Binary Relations
Chapter 9

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Binary relations

Definition
A *binary relation R* from a set A to a set B is a subset $R \subseteq A \times B$.

Example

1. Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$
2. $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.
3. We can represent relations from a set A to a set B graphically or using a table:
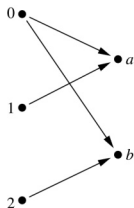
R	a	b
0	×	×
1	×	
2		×

Relations are more general than functions. A function from A to B is a relation where for each element x of A there is exactly one element y of B related to x.

Binary relation on a set

Definition
A binary relation R on a set A is a subset of $A \times A$ or a relation from A to A.

Example
1. Suppose that $A = \{a, b, c\}$.
2. Then $R = \{(a, a), (a, b), (a, c)\}$ is a binary relation on A.

Example
1. Let $A = \{1, 2, 3, 4\}$.
2. The ordered pairs in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ are $(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3)$, and $(4, 4)$.

Binary relation on a set

Example

How many relations are there on a finite set A?

Solution:

1. Because a relation on A is the same thing as a subset of $A \times A$, we count the subsets of $A \times A$.

2. Since $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are $2^{|A|^2}$ subsets of $A \times A$.

3. Therefore, there are $2^{|A|^2}$ relations on a set A.

Binary relations on a set

Example
Consider these relations on the set of integers:

$$R_1 = \{(a, b) \mid a \leq b\} \qquad\qquad R_4 = \{(a, b) \mid a = b\}$$
$$R_2 = \{(a, b) \mid a > b\} \qquad\qquad R_5 = \{(a, b) \mid a = b + 1\}$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\} \qquad R_6 = \{(a, b) \mid a + b \leq 3\}$$

> Note that these relations are on an infinite set and each of these relations is an infinite set.

Which of these relations contain each of the pairs:

❶ (1,1)　　　❷ (1, 2)　　　❸ (2,1)　　　❹ (1, −1)　　　❺ (2,2)

Solution: Checking the conditions that define each relation, we see that the pair

❶ (1,1) is in R_1, R_3, R_4, and R_6:

❷ (1,2) is in R_1 and R_6:

❸ (2,1) is in R_2, R_5, and R_6:

❹ (1, −1) is in R_2, R_3, and R_6:

❺ (2,2) is in R_1, R_3, and R_4.

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Reflexive relations

Definition

The relation R on A is *reflexive* iff $(a, a) \in R$ for every element $a \in A$. That is, R is reflexive if and only if

$$\forall x, \quad x \in A \rightarrow (x, x) \in R$$

Example

The following relations on the integers are reflexive:

$R_1 = \{(a, b) \mid a \leq b\}$

$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$

$R_4 = \{(a, b) \mid a = b\}$

> If $A = \varnothing$ then the empty relation is reflexive vacuously. That is, the empty relation on an empty set is reflexive!

The following relations are not reflexive:

$R_2 = \{(a, b) \mid a > b\}$ (note that $3 \not> 3$)

$R_5 = \{(a, b) \mid a = b + 1\}$ (note that $3 \neq 3 + 1$)

$R_6 = \{(a, b) \mid a + b \leq 3\}$ (note that $4 + 4 \not\leq 3$)

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Symmetric relations

Definition

The relation R on A is *symmetric* iff $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$. That is, R is symmetric if and only if
$$\forall x, \ \forall y, \ \ (x, y) \in R \ \rightarrow \ (y, x) \in R$$

Example

The following relations on the integers are symmetric:

$R_3 = \{(a, b) \mid |a| = |b|\}$

$R_4 = \{(a, b) \mid a = b\}$

$R_6 = \{(a, b) \mid a + b \leq 3\}$

The following relations are not symmetric:

$R_1 = \{(a, b) \mid a \leq b\}$ (note that $3 \leq 4$, but $4 \nleq 3$)

$R_2 = \{(a, b) \mid a > b\}$ (note that $4 > 3$, but $3 \ngtr 4$)

$R_5 = \{(a, b) \mid a = b + 1\}$ (note that $4 = 3 + 1$, but $3 \neq 4 + 1$)

Antisymmetric relations

Definition

The relation R on A is *antisymmetric* if for all $a, b \in A$ if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ holds. That is: R is antisymmetric iff

$$\forall x, \ \forall y, \ \ ((x, y) \in R \land (y, x) \in R) \ \rightarrow \ x = y$$

Note: if x and y are distinct $(x \neq y)$ then R can not have both (x, y) and (y, x).

