CHAPTER 4
Propositional Proofs

Formal proofs help us to develop reasoning skills and are a convenient way to test arguments of various systems. From now on, formal proofs will be our main method of testing arguments.

4.1 Two sample proofs

A formal proof breaks a complicated argument into a series of small steps. Since most steps are based on our S- and I-rules (see Sections 3.10–13), you may want to review these now and then as you learn to do formal proofs.

Let's start with a proof in English. Suppose that we want to show that the butler committed the murder. We know these facts:

1. The only people in the mansion were the butler and the maid.
2. If the only people in the mansion were the butler and the maid, then the butler or the maid did it.
3. If the maid did it, then she had a motive.
4. The maid didn't have a motive.

Using an indirect strategy, we first assume that the butler didn't do it. Then we show that this leads to a contradiction:

5. Assume: The butler didn't do it.
6. ∴ Either the butler or the maid did it. {from 1 and 2}
7. ∴ The maid did it. {from 5 and 6}
8. ∴ The maid had a motive, {from 3 and 7}

Given our premises, we get a contradiction (between 4 and 8) if we assume that the butler *didn't* do it. So our premises entail that the butler *did* it. If we can be confident of the premises, then we can be confident that the butler did it.

This formal proof mirrors in symbols how we reasoned in English:

	1	T	**Valid**
*	2	(T ⊃ (B ∨ M))	
*	3	(M ⊃ H)	
	4	~H ⇦	
		[∴ B	
	5	┌ asm: ~B	
*	6	│ ∴ (B ∨ M) {from 1 and 2}	
	7	│ ∴ M {from 5 and 6}	
	8	└ ∴ H {from 3 and 7} ⇦	
	9	∴ B {from 5; 4 contradicts 8}	

First we block off the conclusion "B" (showing that we can't use it to derive further lines) and assume its opposite "~B." Then we derive further lines using the S- and I-rules until we get a contradiction. We get line 6 from 1 and 2 by the rule that goes "If-then, affirm first, so affirm second." We get 7 by the rule that goes "At least one is true, this one isn't, so the other is." And 8 follows by the rule that goes "If-then, affirm first, so affirm second." But 4 and 8 contradict. So 9 follows, using **RAA,** our new *reductio ad absurdum* (reduction to absurdity) rule, which says that an assumption that leads to a contradiction must be false. As we apply RAA, we block off lines 5 through 8 to show that we can't use them in deriving further lines. Thus we prove the argument valid.

There can be various ways to do a proof. Instead of deriving "H" in line 8, we could have used 3 and 4 to get "~M," which would contradict 7. Inferences and contradictions can use any earlier lines that aren't already blocked off.[1]

We starred lines 2, 3, and 6 when we used them. Starred lines largely can be ignored in deriving further steps. Here are the starring rules—with examples:

Star any wff that you simplify using an S-rule.	Star the *longer* wff used in an I-rule inference.
* (A · B) ——— ∴ A ∴ B	* (A ⊃ B) A ——— ∴ B

Starred lines are redundant, since shorter lines have the same information. When doing proofs, focus on deriving things from *unstarred* complex wffs (where a "complex wff" is anything longer than a letter or its negation).[2]

Here's another formal proof. Again, we first assume that the conclusion is false; then we show that, given our premises, it couldn't be false, since this leads to a contradiction:

```
*  1     ((A · B) ⊃ C)   Valid
   2     B  ⇐
       [ ∴ (C ∨ ~A)
*  3   ┌ asm: ~(C ∨ ~A)
   4   │ ∴ ~C   {from 3}
   5   │ ∴ A   {from 3}
*  6   │ ∴ ~(A · B)   {from 1 and 4}
   7   └ ∴ ~B   {from 5 and 6}  ⇐
   8   ∴ (C ∨ ~A)   {from 3; 2 contradicts 7}
```

1 Each derived step in this book has a brief justification saying what pervious steps were used. Some people like to refer to the inference rule (like "I-rules" or "I-5" or "MP") too. Other people like to skip justifications—since it's usually easy to tell where lines are from.

2 Starring isn't a necessary part of proofs; but you may find it helpful—because it focuses your attention away from redundant lines that aren't going to help you derive further steps.

We start by blocking off the conclusion (showing that we can't use it to derive further lines) and assuming its opposite. Then we derive further lines using the S- and I-rules until we get a contradiction. Line 3 simplifies into lines 4 and 5 (and then gets starred). Lines 1 and 4 give us 6 (and then 1 gets starred). Lines 5 and 6 give us 7 (and then 6 gets starred). But 2 and 7 contradict. Since assumption 3 leads to a contradiction, we apply RAA. We block off lines 3 through 7 to show that we can't use them in deriving further lines.[1] We derive our original conclusion in line 8. Thus we prove the argument valid.

If we try to prove an invalid argument, we won't succeed; instead, we'll be led to refute the argument. We'll see invalid arguments later.

4.2 Easier proofs

Now we'll learn the rules behind formal proofs. We'll use these inference rules, which hold regardless of what pairs of contradictory wffs replace "P"/"~P" and "Q"/"~Q" (here "→" means we can infer whole lines from left to right):

S-rules (Simplifying): S-1 to S-6	*I-rules (Inferring): I-1 to I-6*
(P·Q)→P, Q	~(P·Q), P → ~Q
~(P∨Q) → ~P, ~Q	~(P·Q), Q → ~P
~(P⊃Q) → P, ~Q	
~~ P → P	(P∨Q), ~P → Q
	(P∨Q), ~Q → P
(P≡Q) → (P⊃Q), (Q⊃P)	(P⊃Q), ~P → Q
~(P≡Q) → (P∨Q), ~(P·Q)	(P⊃Q), ~O → ~P

Read "(P·Q)→P, Q" as "from '(P·Q)' one may derive 'P' and also 'Q.'"

Three rules are new. Rule S-4 eliminates "~~" from the beginning of a wff. S-5 breaks a biconditional into two conditionals. S-6 breaks up the denial of a biconditional; since "(P≡Q)" says that P and Q have the same truth value,
"~(P≡Q)" says that P and Q have different truth values—so one or the other is true, but not both. None of these three rules is used very much.[1]

Here are key definitions:

- A **premise** is a line consisting of a wff by itself (with no "asm:" or "∴").
- An **assumption** is a line consisting of "asm:" and then a wff.
- A **derived step** is a line consisting of "∴" and then a wff.

[1] Since the problem is done, why bother to block off lines 3 to 7? The answer is that blocking off will become important later on, when we get to multiple-assumption arguments.

[1] S-1 to S-6 also work in the other direction (for example, "(A·B)" follows from "A" and "B"); but our proofs and software use the S-rules only to simplify. The LogiCola software does, however, let you use two additional rules for the biconditional: (1) Given "(A≡B)": if you have one side true you can get the other side true—and if you have one side false you can get the other side false. (2) Given "~(A≡B)": if you have one side true you can get the other side false—and if you have one side false you can get the other side true.

- A **formal proof** is a vertical sequence of zero or more premises followed by one or more assumptions or derived steps, where each derived step follows from previously not-blocked-off lines by RAA or one of the inference rules listed above, and each assumption is blocked off using RAA.[2]
- Two wffs are **contradictory** if they are exactly alike except that one starts with an additional "~."
- A **simple wff** is a letter or its negation; any other wff is **complex**.

Rule RAA says an assumption is false if it leads to contradictory wffs; these wffs may occur anywhere in the proof (as premises or assumptions or derived steps), so long as neither is blocked off. Here's a more precise formulation:

> **RAA:** Suppose that some pair of not-blocked-off lines have contradictory wffs. Then block off all the lines from the last not-blocked-off assumption on down and infer a step consisting in "∴" followed by a contradictory of that assumption.

Here's another example of a formal proof:

```
    1    A  ⇦      Valid
 *  2    (A ⊃ B)
       [ ∴ (A · B)
 *  3  ┌ asm: ~(A · B)
    4  │ ∴ B   {from 1 and 2}
    5  └ ∴ ~A   {from 3 and 4}   ⇦
    6  ∴ (A · B)   {from 3; 1 contradicts 5}
```

First we assume the contradictory of the conclusion. Then we derive things until we get a contradiction; here line 5 contradicts premise 1. Finally, we block off the wffs from the assumption on down and derive the opposite of the assumption; this gives us our original conclusion.

