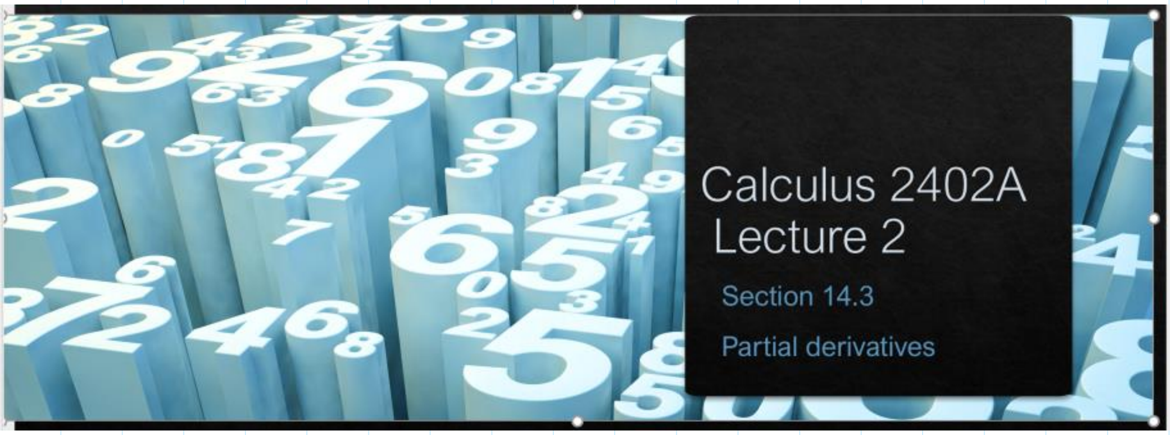
Lecture 2

September 11, 2020 2:06 PM




Definitions: Given the function $z = f(x,y)$, the first partial derivatives of $f$ with respect to $x$ and $y$, respectively, are
(wrt)

$$\frac{\partial z}{\partial x} \equiv \frac{\partial f}{\partial x} \equiv f_1(x,y) \equiv f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} \equiv f_2(x,y) \equiv f_y(x,y) = \lim_{k\to} \frac{f(x,y+k) - f(x,y)}{k}$$

In the above definitions, $f_1(x,y)$ means the partial derivative of $f$ wrt the 1st argument, i.e $x$ and $f_2(x,y)$ means the partial derivative of $f$ wrt the 2nd argument, i.e., $y$.

Rule of finding partial derivatives of $z = f(x,y)$

1. To obtain $f_x$, we consider $y$ as a constant and differentiate $f(x,y)$ wrt $x$.
2. To obtain $f_y$, we consider $x$ as a constant and differentiate $f(x,y)$ wrt $y$.

Ex1: If $z = f(x,y) = xy + x^2$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(2,0)$.

Solution

$y$ is fixed

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(xy + x^2) = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial x}(x^2)$$

$$= y\underbrace{\frac{\partial}{\partial x}(x)}_{1} + 2x$$

$$= y + 2x \quad //$$

fixed  fixed

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(xy + x^2)$$

fixed

$$= \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial y}(x^2)$$

$$= x \frac{\partial}{\partial y}(y) + 0$$

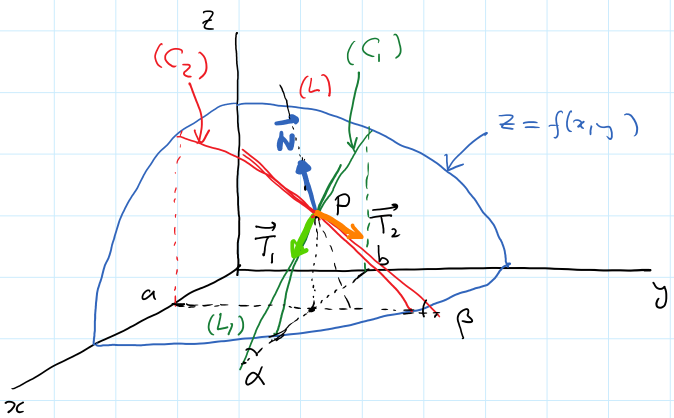
$$= x \quad //$$

At the point $(2,0)$,

$$\left.\frac{\partial z}{\partial x}\right|_{(2,0)} = y + 2x \Big|_{(2,0)} = 0 + 2(2) = 4 \quad // \text{ Ans.}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(2,0)} = x \Big|_{(2,0)} = 2 \quad // \text{ Ans.}$$

