Sequences and Series. (Chapter 11).

数列时间…… 我又可以了.

$a_1, a_2, a_3 \cdots\cdots a_n$.

first term, second term, the n-th term of the sequences.

If a sequence does not have a last one, (goes forever), then it is called an infinite sequences.

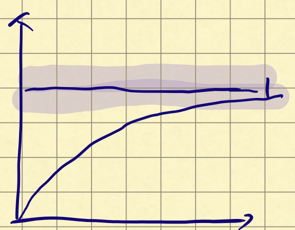
the sequence $a_1, a_2 \cdots\cdots a_n$ is denoted as

$\{a_1, a_2, a_3 \cdots\cdots\}$ or $\{a_n\}$ or $\{a_n\}_{n\geq 1}^{\infty}$

e.g. 1. $\{n\} = \{1, 2, 3, \cdots\cdots\}$.

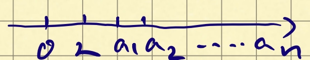
2. $\{\sqrt{n}\} = \{1, \sqrt{2}, \sqrt{3}, \cdots\cdots\}$.

$\frac{n}{n+1}$ <= the sequence converses to 1 as $n \to \infty$.



when $n \geq 100$, all points are inside the strip $(L-\varepsilon, L+\varepsilon)$.

The sequence $\{a_n\}$ is bounded below / above of M; if $a_n \geq L$ / $\leq$ for any value of n.

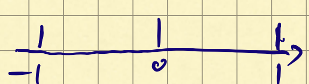


$\Downarrow$

$|a_n| \leq k$ $(-k \leq a_n \leq k)$.

e.g. $\left\{ \frac{(-1)^{n+1}}{n+1} \right\}$ is a bounded sequence.

k=1. all a_n is inside $[-1, 1]$.



(ii) the sequence $\{a_n\}$ is $\begin{matrix}+ve\\ -ve\end{matrix}$ if all a_n is $\begin{matrix}\geq 0\\ \leq 0\end{matrix}$

(iii) $\{a_n\}$ is $\begin{matrix}\text{increasing}\\ \text{decreasing}\end{matrix}$ if a_{n+1} $\begin{matrix}\geq\\ \leq\end{matrix}$ a_n.

(iv) $\{a_n\}$ is alternating if any two con terms. have opposit signs. e.g. $\left\{\frac{(-1)^{n+1}}{n}\right\}$.

For $\{a_n\}$ and $\{b_n\}$.

(i). $\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.

$\qquad\qquad - \qquad = \qquad -$

(ii) $k\lim_{n\to\infty} a_n = \lim_{n\to\infty} ka_n$.

(iii) $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.

$\qquad\qquad / \qquad = \qquad /$

(iv) If $a_n \leq b_n$ for any value of n, then $\lim_{n\to\infty} a_n < \lim_{n\to\infty} b_n$.

(v) If $a_n < c_n < b_n$ for any values of n. and if $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = L$ then $\lim_{n\to\infty} c_n = L$ (squeeze).

$a_n = \frac{2n}{n^2+1}$ $\qquad a_n = \frac{(n!)^2}{(2n)!} =$

$\lim_{n\to\infty} a_n = 0$.

The monotonic sequence: either decreasing or increasing.

Infinite Series (Sec 11.2).

Consider the infinite sequence. $\{a_n\}.=\{a_1, a_2 \cdots a_n\}$.

Infinity sum $a_1+a_2+\cdots+a_m$.

$\sum_{n=1} a_n$. we define a new sequence,

$\{S_n\}$ which is called the sequence of partial sums in which $S_n = \sum_{n=1}^{n} a_n$.

e.g. $S_n = \frac{a(1-r^n)}{1-r}$

$|r|<1 \Rightarrow \lim_{n\to\infty} r^n = 0 \Rightarrow$ Converge to $\frac{a}{1-r}$.

$|r|>1 \Rightarrow \lim_{n\to\infty} |r^n| = \infty \Rightarrow |S_n| \to \infty$

divergent.

$r=1 \Rightarrow S_n = a+a+\cdots a = na \quad \lim_{n\to\infty} S_n = \infty$

divergent

$r=-1 \Rightarrow S_n = a-a+a \cdots\cdots \begin{cases} \text{if } n\%2=1 & S_n = a \\ \text{if } n\%2=0 & S_n = 0 \end{cases}$

divergent

e.g. the value of $0.\dot{9}$

$x = 9\cdot\frac{1}{10} + 9\cdot\frac{1}{10}^2 + 9\cdot\frac{1}{10}^3 + \cdots + 9\cdot\frac{1}{10}^n$.

