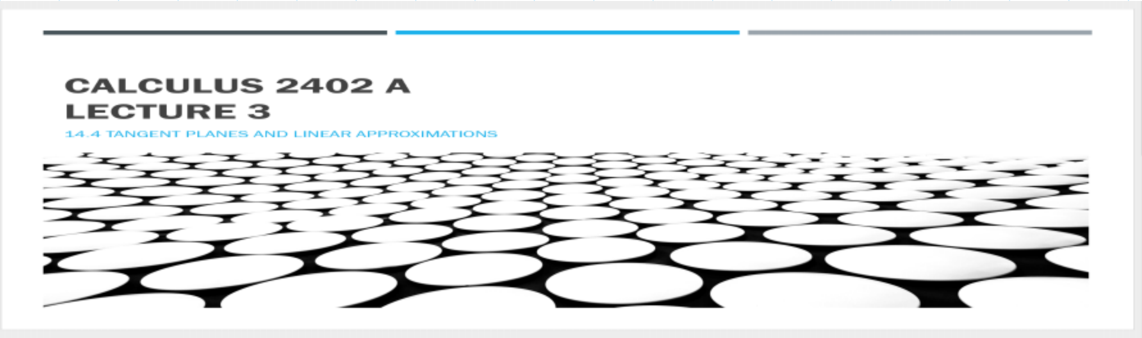

The tangent plane to surface $z = f(x,y)$ at the point $P(a,b,f(a,b))$ is

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b) \qquad (1)$$

Replacing z by $f(x,y)$, (1) becomes

$$f(x,y) - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b) \qquad (2)$$

We note that when (x,y) is close to (a,b) then

$$f(x,y) \approx L(x,y)$$

where

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \qquad (3)$$

$L(x,y)$ is called a linear approximation of f at (a,b).

Ex1: Find the tangent plane to the elliptic paraboloid $z = f(x,y) = 2x^2 + y^2$ at the point $(1,1,3)$.

$$(\underset{a}{1}, \underset{b}{1}, \underset{f(a,b)}{3})$$

Solution

$$f_x(x,y) = 4x \qquad \Rightarrow \qquad f_x(1,1) = 4(1) = 4$$
$$f_y(x,y) = 2y \qquad \qquad f_y(1,1) = 2(1) = 2$$

$\therefore$ The equation of the tangent plane at $(1,1,3)$ is

$$z - 3 = 4(x-1) + 2(y-1)$$
$$z - 3 = 4x + 2y - 4 - 2$$
$$\therefore \quad z = 4x + 2y - 3 \qquad // \text{Ans.}$$

Sketch the level curves of $f(x,y) = 2x^2 + y^2$, we have Fig. 1 as shown below

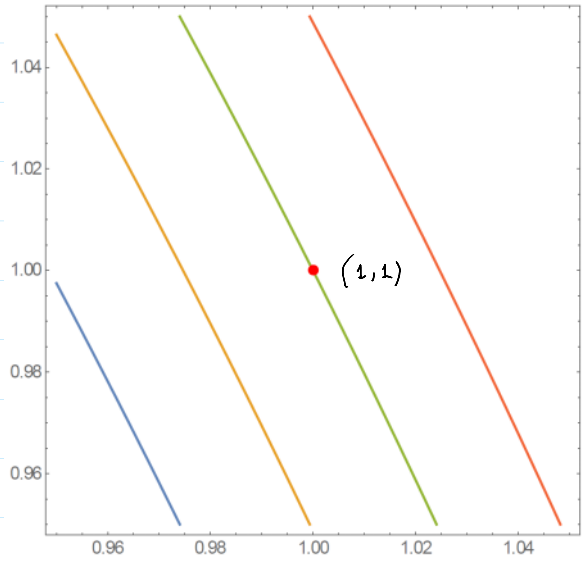

Fig 1

Let's sketch the level curves of the tangent plane $z = 4x + 2y - 3$ which are shown in Fig. 2

