where $b$ is a constant that is independent of both $L$ and $K$. Assumption (i) shows that $\alpha > 0$ and $\beta > 0$.

Notice from Equation 9 that if labor and capital are both increased by a factor $m$, then

$$P(mL, mK) = b(mL)^{\alpha}(mK)^{\beta} = m^{\alpha+\beta}bL^{\alpha}K^{\beta} = m^{\alpha+\beta}P(L, K)$$

If $\alpha + \beta = 1$, then $P(mL, mK) = mP(L, K)$, which means that production is also increased by a factor of $m$. That is why Cobb and Douglas assumed that $\alpha + \beta = 1$ and therefore

$$P(L, K) = bL^{\alpha}K^{1-\alpha}$$

This is the Cobb-Douglas production function that we discussed in Section 14.1.

## 14.3 EXERCISES

**1.** The temperature $T$ (in °C) at a location in the Northern Hemisphere depends on the longitude $x$, latitude $y$, and time $t$, so we can write $T = f(x, y, t)$. Let's measure time in hours from the beginning of January.

(a) What are the meanings of the partial derivatives $\partial T/\partial x$, $\partial T/\partial y$, and $\partial T/\partial t$?

(b) Honolulu has longitude 158°W and latitude 21°N. Suppose that at 9:00 AM on January 1 the wind is blowing hot air to the northeast, so the air to the west and south is warm and the air to the north and east is cooler. Would you expect $f_x(158, 21, 9)$, $f_y(158, 21, 9)$, and $f_t(158, 21, 9)$ to be positive or negative? Explain.

**2.** At the beginning of this section we discussed the function $I = f(T, H)$, where $I$ is the heat index, $T$ is the temperature, and $H$ is the relative humidity. Use Table 1 to estimate $f_T(92, 60)$ and $f_H(92, 60)$. What are the practical interpretations of these values?

**3.** The wind-chill index $W$ is the perceived temperature when the actual temperature is $T$ and the wind speed is $v$, so we can write $W = f(T, v)$. The following table of values is an excerpt from Table 1 in Section 14.1.

Wind speed (km/h); rows: Actual temperature (°C)

| $T$ \ $v$ | 20 | 30 | 40 | 50 | 60 | 70 |
|---|---|---|---|---|---|---|
| −10 | −18 | −20 | −21 | −22 | −23 | −23 |
| −15 | −24 | −26 | −27 | −29 | −30 | −30 |
| −20 | −30 | −33 | −34 | −35 | −36 | −37 |
| −25 | −37 | −39 | −41 | −42 | −43 | −44 |

(a) Estimate the values of $f_T(-15, 30)$ and $f_v(-15, 30)$. What are the practical interpretations of these values?

(b) In general, what can you say about the signs of $\partial W/\partial T$ and $\partial W/\partial v$?

(c) What appears to be the value of the following limit?

$$\lim_{v \to \infty} \frac{\partial W}{\partial v}$$

**4.** The wave heights $h$ in the open sea depend on the speed $v$ of the wind and the length of time $t$ that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in the following table.

Duration (hours); rows: Wind speed (knots)

| $v$ \ $t$ | 5 | 10 | 15 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|---|---|
| 10 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 15 | 4 | 4 | 5 | 5 | 5 | 5 | 5 |
| 20 | 5 | 7 | 8 | 8 | 9 | 9 | 9 |
| 30 | 9 | 13 | 16 | 17 | 18 | 19 | 19 |
| 40 | 14 | 21 | 25 | 28 | 31 | 33 | 33 |
| 50 | 19 | 29 | 36 | 40 | 45 | 48 | 50 |
| 60 | 24 | 37 | 47 | 54 | 62 | 67 | 69 |

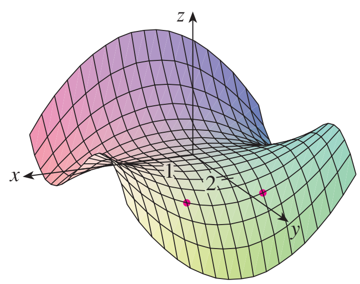
(a) What are the meanings of the partial derivatives $\partial h/\partial v$ and $\partial h/\partial t$?