In this next example, it's easier to assume the simpler contradictory of the conclusion—to assume "C" instead of "~~C":

```
 *  1    (A · ~B)   Valid
 *  2    (C ⊃ B)
       [ ∴ ~C
    3  ┌ asm: C  ⇦
    4  │ ∴ A   {from 1}
    5  │ ∴ ~B   {from 1}
    6  └ ∴ ~C   {from 2 and 5}   ⇦
    7  ∴ ~C   {from 3; 3 contradicts 6}
```

[2] By this definition, the stars, line numbers, blocked off original conclusion, and justifications aren't strictly part of the proof; instead, they are unofficial helps.

Line 6 contradicts the assumption; line 4 wasn't needed and could be omitted.

Don't use the original conclusion to derive further steps or be part of a contradiction; blocking off the original conclusion reminds us not to use it in the proof. And be sure to assume a genuine contradictory of the conclusion:

Right:	*Wrong:*
[∴ (~A ⊃ B)	[∴ (~A ⊃ B)
asm: ~(~A ⊃ B)	asm: (A ⊃ B)

To form a contradictory, you may add a squiggle to the beginning. If the wff already begins with a squiggle (and not with a left-hand parenthesis), you may instead omit the squiggle.

For now, I suggest this strategy for constructing a proof for an argument:

1. START: Block off the conclusion and add "asm:" followed by the conclusion's simpler contradictory.
2. S&I: Go through the unstarred, complex,[1] not-blocked-off wffs and use these to derive new wffs using the S- and I-rules. Star any wff you simplify using an S-rule, or the longer wff used in an I-rule inference.
   Note: While you needn't derive wffs again that you already have, you can star wffs that would give you what you already have.
3. RAA: When some pair of not-blocked-off lines contradict, apply RAA and derive the original conclusion. Your proof is done!

This strategy can prove most valid propositional arguments. We'll see later that some arguments need multiple assumptions and a more complex strategy.

4.2a Exercise—LogiCola F (TE & TH) and GEV

Prove each of these arguments to be valid (all are valid).

(A ∨ B)
∴ (~A ⊃ B)

```
* 1     (A ∨ B)   Valid
      [ ∴ (~A ⊃ B)
* 2   ┌ asm: ~(~A ⊃ B)
  3   │ ∴ ~A    {from 2}
  4   │ ∴ ~B    {from 2}
  5   └ ∴ B   {from 1 and 3}
  6   ∴ (~A ⊃ B)   {from 2; 4 contradicts 5}
```

1. (A ⊃ B)
   ∴ (~B ⊃ ~A)
2. A
   ∴ (A ∨ B)
3. (A ⊃ B)
   (~A ⊃ B)
   ∴ B
4. ((A ∨ B) ⊃ C)
   ∴ (~C ⊃ ~B)
5. (A ∨ B)
   (A ⊃ C)
   (B ⊃ D)
   ∴ (C ∨ D)
6. (A ⊃ B)
   (B ⊃ C)
   ∴ (A ⊃ C)
7. (A ≡ B)
   ∴ (A ⊃ (A · B))
8. ~(A ∨ B)
   (C ∨ B)
   ~(D · C)
   ∴ ~D
9. (A ⊃ B)
   ~B
   ∴ (A ≡ B)
10. (A ⊃ (B ⊃ C))
    ∴ ((A · B) ⊃ C)

[1] Recall that a "complex wff" is anything longer than a letter or its negation.

4.2b Exercise—LogiCola F (TE & TH) and GEV

First appraise intuitively. Then translate into logic (using the letters given) and prove to be valid (all are valid).

1. If Heather saw the butler putting the tablet into the drink and the tablet was poison, then the butler killed the deceased.
   Heather saw the butler putting the tablet into the drink.
   ∴ If the tablet was poison, then the butler killed the deceased. [Use H, T, and B.]

2. If we had an absolute proof of God's existence, then our will would be irresistibly attracted to do right.
   If our will were irresistibly attracted to do right, then we'd have no free will.
   ∴ If we have free will, then we have no absolute proof of God's existence. [Use P, I, and F. This argument is from Immanuel Kant and John Hick, who used it to explain why God doesn't make his existence more evident.]

3. If racism is clearly wrong, then either it's factually clear that all races have equal abilities or it's morally clear that similar interests of all beings ought to be given equal consideration.
   It's not factually clear that all races have equal abilities.
   If it's morally clear that similar interests of all beings ought to be given equal consideration, then it's clear that similar interests of animals and humans ought to be given equal consideration.
   ∴ If racism is clearly wrong, then it's clear that similar interests of animals and humans ought to be given equal consideration. [Use W, F, M, and A. This argument is from Peter Singer, who fathered the animal liberation movement.]

4. The universe is orderly (like a watch that follows complex laws).
   Most orderly things we've examined have intelligent designers.
   We've examined a large and varied group of orderly things.
   If most orderly things we've examined have intelligent designers and we've examined a large and varied group of orderly things, then probably most orderly things have intelligent designers.
   If the universe is orderly and probably most orderly things have intelligent designers, then the universe probably has an intelligent designer.
   ∴ The universe probably has an intelligent designer. [Use U, M, W, P, and D. This is a form of the argument from design for the existence of God.]

5. If God doesn't want to prevent evil, then he isn't all good.
   If God isn't able to prevent evil, then he isn't all powerful.
   Either God doesn't want to prevent evil, or he isn't able.
   ∴ Either God isn't all powerful, or he isn't all good. [Use W, G, A, and P. This form of the problem-of-evil argument is from the ancient Greek Empiricus.]

6. If Genesis gives the literal facts, then birds were created before humans. (Genesis 1:20–26)
   If Genesis gives the literal facts, then birds were not created before humans. (Genesis 2:5–20)

∴ Genesis doesn't give the literal facts. [Use L and B. Origen, an early Christian thinker, gave similar textual arguments against taking Genesis literally.]

7. The world had a beginning in time.
If the world had a beginning in time, there was a cause for the world's beginning.
If there was a cause for the world's beginning, a personal being caused the world.
∴ A personal being caused the world. [Use B, C, and P. This "kalam argument" for the existence of God is from William Craig and James Moreland; they defend premise 1 by various considerations, including the big-bang theory, the law of entropy, and the impossibility of an actual infinite.]

8. If the world had a beginning in time and it didn't just pop into existence without any cause, then the world was caused by God.
If the world was caused by God, then there is a God.
There is no God.
∴ Either the world had no beginning in time, or it just popped into existence without any cause. [Use B, P, C, and G. This argument is from J.L.Mackie, who based his "There is no God" premise on the problem-of-evil argument.]

9. Closed systems tend toward greater entropy (a more randomly uniform distribution of energy). (This is the second law of thermodynamics.)
If closed systems tend toward greater entropy and the world has existed through endless time, then the world would have achieved almost complete entropy (for example, everything would be about the same temperature).
The world has not achieved almost complete entropy.
If the world hasn't existed through endless time, then the world had a beginning in time.
∴ The world had a beginning in time. [Use G, E, C, and B. This argument is from William Craig and James Moreland.]

10. If time stretches back infinitely, then today wouldn't have been reached.
If today wouldn't have been reached, then today wouldn't exist.
Today exists.
If time doesn't stretch back infinitely, then there was a first moment of time.
∴ There was a first moment of time. [Use I, R, T, and F.]

11. If there are already laws preventing discrimination against women, then if the Equal Rights Amendment (ERA) would rob women of many current privileges then passage of the ERA would be against women's interests and women ought to work for its defeat.
The ERA would rob women of many current privileges (like draft exemption).
∴ If there are already laws preventing discrimination against women, then women ought to work for the defeat of the ERA. [Use L, R, A, and W.]

12. If women ought never to be discriminated against, then we should work for current laws against discrimination and also prevent future generations from imposing discriminatory laws against women.
    The only way to prevent future generations from imposing discriminatory laws against women is to pass an Equal Rights Amendment (ERA).
    If we should prevent future generations from imposing discriminatory laws against women and the only way to do this is to pass an ERA, then we ought to pass an ERA.
    ∴ If women ought never to be discriminated against, then we ought to pass an ERA. [Use N, C, F, O, and E.]

13. If the claim that knowledge-is-impossible is true, then we understand the word "know" but there are no cases of knowledge.
    If we understand the word "know," then we understand "know" either from a verbal definition or from experienced examples of knowledge.
    If we understand "know" from a verbal definition, then there's an agreed-upon definition of "know."
    There's no agreed-upon definition of "know."
    If we understand "know" from experienced examples of knowledge, then there are cases of knowledge.
    ∴ The claim that knowledge-is-impossible is false. [Use I, U, C, D, E, and A. This is a form of the paradigm-case argument.]