Example

The following relations on the integers are antisymmetric:

$R_1 = \{(a, b) \mid a \leq b\}$

$R_2 = \{(a, b) \mid a > b\}$

$R_4 = \{(a, b) \mid a = b\}$

$R_5 = \{(a, b) \mid a = b + 1\}$

> For any integer, if a $a \leq b$ and $b \leq a$ then $a = b$.

The following relations are not antisymmetric:

$R_3 = \{(a, b) \mid |a| = |b|\}$ (note that both $(1, -1)$ and $(-1, 1)$ belong to R

$R_6 = \{(a, b) \mid a + b \leq 3\}$ (note that both $(1, 2)$ and $(2, 1)$ belong to R_6)

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Transitive relations

Definition

The relation R on A is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$. That is, R is transitive if and only if

$$\forall x \forall y \forall z[(x, y) \in R \land (y, z) \in R \to (x, z) \in R]$$

Example

The following relations on the integers are transitive:

$R_1 = \{(a, b) \mid a \le b\}$

$R_2 = \{(a, b) \mid a > b\}$

$R_3 = \{(a, b) \mid |a| = |b|\}$

$R_4 = \{(a, b) \mid a = b\}$

> For any integer, if a $a \le b$ and $b \le c$ then $a \le c$.

The following relations are not transitive:

$R_5 = \{(a, b) \mid a = b + 1\}$ (note that both $(3, 2)$ and $(4, 3)$ belong to R_5, but not $(4, 2)$

$R_6 = \{(a, b) \mid a + b \le 3\}$ (note that both $(2, 1)$ and $(1, 2)$ belong to R_6, but not $(2, 2)$

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Combining relations

Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.

Example

Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined using basic set operations to form new relations:

1. $R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$
2. $R_1 \cap R_2 = \{(1,1)\}$
3. $R_1 - R_2 = \{(2,2),(3,3)\}$
4. $R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$

Combining relations via composition
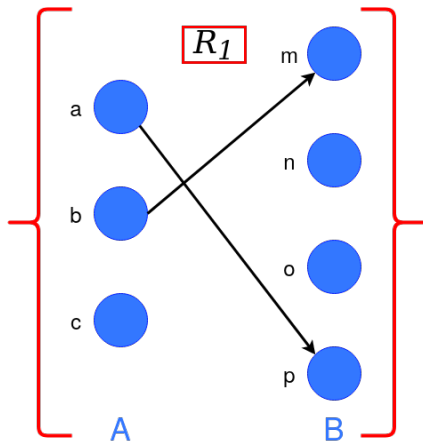
Definition

Suppose:

1. R_1 is a relation from a set A to a set B.
2. R_2 is a relation from B to a set C.

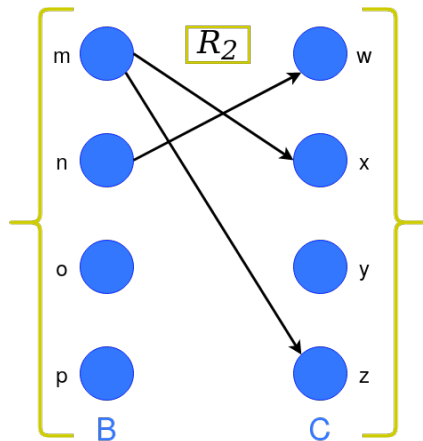Then the *composition* of R_2 with R_1, is a relation from A to C, denoted $R_2 \circ R_1$, where:

1. if (x, y) is a member of R_1 and (y, z) is a member of R_2 then (x, z) is a member of $R_2 \circ R_1$.
2. also, if $(x, z) \in R_2 \circ R_1$ then there exists some $y \in B$ such that $(x, y) \in R_1$ and $(y, z) \in R_2$
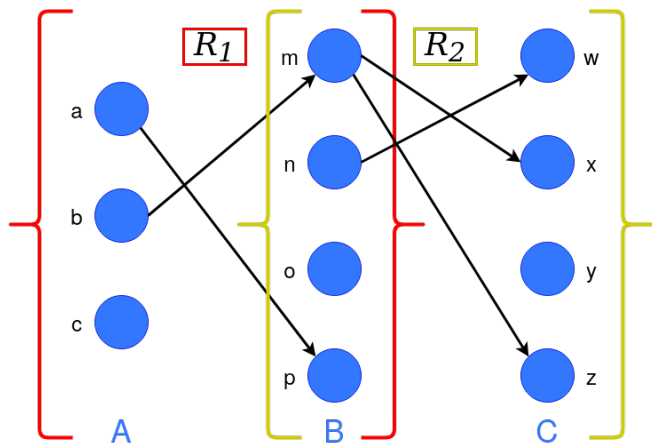
Representing a composition
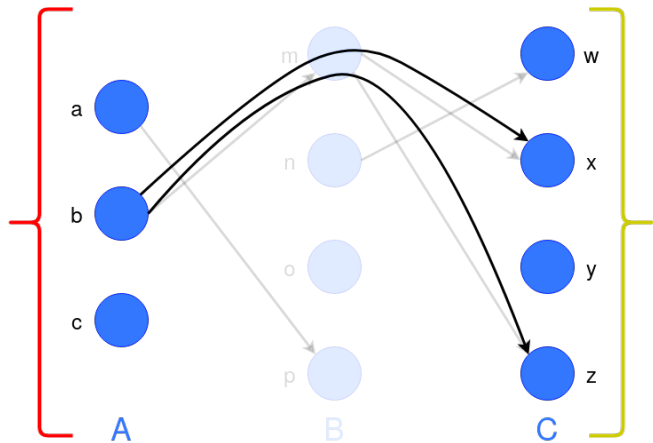
Representing a composition

Representing a composition

$$R_2 \circ R_1 = \{(b, z), (b, x)\}$$

Representing a composition

$$R_2 \circ R_1 = \{(b, z), (b, x)\}$$

Composition of a relation with itself

Definition

Let R be a binary relation on a set A. Then the composition of R with R, denoted $R \circ R$, is a relation on A where:

1. if (x, y) is a member of R
2. and (y, z) is a member of R
3. then (x, z) is a member of $R \circ R$.

Example

Let R be a relation on the set of all people such that (a, b) is in R if person a is parent of person b. Then (a, c) is in $R \circ R$ iff there is a person b such that (a, b) is in R and (b, c) is in R. In other words, (a, c) is in $R \circ R$ iff a is a grandparent of c.

Powers of a relation

Definition

Let R be a binary relation on A. Then the powers R^n of the relation R can be defined inductively by:

❶ Basis Step: $R^1 = R$

❷ Inductive Step: $R^{n+1} = R^n \circ R$

The powers of a transitive relation are subsets of the relation. This is established by the following theorem:

Theorem

The relation R on a set A is transitive iff $R^n \subseteq R$ for all positive integers n.

(see Tutorial 11.)

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Representing relations using matrices

1. A relation between <u>finite sets </u>can be represented using a zero-one matrix.

2. Suppose R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$.

   a. The elements of the two sets can be listed in any particular arbitrary order. When $A = B$, we use the same ordering.

3. The relation R is represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

4. The matrix representing R has a 1 as its (i, j) entry, when a_i is related to b_j, and, a 0 if a_i is not related to b_j.

Examples of matrices representing relations

Example

Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B such that:

$$R = \{(a, b) \mid a \in A, b \in B, \ a > b \ \}$$

What is the matrix representing R (assuming the ordering of elements is the same as the increasing numerical order) ?

Solution:

Because $R = \{(2,1), (3,1),(3,2)\}$, the matrix is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Examples of matrices representing relations

Example

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

Solution:

Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that:
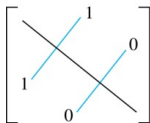
$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), \{(a_3, b_3), (a_3, b_5)\}$

Matrices of relations on sets

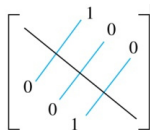1. If R is a reflexive relation, all the elements on the main diagonal of M_R are equal to 1.