$a = \frac{9}{10} \quad r = \frac{1}{10} < 1$

Hence, it has a limit x which is

$$x = \frac{a}{1-r} = \frac{\frac{9}{10}}{1-\frac{1}{10}} = 1$$

e.g. $\sum_{n=0}^{\infty} \frac{3+2^n}{3^{n+2}}$

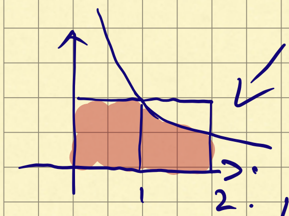
$= \sum_{n=0}^{\infty} \left(\frac{1}{3}\cdot\frac{1}{3^n}\right) + \sum_{n=0}^{\infty} \left(\frac{1}{9}\cdot\left(\frac{2}{3}\right)^n\right)$.

$= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n + \frac{1}{9}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$.

Telescoping Series & Hamonic Series.

Telescoping Series : e.g. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

Hamonic Series: e.g. $\sum_{n=1}^{\infty} \frac{1}{n} \cdot =$



$$\int_1^{1+\frac{1}{n}} \frac{1}{x} dx \le 1 \cdot \frac{1}{n}.$$

$$\ln x \Big|_1^{1+\frac{1}{n}} \le \frac{1}{n}.$$

$$\ln(1+\frac{1}{n}) - \ln 1 \le \frac{1}{n}.$$

$$\ln(1+n) - \ln n \le \frac{1}{n}.$$

$n=1$: $\ln 2 - \ln 1 \le 1$
$n=2$: $\ln 3 - \ln 2 \le \frac{1}{2}$.
$n=3$: $\ln 4 - \ln 3 \le \frac{1}{3}$
$\vdots$
$n=n$: $\ln(n+1) - \ln n \le \frac{1}{n}$.

$$\Rightarrow \ln(n+1) - \ln 1 \le 1 + \cdots + \frac{1}{n}.$$

$$\ln(n+1) \le 1 + \cdots + \frac{1}{n}.$$

If $\sum_{n=1}^{\infty} a_n$ convereage then $\lim_{n\to\infty} a_n = 0$.

$$\lim(S_n - S_{n-1}) = \lim_{n\to\infty} a_\cdot = 0.$$

e.g. Determine $\frac{n}{n+1}$ is converenge or diverses?

$$\frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \quad n\to\infty \Rightarrow 1 \ne 0. \Rightarrow \text{diverses}.$$

If $\sum a$ and $\sum b$ are convergent which converse to A, B

(i) $\sum c a_n = CA$

(ii) $\sum (a_n + b_n) = A + B$

(iii) if $a_n \le b_n$ for any value of n, then $A \le B$.

(11.3). Intergral Test.

Suppose that $a_n = f(n)$ where $f(x)$ is +ve, continuous, descending on an interval $[N, \infty)$

Then $\sum_{n=1}^{\infty} a_n$ convergent if $\int_N^{\infty} f(x)$ convergent
divergent divergent.

e.g. Determine whether $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ convergent or divergent.

Let $f(x) = \frac{1}{x\ln x} \geq 0$ for all $x \geq 2$

$$f'(x) = -\frac{1}{(x\ln x)^2} \cdot (1+\ln x) < 0 \Rightarrow \text{divergent.}$$

$$\int_2^{\infty} \frac{1}{x\ln x} dx = \lim_{b\to\infty} \frac{1}{x\ln x} dx.$$

$u = \ln x.$ $\quad = \ln|u| = \ln(\ln x)|.$

$du = \frac{dx}{x}.$

$$= \lim_{n\to\infty} (\ln|\ln b| - \ln|\ln 2|).$$

$$= \infty. \Rightarrow \text{divergent}.$$

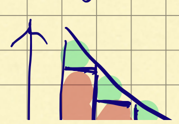
e.g. $\frac{1}{1+n^2}$ For all $x \geq 1$. $f'(x) = \frac{-(1+x^2)'}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2} \leq 0$

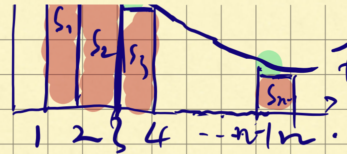
$f(x) \downarrow$.

$$\int_0^{\infty} \frac{1}{1+x^2} dx = \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2} = \lim_{b\to\infty} (\arctan x - \arctan 1)$$
$$= \frac{\pi}{4}.$$

converses by the intergral. Test

The proof of the Intergral Test.





$a_1 + \cdots + a_n = \sum_{n=1}^{n} a_n$ (i).

$a_2 + \cdots + a_n = S_n - a_1$ (ii).

$S_1 = 1 \cdot f(2)$

$\vdots$

$S_n = f(n-1)$.

$S_n - a_1 \le \int_1^n f(x)\,dx$ (iii). (+)

$S_n \le \int_1^n f(x)\,dx$. (iv).

$S_n - a_1 \le \int_1^n f(x)\,dx \le S_{n-1}$.