Math 2155, Fall 2022: Homework 8

Instructions for GradeScope:

You may handwrite your answers or type them. Only type solutions if you are able to type the appropriate symbols. If **typed**, use at least a 12pt font, with **one question per page**. You can then directly submit the pdf file to GradeScope.

If **handwritten**:

- Don't erase or cross out more than a word or two.
- Only write on one side of the page. Use dark, large, neat writing.
- If you have a real scanner, great! Most people will scan their solutions using their phone. Use one of the recommended apps: For iOS, use Scannable by Evernote or Genius Scan. For Android, use Genius Scan. Do *not* just take regular photos.
- When scanning, have good lighting and try to avoid shadows.
- It is best to have **one question per scan**. If the solution is short, **fold the page in half** and scan just the half it is on, so there isn't so much blank space. Or, you can scan and then crop the scan. You don't need to scan parts (a), (b), etc. separately or put them on separate pages.
- It works well to scan each question separately and produce one pdf file with one question per page. Most scanning apps will automatically combine your images into one pdf file.
- You must **check the quality** of your scans afterwards, and rescan if needed.

Make sure you are going to `http://gradescope.ca` not `http://gradescope.com`. You can access it through the course OWL site.

You can **resubmit** your work any number of times until the deadline.

Don't forget to accurately **match questions to pages**. If you do this incorrectly, the grader will not see your solution and will give you zero.

See the GradeScope help website for lots of information: `https://help.gradescope.com/` Select "Student Center" and then either "Scanning Work on a Mobile Device" or "Submitting an Assignment".

Instructions for writing solutions:

- Homework is graded both on **correctness** and on **presentation/style**.
- Your proofs should be written in **complete sentences**, starting with capitals and ending with periods. They should be in **paragraph form**, similar to the proofs in the textbook. For example, you shouldn't put each sentence on a new line. Paragraphs should be used to break longer proofs into logical chunks.
- Don't use informal abbreviations such as three dots for "therefore." Logical symbols can be used when they clarify things to the reader, but English words can often be more effective.
- Include all of the steps that are needed to logically justify every claim you make. Do not include unnecessary steps. Try to be **concise and complete**.
- Do not submit any **rough work**. Do your rough work on scratch paper, and only submit a clearly and neatly written answer. Do not show a table of givens and goals. Do not use Venn diagrams.
- Do not cross out or erase more than a word or two. If you write each final solution on a new page, it's easy to start over on a fresh page when you've made a large error.
- You should do the work **on your own**. Read the course syllabus for the rules about scholastic offences, which include sharing solutions with others, uploading material to a website, viewing material of others or on a website (even if you don't use it), etc. The penalty for cheating on homework will be a grade of **0** on the homework set as well as a penalty of **negative 5%** on the overall course grade.
- We may not grade every question.

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Due **Friday, November 18 at 11:59pm**. You can **resubmit** your work any number of times until the deadline. The deadline is back to being **strict**. Only **some** questions will be graded.

H8Q1 (12 marks): For each of the following relations on the specified set, state whether it is reflexive, transitive, symmetric, and antisymmetric. Then state whether it is a partial order, and if so, whether it is a total order. For all six properties, justify your claims. [Each justification should take just 1 or 2 lines.]

(a) $A_1 = \mathbb{Z}^+$ and $R_1 = \{(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid 1/a \leq 1/b\}$.

(b) $A_2 = \mathbb{Z}$ and $R_2 = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid |a-b| \leq 2\}$.

(c) $A_3 = \{1,2,3\}$ and $R_3 = \emptyset$.

(d) $A_4 = \emptyset$ and $R_4 = \emptyset$.

Solution: (a) R_1 is reflexive: for every $a \in \mathbb{Z}^+$, $1/a \leq 1/a$.

R_1 is transitive: Let $a,b,c \in \mathbb{Z}^+$ and assume $1/a \leq 1/b$ and $1/b \leq 1/c$. Then clearly $1/a \leq 1/c$.

R_1 is not symmetric: Consider $2, 1 \in Z^+$. Then $1/2 \leq 1/1$, but $1/1 \not\leq 1/2$.

R_1 is anti-symmetric: Let $a,b \in \mathbb{Z}^+$ and assume that $1/a \leq 1/b$ and $1/b \leq 1/a$. Then $1/a = 1/b$, so $a = b$.

Since R_1 is reflexive, transitive and anti-symmetric, it is a partial order. It is a total order, since for any $a,b \in \mathbb{Z}^+$, we either have $1/a \leq 1/b$ or $1/b \leq 1/a$.

(b) R_2 is reflexive: for every $a \in \mathbb{Z}$, $|a-a| = 0 \leq 2$.

R_2 is not transitive: Consider $0,2,4 \in \mathbb{Z}$. We have $|0-2| = 2 \leq 2$, so $0 R_2 2$. Similarly, $2 R_2 4$. But $|0-4| = 4 \not\leq 2$, so $0 R_2 4$ is false.

R_1 is symmetric: Let $a,b \in \mathbb{Z}$ and assume that $|a-b| \leq 2$. Then $|b-a| = |a-b| \leq 2$.

