# Calculus

1301

$$\int x^2 \sqrt[3]{x^2}\,dx = \int x^2 \cdot x^{\frac{2}{3}}\,dx = \int x^{\frac{8}{3}}\,dx = \frac{x^{\frac{8}{3}+1}}{\frac{8}{3}+1} + C = \frac{3}{11} x^{\frac{11}{3}} + C$$

$$\int \sec^2 x \tan x \, dx$$

$\tan x = u \qquad \frac{du}{dx} = \sec^2 x \qquad du = \sec^2 x\,dx$

$\int u \, du = \frac{u^2}{2} + C = \frac{\tan^2 x}{2} + C$

$$\int \frac{x^2}{x^3+1}\,dx$$

$x^2+1 = u \qquad \frac{du}{dx} = 3x^2 \qquad \therefore \frac{1}{3}du = x^2 dx$

$\int \frac{1}{u} \cdot \frac{1}{3} du = \frac{1}{3}\int \frac{1}{u} du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln(x^2+1) + C$

$$\int \frac{1}{x^2+2}\,dx = \int \frac{1}{x^2+(\sqrt{2})^2}\,dx = \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C.$$

1. $\boxed{\int u\,dv = uv - \int v\,du}$

$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$

反对幂指三 → $\sin x, \cos x, \tan x, \cot x$.

指 → $e^x, a^x$

反 → 反三角: $\tan^{-1}$, $\cot^{-1}$, $\sin^{-1}$, $\cos^{-1}$

对 → $\ln x$

幂 → $x^n$

Eg: $\int x e^x \, dx$ 转换

let $x = u \qquad e^x dx = dv \Rightarrow \frac{dv}{dx} = e^x \qquad v = e^x \qquad du = dx$

$x e^x - \int e^x dx = x e^x - e^x + C$

Eg: $\int x \cos x \, dx$

let $x = u$ ($\frac{du}{dx} = 1$) $\cos x\,dx = dv \qquad \frac{dv}{dx} = \cos x \qquad v = \sin x$

$x \sin x - \int \sin x\,dx = x\sin x + \cos x + C$

Eg: $\int x^2 \ln x \, dx$

$u = \ln x \qquad \frac{du}{dx} = \frac{1}{x} \qquad du = \frac{1}{x}dx \qquad x^2 dx = dv \qquad \frac{dv}{dx} = x^2$

$v = \frac{x^3}{3} \qquad \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\cdot\frac{1}{x}dx = \frac{x^3}{3}\ln x - \int \frac{x^2}{3}dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$

$$\int_0^{\frac{\pi}{2}} \theta \cos 2\theta \, d\theta$$

$u = \theta \quad \frac{du}{d\theta} = 1 \quad du = d\theta$

$dv = \cos 2\theta \, d\theta \quad \frac{dv}{d\theta} = \cos 2\theta \quad v = \frac{\sin 2\theta}{2}$

$\theta \sin 2\theta - \int \frac{\sin 2\theta}{2} d\theta$

$$= \left[\theta \sin 2\theta\right]_0^{\frac{\pi}{2}} - \left[-\frac{1}{4}\cos 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}\sin\pi - 0 - \left(-\frac{1}{4}\cos\pi + \frac{1}{4}\cos 0\right)$$

$$= -\frac{1}{4} - \frac{1}{4}$$

$$= -\frac{1}{2}$$

Eg: $I = \int e^x \sin x \, dx \qquad I = \frac{1}{2}(-e^x\cos x + e^x \sin x) + C$ Loop.

