UWO CS2214

Tutorial #8

Problem 1 (Summation) Use mathematical induction to show that

$$\Sigma_{j=0}^{2n}(2j+1) = (2\,n+1)^2,$$

for all positive integers n. Provide detailed justifications for your answer.

Solution 1 We shall prove for an arbitrary positive integer n the property $P(n)$ below holds:

$$\Sigma_{j=0}^{2n}(2j+1) = (2\,n+1)^2.$$

Basis step: For $n = 1$, we have

$$\Sigma_{j=0}^{2}(2j+1) = 1+3+5 = 9 = (2+1)^2.$$

Hence the property $P(n)$ holds for $n = 1$.

Recursive step: Let us prove that for all $k \geq 1$ if $P(k)$ holds then so does $P(k+1)$. So let $k \geq 1$, let assume that $P(k)$ holds, that is,

$$\Sigma_{j=0}^{2k}(2j+1) = (2\,k+1)^2,$$

and let us prove that $P(k+1)$ holds as well, that is:

$$\Sigma_{j=0}^{2k+2}(2j+1) = (2\,k+3)^2,$$

We have:

$$\begin{array}{rcl} \Sigma_{j=0}^{2(k+1)}(2j+1) & = & \Sigma_{j=0}^{2k}(2j+1) \quad + \quad 2(2k+1)+1 \quad + \quad 2(2k+2)+1 \\ & = & (2\,k+1)^2 + 8k + 8. \end{array}$$

Since $(2\,k+3)^2 = (2\,k+1)^2 + 8k + 8$, we deduce that $P(k+1)$ holds indeed.

Therefore, we have proved by induction that for all positive integer n, the property $P(n)$ holds.

Problem 2 (Summation) Show by induction that for all $n \geq 1$ we have

$$\sum_{i=1}^{i=n} (i+1) = \frac{n(n+3)}{2} \tag{1}$$

Solution 2 http://www.csd.uwo.ca/~moreno/cs2214_moreno/tut/Problem_1.PDF

Problem 3 (Inequality) Prove by induction that for all $n \geq 3$ we have

$$4^{n-1} > n^2 \tag{2}$$

Solution 3 https://www.iitutor.com/mathematical-induction-inequality/

Step 1: Show it is true for $n = 3$.

LHS $= 4^{3-1} = 16$

RHS $= 3^2 = 9$

LHS > RHS

Therefore it is true for $n = 3$.

Step 2: Assume that it is true for $n = k$.

That is, $4^{k-1} > k^2$.

Step 3: Show it is true for $n = k + 1$.

That is, $4^k > (k+1)^2$.

$$\begin{aligned}
\text{LHS} &= 4^k \\
&= 4^{k-1+1} \\
&= 4^{k-1} \times 4 \\
&> k^2 \times 4 && \text{by the assumption } 4^{k-1} > k^2 \\
&= k^2 + 2k^2 + k^2 && 2k^2 > 2k \text{ and } k^2 > 1 \text{ for } k \geq 3 \\
&> k^2 + 2k + 1 \\
&= (k+1)^2 \\
&= \text{RHS}
\end{aligned}$$

$$\text{LHS} > \text{RHS}$$

Therefore it is true for $n = k + 1$ assuming that it is true for $n = k$.

Therefore $4^{n-1} > n^2$ is true for $n \geq 3$.

Problem 4 (Inequality) Prove by induction that for all $n \geq 3$ we have

$$n^2 \geq 2\,n + 3 \tag{3}$$

Solution 4 *https://www.csm.ornl.gov/~sheldon/ds/ans2.3.2.html*

Problem 5 (Divisibility) Prove by induction that for all $n \geq 1$ the integer $6^n - 1$ is divisible by 5.

Solution 5 `http://home.cc.umanitoba.ca/~thomas/Courses/InductionExamples-Solutions.pdf`

Problem 6 (Incorrect proof) Here is an incorrect proof of the statement:

All people have the same eye color.

Proof by induction: we prove the statement "All members of any non-empty set of people have the same eye color".

1. This is clearly true for any singleton set, that is, any set with a single element.

2. Now, assume we have a non-empty set S of people, and the inductive hypothesis is true for all smaller sets. Choose an ordering on the set, and let S_1 be the set formed by removing the first person, and S_2 be the set formed by removing the last person. All members of S_1 have the same eye color, and also for S_2. However, $S_1 \cap S_2$ has members from both sets, so all members of S have the same eye color.

Explain what is incorrect in the above reasoning.

Solution 6 Let $P(n)$ be the property that any n persons have the same eye color, where n is a positive integer. While $P(1)$ is true, the above reasoning breaks for $P(2)$. Indeed, when applied to $n = 2$, this reasoning considers two sets S_1 and S_2, each of which consisting of a single person so that $S_1 \cap S_2$ is empty.

