

# SETS

## OUTLINE:

- 1) Introduction to sets
- 2) Defining and representing sets
- 3) Set operations
- 4) Sets and logic
- 5) Cardinality
- 6) Cartesian products and relations

# 1. INTRODUCTION TO SETS

# Naive Set Theory

- In this course, we adopt the following **informal definition** of set:
- A **set** is a (unordered) collection of objects, called **elements** or **members** of the set. The set is said to **contain** its elements. The notation  $a \in S$  means that the object  $a$  is an element of the set  $S$ . The notation  $a \notin S$  means that  $a$  is not an element of the set  $S$ .
- Note that we do not specify what an “object” is. The theory that results from this intuitive definition of a set as a collection of objects is called **naive set theory**. This theory leads to logical inconsistencies (paradoxes).

# Russell's Paradox

- Many paradoxes of naive set theory are variations on a common theme.
- The problem is the (implicit) **assumption that any property whatsoever may be used to form a set** (that is, any collection of elements whatsoever can be grouped into a set, without restrictions).
- Here we illustrate the problem in that assumption via Russell's Paradox.
- Preliminary remarks:
  - 1) A set may contain other sets as elements.
  - 2) A set may contain *itself* as an element.

# Russell's Paradox

- Let then  $S$  be the set whose elements are exactly the sets which are not elements of themselves.
- Pop quiz: does  $S$  contain itself?

# Russell's Paradox

- Let's then define  $S$  to be the set whose elements are exactly the sets which are not elements of themselves.
- Pop quiz: does  $S$  contain itself?
  - Assume  $S \in S$ . Then, by definition of  $S$ ,  $S \notin S$ , contradiction.
  - Assume  $S \notin S$ . Then, again by definition of  $S$ ,  $S \in S$ , contradiction.

# Axiomatic vs. Naive

- There is a more proper axiomatic set theory (actually, there are many versions of it) which gets rid of the known paradoxes of naive set theory.
- However, since
  - Naive set theory suffices for the everyday use of set theory concepts in contemporary mathematics (in particular, it does not give rise to inconsistencies within the restricted scope of this course);
  - Naive set theory is much more user-friendly than axiomatic set theory;
  - Naive set theory is a useful stepping stone towards more formal set theories;we are going to adopt naive set theory here (being aware of its limitations).

## 2. DEFINING AND REPRESENTING SETS



# Set equality and subsets

- Two sets  $A$  and  $B$  are equal if and only if they have exactly the same elements:

$$A=B \text{ iff } \forall x (x \in A \leftrightarrow x \in B)$$

- A set  $A$  is a subset of a set  $B$  (notation:  $A \subseteq B$ ) if and only if every element of  $A$  is also an element of  $B$

$$A \subseteq B \text{ iff } \forall x (x \in A \rightarrow x \in B)$$

# Representing sets: the roster method

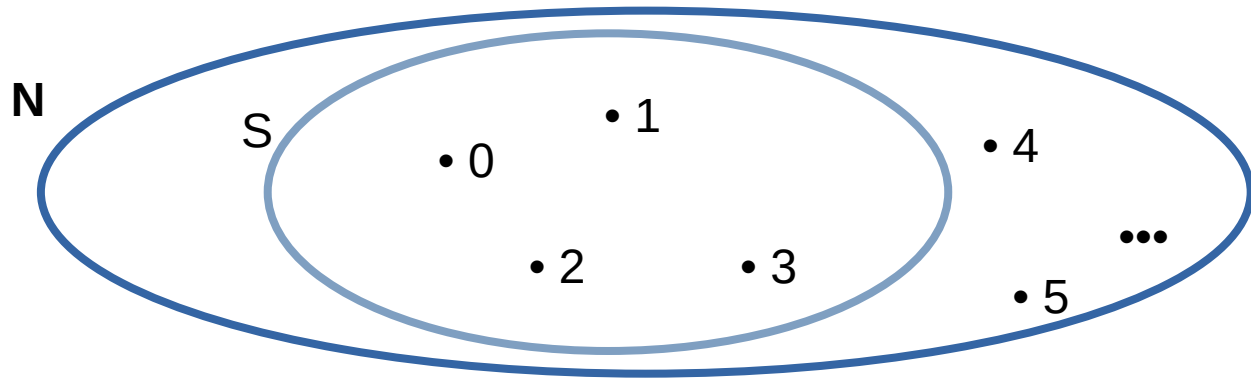
- The elements of a set  $S$  are listed explicitly within curly brackets.  
EX:  $S = \{0,1,2,3\}$
- The order of elements does not matter:  $S = \{0,1,2,3\} = \{2,0,1,3\}$
- Listing a member more than once does not change the set, and it is strongly discouraged:  $S = \{0,1,2,3\} = \{0,0,0,1,1,2,3,3,3,3\}$
- For large sets (finite or infinite), we list enough elements to highlight the pattern and then we use ellipses (...) to leave the others implied:
  - $A = \{a,b,c,\dots,x,y,z\}$  (the set of english letters, i.e. the english alphabet)
  - $\mathbf{Z} = \{\dots,-3,-2,-1,0,1,2,3,\dots\}$  (the set of integer numbers)