(b) Estimate the values of $f_v(40, 15)$ and $f_t(40, 15)$. What are the practical interpretations of these values?

(c) What appears to be the value of the following limit?

$$\lim_{t \to \infty} \frac{\partial h}{\partial t}$$

**5–8** Determine the signs of the partial derivatives for the function $f$ whose graph is shown.



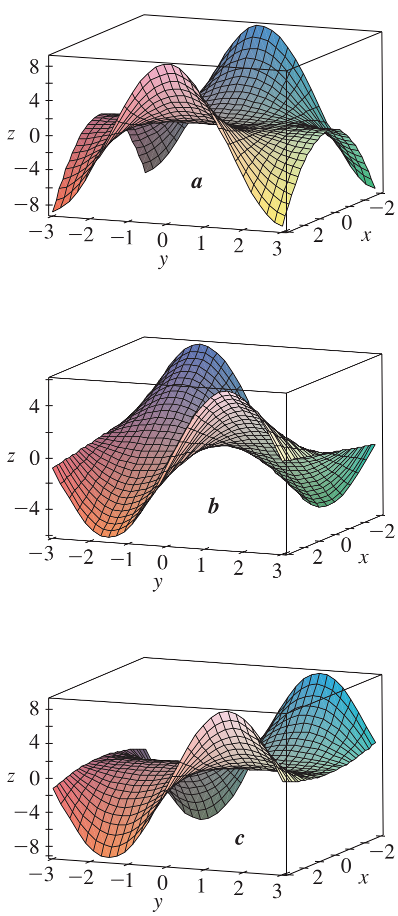

**5.** (a) $f_x(1, 2)$ (b) $f_y(1, 2)$

**6.** (a) $f_x(-1, 2)$ (b) $f_y(-1, 2)$

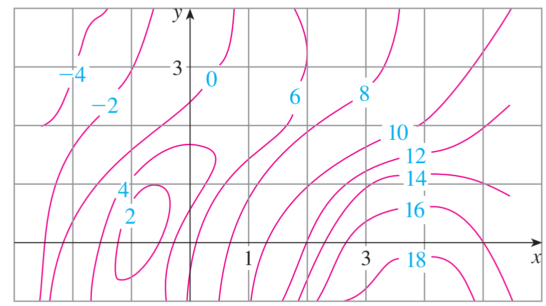
**7.** (a) $f_{xx}(-1, 2)$ (b) $f_{yy}(-1, 2)$

**8.** (a) $f_{xy}(1, 2)$ (b) $f_{xy}(-1, 2)$

**9.** The following surfaces, labeled $a$, $b$, and $c$, are graphs of a function $f$ and its partial derivatives $f_x$ and $f_y$. Identify each surface and give reasons for your choices.




**10.** A contour map is given for a function $f$. Use it to estimate $f_x(2, 1)$ and $f_y(2, 1)$.




**11.** If $f(x, y) = 16 - 4x^2 - y^2$, find $f_x(1, 2)$ and $f_y(1, 2)$ and interpret these numbers as slopes. Illustrate with either hand-drawn sketches or computer plots.

**12.** If $f(x, y) = \sqrt{4 - x^2 - 4y^2}$, find $f_x(1, 0)$ and $f_y(1, 0)$ and interpret these numbers as slopes. Illustrate with either hand-drawn sketches or computer plots.

**13–14** Find $f_x$ and $f_y$ and graph $f$, $f_x$, and $f_y$ with domains and viewpoints that enable you to see the relationships between them.

