1 Propositional Logic - Axioms and Inference Rules

Axioms

Axiom 1.1 [Commutativity]

$$
\begin{aligned}
(p \wedge q) &= (q \wedge p) \\
(p \vee q) &= (q \vee p) \\
(p = q) &= (q = p)
\end{aligned}
$$

Axiom 1.2 [Associativity]

$$
\begin{aligned}
p \wedge (q \wedge r) &= (p \wedge q) \wedge r \\
p \vee (q \vee r) &= (p \vee q) \vee r
\end{aligned}
$$

Axiom 1.3 [Distributivity]

$$
\begin{aligned}
p \vee (q \wedge r) &= (p \vee q) \wedge (p \vee r) \\
p \wedge (q \vee r) &= (p \wedge q) \vee (p \wedge r)
\end{aligned}
$$

Axiom 1.4 [De Morgan]

$$
\begin{aligned}
\neg(p \wedge q) &= \neg p \vee \neg q \\
\neg(p \vee q) &= \neg p \wedge \neg q
\end{aligned}
$$

Axiom 1.5 [Negation]

$$
\neg\neg p = p
$$

Axiom 1.6 [Excluded Middle]

$$
p \vee \neg p = T
$$

Axiom 1.7 [Contradiction]

$$p \wedge \neg p \;=\; F$$

Axiom 1.8 [Implication]

$$p \Rightarrow q \;=\; \neg p \vee q$$

Axiom 1.9 [Equality]

$$(p = q) \;=\; (p \Rightarrow q) \wedge (q \Rightarrow p)$$

Axiom 1.10 [or-simplification]

$$
\begin{aligned}
p \vee p &= p \\
p \vee T &= T \\
p \vee F &= p \\
p \vee (p \wedge q) &= p
\end{aligned}
$$

Axiom 1.11 [and-simplification]

$$
\begin{aligned}
p \wedge p &= p \\
p \wedge T &= p \\
p \wedge F &= F \\
p \wedge (p \vee q) &= p
\end{aligned}
$$

Axiom 1.12 [Identity]

$$p \;=\; p$$

Inference Rules

$$\frac{p_1 \;=\; p_2 \;\;,\;\; p_2 \;=\; p_3}{p_1 \;=\; p_3} \quad \text{Transitivity}$$

$$\frac{p_1 \;=\; p_2}{E(p_1) \;=\; E(p_2) \;\;,\;\; E(p_2) \;=\; E(p_1)} \quad \text{Substitution}$$

$$\frac{q_1 \;\;,\;\; q_2 \;\;,\;\; \ldots \;\;,\;\; q_n \;\;,\;\; q_1 \wedge q_2 \wedge \ldots \wedge q_n \Rightarrow (p_1 = p_2)}{E(p_1) \;=\; E(p_2) \;\;,\;\; E(p_2) \;=\; E(p_1)} \quad \text{Conditional Substitution}$$

2 Propositional Logic - Derived Theorems

Equivalence and Truth

Theorem 2.1 [Associativity of $=$]

$$((p \ = \ q) \ = \ r) \quad = \quad (p \ = \ (q \ = \ r))$$

Theorem 2.2 [Identity of $=$]

$$(T \ = \ p) \quad = \quad p$$

Theorem 2.3 [Truth]

$$T$$

Negation, Inequivalence, and False

Theorem 2.4 [Definition of F]

$$F \quad = \quad \neg T$$

Theorem 2.5 [Distributivity of $\neg$ over $=$]

$$\neg(p \ = \ q) \quad = \quad (\neg p \ = \ q)$$
$$(\neg p \ = \ q) \quad = \quad (p \ = \ \neg q)$$

Theorem 2.6 [Negation of F]

$$\neg F \quad = \quad T$$

Theorem 2.7 [Definition of ¬]

$$(\neg p \;=\; p) \;\;=\;\; F$$
$$\neg p \;\;=\;\; (p \;=\; F)$$

Disjunction

Theorem 2.8 [Distributivity of ∨ over =]

$$(p \;\vee\; (q \;=\; r)) \;\;=\;\; ((p \;\vee\; q) \;=\; (p \;\vee\; r))$$
$$((p \;\vee\; (q \;=\; r)) \;=\; (p \;\vee\; q)) \;\;=\;\; (p \;\vee\; r)$$

