supset Gxz)$
In all cases, if x is greater than y then y isn't greater than x.	$(x)(y)(Gxy \supset \sim Gyx)$
b is greater than a.	Gba
∴ There's something than which nothing is greater.	$\therefore (\exists x)\sim(\exists y)Gyx$

Given these premises, every world with a finite number of beings must have some being unsurpassed in greatness (making the conclusion true). But we can imagine a world with an infinity of beings—in which each being is surpassed in greatness by another. So the argument is invalid.

We can refute the argument by giving another of the same form with true premises and a false conclusion. Let's take the natural numbers (0, 1, 2,…) as the universe of discourse.

Let “a” refer to 0 and “b” refer to 1 and “Gxy” mean “x>y.” On this interpretation, the premises are all true. But the conclusion, which says “There’s a number than which no number is greater,” is false. This shows that the form is invalid.

So relational arguments raise problems about infinity (endless loops and infinite worlds) that other kinds of arguments we’ve studied don’t raise.

6.4a Exercise—LogiCola I (RC & BC)

Say whether each is valid (and give a proof) or invalid (and give a refutation).

(∃x)(∃y)Lxy
∴ (∃y)(∃x)Lxy

*	1	(∃x)(∃y)Lxy	**Valid**
		[∴ (∃y)(∃x)Lxy	
*	2	┌ asm: ~(∃y)(∃x)Lxy	
	3	│ ∴ (y)~(∃x)Lxy	{from 2}
*	4	│ ∴ (∃y)Lay	{from 1}
	5	│ ∴ Lab	{from 4}
*	6	│ ∴ ~(∃x)Lxb	{from 3}
	7	│ ∴ (x)~Lxb	{from 6}
	8	└ ∴ ~Lab	{from 7}
	9	∴ (∃y)(∃x)Lxy	{from 2; 5 contradicts 8}

1. (x)Lxa
   ∴ (x)Lax

2. (∃x)(y)Lxy
   ∴ (∃x)Lxa

3. (x)(y)(Lxy ⊃ x=y)
   ∴ (x)Lxx

4. (x)(∃y)Lxy
   ∴ Laa

5. (x)(y)Lxy
   ∴ (x)(y)((Fx • Gy) ⊃ Lxy)

6. (x)(y)(Uxy ⊃ Lxy)
   (x)(∃y)Uxy
   ∴ (x)(∃y)Lxy

7. (x)Lxx
   ∴ (∃x)(y)Lxy

8. (x)Gaxb
   ∴ (∃x)(∃y)Gxcy

9. (x)(y)Lxy
   ∴ (∃x)Lax

10. Lab
    Lbc
    ∴ (∃x)(Lax • Lxc)

11. (x)Lxx
    ∴ (x)(y)(Lxy ⊃ x=y)

12. (∃x)Lxa
    ~Laa
    ∴ (∃x)(~a=x • Lxa)

13. (x)(y)(z)((Lxy • Lyz) ⊃ Lxz)
    (x)(y)(Kxy ⊃ Lyx)
    ∴ (x)Lxx

14. (x)Lxa
    (x)(Lax ⊃ x=b)
    ∴ (x)Lxb

15. (x)(y)(Lxy ⊃ (Fx • ~Fy))
    ∴ (x)(y)(Lxy ⊃ ~Lyx)

6.4b Exercise—LogiCola I (RC & BC)

First appraise intuitively. Then translate into logic and say whether valid (and give a proof) or invalid (and give a refutation).

1. Juliet loves everyone.
   ∴ Someone loves you. [Use Lxy, j, and u.]

2. Nothing caused itself.
 ∴ There's nothing that caused everything. [Use Cxy.]

3. Alice is older than Betty.
 ∴ Betty isn't older than Alice. [Use Oxy, a, and b. What premise implicit would make this valid?]

4. There's something that everything depends on.
 ∴ Everything depends on something. [Use Dxy.]

5. Everything depends on something.
 ∴ There's something that everything depends on. [Use Dxy.]

6. Romeo loves all females.
 No females love Romeo.
 Juliet is female.
 ∴ Romeo loves someone who doesn't love him. [Use Lxy, r, Fx, and j.]

