Lecture 8

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For a function of three variables $f(x,y,z)$. Let (x_0,y_0,z_0) be a CP of f, ie, $\nabla f(x_0,y_0,z_0) = \vec{0}$. Also, suppose that all 2nd partial derivatives of f are continuous on a neighborhood of (x_0,y_0,z_0). The Hessian of f at (x_0,y_0,z_0) is defined as

$$H(x_0,y_0,z_0) = \begin{bmatrix} f_{xx}(x_0,y_0,z_0) & f_{xy}(x_0,y_0,z_0) & f_{xz}(x_0,y_0,z_0) \\ f_{yx}(x_0,y_0,z_0) & f_{yy}(x_0,y_0,z_0) & f_{zz}(x_0,y_0,z_0) \\ f_{zx}(x_0,y_0,z_0) & f_{zy}(x_0,y_0,z_0) & f_{zz}(x_0,y_0,z_0) \end{bmatrix}$$

then

a) If $H(x_0,y_0,z_0)$ is +ve definite, then f has a local min @ (x_0,y_0,z_0)

b) If $H(x_0,y_0,z_0)$ is -ve definite, then f has a local max @ (x_0,y_0,z_0)

c) If $H(x_0,y_0,z_0)$ is indefinite, then f has (x_0,y_0,z_0) as a saddle point

d) If $\det(H(x_0,y_0,z_0)) = 0$, the test is inconclusive.

In order to apply the 2nd derivative test, the following theorem is very useful. It helps us to determine whether a given symmetric matrix is either +ve definite or -ve definite or indefinite.

<u>Theorem</u>: If A is a symmetric matrix then the $k\times k$ matrix A_k obtained by deleting all except the first k rows and the 1st k columns of A is called the $k\times k$ principal minor of A. By convention $A_n = A$.

(i) A is +ve definite iff $\det(A_k) > 0$ for all k.

(ii) A is -ve definite iff $(-1)^k \det(A_k) > 0$ for all k. In other words,

if k is odd then $\det(A_k) < 0$

if k is even then $\det(A_k) > 0$

(iii) If $\det(A_{2k}) < 0$ for some k, then A is indefinite.

<u>Ex</u>: let $f(x,y,z) = x^2 + y^2 + z^2 - xy + x - 2z$. Find the CP's of f

and classify them.

Solution

$$f_x = 2x - y + 1 = 0 \Rightarrow 2(2y) - y + 1 = 0 \Rightarrow 3y + 1 = 0 \Rightarrow y = -\tfrac{1}{3}$$

$$f_y = 2y - x = 0 \Rightarrow x = 2y \Rightarrow x = -\tfrac{2}{3}$$

$$f_z = 2z - 2 = 0 \Rightarrow z = 1$$

$\therefore \left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$ is a CP.

$$f_{xx} = 2 \quad f_{xy} = -1, \quad f_{xz} = 0 \checkmark$$

$$f_{yy} = 2, \quad f_{yx} = -1, \quad f_{yz} = 0 \checkmark$$

$$f_{zz} = 2, \quad f_{xz} = 0, \quad f_{xy} = 0$$

$$\therefore \quad H\left(-\tfrac{2}{3}, -\tfrac{1}{3}, 1\right) = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} + \\ - \\ + \end{matrix}$$

$$A_1 = [2] \Rightarrow \det(A_1) = 2 > 0$$

$$A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \det(A_2) = (2)(2) - (-1)(-1) = 4 - 1 = 3 > 0$$

$$A_3 = A \Rightarrow \det(A_3) = \det(A) = 2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 2(3) = 6 > 0$$

$\therefore H\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$ is +ve definite. Hence, f has a local minimum value @ $\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$ and

$$f\left(-\tfrac{2}{3}, -\tfrac{1}{3}, 1\right) = \left(-\tfrac{2}{3}\right)^2 + \left(-\tfrac{1}{3}\right)^2 + (1)^2 - \left(-\tfrac{2}{3}\right)\left(-\tfrac{1}{3}\right) + \left(-\tfrac{2}{3}\right) - 2(1)$$

$$= \tfrac{4}{9} + \tfrac{1}{9} + 1 - \tfrac{2}{9} - \tfrac{2}{3} - 2 = -\tfrac{4}{3} \quad // \text{Ans.}$$

The Mean Value Theorem (see Stewart, p. 287)

Let f be a function that satisfies the following conditions

1. f is continuous on a closed interval $[a, b]$
2. f is differentiable on the open interval (a, b)

then there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad (1)$$

Let $\theta = \dfrac{c-a}{b-a}$ then $0 < \theta < 1$

let $h = b - a \Rightarrow c = a + \theta h$

(1) becomes

$$f(a + h) = f(a) + h f'(a + \theta h)$$

The MVT can be generalize to Taylor's theorem

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^n}{n!} f^{(n)}(a) + \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(a+\theta h)$$

for $0 < \theta < 1$ (2)

Taylor series of a function of two variables

Let $F(t) = f(a+ht, b+kt)$

Apply Taylor's theorem to $F(t)$ with $t=1$ and $t=0$

$$F(1) = F(0) + \frac{F'(0)}{1!} + \frac{F''(0)}{2!} + \cdots + \frac{F^{(n)}(0)}{n!} + \frac{1}{(n+1)!} F^{(n+1)}(\theta)$$

$0 < \theta < 1$

(3)