R_1 is not anti-symmetric: For example, $0 R_2 2$ and $2 R_2 0$.

Since R_1 is not transitive, it is not a partial order. Since it is not a partial order, it is not a total order.

(c) R_3 is not reflexive: $(1,1)$ is not in R_3.

R_3 is transitive, symmetric, and anti-symmetric, since all of these properties start by assuming that some elements are related, which never happens.

Since R_3 is not reflexive, it is not a partial order. Since it is not a partial order, it is not a total order.

(d) R_4 satisfies all of the conditions. They all start with "for all x in A, ..." and so they hold vacuously when A is empty.

H8Q2 (6 marks): The following are both partial orders on the specified set (you don't need to prove this). For both, state what the R-smallest element of A is, or explain why it doesn't exist. Also, give the set of R-minimal elements of A. Explain all of your work.

(a) $A = \{\text{bath}, \text{cat}, \text{at}, \text{bat}, \text{decathalon}\}$ and for words u and v in A, we have uRv iff u appears in v. (So cat$R$decathalon and bat$R$bath, but bat is not related to decathalon.)

(b) $A = \{B \subseteq \mathbb{N} \mid B \text{ has either one or two elements}\}$ and R is the $\subseteq$ relation. (So $\{5\}, \{6\}$, and $\{5, 6\}$ are in A, but $\emptyset$ and $\{5, 6, 7\}$ are not in A. And $\{5\}$ is related to $\{5\}$ and to $\{5, 6\}$, but not to $\{6\}$.)

Solution: (a) at is an R-smallest element of A, since at appears in all of the words: b$\underline{\text{at}}$h, c$\underline{\text{at}}$, $\underline{\text{at}}$, b$\underline{\text{at}}$, dec$\underline{\text{at}}$halon.

The set of R-minimal elements is $\{\text{at}\}$. at is in the set since there is no other word that is a subword of at. And no other word is in the set, since it would have at as a subword. (This also follows from Theorem 4.4.6 in the text.)

(b) There is no $\subseteq$-smallest element. If B was $\subseteq$-smallest, then it would have to be a subset of both $\{1\}$ and $\{2\}$, but the only subset of both is the empty set, and it does not have one or two elements.

The set of $\subseteq$-minimal elements is $\{B \subseteq \mathbb{N} \mid B \text{ has exactly one element}\}$. These are all minimal, since any subset of a set B with one element is either equal to B or is empty, so is not in A. On the other hand, if a set B has two elements, then we can choose a one element subset C of B and get that $C \subseteq B$ and $C \neq B$, showing that B is not $\subseteq$-minimal.

It also follows from the second paragraph that there is no R-smallest element, using Theorem 4.4.6 in the text.

H8Q3 (6 marks): Let R and S be partial orders on a set A. Consider the following statements:

Statement 1: If $R \subseteq S$ and a is an S-smallest element of A, then a is an R-smallest element of A.

Statement 2: If $R \subseteq S$ and a is an S-minimal element of A, then a is an R-minimal element of A.

One of these is true and one is false. Prove the correct one and give a counterexample for the other one.

Solution: Statement 1 is not correct. For example, let $A = \{1, 2\}$, let $S = \{(1, 1), (1, 2), (2, 2)\}$ and let $R = i_A$. Then R and S are partial orders on A, with $R \subseteq S$. The element 1 is S-smallest, since $1S1$ and $1S2$, but it is not R-smallest, since $1R2$ does not hold.

Proof of Statement 2. Assume that $R \subseteq S$ and that a is an S-minimal element of A. We need to prove that if bRa, then $b = a$. So let $b \in A$ be arbitrary, and assume bRa. Since $R \subseteq S$, we also have bSa. And since a is S-minimal, this implies that $b = a$, as required. So a is R-minimal. $\square$

H8Q4 (6 marks): Let S and T be relations on a set A. Prove the following:

Theorem 1: If A is non-empty and T is reflexive, then $S \setminus T$ is not reflexive.

Theorem 2: If S and T are symmetric, then $S \triangle T$ is symmetric.

Solution:

Proof of Theorem 1. Assume A is non-empty, and choose $a \in A$. Assume that T is reflexive. Then $(a, a) \in T$. Therefore, $(a, a) \notin S \setminus T$. Therefore, $S \setminus T$ is not reflexive. $\square$

Proof of Theorem 2. Assume that S and T are symmetric. Let $(x, y) \in S \triangle T$. Then $(x, y) \in S \setminus T$ or $(x, y) \in T \setminus S$.

Case 1: Assume $(x, y) \in S \setminus T$. So $(x, y) \in S$ and $(x, y) \notin T$. Since $(x, y) \in S$ and S is symmetric, $(y, x) \in S$. Since $(x, y) \notin T$ and T is symmetric, $(y, x) \notin T$ (because if $(y, x) \in T$, then $(x, y) \in T$, by symmetry). Therefore, $(x, y) \in S \triangle T$.

Case 2: This case is exactly the same as the previous case, with S and T swapped.

The two cases are exhaustive, and in either case it follows that $(y, x) \in S \triangle T$. So $S \triangle T$ is symmetric. $\square$