§ 2.1.
Cont

nested quantifiers.

$u = \mathbb{Z}$.

$\forall x\ \underbrace{\exists y\ x+y=5}$; (T)

proof: $y = 5 - x$.

$\exists y\ \underbrace{\forall x\ \ x+y=5}_{P(y)}$. (F)

=> you cannot find a specific y that meet the statement

$\exists x\ \exists y\ \ x+y=5$ (T) * when the quantifier is different, be careful with the order.

$\exists y\ \exists x\ \ x+y=5$ (T)

$\forall x\ \forall y\ \ x+y=5$ (F)

$\forall y\ \forall x\ \ x+5=5$ (F)

§ 2.2.

equivalences individing quantifiers. like quantifiers commute:

$\forall x \forall y\ P(x,y)$ iff $\forall y \forall x\ P(x,y)$

$\exists x \exists y\ P(x,y)$ iff $\exists y \exists x\ P(x,y)$.

Ex: $\forall x \in \mathbb{R}\ \ \forall n \in \mathbb{N}\ \ \ x+n \geq x$ (T)

$\forall n \in \mathbb{N}\ \ \forall x \in \mathbb{R}\ \ \ x+n \geq x$ (T).

"Nobody is perfect"

$\neg(\exists x\ \ P(x))$

$\forall x\ \ \neg P(x)$.

Not everyone is perfect.

$\exists x\ \ \neg P(x)$.

$\neg \forall x\ \ P(x)$.

=> $\exists x\ \ \neg P(x)$ iff $\neg \forall x\ \ P(x)$.

Ex: negate $A \subseteq B$.

$\neg \forall x\ \ x \in A \to x \in B$.

iff $\exists x\ \ \neg(x \in A \to x \in B)$.

iff $\exists x\ \ \neg(x \notin A \vee x \in B)$

iff $\exists x \ (x \in A \wedge x \notin B)$.

$(0, 1, 4, 9, 16 \cdots)$.

Ex: Negate: Every natural number is a perfect square

$U = \mathbb{N} \quad \neg(\forall n \ S(n))$.

iff $\neg(\forall n \ \exists k \in \mathbb{N} \ \ n = k^2)$

iff $\exists n \ \neg(\exists k \in \mathbb{N} \ \ n = k^2)$

iff $\exists n \ \forall k \in \mathbb{N} \ \ n \neq k^2$ (T) e.g. $k = 3$.

Bounded quantifiers. (nothing to do with free & bound vars)

$\forall x \in A$

$\forall y$ <= both "bound" iff x is "bounded"

$\forall x \in A \ P(x)$: $\forall x \ (x \in A \wedge P(x))$.

$\exists x \in A \ P(x)$: $\exists x \ (x \in A \wedge P(x))$

Ex: $x \in \{2, 3, 4\}$. $x^2 \geq 4$

iff $\forall x \quad x \in \{2, 3, 4\} \wedge x^2 \geq 4$ (T)

$\exists x \in \{2, 3, 4\} \quad x^2 = 9$.

iff $\exists x \quad x \in \{2, 3, 4\} \wedge x^2 = 9$ (T).

* the quantifiers used here are bounded quantifiers.

Note: when we use a universal discourse, all quantifiers are bounded,

Bounded quantifier and negations:

$\neg \exists x \in A \ P(x)$.

iff $\neg(\exists x \ (x \in A \wedge P(x)))$

iff $\forall x \ \neg(x \in A \wedge P(x))$

iff $\forall x \ \ x \notin A \vee \neg P(x)$.

iff $\forall x \ \ x \in A \rightarrow \neg P(x)$

iff $\forall x \in A \ \ \neg P(x)$.

$\neg x \in A \ \ P(x)$

iff $\exists x \in A \ \neg P(x)$.

Note: when $A = \emptyset \longrightarrow x \in A$ is False, $P(x)$ is False.

$\forall x \in A \ \ P(x)$.

$\neg(\exists x \in A \ \ \neg P(x)) \Rightarrow T$

$\exists x \in A \quad P(x) \Rightarrow F.$

Distributive laws:

$\forall x\,(P(x) \wedge Q(x)) \Leftrightarrow (\forall x\, P(x)) \wedge (\forall x\, Q(x))$

$\forall x\,(P(x) \vee Q(x)) \not\Leftrightarrow (\forall x\, P(x)) \vee (\forall x\, Q(x)).$

e.g. $x \in \mathbb{R}$ $\quad P(x): x \geq 0 \quad P(x)\ F$

$\qquad\qquad\quad Q(x): x \leq 0 \quad Q(x)\ F$

$\exists x\,(P(x) \wedge Q(x)) \not\Leftrightarrow (\exists x\, P(x)) \wedge (\exists x\, Q(x)).$

these x could be not the same x.

$\exists x\,(P(x) \vee Q(x)) \Leftrightarrow (\exists x\, P(x)) \vee (\exists x\, Q(x)).$