INDUCTION AND RECURSION

OUTLINE:

(1) Basic induction
(2) Basic recursion
(3) Variations
(4) Structural induction

1. BASIC INDUCTION

Mathematical induction

- The 2 most basic properties which define the set of natural numbers are:
  1) The set has a minimum element *0*
  2) Each element *n* has a successor *n+1*
- Mathematical induction is a proof technique which exploits the construction of the set of natural numbers. In its basic formulation, it says:
- In order to prove that a certain statement holds for any natural number, it is sufficient to
  1) Prove the statement for *0* ("base case");
  2) Prove that, assuming the statement holds for a generic natural number *k* (this assumption is called "inductive hypothesis", IH), then it also holds for *k+1* ("induction step").

Example

- Prove that, for any natural number *n*, the sum of the natural numbers from *0* to *n* is

$$0+1+2+...+n=\frac{n(n+1)}{2}$$

- Base case: for *n* = *0*, the sum of the natural numbers from *0* to *0*, that is just *0*, equals *0(0+1)/2* = *0*.
- Induction step: assume that for an arbitrary *k* we have

$$0+1+2+...+k=\frac{k(k+1)}{2}$$

(this is our inductive hypothesis, IH). We want to show that

$$0+1+...+k+(k+1)=\frac{(k+1)((k+1)+1)}{2}=\frac{(k+1)(k+2)}{2}$$

Example

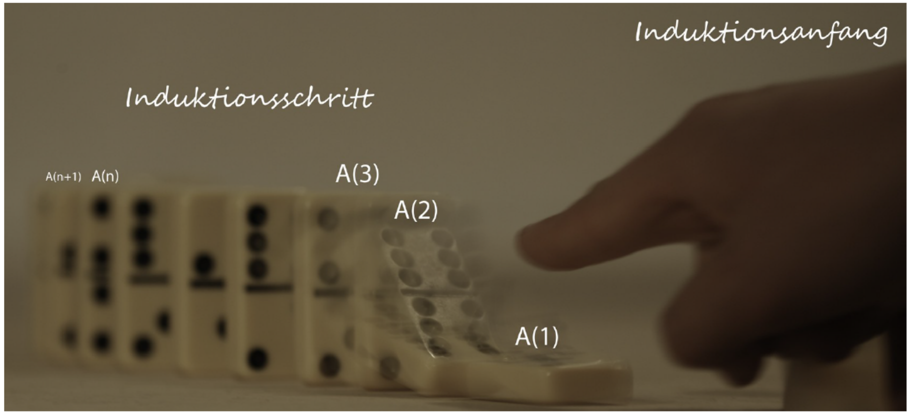
$$0+1+\ldots+k+(k+1) \overset{IH}{=} \frac{k(k+1)}{2}+(k+1) = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

Note that I have actively used IH as a key ingredient to get to the formula I wanted.

Why does this work?

- The intuitive idea behind the induction principle is the same as the idea of domino show: if we can be sure that
  - the first tile falls (base case)
  - provided the k^{th} tile falls (IH), then the $(k+1)^{th}$ tile also falls (inductive step)

  then all tiles fall.



2. BASIC RECURSION

Recursion

- Recursion is mathematical induction’s twin sibling.
- Induction is a proof strategy, exploiting the structure of natural numbers to prove statements.
- Recursion is a definition method, exploiting the structure of natural numbers to define objects (usually functions).
- In its basic form, the recursive definition of an object depending on the natural numbers involves:

  1) Defining the object for the natural *0* (“base case”);
  2) Defining the object for an arbitrary natural number *k+1* in terms of the definition of the object for the natural number *k* (“induction step”).

Example: the factorial.

- The factorial of a natural number n, denoted $n!$, is a natural number defined recursively as follows:
- Base case: $0! = 1$ (direct, explicit definition).
- Inductive step: for a generic natural k, we define $(k+1)! = k! \cdot (k+1)$ (the definition of the factorial of $k+1$ is given in terms of the factorial of the smaller number k).

3. VARIATIONS

Different base cases

- Sometimes, statements only hold from a certain natural number *b* onwards, or a recursive definition makes sense only from a certain natural number *b* onwards. In these situations, induction or recursion cannot be started at *0*, but rather we have to use *b* as our basis step.

Different base cases

- EX: prove that, for any $n \geq 4$, $n! > 2^n$.
- Note that the statement is false for $n = 0,1,2,3$, therefore, we start with the base case $n = 4$.
- Base case: for $n = 4$ we have $4! = 24 > 16 = 2^4$.
- Induction step: assume that, for a generic natural $k \geq 4$, $k! > 2^k$ (IH). Then

$$(k+1)! = k!\cdot(k+1) \overset{IH}{>} 2^k\cdot(k+1) > 2^k\cdot 2 = 2^{k+1}$$

Several base cases

- Sometimes, to inductively prove a statement or to recursively define something on the naturals, we need more than one base case.
- EX: The Fibonacci numbers F_n (a very interesting sequence of numbers with crazily deep properties) are defined recursively as follows:
  - Base case 1: $F_0 = 0$
  - Base case 2: $F_1 = 1$
  - Induction step: $F_{n+2} = F_{n+1} + F_n$

  Note that in this definition the inductive step requires 2 calls of the definition in 2 previous cases, and correspondingly there are 2 base cases to trigger the recursive process.

Strong induction

- Strong induction is a refined form of basic induction in which
  - The basis step works in the same way
  - For the inductive step, we prove that the statement for a generic natural $k+1$ holds if we assume that the statement holds for any natural $\leq k$ (not just for k).

from 0 to k.

An example of strong induction

- Prove the correctness of integer division, i.e., that for all integers $n \geq 0$ (the dividend) and $m > 0$ (the divisor) there are two integers q (the quotient) and r (the remainder), with $0 \leq r < m$, such that $n = mq + r$.
- We proceed by induction on n: to simplify the notation, let $A(n)$ denote the following sentence

  "For all integers $m>0$ there are two integers q and r, with $0 \leq r < m$, such that $n = mq + r$."
- Base case: if $n = 0$, then for all m the assertion is true with $q = r = 0$.

An example of strong induction

- Inductive step: let $k \geq 1$; we have to verify that $A(k)$ follows from the IH that $A(j)$ holds for all integers j between 0 and $k-1$ (included). [Note that, for cleanliness of notation, instead of proving $A(k+1)$ given $A(0),A(1),...,A(k)$, we prove $A(k)$ given $A(0),A(1),...,A(k-1)$.] Let's then pick a positive integer m.
  - Case 1: $m > k$. This is easy, just set $q = 0$ and $r = k$. (here we don't need IH).
  - Case 2: $m \leq k$. Then $k-m$ is a natural number strictly smaller than n (why strictly?), so, by IH, $A(k-m)$ holds, that is, there are integers q and r, with $0 \leq r < m$, such that $k-m = mq+r$. But then $k = m(q+1)+r$, as we required.
- This is an extreme case of induction: to make the inductive step work, we need to assume as IH that all previous instances $A(0)$, $A(1)$, ..., $A(k-1)$ hold; this is because $k-m$ can assume any value between 0 and k.