

Matrix multiplication

Matrix

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We use capital letters, such as A , B and C , to denote matrices.

The entries of a matrix are denoted by the corresponding lowercase letter with “double subscript”, such as a_{ij} . And a_{ij} is called the (i,j) -entry.

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In this way, to describe a matrix, is to know what the entries a_{ij} are.

A special case

Definition If $A = [a_{1j}]$ is a $1 \times n$ matrix (aka, a row vector) and $B = [b_{i1}]$ is an $n \times 1$ matrix (aka, a column vector), the *matrix product* AB is a 1×1 matrix whose entry is given by

$$a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1n}b_{n1}.$$

That is to say,

$$AB = [a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1n}b_{n1}].$$

Example

Find the following product matrices.

(a) AB , where $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and $B^T = \begin{bmatrix} 3 & 4 \end{bmatrix}$.

(b) BA^T , where $A = \begin{bmatrix} 1 & 0 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 & 2 & 5 \end{bmatrix}$.

(c) $C^T C$, where $C^T = \begin{bmatrix} 2 & 1 & -3 \end{bmatrix}$.

Matrix multiplication

Let A be an $m \times n$ matrix and let B be an $n \times s$ matrix (i.e., the number of columns of A equals to the number of the rows of B).

Then the *product* $C = AB$ is an $m \times s$ matrix, defined by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$$

for $1 \leq i \leq m$ and $1 \leq j \leq s$.

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Alternatively, the rows of the matrix A and the columns of B give vectors in \mathbb{R}^n .

Let $\vec{a}_i = (a_{i1}, a_{i2}, \dots, a_{in})$ (the i -th row of A) and $\vec{b}^j = (b_{1j}, b_{2j}, \dots, b_{nj})$ (the j -th column of B) be two vectors in \mathbb{R}^n . Then $c_{ij} = \vec{a}_i \cdot \vec{b}^j$.

Lecture Note Example 7.6.

Consider the matrices shown here:

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & -2 & 5 \\ 0 & 3 & 4 & -1 \\ 5 & -1 & 2 & 4 \end{bmatrix}$$

How many different matrix products of the form M_1, M_2 are defined, where each of M_1 and M_2 is either one of the given matrices or the transpose of one of the given matrices?

Examples

Given the pairs of matrices A and B , find the product AB and BA , if defined.

(1)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -1 & -2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$$

(2)

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 3 & 6 & 5 \\ 2 & -4 & 7 \end{bmatrix} \quad B = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(3) Find CC^T , where $C^T = [2 \quad 3 \quad -1 \quad 4]$.

(4) Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

What is AI_n ? How about $I_m A$?

(5) Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

Find AB and BA . Show that $AB \neq BA$.

Definition Let A be a square matrix of order n (i.e., an $n \times n$ matrix). Then

$$A^1 = A$$

$$A^2 = AA$$

$$A^3 = AA^2$$

$$\vdots$$

$$A^k = AA^{k-1} \quad (k \geq 2)$$

Examples

(1) Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$$

Find matrices A^1 , A^2 , A^3 and A^4 .

(2) Consider the following matrices (the first two are diagonal matrices)

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Find I^k , A^k and B^k for $k \geq 1$.

Theorem Properties of Operations for Matrices

Let A , B and C be matrices. Let a and b be scalars. Assume that the dimensions of the matrices are such that each operation is defined.

1. $A + B = B + A$ (matrix addition is commutative)
2. $A + (B + C) = (A + B) + C$ (matrix addition is also associative)

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5. $A(BC) = (AB)C$ (matrix multiplication is associative)
6. $A I_n = A$ and $I_m A = A$, where A is an $m \times n$ matrix

7. $A(B + C) = AB + AC$ and $(B + C)A = BA + CA$
(matrix multiplication is distributive over matrix addition)

8. $a(B + C) = aB + aC$
(scalar multiplication of matrices is distributive over matrix addition)

9. $(a + b)C = aC + bC$
(scalar multiplication of matrices is also distributive over scalar addition)

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10. $(ab)C = a(bC)$

11. $1A = A$ Note: 1 is the scalar 1.

12. $A0 = 0$ and $0A = 0$, where 0 denotes a zero matrix and the two zero matrices have appropriate dimensions.

13. $a0 = 0$

14. $a(AB) = (aA)B = A(aB)$

15. $(A + B)^T = A^T + B^T$

(matrix transposition is distributive over matrix addition)

16. $(AB)^T = B^T A^T$

(matrix transposition is distributive over matrix multiplication, but the order of multiplication is reversed)

Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 3 & 2 \\ 0 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 1 & 0 \\ 0 & 3 & 3 \end{bmatrix}$$

Find $A(BC)$, $(AB)C$, $A(B + C)$ and $AB + AC$.