# Representing sets: set-builder method

- The elements of a set  $S$  are implicitly defined as all and only the objects satisfying a property:
  - $S = \{x \mid x \text{ is an integer between 0 and 3 (included)}\} = \{x \mid x \in \mathbf{Z} \text{ and } 0 \leq x \leq 3\} = \{0,1,2,3\}$ 
    - We are saying: “ $S$  is the set of the objects  $x$  **such that** (  $\mid$  )  $x$  is an integer and  $0 \leq x \leq 3$ ”
  - $T = \{2x \mid x \in S\} = \{0,2,4,6\}$ 
    - We are saying: “ $T$  is the set of the objects of the form  $2x$  **for** (  $\mid$  )  $x$  in  $S$ ”, that is, “ $T$  is the set of the doubles of the elements of  $S$ ”
  - $\mathbf{Q} = \{a/b \mid a \in \mathbf{Z}, b \in \mathbf{Z} \text{ and } b \neq 0\}$  is the set of rational numbers
- The variable  $x$  is just a label (placeholder), which can be changed to any other symbol, especially to avoid clashes of notation:
  - $A = \{x \mid x \text{ is a letter of the English alphabet}\}$  **not ideal, since the label  $x$  clashes with the English letter ‘x’**
  - **Better:**  $A = \{@ \mid @ \text{ is a letter of the English alphabet}\}$

# Representing sets: Venn diagrams

- This is a graphical representation, useful to illustrate simple set relationships at a glance.
- Objects are drawn as dots and sets as closed curves encircling their elements



# Empty set and universal set

- The empty set is the set with no elements. It is usually denoted  $\emptyset$ , or  $\{\}$ 
  - Not to be confused with  $\{\emptyset\}$ , which is the set containing the empty set (and hence it has one element)
- The universal set is the set of all the objects under consideration.
  - Often denoted  $U$ . In Venn diagrams, represented with a rectangle
  - May be explicitly stated or left implicit
  - Depends on the context

# Some important sets of numbers

- $\mathbb{N}$  or  $\mathbf{N} = \{0,1,2,3,\dots\}$  = the set of natural numbers
- $\mathbb{Z}$  or  $\mathbf{Z} = \{\dots,-2,-1,0,1,2,\dots\}$  = the set of integer numbers
- $\mathbf{Z}^+ = \{1,2,3,\dots\}$  = the set of positive integers
- $\mathbf{Z}^- = \{\dots,-3,-2,-1\}$  = the set of negative integers
- $\mathbb{Q}$  or  $\mathbf{Q} = \{a/b \mid a \in \mathbf{Z}, b \in \mathbf{Z} \text{ and } b \neq 0\}$  = the set of rational numbers
- $\mathbf{Q}^+$  = the set of positive rational numbers,  $\mathbf{Q}^-$  = the set of negative rational numbers
- $\mathbb{R}$  or  $\mathbf{R}$  = the set of real numbers (very complicated definition)
- $\mathbf{R}^+$  = the set of positive real numbers,  $\mathbf{R}^-$  = the set of negative real numbers
- $\mathbb{C}$  or  $\mathbf{C} = \{a + ib \mid a \in \mathbf{R}, b \in \mathbf{R}\}$  = the set of complex numbers;  $i = \sqrt{-1}$