Geometric meaning of partial derivatives




$$\left.\frac{\partial z}{\partial x}\right|_{(a,b)} = \text{Slope of the tangent line } (L_1) \text{ to } (C_1) \text{ at } P(a,b,f(a,b))$$
$$= \tan\alpha$$

$$\left.\frac{\partial z}{\partial y}\right|_{(a,b)} = \text{Slope of the tangent line } (L_2) \text{ to } (C_2) \text{ at } P.$$
$$= \tan\beta$$

Let $\vec{T_1}$ and $\vec{T_2}$ be tangent vectors to $(C_1)$ & $(C_2)$ to the surface $z = f(x,y)$ at the point $P(a, b, f(a,b))$. Then

$$\vec{T_1} = (1, 0, f_x(a,b))$$

and $$\vec{T_2} = (0, 1, f_y(a,b))$$

Then the normal vector $\vec{N}$ to the surface $z=f(x,y)$ at P is

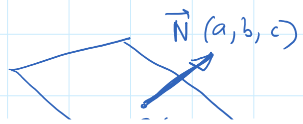
$$\vec{N} = \vec{T_1} \times \vec{T_2}$$ (see Stewart, chapter 12 about the cross-product)

$$= \begin{vmatrix} \overset{+}{\hat{i}} & \overset{-}{\hat{j}} & \overset{+}{\hat{k}} \\ 1 & 0 & f_x(a,b) \\ 0 & 1 & f_y(a,b) \end{vmatrix}$$

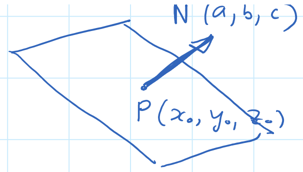
$$= \hat{i} \begin{vmatrix} 0 & f_x(a,b) \\ 1 & f_y(a,b) \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & f_x(a,b) \\ 0 & f_y(a,b) \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= \hat{i}\,(-f_x(a,b)) - \hat{j}\,(f_y(a,b)) + 1\,\hat{k}$$

$$= \underbrace{-f_x(a,b)}_{a}\,\hat{i} - \underbrace{f_y(a,b)}_{b}\,\hat{j} + \underbrace{1}_{c}\,\hat{k}$$



$$= \underbrace{-f_x(a,b)}_{a}\, i \;\; \underbrace{- f_y(a,b)}_{b}\, j + \underbrace{1}_{c}\, k$$




$$P(\underset{x_0}{a},\ \underset{y_0}{b},\ \underset{z_0}{f(a,b)})$$

$$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$

$\therefore$ The equation of the tangent plane to the surface $z = f(x,y)$ at $P(a, b, f(a,b))$ is

$$-f(a,b)\,(x-a) - f_y(a,b)\,(y-b) + 1\,(z - f(a,b)) = 0$$

$$\boxed{z = f(a,b) + (x-a)\, f_x(a,b) + (y-b)\, f_y(a,b)}$$

and equations of the normal line (L) to the surface $z = f(x,y)$ at $P(a, b, f(a,b))$ is

$$\boxed{\frac{x-a}{-f_x(a,b)} = \frac{y-b}{-f_y(a,b)} = \frac{z - f(a,b)}{1}}$$

Ex: Find an equation of the tangent plane to the given surface $z = f(x,y) = y \ln x$ at the point $(\underset{a}{1}, \underset{b}{4}, \underset{f(a,b)}{0})$.

check: $f(a,b) = f(1,4) = (4)\cancel{\ln(1)}^{0} = 0$ ✓

$f_x = y\left(\frac{1}{x}\right) = \frac{y}{x}$

$f_y = (1) \ln x = \ln x$

$\therefore$ At $(1, 4, 0)$

$f_x(1,4) = \frac{4}{1} = 4$

$f_y(1,4) = \ln(1) = 0$

$\therefore$ The equation of the tangent plane is

$-4(x-1) + (0)(y-4) + (1)(z-0) = 0$

$-4x + z + 4 = 0$

$\Rightarrow \quad z = 4x - 4 \qquad$ // Ans.

Like functions of one variable, we may define higher partial derivatives. Given $z = f(x,y)$, 2nd derivatives are

$$\frac{\partial^2 z}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \equiv \frac{\partial^2 f}{\partial x^2} \equiv f_{11}(x,y) \equiv f_{xx}(x,y)$$

$$\frac{\partial^2 z}{\partial y^2} \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \equiv \frac{\partial^2 f}{\partial y^2} \equiv f_{22}(x,y) \equiv f_{yy}(x,y)$$

$$\frac{\partial^2 z}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \equiv \frac{\partial^2 f}{\partial x \partial y} \equiv f_{21}(x,y) \equiv f_{yx}(x,y)$$

$$\frac{\partial^2 z}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \equiv \frac{\partial^2 f}{\partial y \partial x} \equiv f_{12}(x,y) \equiv f_{xy}(x,y)$$

The last two partial derivatives are called mixed partial derivatives of $f$ wrt $x$ and $y$.