2. $\int \sin^m x \cos^n x \, dx$

$\boxed{\text{如果}m\text{是奇数} \quad t = \cos x}$

Eg: $\int \sin^3 x \cos^2 x \, dx$ let $\cos x = t$ $\quad \frac{dt}{dx} = -\sin x \quad dt = -\sin x\,dx$, $-dt = \sin x\,dx$

$= \int \sin^2 x \sin x \cos^2 x \, dx$

$= \int (1-t^2)\, t^2 (-dt) = -\int (t^2 - t^4)\, dt = -\frac{t^3}{3} + \frac{t^5}{5} + C$

$\boxed{\text{如果}n\text{是奇数} \quad t = \sin x}$

Eg: $\int \sin^2 x \cos^3 x \, dx$ let $\sin x = t$ $\quad \frac{dt}{dx} = \cos x \quad dt = \cos x\,dx$

$\boxed{\text{如果}m, n\text{都是偶：} \quad \cos^2 x = \frac{1+\cos 2x}{2} \quad \sin^2 x = \frac{1-\cos 2x}{2}}$

$$\int \cos^2 x \sin^2 x \, dx = \int \left(\frac{1+\cos 2x}{2} \cdot \frac{1-\cos 2x}{2}\right) dx = \frac{1}{4}\int (1 - \cos^2 2x)\, dx = \frac{1}{4}\int \left(1 - \frac{1+\cos 4x}{2}\right) dx$$

$$= \frac{1}{4}\int \left(1 - \frac{1}{2} - \frac{1}{2}\cos 4x\right) dx = \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) dx = \frac{x}{8} - \frac{1}{8}\cdot\frac{\sin 4x}{4} + C.$$

## 3. Trigonometric substitution 7.3

1) $\sqrt{a^2-x^2}$ let $x=a\sin\theta$ $\left(-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\right)$

$\sqrt{a^2-a^2\sin^2\theta}=\sqrt{a^2(1-\sin^2\theta)}=\sqrt{a^2(\cos^2\theta)}=a\cos\theta$

2) $\sqrt{a^2+x^2}$ let $x=a\tan\theta$ $\left(-\frac{\pi}{2}<\theta<\frac{\pi}{2}\right)$

$\sqrt{a^2+a^2\tan^2\theta}=\sqrt{a^2(1+\tan^2\theta)}=\sqrt{a^2\sec^2\theta}=a\sec\theta$

3) $\sqrt{x^2-a^2}$ let $x=a\sec\theta$ $\left(0\le\theta\le\frac{\pi}{2}\text{ or }\pi\le\theta\le\frac{3\pi}{2}\right)$ ☆

$\hookrightarrow\sqrt{a^2\sec^2\theta-a^2}=\sqrt{a^2(\sec^2\theta-1)}=\sqrt{a^2\tan^2\theta}=a\tan\theta$

4) $\sqrt{x\pm a}$ 令 $\sqrt{x\pm a}=t$ $x\pm a=t^2$

Eg: $\int\frac{\sqrt{9-x^2}}{x^2}dx$

let $x=3\sin\theta$ $\left(-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\right)$ $\frac{dx}{d\theta}=(3\sin\theta)'_\theta=3\cos\theta$ $dx=3\cos\theta d\theta$

$=\int\frac{\sqrt{9-9\sin^2\theta}}{(3\sin\theta)^2}3\cos\theta d\theta$

$=\int\frac{3\cos\theta\cdot3\cos\theta}{9\sin^2\theta}d\theta=\int\frac{\cos^2\theta}{\sin^2\theta}d\theta=\int\frac{1-\sin^2\theta}{\sin^2\theta}d\theta=\int\left(\frac{1}{\sin^2\theta}-1\right)d\theta=\int(\csc^2\theta-1)d\theta.$

$=-\cot\theta-\theta+C=-\frac{\sqrt{9-x^2}}{x}-\sin^{-1}\frac{x}{3}+C$

$1+\tan^2\theta=\sec^2\theta$

$1+\cot^2\theta=\csc^2\theta$

Eg: $\int_0^1\frac{1}{\sqrt{4-x^2}}dx$

let $x=2\sin\theta$ $\left(-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\right)$ $\frac{dx}{d\theta}=2\cos\theta$ $dx=2\cos\theta d\theta$