**13.** $f(x, y) = x^2y^3$

**14.** $f(x, y) = \dfrac{y}{1 + x^2y^2}$

**15–40** Find the first partial derivatives of the function.

**15.** $f(x, y) = x^4 + 5xy^3$

**16.** $f(x, y) = x^2y - 3y^4$

**17.** $f(x, t) = t^2e^{-x}$

**18.** $f(x, t) = \sqrt{3x + 4t}$

**19.** $z = \ln(x + t^2)$

**20.** $z = x \sin(xy)$

**21.** $f(x, y) = \dfrac{x}{y}$

**22.** $f(x, y) = \dfrac{x}{(x + y)^2}$

**23.** $f(x, y) = \dfrac{ax + by}{cx + dy}$

**24.** $w = \dfrac{e^v}{u + v^2}$

**25.** $g(u, v) = (u^2v - v^3)^5$

**26.** $u(r, \theta) = \sin(r \cos\theta)$

**27.** $R(p, q) = \tan^{-1}(pq^2)$

**28.** $f(x, y) = x^y$

**29.** $F(x, y) = \displaystyle\int_y^x \cos(e^t)\, dt$

**30.** $F(\alpha, \beta) = \displaystyle\int_\alpha^\beta \sqrt{t^3 + 1}\, dt$

**31.** $f(x, y, z) = x^3yz^2 + 2yz$

**32.** $f(x, y, z) = xy^2e^{-xz}$

**33.** $w = \ln(x + 2y + 3z)$

**34.** $w = y \tan(x + 2z)$

**35.** $p = \sqrt{t^4 + u^2 \cos v}$

**36.** $u = x^{y/z}$

**37.** $h(x, y, z, t) = x^2y \cos(z/t)$

**38.** $\phi(x, y, z, t) = \dfrac{\alpha x + \beta y^2}{\gamma z + \delta t^2}$

**39.** $u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

**40.** $u = \sin(x_1 + 2x_2 + \cdots + nx_n)$

**41–44** Find the indicated partial derivative.

**41.** $R(s, t) = te^{s/t}$; $R_t(0, 1)$

**42.** $f(x, y) = y \sin^{-1}(xy)$; $f_y\left(1, \frac{1}{2}\right)$

**43.** $f(x, y, z) = \ln \dfrac{1 - \sqrt{x^2 + y^2 + z^2}}{1 + \sqrt{x^2 + y^2 + z^2}}$; $f_y(1, 2, 2)$

**44.** $f(x, y, z) = x^{yz}$; $f_z(e, 1, 0)$

**45–46** Use the definition of partial derivatives as limits (4) to find $f_x(x, y)$ and $f_y(x, y)$.

**45.** $f(x, y) = xy^2 - x^3y$

**46.** $f(x, y) = \dfrac{x}{x + y^2}$

**47–50** Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$.

**47.** $x^2 + 2y^2 + 3z^2 = 1$

**48.** $x^2 - y^2 + z^2 - 2z = 4$

**49.** $e^z = xyz$

**50.** $yz + x \ln y = z^2$

**51–52** Find $\partial z/\partial x$ and $\partial z/\partial y$.

**51.** (a) $z = f(x) + g(y)$ (b) $z = f(x + y)$

**52.** (a) $z = f(x)g(y)$ (b) $z = f(xy)$
(c) $z = f(x/y)$

**53–58** Find all the second partial derivatives.

**53.** $f(x, y) = x^4y - 2x^3y^2$

**54.** $f(x, y) = \ln(ax + by)$

**55.** $z = \dfrac{y}{2x + 3y}$

**56.** $T = e^{-2r} \cos \theta$

**57.** $v = \sin(s^2 - t^2)$

**58.** $w = \sqrt{1 + uv^2}$

**59–62** Verify that the conclusion of Clairaut's Theorem holds, that is, $u_{xy} = u_{yx}$.

**59.** $u = x^4y^3 - y^4$

**60.** $u = e^{xy} \sin y$

**61.** $u = \cos(x^2y)$

**62.** $u = \ln(x + 2y)$

**63–70** Find the indicated partial derivative(s).

**63.** $f(x, y) = x^4y^2 - x^3y$; $f_{xxx}$, $f_{xyx}$

**64.** $f(x, y) = \sin(2x + 5y)$; $f_{yxy}$

**65.** $f(x, y, z) = e^{xyz^2}$; $f_{xyz}$

**66.** $g(r, s, t) = e^r \sin(st)$; $g_{rst}$

**67.** $W = \sqrt{u + v^2}$; $\dfrac{\partial^3 W}{\partial u^2\, \partial v}$

**68.** $V = \ln(r + s^2 + t^3)$; $\dfrac{\partial^3 V}{\partial r\, \partial s\, \partial t}$

**69.** $w = \dfrac{x}{y + 2z}$; $\dfrac{\partial^3 w}{\partial z\, \partial y\, \partial x}$, $\dfrac{\partial^3 w}{\partial x^2\, \partial y}$

