Undecidability

Sections 21.1 – 21.3

Problems / Languages

The Problem View	The Language View
Does TM M halt on w?	$H = \{<M, w> : M \text{ halts on } w\}$
Does TM M not halt on w?	$\neg H = \{<M, w> : M \text{ does not halt on } w\}$
Does TM M halt on the empty tape?	$H_\varepsilon = \{<M> : M \text{ halts on } \varepsilon\}$
Is there any string on which TM M halts?	$H_{ANY} = \{<M>$: there exists at least one string on which TM M halts $\}$
Does TM M accept all strings?	$A_{ALL} = \{<M> : L(M) = \Sigma^*\}$
Do TMs M_a and M_b accept the same languages?	EqTMs $= \{<M_a, M_b> : L(M_a) = L(M_b)\}$
Is the language that TM M accepts regular?	TMreg $= \{<M>:L(M)$ is regular$\}$

Reduction

- **Example:** Computing a function

  multiply(x, y) =
  1. *answer* := 0.
  2. For i := 1 to y do:
     answer = ***sum***(*answer* + x).
  3. Return *answer.*

- Computing ***multiply*** reduces to computing ***sum****.*
  or
- If we can do ***sum*** then we can do ***multiply****.*

Using Reduction for Undecidability

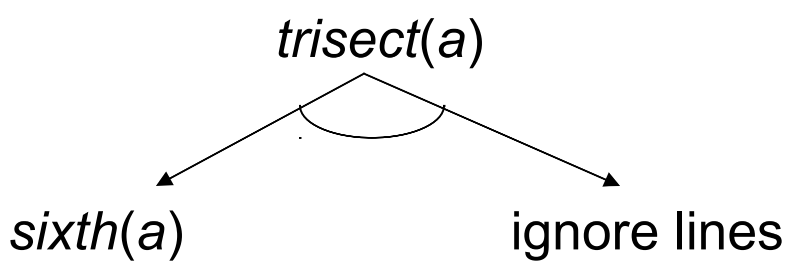
Theorem: There exists no general procedure to solve the following problem:

> Given an angle *A*, divide *A* into sixths using only a straightedge and a compass.

Proof: Suppose that there were such a procedure, which we'll call *sixth*. Then we could trisect an arbitrary angle:

trisect(*a*: angle) =

1. Divide *a* into six equal parts by invoking *sixth*(*a*).
2. Ignore every other line, thus dividing *a* into thirds.



sixth exists → *trisect* exists.

But we know that *trisect* does not exist. So, *sixth* cannot exist either.

Turing Reduction

A ***reduction*** *R* from L_1 to L_2 is one or more Turing machines such that:

If there exists a Turing machine *Oracle* that decides (or semidecides) L_2, then the Turing machines in *R* can be composed with *Oracle* to build a deciding (or a semideciding) Turing machine for L_1.

$L_1 \leq L_2$ means that L_1 is **reducible** to L_2.

Using Reduction for Undecidability

Assume:

$(L_1 \leq L_2) \wedge (L_2 \text{ is in D}) \rightarrow (L_1 \text{ is in D})$

If (L_1 is in D) is false, then at least one of the two antecedents of that implication must be false. So:

If	$(L_1 \leq L_2)$ is true,
then	(L_2 is in D) must be false.

Using Reduction for Undecidability

Showing that L_2 is not in D:

L_1 (known not to be in D) L_1 in D But L_1 not in D

R ↓ ↑ ↓

L_2 (a new language whose decidability we are trying to determine) if L_2 in D So L_2 not in D

To Use Reduction for Undecidability

0. Assume *Oracle* that decides L_2 exists

1. Choose a language L_1:
   - that is already known not to be in D, and
   - that can be reduced to L_2.

2. Define the reduction *R*.

3. Describe the composition *C* of *R* with *Oracle*:

$$C(x) = Oracle(R(x))$$

4. Show that *C* does correctly decide L_1 if *Oracle* exists. We do this by showing:
   - *R* can be implemented by Turing machines,
   - *C* is correct:
     - If $x \in L_1$, then $C(x)$ accepts, and
     - If $x \notin L_1$, then $C(x)$ rejects.

To Use Reduction for Undecidability

The Overall Strategy to show L_2 is undecidable:

- Assume that L_2 is decidable (*Oracle* exists).
- Pick an undecidable L_1 and reduce L_1 to L_2 (prove $L_1 \leq L_2$).
- Use *Oracle* and the reduction from L_1 to L_2 to show that L_1 is decidable.
- Conclude that L_2 is undecidable.