14. If *p* is the greatest prime, then *n* (we may stipulate) is one plus the product of all the primes less than *p*.
    If *n* is one plus the product of all the primes less than *p,* then either *n* is prime or else *n* isn't prime but has prime factors greater than *p*.
    If *n* is prime, then *p* isn't the greatest prime.
    If *n* has prime factors greater than *p,* then *p* isn't the greatest prime.
    ∴ *p* isn't the greatest prime. [Use G, N, P, and F. This proof that there's no greatest prime number is from the ancient Greek mathematician Euclid.]

4.3 Easier refutations

If we try to prove an invalid argument, we won't succeed; instead, we'll be led to *refute* the argument. This version of an earlier example is invalid because it drops a premise; if we try a proof, we get no contradiction:

The only people in the mansion were the butler and the maid.
If the only people in the mansion were the butler and the maid, then the butler or the maid did it.
If the maid did it, then she had a motive.
∴ The butler did it.

```
    1   T                          Invalid
*   2   (T ⊃ (B ∨ M))          ┌──────────────┐
*   3   (M ⊃ H)                │ T, M, H, ~B  │
      [ ∴ B                    └──────────────┘
    4   asm: ~B
*   5   ∴ (B ∨ M)   {from 1 and 2}
    6   ∴ M   {from 4 and 5}
    7   ∴ H   {from 3 and 6}
```

We can show the argument to be invalid by giving a **refutation**—a set of truth conditions making the premises all true and conclusion false. To get the refutation, we take the simple wffs (letters or their negation) from not-blocked-off lines and put them in a box (their order doesn't matter); here the simple wffs come from lines 1, 4, 6, and 7. If the refutation has a letter by itself (like "T" or "M"), then we mark that letter *true* ("1") in the argument; if it has the negation of a letter (like "~B"), then we mark that letter *false* ("0"):

$$\begin{array}{ll} T^1 = 1 & \textbf{Invalid} \\ (T^1 \supset (B^0 \vee M^1)) = 1 & \\ (M^1 \supset H^1) = 1 & \\ \therefore B^0 = 0 & \end{array}$$

Our truth conditions make the premises all true and conclusion false—showing the argument to be invalid. We also could write the refutation in these ways:

$\boxed{T=1,\ M=1,\ H=1,\ B=0}$ $\qquad$ $\boxed{T^1,\ M^1,\ H^1,\ B^0}$

Lawyers often use a similar form of reasoning to defend a client. They sketch a possible situation consistent both with the evidence and with their client's innocence; they conclude that the evidence doesn't establish their client's guilt. For the present example, a lawyer might argue as follows:

> We know that the only people in the mansion were the butler and the maid. But it's possible that the maid did the killing—not the butler—and that the maid had a motive. Since the known facts are consistent with this possibility, these facts don't establish that the butler did it.

Here's another invalid argument and its refutation:

$$\begin{array}{llll} & 1 & (A^0 \supset B^1) = 1 & \textbf{Invalid} \\ * & 2 & (C^0 \vee B^1) = 1 & \\ & & [\,\therefore (C^0 \vee A^0) = 0 & \boxed{B,\ \sim A,\ \sim C} \\ * & 3 & \text{asm: } \sim(C \vee A) & \\ & 4 & \therefore \sim C \quad \{\text{from } 3\} & \\ & 5 & \therefore \sim A \quad \{\text{from } 3\} & \\ & 6 & \therefore B \quad \{\text{from 2 and 4}\} & \end{array}$$

Again, we derive whatever we can. Since we don't get a contradiction, we take any simple wffs (letters or their negation) that aren't blocked off and put them in a box. This box gives truth conditions making the premises all true and conclusion false. This shows that the argument is invalid.

You may be tempted to use line 1 with 5 or 6 to derive a further conclusion—and a contradiction. But we can't derive anything validly here:

$(A \supset B)$	$(A \supset B)$
$\sim A$	B
nil	nil

If we misapply the I-rules, we could incorrectly "prove" the invalid argument to be valid.

Let me summarize. Suppose we want to show that, given certain premises, the butler must be guilty. We assume that he's innocent and try to show that this leads to a contradiction. If we get a contradiction, then his innocence is impossible—and so he must be guilty. But if we get no contradiction, then we may be able to show how the premises could be true while yet he is innocent—thus showing that the argument against him is invalid.

I suggest this strategy for proving or refuting a propositional argument:

1. START: Block off the conclusion and add "asm:" followed by the conclusion's simpler contradictory.
2. S&I: Go through the unstarred, complex, not-blocked-off wffs and use these to derive new wffs using the S- and I-rules. Star any wff you simplify using an S-rule, or the longer wff used in an I-rule inference. If you get a contradiction, apply RAA (step 3). If you can derive nothing further and yet have no contradiction, then refute (step 4).
3. RAA: Since you have a contradiction, apply RAA. You've proved the argument valid.
4. REFUTE: You have no contradiction and yet can't derive anything else. Draw a box containing any simple wffs (letters or their negation) that aren't blocked off. In the original argument, mark each letter "1" or "0" or "?" depending on whether you have the letter or its negation or neither in the box. If these truth conditions make the premises all true and conclusion false, then this shows the argument to be invalid.

This strategy can prove or refute most propositional arguments. We'll see later that some arguments need a more complex strategy and multiple assumptions.

When we plug in the values of our refutation, we should get the premises all true and conclusion false. If that doesn't happen, then we did something wrong. The faulty line (a 0 or ? premise, or a 1 or ? conclusion) is the source of the problem; maybe we derived something incorrectly from this line, or didn't derive something we should have derived. So our strategy can sometimes tell us when something went wrong and where to look to fix the problem.

4.3a Exercise—LogiCola GEI

Prove each of these arguments to be invalid (all are invalid).

$(A \supset B)$
$\therefore (B \supset A)$

1	$(A^0 \supset B^1) = 1$	**Invalid**
	$[\therefore (B^1 \supset A^0) = 0$	
* 2	asm: $\sim(B \supset A)$	$\boxed{B, \sim A}$
3	$\therefore B$ {from 2}	
4	$\therefore \sim A$ {from 2}	

1. $(A \vee B)$
   $\therefore A$

2. $(A \supset B)$
   $(C \supset B)$
   $\therefore (A \supset C)$

3. $\sim(A \cdot \sim B)$
   $\therefore \sim(B \cdot \sim A)$

4. $(A \supset (B \cdot C))$
   $(\sim C \supset D)$
   $\therefore ((B \cdot \sim D) \supset A)$

5. $((A \supset B) \supset (C \supset D))$
   $(B \supset D)$
   $(A \supset C)$
   $\therefore (A \supset D)$

6. $(A \equiv B)$
   $(C \supset B)$
   $\sim(C \cdot D)$
   D
   $\therefore \sim A$

7. $((A \cdot B) \supset C)$
   $\therefore (B \supset C)$

8. $((A \cdot B) \supset C)$
   $((C \vee D) \supset \sim E)$
   $\therefore \sim(A \cdot E)$

9. $\sim(A \cdot B)$
   $(\sim A \vee C)$
   $\therefore \sim(C \cdot B)$

10. $\sim(\sim A \cdot \sim B)$
    $\sim C$
    $(D \vee \sim A)$
    $((C \cdot \sim E) \supset \sim B)$
    $\sim D$
    $\therefore \sim E$

4.3b Exercise—LogiCola GEC

First appraise intuitively. Then translate into logic (using the letters given) and say whether valid (and give a proof) or invalid (and give a refutation).

1. If the butler shot Jones, then he knew how to use a gun.
   If the butler was a former marine, then he knew how to use a gun.
   The butler was a former marine.
   ∴ The butler shot Jones. [Use S, K, and M.]

2. If virtue can be taught, then either there are professional virtue-teachers or there are amateur virtue-teachers.
   If there are professional virtue-teachers, then the Sophists can teach their students to be virtuous.
   If there are amateur virtue-teachers, then the noblest Athenians can teach their children to be virtuous.
   The Sophists can't teach their students to be virtuous and the noblest Athenians (such as the great leader Pericles) can't teach their children to be virtuous.
   ∴ Virtue can't be taught. [Use V, P, A, S, and N. This is from Plato's *Meno*.]

3. It would be equally wrong for a sadist (through a drug injection that would blind you but not hurt your mother) to have blinded you permanently before or after your birth.
   If it would be equally wrong for a sadist (through such a drug injection) to have blinded you permanently before or after your birth, then it's false that one's moral right to equal consideration begins at birth.

If infanticide is wrong and abortion isn't wrong, then one's moral right to equal consideration begins at birth.
Infanticide is wrong.
∴ Abortion is wrong. [Use E, R, I, and A.]

