9.2 Direction Fields and Euler's Method

Unfortunately, it’s impossible to solve most differential equations in the sense of obtaining an explicit formula for the solution.

In this section we show that, despite the absence of an explicit solution, we can still learn a lot about the solution through a graphical approach (direction fields) or a numerical approach (Euler’s method).

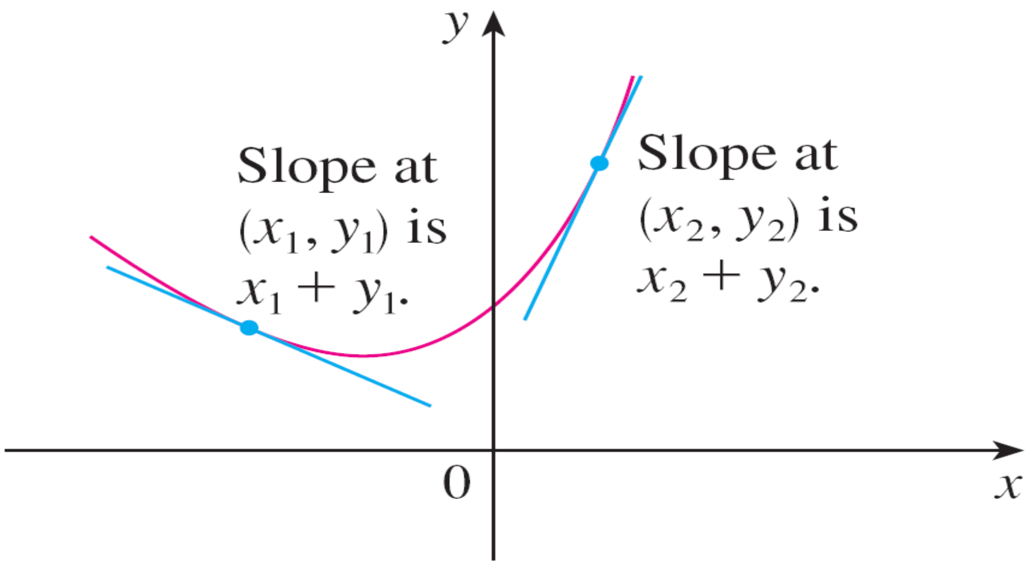
1. Direction Fields

Suppose we are asked to sketch the graph of the solution of the initial-value problem

$$y' = x + y \qquad y(0) = 1$$

We don’t know a formula for the solution, so how can we possibly sketch its graph? Let’s think about what the differential equation means.

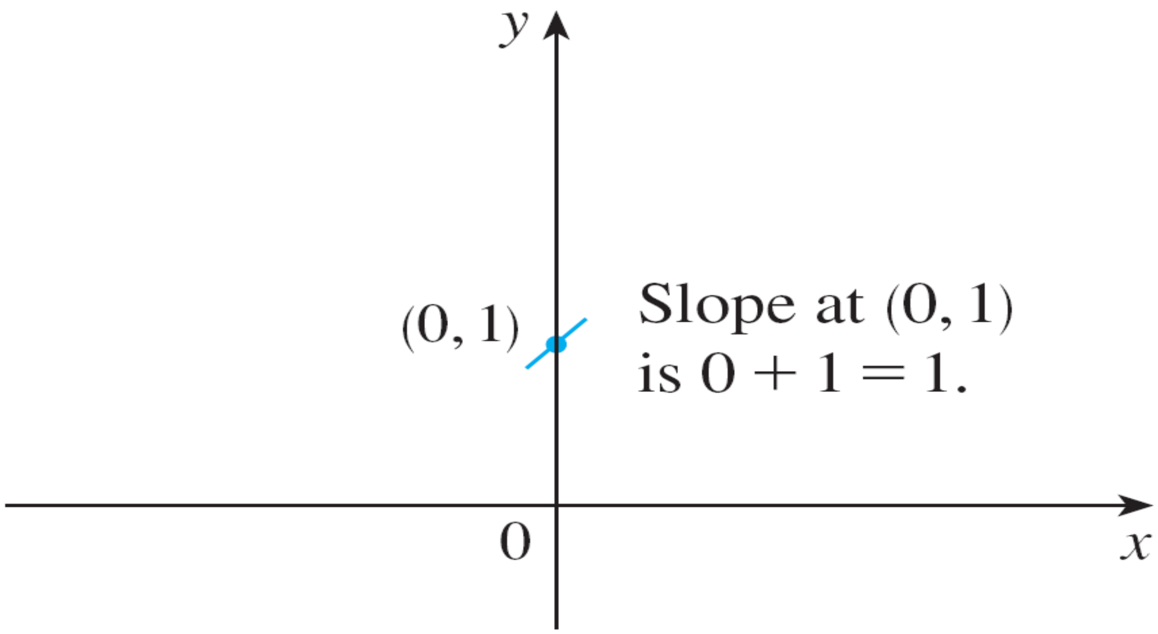
The equation $y' = x + y$ tells us that the slope at any point (x, y) on the graph (called the *solution curve*) is equal to the sum of the x- and y-coordinates of the point (see Figure 1).



