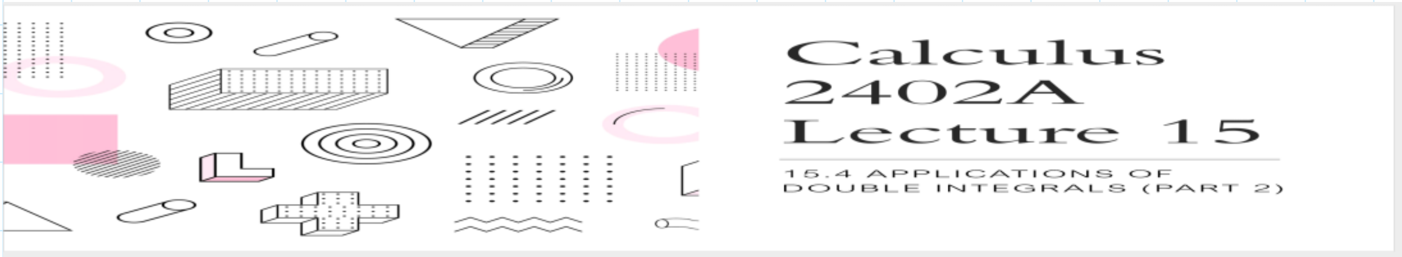
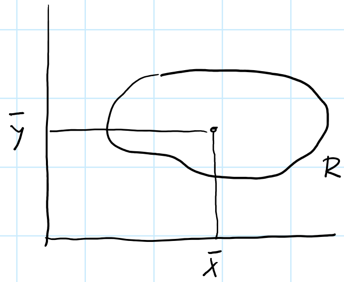
Lecture 15

October 14, 2020 10:52 AM




$$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}$$



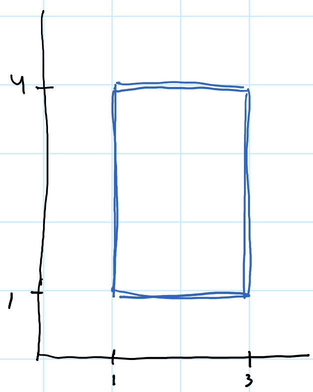

where $M_y = \iint_R x\,\rho(x,y)\,dA$

$M_x = \iint_R y\,\rho(x,y)\,dA$

$m = \iint_R \rho(x,y)\,dA$

Ex1: Find the CM of the lamina R defined by $1 \le x \le 3$, $1 \le y \le 4$ with the mass density $\rho(x,y) = ky^2$.

Solution




Last time, we got

$m = 42k$

$M_y = \iint_R x \underbrace{\boxed{\rho(x,y)}}_{ky^2} dA = k \int_1^4 \left( \int_1^3 \boxed{x\,y^2}\, dx \right) dy$ ← separable

$= k \left( \int_1^3 x\,dx \right) \left( \int_1^4 y^2\,dy \right)$

$= k \left( \frac{x^2}{2} \right) \Big|_1^3 \left( \frac{y^3}{3} \right) \Big|_1^4$

$= \frac{k}{6}(9-1)(64-1) = \frac{k}{\cancel{6}_{3}}(\cancel{8}^{4})(\cancel{63}^{21}) = 84k$ //

$M_x = \iint_R y \underbrace{\boxed{\rho(x,y)}}_{ky^2} dA = k \int_1^4 \left( \int_1^3 y^3\,dx \right) dy$

$= k \underbrace{\left( \int_1^3 dx \right)} \left( \int_1^4 y^3\,dy \right)$

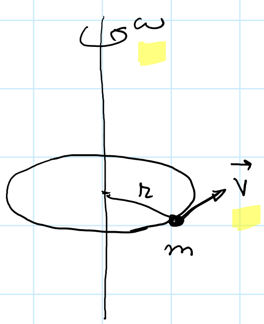
$= k\,(2) \left( \frac{y^4}{4} \right) \Big|_1^4 = \frac{k}{2}(4^4 - 1) = \frac{k}{2}(256-1) = \frac{255k}{2}$ //

$$= k\,(2)\left(\frac{y^4}{4}\right)\Big|_1^4 = \frac{k}{2}\left(4^4-1\right) = \frac{k}{2}\left(256-1\right) = \frac{255k}{2} \quad //$$

$$\therefore \quad \bar{x} = \frac{M_y}{m} = \frac{84\cancel{k}}{42\cancel{k}} = 2 \quad // \text{Ans.}$$

$$\bar{y} = \frac{M_x}{m} = \frac{255\cancel{k}}{2}\,\frac{1}{42\cancel{k}} = \frac{85}{28} \quad // \text{Ans.}$$

