Lines in $\mathbb{R}^2$

Point-parallel form
Two-point form
Standard form
Parametric form
Point-normal form

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Goal: use vector equations to describe lines in $\mathbb{R}^2$.

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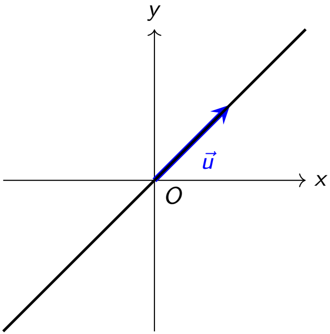
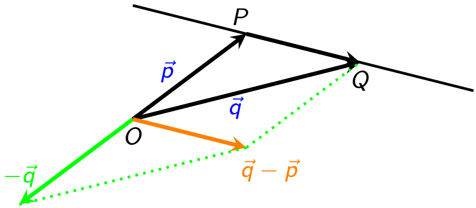


Do you still remember the translating a vector to a point?

Theorem Let $\overrightarrow{PQ}$ be a directed line segment from P to Q, where P and Q are two distinct points in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $\overrightarrow{PQ}$ is equivalent to the vector $\vec{q} - \vec{p}$, where $\vec{q} = \overrightarrow{OQ}$ and $\vec{p} = \overrightarrow{OP}$ and O denotes the origin.



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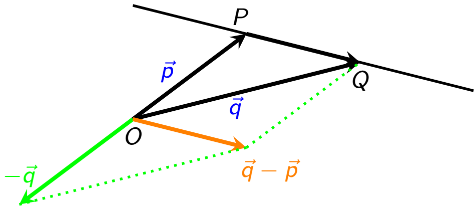

• The process of replacing $\overrightarrow{PQ}$ by the vector $\vec{q} - \vec{p}$ is called *translating* P *to the origin*.

• The process of replacing the vector $\vec{v}$ with an equivalent directed line segment which starts at some point P is called translating $\vec{v}$ to P.

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For example, $2x + y = 1$ is a line in $\mathbb{R}^2$. Points $P(0,1)$ and $Q(\frac{1}{2},0)$ are on the line. Let $\vec{p} = (0,1)$ and $\vec{q} = (\frac{1}{2},0)$ be the two vectors ending at P and Q, respectively. Then the vector $\vec{q} - \vec{p} = (\frac{1}{2},-1)$ is *equivalent* to the directed line segment $\overrightarrow{PQ}$. Hence, the vector $(\frac{1}{2},-1)$ is a direction vector of the line $2x + y = 1$.

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A property: if $\vec{v}$ is the direction vector of a line L, so is $-\vec{v}$. Indeed, any scalar multiple $c\vec{v}$ for which $c \neq 0$ is also a direction vector of L.

Point-parallel form

Let $\vec{v} = (v_1, v_2)$ be a vector in $\mathbb{R}^2$. Then the line L determined by $\vec{v}$ is given by $t\vec{v}$ and t is a parameter. Since $t\vec{v} = (tv_1, tv_2)$ is a vector with parameter t, we denote by $\vec{x}(t) = t\vec{v}$.

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Definition The *point-parallel form* equation of the line L which passes through point $P(p_1, p_2)$ and has direction vector $\vec{v}$ is given by

$$\vec{x}(t) = \vec{p} + t\vec{v}$$

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Example Write a point-parallel form of each of the following line.

1. The line through $P(2, 3)$ and $Q(-1, 4)$;
2. The line through the origin with a direction vector $\vec{v} = (-2, 1)$.
3. The line through $P(2, 3)$ with the direction vector $\vec{v} = (2, -4)$.

Definition The line L with point-parallel form equation $\vec{x}(t) = (p_1, p_2) + t(v_1, v_2)$ has *parametric equations*

$$\begin{aligned} x &= p_1 + tv_1 \\ y &= p_2 + tv_2. \end{aligned}$$

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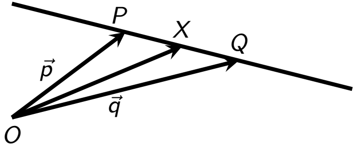
Definition The *two-point form* of equation for the line through points P and Q is

$$\vec{x}(t) = (1 - t)\vec{p} + t\vec{q}.$$



Example Write the two-point form of equation for the lines in the previous example.

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Definition Let L be any line in $\mathbb{R}^2$. If $\vec{n} = (n_1, n_2)$ is a normal for line L, and $P(p_1, p_2)$ is a point on line L, then an equation for line L in *point-normal form* is

$$(n_1, n_2) \cdot (\vec{x} - (p_1, p_2)) = 0.$$

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Example 1. Find an equation in point-normal form for the line

$$\vec{x}(t) = (0, 1) + t(2, -1).$$

2. Find an equation in point-normal form for the line with parametric form

$$x = 3 + t \text{ and } y = 2t - 4.$$

3. Find a point-normal form for the line contains the point $P(1, 2)$ with normal vector $\vec{n} = (-4, 3)$.

4. Find parametric equations for the line through the point $(2, 2)$ and parallel to the vector $(-3, -1)$.

5. Give a point-parallel form for the line given by $y = 3x + 1$.

6. What is the standard form for each line above?

Lines and planes in $\mathbb{R}^3$