A solution of $y' = x + y$

Figure 1

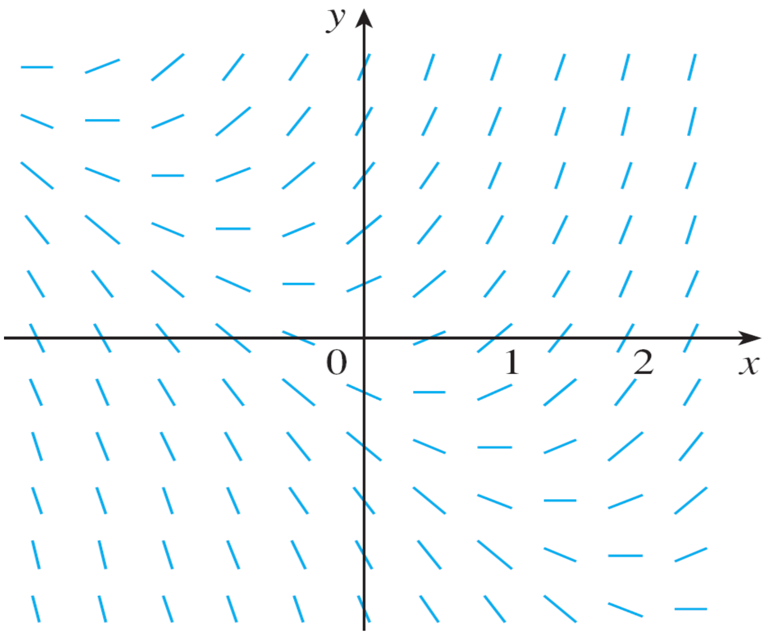
In particular, because the curve passes through the point (0, 1), its slope there must be 0 + 1 = 1. So a small portion of the solution curve near the point (0, 1) looks like a short line segment through (0, 1) with slope 1. (See Figure 2.)



Beginning of the solution curve through (0, 1)

Figure 2

As a guide to sketching the rest of the curve, let's draw short line segments at a number of points (x, y) with slope $x + y$. The result is called a *direction field* and is shown in Figure 3.



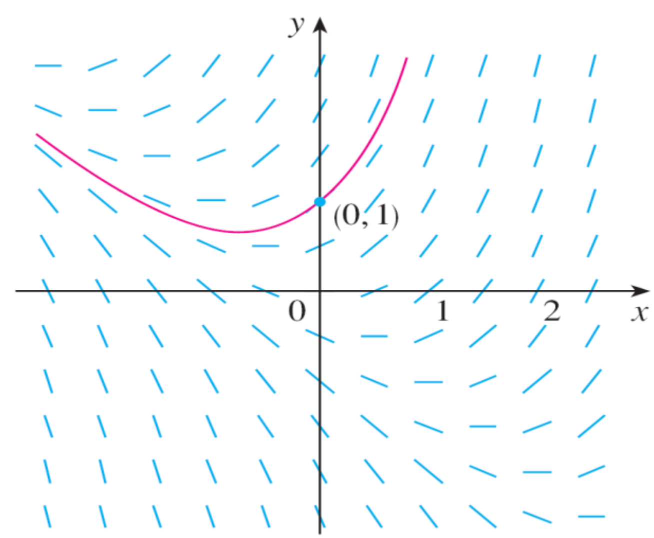
Direction field for $y' = x + y$

Figure 3

For instance, the line segment at the point (1, 2) has slope 1 + 2 = 3.

