Math 2155, Fall 2021: Homework 6

Instructions for GradeScope:

You may handwrite your answers or type them. Only type solutions if you are able to type the appropriate symbols. If **typed**, use at least a 12pt font, with **one question per page**. You can then directly submit the pdf file to GradeScope.

If **handwritten**:

- Don't erase or cross out more than a word or two.

- Only write on one side of the page. Use dark, large, neat writing.

- If you have a real scanner, great! Most people will scan their solutions using their phone. Use one of the recommended apps: For iOS, use Scannable by Evernote or Genius Scan. For Android, use Genius Scan. Do *not* just take regular photos.

- When scanning, have good lighting and try to avoid shadows.

- It is best to have **one question per scan**. If the solution is short, **fold the page in half** and scan just the half it is on, so there isn't so much blank space. Or, you can scan and then crop the scan. You don't need to scan parts (a), (b), etc. separately or put them on separate pages.

- It works well to scan each question separately and produce one pdf file with one question per page. Most scanning apps will automatically combine your images into one pdf file.

- You must **check the quality** of your scans afterwards, and rescan if needed.

Make sure you are going to `http://gradescope.ca` not `http://gradescope.com`. You can access it through the course OWL site.

You can **resubmit** your work any number of times until the deadline.

See the GradeScope help website for lots of information: `https://help.gradescope.com/`
Select "Student Center" and then either "Scanning Work on a Mobile Device" or "Submitting an Assignment".

Instructions for writing solutions:

- Homework is graded both on **correctness** and on **presentation**.

- Your proofs should be written in **complete sentences**, starting with capitals and ending with periods. They should be in **paragraph form**, similar to the proofs in the textbook. For example, you shouldn't put each sentence on a new line. Paragraphs should be used to break the proof into logical chunks.

- Don't use informal abbreviations such as three dots for "therefore." Logical symbols can be used when they clarify things to the reader, but English words can often be more effective.

- Include all of the steps that are needed to logically justify every claim you make. Do not include unnecessary steps. Try to be **concise and complete**.

- Do not submit any **rough work**. Do your rough work on scratch paper, and only submit a clearly and neatly written answer. Do not cross out or erase more than a word or two. If you write each final solution on a new page, it's easy to start over on a fresh page when you've made a large error.

- You should do the work **on your own**. Read the course syllabus for the rules about scholastic offences, which include sharing solutions with others, uploading material to a website, viewing material of others or on a website (even if you don't use it), etc. The penalty for cheating on homework will be a grade of **0** on the homework set as well as a penalty of **negative 5%** on the overall course grade.

- We may not grade every question.

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Due **Friday, October 22 at 11:59pm**. You can **resubmit** your work any number of times until the deadline.

H6Q1: Let $\mathcal{F}$ and $\mathcal{G}$ be non-empty families of sets. Consider the following statements:

Statement 1: $\bigcap \mathcal{F}$ and $\bigcap \mathcal{G}$ are disjoint.

Statement 2: There exist $A \in \mathcal{F}$ and $B \in \mathcal{G}$ such that A and B are disjoint.

(a) Does Statement 1 imply Statement 2? If so, give a proof. Otherwise, give a counterexample.

(b) Does Statement 2 imply Statement 1? If so, give a proof. Otherwise, give a counterexample.

Solution: (a) Statement 1 does not imply Statement 2. For example, if $\mathcal{F} = \{\{1, 2\}\}$ and $\mathcal{G} = \{\{1\}, \{2\}\}$, then $\bigcap \mathcal{F} = \{1, 2\}$ and $\bigcap \mathcal{G} = \emptyset$, which are disjoint. But neither of the sets in $\mathcal{G}$ is disjoint from the only set $\{1, 2\}$ in $\mathcal{F}$.

(b) Statement 2 does imply Statement 1.

Proof. Let $A \in \mathcal{F}$ and $B \in \mathcal{G}$ be disjoint. To get a contradiction, suppose that x is in both $\bigcap \mathcal{F}$ and $\bigcap \mathcal{G}$. Since x is in $\bigcap \mathcal{F}$, we have $x \in C$ for every $C \in \mathcal{F}$. In particular, $x \in A$. Similarly, since $x \in \bigcap \mathcal{G}$, we have $x \in B$. Therefore, $x \in A \cap B$, which contradicts the assumption that A and B are disjoint. Therefore, we must have that $\bigcap \mathcal{F}$ and $\bigcap \mathcal{G}$ are disjoint. $\square$

H6Q2: Prove the following theorem:

Theorem: Let a, b, and x be real numbers with $a < b$. Then $(x - a)(x - b) \geq 0$ if and only if $x \leq a$ or $x \geq b$.

Solution: $(\leftarrow)$ Suppose $x \leq a$ or $x \geq b$.

Case 1: $x \leq a$. Then $x \leq a < b$, so $x < b$. Hence both $x - a \leq 0$ and $x - b < 0$, so we have $(x - a)(x - b) \geq 0$, as desired.

Case 2: $x \geq b$. Then $a < b \leq x$, so $a < x$. Hence $x - a > 0$ and $x - b \geq 0$, so again we have $(x - a)(x - b) \geq 0$.

We have shown that if $x \leq a$ or $x \geq b$, then $(x - a)(x - b) \geq 0$.

