Lecture 27 — Section 4.2

Determinants

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How to use these lectures

1) Download a copy of slides from OWL.

2) Have slides at hand while listening to lecture.

3) Skip forward or revisit parts as needed.

Motivation: 2×2 Determinants

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : 2 \times 2 \text{ matrix} \quad \rightsquigarrow \quad \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Recall

the determinant satisfies the following properties:

— *A is invertible iff* $\det(A) \neq 0$,

— *if one row of A is a multiple of another, then* $\det(A) = 0$

Goal

define $\det(A)$ *with similar properties for any square matrix A*

3×3 matrices

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} : 3 \times 3 \text{ matrix}$$

Definition

the determinant of A is given by

$$\det(A) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

this is the cofactor expansion along the first row.

Example: diagonal matrix

Example

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \quad \det(D) &= \lambda_1 \begin{vmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ 0 & \lambda_3 \end{vmatrix} + 0 \begin{vmatrix} 0 & \lambda_2 \\ 0 & 0 \end{vmatrix} \\ &= \lambda_1(\lambda_2\lambda_3) - 0 + 0 \\ &= \lambda_1\lambda_2\lambda_3 \end{aligned}$$

Claim

determinant of a diagonal matrix is the product of diagonal entries

Example: two identical rows

Example

$$\begin{aligned}\begin{vmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{vmatrix} &= 1\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix} - 1\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix} + 1\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix}\\ &= 1(1-1) - 1(1-1) + 1(1-1)\\ &= 0 - 0 + 0\\ &= 0\\ &\neq \text{product of diagonal entries}\end{aligned}$$

Claim

if one row/column of A is a multiple of another, then $\det(A) = 0$

Other expansions

Definition

the cofactor expansion of $\det(A)$ *along the first column is given by*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

Example

$$\begin{aligned} \begin{vmatrix} \lambda_1 & a & b \\ 0 & \lambda_2 & c \\ 0 & 0 & \lambda_3 \end{vmatrix} &= \lambda_1 \begin{vmatrix} \lambda_2 & c \\ 0 & \lambda_3 \end{vmatrix} - 0 \begin{vmatrix} a & b \\ 0 & \lambda_3 \end{vmatrix} + 0 \begin{vmatrix} a & b \\ \lambda_2 & c \end{vmatrix} \\ &= \lambda_1(\lambda_2\lambda_3) - 0 + 0 \\ &= \lambda_1\lambda_2\lambda_3 \end{aligned}$$

Claim

determinant of a triangular matrix is product of the diagonal entries

Special Expansion

Claim

there is another special method for 3×3 matrices

Step 1: duplicate first two columns of A

$$\Rightarrow \quad (A \mid *) = \left(\begin{array}{ccc|cc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array}\right)$$

Step 2: multiply six diagonal terms and combine

$$\left(\begin{array}{ccc|cc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array}\right) : \searrow \qquad \left(\begin{array}{ccc|cc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array}\right) : \nearrow$$

$$\Rightarrow \quad \det(A) = a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - a_3 b_2 c_1 - b_3 c_2 a_1 - c_3 a_2 b_1$$

WARNING

Warning

the previous method does not generalize to arbitrary matrices!!

Cofactor expansion

$A = (a_{i,j}) \in \mathbb{R}^{n,n} \quad \Rightarrow \quad a_{i,j}$ is entry in row i, column j

$A_{i,j} \in \mathbb{R}^{n-1,n-1}$: remove row i, column j

$C_{i,j} = (-1)^{i+j} \det(A_{i,j}) \in \mathbb{R}$: cofactor

Definition

the cofactor expansion of $\det(A)$ *along row* i *is given by*

$$\det(A) = a_{i,1}C_{i,1} + a_{i,2}C_{i,2} + \cdots + a_{i,n}C_{i,n}$$

and along column j *it is given by*

$$\det(A) = a_{1,j}C_{1,j} + a_{2,j}C_{2,j} + \cdots + a_{n,j}C_{n,j}.$$

Example: 3×3 revisited

Example

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$\Rightarrow \quad A_{1,1} = \begin{bmatrix} b_2 & c_2 \\ b_3 & c_3 \end{bmatrix}, \; A_{1,2} = \begin{bmatrix} a_2 & c_2 \\ a_3 & c_3 \end{bmatrix}, \; A_{1,3} = \begin{bmatrix} a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \quad \det(A) &= a_{1,1}C_{1,1} + a_{1,2}C_{1,2} + a_{1,3}C_{1,3} \\ &= a_1|A_{1,1}| - b_1|A_{1,2}| + c_1|A_{1,3}| \end{aligned}$$

Remark on signs

Claim

the entry in row i, column j of the matrices

$$[+], \quad \begin{bmatrix} + & - \\ - & + \end{bmatrix}, \quad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}, \quad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}, \ldots$$

is the sign of $(-1)^{i+j}$

Example: Expanding about other rows and columns

Recall

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}, \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}, \begin{bmatrix} + & - \\ - & + \end{bmatrix} : \textit{sign matrices}$$

Example

$$\begin{vmatrix} a & 0 & * & 0 \\ * & b & * & 0 \\ 0 & 0 & c & 0 \\ * & * & * & d \end{vmatrix} \overset{\text{col } 4}{=} -0+0-0+d\begin{vmatrix} a & 0 & * \\ * & b & * \\ 0 & 0 & c \end{vmatrix} = d\begin{vmatrix} a & 0 & * \\ * & b & * \\ 0 & 0 & c \end{vmatrix}$$

$$\overset{\text{row } 3}{=} d\left(0-0+c\begin{vmatrix} a & 0 \\ * & b \end{vmatrix}\right) = dc\begin{vmatrix} a & 0 \\ * & b \end{vmatrix}$$

$$\overset{\text{row } 1}{=} cd(ab-0) = abcd$$