(BINARY) RELATIONS

OUTLINE:

1) Introduction to binary relations
2) Properties of relations
3) Combining relations
4) Representing relations
5) Equivalence relations
6) Partial orderings

1. INTRODUCTION TO BINARY RELATIONS

Binary relations

- A binary relation from a set A to a set B is a subset $R \subseteq A \times B$.
  - EX: $A = \{-2,-1,0,1,2,3\}$, $B = \{0,1,2,3\}$, $R = \{(a,b) \in AxB \mid a > b\} = \{(1,0),(2,0),(2,1)(3,0),(3,1),(3,2)\}$
  - EX: $A = \{-2,-1,0,1,2,3\}$, $B = \{0,1,2,3\}$, $R = \{(a,b) \in AxB \mid b = a^2\} = \{(-1,1),(0,0),(1,1)\}$
  - EX: $A = \{-2,-1,0,1,2,3\}$, $B = \{0,1,2,3\}$, $R = \{(a,b) \in AxB \mid b < a^2+3\} = \{(-2,0),(-2,1),(-2,2),(-2,3),(-1,0),(-1,1),(-1,2),(-1,3),(0,0),(0,1),(0,2),(1,0)...\} = AxB \setminus \{(0,3)\}$
  - EX: $A = \{-2,-1,0,1,2,3\}$, $B = \{0,1,2,3\}$, $R = \{(a,b) \in AxB \mid b > a^2+3\} = \emptyset$
  - EX: $A = \{-2,-1,0,1,2,3\}$, $B = \{0,1,2,3\}$, $R = \{(-1,3),(-1,0),(1,2),(3,0)\} \subseteq AxB$ (kinda “random” relation, constructed specifying the couples one by one.)

Binary relations

- A binary relation on a set A is a subset $R \subseteq A x A$.
  - EX: $A = \{0,1,2,3\}$, $R = \{(a,b) \in A x A \mid a \text{ is a multiple of } b\} = \{(0,0),(0,1),(0,2),(0,3),(1,1),(2,1),(2,2),(3,1),(3,3)\}$
  - EX: $A = \mathbf{N}$, $R = \{(a,b) \in \mathbf{N} x \mathbf{N} \mid a = b\} = \{(0,0),(1,1),(2,2),\ldots\}$ (infinite set)
  - EX: $A = \mathbf{N}$, $R = \{(a,b) \in \mathbf{N} x \mathbf{N} \mid a+b \text{ is even}\} = \{(0,0),(0,2),(0,4),\ldots,(1,1),(1,3),\ldots\}$ (infinite set)
  - EX: $A = \mathbf{N}$, $R = \{(a,b) \in \mathbf{N} x \mathbf{N} \mid a \le b\} = \{(0,0),(0,1),(0,2),\ldots,(1,1),(1,2),\ldots\}$ (infinite set)

Notation for binary relations

- If $R \subseteq A \times B$ is a binary relation, there are several ways to denote its elements:
  - $(a,b) \in R$ (set-theoretic notation);
  - $R(a,b)$ (logical notation, using the predicate corresponding to the set R); EX: Equal(a,b);
  - aRb (logical infix notation); EX: $a=b$;

Binary relations on a finite set

- If A is a finite set with |A| = n, then how many distinct binary relations are there on AxA?

Binary relations on a finite set

- If A is a finite set, then how many distinct binary relations are there on AxA?
- Theorem 1: a set with m elements has 2^m subsets (prove this by induction on m)
- Theorem 2: for any finite sets S and T, $|S\times T| = |S|\cdot|T|$ (prove this using the definition of cartesian product)
- The binary relations on A are the subsets of AxA. Since by Theorem 1 $|A\times A|=|A|^2$, by Theorem 2 the number of distinct binary relations on A is

$$2^{|A|^2}$$

2. PROPERTIES OF RELATIONS

Reflexivity

- A relation $R \subseteq A \times A$ is reflexive if $\forall a\ (a \in A \rightarrow (a,a) \in R)$ i.e.,
- $\forall a \in A, (a,a) \in R$
- EX: on the integers **Z**, which of the following relations are reflexive?
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x = y\} = \{(x,x) \mid x \in \mathbf{Z}\} = \{\ldots, (-2,-2),(-1,-1),(0,0),(1,1),(2,2),\ldots\}$ YES
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x \leq y\}$ YES: for any $a \in \mathbf{Z}$, $a \leq a$, so $(a,a) \in R$
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid |x| = |y|\}$ YES
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x < y\}$ NO
  - $R = \mathbf{Z} \times \mathbf{Z}$ YES
  - $R = \varnothing$ NO
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x \text{ is a multiple of } y\}$ YES

Irreflexivity

- A relation $R \subseteq A \times A$ is irreflexive if $\forall a\ (a \in A \rightarrow (a,a) \notin R)$
- EX: on the integers **Z**, which of the following relations are irreflexive?
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x = y\}$ NO
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x \leq y\}$ NO
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x \neq y\}$ YES
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x < y\}$ YES
  - $R = \mathbf{Z} \times \mathbf{Z}$ NO
  - $R = \varnothing$ YES
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x \text{ is a multiple of } y\}$ NO

Question

- Is the following relation on the integers reflexive or irreflexive?