7. In all cases, if a first thing caused a second then the first exists before the second.
 Nothing exists before it exists.
 ∴ Nothing caused itself. [Use Cxy and Bxy (for "x exists before y exists").]

8. Everyone hates my enemy.
 My enemy doesn't hate anyone besides me.
 ∴ My enemy is me. [Use Hxy, e, and m.]

9. Not everyone loves everyone.
 ∴ Not everyone loves you. [Use Lxy and u.]

10. There's someone that everyone loves.
 ∴ Some love themselves. [Use Lxy.]

11. Andy shaves all and only those who don't shave themselves.
 ∴ It is raining. [Use Sxy, a, and R.]

12. No one hates themselves.
 I hate all logicians.
 ∴ I am not a logician. [Use Hxy, i, and Lx.]

13. Juliet loves everyone besides herself.
 Juliet is Italian.
 Romeo is my logic teacher.
 My logic teacher isn't Italian.
 ∴ Juliet loves Romeo. [Use j, Lxy, Ix, r, and m.]

14. Romeo loves either Lisa or Colleen.
 Romeo doesn't love anyone who isn't Italian.
 Colleen isn't Italian.
 ∴ Romeo loves Lisa. [Use Lxy, r, 1, and c.]

15. Everyone loves all lovers.
 Romeo loves Juliet.
 ∴ I love you. [Use Lxy, r, j, i, and u. This one is difficult.]

16. Everyone loves someone.
∴ Some love themselves. [Use Lxy.]

17. Nothing caused itself.
This chemical brain process caused this pain.
∴ This chemical brain process isn't identical to this pain. [Use Cxy, b, and p.]

18. For every positive contingent truth, something explains why it's true.
The existence of the world is a positive contingent truth.
If something explains the existence of the world, then some necessary being explains the existence of the world.
∴ Some necessary being explains the existence of the world. [Cx, Exy, e, and NX. This argument for the existence of God is from Richard Taylor.]

19. That girl is Miss Novak.
∴ If you don't like Miss Novak, then you don't like that girl. [Use t, m, u, and Lxy. This is from the movie, *The Little Shop around the Corner:* "If you don't like Miss Novak, I can tell you right now that you won't like that girl. Why? Because it is Miss Novak."]

20. Everyone who is wholly good prevents every evil that he can prevent.
Everyone who is omnipotent can prevent every evil.
If someone prevents every evil, then there's no evil.
There's evil.
∴ Either God isn't omnipotent, or God isn't wholly good. [Use Gx, Ex, Cxy (for "x can prevent y"), Pxy (for "x prevents y"), Ox, and g. This argument is from J.L.Mackie.]

21. Your friend is wholly good.
Your knee pain is evil.
Your friend can prevent your knee pain.
Your friend doesn't prevent your knee pain (since he could prevent it only by amputating your leg—which would bring about a worse situation).
∴ "Everyone who is wholly good prevents every evil that he can prevent" is false. [Use f, Gx, k, Ex, Cxy, and Pxy. Alvin Plantinga thus attacked premise 1 of the previous argument; he proposed instead roughly this: "Everyone who is wholly good prevents every evil that he knows about if he can do so without thereby eliminating a greater good or bringing about a greater evil."]

22. For everything contingent, there's some time at which it fails to exist.
∴ If everything is contingent, then there's some time at which everything fails to exist. [Use Cx for "x is contingent"; Ext for "x exists at time t"; t for a time variable; and t′, t″, t‴,…for time constants. This is a critical step in St Thomas Aquinas's third argument for the existence of God.]

23. If everything is contingent, then there's some time at which everything fails to exist.

If there's some time at which everything fails to exist, then there's nothing in existence now.
There's something in existence now.
Everything that isn't contingent is necessary.
∴ There's a necessary being. [Besides the letters for the previous argument, use Nx for "x is necessary" and n for "now." This continues St Thomas's argument; here premise 1 is from the previous argument.]