We also note that

$$F(1) = f(a+h, b+k)$$

$$F'(t) = h f_x(a+ht, b+kt) + k f_y(a+ht, b+kt)$$

$$F''(t) = h\left[h f_{xx}(a+ht, b+kt) + k f_{xy}(a+ht, b+kt)\right] + k\left[h f_{yx}(a+ht, b+kt) + k f_{yy}(a+ht, b+kt)\right]$$

$$= h^2 f_{xx}(a+ht, b+kt) + 2hk f_{xy}(a+ht, b+kt) + k^2 f_{yy}(a+ht, b+kt)$$

Set $t=0$,

$$F'(0) = h f_x(a,b) + k f_y(a,b)$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b)$$

$$F''(0) = h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b)$$

$$\vdots$$

$$F^{(n)}(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a,b)$$

Then (3) becomes

$$f(a+h, b+k) = f(a,b) + h f_x(a,b) + k f_y(a,b) + \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right] + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(a,b) + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a,b)$$

$$+ \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(a+\theta h, b+\theta k) \quad (4)$$

$0 < \theta < 1$

This is called Taylor's formula of $f(x,y)$. Let $n \to \infty$, (4) becomes a Taylor series.

N.B

(i) If $n = 1$, (4) becomes

$$f(a+h, b+k) = f(a,b) + h f_x(a,b) + k f_y(a,b) + \frac{1}{2!}\left[h^2 f_{xx}(a+\theta h, b+\theta k) + 2hk f_{xy}(a+\theta h, b+\theta k) + k^2 f_{yy}(a+\theta h, b+\theta k)\right] \quad (5)$$

$$0 < \theta < 1$$

We note that when h, k are very small

$$f(a+h, b+k) \approx f(a,b) + h f_x(a,b) + k f_y(a,b)$$

we have a linear approximation of f (Sec 14.4).

(ii) If $n = 2$, (4) becomes

$$f(a+h, b+k) = f(a,b) + h f_x(a,b) + k f_y(a,b) + \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right] + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(a+\theta h, b+\theta k), \quad 0 < \theta < 1$$

When h, k are very small,

$$f(a+h, b+k) \simeq f(a,b) + h f_x(a,b) + k f_y(a,b) + \frac{1}{2}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right]$$

we have a quadratic approximation.

We also note that if (a,b) is a C.P ($f_x(a,b) = 0$, $f_y(a,b) = 0$) then

$$f(a+h, b+k) - f(a,b) \simeq \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + f_{yy}(a,b)\right]$$

$$\simeq \frac{1}{2}(h, k) \underbrace{\begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}}_{H(a,b)} \begin{bmatrix} h \\ k \end{bmatrix}$$

$f(a+h, b+k) - f(a,b) \geqslant 0$ if $H(a,b)$ is +ve definite

$f(a+h, b+k) - f(a,b) \leqslant 0$ if $H(a,b)$ is -ve definite

If sometimes $f(a+h, b+k) - f(a,b) \geqslant 0$ and sometimes $f(a+h, b+k) - f(a,b) \leq 0$ when $H(a,b)$ is indefinite.

We rediscover the 2nd Derivative Test of fns of two variables.

Ex: Obtain Taylor's polynomial of degree 2 in the Taylor expansion of $f(x,y) = e^{x+y}$ by

a) using Taylor expansion (4),
b) expanding e^{x+y} in a series of $x+y$,
c) multiplying the series expansion of e^x and e^y.

Solution

a) Here $a=0$, $b=0$, $h=x$, $k=y$

from (4)

$$f(x,y)=e^{x+y} = f(0,0) + x\, f_x(0,0) + y\, f_y(0,0) + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy\, f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] \quad (*)$$

$$\begin{aligned} f_x(x,y) &= e^{x+y} & \Rightarrow\ & f_x(0,0)=1 \\ f_y(x,y) &= e^{x+y} & & f_y(0,0)=1 \\ f_{xx}(x,y) &= e^{x+y} & & f_{xx}(0,0)=1 \\ f_{yy}(x,y) &= e^{x+y} & & f_{yy}(0,0)=1 \\ f_{xy}(x,y) &= e^{x+y} & & f_{xy}(0,0)=1 \end{aligned}$$

Subst these into $(**)$,

$$\begin{aligned} e^{x+y} &= 1 + x + y + \frac{1}{2}\left[x^2 + 2xy + y^2\right] \\ &= 1 + x + y + \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 \quad // \text{Ans.} \end{aligned}$$

b) See Stewart, Sec 11.10

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\begin{aligned} \therefore\ e^{x+y} &= \sum_{n=0}^{n} \frac{(x+y)^n}{n!} \\ &\simeq \sum_{n=0}^{2} \frac{(x+y)^n}{n!} \\ &\simeq 1 + (x+y) + \frac{1}{2!}(x+y)^2 \\ &\simeq 1 + x + y + \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 \quad // \text{Ans.} \end{aligned}$$

c)
$$\begin{aligned} e^x e^y &= \left(1 + x + \frac{x^2}{2!} + \dots\right)\left(1 + y + \frac{y^2}{2!} + \dots\right) \\ &\simeq 1 + x + y + \frac{y^2}{2!} + xy + \frac{x^2}{2!} \\ &\simeq 1 + x + y + \frac{y^2}{2} + xy + \frac{x^2}{2} \quad // \text{Ans.} \end{aligned}$$

See you on Monday!