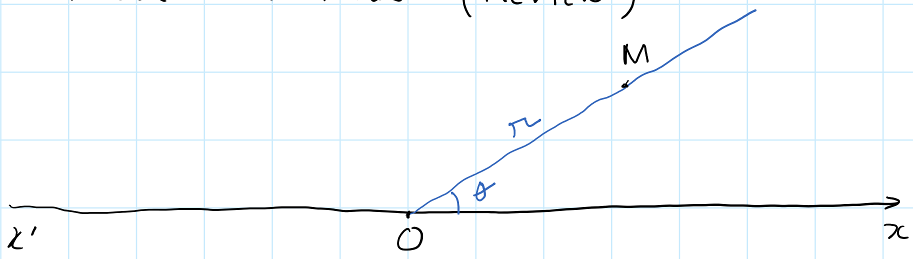
Lecture 12

October 5, 2020 11:04 AM

Polar Coordinates (Review)



Consider an x-axis with a reference O on it. Then any point M in the plane can be defined by

(i) the angle $\widehat{xOM}$, denoted by θ

(ii) the distance OM, denoted by r

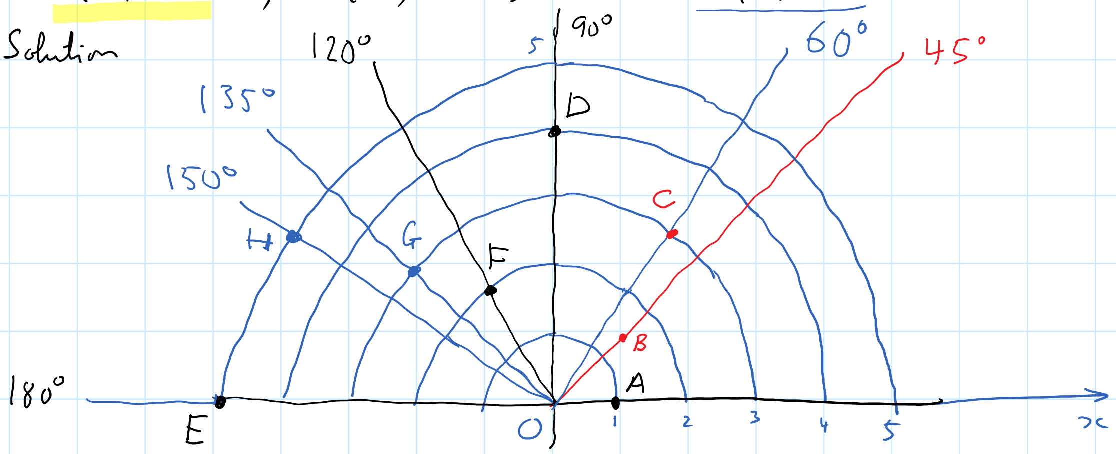
The pair (r, θ) is called polar coordinates of M.

The $x'Ox$ axis is called the polar axis, and O is called the pole.

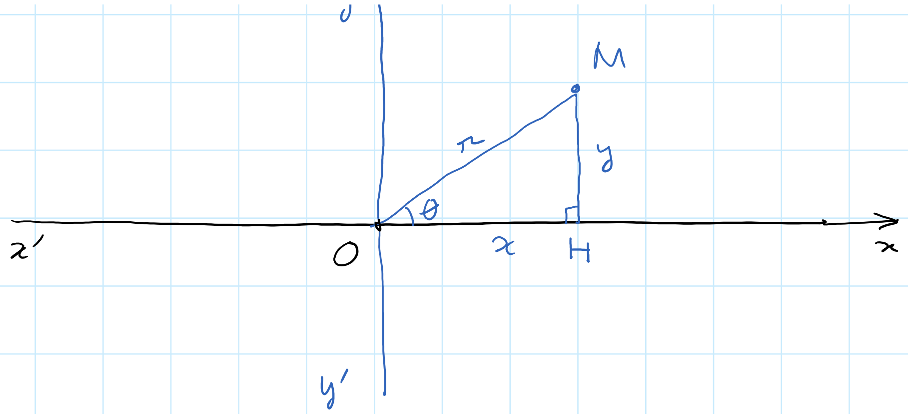
Ex1: Sketch the following points whose polar coordinates are