The direction field allows us to visualize the general shape of the solution curves by indicating the direction in which the curves proceed at each point.

Now we can sketch the solution curve through the point (0, 1) by following the direction field as in Figure 4.



The solution curve through (0, 1)

Figure 4

Notice that we have drawn the curve so that it is parallel to nearby line segments.

In general, suppose we have a first-order differential equation of the form

$$y' = F(x, y)$$

where $F(x, y)$ is some expression in x and y. The differential equation says that the slope of a solution curve at a point (x, y) on the curve is $F(x, y)$.

If we draw short line segments with slope $F(x, y)$ at several points (x, y), the result is called a **direction field** (or **slope field**).

These line segments indicate the direction in which a solution curve is heading, so the direction field helps us visualize the general shape of these curves.

Example 1

(a) Sketch the direction field for the differential equation

$y' = x^2 + y^2 - 1.$

(b) Use part (a) to sketch the solution curve that passes through the origin.

Solution:

(a) We start by computing the slope at several points in the following chart:

x	-2	-1	0	1	2	-2	-1	0	1	2	...
y	0	0	0	0	0	1	1	1	1	1	...
$y' = x^2 + y^2 - 1$	3	0	-1	0	3	4	1	0	1	4	...

cont'd

Now we draw short line segments with these slopes at these points. The result is the direction field shown in Figure 5.

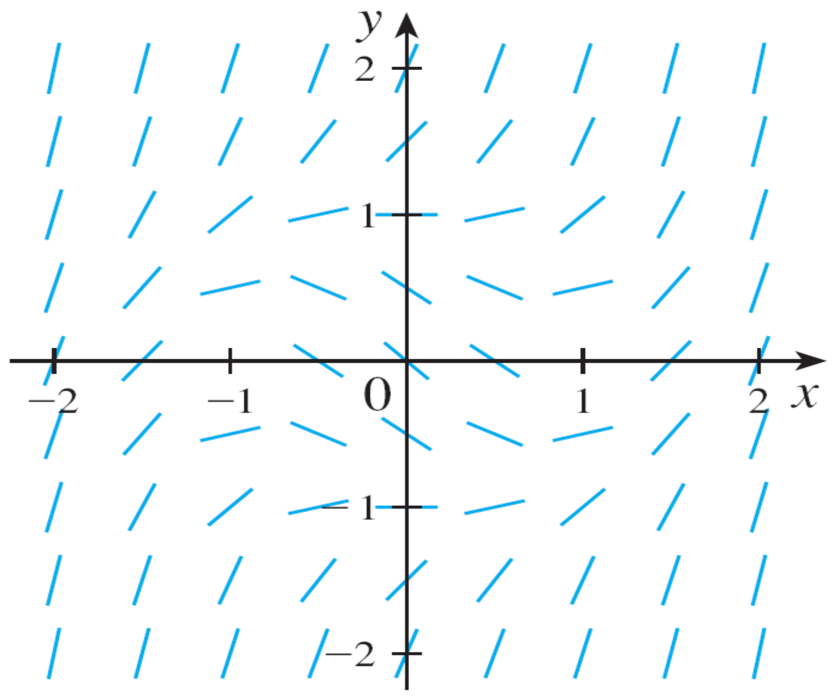


Figure 5

(b) We start at the origin and move to the right in the direction of the line segment (which has slope –1).

We continue to draw the solution curve so that it moves parallel to the nearby line segments. The resulting solution curve is shown in Figure 6.

Returning to the origin, we draw the solution curve to the left as well.

