Calculus 2402A
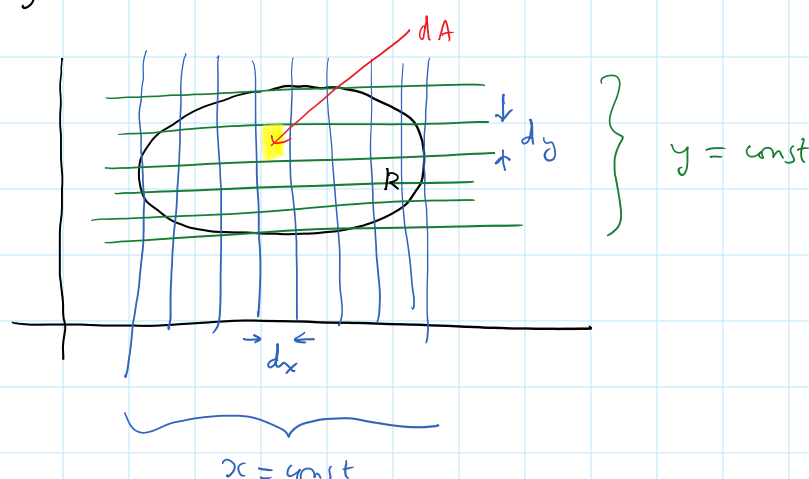Lecture 13
15.3 Double Integrals in Polar
Coordinates

The region R over which a double integral is evaluated may be defined in terms of polar coordinates r, θ. Recall that the area element dA in Cartesian coordinates is $dxdy$, obtained by the lines $x = \text{const}$, $x+dx = \text{const}$, $y = \text{const}$, $y+dy = \text{constant}$.

In polar coordinates, we construct the curves $r = \text{const}$, $r+dr = \text{const}$, $\theta = \text{const}$, $\theta + d\theta = \text{const}$ to obtain the area element dA.
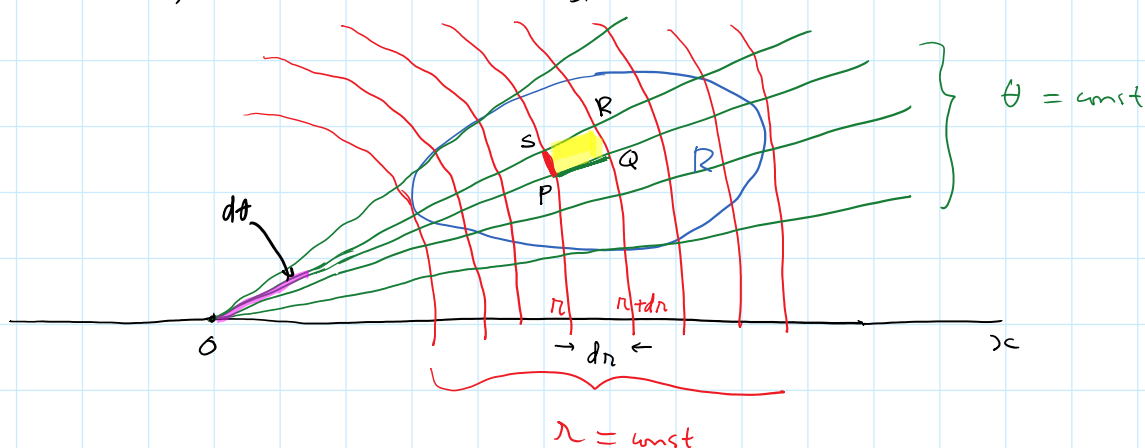
In polar coordinates $r = \text{const}$ is a family of concentric circles centered at the pole and $\theta = \text{const}$ is a family of rays (half-lines) originated from the pole. Then the area element dA is just the area of the parallelogram PQRS in the above figure.

