

# Matrix multiplication

# Matrix

An  $m \times n$  matrix is a rectangular array of numbers

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

We use capital letters, such as  $A$ ,  $B$  and  $C$ , to denote matrices.

The entries of a matrix are denoted by the corresponding lowercase letter with “double subscript”, such as  $a_{ij}$ . And  $a_{ij}$  is called the  $(i, j)$ -entry.

# Matrix

An  $m \times n$  matrix is a rectangular array of numbers

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

We use capital letters, such as  $A$ ,  $B$  and  $C$ , to denote matrices.

The entries of a matrix are denoted by the corresponding lowercase letter with “double subscript”, such as  $a_{ij}$ . And  $a_{ij}$  is called the  $(i, j)$ -entry.

In this way, to describe a matrix, is to know what the entries  $a_{ij}$  are.

## A special case

**Definition** If  $A = [a_{1j}]$  is a  $1 \times n$  matrix (aka, a row vector) and  $B = [b_{i1}]$  is an  $n \times 1$  matrix (aka, a column vector), the *matrix product*  $AB$  is a  $1 \times 1$  matrix whose entry is given by

$$a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1n}b_{n1}.$$

That is to say,

$$AB = [a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1n}b_{n1}].$$

## Example

Find the following product matrices.

(a)  $AB$ , where  $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$  and  $B^T = \begin{bmatrix} 3 & 4 \end{bmatrix}$ .

(b)  $BA^T$ , where  $A = \begin{bmatrix} 1 & 0 & 3 & 2 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & -3 & 2 & 5 \end{bmatrix}$ .

(c)  $C^T C$ , where  $C^T = \begin{bmatrix} 2 & 1 & -3 \end{bmatrix}$ .

# Matrix multiplication

Let  $A$  be an  $m \times n$  matrix and let  $B$  be an  $n \times s$  matrix (i.e., the number of columns of  $A$  equals to the number of the rows of  $B$ ).

Then the *product*  $C = AB$  is an  $m \times s$  matrix, defined by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$$

for  $1 \leq i \leq m$  and  $1 \leq j \leq s$ .

# Matrix multiplication

Let  $A$  be an  $m \times n$  matrix and let  $B$  be an  $n \times s$  matrix (i.e., the number of columns of  $A$  equals to the number of the rows of  $B$ ).

Then the *product*  $C = AB$  is an  $m \times s$  matrix, defined by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$$

for  $1 \leq i \leq m$  and  $1 \leq j \leq s$ .

Alternatively, the rows of the matrix  $A$  and the columns of  $B$  give vectors in  $\mathbb{R}^n$ .

Let  $\vec{a}_i = (a_{i1}, a_{i2}, \dots, a_{in})$  (the  $i$ -th row of  $A$ ) and  $\vec{b}^j = (b_{1j}, b_{2j}, \dots, b_{nj})$  (the  $j$ -th column of  $B$ ) be two vectors in  $\mathbb{R}^n$ . Then  $c_{ij} = \vec{a}_i \cdot \vec{b}^j$ .

Lecture Note Example 7.6.

Consider the matrices shown here:

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & -2 & 5 \\ 0 & 3 & 4 & -1 \\ 5 & -1 & 2 & 4 \end{bmatrix}$$

How many different matrix products of the form  $M_1 M_2$  are defined, where each of  $M_1$  and  $M_2$  is either one of the given matrices or the transpose of one of the given matrices?



# Examples

Given the pairs of matrices  $A$  and  $B$ , find the product  $AB$  and  $BA$ , if defined.

(1)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -1 & -2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$$

(2)

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 3 & 6 & 5 \\ 2 & -4 & 7 \end{bmatrix} \quad B = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(3) Find  $CC^T$ , where  $C^T = [2 \quad 3 \quad -1 \quad 4]$ .

(4) Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

What is  $AI_n$ ? How about  $I_m A$ ?