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

2. R is a symmetric relation, if and only if $m_{ij} = 1$ whenever $m_{ji} = 1$.

(a) Symmetric

(b) Antisymmetric

3. R is an antisymmetric relation, if and only if $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.

Example of a relation on a set

Example

Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution:

1. Because all the diagonal elements are equal to 1, R is reflexive.
2. Because M_R is symmetric, R is symmetric and not antisymmetric because both $m_{1,2}$ and $m_{2,1}$ are 1.

Matrices for combinations of relations

1. The matrix of the union of two relations is the join (Boolean OR) between the matrices of the component relations:

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$$

2. The matrix of the intersection of two relations is the meet (Boolean AND) between the matrices of the component relations:

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$$

3. The matrix of the composite relation $R_1 \circ R_2$ is the Boolean product of the matrices of the component relations:

$$M_{R_1 \circ R_2} = M_{R_1} \odot M_{R_2}$$

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Representing relations using directed graphs (a.k.a. *digraphs*)

Definition
A *directed graph*, or *digraph*, consists of a set V of *vertices* or *nodes* together with a set E of ordered pairs of elements of V called (directed) *edges* or *arcs*.

1. The vertex a is called the *initial vertex* of the edge (a, b), and the vertex b is called the *terminal vertex* of this edge.
2. An edge of the form (a, a) is called a *loop*.

Example

The directed graph with vertices a, b, c, and d, and edges $(a, b), (a, d), (b, b), (b, d), (c, a), (c, b)$, and (d, b) is shown here:
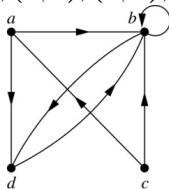
Examples of digraphs representing relations

Example

What are the ordered pairs in
the relation represented by this directed graph?
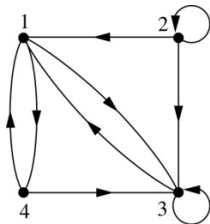
Solution:

The ordered pairs in the relation are:

❶ $(1, 3)$ ❹ $(2, 2)$ ❼ $(3, 3)$

❷ $(1, 4)$ ❺ $(2, 3)$ ❽ $(4, 1)$

❸ $(2, 1)$ ❻ $(3, 1)$ ❾ $(4, 3)$
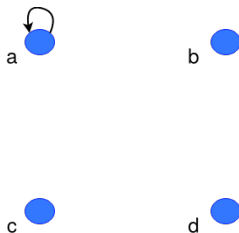
Determining which properties a relation has from its digraph

1. *Reflexivity*: A loop must be present at all vertices.

2. *Symmetry*: If (x, y) is an edge, then so is (y, x).

3. *Antisymmetry*: If (x, y) with $x \neq y$ is an edge, then (y, x) is not an edge.

4. *Transitivity*: If (x, y) and (y, z) are edges, then so is (x, z).

Determining which properties a relation has from its digraph – Example 1

1. *Reflexive?*

   No, not every vertex has a loop

2. *Symmetric?*

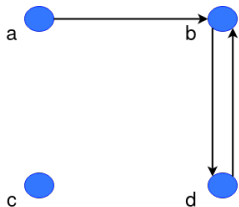   Yes (trivially), there is no edge from one vertex to another

3. *Antisymmetric?*

   Yes (trivially), there is no edge from one vertex to another

4. *Transitive?*

   Yes, (trivially) since there is no edge from one vertex to another

Determining which properties a relation has from its digraph – Example 2

1. *Reflexive?*

   No, there are no loops

2. *Symmetric?*

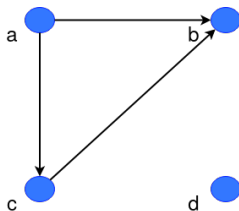   No, there is an edge from *a* to *b*, but not from *b* to *a*

3. *Antisymmetric?*

   No, there is an edge from *d* to *b* and *b* to *d*

4. *Transitive?*

   No, there are edges from *a* to *b* and from *b* to *d*, but there is no edge from *a* to *d*

