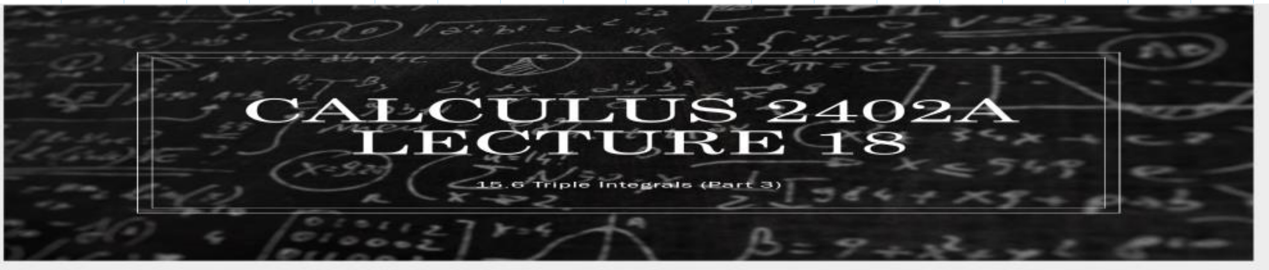
Lecture 18

October 21, 2020 10:58 AM



Applications of triple integrals

1. Mass If $\rho(x,y,z)$ is the mass density of an object D then the mass of D is

$$m = \iiint_D \rho(x,y,z)\,dV$$

2. Volume If $F(x,y,z) \equiv 1$ throughout D then $\iiint_D F(x,y,z)\,dV$ reduces to $\iiint_D dV$ which is the volume of D.

3. The C.M $(\bar{x}, \bar{y}, \bar{z})$ of an object D and centroid

$$\bar{x} = \frac{\iiint_D x\,\rho(x,y,z)\,dV}{\underbrace{\iiint_D \rho(x,y,z)\,dV}_{=m}} = \frac{M_{yz}}{m}$$

where M_{yz} is the moment of D about the yz-plane

$$\bar{y} = \frac{\iiint_D y\,\rho(x,y,z)\,dV}{\iiint_D \rho(x,y,z)\,dV} = \frac{M_{zx}}{m}$$

where M_{zx} is the moment of D about the zx-plane

$$\bar{z} = \frac{\iiint_D z\,\rho(x,y,z)\,dV}{\iiint_D \rho(x,y,z)\,dV} = \frac{M_{xy}}{m}$$

where M_{xy} is the moment of D about the xy-plane

If $\rho(x,y,z) = \rho_0$ (a constant) then the CM is called a centroid

$$\bar{x} = \frac{\iiint_D x\,dV}{V}$$

$$\bar{y} = \frac{\iiint_D y\,dV}{V}$$

$$\overline{\qquad\qquad}$$
$$V$$

$$\bar{z} = \frac{\iiint_D z\, dV}{V} \quad \text{where} \quad V = \iiint_D dV \text{ which is the volume of } D$$

4. Moments of inertia

$$I_x = \iiint_D (y^2+z^2)\,\rho(x,y,z)\,dV$$ Moment of inertia of D about the x-axis

$$I_y = \iiint_D (x^2+z^2)\,\rho(x,y,z)\,dV$$ moment of inertia of D about the y-axis

$$I_z = \iiint_D (x^2+y^2)\,\rho(x,y,z)\,dV$$ moment of inertia of D about the z-axis

5. Joint Probability Density Functions

Consider the three random variables X, Y, and Z. We define

$$P\big((X,Y,Z)\in D\big) = \iiint_D f(x,y,z)\,dV$$

where $f(x,y,z)$ is called the joint probability density function where f must satisfy

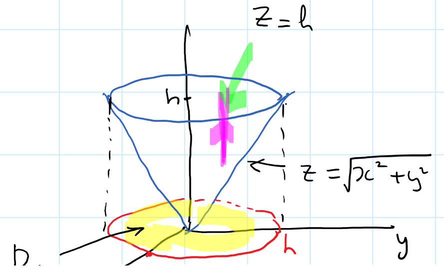
(i) $f(x,y,z) \geq 0$ for $\forall (X,Y,Z) \in \mathbb{R}^3$

(ii) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dx\,dy\,dz = 1$$

or $$\iiint_{\mathbb{R}^3} f(x,y,z)\,dV = 1$$

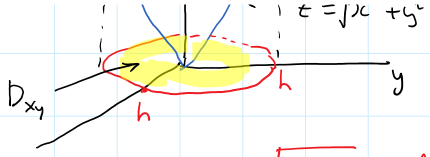
Ex1: Find the moment of inertia about the z-axis of the solid cone $\sqrt{x^2+y^2} \leq z \leq h$ with mass density $\rho = k$ (a constant).