$A(1, 0^\circ)$, $B(\sqrt{2}\ 45^\circ)$, $C(3, 60^\circ)$, $D(4, 90^\circ)$, $E(5, 180^\circ)$, $F(2, 120^\circ)$, $G(3, 135^\circ)$ and $H(5, 150^\circ)$.

Solution



Relationship between Cartesian coordinates (x, y) and polar coordinates (r, θ).



Adding the y-axis to the x-axis at O, consider the right $\triangle OMH$

$$x = OH = \underbrace{OM}_{r} \cos \underbrace{\widehat{xOM}}_{\theta} = r\cos\theta$$

$$y = MH = \underbrace{OM}_{r} \sin \underbrace{\widehat{xOM}}_{\theta} = r\sin\theta$$

$$\therefore \quad \boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

Also from the same right $\triangle OMH$,

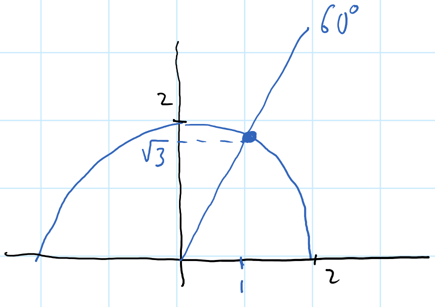
$$r = OM = \sqrt{OH^2 + HM^2} = \sqrt{x^2+y^2}$$

$$\tan\theta = \frac{MH}{OH} = \frac{y}{x}$$

$$\therefore \quad \boxed{\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}}$$

<u>Ex 2</u>: Convert the polar point $(\underset{\underset{r}{\uparrow}}{2}, \underset{\underset{\theta}{\uparrow}}{\frac{\pi}{3}}) \equiv (2, 60°)$ to Cartesian coordinates.

<u>Solution</u>



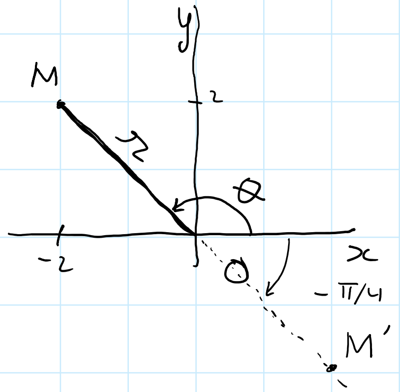
$$x = r\cos\theta = 2\underbrace{\cos\left(\frac{\pi}{3}\right)}_{\frac{1}{2}} = 2\left(\frac{1}{2}\right) = 1$$

$$y = r\sin\theta = 2\underbrace{\sin\left(\frac{\pi}{3}\right)}_{\frac{\sqrt{3}}{2}} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$\therefore$ The point is $(1, \sqrt{3})$ in Cartesian coordinates. // Ans.

<u>Ex3</u>: Convert the Cartesian point $(\underset{\substack{\uparrow \\ x}}{-2}, \underset{\substack{\uparrow \\ y}}{2})$ to polar coordinates.

Solution



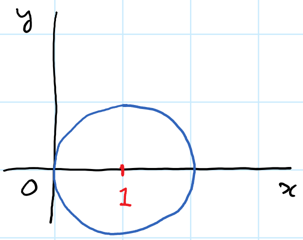
$$r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + (2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2}{-2}\right) = \tan^{-1}(-1) = \frac{3\pi}{4}$$

Warning: your calculator may give you $\theta = -\frac{\pi}{4}$ instead of $\frac{3\pi}{4}$.

<u>Ex4</u>: Write the polar equation of the circle passing thru the origin $(0,0)$ and its center is located at the point $(1,0)$.