  $R = \{(x,y) \in \mathbf{Z} \mid x \text{ and } y \text{ are coprime (i.e., } \gcd(x,y)=1)\}$

Question

- Is the following relation on the integers reflexive or irreflexive?
- R = {(x,y)∈**Z**x**Z** | x and y are coprime (i.e., gcd(x,y)=1)}
- Neither! In fact,
  - gcd(1,1) = 1, so (1,1)∈R, therefore R cannot be irreflexive
  - gcd(2,2) = 2, so (2,2)∉R, therefore R cannot be reflexive

Symmetry

- A relation $R \subseteq A\text{x}A$ is symmetric if $\forall a,b \in A\ ((a,b) \in R \rightarrow (b,a) \in R)$
- EX: on the integers **Z**, which of the following relations are reflexive?
  - $R = \{(x,y) \in \mathbf{Z}\text{x}\mathbf{Z} \mid x = y\}$ YES: if x = y, is it true that y = x? Yes
  - $R = \{(x,y) \in \mathbf{Z}\text{x}\mathbf{Z} \mid x \leq y\}$ NO: counterexample: $0 \leq 1$ but $\neg(1 \leq 0)$
  - $R = \{(x,y) \in \mathbf{Z}\text{x}\mathbf{Z} \mid |x| = |y|\}$ YES
  - $R = \{(x,y) \in \mathbf{Z}\text{x}\mathbf{Z} \mid x < y\}$ NO: counterexample: $0 < 1$ but $\neg(1 < 0)$
  - $R = \mathbf{Z}\text{x}\mathbf{Z}$ YES (any couple of integers is in R)
  - $R = \varnothing$ Is it true that if $(a,b) \in \varnothing$, then $(b,a) \in \varnothing$? Yes! There are no couples satisfying the premise, so symmetry is vacuously true
  - $R = \{(x,y) \in \mathbf{Z}\text{x}\mathbf{Z} \mid x \text{ is a multiple of } y\}$ if a is a multiple of b, is b necessarily a multiple of a? NO: Counterex: 6 is a multiple of 2, but 2 is not a multiple of 6
  - $R = \{(x,y) \in \mathbf{Z}\text{x}\mathbf{Z} \mid x+y < 2\}$ YES: if $x+y < 2$, also $y+x < 2$

Antisymmetry

- A relation $R \subseteq A \times A$ is antisymmetric if $\forall a,b \in A\ ((a,b) \in R \wedge (b,a) \in R \rightarrow a=b)$ that is, the only way for both (a,b) and (b,a) to belong to R is if a=b
- EX: on the integers **Z**, which of the following relations are antisymmetric?
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x = y\}$ YES
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x \leq y\}$ YES: if $a \leq b$ and $b \leq a$, then $a=b$
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid |x| = |y|\}$ NO: counterex: $(1,-1) \in R \wedge (-1,1) \in R$, but $1 \neq -1$
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x < y\}$ YES: there are no x,y such that both $x < y$ and $y < x$, i.e. the premise is never satisfied, so antisymmetry is vacuously true
  - $R = \mathbf{Z} \times \mathbf{Z}$ NO: counterex: $(1,-1) \in R \wedge (-1,1) \in R$, but $1 \neq -1$
  - $R = \varnothing$ YES: vacuously
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x \text{ is a multiple of } y\}$ YES: if x is a multiple of y and y is a multiple of x, then $x = y$
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x+y < 2\}$ NO: counterex: $(1,-1) \in R \wedge (-1,1) \in R$, but $1 \neq -1$

Asymmetry

- A relation $R \subseteq A \times A$ is asymmetric if $\forall a,b \in A\ ((a,b) \in R \rightarrow (b,a) \notin R)$
- EX: on the integers **Z**, which of the following relations are asymmetric?
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x = y\}$ NO: it is symmetric
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x \leq y\}$ NO: reflexivity is incompatible with asymmetry
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid |x| = |y|\}$ NO: it is symmetric
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x < y\}$ YES: if $x < y$, then surely $\neg(y < x)$
  - $R = \mathbf{Z} \times \mathbf{Z}$ NO: reflexivity (or symmetry) is incompatible with asymmetry
  - $R = \varnothing$ YES, vacuously
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x \text{ is a multiple of } y\}$ NO: reflexivity is incompatible with asymmetry
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x+y < 2\}$ NO: it is symmetric

Asymmetric vs antisymmetric

- A relation $R \subseteq AxA$ is asymmetric iff it is both antisymmetric and irreflexive [homework: prove this]
- EX: on the integers **Z**,
  - R = $\{(x,y) \in \mathbf{Z} \mid x \leq y\}$ is antisymmetric but not irreflexive, hence not asymmetric
  - R = $\{(x,y) \in \mathbf{Z} \mid x < y\}$ is antisymmetric and also irreflexive, hence asymmetric

Transitivity

- A relation $R \subseteq A x A$ is transitive if $\forall a,b,c \in A\ ((a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R)$
- EX: on the integers **Z**, which of the following relations are transitive?
  - $R = \{(x,y) \in \mathbf{Z} x \mathbf{Z} \mid x = y\}$ YES: if x=y and y=z, then x=z
  - $R = \{(x,y) \in \mathbf{Z} x \mathbf{Z} \mid x \leq y\}$ YES: if $x \leq y$ and $y \leq z$, then $x \leq z$
  - $R = \{(x,y) \in \mathbf{Z} x \mathbf{Z} \mid |x| = |y|\}$ YES: if $|x| = |y|$ and $|y| = |z|$, then also $|x| = |z|$
  - $R = \{(x,y) \in \mathbf{Z} x \mathbf{Z} \mid x < y\}$ YES: if $x<y$ and $y<z$, then $x<z$
  - $R = \mathbf{Z} x \mathbf{Z}$ YES: R contanins any possible pair of inregers
  - $R = \varnothing$ YES, vacuously
  - $R = \{(x,y) \in \mathbf{Z} x \mathbf{Z} \mid x \text{ is a multiple of } y\}$ YES: if x is a multiple of y and y is a multiple of z, then x is a multiple of z
  - $R = \{(x,y) \in \mathbf{Z} x \mathbf{Z} \mid x+y < 2\}$ NO: counterex: $(2,-1) \in R \wedge (-1,0) \in R$, but (2,0) does not belong to R

