Properties of determinants (1)

Recap

Definition of determinant

Let $A = [a_{ij}]$ be a square matrix of order n.

The expansion along the i-th row is given by

$$\det A = (-1)^{i+1} a_{i1} \det A_{i1} + (-1)^{i+2} a_{i2} \det A_{i2} + \ldots + (-1)^{i+n} a_{in} \det A_{in}$$

The expansion along the j-th column is given by

$$\det A = (-1)^{1+j} a_{1j} \det A_{1j} + (-1)^{2+j} a_{2j} \det A_{2j} + \ldots + (-1)^{n+j} a_{nj} \det A_{nj}$$

Recap

Definition of determinant

Let $A = [a_{ij}]$ be a square matrix of order n.

The expansion along the i-th row is given by

$$\det A = (-1)^{i+1} a_{i1} \det A_{i1} + (-1)^{i+2} a_{i2} \det A_{i2} + \ldots + (-1)^{i+n} a_{in} \det A_{in}$$

The expansion along the j-th column is given by

$$\det A = (-1)^{1+j} a_{1j} \det A_{1j} + (-1)^{2+j} a_{2j} \det A_{2j} + \ldots + (-1)^{n+j} a_{nj} \det A_{nj}$$

A note: Expand a row or a column of a square matrix A which contains the most zeros to find $\det A$.

Theorem A Let A and B be the same matrices except that in a certain row (or column) of B each entry of A is multiplied by a scalar c. Then $\det B = c \det A$.

Theorem A Let A and B be the same matrices except that in a certain row (or column) of B each entry of A is multiplied by a scalar c. Then $\det B = c \det A$.

Proof. The matrices A and B are given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \text{ and } B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ ca_{i1} & ca_{i2} & \dots & ca_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Theorem A Let A and B be the same matrices except that in a certain row (or column) of B each entry of A is multiplied by a scalar c. Then $\det B = c \det A$.

Proof. The matrices A and B are given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \text{ and } B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ ca_{i1} & ca_{i2} & \dots & ca_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Expansion along the i-th row of B, we have

$$\det B = (-1)^{i+1}(ca_{i1}) \det A_{i1} + \ldots + (-1)^{i+n}(ca_{in}) \det A_{in} = c \det A.$$

Examples

(1) Compute $\det A$ and $\det B$

$$A = \begin{bmatrix} 2 & 0 & 8 \\ 1 & -6 & 2 \\ 3 & 9 & 12 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \frac{1}{2} & -\frac{1}{6} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{4}{3} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

(2) Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$ such that $\det A = -3$. Find $\det B$ and $\det C$, where

$$B = \begin{bmatrix} 3a & 9b & 3c \\ d & 3e & f \\ g & 3h & i \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 2g & 2h & 2k \end{bmatrix}.$$

Theorem B Let A be an $n \times n$ matrix. Then $\det(cA) = c^n \det A$. In particular, $\det(-A) = (-1)^n \det A$.

Theorem B Let A be an $n \times n$ matrix. Then $\det(cA) = c^n \det A$. In particular, $\det(-A) = (-1)^n \det A$.

This is because there are n copies of c factored out in $\det(cA)$ from each row. Hence $\det(cA) = c^n \det A$.

Example Find the determinant of $-3I$, where I is the identity matrix of order 17.

Theorem B Let A be an $n \times n$ matrix. Then $\det(cA) = c^n \det A$. In particular, $\det(-A) = (-1)^n \det A$.

This is because there are n copies of c factored out in $\det(cA)$ from each row. Hence $\det(cA) = c^n \det A$.

Example Find the determinant of $-3I$, where I is the identity matrix of order 17. $\det(-3I) = (-3)^{17}$

Note that $\det(cA) \neq c \det A$ in general.

Theorem C Let A be a square matrix. If A has two identical rows (or two columns), then $\det A = 0$.

Theorem A and Theorem C imply the following statement.

Corollary If a square matrix A has a row that is a scalar multiple of another row (or a column is a scalar multiple of another column), then $\det A = 0$.