Solution



$$I_z = \iiint_{\text{Cone}} (x^2+y^2)\,\underbrace{\rho(x,y,z)}_{k}\,dV$$

$$= k \iiint_{\text{Cone}} (x^2+y^2)\,dV$$



Cone

Set $\sqrt{x^2+y^2} = h$

$\therefore \; x^2+y^2 = h^2$

The cone is of type I region so we integrate wrt z first.

$$I_z = k\iint_{D_{xy}} \left(\int_{z=\sqrt{x^2+y^2}}^{z=h} (x^2+y^2)\, dz \right) dA$$

$$I_z = k\iint_{D_{xy}} (x^2+y^2)\left(h - \sqrt{x^2+y^2}\right) dA$$

where D_{xy} is the projection of the cone onto the xy-plane.

// Separable

$$I_z \underset{\substack{\uparrow \\ \text{polar coordinates}}}{=} k \int_0^{2\pi} \int_0^{h} \boxed{r^2 (h - r)\, r}\, dr\, d\theta$$

$$= k \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{h} (h r^3 - r^4)\, dr \right)$$

$$= k\,(2\pi) \left(h\frac{r^4}{4} - \frac{r^5}{5} \right) \Bigg|_0^{h}$$

$$= k(2\pi) \left(\frac{h^5}{4} - \frac{h^5}{5} \right)$$

$$= 2\pi k \left(\frac{h^5}{20} \right) = \frac{\pi k h^5}{10} \quad // \text{Ans.}$$

Ex2: The joint probability density function for the random variables X, Y, and Z is defined as

$$f(x,y,z) = \begin{cases} Cxyz & \text{if } 0 \le x \le 2,\ 0 \le y \le 2,\ 0 \le z \le 2 \\ 0 & \text{, otherwise} \end{cases}$$

a) Find the value of C

b) Find $P(X \le 1, Y \le 1, Z \le 1)$

c) Find $P(X + Y + Z \le 1)$

Solution

$$\iiint_{\mathbb{R}^3} f(x,y,z)\, dV = 1$$

$$\int_0^2 \int_0^2 \int_0^2 C \boxed{xyz}\, dx\, dy\, dz = 1$$

$$C \left(\int_0^2 x\, dx \right) \left(\int_0^2 y\, dy \right) \left(\int_0^2 z\, dz \right) = 1$$

$$C\left(\int_0^2 x\,dx\right)\left(\int_0^2 y\,dy\right)\left(\int_0^2 z\,dz\right) = 1$$

$$c\left(\frac{x^2}{2}\right)\Big|_0^2 \left(\frac{y^2}{2}\right)\Big|_0^2 \left(\frac{z^2}{2}\right)\Big|_0^2 = 1$$

$$c(2)(2)(2) = 1 \Rightarrow \boxed{C = \frac{1}{8}} \quad // \text{Ans.}$$

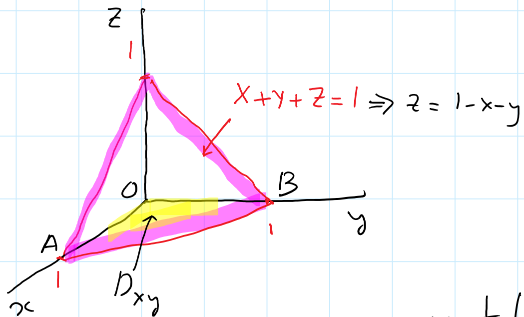
b) $$P(X \leq 1, Y \leq 1, Z \leq 1) = \int_0^1 \int_0^1 \int_0^1 \frac{xyz}{8}\,dz\,dy\,dx$$ separable

$$= \frac{1}{8}\left(\int_0^1 x\,dx\right)\left(\int_0^1 y\,dy\right)\left(\int_0^1 z\,dz\right)$$

$$= \frac{1}{8}\left(\frac{x^2}{2}\right)\Big|_0^1 \left(\frac{y^2}{2}\right)\Big|_0^1 \left(\frac{z^2}{2}\right)\Big|_0^1$$

