Undecidability

COMPSCI 3331

Outline

- Undecidability: definitions.
- Undecidability by diagonalization.
- Basic undecidability result: Halting problem.
- Universal TMs.
- Reductions.
- PCP and CFG undecidability.

Undecidability

- A problem is undecidable if there is no TM which solves it.
- The fact that **there exist undecidable problems** is one of the major contributions of theoretical computer science.
- We show that new problems are undecidable from existing undecidable problems by **reduction**.

Problems and Languages

- A **problem** P assigns each word $w \in \Sigma^*$ a yes/no answer. Let $P(w)$ denote this yes/no answer.
- e.g., Primality: given a word $w \in \{0, 1\}^*$, is w the binary representation of a prime number?
  - P_{prime} denotes this function.
  - $P_{\text{prime}}(101) = \text{yes}$
  - $P_{\text{prime}}(10110) = \text{no}$

MAX

list, num → is num the max in the list?

Undecidability

- We translate **problems** into **languages**: For every problem P, we associate it with

$$L_P = \{\underbrace{x \in \Sigma^*}\ :\ P(x) = \text{ yes }\}.$$

- e.g., primality:

$$\begin{aligned} L_{\text{prime}} &= \{x \in \Sigma^*\ :\ x \text{ encodes a prime number }\}. \\ &= \{10, 11, 101, 111, \ldots,\} \end{aligned}$$

Undecidability

- From last time:
  - a language L is recursively enumerable (r.e.) if there exists a TM M which *accepts* L.
  - The language L is recursive if there exists a TM M which *recognizes* a language L (i.e., M **always halts**).
- A problem P is **undecidable** if L_P is **not** recursive.
- We also say that any language L which is not recursive is an undecidable language.

First Undecidable Language

- By **diagonalization** (Cantor).
- Need to define an ordering of TMs: $M_1, M_2, M_3, \ldots$.
- To do this, we will encode a TM as a word over $\{0, 1\}$.
- The i-th TM will be the i-th word over $\{0, 1\}$.

Encoding TMs

Let $\Sigma = \{0, 1\}$. Let $M = (Q, \{0, 1\}, \Gamma, \delta, q_1, B, F)$ be a TM. We can rename the states and tape alphabet so that:

- $Q = \{q_1, q_2, q_3, \dots, q_r\}$ for some $r \geq 1$. We can also assume that $F = \{q_r\}$.
- $\Gamma = \{\alpha_1, \alpha_2, \dots, \alpha_s\}$ for some $s \geq 3$. We assume $\alpha_1 = 0, \alpha_2 = 1$, and $\alpha_3 = B$.

Consider a transition $\delta(q_i, \alpha_j) = (q_k, \alpha_\ell, D)$. We encode this **single** transition as the word

$$0^i 1 0^j 1 0^k 1 0^\ell 1 0^{m(D)}$$

where $m(D)$ is $1, 2, 3$ if D is L, S, R, respectively.

(state from) 1 (symb. read) 1 (state to) 1 (symb. write) 1 (dir.)

Encoding TMs

We now encode the **entire** TM $M = (Q, \Sigma, \Gamma, \delta, q_1, B, F)$.
Let $C_1, C_2, \ldots, C_m$ be the encodings of the m transitions of the TM. Then we encode M as

$$e(M) = C_1 11 C_2 11 C_3 11 \cdots 11 C_m$$

- Can we decode the TM based on $e(M)$?
- How can we guarantee that there are only finitely many transitions?

Ordering words and TMs

Let $\leq_\ell$ be the total lexicographical ordering of words over $\{0,1\}$:

- if x is shorter than y, then $x \leq_\ell y$.
- if x, y have the same length, but x comes before y in lexicographical order, then $x \leq_\ell y$.

So:

- $00 \leq_\ell 111$
- $00 \leq_\ell 01$
- $00101 \leq_\ell 00110$.

Ordering words and TMs

$\leq_\ell$ imposes a total order on all words over $\{0, 1\}$:

$\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, 0000, \ldots$

- Let w_i be the i-th word in this order. $w_1 = \varepsilon; w_2 = 0$, etc.
- We can design a TM that starts with i (in binary) on its tape and halts with w_i on its tape.
- Order TMs: M_i is the TM with $e(M_i) = w_i$. (Recall that $e(M) \in \{0, 1\}^*$.)