<u>Soln</u>



The equation of the circle in Cartesian coordinates is

$$(x-1)^2 + y^2 = 1$$

$$x^2 - 2x + \cancel{1} + y^2 = \cancel{1}$$

$$\underbrace{x^2 + y^2}_{r^2} - \underbrace{2x}_{r\cos\theta} = 0$$

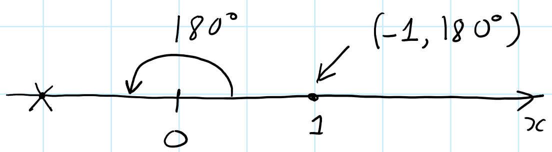
$$r^2 - 2r\cos\theta = 0$$

$r(r - 2\cos\theta) = 0 \Rightarrow$ ~~$r = 0$ which is the pole~~

$r = 2\cos\theta$ which is the polar equation we are looking for.

$$\boxed{r = 2\cos\theta}$$ // Ans.

<u>N.B</u>: Polar representation is NOT unique because $(1, 0°)$ can be written as $(1, 360°)$ or $(1, 720°)$ and so on or it can be written as $(-1, 180°)$



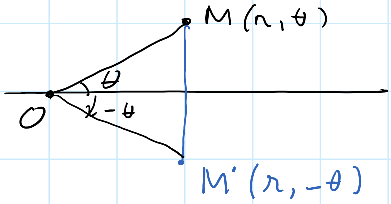
Ex 5: Sketch the polar curves $r = f(\theta) = 1 + \cos\theta$.

Solution

Since $\cos\theta$ has a period 2π (or $360°$), we will construct a table of values for θ from $0°$ to $360°$. However, before constructing the table of values, we should check if the curve has any symmetry.

If the curve has the x-symmetry, then

$$f(-\theta) = f(\theta) \quad \text{for } \forall\, \theta$$



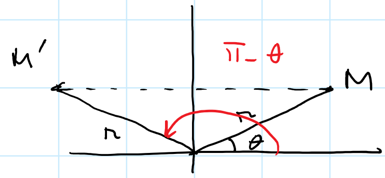
In this example, $f(\theta) = 1 + \cos\theta$

$$\therefore\ f(-\theta) = 1 + \underbrace{\cos(-\theta)}_{\cos(\theta)} = 1 + \cos\theta = f(\theta)$$

$\therefore$ The curve is symmetric about the x-axis.

If the curve has the y-symmetry, then

$$f(\pi - \theta) = f(\theta) \quad \text{for } \forall\theta$$

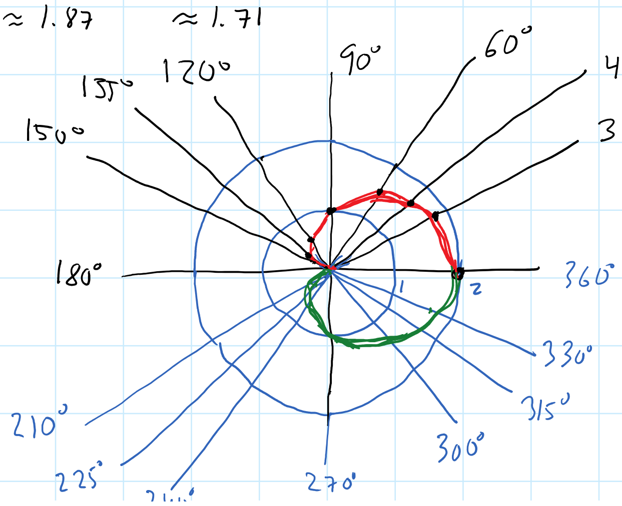


$$f(\pi - \theta) = 1 + \underbrace{\cos(\pi - \theta)}_{-\cos\theta} = 1 - \cos\theta \neq f(\theta)$$

$\therefore$ The curve does NOT have the y-symmetry.

Since the curve has the x-symmetry, we construct a table of values for θ from $0°$ to $180°$ (instead of θ from $0°$ to $360°$).

