MATRICES

OUTLINE:

1) Definition
2) Matrix operations

1. DEFINITION

Matrices

- A matrix is a rectangular array of numbers.
- A matrix with m rows and n columns is called an m x n matrix. The couple (m,n) is called the size of the matrix.
- A matrix with the same number of rows and columns is called a square matrix.
- Two matrices are equal if they have the same number of rows, the same number of columns, and the entries in corresponding position are equal.

Matrices

- A matrix A with entries a_{ij} is denoted $A = [a_{ij}]$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- The entry (i,j) of a matrix A can be also denoted $(A)_{ij}$
- The i^{th} row of A is the 1 x n matrix $[a_{i1}, a_{i2}, \ldots, a_{in}]$
- The j^{th} column of A is the m x 1 matrix $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$

2. MATRIX OPERATIONS

Matrix sum

- Let A = [a_{ij}] and B = [b_{ij}] be two m x n matrices (same size)
- The sum of A and B, denoted by A+B, is the m x n matrix A+B = [$a_{ij}+b_{ij}$] that has $a_{ij}+b_{ij}$ as its (i,j)th element (i.e., we add entries in corresponding positions).
- EX: $$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

- The sum of two matrices is undefined when the matrices have different sizes

Product with a scalar

- Let $A = [a_{ij}]$ be a m x n matrix and let s be a scalar (i.e. a number)
- The product of s and A, denoted by sA, is the m x n matrix $sA = [sa_{ij}]$ which has sa_{ij} as its $(i,j)^{th}$ element (i.e., we multiply each entry of A by s).
- EX:

$$-2\begin{bmatrix} 0 & 1 & -2 \\ -1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 4 \\ 2 & -4 & -6 \end{bmatrix}$$

Matrix product

- Let A = [a_{ij}] be Let A be an m x k matrix and B = [b_{ij}] be a k x n matrix (i.e., the number of columns of A is the same as the number of rows of B)
- The matrix product of A and B, denoted by AB, is the m x n matrix (same number of rows as A, same number of columns as B)

$$AB = \left[\sum_{r=1}^{k} a_{ir}b_{rj}\right] = [a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}]$$

- The product of two matrices is undefined when the number of columns of the first matrix is not the same as the number of rows of the second one.

Matrix product

- EX:

$$\begin{bmatrix} 2 & 0 & 1 \\ 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -2 \\ -1 & -2 & 0 & 4 \end{bmatrix} =$$

$$\begin{bmatrix} 2\cdot 1 + 0\cdot 0 + 1\cdot(-1) & 2\cdot 2 + 0\cdot 1 + 1\cdot(-2) & 2\cdot 3 + 0\cdot 2 + 1\cdot 0 & 2\cdot 0 + 0\cdot(-2) + 1\cdot 4 \\ 3\cdot 1 + (-1)\cdot 0 + 4\cdot(-1) & 3\cdot 2 + (-1)\cdot 1 + 4\cdot(-2) & 3\cdot 3 + (-1)\cdot 2 + 4\cdot 0 & 3\cdot 0 + (-1)\cdot(-2) + 4\cdot 4 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 2 & 6 & 4 \\ -7 & -15 & -11 & 18 \end{bmatrix}$$

Matrix product

- Matrix product is associative but not commutative!
- If A and B are 2 matrices, $A_{(6,3)} \times B_{(3,2)}$.
  - it may happen that AB is defined and BA is not,
  - it may happen that AB and BA are both defined but different
  - It may happen (rarely) that AB and BA are both defined and AB = BA
- EX: find examples of each of these cases

Identity matrix and matrix powers

- For any $n \geq 1$ there is a special n x n matrix, called the identity matrix of size n, and defined as $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

- Identity matrices are neutral elements for matrix product:
  - if A is any k x n matrix, then $AI_n = A$
  - if B is any n x k matrix, then $I_nB = B$

- If A is a n x n matrix, its non-negative powers are defined:

$A^0 = I_n$, $A^1 = A$, $A^2 = AA$, $A^3 = AAA$, ...

Matrix transposition

- Let $A = [a_{ij}]$ be a m x n matrix. The transpose of A is the n x m matrix $A^T = [a_{ji}]$, that is, the matrix obtained interchanging the rows and columns of A
- EX: if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Symmetric matrices

- A matrix A is called symmetric if $A^T = A$: in other words, for all i,j $(A)_{ij} = (A)_{ji}$ (note that this condition may hold only if A is a square matrix)
- EX: Identity matrices of any sizes are symmetric
- EX: The matrix representing a symmetric relation is symmetric (surprise surprise)
- EX: $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 4 & -3 \end{bmatrix}$ is symmetric