Math 2155, Fall 2022: Homework 10

Instructions for GradeScope:

You may handwrite your answers or type them. Only type solutions if you are able to type the appropriate symbols. If **typed**, use at least a 12pt font, with **one question per page**. You can then directly submit the pdf file to GradeScope.

If **handwritten**:

- Don't erase or cross out more than a word or two.
- Only write on one side of the page. Use dark, large, neat writing.
- If you have a real scanner, great! Most people will scan their solutions using their phone. Use one of the recommended apps: For iOS, use Scannable by Evernote or Genius Scan. For Android, use Genius Scan. Do *not* just take regular photos.
- When scanning, have good lighting and try to avoid shadows.
- It is best to have **one question per scan**. If the solution is short, **fold the page in half** and scan just the half it is on, so there isn't so much blank space. Or, you can scan and then crop the scan. You don't need to scan parts (a), (b), etc. separately or put them on separate pages.
- It works well to scan each question separately and produce one pdf file with one question per page. Most scanning apps will automatically combine your images into one pdf file.
- You must **check the quality** of your scans afterwards, and rescan if needed.

Make sure you are going to `http://gradescope.ca` not `http://gradescope.com`. You can access it through the course OWL site.

You can **resubmit** your work any number of times until the deadline.

Don't forget to accurately **match questions to pages**. If you do this incorrectly, the grader will not see your solution and will give you zero.

See the GradeScope help website for lots of information: `https://help.gradescope.com/` Select "Student Center" and then either "Scanning Work on a Mobile Device" or "Submitting an Assignment".

Instructions for writing solutions:

- Homework is graded both on **correctness** and on **presentation/style**.
- Your proofs should be written in **complete sentences**, starting with capitals and ending with periods. They should be in **paragraph form**, similar to the proofs in the textbook. For example, you shouldn't put each sentence on a new line. Paragraphs should be used to break longer proofs into logical chunks.
- Don't use informal abbreviations such as three dots for "therefore." Logical symbols can be used when they clarify things to the reader, but English words can often be more effective.
- Include all of the steps that are needed to logically justify every claim you make. Do not include unnecessary steps. Try to be **concise and complete**.
- Do not submit any **rough work**. Do your rough work on scratch paper, and only submit a clearly and neatly written answer. Do not show a table of givens and goals. Do not use Venn diagrams.
- Do not cross out or erase more than a word or two. If you write each final solution on a new page, it's easy to start over on a fresh page when you've made a large error.
- You should do the work **on your own**. Read the course syllabus for the rules about scholastic offences, which include sharing solutions with others, uploading material to a website, viewing material of others or on a website (even if you don't use it), etc. The penalty for cheating on homework will be a grade of **0** on the homework set as well as a penalty of **negative 5%** on the overall course grade.
- We may not grade every question.

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Due **Saturday, December 3 at 11:59pm**. Last one!

You can **resubmit** your work any number of times until the deadline.

H10Q1 (6 marks): Consider the function f with domain $A = \mathbb{R} \setminus \{1\}$ defined by $f(x) = \frac{5x}{x-1}$. Find a set $B \subseteq \mathbb{R}$ such that $f : A \to B$ is one-to-one and onto. Prove that it is one-to-one and onto and find $f^{-1} : B \to A$.

Hint: you might find it easiest to use Theorems 5.3.4 and 5.3.5 from the text, like the solution of Exercise 3 from Section 5.3 at the back of the book.

Solution: First observe that there does not exist an $x \in \mathbb{R} \setminus \{1\}$ such that $f(x) = 5$. For if $\frac{5x}{x-1} = 5$, then $5x = 5x - 5$, which is impossible. So letting $B = \mathbb{R} \setminus \{5\}$, we have shown that $f : A \to B$ is a well-defined function.

Consider the function $g : B \to A$ defined by $g(y) = \frac{y}{y-5}$. Since $5 \notin B$, $g(y)$ is a well-defined real number for $y \in B$. Note that g does not take on the value 1, since if $\frac{y}{y-5} = 1$, then $y = y - 5$, which is impossible. So $g(y) \in A$ for $y \in B$.

We first show that $g \circ f = i_A$. Let $x \in A$. Then

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{5x}{x-1}\right) = \frac{\frac{5x}{x-1}}{\frac{5x}{x-1} - 5} = \frac{5x}{5x - 5(x-1)} = \frac{5x}{5} = x.$$

So $g \circ f = i_A$.

