Why do we need the convergent of an infinity series.

1) convert the series $\sum_{n=0}^{\infty}(\frac{1}{2})^n$.

$$\sum_{n=0}^{\infty}(\tfrac{1}{2})^n = 1+\tfrac{1}{2}+\cdots+(\tfrac{1}{2})^{n-1}$$
$$= 2-(\tfrac{1}{2})^{n-1}.$$

$n\to\infty$. converges to 2. ← 趋近于某个值.

2) $\sum_{n=0}^{\infty} 2^n = 2^n-1$

$n\to\infty$ diverses.

3) $\sum_{n=0}^{\infty}(-1)^n$.

assume the series converges and its limit is S.

then $S = 1-1+1-1+1\cdots\cdots$

$$= 1-\underbrace{(1-1+1\cdots\cdots)}_{S}$$
$$= \tfrac{1}{2}.$$

$S = (1-1)+(1-1)+\cdots+$

$S=0$.

$S = 1+(-1+1)+(-1+1)+\cdots$

$S=1$.

Because all the series are divergent, so it could end up at any value.

If $a_n \le b_n$ for any value of n, b_n converges

a_n converges

If $a_n \ge b_n$ for any value of n; a_n diverges

b_n diverges.

e.g. 1. $\sum \frac{n^2+1}{n^3+1}$ converges or diverges?

$$a_n = \frac{n^2+1}{n^3+1} \ge \frac{n^2}{n^3+1} \ge \frac{n^2}{n^3+n^3} = \underbrace{\frac{1}{2n}}_{b_n}$$

$b_n = \frac{1}{2}\sum\frac{1}{n}$ ← divergent ← harmonity series

=> the original series diverges. by the comparision test

The limit comparison test: Suppose that $\sum a_n$ and $\sum b_n$ are two series of positive term.

$\lim_{n\to\infty} \frac{a_n}{b_n} = L$, where L is finite number ($\neq 0$).

then either the both series converges. or diverges

e.g. $\sum \frac{n^2+1}{n^3+1}$.

$n \Rightarrow \infty$: $n^2+1 \to n^2$, $n^3+1 \to n^3$ $\Rightarrow a_n \Rightarrow \frac{n^2}{n^3} = \underline{\frac{1}{n}}$. b_n.

b_n is a diverges series because it is a harmonic series.

$$\frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2+1}{n^3+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3(1+\frac{1}{n^2})}{n^3(1+\frac{1}{n^3})} = 1 \Leftarrow \text{a finite number}$$

=> a_n is diverges.

P series. $P=2>1$.

$\sum \frac{n^2-1}{n^4+1}$ $n\to\infty$ $\Rightarrow \frac{1}{n^2}$. converges.

$$\frac{a_n}{b_n} = \frac{\frac{n^2-1}{n^4+1}}{\frac{1}{n^2}} = \frac{n^4(1-\frac{1}{n^2})}{n^4(1+\frac{1}{n^4})} = 1 \quad n\to\infty.$$

$\Downarrow$

converges.

Let $0<m<L<M$

$\frac{a_n}{b_n} = L$ $n\to\infty$, there exist a number N such that $n>N$, we have $m < \frac{a_n}{b_n} < M$.

$\Downarrow$

$$m b_n < a_n < M b_n.$$

if series b_n converges. Mb_n converges => a_n converges diverges mbn diverges diverges.