Turing Machines

COMPSCI 3331

Turing Machines: Outline

- Motivation.
- Formal Definitions.
- Examples.

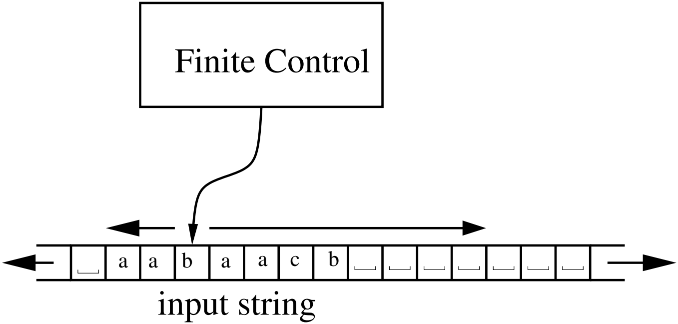
Turing Machines

- Both regular languages and CFLs can't define some languages.
- Turing machines (TMs): a formal model capable of accepting more languages.
- TMs represent our notion of **what is computable**.

Turing Machines

- Basic concept is the same: finite control, input is read sequentially (from a “tape”).
- However, now the **input tape is read/write**.
  - For DFAs, NFAs, PDAs, the input tape was read-only.
- A TM can move either way on the input tape.
  - For DFAs, NFAs, PDAs, could only move to right (or stay in the same place).

Turing Machines



Alan Turing (1912–1954)

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"[Any person] provided with paper, pencil, and [eraser], and subject to strict discipline, is in effect **a universal Turing Machine.**" (1948)

Turing Machines

A Turing Machine is a seven-tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ where

- Q is the finite set of states,
- Σ, Γ are the input and tape alphabets ($\Sigma \subseteq \Gamma$),
- $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$ is the transition function .
- $q_0 \in Q$ is the start state; $F \subseteq Q$ is the set of final states.
- $B \in \Gamma - \Sigma$ is the blank symbol.

Transition Function

$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R, S\}$$

- $\delta(q, \alpha) = (q', \beta, D)$.
- If we are in state q and currently see tape symbol $\alpha \in \Gamma$ on the tape, we
  - (a) go to state $q' \in Q$.
  - (b) rewrite α by β in the current cell of the tape.
  - (c) move the input head in direction D on the tape: L (left), R (right) or S (stationary).

Representing TMs

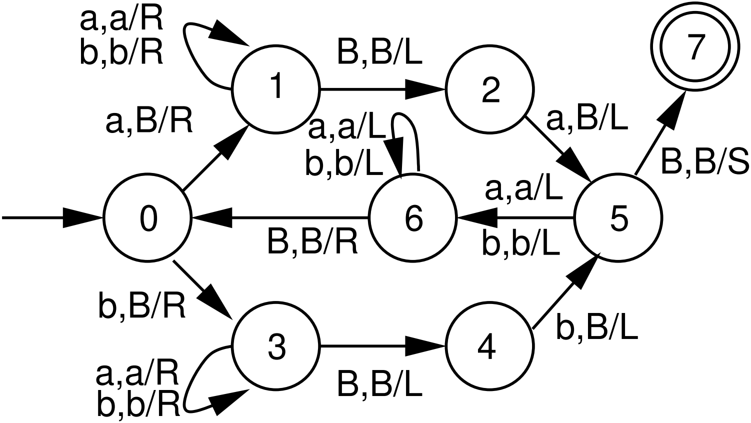
We can represent the transition $\delta(q, \alpha) = (q', \beta, D)$ as an arc:

Computation of a Turing Machine

How does a TM compute?

- The input word x is initially written on the tape, and we start in state q_0. We point at the left-most symbol of x.
- Based on the current symbol on the tape and the current state, we make the move based on the transition function.
- We keep making moves as long as possible.
- We **can** move off the region occupied by x on the tape (these cells contain the blank symbol by default).
- If the TM enters an final state, the word is accepted.
- Otherwise, the string is not accepted.

Example of a TM



Instantaneous Description of a TM

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ be a TM. An instantaneous description (ID) of M is a word from $\Gamma^* Q \Gamma^*$.

Let $x_1 x_2 \cdots x_{i-1} q x_i x_{i+1} \cdots x_n \in \Gamma^* Q \Gamma^*$. This means that

- The non-blank symbols on the input tape from left-to-right are $x_1 x_2 x_3 \cdots x_n$.
  - (Symbols may be a blank if $i = 1$ or $i = n$.)
- The TM M's head is currently pointing at x_i.
- The TM M is currently in state q.

Moves of a TM

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ be a TM. We denote by $\vdash_M$ the relation between IDs given by the transition function. $\delta(q, x_i) = (q', \beta, L)$. Then we have the following cases:

- If $i > 1$, then

$$x_1 x_2 \cdots x_{i-1} q x_i x_{i+1} \cdots x_n \vdash_M x_1 x_2 \cdots x_{i-2} q' x_{i-1} \beta x_{i+1} \cdots x_n.$$

- If $i = 1$, then

$$q x_1 x_2 \cdots x_n \vdash_M q B \beta x_2 \cdots x_n.$$

Moves of a TM

If $\delta(q, x_i) = (q', \beta, R)$, we have two cases:

- If $i < n$, then

$$x_1 x_2 \cdots x_{i-1} q x_i x_{i+1} \cdots x_n \vdash_M x_1 x_2 \cdots x_{i-1} \beta q' x_{i+1} \cdots x_n.$$

- If $i = n$, then

$$x_1 x_2 \cdots q x_n \vdash_M x_1 x_2 \cdots \beta q B.$$

Moves of a TM

If $\delta(q, x_i) = (q', \beta, S)$,

$$x_1 x_2 \cdots x_{i-1} q x_i x_{i+1} \cdots x_n \vdash_M x_1 x_2 \cdots x_{i-1} q' \beta x_{i+1} \cdots x_n.$$

We denote by $\vdash_M^*$ the fact that two IDs are related by zero or more applications of $\vdash$.

Language Acceptance

The language **accepted** by a TM $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ is defined as follows:

$$L(M) = \{w \in \Sigma^* \ : \ \exists x_1, x_2 \in \Gamma^*, q_f \in F, q_0 w \vdash_M^* x_1 q_f x_2\}.$$

Examples:

- $L = \{a^n b^{n^2} \ : \ n \geq 0\}$.
- $L = \{a^n b^n c^n \ : \ n \geq 0\}$.

Halting and Crashing

- We say that a TM **halts** if it enters a state and has no next move.
- Informally, we say that a TM **crashes** if it enters a state that is not final and then has no next move (i.e., halts and rejects).
- For any TM, we can assume that when it enters a final state, it halts.
- That is, for every final state $q_f \in F$, $\delta(q_f, \alpha)$ is undefined for all $\alpha \in \Gamma$.

Some questions...

- What kinds of languages can TMs accept?
- What kinds of languages can't be accepted by a TM?
- Can every CFL be accepted by a TM?
- What about nondeterminism for TMs?