Mapping Reductions

L_1 is ***mapping reducible*** to L_2 ($L_1 \leq_M L_2$) iff there exists some computable function f such that:

$$\forall x \in \Sigma^* \ (x \in L_1 \leftrightarrow f(x) \in L_2).$$

To decide whether x is in L_1, we transform it, using f, into a new object and ask whether that object is in L_2.

Note: mapping reduction is a particular case of Turing reduction.

$H_\varepsilon = \{<M> : \text{TM } M \text{ halts on } \varepsilon\}$

H_ε is in SD. *T* semidecides it:

$T(<M>) =$
1. Run M on ε.
2. Accept.

T accepts $<M>$ iff M halts on ε, so T semidecides H_ε.

H_ε = {<M> : TM M halts on ε}

Theorem: H_ε = {<M> : TM M halts on ε} is not in D.

Proof: by reduction from H:

$$
\begin{array}{ll}
 & H = \{<M, w> : \text{TM } M \text{ halts on input string } w\} \\
 & \big\downarrow R \\
(?\mathit{Oracle}) & H_\varepsilon = \{<M> : \text{TM } M \text{ halts on } \varepsilon\}
\end{array}
$$

R is a mapping reduction from H to H_ε:

R(<M, w>) =

1. Construct <$M\#$>, where $M\#(x)$ operates as follows:
   - 1.1. Erase the tape (ignore its input)
   - 1.2. Write w on the tape.
   - 1.3. Run M on w.
2. Return <$M\#$>.

Proof, Continued

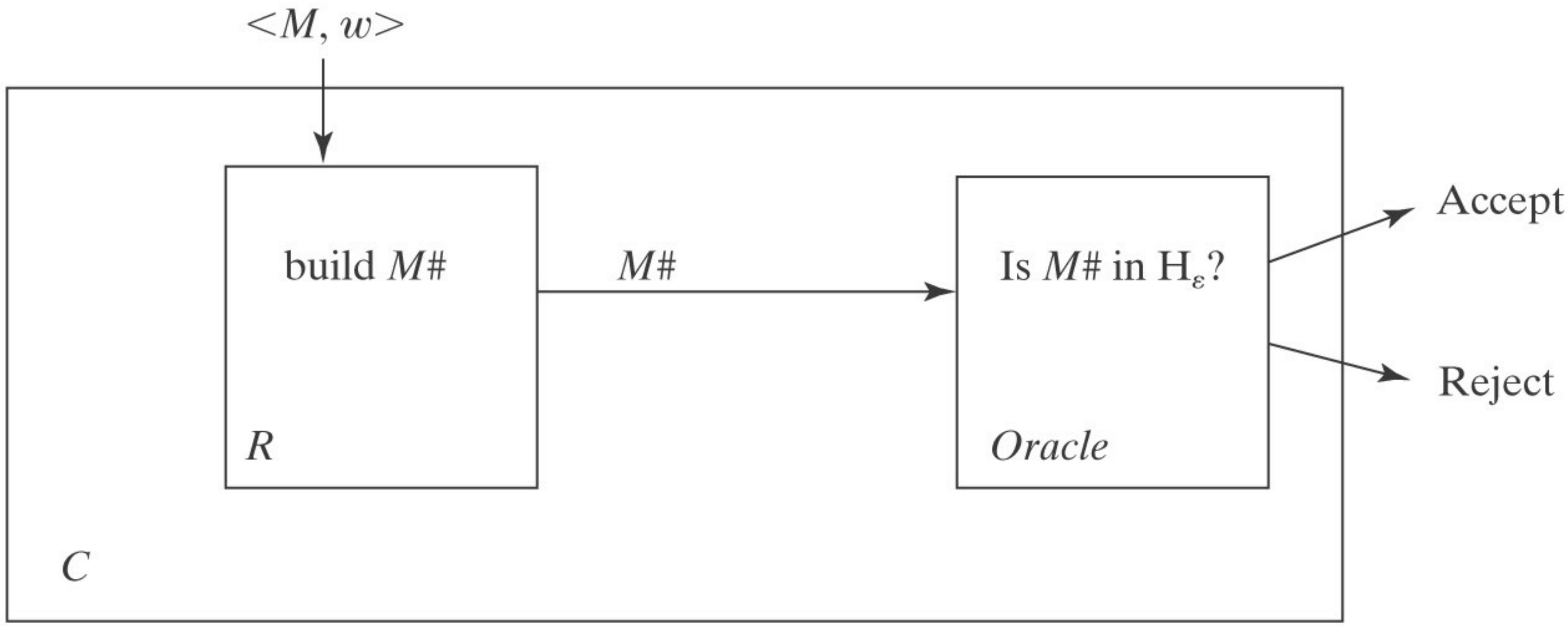
R(<M, w>) =

1. Construct <M#>, where M#(x) operates as follows:
    1.1. Erase the tape.
    1.2. Write w on the tape.
    1.3. Run M on w.
2. Return <M#>.

