Math 2155, Fall 2022: Homework 9

Instructions for GradeScope:

You may handwrite your answers or type them. Only type solutions if you are able to type the appropriate symbols. If **typed**, use at least a 12pt font, with **one question per page**. You can then directly submit the pdf file to GradeScope.

If **handwritten**:

- Don't erase or cross out more than a word or two.
- Only write on one side of the page. Use dark, large, neat writing.
- If you have a real scanner, great! Most people will scan their solutions using their phone. Use one of the recommended apps: For iOS, use Scannable by Evernote or Genius Scan. For Android, use Genius Scan. Do *not* just take regular photos.
- When scanning, have good lighting and try to avoid shadows.
- It is best to have **one question per scan**. If the solution is short, **fold the page in half** and scan just the half it is on, so there isn't so much blank space. Or, you can scan and then crop the scan. You don't need to scan parts (a), (b), etc. separately or put them on separate pages.
- It works well to scan each question separately and produce one pdf file with one question per page. Most scanning apps will automatically combine your images into one pdf file.
- You must **check the quality** of your scans afterwards, and rescan if needed.

Make sure you are going to `http://gradescope.ca` not `http://gradescope.com`. You can access it through the course OWL site.

You can **resubmit** your work any number of times until the deadline.

Don't forget to accurately **match questions to pages**. If you do this incorrectly, the grader will not see your solution and will give you zero.

See the GradeScope help website for lots of information: `https://help.gradescope.com/` Select "Student Center" and then either "Scanning Work on a Mobile Device" or "Submitting an Assignment".

Instructions for writing solutions:

- Homework is graded both on **correctness** and on **presentation/style**.
- Your proofs should be written in **complete sentences**, starting with capitals and ending with periods. They should be in **paragraph form**, similar to the proofs in the textbook. For example, you shouldn't put each sentence on a new line. Paragraphs should be used to break longer proofs into logical chunks.
- Don't use informal abbreviations such as three dots for "therefore." Logical symbols can be used when they clarify things to the reader, but English words can often be more effective.
- Include all of the steps that are needed to logically justify every claim you make. Do not include unnecessary steps. Try to be **concise and complete**.
- Do not submit any **rough work**. Do your rough work on scratch paper, and only submit a clearly and neatly written answer. Do not show a table of givens and goals. Do not use Venn diagrams.
- Do not cross out or erase more than a word or two. If you write each final solution on a new page, it's easy to start over on a fresh page when you've made a large error.
- You should do the work **on your own**. Read the course syllabus for the rules about scholastic offences, which include sharing solutions with others, uploading material to a website, viewing material of others or on a website (even if you don't use it), etc. The penalty for cheating on homework will be a grade of **0** on the homework set as well as a penalty of **negative 5%** on the overall course grade.
- We may not grade every question.

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Due **Friday, November 25 at 11:59pm**. You can **resubmit** your work any number of times until the deadline.

H9Q1 (5 marks): Prove: For all $n \in \mathbb{Z}$, if $n \equiv 0 \pmod 3$, then $n^2 \equiv 0 \pmod 6$ or $n^2 \equiv 3 \pmod 6$.

Solution: Let $n \in \mathbb{Z}$. Suppose that $n \equiv 0 \pmod 3$. Then $n = 3k$ for some $k \in \mathbb{Z}$. We consider two cases.

Case 1: k is even. Then $k = 2m$ for some $m \in \mathbb{Z}$, and $n = 3k = 6m$. Then $n^2 = 36m^2 = 6(6m^2)$. Since $6m^2$ is an integer, we see that $6 \mid n^2$, so $n^2 \equiv 0 \pmod 6$.

Case 2: k is odd. Then $k = 2m + 1$ for some $m \in \mathbb{Z}$, and $n = 3k = 6m + 3$. Then $n^2 = 36m^2 + 36m + 9 = 6(6m^2 + 6m + 1) + 3$. Therefore, $n^2 - 3 = 6(6m^2 + 6m + 1)$. Since $6m^2 + 6m + 1$ is an integer, we see that $6 \mid n^2 - 3$, so $n^2 \equiv 3 \pmod 6$.

The cases are exhaustive, so we have proved the claim.