$$= \frac{1}{8}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{64} \quad // \text{Ans.}$$

c) $$P(X+Y+Z \leq 1) = \int_0^1 \int_0^{1-x} \int_{z=0}^{z=1-x-y} \frac{xyz}{8}\,dz\,dy\,dx$$



$$= \frac{1}{8}\int_0^1 \int_0^{1-x} xy\left(\frac{z^2}{2}\right)\Big|_0^{1-x-y} dy\,dx$$

$$= \frac{1}{8}\left(\frac{1}{2}\right)\int_0^1 \int_0^{1-x} xy\,(1-x-y)^2\,dy\,dx$$

$$= \frac{1}{8}\left(\frac{1}{2}\right)\int_0^1 \int_0^{1-x} xy\left[(1-x)^2 - 2(1-x)y + y^2\right]dy\,dx$$

$$= \frac{1}{8}\frac{1}{2}\int_0^1 \int_0^{1-x}\left[x(1-x)^2 y - 2x(1-x)y^2 + xy^3\right]dy\,dx$$

$$= \frac{1}{8}\frac{1}{2}\int_0^1 \left[x(1-x)^2\frac{y^2}{2} - 2x(1-x)\frac{y^3}{3} + x\frac{y^4}{4}\right]\Big|_{y=0}^{1-x} dx$$

$$= \frac{1}{16}\int_0^1 \left[x(1-x)^2\frac{(1-x)^2}{2} - 2x(1-x)\frac{(1-x)^3}{3} + \frac{x}{4}(1-x)^4\right]dx$$

$$= \frac{1}{16}\int_0^1 \left[\frac{x(1-x)^4}{2} - \frac{2}{3}x(1-x)^4 + \frac{1}{4}x(1-x)^4\right]dx$$

$$= \frac{1}{16}\int_0^1 \frac{x(1-x)^4}{12}\,dx$$

$$= \frac{1}{16}\frac{1}{12}\int_0^1 x(1-x)^4\,dx \qquad \boxed{u = 1-x} \Rightarrow x = 1-u$$

$$du = -dx$$

$$= \frac{1}{16}\frac{1}{12}\int_1^0 u^4(1-u)(-du)$$

$$\underset{\substack{\uparrow \\ \text{Switching the limits}}}{=} \frac{1}{16}\frac{1}{12}\int_0^1 (u^4 - u^5)\,du = \frac{1}{16}\frac{1}{12}\left(\frac{u^5}{5} - \frac{u^6}{6}\right)\Big|_0^1$$

$$= \frac{1}{16}\frac{1}{12}\left(\frac{1}{5} - \frac{1}{6}\right)$$

$$= \ldots \quad // \text{Ans.}$$

Switching the limits

$$= \frac{1}{16} \frac{1}{12} \left(\frac{1}{5} - \frac{1}{6}\right)$$

$$= \frac{1}{16} \frac{1}{12} \left(\frac{1}{30}\right) = \frac{1}{5760} \quad // \text{Ans.}$$

On Friday, we have Midterm Review (no new topics)

Office hour (tomorrow) at 1:00 PM

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Midterm Exam on Saturday Oct 24/2020 at 2:00 PM

You are supposed to write the exam from 2:00PM – 5:00PM. However, I give you extra 30 minutes for printing the exam, scanning and upload your answers to OWL. Hence you must submit your exam by 5:30 PM.

The 1st page of the EXAM BOOKLET

Student's Name: ________	Student Number: ________

8. You have to check the box $\boxed{\checkmark}$

I pledge on my honour that I have neither given nor received aid on this exam.

How to obtain the exam? Go to "Tests & Quizzes" and click this tab (by 2:00 pm) you should see

Calculus 2402A Midterm Exam

and you can print it.

How to submit your Exam?

You submit your complete exam to BOTH

(i) Tests and Quizzes (OWL)

(ii) gradescope (OWL)

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IF you DO NOT HAVE A PRINTER

You use blank papers to write your answers.

For the 1st page, you must write your name & Student number exactly as you did write in the box of the exam booklet and you must check the pledge on honour □.

You must keep the number of pages the same as the number of pages of the exam booklet.

You also keep your answers corresponding to the questions in the exam booklet. For example, if page 2 of the exam booklet has 5 questions then your answers of these questions also are on page 2 of your answer sheets.

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See you either in office hours (tomorrow at 1:00pm) or Friday (Midterm Review)