Examples

Compute $\det A$

$$A = \begin{bmatrix} 2 & 1 & 3 & -4 \\ -1 & 0 & 1 & 2 \\ -3 & 2 & -1 & 6 \\ 4 & 1 & 4 & -8 \end{bmatrix}$$

Find $\det A$.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 20 & 1 & -2 & 3 \\ 2 & 0 & -2 & 5 \\ 13 & -7 & 14 & -21 \end{bmatrix}$$

Find the determinant of the following matrices.

$$\begin{bmatrix} 0 & 2 & 4 \\ 1 & 13 & 26 \\ 0 & 4 & 16 \end{bmatrix}, \qquad \begin{bmatrix} 42 & 5 & 1 \\ 84 & 0 & 0 \\ 63 & 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 4 & 2 & 0 \\ 0 & 0 & 7 & 2 & 3 \\ -2 & 8 & 4 & 2 & 2 \\ 0 & 1 & 2 & 1 & 0 \end{bmatrix}$$

Find $\det A + (\det B)(\det C) + \det D$, where C is the identity matrix of order 6, and A, B, D are the following matrices

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 7 & 3 & 0 & 1 \\ 2 & 1 & 0 & 3 \\ 5 & 2 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 5 & -4 \\ 0 & -3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & -5 & 2 & -3 \\ -2 & 3 & 3 & -1 \\ -1 & 5 & -2 & 3 \\ 0 & -5 & 0 & 6 \end{bmatrix}$$

Properties of determinant (2)

Theorem D Let A be a square matrix of order n for $n \geq 2$. Let B be the matrix obtained either by interchanging two rows of A, or by interchanging two columns of A. Then $\det B = -\det A$.

Theorem D Let A be a square matrix of order n for $n \geq 2$. Let B be the matrix obtained either by interchanging two rows of A, or by interchanging two columns of A. Then $\det B = -\det A$.

For instance, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$.

By definition, $\det A = ad - bc$ and $\det B = bc - ad = -\det A$.

Find det A and det B.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -4 \\ 0 & 0 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

Theorem E If a scalar multiple of one row (or one column) is added to another row (or column) of a square matrix, then the determinant is unchanged.

Theorem E If a scalar multiple of one row (or one column) is added to another row (or column) of a square matrix, then the determinant is unchanged.

Example. Find the determinants.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 5 \\ -2 & -4 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & 3 & -1 \\ -3 & -2 & 0 & 2 \\ 2 & 1 & 2 & -1 \\ 1 & 0 & 1 & -2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 3 & 1 & 4 & 1 \\ 2 & 1 & -1 & 2 \\ 1 & 1 & -2 & 1 \end{bmatrix}$$

We have seen how determinants of square matrices behave with respect to three elementary operations:

- If a row is multiplied by a non-zero scalar, the determinant is multiplied by the same scalar;
- If two rows are interchanged, the sign of the determinant is reversed;
- If a scalar multiple of one row is added to another row, the determinant is unchanged.

Theorem F A square matrix A is invertible if and only if $\det A$ is not zero.

Theorem F A square matrix A is invertible if and only if $\det A$ is not zero.

Recall the definition:

A square matrix A is invertible if there is a matrix B of the same order such that $AB = I$ (or $BA = I$).

Theorem F A square matrix A is invertible if and only if $\det A$ is not zero.

Recall the definition:

A square matrix A is invertible if there is a matrix B of the same order such that $AB = I$ (or $BA = I$).

Equivalently, a square matrix is invertible if and only if the RREF of A is the identity matrix.

The determinant of a square RREF is 1 if and only if it is the identity matrix. Together with the relations between elementary row operations and determinants, the Theorem holds.

Examples

(1) For what value of k does the matrix A below have no inverse?

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 5 \\ 0 & 6 & k+2 \end{bmatrix}$$

(2) (Lecture note Example 11.18)

Prove that $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ are both invertible, but the matrix $A - B$ is not.

(3) Find the invertible matrices by computing the determinants of the following matrices.

$$A = \begin{bmatrix} 1 & 2 & 5 & -4 \\ -1 & 3 & -5 & -15 \\ 3 & 1 & 15 & 7 \\ 2 & 6 & 3 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 5 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 2 & -2 \\ 6 & 0 & 3 \\ 12 & 1 & -3 \end{bmatrix}$$

(4) Let $A = \begin{bmatrix} a & b & c \\ d & 2 & f \\ g & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} a+2 & b & c \\ d & 2 & f \\ g & 0 & 1 \end{bmatrix}$. If $\det B = 5$, what is the value of $\det A$?