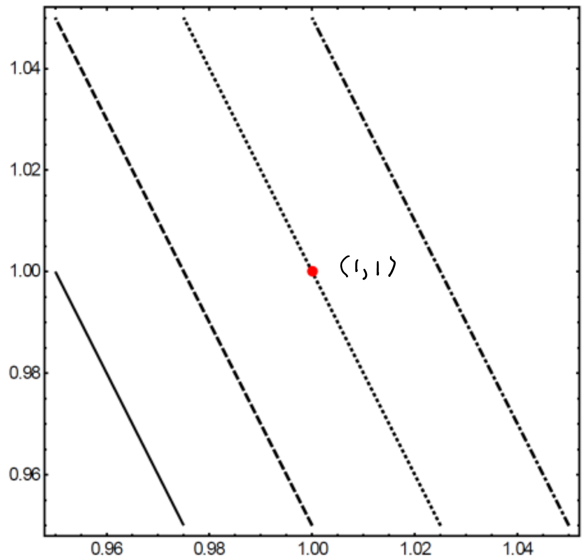

Fig 2

Combining Fig 1 & Fig 2, we obtain Fig. 3

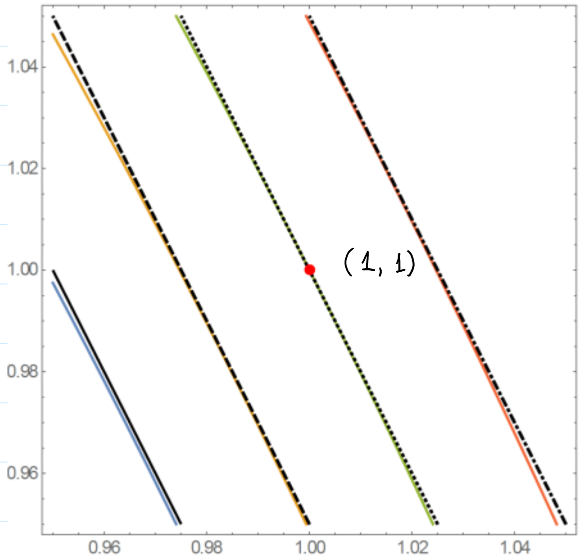

Fig. 3

From Fig. 3, we note that the level curves of the surface $z = f(x, y) = 2x^2 + y^2$ are close to those of the tangent plane $z = 4x + 2y - 3$. Hence, we can use the tangent plane as an approximation to the surface $z = f(x,y) = 2x^2 + y^2$ near the point $(1,1)$.

As an example, let's find the tangent plane approximation (or linear approximation) to the surface $z = f(x,y) = 2x^2 + y^2$ at the point $(0.99, 1.01)$.

Using linear approximation

$$L(0.99, 1.01) = 4(0.99) + 2(1.01) - 3$$
$$= 2.98 \quad //$$

and the exact value is

$$f(0.99, 1.01) = 2(0.99)^2 + (1.01)^2$$
$$= 2.9803 \quad //$$

$\therefore$ The relative error is

$$\left| \frac{2.9803 - 2.98}{2.9803} \right| \times 100 = 0.01\,\%$$

which is a very good approximation.

Thus, as Δx and Δy are very small

$$f(x + \Delta x, y + \Delta y) \simeq L(x + \Delta x, y + \Delta y)$$
$$\simeq f(x,y) + f_x(x,y)\,\Delta x + f_y(x,y)\,\Delta y$$

or

$$\underbrace{f(x + \Delta x, y + \Delta y) - f(x,y)}_{\Delta z} \simeq \underbrace{f_x(x,y)\,\Delta x + f_y(x,y)\,\Delta y}_{dz}$$

$$\Delta z \simeq dz$$

where

$$dz = f_x(x,y)\,\Delta x + f_y(x,y)\,\Delta y$$

replacing Δx by dx and Δy by dy then

$$\boxed{dz = f_x(x,y)\,dx + f_y(x,y)\,dy} \qquad (4)$$

dz is called the total differential of $z = f(x,y)$.

Functions of three or more variables

This can be generalized to functions of 3 or more variables.