4. If you hold a moral belief and don't act on it, then you're inconsistent.
If you're inconsistent, then you're doing wrong.
∴ If you hold a moral belief and act on it, then you aren't doing wrong. [Use M, A, I, and W. Is the conclusion plausible? What more plausible conclusion follows from these premises?]

5. If Socrates escapes from jail, then he's willing to obey the state only when it pleases him.
If he's willing to obey the state only when it pleases him, then he doesn't really believe what he says and he's inconsistent.
∴ If Socrates really believes what he says, then he won't escape from jail. [Use E, W, R, and I. This argument is from Plato's *Crito.* Socrates had been jailed and sentenced to death for teaching philosophy. He discussed with his friends whether he ought to escape from jail instead of suffering the death penalty.]

6. Either Socrates's death will be perpetual sleep, or if the gods are good then his death will be an entry into a better life.
If Socrates's death will be perpetual sleep, then he shouldn't fear death.
If his death will be an entry into a better life, then he shouldn't fear death.
∴ He shouldn't fear death. [Use P, G, B, and F. This argument is from Plato's *Crito*—except for which dropped premise?]

7. If predestination is true, then God causes us to sin.
If God causes us to sin and yet damns sinners to eternal punishment, then God isn't good.
∴ If God is good, then either predestination isn't true or else God doesn't damn sinners to eternal punishment. [Use P, C, D, and G. This attacks the views of the American colonial thinker Jonathan Edwards.]

8. If determinism is true, then we have no free will.
If Heisenberg's interpretation of quantum physics is correct, some events aren't causally necessitated by prior events.
If some events aren't causally necessitated by prior events, determinism is false.
∴ If Heisenberg's interpretation of quantum physics is correct, then we have free will. [Use D, F, H, and E.]

9. Government's function is to protect life, liberty, and the pursuit of happiness.
The British colonial government doesn't protect these.
The only way to change it is by revolution.

If government's function is to protect life, liberty, and the pursuit of happiness and the British colonial government doesn't protect these, then the British colonial government ought to be changed.
If the British colonial government ought to be changed and the only way to change it is by revolution, then we ought to have a revolution.
∴ We ought to have a revolution. [Use G, B, O, C, and R. This summarizes the reasoning behind the American Declaration of Independence. Premise 1 was claimed to be self-evident, premises 2 and 3 were backed by historical data, and premises 4 and 5 were implicit conceptual bridge premises.]

10. The apostles' teaching either comes from God or is of human origin.
If it comes from God and we kill the apostles, then we will be fighting God.
If it's of human origin, then it'll collapse of its own accord.
If it'll collapse of its own accord and we kill the apostles, then our killings will be unnecessary.
∴ If we kill the apostles, then either our killings will be unnecessary or we will be fighting God. [Use G, H, K, F, C, and U. This argument, from Rabbi Gamaliel in Acts 5:34–9, is perhaps the most complex reasoning in the Bible.]

11. If materialism (the view that only matter exists) is true, then idealism is false.
If idealism (the view that only minds exist) is true, then materialism is false.
If mental events exist, then materialism is false.
If materialists *think* their theory is true, then mental events exist.
∴ If materialists *think* their theory is true, then idealism is true. [Use M, I, E, and T.]

12. If determinism is true and cruelty is wrong, then the universe contains unavoidable wrong actions.
If the universe contains unavoidable wrong actions, then we ought to regret the universe as a whole.
If determinism is true and regretting cruelty is wrong, then the universe contains unavoidable wrong actions.
∴ If determinism is true, then either we ought to regret the universe as a whole (the pessimism option) or else cruelty isn't wrong and regretting cruelty isn't wrong (the "nothing matters" option). [Use D, C, U, O, and R. This sketches the reasoning in William James's "The Dilemma of Determinism." James thought that when we couldn't prove one side or the other to be correct (as on the issue of determinism) it was more rational to pick our beliefs in accord with practical considerations. He argued that these weighed against determinism.]

13. If a belief is proved, then it's worthy of acceptance.
If a belief isn't disproved but is of practical value to our lives, then it's worthy of acceptance.
If a belief is proved, then it isn't disproved.

∴ If a belief is proved or is of practical value to our lives, then it's worthy of acceptance. [Use P, W, D, and V.]

14. If you're consistent and think that stealing is normally permissible, then you'll consent to the idea of others stealing from you in normal circumstances.
You don't consent to the idea of others stealing from you in normal circumstances.
∴ If you're consistent, then you won't think that stealing is normally permissible. [Use C, N, and Y.]

15. If the meaning of a term is always the object it refers to, then the meaning of "Fido" is Fido.
If the meaning of "Fido" is Fido, then if Fido is dead the meaning of "Fido" is dead.
If the meaning of "Fido" is dead, then "Fido is dead" has no meaning. "Fido is dead" has meaning.
∴ The meaning of a term isn't always the object it refers to. [Use A, B, F, M, and H. This is from Ludwig Wittgenstein, except for which dropped premise?]

16. God is all powerful.
If God is all powerful, then he could have created the world in any logically possible way and the world has no necessity.
If the world has no necessity, then we can't know the way the world is by abstract speculation apart from experience.
∴ We can't know the way the world is by abstract speculation apart from experience. [Use A, C, N, and K. This is from the medieval William of Ockham.]

17. If God changes, then he changes for the worse or for the better.
If he changes for the better, then he isn't perfect.
If he's perfect, then he doesn't change for the worse.
∴ If God is perfect, then he doesn't change. [Use C, W, B, and P.]

18. If belief in God has scientific backing, then it's rational.
No conceivable scientific experiment could decide whether there is a God.
If belief in God has scientific backing, then some conceivable scientific experiment could decide whether there is a God. ∴ Belief in God isn't rational. [Use B, R, and D.]

19. Every event with finite probability eventually takes place.
If the world powers don't get rid of their nuclear weapons, then there's a finite probability that humanity will eventually destroy the world.
If every event with finite probability eventually takes place and there's a finite probability that humanity will eventually destroy the world, then humanity will eventually destroy the world.

∴ Either the world powers will get rid of their nuclear weapons, or humanity will eventually destroy the world. [Use E, R, F, and H.]

20. If the world isn't ultimately absurd, then conscious life will go on forever and the world process will culminate in an eternal personal goal.
If there is no God, then conscious life won't go on forever.
∴ If the world isn't ultimately absurd, then there is a God. [Use A, F, C, and G. This argument is from the Jesuit scientist, Pierre Teilhard de Chardin.]

21. If it rained here on this date 500 years ago and there's no way to know whether it rained here on this date 500 years ago, then there are objective truths that we cannot know.
If it didn't rain here on this date 500 years ago and there's no way to know whether it rained here on this date 500 years ago, then there are objective truths that we cannot know.
There's no way to know whether it rained here on this date 500 years ago.
∴ There are objective truths that we cannot know. [R, K, and O]

22. If you know that you don't exist, then you don't exist.
If you know that you don't exist, then you know some things.
If you know some things, then you exist.
∴ You exist. [Use K, E, and S.]

23. We have an idea of a perfect being.
If we have an idea of a perfect being, then this idea is either from the world or from a perfect being.
If this idea is from a perfect being, then there is a God.
∴ There is a God. [Use I, W, P, and G. This is from René Descartes, except for which dropped premise?]

24. The distance from A to B can be divided into an infinity of spatial points.
One can cross only one spatial point at a time.
If one can cross only one spatial point at a time, then one can't cross an infinity of spatial points in a finite time.
If the distance from A to B can be divided into an infinity of spatial points and one can't cross an infinity of spatial points in a finite time, then one can't move from A to B in a finite time.
If motion is real, then one can move from A to B in a finite time.
∴ Motion isn't real. [Use D, O, C, M, and R. This argument is from the ancient Greek Zeno of Elea, who denied the reality of motion.]

25. If the square root of 2 equals some fraction of positive whole numbers, then (we stipulate) the square root of 2 equals x/y and x/y is simplified as far as it can be.
If the square root of 2 equals x/y, then $2=x^2/y^2$.
If $2=x^2/y^2$, then $2y^2=x^2$.

If $2y^2=x^2$, then x is even.
If x is even and $2y^2=x^2$, then y is even.
If x is even and y is even, then x/y isn't simplified as far as it can be.
∴ The square root of 2 doesn't equal some fraction of positive whole numbers. [Use F, F′ S, T, T′, X, and Y.]

4.4 Multiple assumptions

We get stuck if we apply our present proof strategy to the following argument:

If the butler was at the party, then he fixed the drinks and poisoned the deceased.	1 (A ⊃ (F · P))
If the butler wasn't at the party, then the detective would have seen him leave the mansion and would have reported this.	2 (~A ⊃ (S · R))
The detective didn't report this.	3 ~R
∴ The butler poisoned the deceased.	[∴ P
	4 asm: ~P

After the assumption, we can't apply the S- or I-rules or RAA; and we don't have enough simple wffs for a refutation. So we're stuck. What can we do?