3. COMBINING RELATIONS

Set-theoretic operations

- Relations are sets, therefore they can be combined using the set operations $\cap, \cup, ^{C}, \setminus$
- EX: on $S = \{0,1,2,3\}$, let $R_1 = \{(0,0),(1,1),(2,1),(2,2),(3,1),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3)\}$. Then
  - $R_1 \cup R_2 = \{(0,0),(1,1),(1,2),(1,3),(2,1),(2,2),(3,1),(3,3)\}$
  - $R_1 \cap R_2 = \{(1,1)\}$
  - $R_1^C = SxS \setminus R_1 = \{(0,1),(0,2),(0,3),(1,0),(1,2),(1,3),(2,0),(2,3),(3,0),(3,2)\}$
    note that the universe is the cartesian product of the set on which the relation is defined
  - $R_2 \setminus R_1 = \{(1,2),(1,3)\}$

Inverse relation

- The inverse of a relation $R \subseteq AxB$ is the relation $R^{-1} = \{(b,a) \in BxA \mid (a,b) \in R\} \subseteq BxA$
  - EX: Let $A = \{a,b,c\}$ and $B = \{0,1\}$ and let $R = \{(a,0),(b,1),(c,1)\} \subseteq AxB$ be a binary relation from A to B. Then $R^{-1} = \{(0,a),(1,b),(1,c)\} \subseteq BxA$
  - EX: Let $R = \{(0,1),(1,1),(1,2),(1,3)\}$ be a binary relation on $S = \{0,1,2,3\}$. Then $R^{-1} = \{(1,0),(1,1),(2,1),(3,1)\} \subseteq SxS$

Composition of relations

- The composition of a relation $R_2 \subseteq B \times C$ with a relation $R_1 \subseteq A \times B$ is the relation $R_2 \circ R_1 \subseteq A \times C$ defined as

  $$R_2 \circ R_1 = \{(a,c) \in A \times C \mid \exists b \in B\ ((a,b) \in R_1 \wedge (b,c) \in R_2)\}$$

- EX: Let $A = \{0,1,2\}$, $B = \{m,n,o,p\}$, $C = \{w,x,y\}$.
  Let $R_1 = \{(1,p),(2,m)\} \subseteq A \times B$, $R_2 = \{(m,x),(m,y),(o,w)\} \subseteq B \times C$.
  Then $R_2 \circ R_1 = \{(2,x),(2,y)\}$

Powers of relations

- A binary relation $R \subseteq S x S$ can be composed with itself
- EX: If S is the set of humans and $R = \{(x,y) \in SxS \mid x$ is a child of $y\}$, then
  $R^2 = R \circ R = \{(x,y) \in SxS \mid x$ is a grandchild of $y\}$,
  $R^3 = R \circ R \circ R = \{(x,y) \in SxS \mid x$ is a great-grandchild of $y\}$,
  and also $R^{-1} = \{(x,y) \in SxS \mid y$ is a child of $x\} = \{(x,y) \in SxS \mid x$ is a parent of $y\}$
  $R^{-2} = (R^{-1})^2 = R^{-1} \circ R^{-1} = (R^2)^{-1} = \{(x,y) \in SxS \mid x$ is a grandparent of $y\}$

4. REPRESENTING RELATIONS

Representation via matrices

- A relation between finite sets can be represented using a matrix of 0s and 1s:
- If $R \subseteq AxB$ with $A = \{a_1, \ldots a_n\}$ and $B = \{b_1, \ldots b_k\}$, then the matrix of R is the $n \times k$ matrix $M_R = [m_{ij}]$ with

  $m_{ij} = 1$ if $(a_i, b_j) \in R$

  $m_{ij} = 0$ if $(a_i, b_j) \notin R$

- Note that the matrix depends on the choice of an ordering of the elements of A and an ordering of the elements of B. Any ordering is acceptable, but when A = B we use the same ordering.

Example

- Let A = {a,b,c} and B = {‘vowel’, ‘consonant’}
- Let R = {(a,‘vowel’),(b,‘consonant’),(c,‘consonant’)}
- The matrix of R is

$$M_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

- The ordering A = {b,a,c}, B = {‘vowel’, ‘consonant’} would produce a different matrix:

$$M_R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example

- Let A = {a,b,c} and B = {0,1,2,3}
- Let R be the relation on AxB represented by the matrix

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Describe R with the roster method

Example

- Let A = {a,b,c} and B = {0,1,2,3}
- Let R be the relation on AxB represented by the matrix

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Describe R with the roster method

R = {(a,0),(a,2),(b,1),(b,2)}

Matrices and relation properties

- Remember that for binary relations $R \subseteq SxS$ over a set S we use the same ordering on the 2 copies of S
- A relation $R \subseteq SxS$ is reflexive iff all the elements on the main diagonal of M_R are 1
- A relation $R \subseteq SxS$ is irreflexive iff all the elements on the main diagonal of M_R are 0
- A relation $R \subseteq SxS$ is symmetric iff M_R is a symmetric matrix (i.e., $m_{ij} = m_{ji}$ for all indices i and j)
- A relation is antisymmetric iff, for any indices $i \neq j$, $(m_{ij} = 0 \vee m_{ji} = 0)$

Example

- Let S = {0,1,2,3} and $R \subseteq S \times S$ be defined by the matrix

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

- What properties of R can we deduce from M_R?