# Interval notation

- Used to concisely represent **ranges** of **real** numbers
- $[a,b] = \{x \in \mathbf{R} \mid a \leq x \leq b\}$  “closed interval”: the extrema are included
- $(a,b] = \{x \in \mathbf{R} \mid a < x \leq b\}$
- $[a,b) = \{x \in \mathbf{R} \mid a \leq x < b\}$
- $(a,b) = \{x \in \mathbf{R} \mid a < x < b\}$  “open interval”: the extrema are excluded
- Note that degenerate cases are possible:
  - if  $a = b$ , then  $[a,b] = \{a\}$  (set with single element) and  $(a,b] = [a,b) = (a,b) = \emptyset$ ;
  - if  $b < a$ , then  $[a,b] = (a,b] = [a,b) = (a,b) = \emptyset$
- $a$  or  $b$  can be infinite (as an excluded extremum): e.g.,  $(-\infty, 4) = \{x \in \mathbf{R} \mid x < 4\}$ ;  
 $(-\infty, +\infty) = \mathbf{R}$

# Subsets and power set

- Recall the notion of subset:  $A \subseteq B$  iff  $\forall x (x \in A \rightarrow x \in B)$
- The empty set is a subset of any set:  $\forall A, \emptyset \subseteq A$
- A set is always a subset of itself:  $A \subseteq A$
- A subset of A different from A itself is a **proper subset** of A (specific notation:  $\subset$ ); e.g.,  $\emptyset \subset A$
- The power set of a set A is the set of all subsets of A, including  $\emptyset$  and A (various notations:  $P(A)$ ,  $\wp(A)$ ,  $2^A$ )

$$P(A) = \{ S \mid S \subseteq A \}$$



# Examples

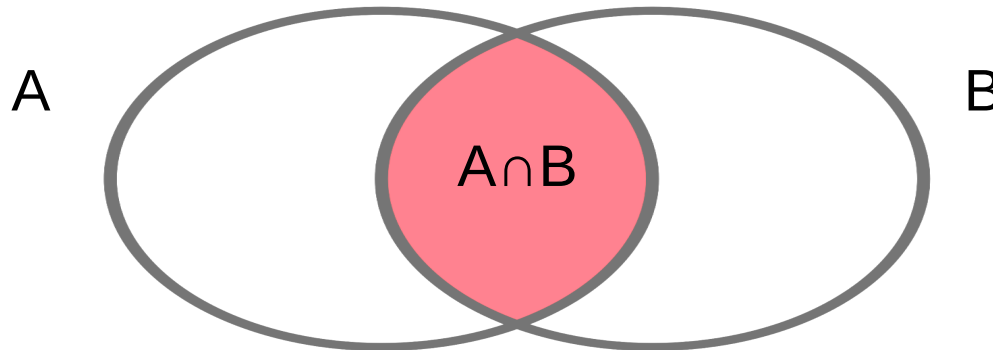
- $P(\emptyset) = \{S \mid S \subseteq \emptyset\} = \{\emptyset\}$
- $P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$
- $P(\{0,1\}) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$
- $P(\{0,1,2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$
- In general, if  $A$  has  $n$  elements,  $P(A)$  has  $2^n$  elements (hence the name power set) [EX: prove this statement by induction on  $n$ ]

# 3. SET OPERATIONS

# Intersection

- The **intersection** of 2 sets A and B is the set of the objects which are elements of both A and B (notation:  $A \cap B$ ).

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$



# Examples

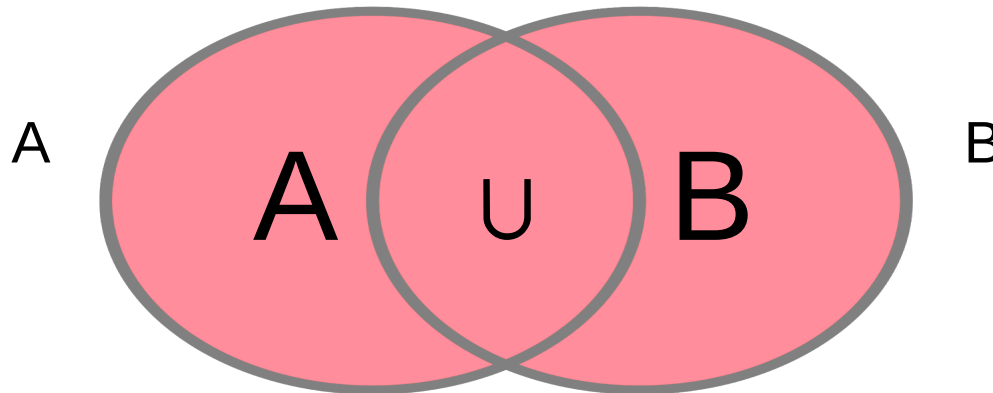
- $\{0,1,2\} \cap \{1,2,3\} = \{1,2\}$
- $\{0,1,2\} \cap \{4,5,6\} = \emptyset$ . Sets with empty intersection are said to be disjoint
- $[1,5] \cap (2,6] = \{x \in \mathbf{R} \mid 1 \leq x \leq 5 \text{ and } 2 < x \leq 6\} = (2,5]$
- $\{x \in \mathbf{N} \mid x \text{ odd}\} \cap \{x^2 \mid x \in \{0,1,2,3\}\} = \{x \in \mathbf{N} \mid x \text{ odd}\} \cap \{0,1,4,9\} = \{1,9\}$
- If  $A \subseteq B$ , then  $A \cap B = A$  [QUESTION: does the converse (if  $A \cap B = A$ , then  $A \subseteq B$ ) hold?]
- Intersection is associative:  $(A \cap B) \cap C = A \cap (B \cap C)$ . Therefore, usually brackets are omitted:  $(A \cap B) \cap C = A \cap (B \cap C) = A \cap B \cap C$
- This allows to intersect an arbitrary (even infinite) number of sets:

$$\bigcap_i A_i = \{x \mid \forall i (x \in A_i)\}$$

# Union

- The **union** of 2 sets A and B is the set of the objects which are elements of A or of B (notation:  $A \cup B$ ).

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$



# Examples

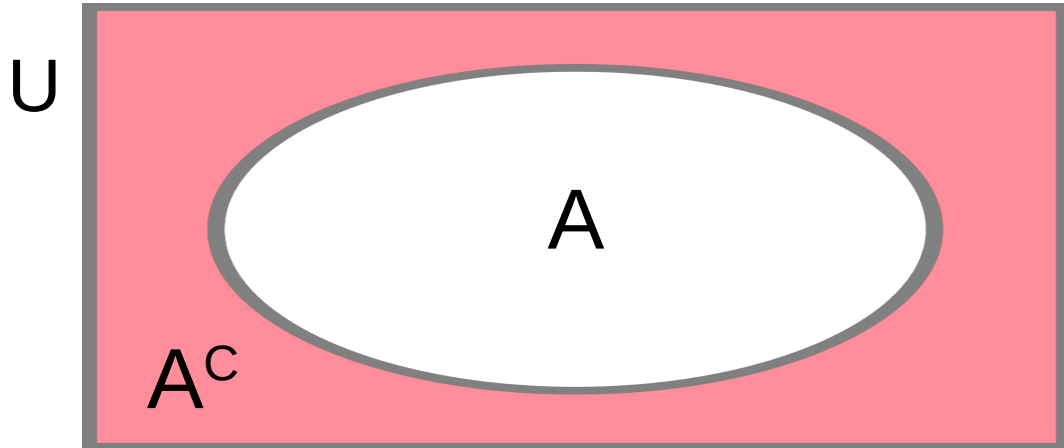
- $\{0,1,2\} \cup \{1,2,3\} = \{0,1,2,3\}$
- $\{0,1,2\} \cup \{4,5,6\} = \{0,1,2,4,5,6\}$
- $[1,5] \cup (2,6] = \{x \in \mathbf{R} \mid 1 \leq x \leq 5 \text{ or } 2 < x \leq 6\} = [1,6]$
- $\{x \in \mathbf{N} \mid x \text{ odd}\} \cup \{x^2 \mid x \in \{0,1,2,3\}\} = \{x \in \mathbf{N} \mid x \text{ odd}\} \cup \{0,1,4,9\} = \{x \in \mathbf{N} \mid x \text{ odd}\} \cup \{0,4\}$
- If  $A \subseteq B$ , then  $A \cup B = B$  [QUESTION: does the converse (if  $A \cup B = B$ , then  $A \subseteq B$ ) hold?]
- Union is associative:  $(A \cup B) \cup C = A \cup (B \cup C)$ . Therefore, usually brackets are omitted:  
 $(A \cup B) \cup C = A \cup (B \cup C) = A \cup B \cup C$
- This allows to unite an arbitrary (even infinite) number of sets:

$$\bigcup_i A_i = \{x \mid \exists i (x \in A_i)\}$$

# Complement

- Given a universe  $U$ , the **complement** of a set  $A$  (with respect to  $U$ ) is the set of the elements (of  $U$ ) which are not in  $A$  (notation:  $A^c$  or  $\bar{A}$ )