On our expanded proof strategy, when we get stuck we'll make another assumption. This assumption may lead to a contradiction; if so, we can apply RAA to derive the opposite and perhaps use this to complete the proof.

Let's focus on breaking up line 1. This line has two sides: "A" and "(F·P)." It would be useful to assume one side or its negation, in order to carry on the proof further; so we arbitrarily assume "~A" and see what happens. Since "~A" leads to a contradiction, we derive "A"—and having "A" lets us complete the proof in the normal way. Our multiple-assumption proof looks like this:

To break up line 1, we assume one side or its negation.

Since "~A" leads to a contradiction, we derive "A" and finish the proof.

```
*  1     (A ⊃ (F · P))    Valid
   2     (~A ⊃ (S · R))
   3     ~R
      [ ∴ P
   4   ┌ asm: ~P
   5   │ ┌ asm: ~A    {break up 1}
   6   │ │ ∴ (S · R)   {from 2 and 5}
   7   │ │ ∴ S    {from 6}
   8   │ └ ∴ R    {from 6}
   9   │ ∴ A    {from 5; 3 contradicts 8}
*  10  │ ∴ (F · P)    {from 1 and 9}
   11  │ ∴ F    {from 10}
   12  └ ∴ P    {from 10}
   13  ∴ P    {from 4; 4 contradicts 12}
```

The novel steps here are 5 through 9, which are used to derive "A." Study these steps careful before reading further.

Let's go through this again, but more slowly. Recall that we got stuck after our initial assumption. To carry on the proof further, it would be useful to break up line 1, by assuming one side or its negation. So we arbitrarily assume "~A." Then we use this second assumption "~A" with line 2 to derive further steps and get a contradiction (between 3 and 8):

Note the double stars!
(We have two not-blocked-off assumptions.)

1 (A ⊃ (F · P))
➔ ** 2 (~A ⊃ (S · R))
3 ~R ⇦
[∴ P
4 asm: ~P
5 asm: ~A {break up 1}
➔ ** 6 ∴ (S · R) {from 2 and 5}
7 ∴ S {from 6}
8 ∴ R {from 6} ⇦

As usual, we star lines 2 and 6 when we use S- and I-rules on them. But now we use *two* stars—since now we have *two* not-blocked-off assumptions. This expanded starring rule covers multiple-assumptions proofs:

When you star, use one star for each not-blocked-off assumption.

Multiple stars mean "You can ignore this line for now, but you may have to use it later."

Let's get back to our proof. Since "asm: ~A" leads to a contradiction (between 3 and 8), we use RAA to derive "A"; notice how we do it:

When you get a contradiction:

- block off the lines from the last assumption on down,
- derive the opposite of this last assumption, and
- erase star strings with more stars than the number of remaining assumptions.

1 (A ⊃ (F · P))
2 (~A ⊃ (S · R))
3 ~R ⇦
[∴ P
4 asm: ~P
5 asm: ~A {break up 1}
6 ∴ (S · R) {from 2 and 5}
7 ∴ S {from 6}
8 ∴ R {from 6} ⇦
9 ∴ A {from 5; 3 contradicts 8}

Since our second assumption (line 5) led to a contradiction, we block off from this assumption on down (lines 5 to 8); now we can't use these lines, which depend on an outdated assumption, to derive further steps or get a contradiction. In line 9, we derive "A" (the opposite of assumption "~A"). Finally, we erase "**" from lines 2 and 6—since now we have only one not-blocked-off assumption; we may have to use these formerly starred lines again.[1]

Now that we have "A" in line 9, we use this with line 1 to derive further steps and get a second contradiction; then we apply RAA to complete the proof:

```
*  1      (A ⊃ (F · P))    Valid
   2      (~A ⊃ (S · R))
   3      ~R
       [ ∴ P
   4   ┌ asm: ~P  ⇦
   5   │ ┌ asm: ~A   {break up 1}
   6   │ │ ∴ (S · R)   {from 2 and 5}
   7   │ │ ∴ S   {from 6}
   8   │ └ ∴ R   {from 6}
   9   │ ∴ A   {from 5; 3 contradicts 8}
*  10  │ ∴ (F · P)   {from 1 and 9}
   11  │ ∴ F   {from 10}
   12  └ ∴ P   {from 10}  ⇦
   13  ∴ P   {from 4; 4 contradicts 12}
```

Study this example carefully. When you understand it well, you're ready to learn the general strategy for doing multiple-assumption proofs.

4.5 Harder proofs

The most difficult part of multiple-assumption proofs is knowing *when* to make another assumption and *what* to assume.

First use the S- and I-rules and RAA to derive everything you can. You may get stuck; this means that you can't derive anything further but yet need to do something to get a proof or refutation. When you get stuck, try to make another assumption. Look for an unstarred complex wff for which you don't already have one side or its negation. This wff will have one of these forms:

~(A·B)   (A∨B)   (A⊃B)

Assume either side or its negation. Here we could use any of these:

asm: A   asm: ~A   asm: B   asm: ~B

[1] Since lines 5–8 are now blocked off, the lines that made the doubly starred lines redundant are no longer available. So we erase "**" when the second assumption dies.

While any of the four will work, our proof will go differently depending on which we use. Suppose that we want to break up "(A⊃B)"; compare what happens if we assume "A" or assume "~A":

	(A⊃B)	(A⊃B)	
(immediate gratification)	asm: A	asm: ~A	*(delayed gratification)*
	∴ B	…	

In the first case, we assume "A" and get immediate gratification; we can use an I-rule on "(A⊃B)" right away to get "B." In the second case, we assume "~A" and get delayed gratification; we'll be able to use an I-rule on "(A⊃B)" only later, after the "~A" assumption dies (if it does) and we derive "A." The "delayed gratification" approach tends to produce shorter proofs; it saves an average of one step, with all the gain coming on invalid arguments. So sometimes a proof is simpler if you assume one thing rather than another.

Follow the same strategy on wffs that are more complicated. To break up "((A·B)⊃(C·D))," we could make any of these four assumptions:

asm: (A·B) asm: ~(A·B) asm: (C·D) asm: ~(C·D)

Assume one side or its negation; never assume the denial of a whole line.

Our final proof strategy can prove or refute any propositional argument:

1. START: Block off the conclusion and add "asm:" followed by the conclusion's simpler contradictory.
2. S&I: Go through the unstarred, complex, not-blocked-off wffs and use these to derive whatever new wffs you can using the S- and I-rules. Star (with one star for each not-blocked-off assumption) any wff you simplify using an S-rule, or the longer wff used in an I-rule inference. If you get a contradiction, apply RAA (step 3). If you can derive nothing further and yet have no contradiction, then make another assumption if you can (step 4); otherwise, refute (step 5).
   Note: While you needn't derive wffs again that you already have, you can star wffs that would give you what you already have.
3. RAA: Since you have a contradiction, apply RAA. If all assumptions are now blocked off, you've proved the argument valid. Otherwise, erase star strings having more stars than the number of not-blocked-off assumptions. Then return to step 2.
4. ASSUME: Make another assumption if you have an unstarred, not-blocked-off wff of one of these forms for which you don't already have one side or its negation:

   ~(A·B) (A∨B) (A⊃B)

   Assume one side or its negation—and then return to step 2.

Note: Don't make an assumption from a wff if you already have one side or its negation. For example, don't make an assumption from "(A⊃B)" if you already have "~A" or "B." In this case, the wff is already "broken up."

5. REFUTE: Here all complex, not-blocked-off wffs are either starred or already broken up—and yet you have no contradiction. Draw a box containing any simple wffs (letters or their negation) that aren't blocked off. In the original argument, mark each letter "1" or "0" or "?" depending on whether you have the letter or its negation or neither in the box. These truth conditions should make the premises all true and conclusion false—thus showing the argument to be invalid.

