§4.3. More on relation

Notation $R \subseteq A \times B$ write xRy to mean $(x,y) \in R$.

Def: If $R \subseteq A \times A$. then R is a relation on A.

Ex 4.3.1:

1. $A = P(\{1,2\})$

   $R = \{(B,C) \in A \times A \mid B \subseteq C\}$.

2. A any set:

   $i_A = \{(x,y) \in A \times A \mid x = y\}$.

   $= \{(x,x) \in A \times A\}$.

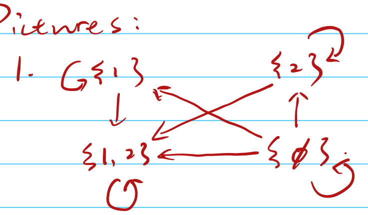
3. Fix $r = \mathbb{R}^+$

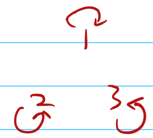
   $D_r = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid |x-y| < r\}$.

   e.g. $r = 1$.

   then $D_r = \{(x,y) \mid |x-y| < 1\}$.

Pictures:

1. 

2. $A = \{1, 2, 3\}$. 

Def: Let R be a relation on a set A.

1. R is reflexive (on A) iff $\forall x \in A\, (xRx)$
2. R is symmetric iff $\forall x, y \in A\, (xRy \to yRx)$
3. R is transitive iff $\forall x, y, z \in A\, (xRy \land yRz \to xRz)$

Ex 1 is reflexive, not symmetric, transitive.

2 is reflexive, symmetric, and transitive

3 is reflexive, symmetric, not transitive

$(x - x = 0 < 1)$ $|x-y| = |y-x| < 1$ e.g. $x = 2$ $y = 3/2$ $z = 1$

$x - y = 1/2$ ✓

$y - z = 1/2$ ✓

$x - z = 1$ ✗

4. $<$ relation in $\mathbb{R}$ $(\{(x,y) \mid x < y\})$.

   it is not reflexive, not symmetric, but transitive

Therom 4.3.4

$R \subseteq A \times A$.

1. R is reflexive iff $i_A \subseteq R$.
2. R is symmetric iff $R = R^{-1}$
3. R is transitive iff $R \circ R \subseteq R$

Proof of 3:

$\rightarrow$: Assume R is transitive. Let $(x,z) \in R \circ R$. Then there is $y \in A$ that xRy and yRz. Since R is transitive, xRz. That is, $(x,z) \in R$. So $R \circ R \subseteq R$.

$\leftarrow$: Assume $R \circ R \subseteq R$. Let $x, y, z \in A$, xRy, yRz. $(x,z) \in R \circ R$. Since $R \circ R \subseteq R$, $(x,z) \in R$, that is xRz. So R is transitive

§ 4.4. Ordinary Relation.

Def 4.4.1. $R \subseteq A \times A$.

R is antisymmetric iff $\forall x, y \in A\, (xRy \wedge yRx \rightarrow x = y)$

This is NOT not-symmetric.