$$A^c = \{ x \in U \mid x \notin A \}$$



# Examples

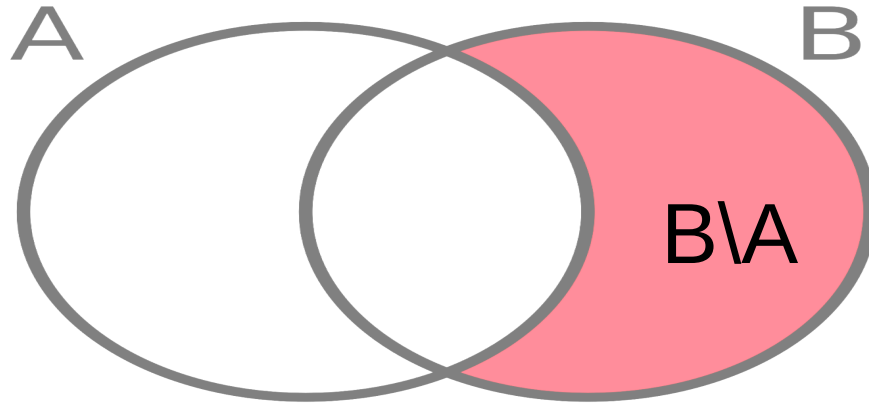
- For any universe  $U$ ,  $U^c = \emptyset$  and  $\emptyset^c = U$
- For any universe  $U$  and any set  $A \subseteq U$ ,  $(A^c)^c = A$  [EX: prove this]
- In the universe  $\mathbf{N}$  of natural numbers,  $\{x \in \mathbf{N} \mid x \text{ odd}\}^c = \{x \in \mathbf{N} \mid x \text{ even}\}$  and  $\{x \in \mathbf{N} \mid x \text{ even}\}^c = \{x \in \mathbf{N} \mid x \text{ odd}\}$
- In the universe  $\mathbf{R}$  of real numbers,  $[1, 5)^c = \{x \in \mathbf{R} \mid \neg(1 \leq x < 5)\}$   
 $= \{x \in \mathbf{R} \mid x < 1 \vee x \geq 5\} = (-\infty, 1) \cup [5, +\infty)$



# Difference

- Given 2 sets A and B, the **difference** of A in B is the set of the elements of B which are not in A (notation:  $B \setminus A$  or  $B - A$ ). Note that it is not required that  $A \subseteq B$ .

$$B \setminus A = \{x \mid x \in B \wedge x \notin A\} = \{x \in B \mid x \notin A\}$$



# Examples

- The difference with respect to the universe is the complement
- $B \setminus A = \{x \mid x \in B \wedge x \notin A\} = \{x \mid x \in B \wedge (x \in U \wedge x \notin A)\} = B \cap A^c$
- $\{0,1,2,3,4\} \setminus \{2,3\} = \{0,1,4\}$
- $\{0,1,2,3,4\} \setminus \{5,6,8\} = \{0,1,2,3,4\}$
- $\{0,1,2,3,4\} \setminus \mathbf{N} = \emptyset$
- $\{0,1,2,3,4\} \setminus \emptyset = \{0,1,2,3,4\}$

# Notable set identities ( $U$ = universe)

- *Identity laws:*  $A \cap U = A$ ;  $A \cup \emptyset = A$
- *Domination laws:*  $A \cup U = U$ ;  $A \cap \emptyset = \emptyset$
- *Law of disjointness (unofficial name):*  $A \cap A^c = \emptyset$
- *Law of partition (unofficial name):*  $A \cup A^c = U$
- *Idempotent laws:*  $A \cap A = A$ ;  $A \cup A = A$
- *Complementation law:*  $(A^c)^c = A$
- *Associative laws:*  $A \cap (B \cap C) = (A \cap B) \cap C$ ;  $A \cup (B \cup C) = (A \cup B) \cup C$
- *Commutative laws:*  $A \cap B = B \cap A$ ;  $A \cup B = B \cup A$
- *Distributive laws:*  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ ;  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- *Absorption laws:*  $A \cap (A \cup B) = A$ ;  $A \cup (A \cap B) = A$
- *De Morgan's laws:*  $(A \cap B)^c = A^c \cup B^c$ ;  $(A \cup B)^c = A^c \cap B^c$

# Proving set identities

- 3 ways:
  - 1) Prove that the LHS set is included in the RHS set and vice-versa
  - 2) Use set-builder notation and logic
  - 3) Use membership tables (deprecated by Prof. Pasini)

# Proving set identities

1) Prove that the LHS set is included in the RHS set and vice-versa

- EX:  $(A^c)^c = A$ .
- $(A^c)^c \subseteq A$ : if  $x \in (A^c)^c$ , then  $x \notin A^c$  (def. of complement), so  $x \in A$  (def. of complement).
- $A \subseteq (A^c)^c$ : if  $x \in A$ , then  $x \notin A^c$  (def. of complement), so  $x \in (A^c)^c$  (def. of complement).