Problem 7 (Counting tree leaves) The set of leaves and the set of internal vertices of a full binary tree are defined recursively as follows:

Basis step: The root r is a leaf of the full binary tree with exactly one vertex r. This tree has no internal vertices.

Recursive step: The set of leaves of the tree $T = T_1 \cdot T_2$ is the union of the sets of leaves of T_1 and T_2. The internal vertices of T are the root r of T and the union of the set of internal vertices of T_1 and the set of internal vertices f T_2.

Use structural induction to prove that $\ell(T)$, the number of leaves of a full binary tree T, is 1 more than $i(T)$, the number of internal vertices of T.

Solution 7 We shall prove that, for an arbitrary full binary tree T, its number of leaves $\ell(T)$ satisfies the property $\mathcal{P}(T)$ below:

$$\ell(T) = i(T) + 1.$$

Basis step: The root r is a leaf and has no internal vertices, that is, $\ell(T) = 1$ and $i(T) = 0$, hence it satisfies $\ell(T) = i(T) + 1$.

Recursive step: Let $T = T_1 \cdot T_2$ be a full binary tree built from two full binary trees T_1, T_2. We shall prove that, if $\mathcal{P}(T_1)$ and $\mathcal{P}(T_2)$ both hold, then so does $\mathcal{P}(T)$. So, let us assume that $\mathcal{P}(T_1)$ and $\mathcal{P}(T_2)$ both hold. By definition of $\ell(T)$, we have:

$$\ell(T) = \ell(T_1) + \ell(T_2).$$

By induction hypothesis, we have:

$$\ell(T_1) = i(T_1) + 1 \quad \text{and} \quad \ell(T_2) = i(T_2) + 1$$

By definition of $i(T)$, we have:

$$i(T) = i(T_1) + i(T_2) + 1$$

Putting everything together:

$$\begin{array}{rcl} \ell(T) & = & \ell(T_1) + \ell(T_2) \\ & = & i(T_1) + 1 + i(T_2) + 1 \\ & = & i(T) + 1. \end{array}$$

Hence, we have proved that $\mathcal{P}(T)$ holds.

Therefore, we have proved by induction that for all binary trees we have the number of leaves is 1 more than the number of internal vertices.

Problem 8 Consider the set S of strings over the alphabet $\{a, b\}$ defined inductively as follows:

- Base case: the empty word λ and the word a belong to S
- Inductive rule: if ω is a string of S then both $\omega\, b$ and $\omega\, b\, a$ belong to S as well.

1. Prove that if a string ω belongs to S, then ω does not have two or more consecutive a's.
2. Prove that for any $n \geq 0$, if ω is a string of length n over the alphabet $\{a, b\}$ that does not have two or more consecutive a's, then ω is a string of S.

Solution 8

1. Let ω be any word over the alphabet $\{a, b\}$. Denote by $P(\omega)$ the property that ω does not have two or more consecutive a's. Consider first a word ω in the base case. Thus, ω is either λ or a. Hence, the property $P(\omega)$ clearly holds for ω Consider now a word ω obtained by applying the inductive rule. Hence ω is either of the $\omega'\, b$ or $\omega'\, b\, a$. We want to prove that if $P(\omega')$ holds then do does $P(\omega)$. Clearly, if ω would have two or more consecutive a's the same would need to hold for ω, which would be a contradiction. Hence $P(\omega)$ holds.

2. Let $n \geq 0$. Denote by $Q(n)$ the property that any word over the alphabet $\{a, b\}$ with length n not having two or more consecutive a's belongs to S. Consider first $n = 0$. The only word of length zero is the empty word λ which (1) does not have two or more consecutive a's, and (2) belongs to S. Hence $Q(0)$ holds. Let $k \geq 0$. Assume that $Q(0), \ldots, Q(k)$ holds and let us prove that $Q(k+1)$ holds as well. Hence, we consider any word ω with length $k + 1$ and which does not have two or more consecutive a's. Either ω has he form $\omega'\, b$ or the form $\omega''\, b\, a$ where ω' has length k and ω'' has length $k - 1$. Neither ω' nor ω'' can have two or more consecutive a's. Hence by inductive hypothesis, they belong to S. Thus, by the inductive rule defining S, it follows that $\omega'\, b$ or the form $\omega''\, b\, a$ belong to S as well. Therefore, we have proved that $Q(k + 1)$ holds as well.

Problem 9 (Exponential growth of the Fibonacci numbers) Recall that $F_0 = 1$, $F_1 = 1$ and that for all $n \geq 2$ we have $F_n = F_{n-1} + F_{n-2}$. Prove that $F_n > (\frac{2}{3})^{n-2}$ for all $n \geq 0$.

Solution 9 Last two slides.