## Moments of inertia and radius of gyration

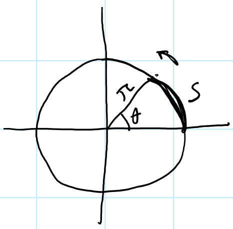



Consider a point mass $m$ rotating about an axis with angular velocity $\omega$. Then its Kinetic energy (KE) is

$$KE = \frac{1}{2} m v^2 \qquad \left(v^2 = \vec{v}\cdot\vec{v}\right)$$

$$= \frac{1}{2} m (r\omega)^2$$

$$= \frac{1}{2} \boxed{mr^2}\, \omega^2$$




$$s = r\theta$$

$$\frac{ds}{dt} = r\,\frac{d\theta}{dt}$$

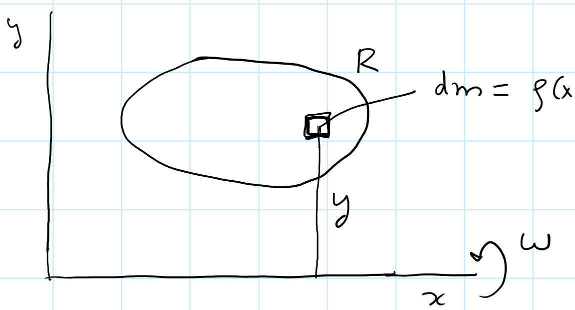
$$v = r\omega$$

$$I = mr^2$$

$I$ is called the moment of inertia of the point mass about the axis of rotation.

$$\therefore \quad \boxed{KE = \frac{1}{2} I \omega^2}$$

Consider a lamina $R$ with mass density $\rho(x,y)$. We want to compute the moment of inertia of $R$ about the $x$-axis.




$dm = \rho(x,y)\, dA$

$$d(I_x) = dm\,(y)^2$$

$$= \rho(x,y)\, dA\; y^2$$

$$= y^2 \rho(x,y)\, dA$$

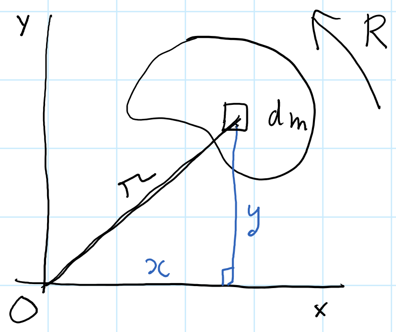
$$\therefore \quad I_x = \iint_R d(I_x)$$

$$\boxed{I_x = \iint_R y^2 \rho(x,y)\, dA}$$

Similarly,

$$I_y = \iint_R x^2 \rho(x,y)\, dA$$

is called the moment of inertia of R about the y-axis.




The moment of inertia of the lamina R about the origin (or the polar moment of R) is

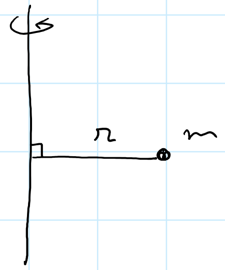
$$I_0 = \iint_R r^2 \rho(x,y)\, dA$$
$$= \iint_R (x^2+y^2)\, \rho(x,y)\, dA$$
$$= \underbrace{\iint_R y^2 \rho(x,y)\, dA}_{I_x} + \underbrace{\iint_R x^2 \rho(x,y)\, dA}_{I_y}$$

$$\therefore \quad \boxed{I_0 = I_x + I_y}$$

## Radius of gyration

For a point mass $m$, its moment of inertia about the axis of rotation is

$$I = mr^2$$




We would like to have the same formula which also applies to a rigid body. Suppose the rigid body has moment of inertia $I$ and mass $M$. Then we define

$$I = MR^2$$

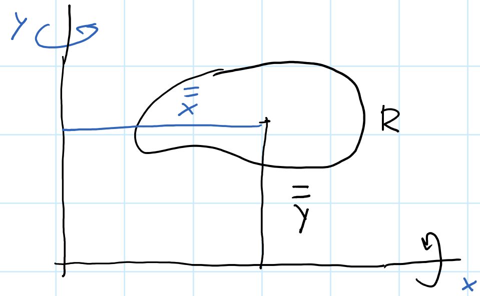
$$\therefore \quad R = \sqrt{\frac{I}{M}}$$

This distance $R$ is called the radius of gyration of the lamina $R$ about the axis of rotation.

Physically, a rotating rigid body is considered as a point mass $M$

rotating about the axis of rotation at a distance R without changing the rotational KE of the rigid body.

$$I_x = M\bar{\bar{y}}^2$$
$$I_y = M\bar{\bar{x}}^2$$




where $\bar{\bar{x}}$ is the radius of gyration of the lamina R about the y-axis and $\bar{\bar{y}}$ is the radius of gyration of the lamina R about the x-axis.