Theorem 2.9 [Distributivity of ∨ over ∨]

$$p \;\vee\; (q \;\vee\; r) \;=\; (p \;\vee\; q) \;\vee\; (p \;\vee\; r)$$

Conjunction

Theorem 2.10 [Mutual definition of ∧ and ∨]

$$(p \;\wedge\; q) \;\;=\;\; (p \;=\; (q \;=\; (p \;\vee\; q)))$$
$$(p \;\wedge\; q) \;\;=\;\; ((p \;=\; q) \;=\; (p \;\vee\; q))$$
$$((p \;\wedge\; q) \;=\; p) \;\;=\;\; (q \;=\; (p \;\vee\; q))$$
$$((p \;\wedge\; q) \;=\; (p \;=\; q)) \;\;=\;\; (p \;\vee\; q)$$
$$(((p \;\wedge\; q) \;=\; p) \;=\; q) \;\;=\;\; (p \;\vee\; q)$$

Theorem 2.11 [Distributivity of ∧ over ∧]

$$p \;\wedge\; (q \;\wedge\; r) \;=\; (p \;\wedge\; q) \;\wedge\; (p \;\wedge\; r)$$

Theorem 2.12 [Absorption]

$$p \wedge (\neg p \vee q) = p \wedge q$$
$$p \vee (\neg p \wedge q) = p \vee q$$

Theorem 2.13 [Distributivity of $\wedge$ over $=$]

$$
\begin{aligned}
(p \wedge q) &= ((p \wedge \neg q) = \neg p) \\
((p \wedge q) = (p \wedge \neg q)) &= \neg p \\
p \wedge (q = p) &= (p \wedge q)
\end{aligned}
$$

Theorem 2.14 [Replacement]

$$(p = q) \wedge (r = p) = (p = q) \wedge (r = q)$$

Theorem 2.15 [Definition of $=$]

$$(p = q) = (p \wedge q) \vee (\neg p \wedge \neg q)$$

Theorem 2.16 [Exclusive or]

$$\neg(p = q) = (\neg p \wedge q) \vee (p \wedge \neg q)$$

Implication

Theorem 2.17 [Definition of Implication]

$$
\begin{aligned}
(p \Rightarrow q) &= ((p \vee q) = q) \\
((p \Rightarrow q) = (p \vee q)) &= q \\
(p \Rightarrow q) &= ((p \wedge q) = p) \\
((p \Rightarrow q) = (p \wedge q)) &= p
\end{aligned}
$$

Theorem 2.18 [Contrapositive]

$$(p \Rightarrow q) \;=\; (\neg q \Rightarrow \neg p)$$

Theorem 2.19 [Distributivity of $\Rightarrow$ over $=$]

$$p \Rightarrow (q = r) \;=\; ((p \Rightarrow q) = (p \Rightarrow r))$$

Theorem 2.20 [Shunting]

$$p \wedge q \Rightarrow r \;=\; p \Rightarrow (q \Rightarrow r)$$

Theorem 2.21 [Elimination/Introduction of $\Rightarrow$]

$$
\begin{aligned}
p \wedge (p \Rightarrow q) \;&=\; p \wedge q \\
p \wedge (q \Rightarrow p) \;&=\; p \\
p \vee (p \Rightarrow q) \;&=\; T \\
p \vee (q \Rightarrow p) \;&=\; \neg q \vee p \\
(p \vee q) \Rightarrow (p \wedge q) \;&=\; (p = q) \\
p \Rightarrow F \;&=\; \neg p \\
F \Rightarrow p \;&=\; T
\end{aligned}
$$

Theorem 2.22 [Right Zero of $\Rightarrow$]

$$(p \Rightarrow T) \;=\; T$$

Theorem 2.23 [Left Identity of $\Rightarrow$]

$$(T \Rightarrow p) \;=\; p$$

Theorem 2.24 [Weakening/Strengthening]

$$
\begin{aligned}
p &\;\Rightarrow\; p \,\vee\, q \\
p \,\wedge\, q &\;\Rightarrow\; p \\
p \,\wedge\, q &\;\Rightarrow\; p \,\vee\, q \\
p \,\vee\, (q \,\wedge\, r) &\;\Rightarrow\; p \,\vee\, q \\
p \,\wedge\, q &\;\Rightarrow\; p \,\wedge\, (q \,\vee\, r)
\end{aligned}
$$