$(\rightarrow)$ We will prove the contrapositive. Suppose neither $x \leq a$ nor $x \geq b$, so $x > a$ and $x < b$. Then $x - a > 0$ and $x - b < 0$, so their product $(x - a)(x - b)$ is negative and non-zero, which is the negation of $(x - a)(x - b) \geq 0$. So we have shown that $(x - a)(x - b) \geq 0$ implies $x \leq a$ or $x \geq b$.

We have now shown both directions, so $(x - a)(x - b) \geq 0$ if and only if $x \leq a$ or $x \geq b$. $\square$

H6Q3: Suppose A, B, and C are sets. Prove that $A \cup B \subseteq A \cup C$ iff $B \setminus A \subseteq C \setminus A$.

Solution: ($\rightarrow$) Assume that $A \cup B \subseteq A \cup C$ and let $x \in B \setminus A$. Then $x \in B$ and $x \notin A$. Since $x \in B$, $x \in A \cup B$. Since $A \cup B \subseteq A \cup C$, $x \in A \cup C$. However, $x \notin A$, so we must have $x \in C$. Therefore, $x \in C \setminus A$. Since x was arbitrary, $B \setminus A \subseteq C \setminus A$.

($\leftarrow$) Assume that $B \setminus A \subseteq C \setminus A$ and let $x \in A \cup B$. We will show that $x \in A \cup C$.

Case 1: $x \in A$. Then clearly $x \in A \cup C$.

Case 2: $x \notin A$. Since $x \in A \cup B$, we must have $x \in B$. And since $x \notin A$, we have $x \in B \setminus A$. Since $B \setminus A \subseteq C \setminus A$, we see that $x \in C \setminus A$. In particular, $x \in C$, so we have $x \in A \cup C$.

These cases are exhaustive, so we have shown that $x \in A \cup C$. Since x was arbitrary, $A \cup B \subseteq A \cup C$. $\square$

H6Q4: Consider the following statements:

Statement 3: Let a, b and n be natural numbers. If a is not a multiple of n and b is not a multiple of n, then ab is not a multiple of n.

Statement 4: Let a, b and n be natural numbers. If a is a multiple of n or b is a multiple of n, then ab is a multiple of n.

One of these is true and one is false. Prove the correct one and give a counterexample for the other one.

Solution: Statement 3 is false. For example, take $a = b = 2$ and $n = 4$. Then neither a not b is a multiple of 4, but $ab = 4$ is a multiple of 4.

Statement 4 is correct.

Proof. Suppose that a is a multiple of n or b is a multiple of n.

Case 1: a is a multiple of n. Then $a = kn$ for some integer k. Therefore $ab = bkn$. Since bk is an integer, ab is a multiple of n.

Case 2: b is a multiple of n. Then $b = kn$ for some integer k. Therefore $ab = akn$. Since ak is an integer, ab is a multiple of n.

We have shown that ab is a multiple of n when either a is a multiple of n or b is a multiple of n. $\square$

H6Q5 (Bonus question): Prove that there are no rational numbers x and y such that $x^2+y^2=3$.

(You can use facts about prime factorizations of integers.)

Solution:

Lemma. If $k \in \mathbb{Z}$ and 3 does not divide k, then the remainder when k^2 is divided by 3 is 1.

Proof. If 3 does not divide k, then we either have that $k = 3m+1$ or $k = 3m+2$ for some $m \in \mathbb{Z}$. In either case, you can check that $k^2 = 3r+1$ for some integer r. $\square$

Proof of bonus question. Suppose that there exist rational numbers x and y such that $x^2+y^2=3$. Write $x = p/q$ and $y = r/s$ for $p, q, r, s \in \mathbb{Z}$ with q and s nonzero. Multiplying through by $(qs)^2$ gives $(ps)^2 + (rq)^2 = 3(qs)^2$. Letting $a = ps, b = rq$ and $c = qs$, this means that we have found integers a, b and c with $a^2 + b^2 = 3c^2$ and $c \neq 0$.

Note that we can't have $a = 0$, since otherwise we would have $3 = (b/c)^2$ which would mean that $\sqrt{3}$ is rational. (It's an easy exercise that this is not possible.) Similarly, we can't have $b = 0$.

Let (a, b, c) be an integer solution to that equation with $c \neq 0$ and $|a|$ as small as possible. We see that $3 \mid a^2 + b^2$. However, if one or both of a and b is not divisible by 3, then (by the Lemma), the remainder when $a^2 + b^2$ is divided by 3 is 1 or 2. So it follows that $3 \mid a$ and $3 \mid b$. Thus $9 \mid a^2$ and $9 \mid b^2$, so $9 \mid a^2 + b^2$. Therefore, $9 \mid 3c^2$, i.e., $3c^2 = 9n$ for some integer n. Dividing by 3, it follows that $3 \mid c^2$. By prime factorization, it follows that $3 \mid c^2$.

Now we know that all of a, b and c are divisible by 3, so it follows that $(a/3, b/3, c/3)$ is another solution to the equation. But $|a/3| < |a|$ (since $a \neq 0$), so this contradicts the choice of (a, b, c). This contradiction shows that there can be no rational solutions to $x^2 + y^2 = 3$. $\square$

Note: If you had a proof and it seems to work equally well for the equation $x^2 + y^2 = 5$, then your argument is not correct, since we have rational numbers $x = 1$ and $y = 2$ that satisfy that equation.