(5) Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

Find  $AB$  and  $BA$ . Show that  $AB \neq BA$ .

**Definition** Let  $A$  be a square matrix of order  $n$  (i.e., an  $n \times n$  matrix).  
Then

$$A^1 = A$$

$$A^2 = AA$$

$$A^3 = AA^2$$

$$\vdots$$

$$A^k = AA^{k-1} \quad (k \geq 2)$$

# Examples

(1) Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$$

Find matrices  $A^1$ ,  $A^2$ ,  $A^3$  and  $A^4$ .

(2) Consider the following matrices (the first two are diagonal matrices)

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Find  $I^k$ ,  $A^k$  and  $B^k$  for  $k \geq 1$ .

## **Theorem** Properties of Operations for Matrices

Let  $A$ ,  $B$  and  $C$  be matrices. Let  $a$  and  $b$  be scalars. Assume that the dimensions of the matrices are such that each operation is defined.

1.  $A + B = B + A$  (matrix addition is commutative)
2.  $A + (B + C) = (A + B) + C$  (matrix addition is also associative)

## **Theorem** Properties of Operations for Matrices

Let  $A$ ,  $B$  and  $C$  be matrices. Let  $a$  and  $b$  be scalars. Assume that the dimensions of the matrices are such that each operation is defined.

1.  $A + B = B + A$  (matrix addition is commutative)
2.  $A + (B + C) = (A + B) + C$  (matrix addition is also associative)
3.  $A + 0 = A$ , where  $0$  denotes the zero matrix with the same dimensions as  $A$ .
4.  $A + (-A) = 0$ , where again  $0$  is the zero matrix with the same dimensions as  $A$ .

## **Theorem** Properties of Operations for Matrices

Let  $A$ ,  $B$  and  $C$  be matrices. Let  $a$  and  $b$  be scalars. Assume that the dimensions of the matrices are such that each operation is defined.

1.  $A + B = B + A$  (matrix addition is commutative)
2.  $A + (B + C) = (A + B) + C$  (matrix addition is also associative)
3.  $A + 0 = A$ , where  $0$  denotes the zero matrix with the same dimensions as  $A$ .
4.  $A + (-A) = 0$ , where again  $0$  is the zero matrix with the same dimensions as  $A$ .
5.  $A(BC) = (AB)C$  (matrix multiplication is associative)
6.  $AI_n = A$  and  $I_mA = A$ , where  $A$  is an  $m \times n$  matrix

7.  $A(B + C) = AB + AC$  and  $(B + C)A = BA + CA$   
(matrix multiplication is distributive over matrix addition)

8.  $a(B + C) = aB + aC$   
(scalar multiplication of matrices is distributive over matrix addition)

9.  $(a + b)C = aC + bC$   
(scalar multiplication of matrices is also distributive over scalar addition)



7.  $A(B + C) = AB + AC$  and  $(B + C)A = BA + CA$   
(matrix multiplication is distributive over matrix addition)

8.  $a(B + C) = aB + aC$   
(scalar multiplication of matrices is distributive over matrix addition)

9.  $(a + b)C = aC + bC$   
(scalar multiplication of matrices is also distributive over scalar addition)

10.  $(ab)C = a(bC)$

11.  $1A = A$  Note: 1 is the scalar 1.

12.  $A0 = 0$  and  $0A = 0$ , where 0 denotes a zero matrix and the two zero matrices have appropriate dimensions.

13.  $a0 = 0$

14.  $a(AB) = (aA)B = A(aB)$

15.  $(A + B)^T = A^T + B^T$

(matrix transposition is distributive over matrix addition)

16.  $(AB)^T = B^T A^T$

(matrix transposition is distributive over matrix multiplication, but the order of multiplication is reversed)

Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 3 & 2 \\ 0 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 1 & 0 \\ 0 & 3 & 3 \end{bmatrix}$$

Find  $A(BC)$ ,  $(AB)C$ ,  $A(B + C)$  and  $AB + AC$ .