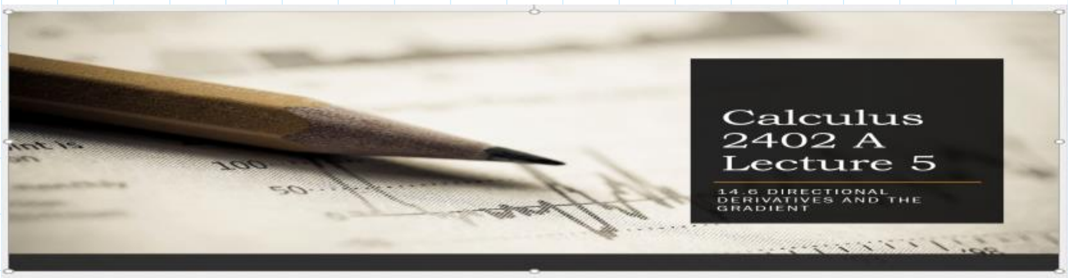
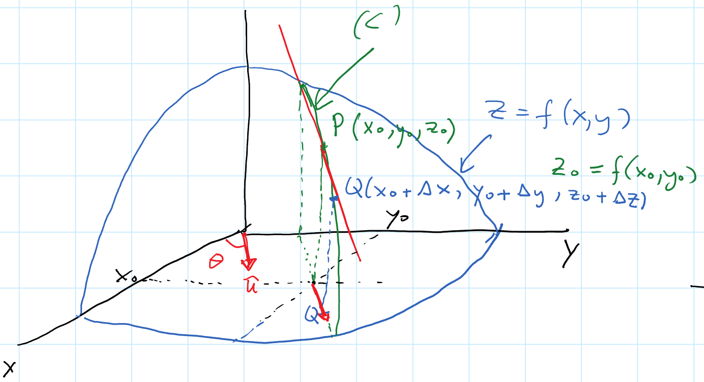
Lecture 5

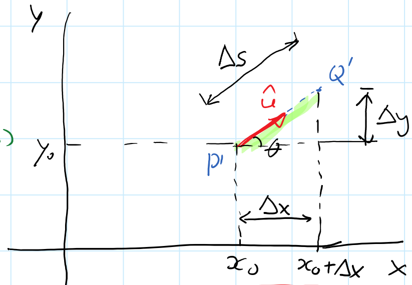
September 18, 2020 10:26 AM



Directional derivatives and the gradient (Sec 14.6)

Imagine that we are in a room which is very cold in the winter. Let's put a fireplace at the centre of the room. Then the temperature $T(x,y)$ is a function of x and y (a location on the floor). If you feel cold you likely move forward to the fireplace to obtain the fastest rate of change in temperature. If you feel quite hot, you likely move away from the fireplace to obtain the fastest decreasing rate in temperature. Thus, the rate of change of temperature with distance depends on the direction you move. This kind of rate of change is called a directional derivative.





$$\boxed{\begin{aligned}\Delta x &= \Delta s \cos\theta \\ \Delta y &= \Delta s \sin\theta\end{aligned}}$$

Suppose that we want to find the rate of change of $z = f(x,y)$ at (x_0, y_0) in the direction of a unit vector $\hat{u} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$. The vertical plane passing thru $P(x_0, y_0, z_0)$ where $z_0 = f(x_0,y_0)$ in the $\hat{u}$ direction cuts the surface $z = f(x,y)$ along a curve (C). The slope of the tangent line to (C) at P is the rate of change of $z = f(x,y)$ in the $\hat{u}$ direction. We have

$\overrightarrow{P'Q'} \parallel \hat{u}$ and if we let $P'Q' = \Delta s$ then

$$\begin{aligned}\overrightarrow{P'Q'} &= \Delta s\,\hat{u} \\ &= \Delta s(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) \\ &= (\Delta s\cos\theta\,\hat{\imath} + \underbrace{\Delta s\sin\theta}_{\Delta y}\hat{\jmath})\end{aligned}$$

$$= (\Delta s \cos\theta \, \hat{\imath} + \underbrace{\Delta s \sin\theta}_{\Delta y} \, \hat{\jmath})$$