**70.** $u = x^a y^b z^c$; $\dfrac{\partial^6 u}{\partial x\, \partial y^2\, \partial z^3}$

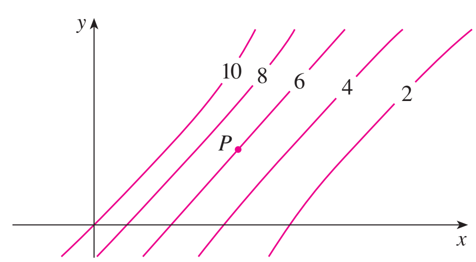
**71.** If $f(x, y, z) = xy^2z^3 + \arcsin\left(x\sqrt{z}\right)$, find $f_{xzy}$. [*Hint:* Which order of differentiation is easiest?]

**72.** If $g(x, y, z) = \sqrt{1 + xz} + \sqrt{1 - xy}$, find $g_{xyz}$. [*Hint:* Use a different order of differentiation for each term.]

**73.** Use the table of values of $f(x, y)$ to estimate the values of $f_x(3, 2)$, $f_x(3, 2.2)$, and $f_{xy}(3, 2)$.

| $x$ \ $y$ | 1.8 | 2.0 | 2.2 |
|---|---|---|---|
| 2.5 | 12.5 | 10.2 | 9.3 |
| 3.0 | 18.1 | 17.5 | 15.9 |
| 3.5 | 20.0 | 22.4 | 26.1 |

**74.** Level curves are shown for a function $f$. Determine whether the following partial derivatives are positive or negative at the point $P$.
(a) $f_x$ (b) $f_y$ (c) $f_{xx}$
(d) $f_{xy}$ (e) $f_{yy}$




**75.** Verify that the function $u = e^{-\alpha^2 k^2 t} \sin kx$ is a solution of the *heat conduction equation* $u_t = \alpha^2 u_{xx}$.

**76.** Determine whether each of the following functions is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$.
(a) $u = x^2 + y^2$ (b) $u = x^2 - y^2$
(c) $u = x^3 + 3xy^2$ (d) $u = \ln \sqrt{x^2 + y^2}$
(e) $u = \sin x \cosh y + \cos x \sinh y$
(f) $u = e^{-x} \cos y - e^{-y} \cos x$

**77.** Verify that the function $u = 1/\sqrt{x^2 + y^2 + z^2}$ is a solution of the three-dimensional Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$.

**78.** Show that each of the following functions is a solution of the wave equation $u_{tt} = a^2 u_{xx}$.
(a) $u = \sin(kx) \sin(akt)$ (b) $u = t/(a^2t^2 - x^2)$
(c) $u = (x - at)^6 + (x + at)^6$
(d) $u = \sin(x - at) + \ln(x + at)$

**79.** If $f$ and $g$ are twice differentiable functions of a single variable, show that the function

$$u(x, t) = f(x + at) + g(x - at)$$

is a solution of the wave equation given in Exercise 78.

**80.** If $u = e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n}$, where $a_1^2 + a_2^2 + \cdots + a_n^2 = 1$, show that

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = u$$

**81.** The *diffusion equation*

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}$$

where $D$ is a positive constant, describes the diffusion of heat through a solid, or the concentration of a pollutant at time $t$ at a distance $x$ from the source of the pollution, or the invasion of alien species into a new habitat. Verify that the function

$$c(x, t) = \frac{1}{\sqrt{4\pi D t}}\, e^{-x^2/(4Dt)}$$

is a solution of the diffusion equation.