Ex 3: Find the 2nd derivatives of $f(x,y) = x^3y^4 + x^4y^3$.

Solution

$$f_x = \frac{\partial f}{\partial x} = (3x^2)y^4 + (4x^3)y^3 = 3x^2y^4 + 4x^3y^3$$

$$f_y = \frac{\partial f}{\partial y} = (x^3)(4y^3) + (x^4)(3y^2) = 4x^3y^3 + 3x^4y^2$$

$$f_{xx} = \frac{\partial}{\partial x}(3x^2y^4 + 4x^3y^3)$$
$$= 6xy^4 + 12x^2y^3 \quad // \text{ Ans.}$$

$$f_{yy} = \frac{\partial}{\partial y}(4x^3y^3 + 3x^4y^2)$$
$$= 4x^3(3y^2) + 3x^4(2y)$$
$$= 12x^3y^2 + 6x^4y \quad // \text{ Ans.}$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(3x^2y^4 + 4x^3y^3)$$
$$= (3x^2)(4y^3) + 4x^3(3y^2)$$
$$= 12x^2y^3 + 12x^3y^2 \quad // \text{ Ans.}$$

$$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(4x^3y^3 + 3x^4y^2)$$
$$= 12x^2y^3 + 12x^3y^2 \quad // \text{ Ans.}$$

We note that $f_{xy} = f_{yx}$. This is NOT a coincidence but it is the content of the following theorem.

Clairault's Theorem: Suppose $f$, $f_x$, $f_y$, $f_{xy}$, $f_{yx}$ are continuous in a neighborhood of $(a,b)$ then

$$\boxed{f_{xy}(a,b) = f_{yx}(a,b)}$$

What is a differential equation (DE)?

A DE is an equation consisting of an unknown function and its derivatives.

There are 2 types of DEs

(i) If the unknown function is a function of one variable ONLY then this DE is called an Ordinary Differential Equation (ODE).

(ii) If the unknown function is a functions of two or more independent variables then this DE is called a Partial Differential Equation (PDE).

The following PDEs are popular and useful in Science and Physics.

(i) Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

where $u(x,y)$ is the unknown function

(ii) The wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(iv) The heat equation (or the diffusion equation)

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$$

Ex4: Show that the function $u = e^{kx} \sin ky$ is a solution of Laplace's equation.

Solution:

$$u_x = k e^{kx} \sin ky \qquad u_y = k e^{kx} \cos ky$$

$$u_{xx} = k^2 e^{kx} \sin ky \qquad u_{yy} = -k^2 e^{kx} \sin ky$$

Subst these into the LHS of Laplace's eqn

$$\begin{aligned} LHS &= u_{xx} + u_{yy} \\ &= k^2 e^{kx} \sin ky - k^2 e^{kx} \sin ky \\ &= 0 \\ &= RHS \ \checkmark \end{aligned}$$

$\therefore$ $u(x,y) = e^{kx} \sin ky$ is a soln of Laplace's equation. // Ans.

Ex5: Show $u(x,t) = \sin(x - ct)$ is a solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \boxed{c^2 \frac{\partial^2 u}{\partial x^2}}$$

Soln

$$u_t = \cos(x - ct)(-c) = -c \cos(x - ct)$$

$$u_{tt} = c \sin(x - ct)(-c)$$

$$= -c^2 \sin(x - ct) \quad //$$

$$u_x = \cos(x - ct) \underbrace{\frac{\partial}{\partial x}(x - ct)}_{1} = \cos(x - ct)$$

$$u_{xx} = -\sin(x - ct) \underbrace{\frac{\partial}{\partial x}(x - ct)}_{1} = -\sin(x - ct)$$

LHS of the wave equation $= u_{tt}$

$$= -c^2 \sin(x - ct) \quad //$$

RHS of the wave equation $= c^2 u_{xx}$

$$= c^2 (-\sin(x - ct))$$

$$= -c^2 \sin(x - ct) \quad //$$

Because LHS = RHS, $u(x,t) = \sin(x - ct)$ is a solution of the wave equation. //Ans.

In problem 81, p. 966 you are asked to show

$$u(x,t) = \frac{e^{-x^2/4Dt}}{\sqrt{4\pi Dt}}$$

is a solution of the diffusion equation

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$$

which is your home work. //

See you on Monday!