# Proving set identities

2) Use set-builder notation and logic

- EX: prove that  $A \cap (B \cap C) = (A \cap B) \cap C$
- $A \cap (B \cap C) = \{x \mid x \in A \wedge x \in (B \cap C)\}$  (def. of  $\cap$ )  
=  $\{x \mid x \in A \wedge (x \in B \wedge x \in C)\}$  (def. of  $\cap$ )  
=  $\{x \mid (x \in A \wedge x \in B) \wedge x \in C\}$  (associativity of  $\wedge$ )  
=  $\{x \mid x \in (A \cap B) \wedge x \in C\}$  (def. of  $\cap$ )  
=  $(A \cap B) \cap C$  (def. of  $\cap$ )

# Proving set identities

3) Use **membership tables** (deprecated by Prof. Pasini).  
Membership tables are “truth tables” for sets.

- EX:  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

| $A$ | $B$ | $C$ | $B \cap C$ | $A \cup (B \cap C)$ | $A \cup B$ | $A \cup C$ | $(A \cup B) \cap (A \cup C)$ |
|-----|-----|-----|------------|---------------------|------------|------------|------------------------------|
| 1   | 1   | 1   | 1          | 1                   | 1          | 1          | 1                            |
| 1   | 1   | 0   | 0          | 1                   | 1          | 1          | 1                            |
| 1   | 0   | 1   | 0          | 1                   | 1          | 1          | 1                            |
| 1   | 0   | 0   | 0          | 1                   | 1          | 1          | 1                            |
| 0   | 1   | 1   | 1          | 1                   | 1          | 1          | 1                            |
| 0   | 1   | 0   | 0          | 0                   | 1          | 0          | 0                            |
| 0   | 0   | 1   | 0          | 0                   | 0          | 1          | 0                            |
| 0   | 0   | 0   | 0          | 0                   | 0          | 0          | 0                            |

## 4. SETS AND LOGIC



# Relationship between sets and logic

- Each set  $S$  is associated to a predicate  $P_S(x) = "x \in S"$ .
- Conversely, given a domain  $D$  and a predicate  $P$  (interpreted on  $D$ ), we can form the **truth set** of  $P$ , which is the set of elements of  $D$  (thought of as the universe) for which  $P$  is true:  $S_P = \{x \in D \mid P(x)\}$ .
  - EX: If  $D = \mathbf{N}$  (the natural numbers) and  $P(x) = "x \leq 3"$ , then  $S_P = \{x \in \mathbf{N} \mid P(x)\} = \{x \in \mathbf{N} \mid x \leq 3\} = \{0, 1, 2, 3\}$ .
- “Compound” predicates may be formed using the logical connectives and the quantifiers:
  - EX (cont.): If  $Q(x) = "x > 2"$ , and we set  $R(x) = P(x) \wedge Q(x)$ , then  $S_R = \{x \in \mathbf{N} \mid P(x) \wedge Q(x)\} = \{x \in \mathbf{N} \mid 2 < x \leq 3\} = \{3\}$ .
  - The domain can also be encoded in a predicate instead of being explicitly stated: if  $I(x) = "x \text{ is a natural number}"$ , and we set  $R'(x) = I(x) \wedge P(x) \wedge Q(x)$ , then
  - $S_{R'} = \{x \mid I(x) \wedge P(x) \wedge Q(x)\} = \{x \mid x \in \mathbf{N} \wedge 2 < x \leq 3\} = \{x \in \mathbf{N} \mid 2 < x \leq 3\} = S_R$