Theorem 2.25 [Modus Ponens]

$$
p \,\wedge\, (p \,\Rightarrow\, q) \;\;\Rightarrow\;\; q
$$

Theorem 2.26 [Proof by Cases]

$$
\begin{aligned}
(p \,\Rightarrow\, r) \,\wedge\, (q \,\Rightarrow\, r) &\;=\; (p \,\vee\, q \,\Rightarrow\, r) \\
(p \,\Rightarrow\, r) \,\wedge\, (\neg p \,\Rightarrow\, r) &\;=\; r
\end{aligned}
$$

Theorem 2.27 [Mutual Implication]

$$
(p \,\Rightarrow\, q) \,\wedge\, (q \,\Rightarrow\, p) \;\;=\;\; (p \,=\, q)
$$

Theorem 2.28 [Antisymmetry]

$$
(p \,\Rightarrow\, q) \,\wedge\, (q \,\Rightarrow\, p) \;\;\Rightarrow\;\; (p \,=\, q)
$$

Theorem 2.29 [Transitivity]

$$
\begin{aligned}
(p \,\Rightarrow\, q) \,\wedge\, (q \,\Rightarrow\, r) &\;\Rightarrow\; (p \,\Rightarrow\, r) \\
(p \,=\, q) \,\wedge\, (q \,\Rightarrow\, r) &\;\Rightarrow\; (p \,\Rightarrow\, r) \\
(p \,\Rightarrow\, q) \,\wedge\, (q \,=\, r) &\;\Rightarrow\; (p \,\Rightarrow\, r)
\end{aligned}
$$

Theorem 2.30 [Monotonicity of $\vee$]

$$
(p \,\Rightarrow\, q) \;\;\Rightarrow\;\; (p \,\vee\, r \,\Rightarrow\, q \,\vee\, r)
$$

Theorem 2.31 [Monotonicity of $\wedge$]

$$(p \Rightarrow q) \quad \Rightarrow \quad (p \wedge r \Rightarrow q \wedge r)$$

Substitution

Theorem 2.32 [Leibniz]

$$(e = f) \quad \Rightarrow \quad (E(e) = E(f))$$

Theorem 2.33 [Substitution]

$$
\begin{aligned}
(e = f) \wedge E(e) &= (e = f) \wedge E(f) \\
(e = f) \Rightarrow E(e) &= (e = f) \Rightarrow E(f) \\
q \wedge (e = f) \Rightarrow E(e) &= q \wedge (e = f) \Rightarrow E(f)
\end{aligned}
$$

Theorem 2.34 [Replace by T]

$$
\begin{aligned}
p \wedge E(p) &= p \wedge E(T) \\
p \Rightarrow E(p) &= p \Rightarrow E(T) \\
q \wedge p \Rightarrow E(p) &= q \wedge p \Rightarrow E(T)
\end{aligned}
$$

Theorem 2.35 [Replace by F]

$$
\begin{aligned}
p \vee E(p) &= p \vee E(F) \\
E(p) \Rightarrow p &= E(F) \Rightarrow p \\
E(p) \Rightarrow p \vee q &= E(F) \Rightarrow p \vee q
\end{aligned}
$$

Theorem 2.36 [Shannon]

$$E(p) \quad = \quad (p \wedge E(T)) \vee (\neg p \wedge E(F))$$

3 Propositional Logic - Examples and Exercises

4 Predicate Logic - Axioms

Axiom 4.1 [Definition of $\exists$]

$$(m \geq n) \quad \Rightarrow \quad \left(\begin{array}{c} \exists i \; : \; m \leq i < n \; : \; p_i \\ = \\ F \end{array} \right)$$

$$(m < n) \quad \Rightarrow \quad \left(\begin{array}{c} \exists i \; : \; m \leq i < n \; : \; p_i \\ = \\ \left(\begin{array}{c} \exists i \; : \; m \leq i < n - 1 \; : \; p_i \\ \vee \\ p_{n-1} \end{array} \right) \end{array} \right)$$