θ	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$
$r = f(\theta)$	2	$1 + \frac{\sqrt{3}}{2} \approx 1.87$	$1 + \frac{\sqrt{2}}{2} \approx 1.71$	1.5	1	0.5	$1 - \frac{\sqrt{2}}{2} \approx 0.29$	$1 - \frac{\sqrt{3}}{2} \approx 0.13$	0



240°

The curve is called a cardiod (shape of a heart)

Example 6 : Plot the polar curve $r = f(\theta) = \cos(2\theta)$.

Solution

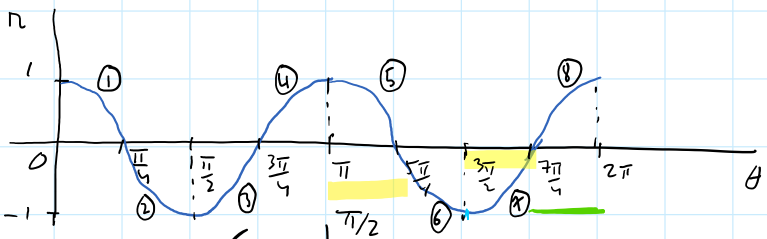
$$f(-\theta) = \cos(2(-\theta)) = \cos(-2\theta) = \cos(2\theta) = f(\theta)$$

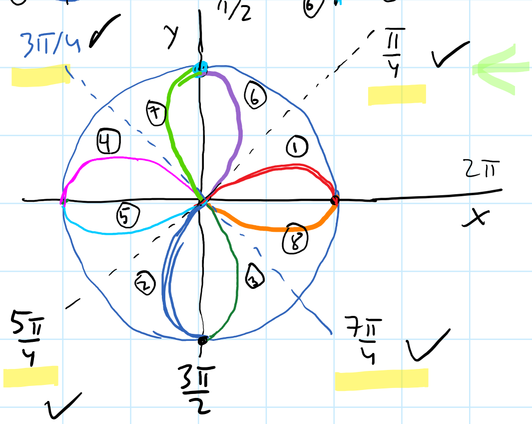
$\therefore$ The curve has the x- symmetry.

$$f(\pi-\theta) = \cos(2(\pi-\theta)) = \cos(2\pi - 2\theta) = \cos(-2\theta) = \cos(2\theta) = f(\theta)$$

$\therefore$ The curve has the y-symmetry as well.

Let's plot $r = \cos(2\theta)$ in Cartesian coordinates





When θ varies from $\frac{\pi}{4}$ to $\frac{\pi}{2}$, r is decreasing from 0 to -1 which is the same as r is increasing from 0 to 1 in the opposite direction, i.e, the direction of $\frac{3\pi}{2}$ and so on.

The curve is a rose of four petals.

<u>Theorem</u>: The tangents to the polar curve $r = f(\theta)$ at the pole satisfy $f(\theta) = 0$.

<u>Ex 7</u>: Find the tangents of the polar curve $f(\theta) = \cos(2\theta)$ at the pole.

<u>Soln</u>

at the pole.

Soln

Solving $f(\theta) = 0$

$$\cos(2\theta) = 0$$

$$2\theta = (2k+1)\frac{\pi}{2} \quad \text{for } k = 0, 1, 2, 3, \ldots$$

$$\therefore \quad \theta = (2k+1)\frac{\pi}{4}$$

Let $k = 0 \Rightarrow \theta = \frac{\pi}{4}$

$k = 1 \Rightarrow \theta = \frac{3\pi}{4}$

$k = 2 \Rightarrow \theta = \frac{5\pi}{4}$

$k = 3 \Rightarrow \theta = \frac{7\pi}{4}$

$\therefore$ There are 4 tangents to the curve at the pole. They are $\theta = \frac{\pi}{4}$, $\theta = \frac{3\pi}{4}$, $\theta = \frac{5\pi}{4}$ and $\theta = \frac{7\pi}{4}$. // Ans.

See you in the next lecture!