Let's take another example. Consider a proof that begins this way:

```
     1    (A ⊃ (B · C))
     2    (B ⊃ (A · C))
       [ ∴ ((A ∨ B) ⊃ C)
*    3    asm: ~((A ∨ B) ⊃ C)
     4    ∴ (A ∨ B)   {from 3}
     5    ∴ ~C   {from 3}
```

It's important to use the S- and I-rules as far as we can; make additional assumptions only as a last resort. But now we're stuck; we can't apply the S- or I-rules or RAA, and we don't have enough values to refute the argument. So we break up line 1 by assuming one side or its negation. We decide to assume "A" (line 6). Then we derive further things (using two stars while there are two not-blocked-off assumptions) until we get a contradiction (between 5 and 9):

```
**   1    (A ⊃ (B · C))
     2    (B ⊃ (A · C))
       [ ∴ ((A ∨ B) ⊃ C)
*    3    asm: ~((A ∨ B) ⊃ C)
     4    ∴ (A ∨ B)   {from 3}
     5    ∴ ~C   {from 3}   ⇦
     6       asm: A   {break up 1}
**   7       ∴ (B · C)   {from 1 and 6}
     8       ∴ B   {from 7}
     9       ∴ C   {from 7}   ⇦
```

Since we have a contradiction, we apply RAA (blocking off lines 6 to 9 and deriving line 10); and we erase each "**" (since now there's only one not-blocked-off assumption):

```
   1     (A ⊃ (B · C))
   2     (B ⊃ (A · C))
        [ ∴ ((A ∨ B) ⊃ C)
*  3     asm: ~((A ∨ B) ⊃ C)
   4     ∴ (A ∨ B)   {from 3}
   5     ∴ ~C   {from 3}   ⇦
   6     ┌ asm: A   {break up 1}
   7     │ ∴ (B · C)   {from 1 and 6}
   8     │ ∴ B   {from 7}
   9     └ ∴ C   {from 7}   ⇦
  10     ∴ ~A   {from 6; 5 contradicts 9}
```

We continue until we get our second contradiction—which finishes our proof:

```
   1     (A ⊃ (B · C))
*  2     (B ⊃ (A · C))
        [ ∴ ((A ∨ B) ⊃ C)
*  3   ┌ asm: ~((A ∨ B) ⊃ C)
*  4   │ ∴ (A ∨ B)   {from 3}
   5   │ ∴ ~C   {from 3}
   6   │ ┌ asm: A   {break up 1}
   7   │ │ ∴ (B · C)   {from 1 and 6}
   8   │ │ ∴ B   {from 7}
   9   │ └ ∴ C   {from 7}
  10   │ ∴ ~A   {from 6; 5 contradicts 9}   ⇦
  11   │ ∴ B   {from 4 and 10}
  12   │ ∴ (A · C)   {from 2 and 11}
  13   └ ∴ A   {from 12}   ⇦
  14   ∴ ((A ∨ B) ⊃ C)   {from 3; 10 contradicts 13}
```

4.5a Exercise—LogiCola GHV

Prove each of these arguments to be valid (all are valid).

```
(B ∨ A)
(B ⊃ A)
∴ ~(A ⊃ ~A)
```

```
*  1     (B ∨ A)   Valid
   2     (B ⊃ A)
        [ ∴ ~(A ⊃ ~A)
*  3   ┌ asm: (A ⊃ ~A)
   4   │ ┌ asm: B   {break up 1}
   5   │ │ ∴ A   {from 2 and 4}
   6   │ └ ∴ ~A   {from 3 and 5}
   7   │ ∴ ~B   {from 4; 5 contradicts 6}
   8   │ ∴ A   {from 1 and 7}
   9   └ ∴ ~A   {from 3 and 8}
  10   ∴ ~(A ⊃ ~A)   {from 3; 8 contradicts 9}
```

1. (A ⊃ B)
   (A ∨ (A · C))
   ∴ (A · B)

2. (((A · B) ⊃ C) ⊃ (D ⊃ E))
   D
   ∴ (C ⊃ E)

3. (B ⊃ A)
   ~(A · C)
   (B ∨ C)
   ∴ (A ≡ B)

4. (A ∨ (D · E))
   (A ⊃ (B · C))
   ∴ (D ∨ C)

5. ((A ⊃ B) ⊃ C)
   (C ⊃ (D · E))
   ∴ (B ⊃ D)

6. (~(A ∨ B) ⊃ (C ⊃ D))
   (~A · ~D)
   ∴ (~B ⊃ ~C)

7. (~A ≡ B)
   ∴ ~(A ≡ B)

8. (A ⊃ (B · ~C))
   C
   ((D · ~E) ∨ A)
   ∴ D

4.5b Exercise—LogiCola GHV

First appraise intuitively. Then translate into logic (using the letters given) and prove to be valid (all are valid).

1. If the butler put the tablet into the drink and the tablet was poison, then the butler killed the deceased and the butler is guilty.
   The butler put the tablet into the drink.
   The tablet was poison.
   ∴ The butler is guilty. [Use P, T, K, and G.]

2. If I'm coming down with a cold and I exercise, then I'll get worse and feel awful.
   If I don't exercise, then I'll suffer exercise deprivation and I'll feel awful.
   ∴ If I'm coming down with a cold, then I'll feel awful. [Use C, E, W, A, and D.]

3. You'll get an A if and only if you either get a hundred on the final exam or else bribe the teacher.
   You won't get a hundred on the final exam.
   ∴ You'll get an A if and only if you bribe the teacher. [Use A, H, and B.]

4. If President Nixon knew about the massive Watergate cover-up, then he lied to the American people on national television and he should resign.
   If President Nixon didn't know about the massive Watergate cover-up, then he was incompetently ignorant and he should resign.
   ∴ Nixon should resign. [Use K, L, R, and I.]

5. Common sense assumes we have moral knowledge.
   There's no disproof of moral knowledge.
   If common sense assumes we have moral knowledge, then if there's no disproof of moral knowledge we should believe that we have moral knowledge.
   Any proof of a moral truth presupposes a more basic moral truth.
   We can't prove moral truths by more basic ones endlessly.
   If any proof of a moral truth presupposes a more basic moral truth and we can't prove moral truths by more basic ones endlessly, then if we should believe that we have moral knowledge we should accept self-evident moral truths.
   ∴ We should accept self-evident moral truths. [Use C, D, B, P, E, and S; this argument defends ethical intuitionism.]

6. Moral judgments express either truth claims or feelings.
   If moral judgments express truth claims, then "ought" expresses either a concept from sense experience or an objective concept that isn't from sense experience.
   "Ought" doesn't express a concept from sense experience.
   "Ought" doesn't express an objective concept that isn't from sense experience.
   ∴ Moral judgments express feelings and not truth claims. [Use T, F, S, and O.]

7. If Michigan either won or tied, then Michigan is going to the Rose Bowl and Gensler is happy.
   ∴ If Gensler isn't happy, then Michigan didn't tie. [Use W, T, R, and H.]

8. There are moral obligations.

If there are moral obligations and moral obligations are explainable, then either there's an explanation besides God's existence or else God's existence would explain moral obligations.
God's existence wouldn't explain moral obligation.
∴ Either moral obligations aren't explainable, or else there's an explanation besides God's existence. [Use M, E, B, and G.]

9. If determinism is true and Dr Freudlov correctly predicts (using deterministic laws) what I'll do, then if she tells me her prediction I'll do something else.
If Dr Freudlov tells me her prediction and yet I'll do something else, then Dr Freudlov doesn't correctly predict (using deterministic laws) what I'll do.
∴ If determinism is true, then either Dr Freudlov doesn't correctly predict (using deterministic laws) what I'll do or else she won't tell me her prediction. [Use D, P, T, and E.]

10. [The parents told their son that their precondition for financing his graduate education was that he leave his girlfriend Suzy. A friend of mine talked the parents out of their demand by using this argument.]
If you make this demand on your son and he leaves Suzy, then he'll regret being forced to leave her and he'll always resent you.
If you make this demand on your son and he doesn't leave Suzy, then he'll regret not going to graduate school and he'll always resent you.
∴ If you make this demand on your son, then he'll always resent you. [Use D, L, F, A, and G; this one is difficult.]

4.6 Harder refutations

Multiple-assumption invalid arguments work much like other invalid arguments—except that we need to make further assumptions before we reach our refutation. Here's an example:

If the butler was at the party, he fixed the drinks and poisoned the deceased.
If the butler wasn't at the party, he was at a neighbor's house.
∴ The butler poisoned the deceased.

	1	$(A^0 \supset (F \cdot P^0)) = 1$	**Invalid**
**	2	$(\sim A^0 \supset N^1) = 1$	$\boxed{N, \sim A, \sim P}$
		[∴ $P^0 = 0$	
	3	asm: ~P	
	4	asm: ~A {break up 1}	
	5	∴ N {from 2 and 4}	

We derive all we can and make additional assumptions when needed. But now we reach no contradiction; instead, we reach a refutation in which the butler was at a neighbor's house, wasn't at the party, and didn't poison the deceased. This refutation makes the premises all true and conclusion false.