Axiom 4.2 [Definition of $\forall$]

$$(m \geq n) \quad \Rightarrow \quad \left(\begin{array}{c} \forall i \; : \; m \leq i < n \; : \; p_i \\ = \\ T \end{array} \right)$$

$$(m < n) \quad \Rightarrow \quad \left(\begin{array}{c} \forall i \; : \; m \leq i < n \; : \; p_i \\ = \\ \left(\begin{array}{c} \forall i \; : \; m \leq i < n - 1 \; : \; p_i \\ \wedge \\ p_{n-1} \end{array} \right) \end{array} \right)$$

Axiom 4.3 [Range Split]

$$(m_1 \leq m_2 \leq m_3) \quad \Rightarrow \quad \left(\begin{pmatrix} \forall i \ : \ m_1 \leq i < m_2 \ : \ p_i \\ \wedge \\ \forall i \ : \ m_2 \leq i < m_3 \ : \ p_i \end{pmatrix} \\ = \\ \forall i \ : \ m_1 \leq i < m_3 \ : \ p_i \right)$$

$$(m_1 \leq m_2) \wedge (n_1 \geq n_2) \quad \Rightarrow \quad \left(\begin{pmatrix} \forall i \ : \ m_1 \leq i < n_1 \ : \ p_i \\ \wedge \\ \forall i \ : \ m_2 \leq i < n_2 \ : \ p_i \end{pmatrix} \\ = \\ \forall i \ : \ m_1 \leq i < n_1 \ : \ p_i \right)$$

$$(m_1 \leq m_2 \leq m_3) \quad \Rightarrow \quad \left(\begin{pmatrix} \exists i \ : \ m_1 \leq i < m_2 \ : \ p_i \\ \vee \\ \exists i \ : \ m_2 \leq i < m_3 \ : \ p_i \end{pmatrix} \\ = \\ \exists i \ : \ m_1 \leq i < m_3 \ : \ p_i \right)$$

$$(m_1 \leq m_2) \wedge (n_1 \geq n_2) \quad \Rightarrow \quad \left(\begin{pmatrix} \exists i \ : \ m_1 \leq i < n_1 \ : \ p_i \\ \vee \\ \exists i \ : \ m_2 \leq i < n_2 \ : \ p_i \end{pmatrix} \\ = \\ \exists i \ : \ m_1 \leq i < n_1 \ : \ p_i \right)$$

Axiom 4.4 [Interchange of Dummies]

$$\forall i \ : \ m_1 \leq i < n_1 \ : \ (\forall j \ : \ m_2 \leq j < n_2 \ : \ p_{i,j})$$
$$=$$
$$\forall j \ : \ m_2 \leq j < n_2 \ : \ (\forall i \ : \ m_1 \leq i < n_1 \ : \ p_{i,j})$$

$$\exists i \ : \ m_1 \leq i < n_1 \ : \ (\exists j \ : \ m_2 \leq j < n_2 \ : \ p_{i,j})$$
$$=$$
$$\exists j \ : \ m_2 \leq j < n_2 \ : \ (\exists i \ : \ m_1 \leq i < n_1 \ : \ p_{i,j})$$

Axiom 4.5 [Dummy Renaming]

$$\forall i \; : \; m \leq i < n \; : \; p_i \;\; = \;\; \forall j \; : \; m \leq j < n \; : \; p_j$$

Axiom 4.6 [Distributivity of $\vee$ over $\forall$]

$$(p \; \vee \; (\forall i \; : \; m \leq i < n \; : \; q_i)) \;\; = \;\; \forall i \; : \; m \leq i < n \; : \; (p \; \vee \; q_i)$$

Axiom 4.7 [Distributivity of $\wedge$ over $\forall$]

$$(m < n) \quad \Rightarrow \quad \left(\begin{array}{c} p \; \wedge \; (\forall i \; : \; m \leq i < n \; : \; q_i) \\ = \\ \forall i \; : \; m \leq i < n \; : \; p \; \wedge \; q_i \end{array} \right)$$

$$\left(\begin{array}{c} \forall i \; : \; m \leq i < n \; : \; p_i \\ \wedge \\ \forall i \; : \; m \leq i < n \; : \; q_i \end{array} \right) \;\; = \;\; \forall i \; : \; m \leq i < n \; : \; (p_i \; \wedge \; q_i)$$