Example

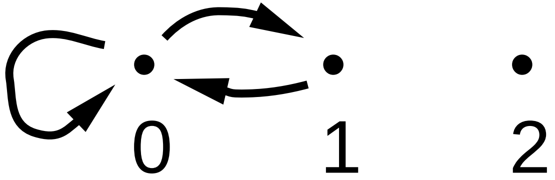
- Let S = {0,1,2,3} and R⊆SxS be defined by the matrix

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

- m_{33} = 0, so R is not reflexive
- m_{11} = 1, so R is not irreflexive
- m_{13} = 1 and m_{31} = 0, so R is not symmetric
- (m_{13} = 1 and m_{31} = 0), (m_{23} = 1 and m_{32} = 0), (m_{42} = 1 and m_{24} = 0), so R is antisymmetric (whenever we have a 1 off the main diagonal, in the symmetric position we have a 0)

Representation via graphs

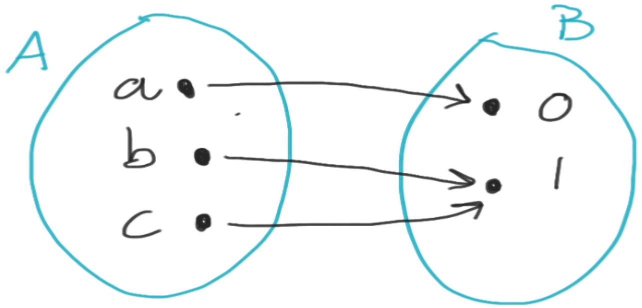
- A directed graph consists of a set V of vertices (aka nodes or points) and a set $E \subseteq V \times V$ of edges. If $(a,b) \in E$, then a is the initial vertex and b is the terminal vertex of the edge (a,b). An edge of the form (a,a) is a loop. Edges are drawn as arrows from their initial to their terminal vertex.
- EX: the graph G=(V,E) with V = {0,1,2} and E = {(0,0),(0,1),(1,0)} is



- Graphs will be studied in detail in next episodes

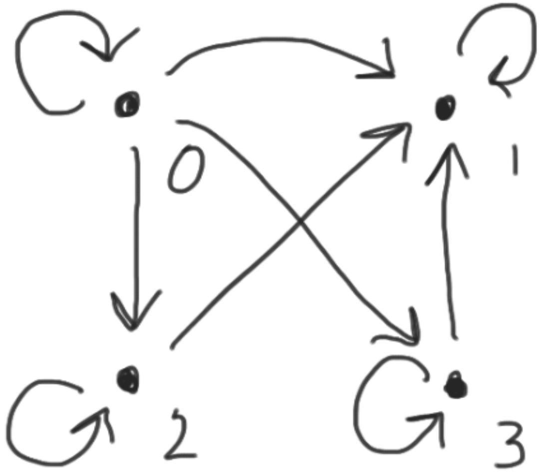
Representation via graphs

- A relation $R \subseteq A \times B$ can be represented as a graph with vertex set $V = A \cup B$ and edge set R. If $A \neq B$, then the elements of A are kept “separate” from the elements of B (usually, Venn diagrams for A and B are also included).
- EX: if $A = \{a,b,c\}$ and $B = \{0,1\}$, the relation $R = \{(a,0),(b,1),(c,1)\} \subseteq A \times B$ can be represented by the graph



Representation via graphs

- EX: if $A = \{0,1,2,3\}$, the relation $R = \{(a,b) \in A \times A \mid a \text{ is a multiple of } b\}$ can be represented by the graph

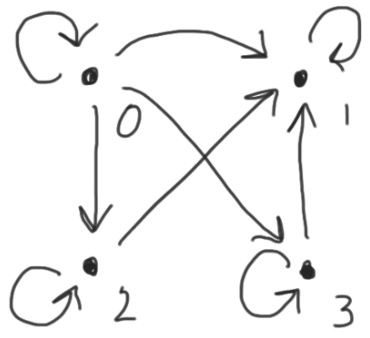


Graphs and relation properties

- A relation is reflexive iff all vertices have a loop
- A relation is irreflexive iff no vertex has a loop
- A relation is symmetric iff whenever (x,y) is an edge, then so is (y,x)
- A relation is antisymmetric iff whenever (x,y) is an edge with x≠y, then (y,x) is not an edge
- A relation is transitive iff whenever (x,y) and (y,z) are edges, then so is (x,z)

Previous example, revisited

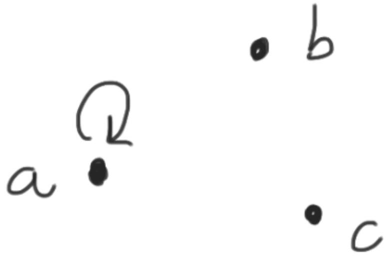
- EX: if A = {0,1,2,3}, the relation R = {(a,b)∈AxA | a is a multiple of b} can be represented by the graph



- Each vertex has a loop, so R is reflexive
- (0,1) is an edge, but (1,0) is not, so R is not symmetric
- Whenever (a,b) is an edge with a≠b, then (b,a) is not an edge (check (0,1) vs (1,0), (0,2) vs (2,0), (0,3) vs (3,0), (2,1) vs (1,2), (3,1) vs (1,3)), so R is antisymmetric
- Whenever (a,b) and (b,c) are edges, so is (a,c) (e.g., (0,2),(2,1) and (0,1)), so R is transitive

Another example (mind trivial cases!)

