For a function of three variables $f(x,y,z)$, let (x_0, y_0, z_0) be a critical point of f. i.e. $\nabla f(x,y,z) = \vec{0}$ Also, suppose that all 2nd partial derivities of f are continuous on a neighborhood of (x_0, y_0, z_0) The Hessian of f at (x_0, y_0, z_0) is defined as follow:

$$H(x_0, y_0, z_0) = \begin{bmatrix} f_{xx}(x_0,y_0,z_0) & f_{xy}(x_0,y_0,z_0). & f_{xz}(x_0,y_0,z_0) \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

then a) If $H(x_0, y_0, z_0)$ is positive definite, then f has a local minimum at (x_0, y_0, z_0).

b) If negative

maximum

c) If $H(x_0, y_0, z_0)$ is indefinite, then f has (x_0, y_0, z_0) as a saddle point.

d) If $\det H(x_0, y_0, z_0) = 0$, the test is inconclusive.

In order to apply 2nd derivtive test, the following therom is very useful. It helps us to determin whether a given symmetric matrix is either positive or negative or indefinite.

Theorem: If A is a symmetric matrix then the $k \times k$ matrix A_k obtain by deleting all except the first k row and first k columns of A is called the $k \times k$ principle minor of A. By conversion $A_n = A$.

1) A is positive definite iff $\det(A_k) > 0$ for all k.

2) negative $(-1)^k \det(A_k) > 0$.

$\Rightarrow$ if k is odd, $A_k < 0$ / even, $A_k > 0$.

3). If $\det(A_{2k}) < 0$, then A is indefinite.

e.g. 1. $f(x,y,z) = x^2 + y^2 + z^2 - xy + x - 2z$. Find the critical point of f and classify them.

$f_x = 2x - y + 1 = 0$ $\qquad x = -\frac{2}{3}$.

$f_y = 2y - x = 0$ $\quad \Rightarrow \quad y = -\frac{1}{3}$

$f_z = 2z - 2 = 0$. $\qquad z = 1$

$\Rightarrow$ the only critical point is $(-\frac{2}{3}, -\frac{1}{3}, 1)$.

$f_{xx} = 2 \quad f_{xy} = -1 \quad f_{xz} = 0$

$f_{yx} = -1 \quad f_{yy} = 2 \quad f_{yz} = 0$.

$f_{zx} = 0 \quad f_{zy} = 0 \quad f_{zz} = 2$.

$$H(-\tfrac{2}{3}, -\tfrac{1}{3}, 1) = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$A_1 = [2]$. $\det A_1 = 2 > 0$

$A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $\det A_2 = 3 > 0$.

$A_3 = H$. $\det A_3 = 2 \cdot \det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 6 > 0$

$\Rightarrow H(-\frac{2}{3}, -\frac{1}{3}, 1)$ is positive defined,

the point is a local minimum and the minimum value is $-\frac{4}{3}$.

The mean value therom:

Let f be a function that satisfy the following conditions

1. f is continuous on a closed interval $[a,b]$.

2. f is differentiable on (a,b).

there is a number $c \in (a,b)$. such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Let $\theta = \frac{c-a}{b-a}$. $\Rightarrow \theta \in (0,1)$

$h = b-a \Rightarrow c = a+\theta h$

$\Rightarrow f(a+h) = f(a) + hf'(a+\theta h)$.

Generalize to Taylor's therom:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^n}{n!}f^{n}(a) + \frac{h^{n+1}}{(n+1)!}f^{n+1}(a+\theta h)$$

Taylor series of a function of two variables.

Let $F(t) = f(a+ht, b+kt)$

Apply Taylor's therom to $F(t)$ with $t=1$ and $t=0$.

$$F(1) = F(0) + F'(0)/1! + F''(0)/2! \cdots F^{(n)}(0)/n! + \frac{F^{n+1}(\theta)}{(n+1)!}$$

$$= f(a+h, b+k).$$

$$F'(t) = hf_x(a+ht, b+kt) + kf_y(a+ht, b+kt).$$

$$F''(t) = h[hf_{xx}(a+ht, b+kt) + kf_{xy}(a+ht, b+kt)]$$
$$+ k[hf_{yx}(a+ht, b+kt) + kf_{yy}(a+ht, b+kt)].$$
$$= h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}.$$

set $t=0$.

$$F'(0) = hf_x(a,b) + kf_y(a,b) = \left(h\frac{d}{dx} + k\frac{d}{dy}\right)f(a,b).$$

$$F''(0) = h^2 f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2 f_{yy}(a,b).$$
$$= \left(h\frac{d}{dx} + k\frac{d}{dy}\right)^2 f(a,b).$$

$\vdots$

$$f^{n}(0) = \left(h\frac{d}{dx} + k\frac{d}{dy}\right)^n f(a,b).$$

$$f(a+h, b+k) = f(a,b) + \frac{\left(h\frac{d}{dx} + k\frac{d}{dy}\right)^1 f(a,b)}{1!} + \frac{\left(h\frac{d}{dx} + k\frac{d}{dy}\right)^2 f(a,b)}{2!}$$

$$+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n} f(a,b)+\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(a+\theta h, b+\theta k)$$

let $n\to\infty$. $f(a+h, b+h)$ becomes the taylor series.

$$f(a+h, b+k)=\sum_{n=0}^{n}\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n} f(a,b)$$
$$+\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(a+\theta h, b+\theta k).$$

(i) $n=1$ If $h,k\to 0$ $f(a+h,b+k)\to \underbrace{f(a,b)+hf_x(a,b)+kf_y(a,b)}_{\text{linear approximation of } f.}$

(ii) $n=2$. If $h,k\to 0$, $f(a+h,b+k)\to \underline{f(a,b)+hf_x(a,b)+kf_y(a,b)}$

$$\underbrace{+\tfrac{1}{2!}\left[h^2 f_{xx}(a,b)+2hk f_{xy}(a,b)+k^2 f_{yy}(a,b)\right]}_{\text{quadratic approximation}}$$

If (a,b) is a critical point,

then $f(a+h,b+k)-f(a,b)\to \frac{1}{2!}(h,k)\underbrace{\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{bmatrix}}_{H(a,b).}\begin{bmatrix} h \\ k\end{bmatrix}.$

$f(a+h,b+k)-f(a,b)\geq 0$

if $H(a,b)$ is positive definite.

------ <0, negative

-------- $\geq 0/<0$ sometimes, ----- indefinite.

e.g. 2 Obtain Taylor's polynomial of degree 2 in the taylor expension of $f(x,y)=e^{x+y}$ by

a) using Taylor expension

b) expending e^{x+y} in a series of $x+y$.

c) multiplying this series expension of e^x and e^y.

Sol: $a=0$, $b=0$ $h=x$ $k=y$.

degree 1

a) $f(x,y)=e^{x+y}=f(0,0)+xf_x(0,0)+yf_y(0,0)$;

$$+ \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{yx}(0,0) + y^2 f_{yy}(0,0)\right]$$

degree 2 →

$$= 1 + x + y + \frac{1}{2}(x^2 + 2xy + y^2).$$

b). $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$e^{x+y} = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} \simeq \sum_{n=0}^{2} \frac{(x+y)^n}{n!} = 1 + (x+y) + \frac{1}{2}(x+y)^2.$$

c). $e^x e^y = \sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot \sum_{n=0}^{\infty} \frac{y^n}{n!}$

$$\simeq (1 + x + \frac{x^2}{2}) \cdot (1 + y + \frac{y^2}{2}).$$