The Foundations: Logic and Proofs

Chapter 1, Part I: Propositional Logic

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Chapter Summary

1. Part I: Propositional Logic
   a. The Language of Propositions
   b. Applications
   c. Logical Equivalences
2. Part II: Predicate Logic
   a. The Language of Quantifiers
   b. Logical Equivalences
   c. Nested Quantifiers
3. Part III: Proofs
   a. Rules of Inference
   b. Proof Methods
   c. Proof Strategy

Plan for Part I

1. The Language of Propositions

1.1 Propositions

1.2 Connectives

1.3 Truth Tables and Compound Propositions

2. Applications

2.1 Translating English to Propositional Logic

2.2 System Specifications

2.3 Logic Puzzles

2.4 Logic Circuits

3. Propositional Equivalences

3.1 Tautologies, Contradictions, and Contingencies

3.2 Logical Equivalence

3.3 Propositional Satisfiability

Plan for Part I

Propositions

1. A *proposition* is a declarative sentence that is either true or false.
2. Examples of propositions:
   a. The Moon is made of green cheese.
   b. Toronto is the capital of Canada.
   c. $1 + 0 = 1$
   d. $0 + 0 = 2$
3. Examples that are not propositions:
   a. Sit down!
   b. What time is it?
   c. $x + 1 = 2$
   d. $x + y = z$

Propositional Logic

1. Constructing Propositions formally
    a. The proposition that is always true, denoted by **T**, and the proposition that is always false, denoted by **F**.
    b. Propositional variables, usually denoted p, q, r, s, take values **T** or **F**.
    c. Compound Propositions; constructed from other propositions by means of *logical connectives*:
        1. Negation $\neg$
        2. Conjunction $\wedge$
        3. Disjunction $\vee$
        4. Implication $\rightarrow$
        5. Biconditional $\leftrightarrow$ iff
2. Examples of formal propositions
    a. **T**,
    b. $p \vee q$,
    c. $(p \vee q) \wedge (p \vee \neg q)$,
    d. $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$.

Plan for Part I

Compound Propositions: Negation

1. The *negation* of a proposition p is denoted by $\neg p$ and has this truth table:

p	¬p
T	F
F	T

2. **Example**: If p denotes "The earth is round.", then $\neg p$ denotes "It is not the case that the earth is round," or more simply "The earth is not round."

Conjunction

1. The *conjunction* of propositions p and q is denoted by $p \wedge q$ and has this truth table:

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

2. **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \wedge q$ denotes "I am at home and it is raining."

Disjunction

1. The *disjunction* of propositions p and q is denoted by $p \vee q$ and has this truth table:

p	q	p ∨q
T	T	T
T	F	T
F	T	T
F	F	F

2. **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \vee q$ denotes "I am at home or it is raining."

In English "or" has two distinct meanings

1. "Inclusive Or'
   a. In the sentence "Students who have taken CS202 or Math120 may take this class," we assume that students need to have taken one of the prerequisites, but may have taken both.
   b. This is the meaning of disjunction.
   c. For $p \vee q$ to be true, either one or both of p and q must be true.
2. "Exclusive Or"
   a. When reading the sentence "Soup or salad comes with this entr(é)e," we do not expect to be able to get both soup and salad.
   b. This is the meaning of Exclusive Or (Xor).
   c. In $p \oplus q$, one of p and q must be true, but not both. The truth table for $\oplus$ is:

p	q	p $\oplus$q
T	T	F
T	F	T
F	T	T
F	F	F

Implication

1. If p and q are propositions, then $p \to q$ is a *conditional statement* or *implication* which is read as "if p, then q" and has this truth table:

p	q	p →q
T	T	T
T	F	F
F	T	T
F	F	T

2. **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \to q$ denotes "If I am at home then it is raining."
3. In $p \to q$, p is the *hypothesis* (*antecedent* or *premise*) and q is the *conclusion* (or *consequence*).

Understanding Implication

1. In $p \rightarrow q$ there does not need to be any connection between the antecedent or the consequent.
2. The "meaning" of $p \rightarrow q$ depends only on the truth values of p and q.
3. These implications are perfectly fine, but would not be used in ordinary English:
   a. "If the moon is made of green cheese, then I have more money than Bill Gates. "
   b. ""If $1 + 1 = 3$, then my grandma wears combat boots."

Understanding Implication (cont)

1. One way to view the logical conditional is to think of an obligation or contract.
   a. "If I am elected, then I will suppress income tax."
2. If the politician is elected and does not suppress income tax, then the voters can say that the politician has broken the campaign pledge.

