The p-integrals Theorem

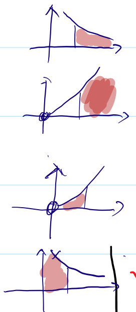
If $0<a<\infty$ (it means a is +ve and finite), then

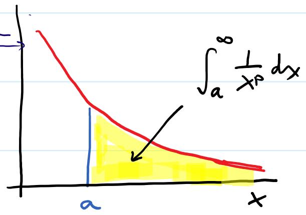
(i) $\displaystyle\int_a^{\infty}\frac{1}{x^p}\,dx$ $\begin{cases}\text{converges if } p>1 \\ \text{diverges if } p\le 1\end{cases}$ 收敛.

面积

(ii) $\displaystyle\int_0^{a}\frac{1}{x^p}\,dx$ $\begin{cases}\text{converges if } p<1 \\ \text{diverges if } p\ge 1\end{cases}$

e.g. $\int_0^1 \frac{1}{x}dx$. $p=1$.





Proof

(i) $$\int_a^{\infty}\frac{1}{x^p}\,dx=\lim_{b\to\infty}\int_a^{b}\frac{1}{x^p}\,dx=\lim_{b\to\infty}\int_a^{b}x^{-p}\,dx$$

$$=\lim_{b\to\infty}\left.\frac{x^{1-p}}{1-p}\right|_{x=a}^{b}=\frac{1}{1-p}\lim_{b\to\infty}\left(\cancel{b^{1-p}}^{\to 0}-a^{1-p}\right)$$ a constant; if $p>1$

where $$\lim_{b\to\infty}b^{1-p}=\begin{cases}\infty & \text{if } 1-p>0 \iff p<1 \\ 0 & \text{if } 1-p<0 \iff p>1\end{cases}$$

$$=\frac{1}{1-p}\left(0-a^{1-p}\right)=-\frac{a^{1-p}}{1-p}\quad \text{a finite number}$$

$\therefore$ $\displaystyle\int_a^{\infty}\frac{1}{x^p}\,dx$ is convergent if $p>1$

and $\displaystyle\int_a^{\infty}\frac{1}{x^p}\,dx$ is divergent if $p<1$

If $p=1$, then $$\int_a^{\infty}\frac{1}{x^p}\,dx=\int_a^{\infty}\frac{1}{x}\,dx=\lim_{b\to\infty}\int_a^{b}\frac{1}{x}\,dx=\lim_{b\to\infty}\ln|x|\Big|_{x=a}^{b}$$

$$=\lim_{b\to\infty}\left[\ln(b)^{\to\infty}-\ln a\right]=\infty$$

积分取极限 { 定值 → convergence
无穷/摆动 → divergence.

$\therefore$ $\displaystyle\int_a^{b}\frac{1}{x^p}\,dx$ is divergent if $p\le 1$. //

(ii) $$\int_0^{a}\frac{1}{x^p}\,dx=\lim_{\alpha\to 0}\int_\alpha^{a}x^{-p}\,dx=\lim_{\alpha\to 0}\left.\frac{x^{1-p}}{1-p}\right|_{x=\alpha}^{a}$$ is a constant

$$=\frac{1}{1-p}\lim_{\alpha\to 0}\left[a^{1-p}-\alpha^{1-p}\right]$$ $\alpha\to 0$.

$\int_0^a \frac{1}{x^p}$, $p\ge 1$

注意α的先前条件：α→0.

where $$\lim_{\alpha\to 0}\alpha^{1-p}=\begin{cases}0 & \text{if } 1-p>0 \iff p<1 \\ \infty & \text{if } 1-p<0 \iff p>1\end{cases}$$

$\therefore$ If $p<1 \Rightarrow \displaystyle\int_0^{a}\frac{1}{x^p}\,dx$ converges to $\dfrac{a^{1-p}}{1-p}$. 理解!

$100=\left(\frac{1}{100}\right)^{-1}$

$\alpha\to\frac{1}{100}$ $1-p\to -1<0$

$\therefore$ If $p<1 \Rightarrow \int_0^a \frac{1}{x^p}\,dx$ converges to $\frac{a^{1-p}}{1-p}$.

$\rightarrow$ If $p>1 \Rightarrow \int_0^a \frac{1}{x^p}\,dx$ diverges.