If *Oracle* exists, C = *Oracle*(R(<M, w>)) decides H:

- C is correct: M# ignores its own input. It halts on everything or nothing. So:
    - <M, w> $\in$ H: M halts on w, so M# halts on everything. In particular, it halts on ε. *Oracle* accepts.
    - <M, w> $\notin$ H: M does not halt on w, so M# halts on nothing and thus not on ε. *Oracle* rejects.

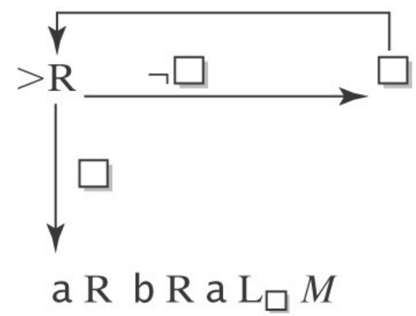
A Block Diagram of *C*



R Can Be Implemented as a Turing Machine

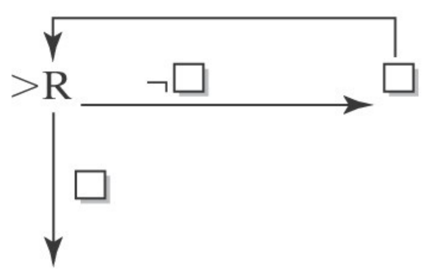
R must construct <*M#*> from <*M*, *w*>. Suppose *w* = `aba`.

M# will be:



So the procedure for constructing *M#* is:

1. Write:



2. For each character *x* in *w* do:
    2.1. Write *x*.
    2.2. If *x* is not the last character in *w*, write R.
3. Write $L_{\square}$ *M*.

Conclusion

R can be implemented as a Turing machine.

C is correct.

So, if *Oracle* exists:

$C = Oracle(R(<M, w>))$ decides H.

But no machine to decide H can exist.

So neither does *Oracle*.

This Result is Somewhat Surprising

If we could decide whether *M* halts on the specific string ε, we could solve the more general problem of deciding whether *M* halts on an arbitrary input.

Clearly, the other way around is true: If we could solve H we could decide whether *M* halts on any one particular string.

But doing a reduction in that direction would tell us nothing about whether H_ε was decidable.

The significant thing that we just saw in this proof is that there also exists a reduction in the direction that does tell us that H_ε is not decidable.

Important Elements in a Reduction Proof

- A clear declaration of the reduction “from” and “to” languages.
- A clear description of *R*.
- If *R* is doing anything nontrivial, argue that it can be implemented as a TM.
- Run through the logic that demonstrates how the “from” language is being decided by the composition of *R* and *Oracle*. You must do both accepting and rejecting cases.
- Declare that the reduction proves that your “to” language is not in D.

The Most Common Mistake: Doing the Reduction Backwards

- Right way: to show that L_2 is not in D:

1. Reduce a known hard one, L_1, to L_2: $L_1 \longrightarrow L_2$
2. Given that L_1 is not in D,
3. Reduce L_1 to L_2, i.e., show how to solve L_1 (the known one) in terms of L_2 (the unknown one)

- Wrong way: reduce L_2 (the unknown one) to L_1 (the known hard):

Example (wrong):
If there exists a machine M_H that solves H, then we could build a machine that solves H_ε as follows:

1. Return (M_H(<M, ε>)).

This proves nothing. It's an argument of the form:

If *False* then …

H_{ANY} = {<*M*> : there exists at least one string on which TM *M* halts}

Theorem: H_{ANY} is in SD.

Proof: by exhibiting a TM *T* that semidecides it.

What about simply trying all the strings in Σ^* one at a time until one halts?

H_{ANY} is in SD

$T(\langle M \rangle)$ =

1. Use dovetailing to try M on all of the elements of Σ^*:

ε	[1]								
ε	[2]	a	[1]						
ε	[3]	a	[2]	b	[1]				
ε	[4]	a	[3]	b	[2]	aa	[1]		
ε	[5]	a	[4]	b	[3]	aa	[2]	ab	[1]

2. If any instance of M halts, halt and accept.

T will accept iff M halts on at least one string. So T semidecides H_{ANY}.