Different Ways of Expressing $p \to q$

1. **if** p, **then** q
2. p **implies** q
3. **if** p, q
4. p **only if** q
5. q **unless** $\neg p$
6. q **when** p
7. q **if** p
8. q **when** p
9. q **whenever** p
10. p **is sufficient for** q
11. q **follows from** p
12. q **is necessary for** p
13. **a necessary condition for** p **is** q
14. **a sufficient condition for** q **is** p

Converse, Contrapositive, and Inverse

1. From $p \rightarrow q$ we can form new conditional statements .
   a. $q \rightarrow p$ is the **converse** of $p \rightarrow q$,
   b. $\neg q \rightarrow \neg p$ is the **contrapositive** of $p \rightarrow q$,
   c. $\neg p \rightarrow \neg q$ is the **inverse** of $p \rightarrow q$.
2. **Example**: Find the converse, inverse, and contrapositive of "It is raining is a sufficient condition for my not going to town."

Converse, Contrapositive, and Inverse

1. From $p \rightarrow q$ we can form new conditional statements .
   a. $q \rightarrow p$ is the **converse** of $p \rightarrow q$
   b. $\neg q \rightarrow \neg\, p$ is the **contrapositive** of $p \rightarrow q$
   c. $\neg\, p \rightarrow \neg\, q$ is the **inverse** of $p \rightarrow q$
2. **Example**: Find the converse, contrapositive, and inverse of "It raining is a sufficient condition for my not going to town."
3. **Solution:**
   a. **converse**: If I do not go to town, then it is raining.
   b. **contrapositive**: If I go to town, then it is not raining.
   c. **inverse**: If it is not raining, then I will go to town.

Biconditional

1. If p and q are propositions, then we can form the *biconditional* proposition$p \leftrightarrow q$, read as "p if and only if q ."
2. The biconditional $p \leftrightarrow q$ denotes the proposition with this truth table:

p	q	p ↔q
T	T	T
T	F	F
F	T	F
F	F	T

3. If p denotes "I am at home." and q denotes "It is raining." then $p \leftrightarrow q$ denotes "I am at home if and only if it is raining."

Expressing the Biconditional

1. Some alternative ways "p if and only if q" is expressed in English:
   a. p **is necessary and sufficient for** q
   b. **if** p **then** q **, and conversely**
   c. p **iff** q

Plan for Part I

Truth Tables For Compound Propositions

1. Construction of a truth table:
    a. It need a row for every possible combination of values for the atomic propositions.
    b. It needs a column for the compound proposition (usually at far right)
    c. It also needs a column for the truth value of each sub-expression that occurs in the compound proposition as it is built up.
    d. This includes the atomic propositions

Example of a Truth Table

Exercise: Construct a truth table for $p \vee q \to \neg r$

p	q	r	$\neg r$	$p \vee q$	$p \vee q \to \neg r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Equivalent Propositions

1. Two propositions are logically *equivalent* if they always have the same truth value. (notation $A \equiv B$)
2. **Example**: Show using a truth table that the *implication* is equivalent to the *contrapositive*.
3. **Solution:**

p	q	¬ p	¬ q	p →q	¬q → ¬ p
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

$$p \to q \equiv \neg q \to \neg p$$

Using a Truth Table to Show Non-Equivalence

1. **Example**: Show using truth tables that neither the *converse* nor *inverse* of an implication are equivalent to the *implication*
2. **Solution:**

p	q	$\neg$ p	$\neg$ q	p $\rightarrow$q	q $\rightarrow$ p	$\neg$ p $\rightarrow\neg$ q
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

NOTE: *converse* and *inverse* are equivalent to each other

Extra exercise:

Prove that *biconditional* $p \leftrightarrow q$ is equivalent to the conjunction of *implication* $p \rightarrow q$ and its *converse* $q \rightarrow p$, which is $(p \rightarrow q) \wedge (q \rightarrow p)$

Problem

1. How many rows are there in a truth table with n propositional variables?

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1. How many rows are there in a truth table with n propositional variables?
2. **Solution**: 2^n We will see how to do this in Chapter 6.
3. Note that this means that with n propositional variables, we can construct 2^{n+1} distinct (i.e., not equivalent) propositions. Indeed, for each row, a given proposition is either true or false.

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

1. $p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$
2. If the intended meaning is $p \vee (q \rightarrow \neg r)$ then parentheses must be used.