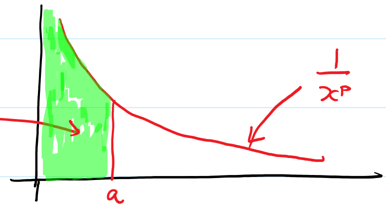
$\rightarrow$ If $p=1$ then $\int_0^a \frac{1}{x^p}\,dx = \int_0^1 \frac{1}{x}\,dx = \lim_{\alpha\to 0} \int_\alpha^1 \frac{1}{x}\,dx$

$$= \lim_{\alpha\to 0} \ln|x| \Big|_\alpha^1 = \lim_{\alpha\to 0} \left(\cancel{\ln|1|}^{\,0} - \ln\alpha\right)$$

$= \infty$ $\qquad\qquad \ln\alpha \to -\infty$

$\therefore \int_0^a \frac{1}{x^p}\,dx$ diverges if $p \geq 1$ /QED/

$\int_0^a \frac{1}{x^p}\,dx$



$\int_0^\infty e^{-x^2}\,dx$ converges? or diverges?

Sometimes we are only interested in the convergence of a given improper integral.

The Comparison Theorem: Suppose that $f(x)$ and $g(x)$ are continuous with $f(x) \leq g(x)$ for any $x \geq a$. Then

a) If $\int_a^\infty g(x)\,dx$ converges then so does $\int_a^\infty f(x)\,dx$.

b) If $\int_a^\infty f(x)\,dx$ diverges then so does $\int_a^\infty g(x)\,dx$.

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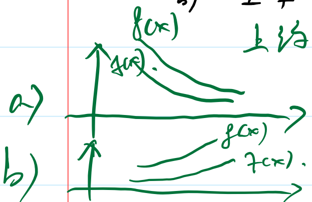

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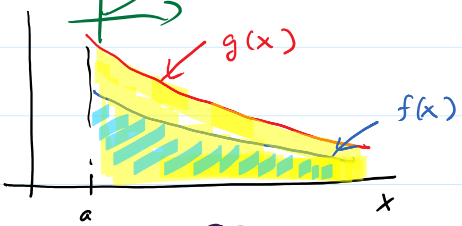
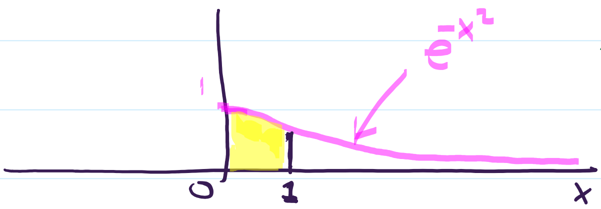


Ex 5: Show that $\int_0^\infty e^{-x^2}\,dx$ is convergent.



$\int_0^1 e^{-x^2}\,dx$. $\forall x \in [0,1]$, $e^{-x^2} \leq e^{-x}$.

$\int_0^1 e^{-x}\,dx = -e^{-x}\Big|_{x=0}^{1}$

$= -\frac{1}{e} + 1$.

$\Rightarrow$ finite number.

Let $I = \int_0^\infty e^{-x^2}\,dx$. Then

$\int_1^\infty e^{-x^2}\,dx$. $\forall x \in [1,+\infty)$ $e^{-x^2} \leq e^{-x}$.

$$\int_1 e \, dx = \lim_{b\to\infty}(-e \quad + e^{-1}) = \frac{1}{e}.$$

$\Rightarrow$ finite number.

$\Rightarrow e^{-x^2}$ is convergent.

Let $I = \int_0^\infty e^{-x^2}\,dx$. Then

拆分 $0 \to 1$, $1 \to \infty$.

$$I = \underbrace{\int_0^1 e^{-x^2}\,dx}_{I_1} + \underbrace{\int_1^\infty e^{-x^2}\,dx}_{I_2}$$

Consider I_1. ① Since e^{-x^2} is continuous on $[0,1]$ and this interval is finite, $I_1 = \int_0^1 e^{-x^2}\,dx$ is finite!

1) continuous

2) finite interval.

Consider I_2. $I_2 = \int_1^\infty e^{-x^2}\,dx$ is an improper integral of type I.

② when $x \geq 1 \Rightarrow x^2 > x$

$$\Rightarrow e^{x^2} > e^x$$

$$\Rightarrow \frac{1}{e^{x^2}} < \frac{1}{e^x}$$

$5 > 3$

$\frac{1}{5} < \frac{1}{3}$

$$x \geq \underset{a}{1}: \quad \underset{f(x)}{e^{-x^2}} < \underset{g(x)}{e^{-x}}$$

找一个熟悉的函数去判断是否收敛.

$$\int_1^\infty g(x)\,dx = \int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx = \lim_{b\to\infty}(-e^{-x})\Big|_{x=1}^{b}$$

$$= -\lim_{b\to\infty}(\cancel{e^{-b}}^{\,0} - e^{-1}) = \frac{1}{e}$$

③ $\therefore$ By the comparison theorem $I_2 = \int_1^\infty e^{-x^2}\,dx$ is convergent.