24. [The great logician Gottlob Frege tried to systematize mathematics. One of his axioms said that *every sentence with a free variable*[1] *determines a set*. So then "x is blue" determines a set: there's a set y containing all and only blue things. While this seems sensible, Bertrand Russell showed that Frege's axiom entails that "x doesn't contain x" determines a set—so there's a set y containing all and only those things that don't contain themselves—and this leads to the self-contradiction "y contains y if and only if y doesn't contain y." The foundations of mathematics haven't been the same since "Russell's paradox."]
If every sentence with a free variable determines a set, then there's a set y such that, for all x, y contains x if and only if x doesn't contain x.
∴ Not every sentence with a free variable determines a set. [Use D for "Every sentence with a free variable determines a set," Sx for "x is a set," and Cyx for "y contains x."]

25. All dogs are animals.
∴ All heads of dogs are heads of animals. [Use Dx, Ax, and Hxy (for "x is a head of y"). Translate "x is a head of a dog" as "for some y, y is a dog and x is a head of y." Augustus DeMorgan in the nineteenth century claimed that this was a valid argument that traditional logic couldn't validate.]

6.5 Definite descriptions

Phrases of the form "the so and so" are called **definite descriptions,** since they're meant to pick out a definite (single) person or thing. This final section sketches Bertrand Russell's influential ideas on definite descriptions. While philosophical discussions about these and about proper names can get complex and controversial, I'll try to keep things fairly simple.

Consider these two sentences and how we've been symbolizing them:

Socrates is bald.	The king of France is bald.
Bs	Bk

The first sentence has a proper name ("Socrates") while the second has a definite description ("the king of France"); both seem to ascribe a *property* (baldness) to a particular *object* or entity. Russell argued that this object-property analysis is misleading in the second

[1] An instance of a variable is "free" in a wff if it doesn't occur as part of a wff that begins with a quantifier using that variable; each instance of "x" is free in "Fx" but not in "(x)Fx."

case;[1] sentences with definite descriptions (like "the king of France") were in reality more complicated and should be analyzed in terms of a complex of predicates and quantifiers: The king of France is bald.

=There's exactly one king of France, and he is bald.
=For some x:x is king of France, there's no y such that y≠x and y is king of France, and x is bald.
$= (\exists x)((Kx \cdot \sim(\exists y)(\sim y{=}x \cdot Ky)) \cdot Bx)$

Russell saw his analysis as having several advantages; I'll mention two.

First, "The king of France is bald" might be false for any of three reasons:

1. There's no king of France;
2. there's more than one king of France; or
3. there's exactly one king of France, and he has hair on his head.

In fact, "The king of France is bald" is false for reason 1: France is a republic and has no king. This accords well with Russell's analysis. By contrast, the object-property analysis suggests that if "The king of France is bald" is false, then "The king of France isn't bald" would have to be true[2]—and so the king of France would have to have hair! So Russell's analysis seems to express better the logical complexity of definite descriptions.

Second, the object-property analysis of definite descriptions can lead us into metaphysical errors, like positing existing things that aren't real. The philosopher Alexius Meinong argued roughly as follows:

"The round square does not exist" is a true statement about the round square.
If there's a true statement about something, then that something has to exist.
∴ The round square exists.
But the round square isn't a real thing. ∴
Some things that exist aren't real things.

For a time, Russell accepted this argument. Later he came to see the belief in non-real existing things as foolish; he rejected Meinong's first premise and appealed to the theory of descriptions to clear up the confusion.

According to Russell, Meinong's error comes from his naïve object-property understanding of the following statement:

The round square does not exist.

This, Russell contended, isn't a true statement ascribing non-existence to some object called "the round square." If it were a true statement about the round square, then the round square would have to exist—which the statement denies. Instead, the statement just denies that there's exactly one round square. So Russell's analysis keeps us from having to accept that there are existing things that aren't real.

[1] He also thought the analysis misleading in the first case; but I don't want to discuss this now.

[2] On Russell's analysis, "The king of France isn't bald" is false too—since it means "There's exactly one king of France, and he isn't bald."