Determining which properties a Relation has from its digraph – Example 3

1. *Reflexive?*

   No, there are no loops

2. *Symmetric?*

   No, for example, there is no edge from *c* to *a*
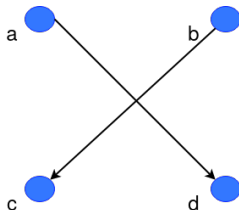
3. *Antisymmetric?*

   Yes, whenever there is an edge from one vertex to another, there is not one going back

4. *Transitive?*

   Yes

Determining which properties a relation has from its digraph – Example 4

1. *Reflexive?*

   No, there are no loops

2. *Symmetric?*

   No, for example, there is no edge from *d* to *a*

3. *Antisymmetric?*
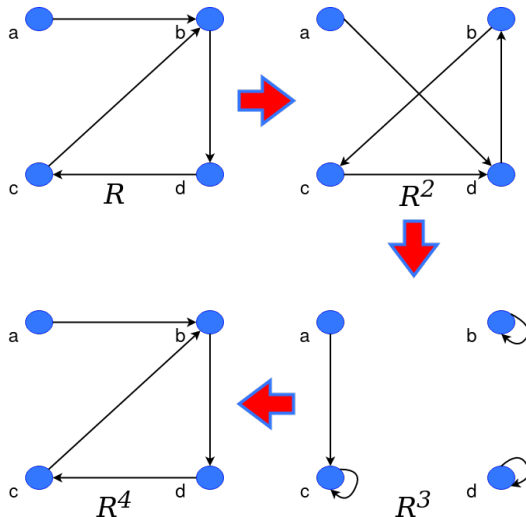
   Yes, whenever there is an edge from one vertex to another, there is not one going back

4. *Transitive?*

   Yes (trivially), there are no two edges where the first edge ends at the vertex where the second edge begins

Example of the powers of a relation

The pair (x, y) is in R^n if there is a path of length n from x to y in R (following the direction of the arrows).

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Equivalence relations

Definition
A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Definition
Two elements a and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Example
Assume C is the set of all cars and a relation R on C such that $R = \{(a, b) \mid a \in C, b \in C \text{ and, } a \text{ and } b \text{ have the same color } \}$.

R is an *equivalence relation* on C.

Strings

Example

Suppose that R is the relation on the set of strings of English letters such that $(a, b) \in R$ if and only if $\ell(a) = \ell(b)$, where $\ell(x)$ is the length of the string x. Is R an equivalence relation?

Solution: Show that all of the properties of an equivalence relation hold:

1. *Reflexivity*: Because $\ell(a) = \ell(a)$, it follows that $(a, a) \in R$ for all strings a.

2. *Symmetry*: Assume $(a, b) \in R$. Since $\ell(a) = \ell(b)$, then $\ell(b) = \ell(a)$ also holds and we have $(b, a) \in R$.

3. *Transitivity*: Suppose that $(a, b) \in R$ and $(b, c) \in R$. Since $\ell(a) = \ell(b)$ and $\ell(b) = \ell(c)$ both hold, then $\ell(a) = \ell(c)$ also holds and we have $(a, c) \in R$.

Yes, R is an equivalence relation.

Congruence modulo m

Example

Let m be an integer with $m > 1$. Show that the relation
$$R = \{(a, b) \mid a \equiv b \pmod{m}\}$$
is an equivalence relation on the set of integers.

Solution: Recall that $a \equiv b \pmod{m}$ if and only if m divides $a - b$.

❶ *Reflexivity*:

$a \equiv a \pmod{m}$ since $a - a = 0$ is divisible by m since $0 = 0 \cdot m$.

❷ *Symmetry*:

ⓐ Suppose that $a \equiv b \pmod{m}$.

ⓑ Then $a - b$ is divisible by m, and so $a - b = km$, where k is an integer.

ⓒ It follows that $b - a = (-k)m$, so $b \equiv a \pmod{m}$.

❸ *Transitivity*:

ⓐ Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$.

ⓑ Then m divides both $a - b$ and $b - c$.

ⓒ Hence, there are integers k and ℓ with $a - b = km$ and $b - c = \ell m$.