# Relationship between sets and logic

- Intersection of sets corresponds to conjunction of predicates:
  - If  $S = \{x \mid P(x)\}$  and  $T = \{x \mid Q(x)\}$ , then  $S \cap T = \{x \mid P(x) \wedge Q(x)\}$
- Union of sets corresponds to disjunction of predicates:
  - If  $S = \{x \mid P(x)\}$  and  $T = \{x \mid Q(x)\}$ , then  $S \cup T = \{x \mid P(x) \vee Q(x)\}$
- Complementation of sets corresponds to negation:
  - If  $S = \{x \mid P(x)\}$ , then  $S^c = \{x \mid \neg P(x)\}$
- We could define other set operations corresponding to the other connectives, but they are not customary.
- Multiple intersection of sets corresponds to universal quantifier
  - If  $S_i = \{x \mid P_i(x)\}$  for  $i = 1, \dots, n$ , then  $\cap S_i = \{x \mid \forall i P_i(x)\}$
- Multiple union of sets corresponds to existential quantifier
  - If  $S_i = \{x \mid P_i(x)\}$  for  $i = 1, \dots, n$ , then  $\cup S_i = \{x \mid \exists i P_i(x)\}$

## 5. CARDINALITY

- We say that a set is finite if it has  $n$  elements for some natural number  $n$ . Otherwise, we say that the set is infinite.
- The cardinality of a finite set  $S$ , denoted  $|S|$ , is the number of its elements.
  - EX:  $|\emptyset| = 0$ ,  $|\{\emptyset\}| = 1$ ,  $|\{0,1,2,3\}| = 4$ ,  $|P(S)| = 2^S$ ,  $|S \times T| = |S| \cdot |T|$
- The cardinality of an infinite set is infinite, but there are various “grades” of infinity:
  - $|\mathbf{N}| = |\mathbf{Z}| = |\mathbf{Q}| < |(0,1)| = |\mathbf{R}| = |\mathbf{C}| = |\mathbf{R}^3| < |P(\mathbf{R})| \dots\dots$

# 6. CARTESIAN PRODUCTS AND RELATIONS

# Tuples

- An **[ordered] n-tuple**  $(a_1, a_2, \dots, a_n)$  is a collection of objects in which one  $(a_1)$  can be identified as the 1<sup>st</sup>, one  $(a_2)$  as the 2<sup>nd</sup>, and so on up to the  $n^{\text{th}}$   $(a_n)$ . In other words, differently than for sets, for tuples the order of the elements matter:
  - $\{0,1,2\} = \{1,2,0\}$  (sets: only the elements matter, not the order in which they appear)
  - $(0,1,2) \neq (1,2,0)$  (3-tuples: although the appearing objects are the same, they appear in different orders)
- Two tuples are **equal** iff they [have the same number of elements and] have the same elements in the same positions:  $(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_k)$  iff  $n=k \wedge a_1=b_1 \wedge a_2=b_2 \wedge \dots \wedge a_n=b_n$
- 2-tuples are also called **couples** or **pairs**, 3-tuples are also called **triplets**.

# Cartesian product

- The **cartesian product** of 2 sets A and B is the set of ordered couples (a,b) for  $a \in A$  and  $b \in B$  (notation:  $A \times B$ ).

$$A \times B = \{ (a, b) \mid a \in A \wedge b \in B \}$$

- More generally, the cartesian product of n sets  $A_1, \dots, A_n$  is the set of n-tuples  $(a_1, \dots, a_n)$  with  $a_i \in A_i$

$$A_1 \times A_2 \times \dots \times A_n = \{ (a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for all } i = 1, \dots, n \}$$

# Examples

- $\{0\} \times \{1,2,3,4\} = \{(0,1),(0,2),(0,3),(0,4)\}$
- $\{a,b,c\} \times \{0,1\} = \{(a,0),(b,0),(c,0),(a,1),(b,1),(c,1)\}$
- $\emptyset \times A = \emptyset$
- $\mathbf{R} \times \mathbf{R} = \mathbf{R}^2$ , which can be also interpreted geometrically: the cartesian product of 2 lines is a plane (the cartesian plane)
- $\{0,1\} \times \{0,1\} \times \{0,1\} = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$
- Cartesian products of infinitely many sets are also defined



# Relations

- A subset  $R$  of the cartesian product  $A \times B$  is called a **binary relation** on  $A$  and  $B$  (or from  $A$  to  $B$ ).
- More generally, a subset  $R$  of the cartesian product  $A_1 \times \dots \times A_n$  is called an  **$n$ -ary relation** on  $A_1, \dots, A_n$ . The number  $n$  is the **arity** of the relation.