Assignment 4

Table of Contents

▲ Question Progress ▲

Part 2 of 3 - Problem 2

Question 2 of 3

30.0 Points

Problem (Structural Induction)

Quadtrees (https://en.wikipedia.org/wiki/Quadtree) play a fundamental in computer science. Here's a recursive definition of full quadtrees, similar to that of full binary trees given in the lectures. The set of *full quadtrees* can be defined recursively by these two steps.

1. BASIS STEP: There is a full quadtree consisting of only a single vertex r.
2. RECURSIVE STEP: If Q_1, Q_2, Q_3, Q_4 are pairwise disjoint full quadtrees, (that is, the quadtrees Q_i and Q_j have no vertex in common, for $1 \le i < j \le 4$) and if r is a vertex not belonging to Q_1, Q_2, Q_3 or Q_4, there is a full quadtree, denoted by (Q_1, Q_2, r, Q_3, Q_4), consisting of r as a root together with edges connecting r to each of the roots of Q_1, Q_2, Q_3, Q_4.

The *number of vertices* $n(Q)$ of a full quadtree Q satisfies the following recursive formula:

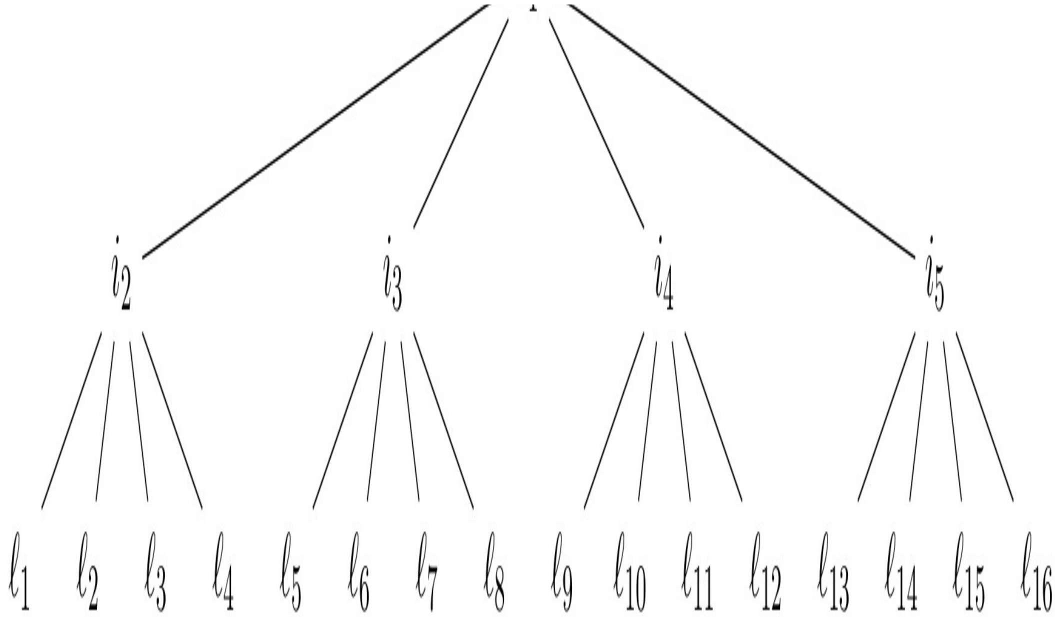
1. BASIS STEP: The number of vertices of a full quadtree Q consisting of only a root r is $n(Q) = 1$.
2. RECURSIVE STEP: If Q_1, Q_2, Q_3, Q_4 are pairwise disjoint full quadtrees, and if r is a vertex not belonging to Q_1, Q_2, Q_3 or Q_4, then the full quadtree $Q = (Q_1, Q_2, r, Q_3, Q_4)$ has the number of vertices: $n(Q) = 1 + n(Q_1) + n(Q_2) + n(Q_3) + n(Q_4)$.

As for full binary trees, a vertex of a full quadtree Q is called a *leaf* if it is connected to at most one other vertex of Q. Similarly, a vertex of a full quadtree Q is called an *internal node* if it is connected to more than one vertex of Q. For a full quadtree Q, we denote by $\ell(Q)$ the number of leaves of Q and by $i(Q)$ the number of internal nodes of Q. Note that we have:

1. if Q consisting of only a root r then $\ell(Q) = 1$ and $i(Q) = 0$
2. If Q_1, Q_2, Q_3, Q_4 are pairwise disjoint full quadtrees, and if r is a vertex not belonging to Q_1, Q_2, Q_3 or Q_4, then the for the full quadtree $Q = (Q_1, Q_2, r, Q_3, Q_4)$ we have: $\ell(Q) = \ell(Q_1) + \ell(Q_2) + \ell(Q_3) + \ell(Q_4)$ and $i(Q) = 1 + i(Q_1) + i(Q_2) + i(Q_3) + i(Q_4)$.

The *height* $h(Q)$ of a full quadtree Q is defined recursively as follows:

1. BASIS STEP: The height of a full quadtree Q consisting of only a root r is $h(Q) = 0$.



The above figure shows a quadtree with

- 21 as number of vertices,
- 5 internal nodes, namely i_1, i_2, i_3, i_4, i_5
- 16 leaves, namely $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6, \ell_7, \ell_8, \ell_9, \ell_{10}, \ell_{11}, \ell_{12}, \ell_{13}, \ell_{14}, \ell_{15}, \ell_{16}$.
- 2 as height.

Answer the following questions:

1. Prove that for any full quadtree Q, we have: $n(Q) = \ell(Q) + i(Q)$.
2. Prove by structural induction that for any full quadtree Q, we have: $\ell(Q) \leq 4^{h(Q)}$.
3. Prove by structural induction that for any full quadtree Q, we have:
   $n(Q) \leq \sum_{j=0}^{h} 4^j$ where $h = h(Q)$.

Click "Browse" to locate your file and then click "Upload" to upload your file.
(Maximum file size: 40MB)
File:
浏览... 未选择文件。

- Gateway
- Help & Support
- Western University