Axiom 4.8 [Distributivity of $\wedge$ over $\exists$]

$$(p \; \wedge \; (\exists i \; : \; m \leq i < n \; : \; q_i)) \;\; = \;\; \exists i \; : \; m \leq i < n \; : \; (p \; \wedge \; q_i)$$

Axiom 4.9 [Distributivity of $\vee$ over $\exists$]

$$(m < n) \quad \Rightarrow \quad \left(\begin{array}{c} p \; \vee \; (\exists i \; : \; m \leq i < n \; : \; q_i) \\ = \\ \exists i \; : \; m \leq i < n \; : \; (p \; \vee \; q_i) \end{array} \right)$$

$$\left(\begin{array}{c} \exists i \; : \; m \leq i < n \; : \; p_i \\ \vee \\ \exists i \; : \; m \leq i < n \; : \; q_i \end{array} \right) \;\; = \;\; \exists i \; : \; m \leq i < n \; : \; (p_i \; \vee \; q_i)$$

Axiom 4.10 [Universality of T]

$$\forall i \; : \; m \leq i < n \; : \; T \;\; = \;\; T$$

Axiom 4.11 [Existence of F]

$$\exists i \; : \; m \leq i < n \; : \; F \;\; = \;\; F$$

Axiom 4.12 [Generalized De Morgan]

$$\neg\,(\exists i \; : \; m \leq i < n \; : \; p_i) \;\; = \;\; \forall i \; : \; m \leq i < n \; : \; \neg p_i$$
$$\neg\,(\forall i \; : \; m \leq i < n \; : \; p_i) \;\; = \;\; \exists i \; : \; m \leq i < n \; : \; \neg p_i$$

Axiom 4.13 [Trading]

$$(m \leq i < n) \;\Rightarrow\; p_i \;\; = \;\; \forall i \; : \; m \leq i < n \; : \; p_i$$
$$(m \leq i < n) \;\wedge\; p_i \;\; \Rightarrow \;\; \exists i \; : \; m \leq i < n \; : \; p_i$$

Axiom 4.14 [Definition of Numerical Quantification]

$$(m \geq n) \;\; \Rightarrow \;\; \left(\begin{array}{c} \mathcal{N}i \; : \; m \leq i < n \; : \; p_i \\ = \\ 0 \end{array} \right)$$

$$(m < n) \;\wedge\; \neg p_{n-1} \;\; \Rightarrow \;\; \left(\begin{array}{c} \mathcal{N}i \; : \; m \leq i < n \; : \; p_i \\ = \\ \mathcal{N}i \; : \; m \leq i < n-1 \; : \; p_i \end{array} \right)$$

$$(m < n) \;\wedge\; p_{n-1} \;\; \Rightarrow \;\; \left(\begin{array}{c} \mathcal{N}i \; : \; m \leq i < n \; : \; p_i \\ = \\ \left(\begin{array}{c} \mathcal{N}i \; : \; m \leq i < n-1 \; : \; p_i \\ + \\ 1 \end{array} \right) \end{array} \right)$$

Axiom 4.15 [Definition of Σ]

$$(m \geq n) \quad \Rightarrow \quad \left(\begin{array}{c} \Sigma i \ : \ m \leq i < n \ : \ e_i \\ = \\ 0 \end{array} \right)$$

$$(m < n) \quad \Rightarrow \quad \left(\begin{array}{c} \Sigma i \ : \ m \leq i < n \ : \ e_i \\ = \\ \left(\begin{array}{c} \Sigma i \ : \ m \leq i < n - 1 \ : \ e_i \\ + \\ e_{n-1} \end{array} \right) \end{array} \right)$$

Axiom 4.16 [Definition of Π]

$$(m \geq n) \quad \Rightarrow \quad \left(\begin{array}{c} \Pi i \ : \ m \leq i < n \ : \ e_i \\ = \\ 1 \end{array} \right)$$

$$(m < n) \quad \Rightarrow \quad \left(\begin{array}{c} \Pi i \ : \ m \leq i < n \ : \ e_i \\ = \\ \left(\begin{array}{c} \Pi i \ : \ m \leq i < n - 1 \ : \ e_i \\ * \\ e_{n-1} \end{array} \right) \end{array} \right)$$