$$dA = \text{area}\,(PQRS)$$
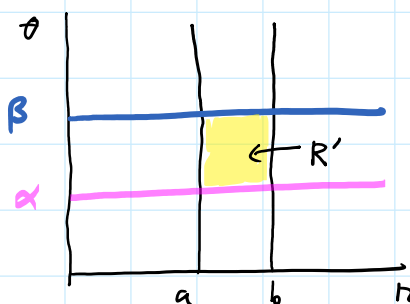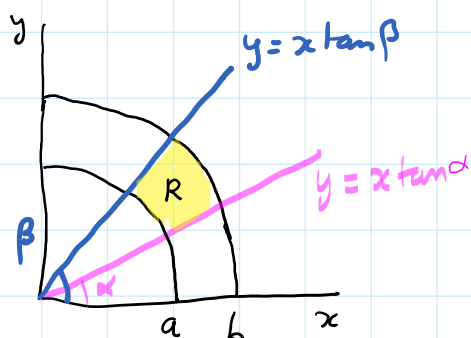$$= (PQ)(PS)$$
$$= (dr)(r\,d\theta)$$

$$\boxed{dA = r\,dr\,d\theta}$$

Then $z = f(x,y)$ can be written as $f(r\cos\theta, r\sin\theta)$ and

$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

where R' is the region in $r\theta$-plane corresponding to the region R in the xy-plane.

In particular, if R is the region bounded by two circles $x^2 + y^2 = a^2$ ($r = a$) and $x^2 + y^2 = b^2$ ($r = b$, $a < b$) and the lines $y = x\tan\alpha$ ($\theta = \alpha$) and $y = x\tan\beta$ ($\theta = \beta$, $\alpha < \beta$), then
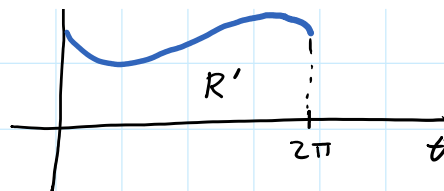
$$\iint_R f(x,y)\,dA = \iint_{R'} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

We note that R' is a rectangle defined by $a \leq r \leq b$, $\alpha \leq \theta \leq \beta$. This explains why a change of variables from Cartesian coordinates to polar coordinates is a necessity in some situations.

More generally, the region R may be bounded by a closed curve enclosing the pole given by $r = r(\theta)$ as shown in the figure below
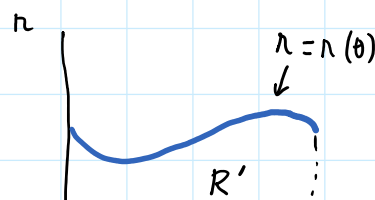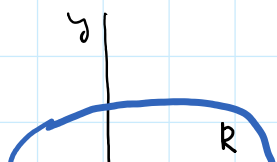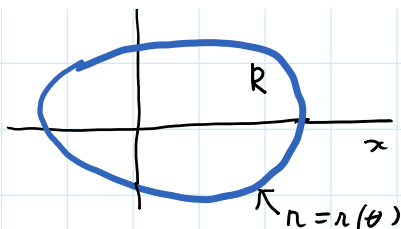
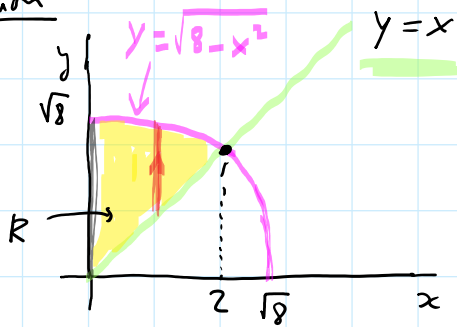$$\iint\limits_{R} f(x,y)\, dxdy = \int_{0}^{2\pi} \int_{0}^{r(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Example: Evaluate $I = \iint\limits_{R} \dfrac{1}{5+x^2+y^2}\, dA$ where R is the

region in the first quadrant bounded by $x^2+y^2 = 8$, $y = x$, $x = 0$.

Solution

We note that R is a type I region. Next, we find the points of intersection of the circle $x^2+y^2 = 8$ and the line $y = x$,

$$x^2 + x^2 = 8$$
$$2x^2 = 8$$
$$x^2 = 4 \Rightarrow x = \pm 2$$

we choose $x = 2$.

