Puzzle: $S = \{A \mid A \notin A\}$ is $S \in S$?

is $S \notin S$?

§3.4

Proofs involving $\land$ and $\leftrightarrow$

Strategy: to prove the goal of the form $p \land q$, prove each part seperately.

Strategy: to use the given of the form $p \land q$, add p and q as givens.

Exercise 2. if $A \subseteq B$, $A \subseteq C$, proof $A \subseteq B \cap C$.

Given

$(A \subseteq B) \land (A \subseteq C)$

$A \subseteq B$

$A \subseteq C$

$\forall x (x \in A \to x \in B)$

$\forall x (x \in A \to x \in C)$

$x \in A$

$x \in B$

$x \in C$

Goal

$A \subseteq B \cap C$

$\forall x (x \in A \to x \in B \cap C)$

$x \in B \cap C$

Proof: Suppose $A \subseteq B$, $A \subseteq C$. Let $x \in A$. Since $x \in A$ and $A \subseteq B$, we have $x \in B$. Since $x \in A$ and $A \subseteq C$, then $x \in C$. Since $x \in B$ and $x \in C$, then $x \in B \cap C$. Since x is arbitrary, $A \subseteq B$ and $A \subseteq C$ implies $A \subseteq B \cap C$. $\square$.

Pattern: So, [p] implies [q]. $\square$

Recall: $P \leftrightarrow Q$ means $(P \to Q) \land (Q \to P)$.

Strategy: To prove a goal $P \leftrightarrow Q$, prove $P \to Q$ and $Q \to P$ seperately.

To use a given $P \leftrightarrow Q$, add both $P \to Q$ and $Q \to P$ to given.

Example: 8: $A \subseteq B$ iff $P(A) \subseteq P(B)$.

Given

1st: $A \subseteq B$

$\forall x (x \in A \to x \in B)$.

Goal

$A \subseteq B \leftrightarrow P(A) \subseteq P(B)$

1st goal: $A \subseteq B \to P(A) \subseteq P(B)$

$\forall y (y \subseteq A \to y \subseteq B)$.

2nd goal: $P(A) \subseteq P(B) \to A \subseteq B$.

2nd: $P(A) \subseteq P(B)$

$\forall x (x \in P(A) \to x \in P(B))$

$\forall x (x \subseteq A \to x \subseteq B)$.

$A \subseteq A \to A \subseteq B$

Proof: ($\to$) Suppose $A \subseteq B$, let $x \in P(A)$. Let $y \in x$. Since $\forall y \in x$ and $x \subseteq A$
So $y \in A$. Since $y \in A$ and $A \subseteq B$, we get $y \in B$. So $x \subseteq B$
Thus, $x \in P(B)$. So $P(A) \subseteq P(B)$. Therefore $A \subseteq B$ implies
$P(A) \subseteq P(B)$

($\leftarrow$) Suppose $P(A) \subseteq P(B)$. Since $A \subseteq A$, so $A \in P(A)$. Since $P(A) \subseteq P(B)$
$A \in P(B)$. That is, $A \subseteq B$. Therefore, $P(A) \subseteq P(B)$ implies $A \subseteq B$.

For every integer n, $6 | n$ iff $2 | n$ and $3 | n$.

$\forall n \in \mathbb{Z} (n|6 \leftrightarrow n|2 \wedge n|3$

$\neq$

$(\forall n \in \mathbb{Z}\, n|6) \longleftrightarrow (\forall n \in \mathbb{Z}\, n|2 \wedge n|3)$

Proof: ($\to$) Suppose $n = 6k$ for some integer k.
So $n = 2 \times 3k$. Since k is an integer, $3k$ is also an integer,
so $2 | n$. Similarly, $n = 3 \times 2k$ and $3 | n$.

($\leftarrow$) Assume $2 | n$ and $3 | n$. Then $n = 2k_1 = 3k_2$ for
some integer k_1 and k_2. So $6(k_1 - k_2) = n$.
Since $k_1 - k_2 \in \mathbb{Z}$, so $6 | n$

Strategy: To prove $P \leftrightarrow Q$. You can prove with intermedinm
states $r_1, r_2 \cdots$: $P \leftrightarrow r_1 \leftrightarrow r_2 \cdots r_n \leftrightarrow Q$.