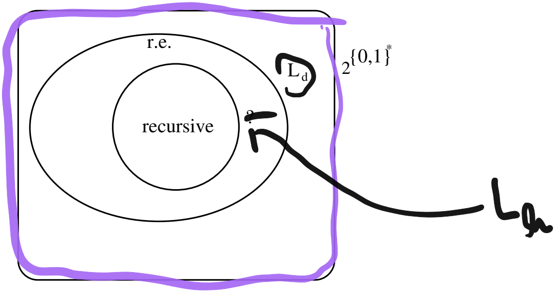
Diagonalization

$W_i \longleftrightarrow e(M_i)$

Define

$$L_d = \{w_i \in \{0,1\}^* \ : \ i \geq 0, w_i \notin L(M_i)\}.$$

Thm. L_d is not r.e.



Is there a language L which is r.e. but not recursive?

An undecidable problem that is r.e.

Consider the following problem (*the halting problem*):

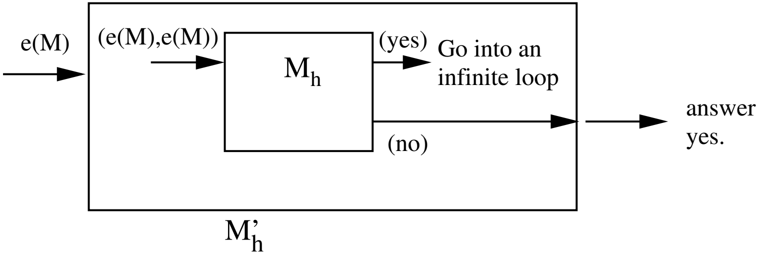
Given a TM M, and a word w, does M accept w?

- The input to this problem is $(e(M), w)$.
- This problem is r.e.: we **can** give a TM which halts and accepts if the answer to the problem is yes (later...)
- This problem is undecidable: it is not recursive.

Halting problem

Thm. The halting problem is undecidable.

Pf. Suppose there is a TM M_h which solves the halting problem $(e(M), w)$ and always halts with a yes/no answer.
Let M'_h be the following TM:



What does M'_h do when it gets $e(M'_h)$ as input?

① M_h' ask M_h → do I stop when I get $e(M_h')$ as input?

Ⓐ if M_h says yes.

— that means $e(M_h') \in L(M_h')$

— action of M_h' → goes into infinite loop.

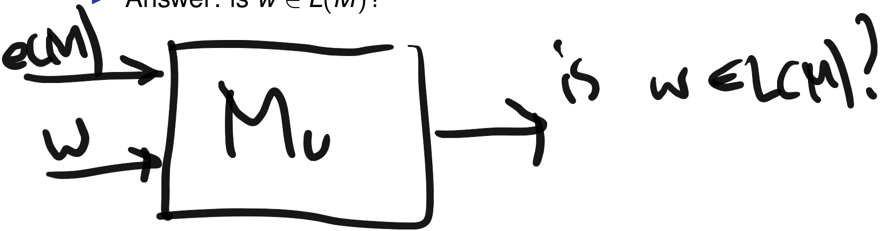
Ⓑ if M_h says no

— that means $e(M_h') \notin L(M_h')$

- action of M_h' → says yes.

Halting Problem

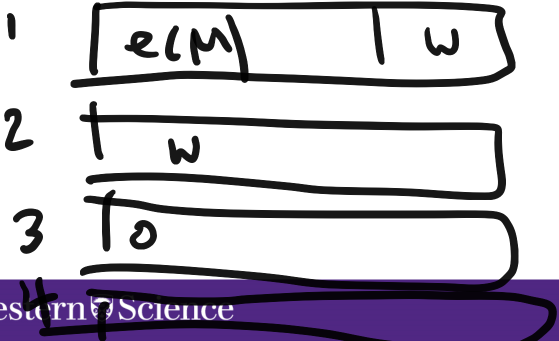
- We have shown that the halting problem "Does M halt on w?" is undecidable.
- Want to show that it is recursively enumerable: There exists a TM M_u such that if M halts on w, M_u halts and says yes.
- If M does not halt on w, M_u may not halt.
- We call this TM M_u a universal TM.
  - M_u gets $e(M), w$ as input.
  - Answer: is $w \in L(M)$?



Universal TM

M_u has at four tapes:

- On tape 1 is the input word $(e(M), w)$.
- Tape 2 will simulate the input tape of M; initially we copy w from tape 1 to tape 2.
- Tape 3 will contain the current state of M; initially we write the code for the start state of M on tape 3 (encoded as 0).
- Tape 4 will be a scratch tape.