- EX: if A = {a,b,c} and R is the relation represented by the graph



- R is neither reflexive nor irreflexive (only a has a loop)
- R is vacuously symmetric (there are no edges (x,y) with x≠y)
- R is vacuously antisymmetric (same reason)
- R is vacuously transitive (same reason)

  Remember vacuous conditionals? (Conditionals with false premise are true)

5. EQUIVALENCE RELATIONS

Equivalence relations

- A relation $R \subseteq AxA$ (same set) is called an equivalence relation if it is reflexive, symmetric, and transitive.
- If R is an equivalence relation, two elements a and b such that aRb are called equivalent. In this case, the notation a~b is often used.
- EX: For any set A, the identity relation $I_A = \{(a,b) \in AxA \mid a=b\} = \{(a,a) \mid a \in A\}$ is an equivalence relation. In fact, it is
  - Reflexive, because any $a \in A$ is equal to itself (a=a)
  - Symmetric, because if a=b then b=a
  - Transitive, because if a=b and b=c, then a=c
- In fact, the identity is the archetypical equivalence relation: the definition of equivalence relation is modelled on the properties of the identity relation

Equivalence classes

- If $R \subseteq A x A$ is an equivalence relation, for any fixed $x \in A$, the subset $\{a \in A \mid a \sim x\} = \{a \in A \mid aRx\} = \{a \in A \mid (a,x) \in R\} \subseteq A$ of the elements in relation with x is called the equivalence class of x, and denoted $[x]_R$, or just $[x]$ if R is clear from the context.
- CAUTION! $[x] = [y]$ for any y such that $y \sim x$, thus in general $[x] = [y]$ does not imply $x = y$.
- When we write an equivalence class as $[x]$, we say that x is a representative of that class. Any element of a class can be used as representative.

EX: Congruence modulo m

- Let $m>1$ be an integer. Remember that, for two integers a and b, $a \equiv b \bmod m$ means that a and b have the same remainder in the integer division by m
- The relation $\{(a,b) \in \mathbf{Z} \times \mathbf{Z} \mid a \equiv b \bmod m\}$ is an equivalence relation on the integers

EX: Congruence modulo m

- Reflexivity: clearly for any $a \in Z$ ($a \equiv a \bmod m$)
- Symmetry: if $a \equiv b \bmod m$ (that is, a and b have the same remainder when divided by m), then $b \equiv a \bmod m$ (that is, b and a have the same remainder when divided by m)
- Transitivity: if $a \equiv b \bmod m$ (that is, a and b have the same remainder, say r, when divided by m), and $b \equiv c \bmod m$ (that is, b and c have the same remainder, which must be r again, when divided by m), then $a \equiv c \bmod m$ (that is, a and c have the same remainder, still r, when divided by m)

EX: Congruence modulo m

- The equivalence class of an integer a modulo m is $[a]_m = \{..., a-3m, a-2m, a-m, a, a+m, a+2m, a+3m, ...\}$
- The difference between consecutive elements in $[a]_m$ is m
- There are exactly m distinct equivalence classes modulo m: $[0]_m$, $[1]_m$,..., $[m-1]_m$
- Of course, other choices of representatives are possible

Concrete ex: Congruence modulo 3

- There are 3 equivalence classes modulo 3:
  - $[0]_3 = \{..., -9, -6, -3, 0, 3, 6, 9, ...\} = [3] = [6] = ...$
  - $[1]_3 = \{..., -8, -5, -2, 1, 4, 7, 10, ...\} = [-8] = [7] = ...$
  - $[2]_3 = \{..., -7, -4, -1, 2, 5, 8, 11, ...\} = [5] = [-7] = ...$
- Notice that the equivalence classes are mutually disjoint, nonempty, and their union is the whole Z. This is a general property of equivalence relations.

Equivalence relations and partitions

- A partition of a set S is a collection $\{A_j \mid j \in J\}$ (where J is a set of indices) of subsets of S which are
  - mutually disjoint (for all $j,k \in J$ with $j \neq k$, $A_j \cap A_k = \emptyset$),
  - nonempty (for all $k \in J$, $A_k \neq \emptyset$),
  - and whose union is S ($\bigcup_{j \in J} A_j = S$)
- If on a set S there is an equivalence relation, the equivalence classes form a partition of S.
- Viceversa, if a set S has a partition $\{A_j \mid j \in J\}$, then the relation $R = \{(x,y) \in SxS \mid x$ and y belong to the same A_k $(k \in J)\}$ is an equivalence relation with the A_j $(j \in J)$ as the equivalence classes.

