Math 224
Practice Exam 1
Solutions

1. Find a basis for the row space, column space, and null space of the matrix given below:

$$A = \begin{bmatrix} 3 & 4 & 0 & 7 \\ 1 & -5 & 2 & -2 \\ -1 & 4 & 0 & 3 \\ 1 & -1 & 2 & 2 \end{bmatrix}$$

Solution. $rref(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. Thus a basis for the row space of A is $\{[1,0,0,1],[0,1,0,1],[0,0,1,1]\}$. Since the first, second, and third columns of $rref(A)$ contain a pivot, a basis for the column space of A is $\{\begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -5 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}\}$.

If we solve $A\mathbf{x} = \mathbf{0}$, we find that x_4 is a free variable, so we set $x_4 = r$. We obtain $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$, so $\{\begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}\}$ is a basis for the nullspace of A.

2. What is the maximum number of linearly independent vectors that can be found in the nullspace of

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 4 & -1 & 5 & 4 \\ 3 & 6 & -1 & 8 & 5 \\ 4 & 8 & -1 & 12 & 8 \end{bmatrix}$$

Solution. $rref(A)$ has three columns with pivots and two columns without pivots. Thus the dimension of the nullspace of A is 2, so at most 2 linearly independent vectors can be found in the nullspace of A.

3. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation defined by

$$T([x_1, x_2, x_3]) = [2x_1 + 3x_2, x_3, 4x_1 - 2x_2].$$

Find the standard matrix representation of T. Is T invertible? If so, find a formula for T^{-1}.

Solution. The standard matrix representation of T is $A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 0 & 1 \\ 4 & -2 & 0 \end{bmatrix}$. $A^{-1} = \begin{bmatrix} 1/8 & 0 & 3/16 \\ 1/4 & 0 & -1/8 \\ 0 & 1 & 0 \end{bmatrix}$. Since A is invertible, T is invertible. $T^{-1}([x_1, x_2, x_3]) = [\frac{1}{8}x_1 + \frac{3}{16}x_3, \frac{1}{4}x_1 - \frac{1}{8}x_3, x_2]$.

4. Is the set $S = \{[x, y]$ such that $y = x^2\}$ a subspace of $\mathbf{R}^2$?

   Solution. No. $[1, 1]$ and $[2, 4]$ are both in S, but $[1, 1] + [2, 4] = [3, 5]$ is not in S, so S is not closed under addition.

5. Use the Cauchy-Schwarz inequality

$$|\mathbf{v} \cdot \mathbf{w}| \leq ||\mathbf{v}|| \cdot ||\mathbf{w}||$$

   to prove the Triangle Inequality

$$||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||.$$

   (Hint: Begin by computing $||\mathbf{v} + \mathbf{w}||^2$ with dot products, and then plug in the Cauchy-Schwarz inequality when the opportunity arises.)

   Solution.

$$\begin{aligned} ||v + w||^2 &= (v + w) \cdot (v + w) \\ &= v \cdot v + v \cdot w + w \cdot v + w \cdot w \\ &= ||v||^2 + 2(v \cdot w) + ||w||^2 \\ &\leq ||v||^2 + 2||v||||w|| + ||w||^2 \text{ by the C.S. Inequality} \\ &= (||v|| + ||w||)^2 \end{aligned}$$

   Taking the square root of both sides, we conclude that

$$||v + w|| \leq ||v|| + ||w||$$

6. Determine whether each of the following statements is True or False. No explanation is necessary.

   (a) If V is a subspace of $\mathbf{R}^5$ and $V \neq \mathbf{R}^5$, then any set of 5 vectors in V is linearly dependent.

   Solution. True. If $V \neq \mathbf{R}^5$, then the dimension of V is at most 4, so at most 4 vectors in V can be linearly independent.

   (b) If A is a 4×7 matrix and if the dimension of the nullspace of A is 3, then for any $\mathbf{b}$ in $\mathbf{R}^4$, the linear system $A\mathbf{x} = \mathbf{b}$ has at least one solution.

Solution. True. Since A has 7 columns and the nullity of A is 3, the rank equation implies that the rank of A is 4. Thus the dimension of the column space of A is 4, so that the column space of A is a 4-dimensional subspace of $\mathbf{R}^4$, i.e. it is all of $\mathbf{R}^4$. Thus any vector $\mathbf{b}$ in $\mathbf{R}^4$ can be written as a linear combination of the columns of A.

(c) Any 4 linearly independent vectors in $\mathbf{R}^4$ are a basis for $\mathbf{R}^4$.

Solution. True. The dimension of the span of any set of 4 linearly independent vectors is 4, so 4 linearly independent vectors in $\mathbf{R}^4$ are a basis for $\mathbf{R}^4$.

(d) If A is an $m \times n$ matrix, then the set of solutions of a linear system $A\mathbf{x} = \mathbf{b}$ must be a linear subspace of $\mathbf{R}^n$.

Solution. False. It will only be a subspace if $\mathbf{b} = \mathbf{0}$.