Universal TM

M_u works as follows:

- After initialization, M_u simulates steps of M.
- M_u searches through the transitions of M and simulates one step of M, changing the input to M on tape 2, updating the head position on tape 2, and the state of M on tape 3.
- M_u faithfully simulates M for acceptance and crashing.
- If M accepts w, then so does M_u.

Universal TM

- The halting problem is r.e.:

$$\{(e(M), w) \ : \ w \in L(M)\}.$$

- However, it is **not** decidable.
- Universal TMs play a big part in problems involving TMs: it is often useful to simulate a TM on a given input.

More Undecidable Problems

We now have two undecidable problems:

- The diagonalization language.
- The halting problem.

Are there more? How can we find them?

Tool: TM computing functions.

- A TM M **computes a function** $f : \Sigma^* \to \Sigma^*$ if, whenever the TM gets [redacted] x as input, it halts and outputs $f(x)$ on the tape.
- There is no concept of acceptance and rejectance. Whatever happens when M halts, we take the tape contents as the result of applying f to the input.
- M is deterministic.
- Examples:
  1. $f(a^n) = n$ (written in binary).
  2. $f(a^n b^m) = c^{n^m}$ for all $n, m \geq 0$.
  3. $f((e(M_1), e(M_2))) = e(M)$ where $L(M) = L(M_1) \cup L(M_2)$.

More Undecidable Problems

- Let P_0 be an undecidable problem.
- We can show that another problem P_1 is undecidable by **reduction.**
- A reduction is a function f which can be computed by a TM and satisfies the following properties:
  - if $x \in P_0$ then $f(x) \in P_1$.
  - if $x \notin P_0$ then $f(x) \notin P_1$.

Using Reductions

- A reduction of problem P_0 to P_1 means that P_0 is **at least as hard** as P_1.
- Suppose that P_1 is decidable. To solve P_0: Take input x, convert it to $f(x)$, and decide whether $f(x) \in P_1$.
- This is an algorithm for P_0. Therefore

$$\cancel{P_1 \text{ decidable} \quad \Rightarrow \quad P_0 \text{ decidable.}}$$

$$\boxed{P_0 \text{ undecidable} \quad \Rightarrow \quad P_1 \text{ undecidable.}}$$

IMPORTANT

$x \in P_0$?

$f(x) \in P_1$?

M_1

yes $\rightarrow x \in P_0$

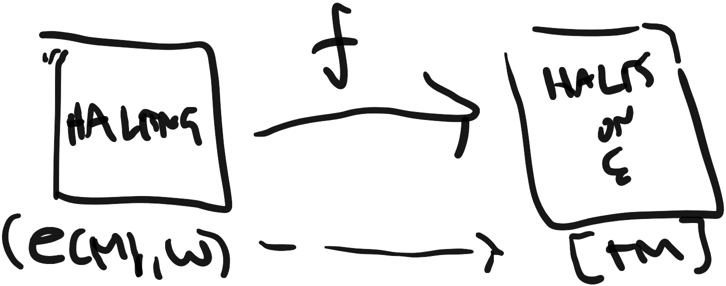
no $\rightarrow x \notin P_0$

“Halts on ε”

Want to show that the following problem is undecidable (*halts on* ε):

> *Given a TM M, is* $\varepsilon \in L(M)$*?*

Use the halting problem show that “halts on ε” is undecidable.



"Halts on ε"

Suppose (M, w) is an instance of the halting problem. We map (M, w) to a new TM M_0. M_0 does the following on input x:

1. If $x \neq \varepsilon$, reject.
2. Otherwise, if $x = \varepsilon$, use M_u to simulate the action of M on w.
3. If M_u determines $w \in L(M)$, halt and accept ε.
4. If M_u determines $w \notin L(M)$, reject ε.

$(M, w) \to M_0$ **is our reduction!**

H.P.

yes

no

$w \in L(M)$

$w \notin L(M)$

"Halts on ε"

What does M_0 do?

- If $w \in L(M)$, then $\varepsilon \in L(M_0)$.
- If $w \notin L(M)$, then $\varepsilon \notin L(M_0)$.

This is a reduction of the halting problem to "halts on ε".
To solve the halting problem:

- Given (M, w), construct M_0.
- Use "halts on ε" to determine whether $\varepsilon \in L(M_0)$.
- This is an algorithm for the halting problem, therefore, an algorithm solving "halts on ε" must not exist.

therefore $\longrightarrow$ "halts on ε" is undecidable.