Plan for Part I

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Plan for Part I

Translating English Sentences

1. Steps to convert an English sentence to a statement in propositional logic
   a. Identify *atomic* propositions and represent using propositional variables.
   b. Determine appropriate logical connectives
2. "If I go to Harry's or to the country, I will not go shopping."
   a. p: I go to Harry's
   b. q: I go to the country.
   c. r: I will go shopping.
   d. If p or q then not r
   e. $(p \vee q) \rightarrow \neg r$

State

English

Symbolic.

Example

1. **Problem:** Translate the following sentence into propositional logic:
2. "You can access the Internet from campus only if you are a computer science major or you are not a freshman."

p q $\neg r$

1. p
2. q
3. r
4. if p then q or not r
5. $p \to (q \vee \neg r)$

Example

1. **Problem:** Translate the following sentence into propositional logic:
2. "You can access the Internet from campus only if you are a computer science major or you are not a freshman."
3. **One Solution**: Let a, c, and f represent respectively "You can access the internet from campus," "You are a computer science major," and "You are a freshman."
4. $a \to (c \vee \neg f)$

Plan for Part I

System Specifications

1. System and Software engineers take requirements in English and express them in a precise specification language based on logic.
2. **Example**: Express in propositional logic: "The automated reply cannot be sent when the file system is full"
3. **Solution**: One possible solution: Let p denote "The automated reply can be sent" and q denote "The file system is full."
4. $q \rightarrow \neg p$

Consistent System Specifications

1. **Definition**: A list of propositions is *consistent* if it is possible to assign truth values to the proposition variables so that each proposition is true.
2. **Exercise**: Are these specifications consistent?
   a. "The diagnostic message is stored in the buffer or it is retransmitted."
   b. "The diagnostic message is not stored in the buffer."
   c. "If the diagnostic message is stored in the buffer, then it is retransmitted."

Consistent System Specifications

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   b. "The diagnostic message is not stored in the buffer."
   c. "If the diagnostic message is stored in the buffer, then it is retransmitted."
3. **Solution**:
   a. Let p denote "The diagnostic message is stored in the buffer."
   b. Let q denote "The diagnostic message is retransmitted".
   c. The specification can be written as: $p \vee q$, $\neg p$, $p \rightarrow q$.
   d. When p is false and q is true all three statements are true. So the specification is consistent.
4. **New Exercise**: What if "The diagnostic message is not retransmitted" is added?
5. **New Solution**: Now we are adding $\neg q$ and there is no satisfying assignment. So the specification is not consistent.

Plan for Part I

Logic Puzzles

1. An island has two kinds of inhabitants, *knights*, who always tell the truth, and *knaves*, who always lie.
2. You go to the island and meet A and B.
   a. A says "B is a knight."
   b. B says "The two of us are of opposite types."
3. **Puzzle :** What are the types of A and B?

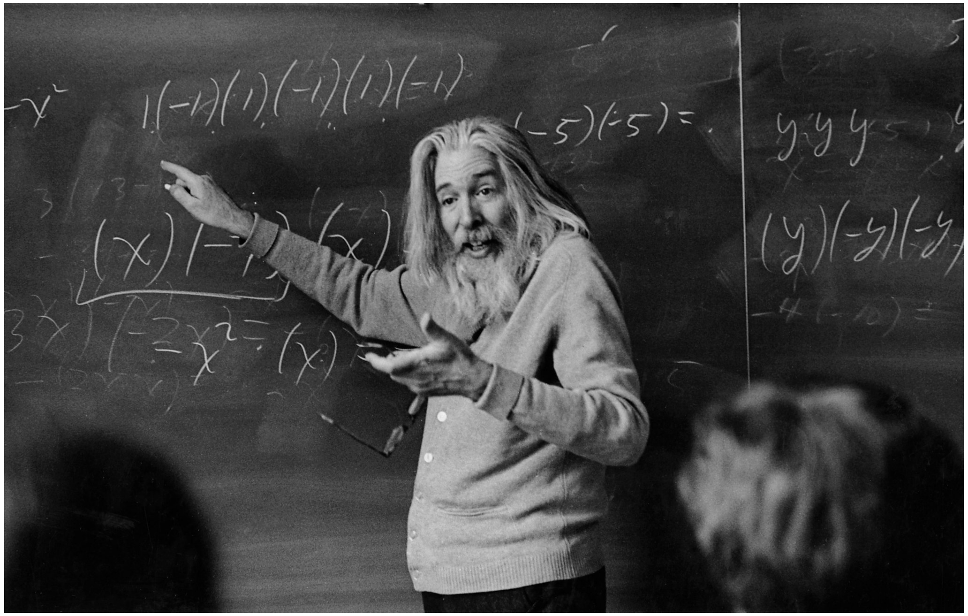

Figure: *"Man can never eliminate the necessity of using his own intelligence, regardless of how cleverly he tries!"*— Raymond M. Smullyan (May 25, 1919 – February 6, 2017), The Gödelian Puzzle Book: Puzzles, Paradoxes and Proofs