**82.** The temperature at a point $(x, y)$ on a flat metal plate is given by $T(x, y) = 60/(1 + x^2 + y^2)$, where $T$ is measured in °C and $x$, $y$ in meters. Find the rate of change of temperature with respect to distance at the point (2, 1) in (a) the $x$-direction and (b) the $y$-direction.

**83.** The total resistance $R$ produced by three conductors with resistances $R_1$, $R_2$, $R_3$ connected in a parallel electrical circuit is given by the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Find $\partial R/\partial R_1$.

**84.** Show that the Cobb-Douglas production function $P = bL^{\alpha}K^{\beta}$ satisfies the equation

$$L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = (\alpha + \beta)P$$

**85.** Show that the Cobb-Douglas production function satisfies $P(L, K_0) = C_1(K_0)L^{\alpha}$ by solving the differential equation

$$\frac{dP}{dL} = \alpha\frac{P}{L}$$

(See Equation 6.)

**86.** Cobb and Douglas used the equation $P(L, K) = 1.01L^{0.75}K^{0.25}$ to model the American economy from 1899 to 1922, where $L$ is the amount of labor and $K$ is the amount of capital. (See Example 14.1.3.)

(a) Calculate $P_L$ and $P_K$.

(b) Find the marginal productivity of labor and the marginal productivity of capital in the year 1920, when $L = 194$ and $K = 407$ (compared with the assigned values $L = 100$ and $K = 100$ in 1899). Interpret the results.

(c) In the year 1920, which would have benefited production more, an increase in capital investment or an increase in spending on labor?

**87.** The *van der Waals equation* for $n$ moles of a gas is

$$\left(P + \frac{n^2a}{V^2}\right)(V - nb) = nRT$$

where $P$ is the pressure, $V$ is the volume, and $T$ is the temperature of the gas. The constant $R$ is the universal gas constant and $a$ and $b$ are positive constants that are characteristic of a particular gas. Calculate $\partial T/\partial P$ and $\partial P/\partial V$.

**88.** The gas law for a fixed mass $m$ of an ideal gas at absolute temperature $T$, pressure $P$, and volume $V$ is $PV = mRT$, where $R$ is the gas constant. Show that

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1$$

**89.** For the ideal gas of Exercise 88, show that

$$T\frac{\partial P}{\partial T}\frac{\partial V}{\partial T} = mR$$

**90.** The wind-chill index is modeled by the function

$$W = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$$

where $T$ is the temperature (°C) and $v$ is the wind speed (km/h). When $T = -15$°C and $v = 30$ km/h, by how much would you expect the apparent temperature $W$ to drop if the actual temperature decreases by 1°C? What if the wind speed increases by 1 km/h?

**91.** A model for the surface area of a human body is given by the function

$$S = f(w, h) = 0.1091w^{0.425}h^{0.725}$$

where $w$ is the weight (in pounds), $h$ is the height (in inches), and $S$ is measured in square feet. Calculate and interpret the partial derivatives.

(a) $\dfrac{\partial S}{\partial w}(160, 70)$  (b) $\dfrac{\partial S}{\partial h}(160, 70)$

**92.** One of Poiseuille's laws states that the resistance of blood flowing through an artery is

$$R = C\frac{L}{r^4}$$

where $L$ and $r$ are the length and radius of the artery and $C$ is a positive constant determined by the viscosity of the blood. Calculate $\partial R/\partial L$ and $\partial R/\partial r$ and interpret them.

**93.** In the project on page 271 we expressed the power needed by a bird during its flapping mode as

$$P(v, x, m) = Av^3 + \frac{B(mg/x)^2}{v}$$

where $A$ and $B$ are constants specific to a species of bird, $v$ is the velocity of the bird, $m$ is the mass of the bird, and $x$ is the fraction of the flying time spent in flapping mode. Calculate $\partial P/\partial v$, $\partial P/\partial x$, and $\partial P/\partial m$ and interpret them.

**94.** The average energy $E$ (in kcal) needed for a lizard to walk or run a distance of 1 km has been modeled by the equation

$$E(m, v) = 2.65m^{0.66} + \frac{3.5m^{0.75}}{v}$$

where $m$ is the body mass of the lizard (in grams) and $v$ is its speed (in km/h). Calculate $E_m(400, 8)$ and $E_v(400, 8)$ and interpret your answers.