As we work out our attempted proof, we can follow the five-step proof strategy of the previous section until one of two things happens:

- Every assumption leads to a contradiction. Then we have a proof of validity.
- We can derive nothing further and all complex wffs are either starred or already broken up (we already have one side or its negation). Then the remaining simple wffs will give a refutation that proves invalidity.

In the invalid case, additional assumptions can help to bring us our refutation.

Invalid arguments often need three or more assumptions, as in this example:

	1	$(A^0 \supset B) = 1$	**Invalid**
	2	$(C^0 \supset D) = 1$	$\boxed{E, \sim A, \sim C, \sim F}$
	3	$(F^0 \supset (C^0 \cdot D)) = 1$	
		$[\therefore (E^1 \supset C^0) = 0$	
*	4	asm: $\sim(E \supset C)$	
	5	$\therefore E$ {from 4}	
	6	$\therefore \sim C$ {from 4}	
	7	asm: $\sim A$ {break up 1}	
	8	asm: $\sim F$ {break up 3}	

We keep going until we can derive nothing further and all complex wffs are either starred (like line 4) or already broken up (like lines 1–3).[1] While our refutation doesn't give us a value for "B" or "D," this is all right, since the refutation still makes the premises all true and conclusion false.

Our proof strategy, if applied correctly, will always give a proof or refutation. How exactly these go may depend on which steps we do first and what we decide to assume; proofs and refutations may differ but still be correct.

4.6a Exercise—LogiCola GHI

Prove each of these arguments to be invalid (all are invalid).

$(A \vee \sim(B \supset C))$
$(D \supset (A \supset B))$
$\therefore (C \supset \sim(D \vee A))$

	1	$(A^1 \vee \sim(B \supset C^1)) = 1$	**Invalid**
	2	$(D^0 \supset (A^1 \supset B)) = 1$	$\boxed{A, C, \sim D}$
		$[\therefore (C^1 \supset \sim(D^0 \vee A^1)) = 0$	
*	3	asm: $\sim(C \supset \sim(D \vee A))$	
	4	$\therefore C$ {from 3}	
	5	$\therefore (D \vee A)$ {from 3}	
	6	asm: A {break up 1}	
	7	asm: $\sim D$ {break up 2}	

1 If you like, you can star a line when it becomes "broken up" (when you have one side or its negation, but not what is needed to infer something). Then you can continue until all assumptions die (then the argument is valid) or until you can derive nothing further and all complex wffs are starred (then the argument is invalid).

1. $\sim(A \cdot B)$
   $\therefore (\sim A \cdot \sim B)$
2. $(\sim A \supset B)$
   $\therefore \sim(A \supset B)$
3. $((A \cdot B) \supset \sim(C \cdot D))$
   C
   $(E \supset B)$
   $\therefore \sim E$
4. $\sim(A \cdot B)$
   $\therefore \sim(A \equiv B)$
5. $(A \supset B)$
   $(C \supset (\sim D \cdot E))$
   $\therefore (D \vee F)$
6. $(\sim A \vee \sim B)$
   $\therefore \sim(A \vee B)$
7. $(A \supset (B \cdot C))$
   $((D \supset E) \supset A)$
   $\therefore (E \vee C)$
8. $(A \supset (B \supset C))$
   $(B \vee \sim(C \supset D))$
   $\therefore (D \supset \sim(A \vee B))$

4.6b Exercise—LogiCola G (HC & MC)

First appraise intuitively. Then translate into logic (using the letters given) and say whether valid (and give a proof) or invalid (and give a refutation).

1. If the maid prepared the drink, then the butler didn't prepare it.
   The maid didn't prepare the drink.
   If the butler prepared the drink, then the butler poisoned the drink and the butler is guilty.
   ∴ The butler is guilty. [Use M, B, P, and G.]

2. If you tell your teacher that you like logic, then your teacher will think that you're insincere and you'll be in trouble.
   If you don't tell your teacher that you like logic, then your teacher will think that you dislike logic and you'll be in trouble.
   ∴ You'll be in trouble. [Use L, I, T, and D.]

3. If we don't get reinforcements, then the enemy will overwhelm us and we won't survive.
   ∴ If we do get reinforcements, then we'll conquer the enemy and we'll survive. [Use R, O, S, and C.]

4. If Socrates didn't approve of the laws of Athens, then he would have left Athens or would have tried to change the laws.
   If Socrates didn't leave Athens and didn't try to change the laws, then he agreed to obey the laws.
   Socrates didn't leave Athens.
   ∴ If Socrates didn't try to change the laws, then he approved of the laws and agreed to obey them. [Use A, L, C, and O. This argument is from Plato's *Crito,* which argued that Socrates shouldn't disobey the law by escaping from jail.]

5. If I hike the Appalachian Trail and go during late spring, then I'll get maximum daylight and maximum mosquitoes.
   If I'll get maximum mosquitoes, then I won't be comfortable.
   If I go right after school, then I'll go during late spring.
   ∴ If I hike the Appalachian Trail and don't go right after school, then I'll be comfortable. [Use A, L, D, M, C, and S.]

6. [Logical positivism says *"Every genuine truth claim is either experimentally testable or true by definition."* This view, while once popular, is self-refuting and hence not very popular today.]
   If LP (logical positivism) is true and is a genuine truth claim, then it's either experimentally testable or true by definition.
   LP isn't experimentally testable.
   LP isn't true by definition.
   If LP isn't a genuine truth claim, then it isn't true.
   ∴ LP isn't true. [Use T, G, E, and D.]

7. If you give a test, then students either do well or do poorly.
   If students do well, then you think you made the test too easy and you're frustrated.
   If students do poorly, then you think they didn't learn any logic and you're frustrated.
   ∴ If you give a test, then you're frustrated. [Use T, W, P, E, F, and L. This is from a class who tried to talk me out of giving a test.]

8. If the world contains moral goodness, then the world contains free creatures and the free creatures sometimes do wrong.
   If the free creatures sometimes do wrong, then the world is imperfect and the creator is imperfect.
   ∴ If the world doesn't contain moral goodness, then the creator is imperfect. [Use M, F, S, W, and C.]

9. We'll find a cause for your action, if and only if your action has a cause and we look hard enough.
   If all events have causes, then your action has a cause.
   All events have causes.
   ∴ We'll find a cause for your action, if and only if we look hard enough. [Use F, H, L, and A.]

10. Herman sees that the piece of chalk is white.
    The piece of chalk is the smallest thing on the desk.
    Herman doesn't see that the smallest thing on the desk is white. (He can't see the whole desk and so can't tell that the piece of chalk is the smallest thing on it.)
    If Herman sees a material thing, then if he sees that the piece of chalk is white and the piece of chalk is the smallest thing on the desk then he sees that the smallest thing on the desk is white.
    If Herman doesn't see a material thing, then he sees a sense datum.
    ∴ Herman doesn't see a material thing, but he does see a sense datum. [Use H, P, H′, M, and S. This argument attacks direct realism—the view that we directly perceive material things and not just sensations or sense data.]

11. If the final loading capacitor in the radio transmitter is arcing, then the SWR (standing wave ratio) is too high and the efficiency is lowered.

If you hear a cracking sound, then the final loading capacitor in the radio transmitter is arcing.

∴ If you don't hear a cracking sound, then the SWR isn't too high. [Use A, H, L, and C.]

12. If we can know that God exists, then we can know God by experience or we can know God by logical inference from experience.
If we can't know God empirically, then we can't know God by experience and we can't know God by logical inference from experience.
If we can know God empirically, then "God exists" is a scientific hypothesis and is empirically falsifiable.
"God exists" isn't empirically falsifiable.
∴ We can't know that God exists. [Use K, E, L, M, S, and F.]

13. If I perceive, then my perception is either delusive or veridical.
If my perception is delusive, then I don't directly perceive a material object.
If my perception is veridical and I directly perceive a material object, then my experience in veridical perception would always differ qualitatively from my experience in delusive perception.
My experience in veridical perception doesn't always differ qualitatively from my experience in delusive perception.
If I perceive and I don't directly perceive a material object, then I directly perceive a sensation.
∴ If I perceive, then I directly perceive a sensation and I don't directly perceive a material object. [Use P, D, V, M, Q, and S. This form of the argument from illusion attacks direct realism—the view that we directly perceive material objects and not just sensations or sense data.]

14. If you're romantic and you're Italian, then Juliet will fall in love with you and will want to marry you.
If you're Italian, then you're romantic.
∴ If you're Italian, then Juliet will want to marry you. [Use R, I, F, and M.]