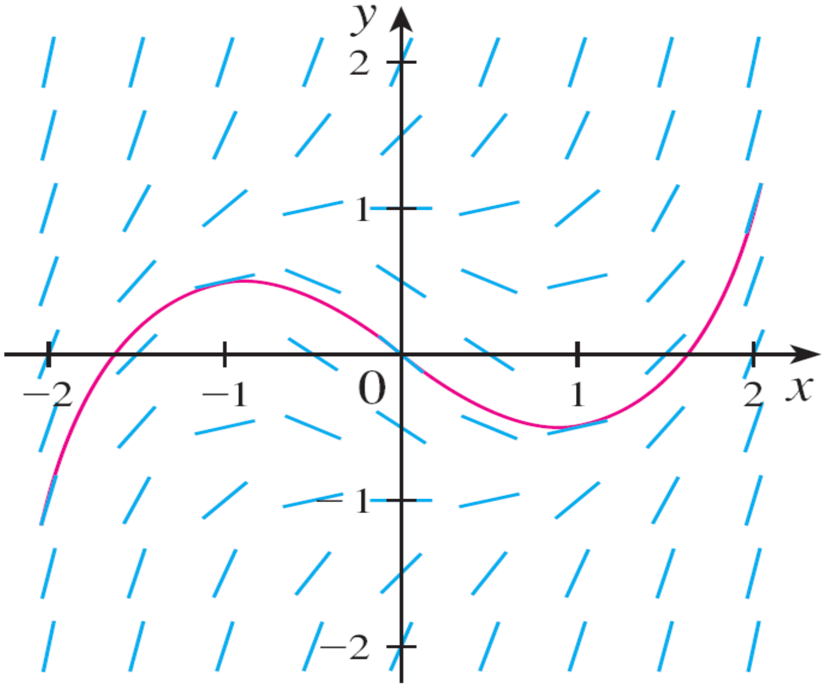


Figure 6

2. Euler's Method

The basic idea behind direction fields can be used to find numerical approximations to solutions of differential equations.

We illustrate the method on the initial-value problem that we used to introduce direction fields:

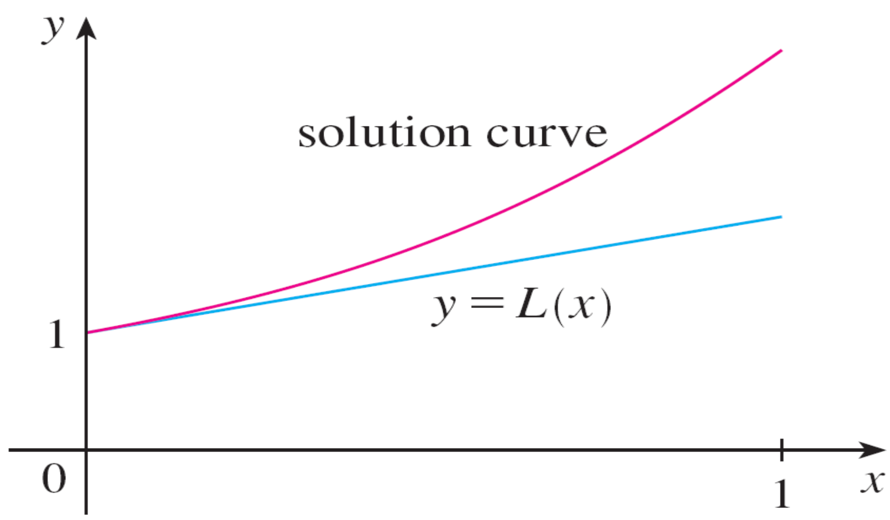
$$y' = x + y \qquad\qquad y(0) = 1$$

The differential equation tells us that $y'(0) = 0 + 1 = 1$, so the solution curve has slope 1 at the point (0, 1).

As a first approximation to the solution we could use the linear approximation $L(x) = x + 1$.

In other words, we could use the tangent line at (0, 1) as a rough approximation to the solution curve (see Figure 7).

Euler's idea was to improve on this approximation by proceeding only a short distance along this tangent line and then making a midcourse correction by changing direction as indicated by the direction field.