*Source:* C. Robbins, *Wildlife Feeding and Nutrition*, 2d ed. (San Diego: Academic Press, 1993).

**95.** The kinetic energy of a body with mass $m$ and velocity $v$ is $K = \frac{1}{2}mv^2$. Show that

$$\frac{\partial K}{\partial m}\frac{\partial^2 K}{\partial v^2} = K$$

**96.** If $a$, $b$, $c$ are the sides of a triangle and $A$, $B$, $C$ are the opposite angles, find $\partial A/\partial a$, $\partial A/\partial b$, $\partial A/\partial c$ by implicit differentiation of the Law of Cosines.

**97.** You are told that there is a function $f$ whose partial derivatives are $f_x(x, y) = x + 4y$ and $f_y(x, y) = 3x - y$. Should you believe it?

**98.** The paraboloid $z = 6 - x - x^2 - 2y^2$ intersects the plane $x = 1$ in a parabola. Find parametric equations for the tangent line to this parabola at the point $(1, 2, -4)$. Use a computer to graph the paraboloid, the parabola, and the tangent line on the same screen.

**99.** The ellipsoid $4x^2 + 2y^2 + z^2 = 16$ intersects the plane $y = 2$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1, 2, 2)$.

**100.** In a study of frost penetration it was found that the temperature $T$ at time $t$ (measured in days) at a depth $x$ (measured in feet) can be modeled by the function

$$T(x, t) = T_0 + T_1 e^{-\lambda x}\sin(\omega t - \lambda x)$$

where $\omega = 2\pi/365$ and $\lambda$ is a positive constant.

(a) Find $\partial T/\partial x$. What is its physical significance?

(b) Find $\partial T/\partial t$. What is its physical significance?

(c) Show that $T$ satisfies the heat equation $T_t = kT_{xx}$ for a certain constant $k$.

(d) If $\lambda = 0.2$, $T_0 = 0$, and $T_1 = 10$, use a computer to graph $T(x, t)$.

(e) What is the physical significance of the term $-\lambda x$ in the expression $\sin(\omega t - \lambda x)$?

**101.** Use Clairaut's Theorem to show that if the third-order partial derivatives of $f$ are continuous, then

$$f_{xyy} = f_{yxy} = f_{yyx}$$

**102.** (a) How many $n$th-order partial derivatives does a function of two variables have?

(b) If these partial derivatives are all continuous, how many of them can be distinct?

(c) Answer the question in part (a) for a function of three variables.

**103.** If

$$f(x, y) = x(x^2 + y^2)^{-3/2}e^{\sin(x^2 y)}$$

find $f_x(1, 0)$. [*Hint:* Instead of finding $f_x(x, y)$ first, note that it's easier to use Equation 1 or Equation 2.]

**104.** If $f(x, y) = \sqrt[3]{x^3 + y^3}$, find $f_x(0, 0)$.

**105.** Let

$$f(x, y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(a) Use a computer to graph $f$.

(b) Find $f_x(x, y)$ and $f_y(x, y)$ when $(x, y) \neq (0, 0)$.

(c) Find $f_x(0, 0)$ and $f_y(0, 0)$ using Equations 2 and 3.

(d) Show that $f_{xy}(0, 0) = -1$ and $f_{yx}(0, 0) = 1$.

CAS (e) Does the result of part (d) contradict Clairaut's Theorem? Use graphs of $f_{xy}$ and $f_{yx}$ to illustrate your answer.

## 14.4 Tangent Planes and Linear Approximations

One of the most important ideas in single-variable calculus is that as we zoom in toward a point on the graph of a differentiable function, the graph becomes indistinguishable from its tangent line and we can approximate the function by a linear function. (See Section 2.9.) Here we develop similar ideas in three dimensions. As we zoom in toward a point on a surface that is the graph of a differentiable function of two variables, the surface looks more and more like a plane (its tangent plane) and we can approximate the function by a linear function of two variables. We also extend the idea of a differential to functions of two or more variables.