5   Predicate Logic - Derived Theorems

Theorem 5.1 [Definition of $\exists$]

$$(m \geq n) \quad \Rightarrow \quad \left(\begin{array}{c} \exists i \ : \ m < i \leq n \ : \ p_i \\ = \\ F \end{array} \right)$$

$$(m < n) \quad \Rightarrow \quad \left(\begin{array}{c} \exists i \ : \ m < i \leq n \ : \ p_i \\ = \\ \left(\begin{array}{c} \exists i \ : \ m + 1 < i \leq n \ : \ p_i \\ \vee \\ p_{m+1} \end{array} \right) \end{array} \right)$$

Theorem 5.2 [Definition of $\forall$]

$$(m \geq n) \quad \Rightarrow \quad \left(\begin{array}{c} \forall i \ : \ m < i \leq n \ : \ p_i \\ = \\ T \end{array} \right)$$

$$(m < n) \quad \Rightarrow \quad \left(\begin{array}{c} \forall i \ : \ m < i \leq n \ : \ p_i \\ = \\ \left(\begin{array}{c} \forall i \ : \ m + 1 < i \leq n \ : \ p_i \\ \wedge \\ p_{m+1} \end{array} \right) \end{array} \right)$$

Theorem 5.3 [Definition of Σ]

$$(m \geq n) \quad \Rightarrow \quad \left(\begin{array}{c} \Sigma i \ : \ m < i \leq n \ : \ e_i \\ = \\ 0 \end{array} \right)$$

$$(m < n) \quad \Rightarrow \quad \left(\begin{array}{c} \Sigma i \ : \ m < i \leq n \ : \ e_i \\ = \\ \left(\begin{array}{c} \Sigma i \ : \ m+1 < i \leq n \ : \ e_i \\ + \\ e_{m+1} \end{array} \right) \end{array} \right)$$

Theorem 5.4 [Definition of Π]

$$(m \geq n) \quad \Rightarrow \quad \left(\begin{array}{c} \Pi i \ : \ m < i \leq n \ : \ e_i \\ = \\ 1 \end{array} \right)$$

$$(m < n) \quad \Rightarrow \quad \left(\begin{array}{c} \Pi i \ : \ m < i \leq n \ : \ e_i \\ = \\ \left(\begin{array}{c} \Pi i \ : \ m+1 < i \leq n \ : \ e_i \\ * \\ e_{m+1} \end{array} \right) \end{array} \right)$$

6   Some Simple Laws of Arithmetic

Throughout this compendium, we assume the validity of all "simple" arithmetic rules. Examples of such rules are all *simplification rules*, e.g. =

$$2 + 3 = 5$$
$$x + x = 2 * x$$
$$x + y - y = x$$
$$(x/3) * 3 = x$$
$$0 * x = 0$$
$$1 * x = x$$
$$x * x = x^2$$
$$0^x = 0$$
$$1^x = 1$$
$$(2 * x + 10 = 20) = (x = 5)$$
$$(x + y < 2 * y) = (x < y)$$
$$(x + y = x + z) = (y = z)$$
$$x * (y + 1) - (x + z) = (x * y - z)$$

Following is a collection of theorems that might be used. The list is not exhaustive but intend to show the level of complexity that you can specify theorems on.

Theorems on $<$ and $\leq$

$$
\begin{aligned}
(x < y) \quad &= \quad (y > x) \\
(x < y) \quad &\Rightarrow \quad \neg(y = x) \wedge \neg(y < x) \\
(x < y) \quad &\Rightarrow \quad (x \leq y) \\
(x < y) \quad &= \quad (x \leq y) \wedge (x \neq y) \\
(x < y) \wedge (y \leq z) \quad &\Rightarrow \quad (x < z) \\
(x \leq y) \quad &= \quad \neg(x > y) \\
(x \leq x) \quad &= \quad T \\
(x \leq y) \wedge (y \leq z) \quad &\Rightarrow \quad (x \leq z) \\
(x \leq y) \wedge (y < z) \quad &\Rightarrow \quad (x < z) \\
(x \leq y) \wedge \neg(x < y) \quad &\Rightarrow \quad (x < y) \\
(x \leq y - 1) \quad &= \quad (x < y) \\
(x \leq y) \vee (y \leq x) \quad &= \quad T \\
(x \leq y) \vee (y < x) \quad &= \quad T \\
(x \leq y) \quad &= \quad (x < y) \vee (x = y)
\end{aligned}
$$