Next we show that $f \circ g = i_B$. Let $y \in B$. Then

$$f(g(y)) = f\left(\frac{y}{y-5}\right) = \frac{5\frac{y}{y-5}}{\frac{y}{y-5} - 1} = \frac{5y}{y - (y-5)} = \frac{5y}{5} = y.$$

So $f \circ g = i_B$.

By Theorem 5.3.4, f is one-to-one and onto. By Theorem 5.3.5, $f^{-1} = g$. So $f^{-1}(y) = \frac{y}{y-5}$. $\square$

Note: We did not explain where the formula for g came from. It was done in rough work. We also didn't explain how we guessed that $B = \mathbb{R} \setminus \{5\}$ is the right set. But note that we needed to prove that f and g were well-defined functions between those sets.

Note: This question can also be done without using those theorems. One can prove that f is one-to-one and onto, and then prove a formula for the inverse.

H10Q2 (4 marks): Suppose that $g : A \to B$ and $h : B \to C$. Prove that if h is not one-to-one and g is onto, then $h \circ g$ is not one-to-one.

Solution: Suppose that h is not one-to-one and g is onto. Since h is not one-to-one, there exist $b_1, b_2 \in B$ with $b_1 \neq b_2$ and $h(b_1) = h(b_2)$. Since g is onto, there exist $a_1, a_2 \in A$ with $g(a_1) = b_1$ and $g(a_2) = b_2$. If $a_1 = a_2$, then $b_1 = g(a_1) = g(a_2) = b_2$, a contradiction, so we must have $a_1 \neq a_2$. We also have that $(h \circ g)(a_1) = h(g(a_1)) = h(b_1) = h(b_2) = h(g(a_2)) = (h \circ g)(a_2)$. This shows that $h \circ g$ is not one-to-one. $\square$

This question was not graded.

H10Q3 (4 marks): Prove that for every $n \in \mathbb{N}$,

$$0 \cdot 3 + 1 \cdot 4 + 2 \cdot 5 + \cdots + n(n+3) = \frac{n(n+1)(n+5)}{3}.$$

Solution: We prove this by induction on n.

Base case: $n = 0$: Then the left-hand-side is $0 \cdot 3 = 0$ and the right-hand-side is $\frac{0(1)(5)}{3} = 0$, so the statement is true.

Induction step: Let $n \in \mathbb{N}$ and assume that

$$0 \cdot 3 + 1 \cdot 4 + 2 \cdot 5 + \cdots + n(n+3) = \frac{n(n+1)(n+5)}{3}.$$

Then

$$\begin{aligned} 0 \cdot 3 + 1 \cdot 4 + \cdots + n(n+3) + (n+1)(n+4) &= \frac{n(n+1)(n+5)}{3} + (n+1)(n+4) \qquad \text{(by inductive hyp.)} \\ &= \frac{n(n+1)(n+5) + 3(n+1)(n+4)}{3} \\ &= \frac{(n+1)(n^2 + 5n + 3n + 12)}{3} \\ &= \frac{(n+1)(n+2)(n+6)}{3}. \end{aligned}$$

So the formula holds with n replaced by $n+1$, as required. $\square$

H10Q4 (6 marks): Find the smallest natural number m such that for all natural numbers $n \geq m$, $n! > n^2 + 2$. Prove that your m works and that no smaller m works.

Solution: When $n = 3$, we have $n! = 3! = 6$ and $n^2 + 2 = 3^2 + 2 = 11$, so the statement is false. So we must have $m \geq 4$. We claim that $m = 4$ works.

We prove by induction on n that for all $n \geq 4$, $n! > n^2 + 2$.

Base case: $n = 4$: Then $n! = 4! = 24$ and $n^2 + 2 = 4^2 + 2 = 18$. Since $24 > 18$, the statement is true.

Induction step: Let $n \geq 4$ and assume that $n! > n^2 + 2$. Then

$$\begin{aligned}
(n+1)! &= (n+1)n! \\
&> (n+1)(n^2+2) && \text{(by inductive hypothesis)} \\
&= n^3 + n^2 + 2n + 2 \\
&\geq 1 + n^2 + 2n + 2 && \text{(since } n \geq 4) \\
&= (n+1)^2 + 2
\end{aligned}$$

So we have proved the inequality with n replaced by $n + 1$, as required. $\square$

Note: There are a variety of ways to organize the algebra and inequalities here. However you do it, it is important to mention where you use the inductive hypothesis and the assumption that $n \geq 4$.