$$I = \int_{0}^{2} \int_{x}^{\sqrt{8-x^2}} \frac{1}{5+x^2+y^2}\, dy\, dx$$

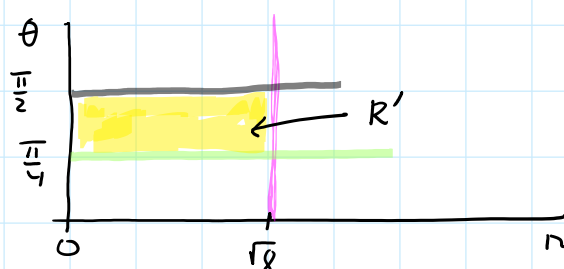$$\int \frac{du}{a^2+u^2} = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right)$$

$$= \int_{0}^{2} \frac{1}{\sqrt{x^2+5}} \arctan\left(\frac{y}{\sqrt{x^2+5}}\right)\Bigg|_{y=x}^{\sqrt{8-x^2}} dx$$

$$= \int_{0}^{2} \frac{1}{\sqrt{x^2+5}} \left[\arctan\left(\frac{\sqrt{8-x^2}}{\sqrt{x^2+5}}\right) - \arctan\left(\frac{x}{\sqrt{x^2+5}}\right) \right] dx$$

This integral is too difficult to proceed further (both Mathematica and Maple fail badly!) So we need to change the double integral from Cartesian coordinates to polar coordinates

The circle $x^2+y^2 = 8 \Rightarrow r = \sqrt{8}$

The line $y = x$ $\implies$ $\theta = \frac{\pi}{4}$

the y-axis $\implies$ $\theta = \frac{\pi}{2}$

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\sqrt{8}} \frac{1}{5 + r^2} \, r \, dr \, d\theta$$

$$= \left(\int_{\frac{\pi}{4}}^{\pi/2} d\theta \right) \left(\frac{1}{2} \int_0^{\sqrt{8}} \frac{2r \, dr}{\underbrace{5 + r^2}_{u}} \right)$$

$$= \left(\frac{\pi}{4} \right) \frac{1}{2} \ln(5 + r^2) \Big|_{r=0}^{\sqrt{8}}$$

$$= \frac{\pi}{8} \Big(\ln(5 + 8) - \ln(5) \Big) = \frac{\pi}{8} \Big(\ln(13) - \ln(5) \Big)$$

$$= \frac{\pi}{8} \ln\left(\frac{13}{5} \right) \qquad // \text{ Ans.}$$

$\underline{Ex2}$: Evaluate $I = \int_0^\infty e^{-x^2} dx$.

$\underline{Soln}$ We already know e^{-x^2} DOES NOT have an antiderivative in a "closed form". However, we can use polar coordinates to evaluate this improper integral.

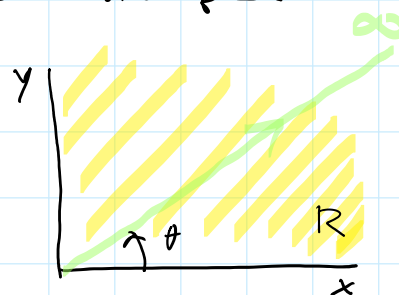To use double integral, we need compute I^2 instead of I.

$$I^2 = \left(\int_0^\infty e^{-x^2} dx \right) \left(\int_0^\infty e^{-y^2} dy \right)$$

$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx \, dy$$

$$= \iint_R e^{-(x^2+y^2)} dA$$

where R is the 1st quadrant of the xy-plane

$$\underset{\uparrow}{=} \int_0^{\frac{\pi}{2}} \int_0^\infty \underbrace{e^{-r^2}}_{= \text{ Separable}} r \, dr \, d\theta$$

$$\overset{=}{\uparrow} \quad \int_0^2 \int_0^{\boxed{e^{-r^2}}} r \, dr \, d\theta$$

Change to polar
coordinates

$$= \left(\int_0^{\pi/2} d\theta \right) \left(\int_0^\infty e^{-r^2} r \, dr \right) \qquad \begin{array}{l} u = r^2 \\ du = 2r\,dr \end{array}$$

$$= \left(\frac{\pi}{2} \right) \left(\int_0^\infty e^{-u} \frac{du}{2} \right)$$

$$= \frac{\pi}{4} \left(-e^{-u} \right) \Big|_0^\infty \quad = \quad \frac{\pi}{4} \left(-\overset{0}{\cancel{e^{-\infty}}} + \overset{1}{e^0} \right) \quad = \quad \frac{\pi}{4}$$

$$\therefore \quad I = \int_0^\infty e^{-x^2} \, dx = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2} \quad \| \text{ Ans.}$$

See you on Friday!