Theorems on properties about $+$ and $-$

$$
\begin{aligned}
(x < y) \quad &\Rightarrow \quad (x < y + 1) \\
(x < y + 1) \quad &= \quad (x \leq y) \\
(x < y) \quad &\Rightarrow \quad (z - y < z - x) \\
(0 < x) \quad &= \quad (-x < 0) \\
(x - 1 < x) \quad &= \quad T \\
(x \leq y - 1) \quad &= \quad (x < y) \\
(x \leq y) \quad &= \quad (x - 1 < y) \\
(x_1 < y_1) \wedge (x_2 < y_2) \quad &\Rightarrow \quad (x_1 + x_2 < y_1 + y_2) \\
(x_1 \leq y_1) \wedge (x_2 \leq y_2) \quad &\Rightarrow \quad (x_1 + x_2 \leq y_1 + y_2)
\end{aligned}
$$

Theorems on properties about $*$ and $/$

$$
\begin{aligned}
(0 < x) &= (0 < 2 * x) \\
(0 < x) &= (x < 2 * x) \\
(0 < x) &= (x \div 2 < x) \\
(0 \leq x/2) &= (0 \leq x) \\
(x = 0) &\Rightarrow (x * y = 0) \\
2 * (x/2) &= x
\end{aligned}
$$

Theorems on equivalence relation

$$
\begin{aligned}
(x = x) &= T \\
(x = y) &= (x \leq y) \wedge (y \leq x)
\end{aligned}
$$

Theorems about odd(n) and even(n)

$$
\begin{aligned}
\mathrm{odd}(x) &\Rightarrow ((x - 1) \div 2 = (x - 1)/2) \\
\mathrm{even}(x) &\Rightarrow (x \div 2 = x/2) \\
\mathrm{odd}(x + 2 * y) &= \mathrm{odd}(x) \\
\mathrm{even}(x + 2 * y) &= \mathrm{even}(x) \\
\mathrm{odd}(x) &= \neg\mathrm{even}(x) \\
\mathrm{odd}(x) &\Rightarrow ((x \geq 1) = (x \geq 0)) \\
\mathrm{odd}(x) \wedge (x = 0) &= F
\end{aligned}
$$

7   Predicate Logic - Examples and Exercises

8   Arrays - Axioms

8.1   Axioms

Axiom 8.1 [Assignment to Array Element]

$$((b;\ i:e)\,[j]\ =\ f)\ \ =\ \ \left(\begin{array}{c} (i=j)\ \Rightarrow\ (e\ =\ f) \\ \wedge \\ (i\neq j)\ \Rightarrow\ (b[j]\ =\ f) \end{array} \right)$$

Axiom 8.2 [Definition of Arithmetic Relations]

$$
\begin{array}{rcl}
(b[i:j]\ =\ x) & = & (\forall k\ :\ i \le k < j+1\ :\ b[k]\ =\ x) \\
(b[i:j]\ <\ x) & = & (\forall k\ :\ i \le k < j+1\ :\ b[k]\ <\ x) \\
(b[i:j]\ >\ x) & = & (\forall k\ :\ i \le k < j+1\ :\ b[k]\ >\ x) \\
(b[i:j]\ \le\ x) & = & (\forall k\ :\ i \le k < j+1\ :\ b[k]\ \le\ x) \\
(b[i:j]\ \ge\ x) & = & (\forall k\ :\ i \le k < j+1\ :\ b[k]\ \ge\ x) \\
(b[i:j]\ \neq\ x) & = & (\forall k\ :\ i \le k < j+1\ :\ b[k]\ \neq\ x) \\
x \in b[i:j] & = & (\exists k\ :\ i \le k < j+1\ :\ x\ =\ b[k])
\end{array}
$$