ⓓ We obtain by adding the equations:
$a - c = (a - b) + (b - c) = km + \ell m = (k + \ell)m$.

ⓔ Therefore, $a \equiv c \pmod{m}$.

Divides

Example
Show that the "divides" relation on the set of positive integers is not an equivalence relation.

Solution:

1. *Reflexive*:

   $a \mid a$ for all a.

2. *Not symmetric*:

   For example, $2 \mid 4$, but $4 \nmid 2$. Hence, the relation is not symmetric.

3. *Transitivite*:

   a. Suppose that a divides b and b divides c.
   b. Then there are positive integers k and ℓ such that $b = ak$ and $c = b\ell$.
   c. Hence, we have $c = a(k\ell)$. So, we have: a divides c.
   d. Therefore, the relation is transitive.

The properties of reflexivity, and transitivity do hold, but the relation is not symmetric. Hence, "divides" is not an equivalence relation.

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Equivalence classes

Definition

Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a, denoted by $[a]_R$.

$$[a]_R := \{s \in A \mid (a, s) \in R\} \equiv \{s \in A \mid s \sim a\}$$

When only one relation is under consideration, we can write $[a]$, without the subscript R, for this equivalence class.

1. If $b \in [a]_R$, then b is a *representative* of this equivalence class. Any element of a class can be used as representative.

2. The equivalence classes of the relation "congruence modulo m" are called the *congruence classes modulo m*. The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\ldots, a - 2m, a - m, a + m, a + 2m, \ldots\}$.

$[0]_4 = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$ $[1]_4 = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$

$[2]_4 = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$ $[3]_4 = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

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Equivalence classes and partitions

Theorem

Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

1. $(a, b) \in R$,
2. $[a] = [b]$,
3. $[a] \cap [b] \neq \varnothing$.

We show $(i) \rightarrow (ii)$. All other implications are proved similarly.

1. Assume that $(a, b) \in R$.
2. Now suppose that $c \in [a]$.
3. Then $(a, c) \in R$.
4. Because $(a, b) \in R$ and R is symmetric, $(b, a) \in R$.
5. Because R is transitive and $(b, a) \in R$ and $(a, c) \in R$, it follows that $(b, c) \in R$.
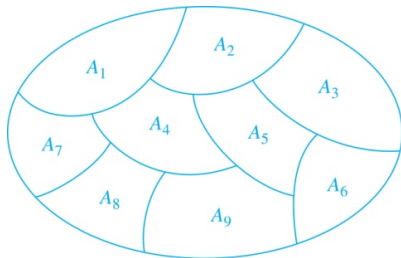6. Hence, $c \in [b]$. Therefore, $[a] \subseteq [b]$.

A similar argument proved $[b] \subseteq [a]$. Hence, we have $[b] = [a]$.

Partition of a set

Definition

A *partition* of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, where $i \in I$ (where I is an index set), forms a partition of S if and only if

1 $A_i \neq \varnothing$ for $i \in I$,

2 $A_i \cap A_j = \varnothing$ when $i \neq j$,

3 and $\bigcup\limits_{i \in I} A_i = S$

A partition of a set

An equivalence relation partitions a set

1. Let R be an equivalence relation on a set A. The union of all the equivalence classes of R is all of A, since an element a of A is in its own equivalence class $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_r = A$$

2. From the previous theorem, it follows that these equivalence classes are either equal or disjoint, so $[a]_R \cap [b]_R = \varnothing$ when $[a]_R \neq [b]_R$.

3. Therefore, the equivalence classes form a partition of A, because they split A into disjoint subsets.

An equivalence relation partitions a set

Theorem

Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets $A_i, i \in I$, as its equivalence classes.

Proof.

We have already shown the first part of the theorem. For the second part, assume that $\{A_i \mid i \in I\}$ is a partition of S. Let R be the relation on S consisting of the pairs (x, y) where x and y belong to the same subset A_i in the partition. We must show that R satisfies the properties of an equivalence relation.

① *Reflexivity*:

For every $a \in S, (a, a) \in R$, because a is in the same subset as itself.

② *Symmetry*:

If $(a, b) \in R$, then b and a are in the same subset of the partition, so $(b, a) \in R$.

