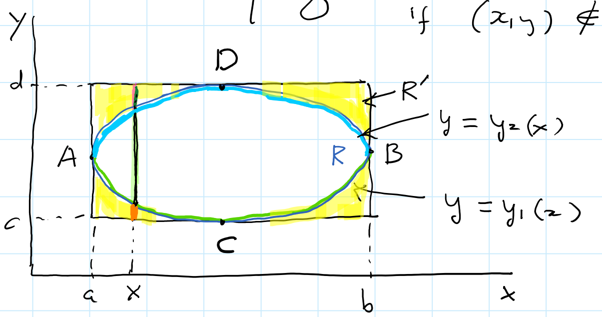
Lecture 11

October 2, 2020 11:07 AM

Double Integrals over a general region (sec 15.2)

Enclose R in a rectangle R' defined by $[a,b] \times [c,d]$ (or $a \le x \le b$, $c \le y \le d$) and define a new function $F(x,y)$ throughout R' as follows

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in R \\ 0 & \text{if } (x,y) \notin R \end{cases}$$



Let the lower arc $\widehat{ACB}$ be defined by $y = y_1(x)$ and the upper arc $\widehat{ADB}$ be defined by $y = y_2(x)$. Then

$$\iint_R f(x,y)\, dA = \iint_{R'} F(x,y)\, dA$$

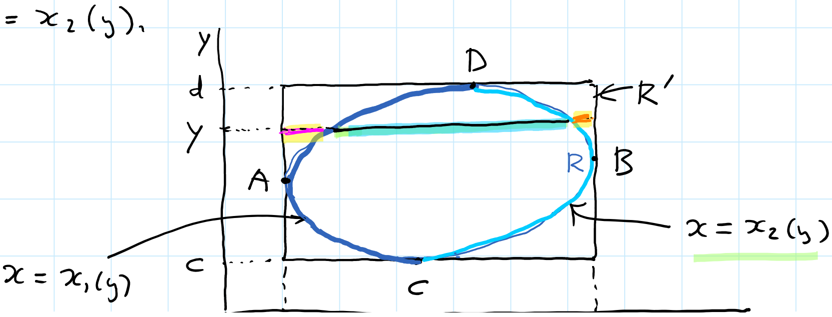
$$= \int_a^b \left(\int_c^d F(x,y)\, dy \right) dx$$

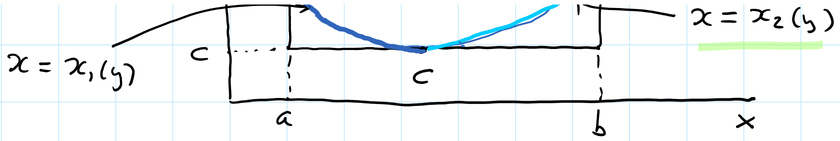
apply Fubini's theorem for a rectangular region

$$= \int_a^b \left[\int_c^{y_1(x)} \cancel{F(x,y)}^{\,0}\, dy + \int_{y_1(x)}^{y_2(x)} \underbrace{F(x,y)}_{=f(x,y)}\, dy + \int_{y_2(x)}^{d} \cancel{F(x,y)}^{\,0}\, dy \right] dx$$

$$= \int_a^b \left(\int_{y_1(x)}^{y_2(x)} f(x,y)\, dy \right) dx \qquad (I)$$

To reverse the order of integration, we let the left arc $\widehat{CAD}$ be defined as $x = x_1(y)$ and the right arc $\widehat{CBD}$ be defined as $x = x_2(y)$.





$$\iint_R f(x,y)\,dA = \iint_{R'} F(x,y)\,dA$$

$$= \int_c^d \left(\int_a^b F(x,y)\,dx \right) dy$$

apply Fubini's theorem to a rectangular region R'

$$= \int_c^d \left[\overbrace{\int_a^{x_1(y)} F(x,y)\,dx}^{0} + \int_{x_1(y)}^{x_2(y)} \overbrace{F(x,y)}^{f(x,y)}\,dx + \overbrace{\int_{x_2(y)}^{b} F(x,y)\,dx}^{0} \right] dy$$

$$= \int_c^d \left(\int_{x_1(y)}^{x_2(y)} f(x,y)\,dx \right) dy \qquad (\text{II})$$

