$S \to aSb$
$S \to aSbb$
$S \to \#$

$L = \{ a^i \# b^j : 1 \le i \le j \le 2i \}$

$\Sigma = \{ a, b, \# \}$

(1) $L(G) \subseteq L$

SUPPOSE $x \in \Sigma^*$ $S \Rightarrow^* x$

PROVE $x \in L$ BY INDUCTION ON THE LENGTH OF THE DERIVATION.

BASE CASE: LENGTH IS ONE
MUST MEAN $S \Rightarrow x = \#$
WE HAVE $x \in L$

INDUCTIVE HYPOTHESIS: IF $S \Rightarrow x$ IN n STEPS THEN $x \in L$ $(S \Rightarrow^n x)$

ASSUME THAT $S \Rightarrow^{n+1} x$

(a) $S \Rightarrow aSb \Rightarrow^n x$
WE CAN WRITE $x = awb$
WHERE $S \Rightarrow^n w$
BY INDUCTIVE HYPOTHESIS $w \in L$
$\Rightarrow w = a^i \# b^j$

WHERE $0 \le i \le j \le 2i$

THUS $x = a(a^i \# b^j)b$
$= a^{i+1} \# b^{j+1}$

$0 \le i+1 \le j+1 \le 2i+1 \le 2i+2$
$= 2(i+1)$

$\Rightarrow x \in L$.

b $S \Rightarrow aSbb \overset{n}{\Rightarrow} x$

WE CAN WRITE $x = awbb$

WHERE $S \overset{n}{\Rightarrow} w$

INDUCTION: $w = a^i \# b^j$
$0 \le i \le j \le 2i$

$x = a^{i+1} \# b^{j+2}$

$0 \le i+1 \le j+2 \le 2i+2$
$= 2(i+1)$

$\Rightarrow x \in L$

THEREFORE $L(G) \subseteq L$ (BY IND)

② $L \subseteq L(G)$:

$x \in L \quad x = a^i \# b^j \quad 0 \le i \le j \le 2i$

$$x = a^i \# b^{i+k} \qquad 0 \le k \le i$$

$$S \Rightarrow^{k} a^k S b^{2k}$$

$$\Rightarrow^{i-k} a^k a^{i-k} S b^{i-k} b^{2k}$$

$$\Rightarrow a^k a^{i-k} \# b^{i-k} b^{2k}$$

$$= a^i \# b^{i+k} = x$$

$x \in L(G)$. $\quad L \subseteq L(G)$

AND WE ARE DONE.