H_{ANY} is not in D

H = {<*M*, *w*> : TM M halts on input string *w*}

$R \downarrow$

(?*Oracle*) H_{ANY} = {<*M*> : there exists at least one string on which TM *M* halts}

R(<*M*, *w*>) =

1. Construct <*M#*>, where *M#*(*x*) operates as follows:
    1.1. Examine *x*.
    1.2. If *x* = *w*, run *M* on *w*, else loop.
2. Return <*M#*>.

If *Oracle* exists, then *C* = *Oracle*(*R*(<*M*, *w*>)) decides H:

- *R* can be implemented as a Turing machine.
- *C* is correct: The only string on which *M#* can halt is *w*. So:
    - <*M*, *w*> $\in$ H: *M* halts on *w*. So *M#* halts on *w*. There exists at least one string on which *M#* halts. *Oracle* accepts.
    - <*M*, *w*> $\notin$ H: *M* does not halt on *w*, so neither does *M#*. So there exists no string on which *M#* halts. *Oracle* rejects.

But no machine to decide H can exist, so neither does *Oracle*.

H_{ANY} is not in D: another reduction

Proof: We show that H_{ANY} is not in D by reduction from H:

H = {<*M*, *w*> : TM M halts on input string *w*}

$\downarrow R$

(?*Oracle*) H_{ANY} = {<*M*> : there exists at least one string on which TM *M* halts}

R(<*M*, *w*>) =

1. Construct the description <*M#*>, where *M#*(*x*) operates as follows:
   - 1.1. Erase the tape.
   - 1.2. Write *w* on the tape.
   - 1.3. Run *M* on *w*.
2. Return <*M#*>.

If *Oracle* exists, then *C* = *Oracle*(*R*(<*M*, *w*>)) decides H:

- *C* is correct: *M#* ignores its own input. It halts on everything or nothing. So:
  - <*M*, *w*> $\in$ H: *M* halts on *w*, so *M#* halts on everything. So it halts on at least one string. *Oracle* accepts.
  - <*M*, *w*> $\notin$ H: *M* does not halt on *w*, so *M#* halts on nothing. So it does not halt on at least one string. *Oracle* rejects.

But no machine to decide H can exist, so neither does *Oracle*.

The Steps in a Reduction Proof

1. Assume *Oracle* exists.

2. Choose an undecidable language to reduce from.

3. Define the reduction *R*.

4. Show that *C* (the composition of *R* with *Oracle*) is correct.

H_{ALL} = {<*M*> : TM *M* halts on all inputs}

We show that H_{ALL} is not in D by reduction from H_ε.

$$H_\varepsilon = \{<M> : \text{TM } M \text{ halts on } \varepsilon\}$$

$$\Big\downarrow R$$

(?*Oracle*) $$H_{ALL} = \{<M> : \text{TM } M \text{ halts on all inputs}\}$$

R(<*M*>) =

1. Construct the description <*M#*>, where *M#*(*x*) operates as follows:
    1.1. Erase the tape.
    1.2. Run *M*.
2. Return <*M#*>.

If *Oracle* exists, then *C* = *Oracle*(*R*(<*M*>)) decides H_ε:

- *R* can be implemented as a Turing machine.
- *C* is correct: *M#* halts on everything or nothing, depending on whether *M* halts on ε. So:
    - <*M*> $\in H_\varepsilon$: *M* halts on ε, so *M#* halts on all inputs. *Oracle* accepts.
    - <*M*> $\notin H_\varepsilon$: *M* does not halt on ε, so *M#* halts on nothing. *Oracle* rejects.

But no machine to decide H_ε can exist, so neither does *Oracle*.

The Membership Question for TMs

We next define a new language:

$$A = \{<M, w> : M \text{ accepts } w\}.$$

Note that A is different from H since it is possible that M halts but does not accept. An alternative definition of A is:

$$A = \{<M, w> : w \in L(M)\}.$$

A = {<*M*, *w*> : *w* ∈ *L*(*M*)}

We show that A is not in D by reduction from H.

H = {<*M, w*> : TM *M* halts on input string *w*}

$\Big\downarrow R$

(?*Oracle*) A = {<*M, w* > : *w* ∈ *L*(*M*) }

R(<*M, w*>) =

1. Construct the description <*M#*>, where *M#*(*x*) operates as follows:
   1.1. Erase the tape.
   1.2. Write *w* on the tape.
   1.3. Run *M* on *w*.
   1.4. **Accept**
2. Return <*M#, w*>.