6. PARTIAL ORDERINGS

Partial orderings

- A relation $R \subseteq A \times A$ (same set) is called a partial ordering, or partial order, if it is reflexive, antisymmetric, and transitive.
- A set S together with a partial ordering R is called a partially ordered set, or poset, and is denoted by (S,R).
- EX: On **Z**, the relation ≤ ("less than or equal to"), i.e. $\{(a,b) \in \mathbf{Z} \times \mathbf{Z} \mid a \leq b\}$ is a partial order. In fact, it is
  - Reflexive, because any $a \in \mathbf{Z}$ is less than or equal to itself ($a \leq a$)
  - Antisymmetric, because if $a \leq b$ and $b \leq a$, then $a = b$
  - Transitive, because if $a \leq b$ and $b \leq c$, then $a \leq c$
- Therefore, $(\mathbf{Z}, \leq)$ is a poset. The same reasoning works for the relation ≥
- In fact, ≤ (or ≥) is the archetypical partial order: the definition of partial order is modelled on the properties of ≤ (or ≥)

Strict orderings

- A relation R⊆AxA (same set) is called a strict (partial) ordering (or order) if it is asymmetric (or equivalently irreflexive and antisymmetric), and transitive.
- A set S together with a partial ordering R is called a strict partially ordered set, or strict poset, and is denoted by (S,R).
- EX: On **Z**, the relation < (“less than”), i.e. $\{(a,b)\in\mathbf{Z}\mathrm{x}\mathbf{Z} \mid a < b\}$ is a strict order. In fact, it is
  - Asymmetric, because if $a<b$, then $\neg(b<a)$
  - Transitive, because if $a<b$ and $b<c$, then $a<c$
- Therefore, (**Z**,<) is a strict poset. The same reasoning works for the relation >
- In fact, < (or >) is the archetypical strict order: the definition of strict order is modelled on the properties of < (or >)

Strict vs non-strict orderings

- Let A be a set. Recall the identity relation on A: $I_A = \{(a,b) \in AxA \mid a=b\}$

(1) Given a (non-strict) partial order $P \subseteq AxA$, there is an induced strict partial order $Q \subseteq AxA$, defined by $Q = P \setminus I_A = \{(a,b) \in P \mid \neg(a=b)\} = \{(a,b) \in P \mid a \neq b)\}$.

(2) Viceversa, given a strict order $R \subseteq AxA$, there is an induced partial order $T \subseteq AxA$, defined by $T = R \cup I_A = \{(a,b) \in AxA \mid (a,b) \in R \vee a=b\}$

- Homework: prove the above points. That is, for (1), show that Q is irreflexive and transitive; for (2), show that T is reflexive, antisymmetric and transitive.

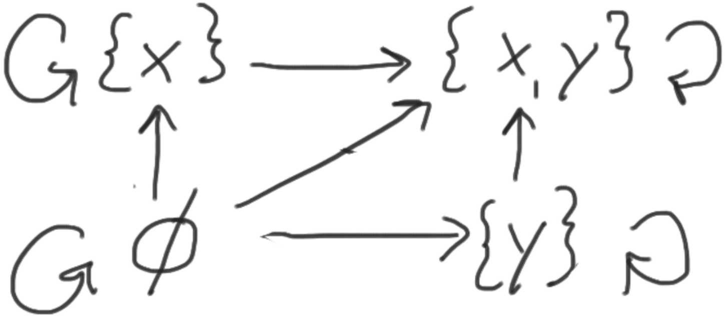
EX: The power set poset

- Let A be a set. The power set P(A) together with the inclusion relation $\subseteq$ is a poset. In fact, by the definition of set inclusion,
  - Reflexivity: every subset S of A is included in itself ($S \subseteq S$)
  - Antisymmetry: if 2 subsets S and T of A satisfy $S \subseteq T$ and $T \subseteq S$, then $S=T$ (this is our favourite technique to show a set equality)
  - Transitivity: if 3 subsets $B,C,D \in P(A)$ satisfy $B \subseteq C$ and $C \subseteq D$, then also $B \subseteq D$
- EX: show that P(A) with the proper inclusion relation $\subset$ is a strict poset

Concrete ex: The power set of {x,y}

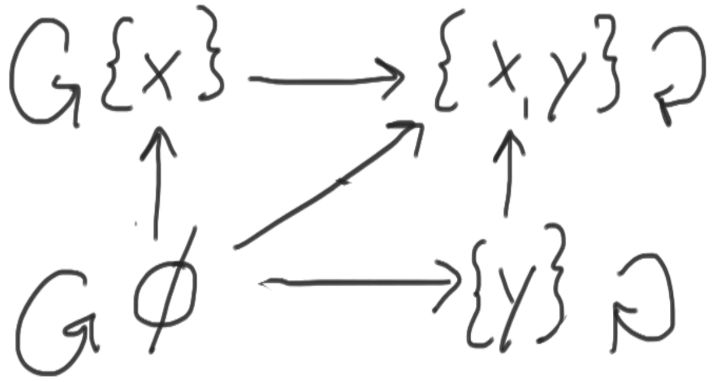
- Let A = {x,y}
- Then P(A) = {Ø, {x}, {y}, {x,y}}
- The inclusion relation is {(Ø,Ø),(Ø,{x}),(Ø,{y}),(Ø,{x,y}),({x},{x}),({x},{x,y}),({y},{y}),({y},{x,y}),({x,y},{x,y})}...correct but not the clearest. In matrix and graph representation:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



