The Biomial Series.

Recall (in high school):

$(1+x)^2 = x^2+2x+1.$

$(1+x)^3 = x^3+3x^2+3x+1$

$(1+x)^4 = x^4+4x^3+6x^2+4x+1.$

$\vdots$

$(1+x)^n = \sum_{m=0}^{\infty} \binom{n}{m} x^m$ where n is a real number

where $\binom{n}{m} = \dfrac{n!}{m!(n-m)!}$

$= \dfrac{n(n-1)\cdots\cdots(n-m+1)}{m!}$

e.g. $\binom{3}{2} = \dfrac{3!}{2!(3-2)!}$

$= \dfrac{3\times2\times1}{2\times1\times1} = 3.$

$\binom{n}{n-1} = n$ (prove it!).

How about $\binom{1/2}{2}$?

$\binom{1/2}{2} = \dfrac{1/2\,(1/2-1)}{2!} = -\dfrac{1}{8}$

$\binom{1/2}{3} = \dfrac{1/2\,(1/2-1)\,(1/2-2)}{3!} = \dfrac{1/2\cdot(-1/2)\cdot(-3/2)}{6} = \dfrac{1}{16}$

$\binom{n}{n} = 1$

$\dfrac{n!}{n!\,0!}$ $\quad 0! = 1 \quad \Rightarrow \quad = 1.$

$* \ \binom{4}{5} = \dfrac{5\times(5-1)\times(5-2)\times(5-3)\times(5-4)\times(5-5).}{4!} = 0.$

$\Downarrow$ ✓

$\binom{n}{n+a} = 0. \quad \binom{n}{n-1} = n.$

$a>0.$

e.g. 1. Find Maclaurn series of $\sqrt{1+x}$

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}}.$$

$$= \sum_{m=0}^{\infty} \binom{\frac{1}{2}}{m} x^m .$$

where $\binom{\frac{1}{2}}{0} = 1$, $\binom{\frac{1}{2}}{1} = \frac{1}{2}$.

$$\binom{\frac{1}{2}}{n} = (-1)^{n-1} \frac{1\cdot 3\cdot 5\cdots (n}{n!\cdot 2^n}$$

$$\sqrt{1+x} = 1 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1\cdot 3\cdot 5\cdots (2n-3)}{n!\cdot 2^n} x^n .$$

Racial Test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{1\cdot 3\cdot 5\cdots (2n-1)\, x^{n+1}}{(n+1)!\cdot 2^{n+1}} \Big/ \frac{1\cdot 3\cdot 5\cdots (2n-3)\, x^n}{n!\, 2^n} \right|$

$$= \lim_{n\to\infty} \left| \frac{2n-1}{(n+1)\cdot 2} x \right| = \lim_{n\to\infty} \left| \frac{2-\frac{1}{n}}{2+\frac{2}{n}} x \right| .$$

$$= \lim_{n\to\infty} \left|\frac{x}{2}\right| \cdot \frac{2}{1+0} = |x| .$$

$|x| < 1$ $x \in (-1, 1)$.

at the endpoints:

$x = 1$: series : $S_n = 1 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1\cdot 3\cdot 5\cdots (2n-3)}{n!\, 2^n} \cdot (1)^n$

$\sqrt{2}$ converses by AST.

$x = -1$: Series. $S_n = 1 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1\cdot 3\cdot 5\cdots (2n-3)}{n!\, 2^n} \cdot (-1)^n$

$\sqrt{0}$

$$= 1 - \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots (2n-3)}{n!\, 2^n}$$

Ratial test (x)

$$a_n = \frac{1}{2^n} \cdot \frac{(2n-1)!}{n!} \cdot \frac{1}{2n-1} .$$

$$= \frac{1}{2^n} \cdot \frac{1\cdot 3\cdot 5\cdots (2n-1)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots n} .$$

$$= \frac{1}{2^n} \cdot \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots (2n-3)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots n} \cdot \frac{1}{2\cdot 4\cdots (2n-4)}$$

to be continued...

e.g. 2. $(1-x^2)^{-\frac{1}{3}}$.

Recall. $(1+x)^{n} = \sum_{n=0}^{\infty} \binom{k}{n} x^n$

Replace x by $-x^2$, then $= \sum_{n=0}^{\infty} (-1)^n \binom{-1/3}{n} x$

$$\binom{-1/3}{0} = 1$$

$$\binom{-1/3}{1} = -\frac{1}{3}.$$

$$\binom{-1/3}{2} = \frac{-\frac{1}{3}(-\frac{1}{3}-1)}{2!} = \frac{(-\frac{1}{3})(-\frac{4}{3})}{2!} = (-1)^2 \frac{1\times 4.}{2!\cdot 3^2}$$

$$\binom{-1/3}{3} = \frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})}{3!} = (-1)^3 \cdot \frac{1\times 4\times 7}{3!\cdot 3^3}$$

$$\vdots$$

$$\binom{-1/3}{n} = \frac{1\times 4\times 7\times \cdots \times (3n-2)}{n!\cdot 3^n}$$

$$(1-x^2)^{-1/3} = 1+\frac{1}{3}x^2+\sum_{n=2}^{\infty}(-1)^n\frac{1\times 4\times 7\cdots\times(3n-2)}{n!\,3^n}$$

e.g. $(1-x^2)^{-1/2}$.