H9Q2 (4 marks): Let $C = \{-2, -1, 0, 1, 2\}$ and let

$$E = \{(x, y) \in C \times C \mid x^2 = y^2\}.$$

E is an equivalence relation, but you do *not* need to prove this. Write down E as a set of ordered pairs, determine C/E and explicitly write down all of the equivalence classes. Briefly explain.

Solution: We see that xEy if and only if $x^2 = y^2$, which is true if and only if $|x| = |y|$. So we have

$$E = \{(0, 0), (1, 1), (1, -1), (-1, 1), (-1, -1), (2, 2), (2, -2), (-2, 2), (-2, -2)\}.$$

So $[0]_E = \{0\}$, $[1]_E = \{1, -1\}$, $[2]_E = \{2, -2\}$, and $C/E = \{[0]_E, [1]_E, [2]_E\}$.

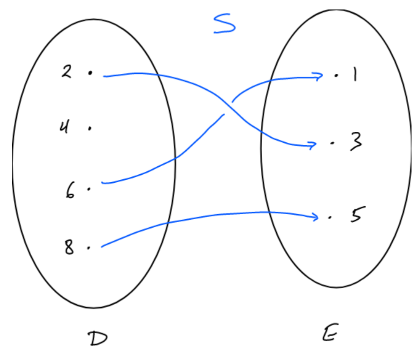
H9Q3 (5 marks): For each relation below, determine whether or not it is a function.

- If it is *not* a function, explain why not.
- If it *is* a function, explain why, and determine whether it is one-to-one and whether it is onto. Explain fully.

(a) $A = \{1, 2, 3\}$, $B = \{7, 8, 9, 10\}$, $R = \{(1, 9), (2, 7), (2, 10), (3, 8)\}$, a relation from A to B.

(b) R^{-1}, a relation from B to A, using the notation from (a).

(c) The relation $S \subseteq D \times E$ given by the following picture:



Solution: (a) R is not a function because $2 \in A$ is paired with more that one element in B, namely 7 and 10. More precisely, $(2, 7) \in R$ and $(2, 10) \in R$.

(b) We have that $R^{-1} = \{(9, 1), (7, 2), (10, 2), (8, 3)\}$. Each of the four elements of B is paired with exactly one element of A, so R^{-1} is a function from B to A. It is onto because every element of A appears at least once as the second component of a pair. It is not one-to-one because both 7 and 10 are paired with 2.

(c) S is not a function $D \to E$ because $4 \in D$ is not paired with anything.

H9Q4 (7 marks): Let $A = \{f \mid f : \mathbb{Z}^+ \to \mathbb{Z}\}$ be the set of all functions from the positive integers to the integers. Let P be the set of all prime numbers. Consider the relation

$$T = \{(f, g) \in A \times A \mid \forall p \in P\ (f(p)^2 = g(p)^2)\}$$

on A.

(a) Let $f : \mathbb{Z}^+ \to \mathbb{Z}$ and $g : \mathbb{Z}^+ \to \mathbb{Z}$ be the functions defined by the formulas $f(n) = 2(-1)^n$ and $g(n) =$ the number of positive divisors of n. Show that fTg.

(b) Prove that T is an equivalence relation.

Solution: (a) Let p be a prime number. Then the only positive divisors of p are 1 and p, so $g(p) = 2$. Therefore, $f(p)^2 = (2(-1)^p)^2 = 4 = g(p)^2$. Since this is true for all p, fTg.

(b) Reflexivity: Let $f \in A$ and let $p \in P$. Then $f(p)^2 = f(p)^2$. Since this is true for all p, fTf.

Symmetry: Let $f, g \in A$ and assume that fTg. Let $p \in P$. Since fTg, we have that $f(p)^2 = g(p)^2$. Therefore, $g(p)^2 = f(p)^2$. Since this is true for all p, gTf.

Transitivity: Let $f, g, h \in A$ and assume that fTg and gTh. Let $p \in P$. Since fTg, we have that $f(p)^2 = g(p)^2$. Since gTh, we have that $g(p)^2 = h(p)^2$. Therefore, $f(p)^2 = h(p)^2$. Since this is true for all p, fTh.