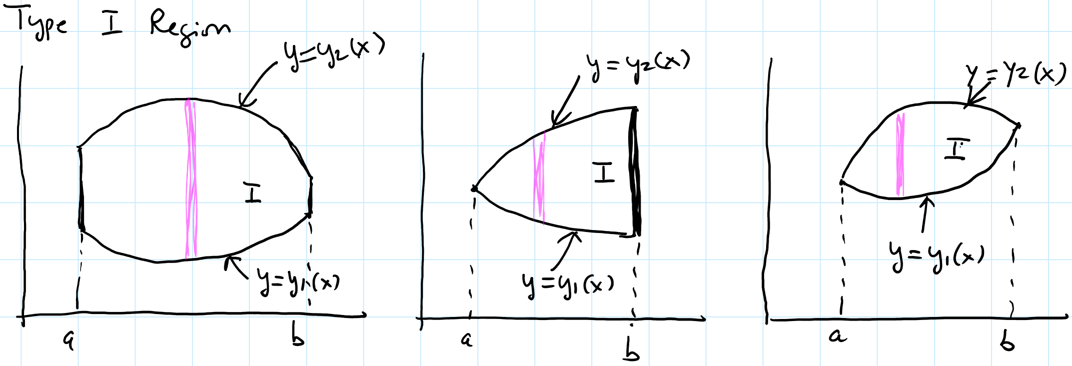
Combining (I) & (II),

$$\boxed{\iint_R f(x,y)\,dA = \int_a^b \left(\int_{y_1(x)}^{y_2(x)} f(x,y)\,dy \right) dx = \int_c^d \left(\int_{x_1(y)}^{x_2(y)} f(x,y)\,dx \right) dy} \qquad (\text{III})$$

which is called Fubini's theorem for a general region.

Region Classification

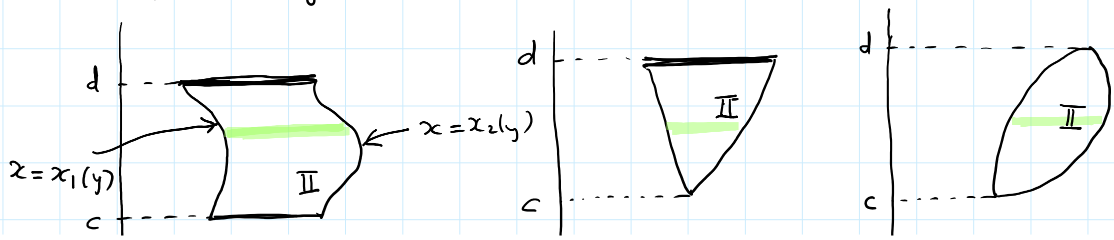
Type I Region

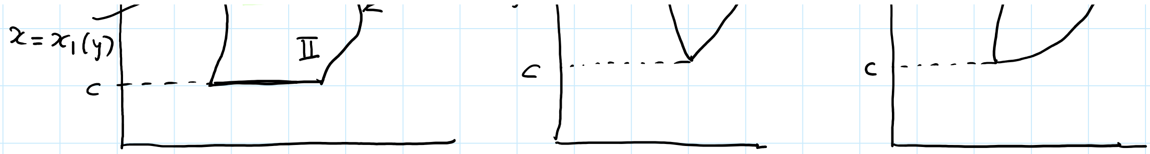


In this case, we integrate wrt y first, then x

$$\iint_R f(x,y)\,dA = \int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx$$

Type II Region



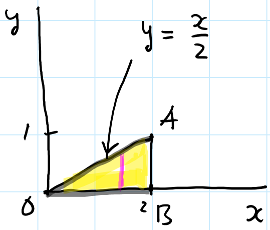


In this case, we integrate wrt x first, then y.

$$\iint_R f(x,y)\,dA = \int_c^d \left(\int_{x_1(y)}^{x_2(y)} f(x,y)\,dx \right) dy$$

Ex1: Evaluate $I = \iint_R (2x-3y)\,dA$ where R is the Δ with vertices $(0,0)$, $(2,1)$ and $(2,0)$.

Solution



R can be consider as type I region or type II region.

If we integrate y first then x, then

$$\iint_R (2x-3y)\,dA = \int_0^2 \int_{y=0}^{x/2} (2x-3y)\,dy\,dx$$

$$\iint_R (2x-3y)\,dA = \int_0^2 \left(2xy - \frac{3y^2}{2}\right)\Big|_{y=0}^{x/2} dx$$

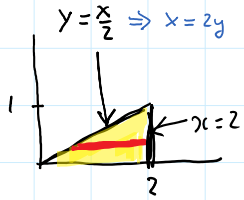
$$= \int_0^2 \left[\cancel{2}x\left(\frac{x}{\cancel{2}}\right) - \frac{3}{2}\left(\frac{x}{2}\right)^2 - 0 \right] dx$$

$$= \int_0^2 \left(x^2 - \frac{3}{8}x^2\right) dx = \int_0^2 \left(\frac{5}{8}x^2\right) dx = \frac{5}{8}\frac{x^3}{3}\Big|_0^2$$

$$= \frac{5}{\cancel{8}}\left(\frac{\cancel{8}}{3}\right) = \frac{5}{3} \quad // \text{Ans.}$$