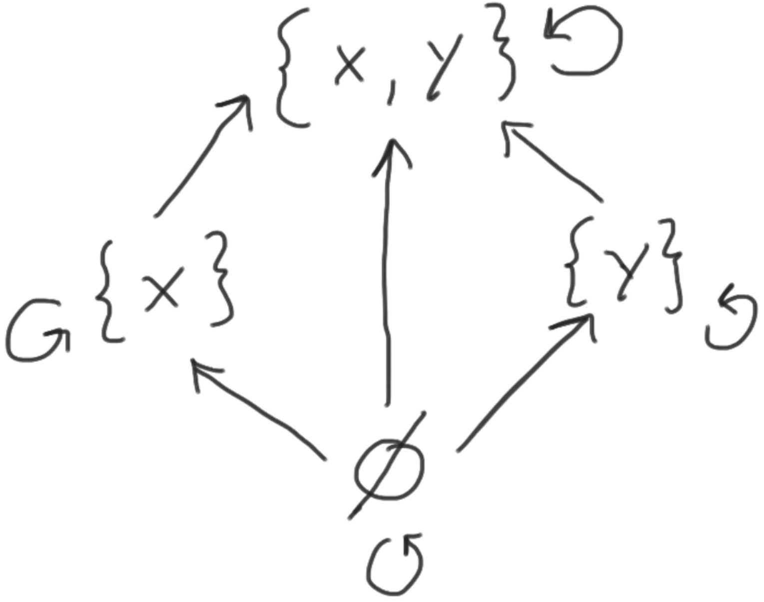
Hasse diagrams

- Hasse diagrams are another type of graph representation specific for partial orders. Suppose you have a partial order R on a set S (we will use ⊆ on P({x,y}) = {Ø, {x}, {y}, {x,y}} as an example).
- Start with a "normal" graph representation of R



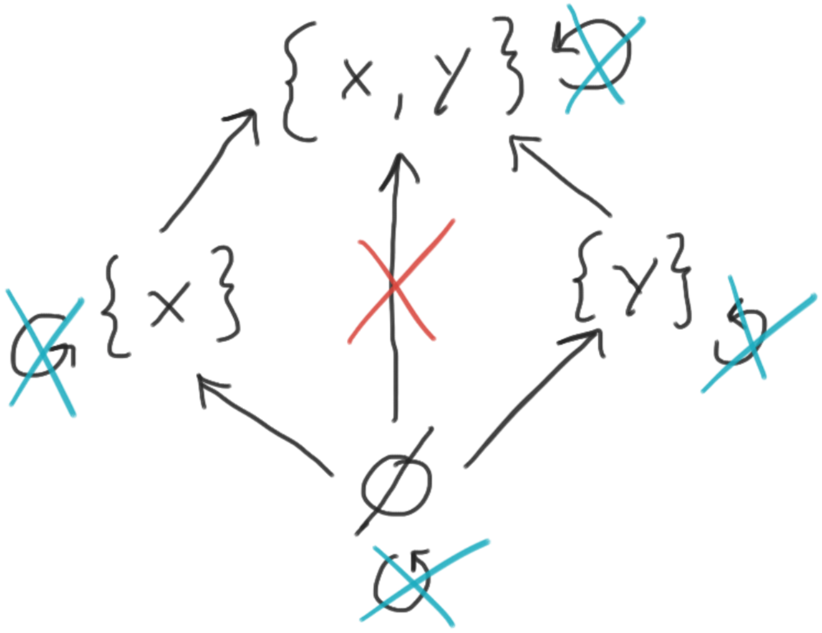
Hasse diagrams

- Rearrange the vertices in such a way that, if vertices a and b satisfy aRb, then b is higher up on the page than b (if a and b are not related, their relative height can be whatever). In our case, {x,y} has to be at the top, {x} and {y} below it (and their relative height does not matter) and Ø must be at the bottom



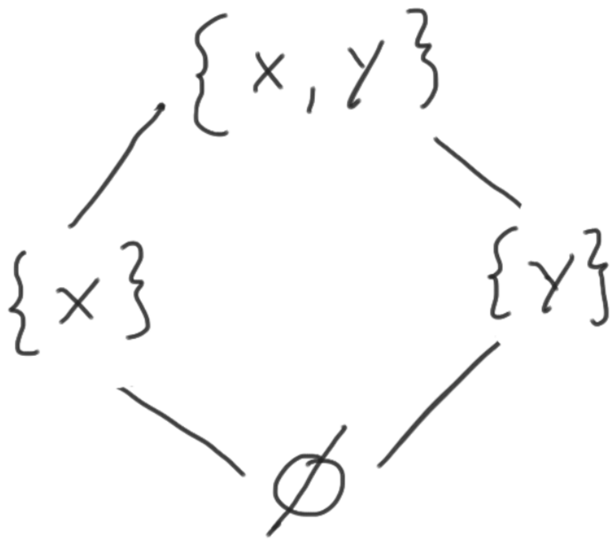
Hasse diagrams

- Remove the edges due to reflexivity (i.e., the loops) and those implied by transitivity. In our case, the edge (Ø,{x,y}) is implied by transitivity from the edges (Ø,{x}) and ({x},{x,y}), so we remove it



Hasse diagrams

- Remove the arrow tips (the direction of edges is implied by the relative height of the vertices)
- That’s your Hasse diagram

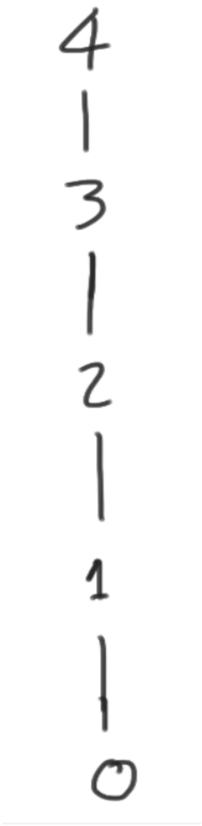


Total orders

- Let (A,R) be a poset. Two elements a,b of A are said to be comparable if aRb or bRa. The elements are called incomparable if neither aRb nor bRa.
- A poset (A,R) in which all elements are comparable is said to be a totally ordered set (other names: linearly ordered set, chain) and R is called a total (or linear) order.
- A totally ordered set such that every nonempty subset has a minimum is called a well-ordered set.
  - EX: (P({x,y}), ⊆) is not a totally ordered set because {x} and {y} are incomparable.
  - EX: (**Z**, ≤) is a totally ordered set, but not a well-ordered set (**Z** itself has no minimum).
  - EX: (**N**, ≤) is a well-ordered set.