Post's Correspondence Problem

Post's Correspondence Problem (PCP) is a simple example of an undecidable problem:

Input: Two lists of words $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ of the same length $n \geq 1$.

Determine: does there exist indices $i_1, i_2, \ldots, i_m$ with $1 \leq i_j \leq n$ and $m \geq 1$ such that

$$u_{i_1} u_{i_2} \cdots u_{i_m} = v_{i_1} v_{i_2} \cdots v_{i_m}.$$

Thm. PCP is undecidable.

PCP: Examples

Example 1:

u abcbb X
v abcbc

NO

i	1	2	3
u_i	ab	abc	bb
v_i	b	ab	cbc

Example 2:

$i_1 = 2$ $i_2 = 3$ $i_3 = 1$

i	1	2	3
u_i	ab	abc	bb
v_i	b	ab	$cbba$

YES

u abc | bb | ab
v ab | cbba | b

PCP is undecidable

Thm. PCP is undecidable.

- Given (M, w), we construct a PCP instance $(u_1, u_2, \ldots, u_n), (v_1, v_2, \ldots, v_n)$
- M accepts w iff the PCP instance has a solution.
- Idea: the solution to PCP will be a proof that M accepts w: a list of the IDs of M showing how w is accepted.
- $q_0 w = \alpha_0 \vdash_M \alpha_1 \vdash_M \cdots \vdash_M \alpha_n$.
- The solution will be of the form

$$\alpha_0 \# \alpha_1 \# \alpha_2 \# \cdots \# \alpha_n.$$

tape contents + state

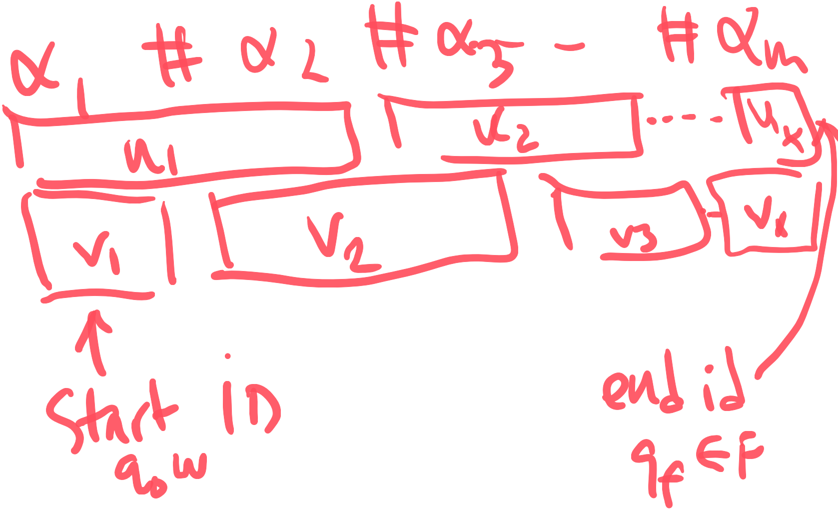
$\boxed{\text{tape}}\,\boxed{q}\,\text{tape}$

$\boxed{\text{HALT}} \xrightarrow{f} \boxed{\text{PCP}}$

$(M, w) \longrightarrow$ instance

$u_i \rightarrow$ simulate one step of TM

$v_i \rightarrow$ simulate one step of TM



Using PCP

Thm. Given CFGs G_1, G_2, it is undecidable whether $L(G_1) \cap L(G_2) = \emptyset$.

Thm. Given a CFG G, it is undecidable whether $L(G) = \Sigma^*$.

Thm. Given a CFG G, it is undecidable whether G is ambiguous.

PCP instance $\longrightarrow$ CFG

$u_1, u_2, \dots, u_n$ $\qquad$ G_1

$v_1, v_2, \dots, v_n$ $\qquad$ G_2.

$G_1 \quad \{u_1, \dots, u_n\} \subseteq \Sigma \quad \{1, \dots, n\}$

$$S \to u_1 S 1$$
$$S \to u_2 S 2$$
$$\vdots$$
$$S \to u_n S n$$
$$S \to \#$$

$$L(G_1) = \{ u_{i_1} u_{i_2} \cdots u_{i_k} \# i_k i_{k-1} \cdots i_1 : 1 \le i_1, i_2 \cdots i_k \le n,\ k \ge 0 \}$$

Emil Post (1897–1954)