③ *Transitivity*:

If $(a, b) \in R$ and $(b, c) \in R$, then a and b are in the same subset of the partition, as are b and c. Since the subsets are disjoint and b belongs to both, the two subsets of the partition must be identical. Therefore, $(a, c) \in R$ since a and c belong to the same subset of the partition.
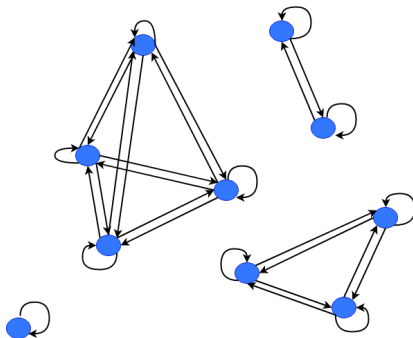
∎

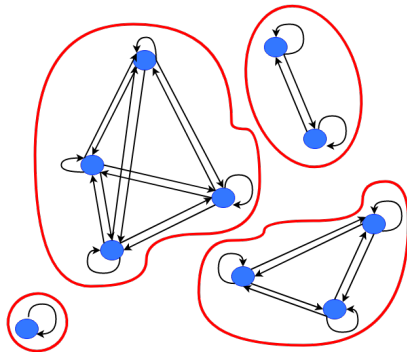# An equivalence relation digraph representation
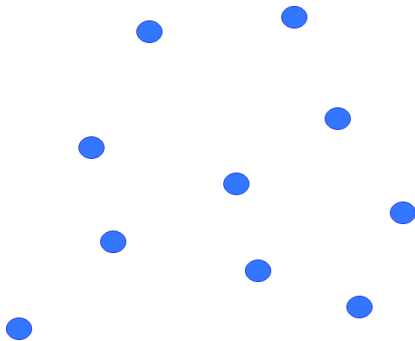
Set A.

An Equivalence Relation digraph representation

Equivalence relation R on set A.

An Equivalence Relation digraph representation

Digraph for equivalence relation R on finite set A is a union of
disjoint sub-graphs (representing disjoint equivalent classes).
Nodes in each distinct subgraph (equivalence class) are fully
interconnected.

Plan for Chapter 9

Plan for Chapter 9

Partial orderings

Definition

A relation R on a set S is called a *partial ordering,* or *partial order*, if it is reflexive, antisymmetric, and transitive.

A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R). Members of S are called *elements* of the poset.

Partial orderings

Example

Show that the "greater than or equal" relation ($\geq$) is a partial ordering on the set of integers.

Solution:

1. *Reflexivity*:

   $a \geq a$ for every integer a.

2. *Antisymmetry*:

   If $a \geq b$ and $b \geq a$, then $a = b$.

3. *Transitivity*:

   If $a \geq b$ and $b \geq c$, then $a \geq c$.

These properties all follow from the order axioms for the integers.)

Partial orderings

Example

Show that the divisibility relation is a partial ordering on the set of positive integers.

Solution:

1. *Reflexivity*:

   $a \mid a$ for all integers a.

2. *Antisymmetry*:

   If a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$.

3. *Transitivity*:

   a. Suppose that a divides b and that b divides c.
   b. Then, there are positive integers k and ℓ such that $b = ak$ and $c = b\ell$.
   c. Hence, $c = a(k\ell)$, so that a divides c.
   d. Therefore, the relation is transitive.

$(Z^+, \mid)$ is a poset.

Partial orderings

Example

Show that the inclusion relation ($\subseteq$) is a partial ordering on the power set of a set S.

Solution:

1. *Reflexivity*:

   $A \subseteq A$ whenever A is a subset of S.

2. *Antisymmetry*:

   If A and B are subsets of S, with $A \subseteq B$ and $B \subseteq A$, then $A = B$.

3. *Transitivity*:

   If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

The properties all follow from the definition of set inclusion.

Comparability

Definition
The elements a and b of a poset $(S, \preceq)$ are *comparable* if either $a \preceq b$ or $b \preceq a$ holds. When a and b are elements of S so that neither $a \preceq b$ nor $b \preceq a$ holds, then a and b are called *incomparable*.

