Chapter 4: Relation

$\mathbb{R}^2 = \{(x,y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$
e.g. $(1,3)$, $(3,2)$
Note: $(1,3) \neq (3,1) \neq \{3,1\}$.
key property of ordered pairs: $(x_1,y_1) = (x_2,y_2)$ iff 1) $x_1 = x_2$, 2) $y_1 = y_2$.
Components can be anything: (Math, 2155), $(\{1,2\}, 2)$
* it is not required to be the same type of data.
The cartesian product $A \times B$
is $\{(a,b) \mid a \in A, b \in B\}$.
$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$.
e.g. $A = \{1,2\}$, $B = \{a,b\}$
$B \times A = \{(a,1), (a,2), (b,1), (b,2)\}$.
$A \times B$.size() = A.size $\cdot$ B.size.

Recall: $\{x \mid P(x)\}$ is the true set of $P(x)$.
Let $P(x,y)$ be a statement with two variables x, y where $x \in A, y \in B$.
the true set of $P(x,y)$ will be $A \times B$.

e.g. Let $D(x,y)$ be "$y \% x = 0$" for $x, y \in \mathbb{Z}$.
The true sets will be $\{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid y\%x=0\} = T$
e.g.2 Let $P(x,y)$ be "$y = 2x+3$". The true set will be $\{(x,y) \in \mathbb{R}^2 \mid y = 2x+3\}$.
=> It is the graph of this function.
e.g.3. $Q(c,p)$ be (course c, professor p)
$S = \{(c,p) \in \{\text{courses}\} \times \{\text{professors}\} \mid c \text{ is taught by } p\}$.

Facts: $T = \{(a,b) \in \mathbb{R}^2 \mid a \in A, b \in B\}$.
=> 1) $T \subseteq A \times B$. 2) $(a,b) \in T$ iff $P(a,b)$ holds.
3) if $P(a,b)$ is always true, $T = A \times B$
false, $T = \emptyset$
4) if $S = \{(x,y) \in \mathbb{R}^2 \mid x \in A, y \in B\}$, then
$P(x,y) \wedge S(x,y) = T \cap S$
$\vee$ $\cup$

=> these can apply to more than two variables, it can be triple or more.

Let A, B, C, D be sets.
1) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
2) $A \times (B \cup C) = (A \times B) \cup (A \times B)$
3) $(A \cap D) \times (B \cap C) = (A \times B) \cap (D \times C)$
if $D = A$
4) $(A \cup D) \times (B \cup C) \neq (A \times B) \cup (D \times C)$
5) $A \times \emptyset = \emptyset = \emptyset \times A$.

Proof of 2): Let $P \in A \times (B \cup C)$, then $P = (a,x)$. $a \in A$, $x \in B \cup C$.
=> if $x \in B$ => Then $P \in A \times B$
if $x \in C$ => $P \in A \times C$
$x \in B \cup C$ iff $(x \in B) \cup (x \in C)$
=> $(a,x) \in (A \times B) \cup (A \times C)$.
So $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.
in another direction: Let $P \in (A \times B) \cup (A \times C)$.

if $P \in A \times B \Rightarrow P = (a,b) \mid a \in A, b \in B$. then $b \in B \cup C$

so $P \in A \times B \in A \times (B \cup C)$

if $P \in A \times C \Rightarrow P = (a,c) \mid a \in A, c \in C$ then $c \in B \cup C$.

so $P \in A \times C \in A \times (B \cup C)$.

So $P \subseteq A \times (B \cup C)$.

$\Rightarrow A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Proof of 5: $A \times \emptyset = \{(a,b) \mid a \in A, \underbrace{b \in \emptyset}_{\text{it is always false.}}\}$.

$= \emptyset$.

$A \times B = B \times A$ iff 1) $A = \emptyset$

2) $B = \emptyset$ in other cases, $A \times B \neq B \times A$.

3) $A = B$.

Proof of 4: $(A \cup D) \times (B \cup C)$ $(x,y) \in A \cup D \times B \cup C$.

$x \in D \setminus A$ $\Rightarrow$ $(x,y) \notin A \times B$ $(x \notin A)$

$y \in B \setminus C$ $(x,y) \notin D \times C$ $(x \notin C)$

assume $A = \emptyset$, $C = \emptyset$. , $B = \{b\}$, $D = \{d\}$,

$(A \cup D) \times (B \cup C) = \{(b.d)\}$.

$A \times D = \emptyset$ $(A \times D) \cup (B \times C) = \emptyset$

$B \times C = \emptyset$

$\Rightarrow (A \cup D) \times (B \cup C) \nsubseteq (A \times B) \cup (D \times C)$.