Logic Puzzles

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2. You go to the island and meet A and B.
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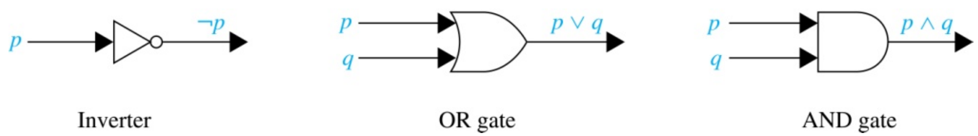
   b. B says "The two of us are of opposite types."
3. **Puzzle :** What are the types of A and B?
4. **Solution:** Let p and q be the statements that A is a knight and B is a knight, respectively. So, then $\neg p$ represents the proposition that A is a knave and $\neg q$ that B is a knave.
   a. If A is a knight, then p is true. Since knights tell the truth, q must also be true. Then $(p \wedge \neg q) \vee (\neg p \wedge q)$ would have to be true, but it is not. So, A is not a knight and therefore $\neg p$ must be true.
   b. If A is a knave, then B must not be a knight since knaves always lie. So, then both $\neg p$ and $\neg q$ hold since both are knaves.

Plan for Part I

Logic Circuits (Studied in depth in CS2209)

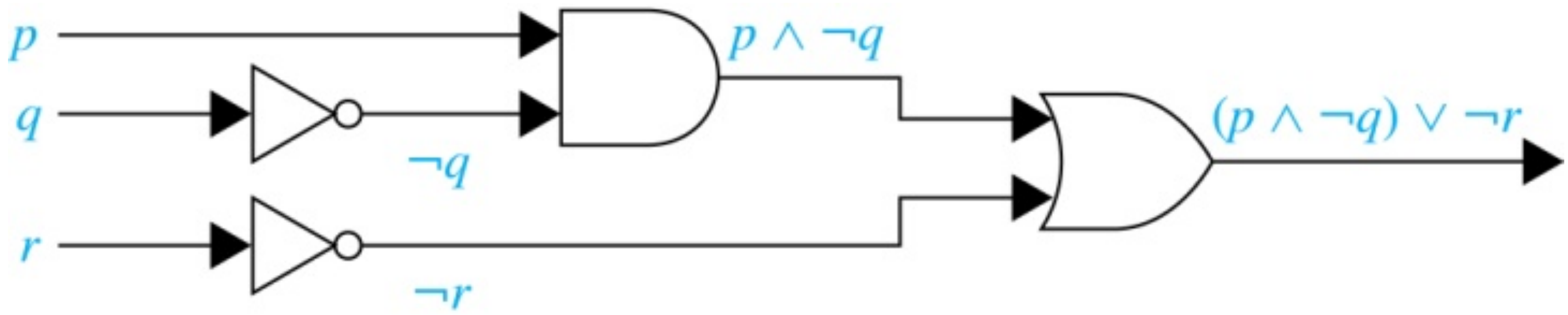
1. Electronic circuits; each input/output signal can be viewed as a 0 or 1.
   a. 0 represents **False**
   b. 1 represents **True**
2. Complicated circuits are constructed from three basic circuits called gates.



   a. The inverter (**NOT gate**) takes an input bit and produces the negation of that bit.
   b. The **OR gate** takes two input bits and produces the value equivalent to the disjunction of the two bits.
   c. The **AND gate** takes two input bits and produces the value equivalent to the conjunction of the two bits.