Hasse diagrams of total orders

- EX: consider the poset A = {0,1,2,3,4} with the total order ≤.
- Its Hasse diagram is a "line".
- This is true for any totally ordered set.

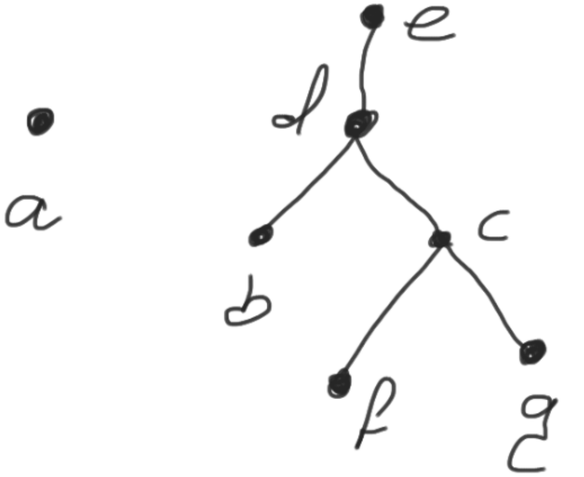


Maximal and minimal elements

- Remember that partial orders are modelled on the $\leq$ relation. In other words, if R is a partial order, we think of aRb as saying that a precedes (or is smaller than) b.
- An element of a poset (S,R) is called maximal if it is not smaller than any other element of S. In symbols, $m \in S$ is maximal iff $\forall x \in S$ (mRx $\rightarrow$ m=x).
- An element of a poset is called minimal if it is not greater than any other element of the poset. In symbols, $m \in S$ is minimal iff $\forall x \in S$ (xRm $\rightarrow$ m=x).
- Maximal and minimal elements need not be unique.
- Maximal and minimal elements are easy to spot using a Hasse diagram. They are the "top" and "bottom" elements in the diagram.

Example

In the following poset (represented via a Hasse diagram)



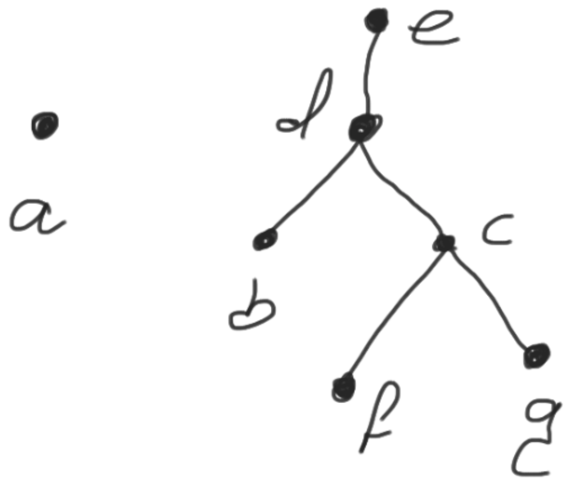
- The maximal elements are a and e
- The minimal elements are a,b,f,g
- (note that an isolated element, like a, is both maximal and minimal)

Maxima and minima

- Remember that partial orders are modelled on the $\leq$ relation. In other words, if R is a partial order, we think of aRb as saying that a precedes (or is smaller than) b.
- The maximum of a poset (S,R) is the element (if it exists) which is greater than any other element. In symbols, $m \in S$ is the maximum of S iff $\forall x \in S$ (xRm).
- The minimum of a poset (S,R) is the element (if it exists) which is smaller than any other element. In symbols, $m \in S$ is the minimum of S iff $\forall x \in S$ (mRx).
- Maximum and minimum need not exist, but if they do, they are unique.

Example

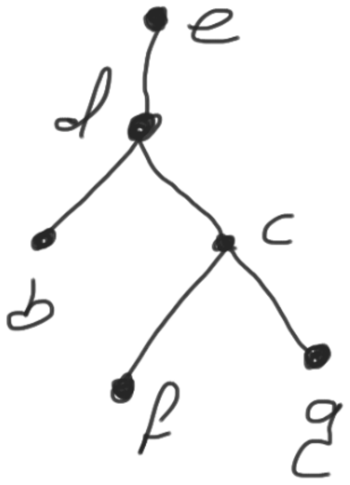
In the following poset (represented via a Hasse diagram)



- There is no maximum
- There is no minimum

Example

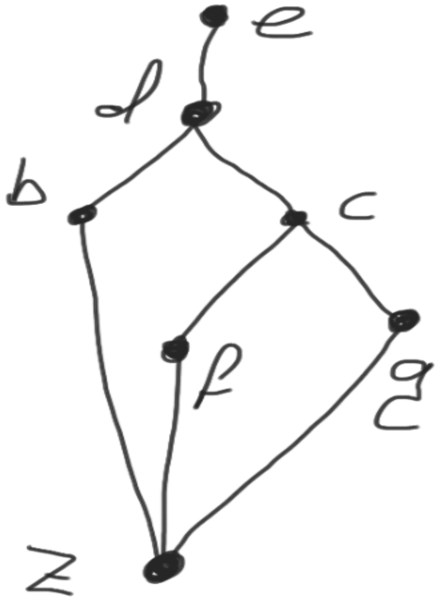
In the following poset (represented via a Hasse diagram)



- The maximum is e
- There is no minimum

Example

In the following poset (represented via a Hasse diagram)



- The maximum is e
- The minimum is z