Then the directional derivative of $z = f(x,y)$ in the direction of $\hat{u}$ is

$$D_{\hat{u}} f(x_0,y_0) = \lim_{\Delta s \to 0} \frac{\Delta f}{\Delta s}$$

$$= \lim_{\Delta s \to 0} \frac{f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)}{\Delta s}$$

$$= \lim_{\Delta s \to 0} \frac{f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y}{\Delta s}$$

$$= \lim_{\Delta s \to 0} \frac{f_x(x_0,y_0)\Delta s\cos\theta + f_y(x_0,y_0)\Delta s \sin\theta + \varepsilon_1 \Delta s \cos\theta + \varepsilon_2 \Delta s \sin\theta}{\Delta s}$$

$$= \lim_{\Delta s \to 0} \frac{\cancel{\Delta s}\left[f_x(x_0,y_0)\cos\theta + f_y(x_0,y_0)\sin\theta + \underset{\downarrow\, 0}{\varepsilon_1}\cos\theta + \underset{\downarrow\, 0}{\varepsilon_2}\sin\theta\right]}{\cancel{\Delta s}}$$

$$\boxed{D_{\hat{u}} f(x_0,y_0) = f_x(x_0,y_0)\cos\theta + f_y(x_0,y_0)\sin\theta} \qquad (\mathrm{I})$$

The above expression can be written as a dot product

$$D_{\hat{u}} f(x_0,y_0) = \underbrace{\left[f_x(x_0,y_0)\hat{\imath} + f_y(x_0,y_0)\hat{\jmath}\right]}_{\nabla f(x_0,y_0)} \cdot \underbrace{\left[\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}\right]}_{\hat{u}}$$

$$\therefore \quad \boxed{D_{\hat{u}} f(x_0,y_0) = \nabla f(x_0,y_0) \cdot \hat{u}} \qquad (\mathrm{II}) \longleftarrow$$

where $\nabla f(x,y) = f_x(x,y)\hat{\imath} + f_y(x,y)\hat{\jmath}$

is called the gradient of f at the point (x,y).

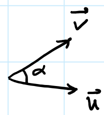
N.B:

(i) If $\hat{u} = \hat{\imath}$ then $\theta = 0 \Rightarrow \cos\theta = 1$ and $\sin\theta = 0$

$$\therefore \quad D_{\hat{\imath}} f(x_0,y_0) = f_x(x_0,y_0)$$

(ii) If $\hat{u} = \hat{\jmath}$ then $\theta = \frac{\pi}{2} \Rightarrow \cos\theta = 0$ and $\sin\theta = 1$

$$\therefore \quad D_{\hat{\jmath}} f(x_0,y_0) = f_y(x_0,y_0)$$



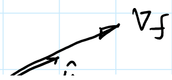
$\vec{u}.\vec{v} = \|\vec{u}\|\,\|\vec{v}\| \cos\alpha$

(II) can be expressed as

$$D_{\hat{u}} f(x_0,y_0) = \|\nabla f(x_0,y_0)\| \underbrace{\|\hat{u}\|}_{1} \cos\alpha$$

$$D_{\hat{u}} f(x_0,y_0) = \|\nabla f(x_0,y_0)\| \cos\alpha \qquad (\mathrm{III})$$

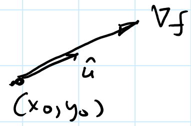
$D_{\hat{u}} f$ is maximum when $\alpha = 0$, ie, $\hat{u}$ has the same direction as ∇f. In this case,



$D_{\hat{u}} f \,|\; = \|\nabla f\|$

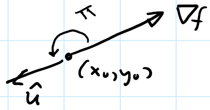
as ∇f. In this case,

$$D_{\hat{u}} f \Big|_{max} = \|\nabla f\|$$



When $\alpha = \pi$, $\cos \pi = -1$, then

$$D_{\hat{u}} f = -\|\nabla f\|$$



In this case, f decreases most rapidly in the direction of $-\nabla f$.

<u>Ex1</u>: Given $f(x,y) = \dfrac{x}{y}$, find the gradient of f at $P(2,1)$ and find the rate of change of f at P in the direction of $\vec{u} = 3\hat{i} + 4\hat{j}$.

Solution

$$D_{\hat{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \hat{u}$$

$$\nabla f = f_x \hat{i} + f_y \hat{j}$$

$$= \frac{1}{y} \hat{i} + \left(\frac{-x}{y^2}\right) \hat{j}$$

$$\therefore \quad \nabla f(2,1) = \frac{1}{(1)} \hat{i} - \frac{(2)}{(1)^2} \hat{j}$$

$$= \hat{i} - 2\hat{j} \quad // \text{Ans.}$$

Since $\vec{u} = 3\hat{i} + 4\hat{j}$ is not a unit vector, we must normalize it as

