PROVE THAT $\forall n \in \mathbb{N}$

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$$

- BASE CASE: FOR $\boxed{n=0}$, $\sum_{i=0}^{0} x^i = x^0 = 1$ ✓

RHS: $\frac{x^{0+1}-1}{x-1} = \frac{x^1-1}{x-1} = 1$

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- INDUCTION STEP: ASSUME THAT, FOR $k \in \mathbb{N}$

$$\boxed{\sum_{i=0}^{k} x^i} = \boxed{\frac{x^{k+1}-1}{x-1}} \quad (\text{I.H})$$

WANT: $\sum_{i=0}^{k+1} x^i = \frac{x^{k+1+1}-1}{x-1} = \frac{x^{k+2}-1}{x-1}$

$$\sum_{i=0}^{k+1} x^i = \boxed{\sum_{i=0}^{k} x^i} + x^{k+1} \overset{IH}{=} \boxed{\frac{x^{k+1}-1}{x-1}} + x^{k+1} =$$

$$= \frac{x^{k+1}-1+(x-1)\cdot x^{k+1}}{x-1} = \frac{\cancel{x^{k+1}}-1+x^{k+2}\cancel{-x^{k+1}}}{x-1}$$ ✓

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- CONCLUSION: BY INDUCTION, $\forall n \in \mathbb{N}$

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$$

LET $F(0) = 0$, $F(1) = 1$,

$$F(n+2) = F(n) + F(n+1) \quad \forall n \in \mathbb{N}$$

PROVE: $\forall n \in \mathbb{N} \setminus \{0\} \; F(n-1) \cdot F(n+1) - F(n)^2 = (-1)^n$

- BASE CASE: $\boxed{n=1}$: $F(0)\,F(2) - F(1)^2 =$

$= 0 \cdot (0+1) - 1^2 = 0 - 1 = (-1)^1$ ✓

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- IND. STEP: ASSUME, FOR A $\underline{k \in \mathbb{N} \setminus \{0\}}$, $\underline{[k \geq 1]}$

$\rightarrow \boxed{F(k-1)\,F(k+1) - F(k)^2} = (-1)^k$ (I.H.)

WANT: $F(k+1-1)\,F(k+1+1) - F(k+1)^2 = (-1)^{k+1}$

$F(k) \cdot \underline{F(k+2)} - F(k+1)^2 = (-1)^{k+1}$

$F(k+1) = F(k) + F(k-1)$

$-F(k-1) = F(k) - F(k+1)$

$F(k)\big(F(k) + F(k+1)\big) - F(k+1)^2 =$

$F(k)^2 + F(k)\,\underline{F(k+1)} - \underline{F(k+1)}^2 =$

$F(k)^2 + F(k+1)\underbrace{\big(F(k) - F(k+1)\big)}_{-F(k-1)}$

$F(k)^2 - F(k+1)\,F(k-1) = -\boxed{\big(F(k+1)\,F(k-1) - F(k)^2\big)}$

$\overset{\text{I.H.}}{=} -\boxed{(-1)^k} = (-1)^{k+1}$ ✓

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- CONCLUSION: BY INDUCTION, $\underbrace{\forall n \in \mathbb{N} \setminus \{0\}}_{\text{EQUIV.}}$

$F(n-1)\,F(n+1) - F(n)^2 = (-1)^n$

$\forall n \in \mathbb{N},\ n \geq 1$