If we reverse the order of integration, then

$y = \frac{x}{2} \Rightarrow x = 2y$



$$I = \int_0^1 \int_{2y}^{2} (2x-3y)\,dx\,dy$$

$$= \int_0^1 (x^2 - 3yx)\Big|_{x=2y}^{2} dy$$

$$I = \int_0^1 \left[(2)^2 - 3y(2) - \left((2y)^2 - 3y(2y) \right) \right] dy$$

The same!

$$= \int_0^1 \left(4 - 6y - (4y^2 - 6y^2)\right) dy = \int_0^1 (2y^2 - 6y + 4)\,dy$$

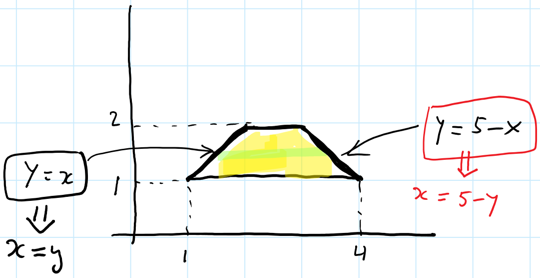
$$= \left(\frac{2}{3}y^3 - 3y^2 + 4y\right)\Big|_0^1 = \frac{2}{3} - 3 + 4 = 1 + \frac{2}{3} = \frac{5}{3} \quad //$$

Ans.

Ex2: Evaluate $I = \iint e^{x+3y}\,dA$ where R is the trapezoid

Ex2 : Evaluate $I = \iint_R e^{x+3y} \, dA$ where R is the trapezoid defined by $y=1$, $y=2$, $y=x$ and $y=5-x$.

Solution



We note that R is a type II region. Therefore, we integrate wrt x first then y.

$$I = \int_1^2 \int_y^{5-y} e^{x+3y} \, dx \, dy$$

$$I = \int_1^2 \left(e^{x+3y}\right)\Big|_{x=y}^{5-y} dy$$

$$= \int_1^2 \left(e^{5-y+3y} - e^{y+3y}\right) dy$$

$$= \int_1^2 \left(e^{5+2y} - e^{4y}\right) dy$$

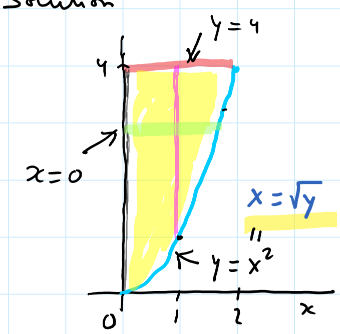
$$= \frac{e^{5+2y}}{2}\Big|_1^2 - \frac{e^{4y}}{4}\Big|_1^2$$

$$= \frac{e^{5+2(2)} - e^{5+2(1)}}{2} - \frac{e^{4(2)} - e^{4(1)}}{4}$$

$$= \frac{e^9 - e^7}{2} - \frac{e^8 - e^4}{4} \quad // \text{Ans.}$$

Ex3 : Evaluate $I = \iint_R x e^{y^2} \, dA$ where R is the region in the 1st quadrant bounded by $y=x^2$, $x=0$, $y=4$.

Solution



Let's integrate wrt y first

$$I = \int_0^2 \int_{x^2}^4 x e^{y^2} \, dy \, dx$$

$$= \int_0^2 x \left(\int_{x^2}^4 e^{y^2} \, dy\right) dx$$

Here, we have a problem because e^{y^2} does NOT have an antiderivative in a "closed form"! So we must switch the order of integration.

$$I = \int_0^4 \int_0^{\sqrt{y}} x e^{y^2} \, dx \, dy$$

$$I = \int_0^4 \int_0^{\sqrt{y}} x e^{y^2} \, dx \, dy$$

$$= \int_0^4 e^{y^2} \left(\int_0^{\sqrt{y}} x \, dx \right) dy$$

$$= \int_0^4 e^{y^2} \left(\frac{x^2}{2} \right) \Big|_{x=0}^{\sqrt{y}} dy$$

$$= \int_0^4 e^{y^2} \left(\frac{(\sqrt{y})^2}{2} - 0 \right) dy$$

$$= \int_0^4 y e^{y^2} dy$$

$$\boxed{u = y^2}$$

$$du = 2y \, dy \Rightarrow y \, dy = \frac{1}{2} du$$

$$= \int_0^{16} e^u \frac{1}{2} du$$

$$= \frac{1}{2} e^u \Big|_0^{16} = \frac{1}{2}(e^{16} - e^0) = \frac{1}{2}(e^{16} - 1) \quad // \text{Ans.}$$

See you on Monday!