The second derivative test of functions of two variables.

Suppose the 2nd particial derivative of f are continuous on a disk where centered (a,b) and suppose that $f_x(a,b)=0$, $f_y(a,b)=0$. i.e. (a,b) is a critical point for f. Let.

$$D = \det \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2.$$

If $D>0$ and $f_x(a,b)>0$, then $f(a,b)$ is a local minimum

$D>0$ and $f_x(a,b)<0$, then $f(a,b)$ is a local maximum.

$D<0$ then f has (a,b) as a saddle point

$D=0$ then (a,b) need further investigation.

the test is inconclusive.

e.g.1 Find the local minimum/maximum/saddle point of

$$f = x^2 + xy + y^2 + y.$$

Sol: $f_x = 2x + y = 0$, $f_y = x + 2y + 1 = 0$ $\Rightarrow$ $x = 1/3$, $y = -2/3$

$f_{xx} = 2$ $f_{xy} = 1$

$f_{yx} = 1$ $f_{yy} = 2$.

$$D = \det \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 > 0$$ $\therefore$ the local minimum at $(1/3, -2/3)$.

and the value is $-1/3$.

We note that $f(x,y)$ can be written as $(x+\frac{y}{2})^2 + \frac{3}{4}(y+\frac{2}{3})^2 - \frac{1}{3}$.

The Hessian.

$[\, f_{xx}(x,y) \quad f_{xy}(x,y) \,]$

$$H(x,y) = \begin{bmatrix} \cdots & \cdots \\ f_{yx}(x,y) & f_{yy}(x,y) \end{bmatrix}.$$ is called the hessian of f.

at the point (x,y). We also define the trace of H. denoted as $Tr(H)$ as

$$Tr(H) = f_{xx}(x,y) + f_{yy}(x,y).$$

$H(x,y)$ is a symmetric matrix,

$$\det H(x,y) = f_{xx} f_{yy} - (f_{xy})^2 . = \underline{\underline{D}}$$

官方表明.

The second derivaties test in terms of Hessian

Suppose (a,b) is a critical point of f. and f is continuous on (a,b). Then:

a) f has a local min value at point (a,b) if $Tr(H(a,b)) > 0$ and $\det H(a,b) > 0$

b) max $Tr(H(a,b)) < 0$ $\det H(a,b) > 0$.

c) saddle point if $\det H(a,b) < 0$

d) inconclusive. $= 0$

Recall: Eigenvalues and Eigenvectors.

A non-zero vector $\vec{v}$ is called an Eigenvector of a matrix A corresponding an Eigenvalue λ if $A\vec{v} = \lambda \vec{v}$

If all Eigen values of A are positive, A is called positive definite

negative negative

some positive some negative indefinited.

$A\vec{v} = \lambda \vec{v}$

$A\vec{v} = \lambda I \vec{v}$ where I is identity vector. $(I\vec{v} = \vec{v})$.

$$A\vec{v} - \lambda I \vec{v} = 0.$$

$$\underbrace{(A - \lambda I)}_{?}(\vec{v}) = 0.$$

$\vec{v}$ → non-zero vector.

$\det|A - \lambda I| = 0 \Leftarrow$ characterstic equation

Consider A as a 2x2 matrix, let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} h \\ k \end{bmatrix}.$$

$$A - \lambda I = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}.$$

$$\det(A - \lambda I) = 0$$

$$(a-\lambda)(d-\lambda) - bc = 0$$

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

$$\left\{ \begin{array}{l} Tr(A) = a+d \\ \det(A) = ad - bc. \end{array} \right.$$

$$\lambda^2 - Tr(A)\lambda + \det A = 0.$$

$$\lambda_1 + \lambda_2 = Tr(A)$$

$$\lambda_1 \lambda_2 = \det A.$$

$\Rightarrow$ A is positive definite if $\det A > 0 \wedge Tr(A) > 0$

negative $> 0 \wedge$ < 0

indefinite if $\det A < 0$.