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{3\hat{i} + 4\hat{j}}{\sqrt{(3)^2 + (4)^2}} = \frac{3\hat{i} + 4\hat{j}}{\sqrt{25}} = \frac{3\hat{i} + 4\hat{j}}{5}$$

$$\therefore \quad D_{\hat{u}} f(2,1) = (1, -2) \cdot \frac{(3,4)}{5}$$

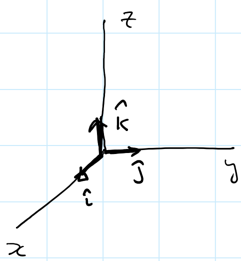
$$= \frac{(1)(3) + (-2)(4)}{5} = \frac{3 - 8}{5} = \frac{-5}{5} = -1 \quad // \text{Ans.}$$

For functions of three variables, we can show that the directional derivative of $f(x,y,z)$ at (x_0, y_0, z_0) in a direction of a unit vector $\hat{u}$ is

$$D_{\hat{u}} f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \hat{u}$$

where

$$\nabla f(x,y,z) = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$$



<u>Ex2</u>: Find the directional derivative of $f(x,y,z) = \sqrt{xyz}$ at the point $(3,2,6)$ in the direction of $\vec{v} = (-1, -2, 2)$.

<u>Solution</u>

$$\nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$$

$$\nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$$

$$f_x = \frac{yz}{2\sqrt{xyz}}, \quad f_y = \frac{xz}{2\sqrt{xyz}}, \quad f_z = \frac{xy}{2\sqrt{xyz}}$$

$$\therefore \quad \nabla f = \frac{yz}{2\sqrt{xyz}}\hat{i} + \frac{xz}{2\sqrt{xyz}}\hat{j} + \frac{xy}{2\sqrt{xyz}}\hat{k}$$

@ $(3,2,6)$,

$$\nabla f(3,2,6) = \frac{(2)(6)}{2\sqrt{(3)(2)(6)}}\hat{i} + \frac{(3)(6)}{2\sqrt{(3)(2)(6)}}\hat{j} + \frac{(3)(2)}{2\sqrt{(3)(2)(6)}}\hat{k}$$

$$= \frac{\cancel{(2)}\cancel{(6)}}{\cancel{(2)}\cancel{(6)}}\hat{i} + \frac{(3)\cancel{(6)}}{(2)\cancel{(6)}}\hat{j} + \frac{\cancel{(3)}(2)}{(2)\cancel{(6)}}\hat{k}$$

$$\nabla f(3,2,6) = \hat{i} + \frac{3}{2}\hat{j} + \frac{1}{2}\hat{k}$$

$$D_{\hat{v}} f(3,2,6) = \nabla f(3,2,6) \cdot \hat{v}$$

To obtain $\hat{v}$, we normalize $\vec{v}$

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{(-1,-2,2)}{\sqrt{(-1)^2+(-2)^2+(2)^2}} = \frac{(-1,-2,2)}{3} \leftarrow$$

$$\therefore \quad D_{\hat{v}} f(3,2,6) = \left(1, \frac{3}{2}, \frac{1}{2}\right) \cdot \frac{(-1,-2,2)}{3}$$

$$= \frac{(1)(-1) + \left(\frac{3}{2}\right)(-2) + \left(\frac{1}{2}\right)(2)}{3}$$

$$= \frac{-1 - 3 + 1}{3} = \frac{-3}{3} = -1 \quad // \text{Ans.}$$

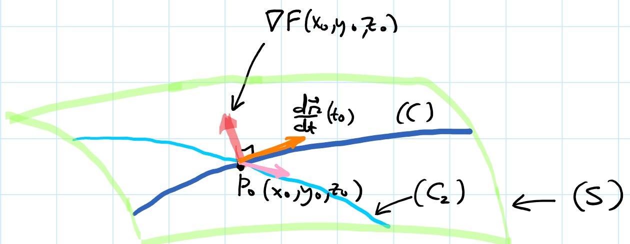
Tangent Plane to Level Surface

Given a function of 3 variables

$$w = F(x,y,z)$$

Then $F(x,y,z) = 0$ is a level surface (S) of w. Let $P_0(x_0,y_0,z_0)$ be a point on (S). Let (C) be any curve on (S) and it passes through P_0. Then parametric eqns of (C) are

$$(C)\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \quad \text{such that at } t = t_0 \text{ then } \begin{cases} x(t_0) = x_0 \\ y(t_0) = y_0 \\ z(t_0) = z_0 \end{cases}$$



Since (C) lies on (S), we must have

$$F(x(t), y(t), z(t)) = 0 \quad (1)$$

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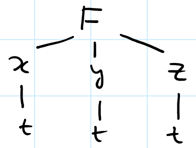
Differentiating (1) wrt t,



$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0 \qquad (2)$$

(2) can be expressed in a dot product

$$\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = 0 \qquad (3)$$

At $t = t_0$, (3) becomes

$$\nabla F(x_0, y_0, z_0) \cdot \frac{d\vec{r}}{dt}(t_0) = 0$$

This means $\nabla F(x_0, y_0, z_0) \perp$ the tangent $\left.\frac{d\vec{r}}{dt}\right|_{t_0}$ to (C) at P_0.

