Matrix equation

SLE

Inverse matrix

Outline

- Express an SLE in terms of a matrix equation
- For a matrix equation $A\vec{x} = \vec{b}$, where A is a square matrix (i.e., an $n \times n$ matrix), we will introduce the notion of inverse matrix to solve the equation.
- Use elementary row operations to find the inverse matrix.

Consider m linear equations with n variables x_1, x_2 …, x_n

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
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Recall that the coefficient matrix of an SLE is a matrix whose i-th row is given by the coefficients in front of variables at the i-th equation. So

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

is the coefficient matrix of the SLE above.

Because of the matrix multiplication, we have

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix}$$

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If we denote $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$, then the matrix equation $A\vec{x} = \vec{b}$ exactly represents the SLE.

For instance, consider the SLE

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$$\begin{aligned} x \ + y + \ z &= 1 \\ x - y + 2z &= 2 \\ 2x \qquad + 3z &= 0 \end{aligned}$$

The corresponding matrix equation is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

We can think of an augmented matrix as an “abbreviation” of a matrix equation.

Some remarks

• When we have a solution of an SLE, we write

$$(x, y, z) = (1, 2, 3)$$

instead of

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

• In a matrix equation $A\vec{x} = \vec{b}$ representing an SLE, A is the coefficient matrix, $\vec{x}$ is a column vector consisting of all variables and $\vec{b}$ is a column vector consisting of all constants.

Matrix equation

Definition Any SLE involving m equations and n variables can be represented by the *matrix form* $A\vec{x} = \vec{b}$ of the SLE, where A is the $m \times n$ coefficient matrix, $\vec{x}$ is the column vector of the unknown variables and $\vec{b}$ is the column vector of right hand side values.

This means that solving an SLE is equivalent to find all $\vec{x}$ such that $A\vec{x} = \vec{b}$.

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Think about a single equation $2x = 3$. We know that $(\frac{1}{2})(2) = 1$ so that $x = (\frac{1}{2})(2x) = (\frac{1}{2})(3) = \frac{3}{2}$.

We would like to mimic this process in a case of matrix equations.

Inverse matrix

That is, if A is a **square matrix**, find a matrix B such that $BA = I$. Thus $\vec{x} = BA\vec{x} = B\vec{b}$. If such B exists, it is called the *inverse matrix* of A.

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Definition Let A be a square matrix. If there exists a matrix B with the same dimensions as A such that

$$AB = BA = I$$

then we say that A is *invertible* (or *nonsingular*) and that B is the *inverse* of A, written $B = A^{-1}$.

If A has no inverse (i.e., if no such matrix B exists), then A is said to be *noninvertible* (or *singular*).

Recap

The *identity matrix* I_n of order n is an $n \times n$ diagonal matrix such that $a_{ii} = 1$.

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

The following matrices are NOT identity matrices.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$. Show that $A^{-1} = B$.

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proof. We need to check $AB = I$ and $BA = I$ by definition.

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \times (-2) + (2)(\frac{3}{2}) & 1 \times 1 + (2)(-\frac{1}{2}) \\ 3 \times (-2) + (4)(\frac{3}{2}) & 3 \times 1 + (4)(-\frac{1}{2}) \end{bmatrix}$$

$$= \begin{bmatrix} -2+3 & 1-1 \\ -6+6 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (-2) \times 1 + 3 & (-2) \times 2 + 4 \\ \frac{3}{2} - \frac{1}{2} \times 3 & \frac{3}{2} \times 2 - \frac{1}{2} \times 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, we have $AB = BA = I$ as required. So $B = A^{-1}$.

Find the inverse matrix

- Remark that not every square matrix is invertible.

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Step 1. We form the following matrix

$$\left[A \mid I_n \right].$$

Step 2. We perform elementary row operations to $\left[A \mid I_n \right]$ such that A becomes an RREF. Say the new matrix is

$$\left[R \mid B \right].$$

Conclusion: if R is an identity matrix, then A is invertible and the right hand side matrix B is the inverse of A. If R is not an identity matrix (for instance, R contains a row of entire zero entries), then A is singular.

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Theorem If the inverse matrix of A exists, then it is unique.

Example. Find A^{-1} if it exists.

$$(a) \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 1 & 2 \end{bmatrix} \qquad\qquad (b) \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$