Since I is a sum of two finite numbers, I is also finite, i.e., $I = \int_0^\infty e^{-x^2}\,dx$ is convergent. // Ans.

Ex 6: Determine if $\int_0^\infty \frac{dx}{\sqrt{x+x^3}}$ is convergent or divergent.

Soln We note that the integral is of both type I and II.

$$\int_0^\infty \frac{dx}{\sqrt{x+x^3}} = \underbrace{\int_0^1 \frac{dx}{\sqrt{x+x^3}}}_{I_1} + \underbrace{\int_1^\infty \frac{dx}{\sqrt{x+x^3}}}_{I_2}$$

Consider I_1. $I_1 = \int_0^1 \frac{dx}{\sqrt{x+x^3}}$ where $f(x) = \frac{1}{\sqrt{x+x^3}}$

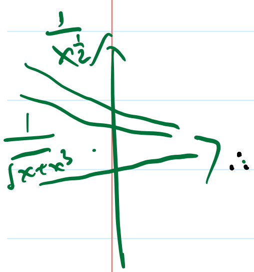
convergent $\Rightarrow$ $p<1$.

Consider I_1. $I_1 = \int_0^1 \frac{dx}{\sqrt{x+x^3}}$ where $f(x) = \frac{1}{\sqrt{x+x^3}}$

$$f(x) = \frac{1}{\sqrt{x+x^3}} \leqslant \frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}}$$

The p-integral test

$$\int_0^a \frac{1}{x^p}\,dx \begin{cases} \text{converges if } p<1 \\ \text{diverges if } p \geq 1 \end{cases}$$



$$\therefore \int_0^1 f(x)\,dx = \int_0^1 \frac{dx}{\sqrt{x+x^3}} \leqslant \boxed{\int_0^1 \frac{1}{x^{1/2}}\,dx} \text{ converges}$$

$p = \frac{1}{2} < 1$

$\therefore$ By the comparison test $I_1 = \int_0^1 \frac{dx}{\sqrt{x+x^3}}\,dx$ converges.

以a为分界

以a设立条件

convergent:

$>a \Rightarrow\ >1$

$<a \Rightarrow\ <1$

Consider I_2 where $I_2 = \int_1^\infty \frac{dx}{\sqrt{x+x^3}}$

$$f(x) = \frac{1}{\sqrt{x+x^3}} \leqslant \frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}}$$

$p = \frac{3}{2} > 1$

The p-integral test

$$\int_a^\infty \frac{dx}{x^p} \begin{cases} \text{converges if } p>1 \\ \text{diverges if } p \leq 1 \end{cases}$$

$$\therefore I_2 = \int_1^\infty \frac{dx}{\sqrt{x+x^3}} \leqslant \int_1^\infty \frac{1}{x^{3/2}}\,dx$$ converges by the p-integral test.

$\therefore$ I_2 is convergent by the Comparison Test.

Hence, $I = I_1 + I_2$ is also finite, i.e. $I = \int_0^\infty \frac{dx}{\sqrt{x+x^3}}$ converges.

//Ans.

Sequences and Series (Chapter 11)

A sequence is an ordered list of numbers

$$a_1, a_2, a_3, a_4, \ldots\ldots, a_{100}, \ldots$$

The number a_1 is called the first term, a_2 is called the 2nd term, and a_n is called the nth term of the sequence. If a sequence DOES NOT have the last term that it is called an infinite sequence.

The sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is denoted as

$$\{a_1, a_2, a_3, \ldots\} \quad \text{or} \quad \{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}$$

Ex1: (i) $\{n\} = \{1, 2, 3, 4, 5, \ldots\}$

(ii) $\{\sqrt{n}\} = \{1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}, \ldots\}$

(ii) $\{\sqrt{n}\} = \{1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}, \ldots\}$

(iii) $\left\{\frac{(-1)^{n+1}}{n}\right\}_{n=1}^{\infty} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots\right\}$

in (i) $a_n = n$, (ii) $a_n = \sqrt{n}$, (iii) $a_n = \frac{(-1)^{n+1}}{n}$

So far a sequence is defined by a single formula as in Ex 1. However, there are some sequences which cannot be defined by a single formula! For example, the Fibonacci sequence which is defined as

$$a_1 = 1, \quad a_2 = 1$$

$$\text{and} \quad a_{n+2} = a_{n+1} + a_n, \quad n \geq 1$$

Then this sequence is

$$\{1, 1, 2, 3, 5, 8, \ldots\}$$