Logic Circuits (Studied in depth in CS2209)

More complicated digital circuits can be constructed by combining these basic circuits to produce the desired output given the input signals by building a circuit for each piece of the output expression and then combining them. For example:



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Plan for Part I

Tautologies, Contradictions, and Contingencies

1. A *tautology* is a proposition which is always true.
   a. Example: $p \vee \neg p$
2. A *contradiction* is a proposition which is always false.
   a. Example: $p \wedge \neg p$
3. A *contingency* is a proposition which is neither a tautology nor a contradiction.
   a. Example: p

p	¬p	p ∨¬p	p ∧¬p
T	F	T	F
F	T	T	F

Plan for Part I

Logical Equivalence

1. Two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a *tautology*.
2. We write this as $p \Leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions.
3. Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree.
4. The truth table below shows that $\neg p \vee q$ is equivalent to $p \rightarrow q$:

p	q	¬p	¬p ∨ q	p→ q
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Figure: *"I did not hear what you said, but I absolutely disagree with you."* –Attributed to Augustus De Morgan (1806-1871) in: August Stern (1994). The Quantum Brain: Theory and Implications. North-Holland/Elsevier. p. 7

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

The truth table below shows that De Morgan's Second Law holds:

p	q	$\neg$p	$\neg$q	(p$\vee$q)	$\neg$(p$\vee$q)	$\neg$p$\wedge\neg$q
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Key Logical Equivalences

1. Identity Laws:
   a. $p \wedge T \equiv p$
   b. $p \vee F \equiv p$
2. Domination Laws:
   a. $p \vee T \equiv T$
   b. $p \wedge F \equiv F$
3. Idempotent laws:
   a. $p \vee p \equiv p$
   b. $p \wedge p \equiv p$
4. Double Negation Law:
   a. $\neg(\neg p) \equiv p$
5. Negation Laws:
   a. $p \vee \neg p \equiv T$
   b. $p \wedge \neg p \equiv F$

Key Logical Equivalences (*cont*)

1. Commutative Laws:
   a. $p \vee q \equiv q \vee p$
   b. $p \wedge q \equiv q \wedge p$
2. Associative Laws:
   a. $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
   b. $(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. Distributive Laws:
   a. $(p \vee (q \wedge r)) \equiv (p \vee q) \wedge (p \vee r)$
   b. $(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$
4. Absorption Laws:
   a. $p \vee (p \wedge q) \equiv p$
   b. $p \wedge (p \vee q) \equiv p$

More Logical Equivalences

TABLE 7 Logical Equivalences Involving Conditional Statements.
$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.
$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

These valid logical equivalences could be used as a valid argument form in proofs (later)

More Logical Equivalences

$$(p \wedge (p \to q)) \to q \equiv T$$

p	q	$p \to q$	$p \wedge (p \to q)$	$(p \wedge (p \to q)) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

1. The tautology above is known as "Modus Podens" (one of the rules of inference)
2. It establishes validity of the following argument form:

if p **and** $p \to q$ **then** q

Constructing New Logical Equivalences

1. We can show that two expressions are logically equivalent by developing a series of logically equivalent statements.
2. To prove that $A \equiv B$ we produce a series of equivalences beginning with A and ending with B:

$$A \equiv A_1$$

$$\vdots$$

$$A_n \equiv B$$

3. Keep in mind that whenever a proposition (represented by a propositional variable) occurs in the equivalences listed earlier, it may be replaced by an arbitrarily complex compound proposition.

Equivalence Proofs

Example: Show that $\neg(p \vee (\neg p \wedge q))$ is logically equivalent to $\neg p \wedge \neg q$

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Solution:

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by 2nd De Morgan law} \\
&\equiv \neg p \wedge (\neg(\neg p) \vee \neg q) && \text{by first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by 2nd distrib. law} \\
&\equiv F \vee (\neg p \wedge \neg q) && \text{by the negation law} \\
&\equiv (\neg p \wedge \neg q) \vee F && \text{by commutative law for disjunction} \\
&\equiv (\neg p \wedge \neg q) && \text{by identity law for F}
\end{aligned}
$$

Equivalence Proofs

Example: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

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Solution:

$$
\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by truth table for } \rightarrow \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by first De Morgan law} \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by associative and commutative} \\
& && \text{laws for disjunction} \\
&\equiv T \vee T && \text{by truth tables} \\
&\equiv T && \text{by domination law}
\end{aligned}
$$

Plan for Part I

Propositional Satisfiability

1. A compound proposition is *satisfiable* if there is an assignment of truth values to its variables that make it true.
2. When no such assignments exist, the compound proposition is *unsatisfiable*.
3. A compound proposition is *unsatisfiable* if and only if its negation is a *tautology*.
4. A compound proposition is *unsatisfiable* if and only if it is a *contradiction*.

Questions on Propositional Satisfiability

Example: Determine the satisfiability of the following compound propositions:

1. $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$

   Solution: Satisfiable. Assign **T** to p, q, and r.

2. $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$

   Solution: Satisfiable. Assign **T** to p and F to q.

3. $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$

   Solution: Not satisfiable. Check each possible assignment of truth values to the propositional variables and none will make the proposition true.