> The symbol $\preceq$ is used to denote the relation in any poset.

Definition
If $(S, \preceq)$ is a poset and any two elements of S are comparable, then S is called a *totally ordered* or *linearly ordered set*, and $\preceq$ is called a *total order* or a *linear order*. (A totally ordered set is also called a *chain*.)

Definition
$(S, \preceq)$ is a well-ordered set if it is a poset such that $\preceq$ is a total ordering <u>and</u> every nonempty subset of S has a least element.

Example
1. $(Z, \leq)$ is a totally ordered set
2. $(Z, |)$ is a partially ordered but not totally ordered set
3. $(N, \leq)$ is a well-ordered set

Plan for Chapter 9

Lexicographic order

Definition

Given two partially ordered sets $(A_1, \preceq_1)$ and $(A_2, \preceq_2)$, the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying when (a_1, a_2) is less than (b_1, b_2), written, $(a_1, a_2) \prec (b_1, b_2)$, which holds either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ and $a_2 \prec_2 b_2$ holds.

This definition can be easily extended to a lexicographic ordering on strings.

Example

Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet. This is the same ordering as that used in dictionaries.

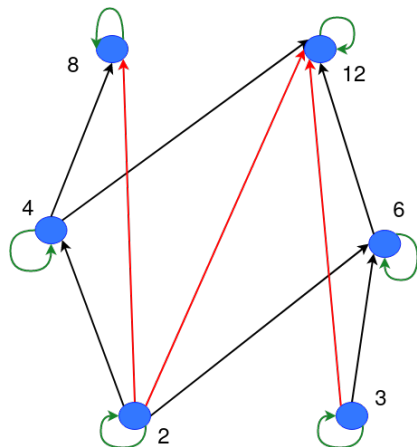1. *discreet* < *discrete*, because these strings differ in the seventh position and $e < t$.

2. *discreet* < *discreetness*, because the first eight letters agree, but the second string is longer.

Partial ordering relation digraph representation
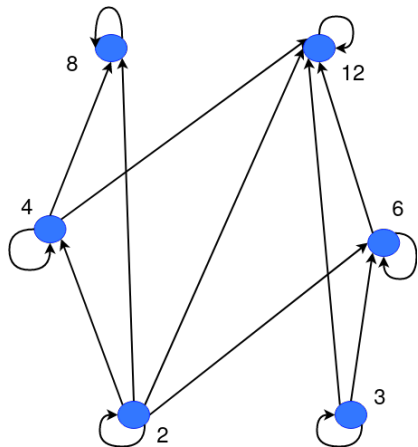
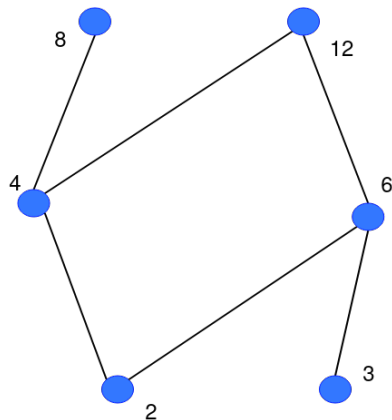poset $R = (X, |)$ for divisibility $|$ on set $X = \{2, 3, 4, 6, 8, 12\}$

Partial ordering relation Hesse diagram

poset $R = (X, |)$ for divisibility $|$ on set $X = \{2, 3, 4, 6, 8, 12\}$

❶ Leave out all edges that are implied by reflexivity (loop)
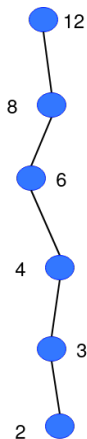❷ Leave out all edges that are implied by transitivity

Partial ordering relation Hesse diagram

poset $R = (X, |)$ for divisibility $|$ on set $X = \{2, 3, 4, 6, 8, 12\}$

Can also drop "direction" assuming that (partial) order is **upward**

Partial ordering relation Hesse diagram

poset $R = (X, \leq)$ for
"less than or equal" on set
$X = \{2, 3, 4, 6, 8, 12\}$

Totally ordered sets are also called "chains"