$\therefore$ $(\bar{\bar{x}}, \bar{\bar{y}})$ is the point at which the mass of the lamina can be concentrated without changing the moments of inertia wrt the coordinate axes.

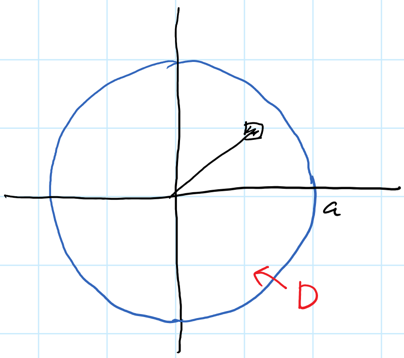
Ex 2: Find the moments of inertia $I_x$, $I_y$, $I_0$ of a homogeneous disk D with mass density $\rho(x,y) = \rho_0$ (a constant), center the origin, radius a.

Solution

$y = r\sin\theta$

$dA = r\,dr\,d\theta$

$$I_x = \iint_D y^2 \underbrace{\boxed{\rho(x,y)}}_{\rho_0}\, dA = \rho_0 \iint_D y^2\, dA$$




$$\underset{\text{polar coordinates}}{\overset{\uparrow}{=}} \rho_0 \int_0^{2\pi}\int_0^a \underbrace{\boxed{r^2\sin^2\theta\,(r)}}_{\text{separable}}\, dr\, d\theta$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \rho_0 \left(\int_0^{2\pi} \sin^2\theta\right)\left(\int_0^a r^3\, dr\right)$$

$$I_x = \rho_0\left(\frac{1}{2}\int_0^{2\pi}(1-\cos 2\theta)\,d\theta\right)\left(\frac{r^4}{4}\right)\Big|_0^a$$

$$= \rho_0\left(\frac{1}{2}\left(\theta\Big|_0^{2\pi} - \overset{0}{\cancel{\frac{1}{2}\sin 2\theta\Big|_0^{2\pi}}}\right)\right)\left(\frac{a^4}{4}\right)$$

$$= \rho_0\left(\frac{1}{\cancel{2}}(\cancel{2}\pi)\right)\frac{a^4}{4} = \frac{\pi\rho_0 a^4}{4} \quad // \text{Ans.}$$

By Symmetry,

$$I_y = I_x = \frac{\pi\rho_0 a^4}{4} \quad // \text{Ans.}$$

$$I_0 = I_x + I_y = \frac{\pi\rho_0 a^4}{2} \quad // \text{Ans.}$$

$$m = \iint_D \boxed{\rho(x,y)}\, dA = \rho_0 \iint_D dA = \rho_0 (\pi a^2) = \pi \rho_0 a^2$$

$\rho(x,y) = \rho_0$

$\iint_D dA =$ area of the disk

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{\cancel{\pi \rho_0 a^4}}{4} \cdot \frac{1}{\cancel{\pi \rho_0 a^2}}} = \sqrt{\frac{a^2}{4}} = \frac{a}{2} \quad // \text{Ans.}$$

By symmetry of the circle,

$$\bar{\bar{y}} = \bar{\bar{x}} = \frac{a}{2} \quad // \text{Ans.}$$

## Probability

A function $f$ is called a probability density of a continuous random variable $X$ if

(i) $f(x) \geq 0$ for $\forall x$

(ii) $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx = 1$

$\therefore$ The probability that $X$ lies between $a$ and $b$ is

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$

Consider a pair of continuous random variables $X$ and $Y$, such as the life times of two components of a machine or the height and weight of an adult female chosen at random. The joint density function of $X$ and $Y$ is a function $f$ of two variables such that the probability that $(X, Y)$ lies in a region $D$ is

$$P((X,Y) \in D) = \iint_D f(x,y)\, dA$$

In particular if $D$ is a rectangular region defined by $[a,b] \times [c,d]$ then

$$P(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

Again, f must satisfy

(i) $f(x,y) \geq 0$ for all $(X,Y)$

(ii) $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$

Expected Values: If X and Y are RVs with joint probability density function f, we define X-mean and Y-mean, also called the expected values of X and Y, to be

$$E[X] = \mu_X = \iint_{\mathbb{R}^2} x\, f(x,y)\,dA$$

$$E[Y] = \mu_Y = \iint_{\mathbb{R}^2} y\, f(x,y)\,dA$$

We note that $E[X]$ resembles the moment $M_y$

$E[Y]$ resembles the moment $M_x$

Ex 3 The joint probability density function for a pair of random variables X and Y is

$$f(x,y) = \begin{cases} Cx(1+y) & \text{if } 0 \leq x \leq 1,\ 0 \leq y \leq 2 \\ 0 & \text{, otherwise} \end{cases}$$