Since (C) is an arbitrary curve passing through P_0, we conclude that $\nabla F(P_0)$ is the normal vector to the level surface (S) at P_0.

$\therefore$ The equation of the tangent plane to the level surface (S) defined by $F(x, y, z) = C$ at $P_0(x_0, y_0, z_0)$ is

$$\boxed{F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0} \leftarrow$$

Ex 3: Find equations of the tangent plane and the normal line to the surface (S) $y = x^2 - z^2$ at the point $(4, 7, 3)$.

Solution

Rewriting the equation of (S) in the form of $F(x, y, z) = 0$ to obtain

$$F(x, y, z) = x^2 - y - z^2 = 0$$

then $\quad \nabla F = 2x\hat{i} - \hat{j} - 2z\hat{k}$

$$\therefore \quad \nabla F(4, 7, 3) = 2(4)\hat{i} - \hat{j} - 2(3)\hat{k}$$
$$= 8\hat{i} - \hat{j} - 6\hat{k}$$

$\therefore$ An equation of the tangent plane to (S) @ $(4, 7, 3)$ is

$$8(x - 4) + (-1)(y - 7) + (-6)(z - 3) = 0$$
$$8x - y - 6z - 32 + 7 + 18 = 0$$
$$8x - y - 6x - 7 = 0 \qquad // \text{Ans.}$$

and the eqns of the normal line are

$$\frac{x - 4}{8} = \frac{y - 7}{-1} = \frac{z - 3}{-6} \qquad // \text{Ans.}$$

or in parametric form

$$\begin{cases} x = 4 + 8t \\ y = 7 - t \end{cases} \qquad // \text{Ans.}$$

$$\begin{cases} x = 4 + 8t \\ y = 7 - t \\ z = 3 - 6t \end{cases} \qquad // \text{Ans.}$$

Ex 5: The temperature at a point (x,y,z) is given by

$$T(x,y,z) = 200\, e^{-x^2-3y^2-9z^2}$$

where T is measured in °C and x,y,z in meters.

a) Find the rate of change of temperature at the point $P(2,-1,2)$ in the direction toward the point $(3,-3,3)$.

b) In which direction does the temperature increase fastest at P?

c) Find the maximum rate of increase at P.

Solution

a)
$$\nabla T = \frac{\partial T}{\partial x}\hat{\imath} + \frac{\partial T}{\partial y}\hat{\jmath} + \frac{\partial T}{\partial z}\hat{k}$$
$$= -400x\, e^{-x^2-3y^2-9z^2}\,\hat{\imath} - 1200y\, e^{-x^2-3y^2-9z^2}\,\hat{\jmath} - 3600z\, e^{-x^2-3y^2-9z^2}\,\hat{k}$$

$$\nabla T(2,-1,2) = \left[-400(2)\hat{\imath} - 1200(-1)\hat{\jmath} - 3600(2)\hat{k}\right] e^{-(2)^2-3(-1)^2-9(2)^2}$$
$$= (-800\hat{\imath} + 1200\hat{\jmath} - 7200\hat{k})\, e^{-43}$$

$$\vec{u} = \overrightarrow{PQ} = (3,-3,3) - (2,-1,2)$$
$$= (3-2, -3-(-1), 3-2)$$
$$= (1,-2,1)$$

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{(1,-2,1)}{\sqrt{(1)^2+(-2)^2+(1)^2}} = \frac{(1,-2,1)}{\sqrt{6}}$$

$$\therefore\; D_{\hat{u}}T(2,-1,2) = \nabla T(2,-1,2)\cdot\hat{u}$$
$$= (-800e^{-43}, 1200e^{-43}, -7200e^{-43})\cdot\frac{(1,-2,1)}{\sqrt{6}}$$
$$= \frac{(-800 - 2400 - 7200)}{\sqrt{6}}\, e^{-43}$$
$$= -\frac{5200\sqrt{6}}{3}\, e^{-43} \;\; °C/m \qquad // \text{Ans.}$$

b) In the direction of $\nabla T(2,-2,2)$ which is the direction of $(-2,3,-18)$ the temperature increases fastest.

c)
$$D_{\hat{u}} f(2,-2,2)\big|_{max} = \|\nabla T(2,-2,2)\|$$
$$= \sqrt{(-800)^2 + (1200)^2 + (-7200)^2}\; e^{-43}$$
$$\approx 7343\, e^{-43} \;\; °C/m \qquad // \text{Ans.}$$

See you on Monday!