$=\int\frac{1}{\sqrt{4-4\sin^2\theta}}\cdot2\cos\theta d\theta$

$=\int\frac{2\cos\theta}{\sqrt{4-4\sin^2\theta}}d\theta=\int\frac{4\cos^2\theta}{4-4\sin^2\theta}d\theta=\int\frac{\cos^2\theta}{\cos^2\theta}d\theta=\int1d\theta=[\theta]_0^{\frac{\pi}{6}}=\frac{\pi}{6}.$

## 4. Partial Fractions.

$\frac{1}{(x-1)(x-2)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}$

$\frac{1}{(x-1)^2(x-2)}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x-2)}$

$\frac{1}{(x-1)^4}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{D}{(x-1)^4}$

$\frac{1}{(x-1)(x^2+5)}=\frac{A}{(x-1)}+\frac{Bx+C}{x^2+5}$

$\frac{1}{(x-1)(x^2+5)^2}=\frac{A}{x-1}+\frac{Bx+C}{x^2+5}+\frac{Dx+E}{(x^2+5)^2}$

$\frac{1}{(x-2)^2(x^2+x-5)^3}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+x-5}+\frac{Ex+F}{(x^2+x-5)^2}+\frac{Gx+H}{(x^2+x-5)^3}$

## 5. Def. Improver Integrals.

$$\int_a^{+\infty} f(x)\,dx$$
$$= \lim_{t\to\infty}\int_a^t f(x)\,dx$$

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \left(\int_{-\infty}^{c} + \int_c^{+\infty}\right) = \lim_{t\to-\infty}\int_t^c f(x)dx + \lim_{t\to\infty}\int_c^t f(x)\,dx.$$

Eg: $\int_0^{\infty} e^{-x}\,dx = 1$

$$= \lim_{t\to\infty}\int_0^t e^{-x}dx = \lim_{t\to\infty} -\left[e^{-x}\right]_0^t = \lim_{t\to\infty}(-e^{-t}+1) = \lim_{t\to\infty}\left(1-\frac{1}{e^t}\right) = 1$$

Eg: $\int_{-\infty}^{0} xe^x\,dx$

let $u = x$ $\quad \frac{du}{dx} = 1 \quad du = dx \quad dv = e^x dx \quad \frac{dv}{dx} = e^x \quad v = e^x$

$$\int_{-\infty}^{0} xe^x\,dx = \lim_{t\to-\infty}\int_t^0 xe^x\,dx$$
$$=$$

若 $\int_1^5 \frac{1}{(x-1)^2}\,dx$ 此时分母不能为0

$$= \lim_{t\to1}\int_t^5 \frac{1}{(x-1)^2}\,dx$$
$$= \int_{-1}^{2} \frac{1}{x-1}\,dx$$
$$= \int_{-1}^{1}\frac{1}{x-1}\,dx + \int_1^2 \frac{1}{x-1}\,dx$$
$$= \lim_{t\to}\int_{-1}^{t}\frac{1}{x-1}\,dx + \lim_{t\to1}\int_t^2\frac{1}{x-1}\,dx$$

1. $\int_0^{\frac{\pi}{2}} \theta \cos(2\theta) d\theta$

2. $\int_0^{\infty} e^{-2x} dx$

3. $\int_0^{1} \ln x \, dx$

4. $\int_0^{k} x e^{\frac{x}{k}} dx = 9$ $k=$ ___

反对幂指三

$$\int_0^{\frac{\pi}{2}} \theta \cos(2\theta) \, d\theta$$

Let $\theta = u$ $\quad \frac{du}{d\theta} = 1 \quad du = d\theta \quad \cos(2\theta) = \frac{dv}{d\theta} \quad \cancel{d\theta} \, dv = \cos(2\theta) d\theta.$

$v = \frac{1}{2}\sin(2\theta) \quad \frac{1}{2}\theta \sin(2\theta) + \int \frac{1}{2}\sin(2\theta) \, d\theta$

$$= \left[\frac{1}{2}\theta \sin(2\theta)\right]_0^{\frac{\pi}{2}} + \left[\frac{1}{4}\cos(2\theta)\right]$$