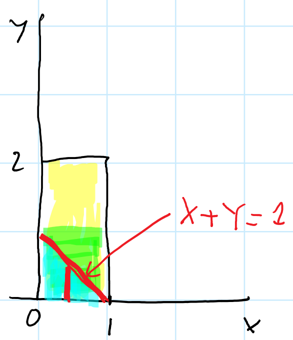
a) Find the value of C

b) Find $P(X \leq 1, Y \leq 1)$

c) Find $P(X + Y \leq 1)$

d) Find $E[X]$, $E[Y]$

Solution




$$\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$$

$$C\int_0^2 \left(\int_0^1 \boxed{x(1+y)}\,dx\right) dy = 1$$

$$C\left(\int_0^1 x\,dx\right)\left(\int_0^2 (1+y)\,dy\right) = 1$$

$$C\left(\frac{x^2}{2}\right)\Big|_0^1 \left(y + \frac{y^2}{2}\right)\Big|_0^2 = 1$$

$$c\left(\frac{x^2}{2}\right)\Big|_0^1\left(y+\frac{y^2}{2}\right)\Big|_0^2 = 1$$

$$c\left(\frac{1}{2}\right)(2+2) = 1 \Rightarrow 2c = 1 \Rightarrow \boxed{c = \frac{1}{2}} \quad // \text{Ans.}$$

b) $$P(X \leq 1, Y \leq 1) = \int_0^1 \int_0^1 \frac{x(1+y)}{2}\, dx\, dy$$

$$= \frac{1}{2}\left(\int_0^1 x\, dx\right)\left(\int_0^1 (1+y)\, dy\right)$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)\Big|_0^1 \left(y+\frac{y^2}{2}\right)\Big|_0^1$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) = \frac{3}{8} \quad // \text{Ans.}$$

c) $$P(X+Y \leq 1) = \int_0^1 \int_0^{1-x} \frac{x(1+y)}{2}\, dy\, dx$$

$$= \frac{1}{2}\int_0^1 x\left(y+\frac{y^2}{2}\right)\Big|_{y=0}^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 x\left[(1-x) + \frac{(1-x)^2}{2}\right] dx$$

$$= \frac{1}{2}\int_0^1 x\left[1 - x + \frac{1}{2} - x + \frac{x^2}{2}\right] dx$$

$$= \frac{1}{2}\int_0^1 x\left(\frac{3}{2} - 2x + \frac{x^2}{2}\right) dx$$

$$= \frac{1}{2}\int_0^1 \left(\frac{3}{2}x - 2x^2 + \frac{x^3}{2}\right) dx$$

$$= \frac{1}{2}\left(\frac{3}{4}x^2 - \frac{2}{3}x^3 + \frac{x^4}{8}\right)\Big|_0^1 = \frac{1}{2}\left(\frac{3}{4} - \frac{2}{3} + \frac{1}{8}\right) = \frac{5}{48} \quad // \text{Ans.}$$

d) $$E[X] = \iint_D x\, \frac{x(1+y)}{2}\, dA = \frac{1}{2}\int_0^2 \int_0^1 x^2(1+y)\, dx\, dy$$

$$= \frac{1}{2}\left(\int_0^1 x^2\, dx\right)\left(\int_0^2 (1+y)\, dy\right)$$

$$= \frac{1}{2}\left(\frac{x^3}{3}\right)\Big|_0^1 \left(y + \frac{y^2}{2}\right)\Big|_0^2 = \frac{1}{2}\left(\frac{1}{3}\right)(2+2) = \frac{2}{3} \quad // \text{Ans.}$$

$$E[Y] = \iint_D y\, \frac{x(1+y)}{2}\, dA = \frac{1}{2}\int_0^2 \int_0^1 x\,(y+y^2)\, dx\, dy$$

$$= \frac{1}{2}\left(\int^1 x\, dx\right)\left(\int^2 (y+y^2)\, dy\right)$$

$$= \frac{1}{2}\left(\int_0^1 x\,dx\right)\left(\int_0^2 (y+y^2)\,dy\right)$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)\Big|_0^1 \left(\frac{y^2}{2}+\frac{y^3}{3}\right)\Big|_0^2$$

$$= \frac{1}{2}\left(\frac{1}{2}\right)\left(2+\frac{8}{3}\right) = \frac{7}{6} \approx 1.17 \quad // \text{Ans.}$$

See you on Friday!