15. If emotions can rest on factual errors and factual errors can be criticized, then we can criticize emotions.
If we can criticize emotions and moral judgments are based on emotions, then beliefs about morality can be criticized and morality isn't entirely non-rational.
∴ If morality is entirely non-rational, then emotions can't rest on factual errors. [Use E, F, W, M, B, and N.]

16. If you backpack over spring break and don't study logic, then you won't know how to do proofs.
If you take the test and don't know how to do proofs, then you'll miss many problems and get a low grade.
∴ If you backpack over spring break, then you'll get a low grade. [Use B, S, K, T, M, and L.]

4.7 Other proof methods

The proof method in this book tries to combine the best features of two other methods: traditional proofs and truth trees. These three approaches, while differing in how they do proofs, can prove all the same arguments.

Traditional proofs use a standard set of inference rules and equivalence rules. The nine inference rules are like our S- and I-rules, in that they let us infer whole lines from previous whole lines:

$(P \cdot Q) \rightarrow P$	$(P \supset Q), P \rightarrow Q$
$P, Q \rightarrow (P \cdot Q)$	$(P \supset Q), \sim Q \rightarrow \sim P$
$(P \vee Q), \sim P \rightarrow Q$	$(P \supset Q), (Q \supset R) \rightarrow (P \supset R)$
$P \rightarrow (P \vee Q)$	$(P \supset Q) \rightarrow (P \supset (P \cdot Q))$

$$((P \supset Q) \cdot (R \supset S)), (P \vee R) \rightarrow (Q \vee S)$$

The sixteen equivalence rules let us replace parts of formulas with equivalent parts (I've dropped outer parentheses here to promote readability):

$P \equiv \sim\sim P$	$(P \supset Q) \equiv (\sim P \vee Q)$
$P \equiv (P \cdot P)$	$(P \cdot (Q \cdot R)) \equiv ((P \cdot Q) \cdot R)$
$P \equiv (P \vee P)$	$(P \vee (Q \vee R)) \equiv ((P \vee Q) \vee R)$
$(P \cdot Q) \equiv (Q \cdot P)$	$(P \cdot (Q \vee R)) \equiv ((P \cdot Q) \vee (P \cdot R))$
$(P \vee Q) \equiv (Q \vee P)$	$(P \vee (Q \cdot R)) \equiv ((P \vee Q) \cdot (P \vee R))$
$\sim(P \cdot Q) \equiv (\sim P \vee \sim Q)$	$(P \equiv Q) \equiv ((P \supset Q) \cdot (Q \supset P))$
$\sim(P \vee Q) \equiv (\sim P \cdot \sim Q)$	$(P \equiv Q) \equiv ((P \cdot Q) \vee (\sim P \cdot \sim Q))$
$(P \supset Q) \equiv (\sim Q \supset \sim P)$	$((P \cdot Q) \supset R) \equiv (P \supset (Q \supset R))$

Our approach uses a simpler and more understandable set of rules.

Traditional proofs can use **indirect proofs** (where we assume the opposite of the conclusion and then derive a contradiction); but they more often use **direct proofs** (where we just derive things from the premises and eventually derive the desired conclusion) or **conditional proofs** (where we prove "$(P \supset Q)$" by assuming "P" and then deriving "Q"). Here's an argument proved two ways:

Traditional proof

1 ((A ⊃ B) · (C ⊃ D))
[∴ ((A · C) ⊃ (B ∨ D))
2 ∴ (A ⊃ B) {from 1}
3 ∴ (~A ∨ B) {from 2}
4 ∴ ((~A ∨ B) ∨ D) {from 3}
5 ∴ (~A ∨ (B ∨ D)) {from 4}
6 ∴ ((~A ∨ (B ∨ D)) ∨ ~C) {from 5}
7 ∴ (~C ∨ (~A ∨ (B ∨ D))) {from 6}
8 ∴ ((~C ∨ ~A) ∨ (B ∨ D)) {from 7}
9 ∴ ((~A ∨ ~C) ∨ (B ∨ D)) {from 8}
10 ∴ (~(A · C) ∨ (B ∨ D)) {from 9}
11 ∴ ((A · C) ⊃ (B ∨ D)) {from 10, giving the original conclusion}

Our proof

* 1 ((A ⊃ B) · (C ⊃ D))
[∴ ((A · C) ⊃ (B ∨ D))
* 2 ┌ asm: ~((A · C) ⊃ (B ∨ D))
* 3 │ ∴ (A ⊃ B) {from 1}
4 │ ∴ (C ⊃ D) {from 1}
* 5 │ ∴ (A · C) {from 2}
6 │ ∴ ~(B ∨ D) {from 2}
7 │ ∴ A {from 5}
8 │ ∴ C {from 5}
9 │ ∴ B {from 3 and 7}
10 └ ∴ ~B {from 6}
11 ∴ ((A · C) ⊃ (B ∨ D)) {from 2; 9 contradicts 10}

Both proofs have the same number of steps; but this can vary, depending on how we do each proof. Our steps generally use shorter formulas; our proofs tend to simplify larger formulas into smaller ones—while traditional proofs tend to manipulate longer formulas (often by substituting equivalents) to get the desired result. Our proofs are easier to do, since they use an automatic proof-strategy that students learn quickly; traditional proofs require guesswork and intuition. Also, our system refutes invalid arguments; it can separate valid from invalid arguments, prove valid ones to be valid, and refute invalid ones. In contrast, the traditional system is only a proof method; if we try to prove an invalid argument, we'll fail but won't necessarily learn that the argument is invalid.

Another common approach is **truth trees,** which decompose formulas into the cases that make them true. Truth trees use simplifying rules and branching rules. The simplifying rules are like our S-rules, in that they let us simplify a formula into smaller parts and then ignore the original formula. These four simplifying rules (which apply to whole lines) are used:

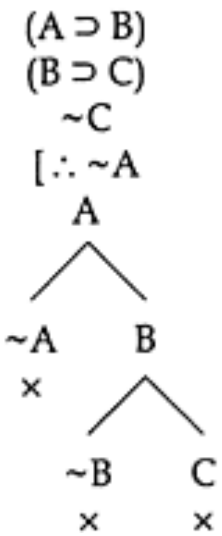
~~P → P
(P · Q) → P, Q
~(P ∨ Q) → ~P, ~Q
~(P ⊃ Q) → P, ~Q

Each form that can't be simplified is branched into the two sub-cases that would make it true; for example, since "~(P·Q)" is true just if "~P" is true or "~Q" is true, it branches into these two formulas. There are five branching rules:

~(P · Q)	(P ∨ Q)	(P ⊃ Q)	(P ≡ Q)	~(P ≡ Q)
~P ~Q	P Q	~P Q	P ~P Q ~Q	P Q ~Q ~P

To test an argument, we write the premises, block off the original conclusion (showing that it is to be ignored in constructing the tree), and add the denial of the conclusion. Then we apply the simplifying and branching rules to each formula, and to each further formula that we get, until every branch either dies (contains a pair of contradictory wffs) or contains only simple wffs (letters or their negation). The argument is valid if and only if every branch dies. Here's an argument proved two ways:

Truth tree



Our proof

```
* 1    (A ⊃ B)
* 2    (B ⊃ C)
  3    ~C
     [ ∴ ~A
  4  ┌ asm: A
  5  │ ∴ B    {from 1 and 4}
  6  └ ∴ ~B   {from 2 and 3}
  7  ∴ ~A   {from 4; 5 contradicts 6}
```

In the truth tree, we write the premises, block off the original "∴ A" conclusion (and henceforth ignore it), and add its contradictory "A." Then we branch "(A⊃B)" into its two sub-cases: "~A" and "B." The left branch dies, since it contains "A" and "~A"; we indicate this by putting "×" at its bottom. Then we branch "(B⊃C)" into its two sub-cases: "~B" and "C." Each branch dies; the left branch has "B" and "~B," while the right has "C" and "~C." Since every branch of the tree dies, no possible truth conditions would make the premises all true and conclusion false, and so the argument is valid.

An argument is invalid if some branch of the tree doesn't die. Then the simple wffs on each live branch give a refutation of the argument—truth conditions making the premises all true and conclusion false.

As compared with traditional proofs, truth trees give a simple and efficient way to decide whether an argument is valid or invalid—a way that uses an automatic strategy instead of guesswork and intuition. But truth trees don't mirror ordinary reasoning very well; they give a mechanical way to test validity instead of a way to help develop reasoning skills. And branching can get messy. Our method avoids these disadvantages but keeps the main advantages of truth trees over traditional proofs.

[1] While somewhat overlapping with syllogistic logic (Chapter 2), quantificational logic is more powerful and flexible because it also includes propositional logic.