Vectors in $\mathbb{R}^m$

Definition For any positive integer $m \geq 2$, we use $\mathbb{R}^m$, also called an *m-space*, to denote the set of all ordered m-tuples $\vec{v} = (v_1, v_2, \ldots, v_m)$, where each value v_i can be any real number (i.e. for all $v_i \in \mathbb{R}$).

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For any $\vec{v} \in \mathbb{R}^m$, we refer to $\vec{v}$ as a *vector*, or an *m-vector*.

The numbers $v_1,\ v_2, \ldots, v_m$ are called the *components* of the m-vector $\vec{v}$.

Example $\vec{u} = (1, 2, 3, 4)$ is a vector in $\mathbb{R}^4$
$\vec{v} = (\sqrt{2}, 0.11111, 0, 6, \frac{1}{7}, -9, 0)$ is a vector in $\mathbb{R}^7$.

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Two m-vectors are *equal* if and only if their corresponding components are identical. That is, for any $\vec{u}$, $\vec{v} \in \mathbb{R}^m$, $\vec{u} = \vec{v}$ if and only if $u_1 = v_1, u_2 = v_2, \ldots,$ and $u_m = v_m$.

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Example $\vec{0} = (0, 0, 0, 0)$ is a zero vector in $\mathbb{R}^4$.

$\vec{0} = (0, 0, 0, 0, 0)$ is a zero vector in $\mathbb{R}^5$.

$(a, b, c, 1, 8, \frac{1}{4})$ and $(-2, 3, 4, d, f, g) \in \mathbb{R}^6$ are equal if and only if

$$\begin{cases} a = -2 \\ b = 3 \\ c = 4 \\ d = 1 \\ f = 8 \\ g = \frac{1}{4}. \end{cases}$$

Definition The *distance* between $\vec{u} = (u_1, u_2, \ldots, u_m)$ and $\vec{v} = (v_1, v_2, \ldots, v_m) \in \mathbb{R}^m$ is defined by

$$d(\vec{u}, \vec{v}) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_m - v_m)^2}$$

Example Let $\vec{u} = (1, -2, 0, 3)$ and $\vec{v} = (2, 0, -1, 4) \in \mathbb{R}^4$, then

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Definition The *length (or norm, magnitude)* of $\vec{v} = (v_1, v_2, \dots, v_m) \in \mathbb{R}^m$ is defined by

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_m^2}$$

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Definition The vector $\vec{v}$ is a *unit vector* if and only if $\|\vec{v}\| = 1$.
Example Show that there are no real numbers a and b for which $\vec{v} = (1, -1, a, b)$ is a unit vector.

Definition Let c be a scalar and let $\vec{v}$ be a vector in $\mathbb{R}^m$. The *scalar multiple* $c\vec{v}$ of $\vec{v}$ by c is the vector

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If two **non-zero** m-vectors $\vec{u}$ and are collinear, so that $\vec{u} = c\vec{v}$, then they are said to have the *same* direction if $c > 0$ and are said to have *opposite* directions if $c < 0$.

Theorem For any m-vector $\vec{v}$ and any scalar c,

$$\|c\vec{v}\| = |c|\,\|\vec{v}\|$$

That is, the magnitude of the scalar multiple $c\vec{v}$ is the magnitude of $\vec{v}$ times the absolute value of the scalar multiplier.

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Examples 1. Find the unit vector which has the opposite direction with $\vec{v} = (-1, 1, 0, -2)$.

2. Find a, b and k such that $\vec{u} = (1, a, b, 5)$ and $\vec{v} = (-2, 1, 4, k)$ are collinear.

3. Find the magnitude of $(4, 8, -20, 12)$.

Definition Let $\vec{u}$ and $\vec{v}$ be two m-vectors.

Then the *sum* of $\vec{u}$ and $\vec{v}$ is the vector

$$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_m + v_m).$$

The *difference* of $\vec{u}$ and $\vec{v}$ is the vector

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The *dot product* of $\vec{u}$ and $\vec{v}$ is denoted by $\vec{u} \cdot \vec{v}$ and is defined by

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \ldots + u_m v_m.$$

Vectors $\vec{u}$ and $\vec{v}$ are *orthogonal* if and only if $\vec{u} \cdot \vec{v} = 0$.

Let $P(p_1, p_2, \ldots, p_m)$ and $Q(q_1, q_2, \ldots, q_m)$ be two distinct points (i.e., m-tuples) in $\mathbb{R}^m$ and let $\vec{p} = (p_1, p_2, \ldots, p_m)$ and $\vec{q} = (q_1, q_2, \ldots, q_m)$ be the vectors in $\mathbb{R}^m$ corresponding to them.

Lines in $\mathbb{R}^m$

Let $P(p_1, p_2, \dots, p_m)$ and $Q(q_1, q_2, \dots, q_m)$ be two distinct points (i.e., m-tuples) in $\mathbb{R}^m$ and let $\vec{p} = (p_1, p_2, \dots, p_m)$ and $\vec{q} = (q_1, q_2, \dots, q_m)$ be the vectors in $\mathbb{R}^m$ corresponding to them.

Then the line L in $\mathbb{R}^m$ passing through P and Q is defined by

Two-point form equation	$\vec{x}(t) = (1-t)\vec{p} + t\vec{q}$
Point-parallel form equation	$\vec{x}(t) = \vec{p} + t\vec{v}$ where $\vec{v}$ is a non-zero vector parallel to $\vec{q} - \vec{p}$
Parametric form equation	$\begin{aligned} x_1 &= p_1 + tv_1 \\ x_2 &= p_2 + tv_2 \\ &\vdots \\ x_m &= p_m + tv_m \end{aligned}$

1. Write equations of the line in $\mathbb{R}^4$ containing $P(1, 2, 3, 4)$ and $Q(2, 0, -1, 1)$ in the following forms: two-point form, point-parallel form, parametric equations.

2. Write parametric equations of the line through $P(1, 2, -2, 1, 3, 2)$ which is parallel to $\vec{v} = (-1, 1, 2, -1, 1, 3)$.

Hyperplanes

Recall that a point-normal form for a plane in $\mathbb{R}^3$ is

$$(n_1, n_2, n_3) \cdot (\vec{x} - (p_1, p_2, p_3)) = 0.$$

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Definition In $\mathbb{R}^m$, $\vec{n} \cdot (\vec{x} - \vec{p}) = 0$ is a point-normal form equation of the *hyperplane* through a point P with normal vector $\vec{n}$.
The standard form equation of this hyperplane is

$$n_1 x_1 + n_2 x_2 + \ldots + n_m x_m = c$$

where the coefficients n_i are the corresponding components of $\vec{n}$ and c is a constant whose value is given by $c = \vec{n} \cdot \vec{p}$.

1. Find the hyperplane in $\mathbb{R}^5$ through $P(1, 0, -2, -1, 3)$ with normal $(2, 1, -3, 4, 0)$ in a point-normal form and the standard form.

2. A hyperplane in $\mathbb{R}^6$ has a standard form

$$x_1 + 2x_2 - 4x_3 - x_4 + 6x_5 - 2x_6 = 2.$$

What is the normal of it? Find any point on this hyperplane.