MATH 1600 Linear Algebra — Winter 2020

Tutorial 6 - Wednesday

Basic matrix arithmetic

Recall that $\mathbb{R}^{n,m}$ denotes the set of all $n \times m$ matrices with real coefficients.

1. Suppose that $A \in \mathbb{R}^{n,m}$, $B \in \mathbb{R}^{l,k}$, and $r \in \mathbb{R}$.
   (a) For what values of l and k is $A + B$ defined and, in that case, what are the dimensions of $A + B$?
   (b) For what values of l and k is $A - B$ defined and, in that case, what are the dimensions of $A - B$?
   (c) For what values of n and m is rA defined and, in that case, what are the dimensions of rA?
2. Consider the following matrices with real coefficients:

$$A = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 2 \\ -2 & -2 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & -2 \\ 2 & 2 \\ 0 & 0 \end{bmatrix}.$$

   Perform the following operations, whenever they are defined.
   (a) $A + B$
   (b) $A - B$
   (c) $B + C$
   (d) $B - C$
3. Find all matrices $A, B \in \mathbb{R}^{2,2}$ that satisfy the following (*Hint:* use matrix operations).

$$2A + B = \begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix}, \quad A - 2B = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}.$$

Special matrices

4. Start by reminding yourself what **square**, **diagonal**, **scalar**, and **triangular** matrices are. And what are the **entries** of a matrix.
   (a) Suppose that $A \in \mathbb{R}^{n,n}$ is such that all of its entries are 0, except the entries of the form a_{ii} for i between 1 and n. What kind of special matrix is A?
   (b) Prove that if A and B are 3×3 diagonal matrices, then $A + B$ is also diagonal. Convince yourself that this works in $\mathbb{R}^{n,n}$ for arbitrary n.
   (c) Either prove or provide a counterexample for the following statements.
      I. All scalar matrices are diagonal.
      II. All diagonal matrices are scalar.
      III. All diagonal matrices are triangular.

Matrix multiplication

5. Suppose that $A \in \mathbb{R}^{n,m}$ and $B \in \mathbb{R}^{l,k}$. For what values of l and k is AB defined and, in that case, what are the dimensions of AB?

6. Recall that I_n denotes the $n \times n$ **identity matrix**. That is, I_n is the scalar $n \times n$ matrix with all non-diagonal entries equal to 0, and all diagonal entries equal to 1. Let

$$A = \begin{bmatrix} 4 & 5 & 7 \\ 3 & 3 & -5000 \end{bmatrix}$$

   Verify that $I_2 A = A = A I_3$. Convince yourself that for any matrix $A \in \mathbb{R}^{n,m}$ we have $I_n A = A = A I_m$.

7. Although matrix addition has the same nice properties as addition of numbers, this is not the case for multiplication. Let us see some of the things that can go wrong. Here all matrices have real coefficients.

   (a) Find two matrices A and B such that AB and BA are not defined.

   (b) Find two matrices A and B such that AB and BA are defined but have different dimensions.

   (c) Find two matrices A and B such that AB and BA are defined and have the same dimensions, but still $AB \neq BA$.

   (d) Find two square matrices A and B such that $(A+B)^2 \neq A^2 + 2AB + B^2$.

   (e) Find two *non-zero* square matrices A and B such that $AB = 0$.

8. Consider the following triangular matrices with real coefficients

$$A = \begin{bmatrix} 1 & 5 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & -3 \\ 0 & 4 & -2 \\ 0 & 0 & -1 \end{bmatrix}.$$

   Check that AB and BA are also triangular, and convince yourself that the product of two triangular matrices is always triangular.

9. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2,2}$. Compute A^{5000}.