For example, if $w = f(x,y,z)$ then

$$\Delta w = f(x+\Delta x,\, y+\Delta y,\, z+\Delta z) - f(x,y,z)$$
$$\simeq dw$$

when $\Delta x, \Delta y, \Delta z$ are very small and

$$dw = f_x(x,y,z)\,dx + f_y(x,y,z)\,dy + f_z(x,y,z)\,dz \qquad (5)$$

Ex: Find the linear approximation of the function

$$f(x,y,z) = \sqrt{x^2+y^2+z^2}$$

at $(3,2,6)$ and use it to approximate the number

$$\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$$

Solution

$$L(x,y,z) = f(3,2,6) + f_x(3,2,6)\,\Delta x + f_y(3,2,6)\,\Delta y + f_z(3,2,6)\,\Delta z \qquad (*)$$

where $f(3,2,6) = \sqrt{3^2+2^2+6^2} = \sqrt{9+4+36} = \sqrt{49} = 7$

$$f_x = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}} \Rightarrow f_x(3,2,6) = \frac{3}{7}$$

$$f_y = \frac{y}{\sqrt{x^2+y^2+z^2}} \Rightarrow f_y(3,2,6) = \frac{2}{7}$$

$$f_z = \frac{z}{\sqrt{x^2+y^2+z^2}} \Rightarrow f_z(3,2,6) = \frac{6}{7}$$

N.B:
$\Delta x = x-3$
$\Delta y = y-2$
$\Delta z = z-6$

Then $(*)$ becomes

$$L(x,y,z) = 7 + \frac{3}{7}(x-3) + \frac{2}{7}(y-2) + \frac{6}{7}(z-6)$$
$$= \frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z \qquad // \text{Ans.}$$

$$\therefore \sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2} \simeq \frac{3}{7}(3.02) + \frac{2}{7}(1.97) + \frac{6}{7}(5.99)$$
$$\simeq 6.991428 \qquad // \text{Ans.}$$

The exact value of $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$ is

6.991523 //

The percentage relative error is

$$\left| \frac{6.991523 - 6.991428}{6.991523} \right| \times 100 = 0.0013\%$$

which is a very good approximation.

<u>Definition</u>: If $z = f(x,y)$ is differentiable at (a,b) then $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$ can be expressed in the form

$$\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and ε_2 tend to zero as $(\Delta x, \Delta y) \to (0,0)$.

<u>Ex</u>: Show $z = f(x,y) = xy - 5y^2$ is differentiable by finding ε_1 and ε_2.

<u>Solution</u>

$$\begin{aligned}
\Delta z &= f(x+\Delta x, y+\Delta y) - f(x,y) \\
&= (x+\Delta x)(y+\Delta y) - 5(y+\Delta y)^2 - (xy - 5y^2) \\
&= xy + y\,\Delta x + x\,\Delta y + (\Delta x)(\Delta y) - 5(y^2 + 2y\,\Delta y + (\Delta y)^2) \\
&\qquad - xy + 5y^2 \\
&= \cancel{xy} + y\,\Delta x + x\,\Delta y + (\Delta x)(\Delta y) - \cancel{5y^2} - 10y\,\Delta y - 5(\Delta y)^2 \\
&\qquad - \cancel{xy} + \cancel{5y^2}
\end{aligned}$$

$$\Delta z = \underbrace{y}_{f_x}\,\Delta x + \underbrace{(x - 10y)}_{f_y}\,\Delta y + \underbrace{(\Delta x)(\Delta y)}_{\varepsilon_1 \Delta x} - \underbrace{5(\Delta y)^2}_{\varepsilon_2 \Delta y}$$

$$\varepsilon_1 = \Delta y$$

$$\varepsilon_2 = -5\Delta y$$

and both ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$. Hence, by the above definition, $f(x,y) = xy - 5y^2$ is differentiable everywhere. //Ans.

The above definition is quite difficult to use so we should use the more useful theorem as follows

<u>Theorem</u>: If the partial derivatives f_x and f_y exist in a neighborhood of (a,b) and continuous at (a,b) then f is differentiable at (a,b).

In the above example, $f(x,y) = xy - 5y^2$. Its partials are

$$f_x = y \quad \text{and} \quad f_y = x - 10y$$

These partials are continuous everywhere. Hence, by the above theorem, f is differentiable everywhere (i.e. the whole xy-plane or $\mathbb{R}^2$). //Ans.

See you on Wednesday.