If *Oracle* exists, then *C* = *Oracle*(*R*(<*M*, *w*>)) decides H:

- *R* can be implemented as a Turing machine.
- *C* is correct: *M#* accepts everything or nothing. So:
  - <*M, w*> ∈ H: *M* halts on *w*, so *M#* accepts everything. In particular, it accepts *w*. *Oracle* accepts.
  - <*M, w* > ∉ H: *M* does not halt on *w*. *M#* gets stuck in step 1.3 and so accepts nothing. *Oracle* rejects.

But no machine to decide H can exist, so neither does *Oracle*.

A_ε, A_{ANY}, and A_{ALL}

Theorem: A_ε = {<*M*> : TM *M* accepts ε} is not in D.

Proof: Analogous to that for H_ε.

Theorem: A_{ANY} = {<*M*> :TM *M* accepts at least one string} is not in D.

Proof: Analogous to that for H_{ANY}.

Theorem: A_{ALL} = {<*M*> : $L(M) = \Sigma^*$} is not in D.

Proof: Analogous to that for H_{ALL}.

EqTMs={<M_a, M_b>: $L(M_a)=L(M_b)$}

$$A_{ALL} = \{<M> : L(M) = \Sigma^*\}$$

$$\Big\downarrow R$$

(*Oracle*) EqTMs = {<M_a, M_b>: $L(M_a) = L(M_b)$}

R(<*M*>) =

1. Construct the description of *M#*(*x*):
    1.1. Accept. (*M#* accepts everything)
2. Return <*M*, *M#*>.

If *Oracle* exists, then *C* = *Oracle*(*R*(<*M*>)) decides A_{ALL}:

- *C* is correct: *M#* accepts everything. So if $L(M) = L(M\#)$, *M* must also accept everything. So:
    - <*M*> $\in A_{ALL}$: $L(M) = L(M\#)$. *Oracle* accepts.
    - <*M*> $\notin A_{ALL}$: $L(M) \neq L(M\#)$. *Oracle* rejects.

But no machine to decide A_{ALL} can exist, so neither does *Oracle*.

Sometimes Mapping Reducibility Isn't Right

Recall that a mapping reduction from L_1 to L_2 is a computable function f where:

$$\forall x \in \Sigma^* \; (x \in L_1 \leftrightarrow f(x) \in L_2).$$

When we use a mapping reduction, we return:

$$Oracle(f(x))$$

Sometimes we need a more general ability to use *Oracle* as a subroutine and then to do other computations after it returns.

{<*M*> : *M* accepts no even length strings}

H = {< *M*, *w*> : TM *M* halts on input string *w*}

$\downarrow R$

(?*Oracle*) L_2 = {<*M*> : *M* accepts no even length strings}

R(<*M*, *w*>) =

1. Construct the description <*M#*>, where *M#*(*x*) operates as follows:
    1.1. Erase the tape.
    1.2. Write *w* on the tape.
    1.3. Run *M* on *w*.
    1.4. Accept.
2. Return <*M#*>.

If *Oracle* exists, then *C* = *Oracle*(*R*(<*M*, *w*>)) decides H:

- *C* is correct: *M#* ignores its own input. It accepts everything or nothing, depending on whether it makes it to step 1.4. So:
    - <*M*, *w*> $\in$ H: *M* halts on *w*. *Oracle*:
    - <*M*, *w*> $\notin$ H: *M* does not halt on *w*. *Oracle*:

Problem:

{<*M*> : *M* accepts no even length strings}

H = {< *M*, *w*> : TM *M* halts on input string *w*}

$\downarrow R$

(?*Oracle*) L_2 = {<*M*> : *M* accepts no even length strings}

R(<*M*, *w*>) =

1. Construct the description <*M#*>, where *M#*(*x*) operates as follows:
   1.1. Erase the tape.
   1.2. Write *w* on the tape.
   1.3. Run *M* on *w*.
   1.4. Accept.
2. Return <*M#*>.

If *Oracle* exists, then *C* = ¬*Oracle*(*R*(<*M*, *w*>)) decides H:

- *R* and ¬ can be implemented as Turing machines.
- *C* is correct:
  - <*M*, *w*> ∈ H: *M* halts on *w*. *M#* accepts everything, including some even length strings. *Oracle* rejects so *C* accepts.
  - <*M*, *w*> ∉ H: *M* does not halt on *w*. *M#* gets stuck. So it accepts nothing, so no even length strings. *Oracle* accepts. So *C* rejects.

But no machine to decide H can exist, so neither does *Oracle*.

Are All Questions about TMs Undecidable?

L = {<M> : TM M contains an even number of states}

L = {<M, w> : M halts on w within 3 steps}.

L_q = {<M, q> : there is some configuration (p, $u\underline{a}v$) of M, with $p \neq q$, that yields a configuration whose state is q}.