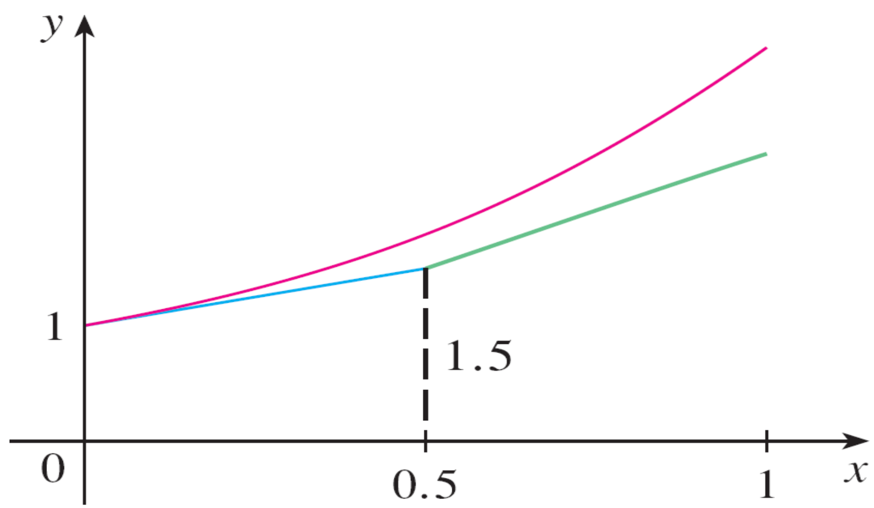


First Euler approximation

Figure 7

Figure 8 shows what happens if we start out along the tangent line but stop when $x = 0.5$. (This horizontal distance traveled is called the *step size.*)

Since $L(0.5) = 1.5$, we have $y(0.5) \approx 1.5$ and we take $(0.5, 1.5)$ as the starting point for a new line segment.



Euler approximation with step size 0.5

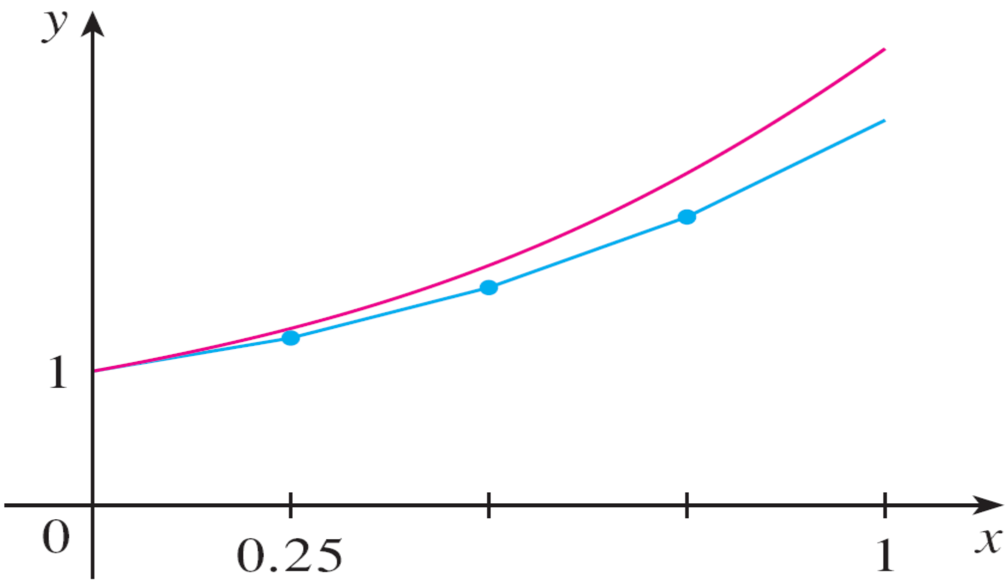
Figure 8

The differential equation tells us that $y'(0.5) = 0.5 + 1.5 = 2$, so we use the linear function

$$y = 1.5 + 2(x - 0.5) = 2x + 0.5$$

as an approximation to the solution for $x > 0.5$ (the green segment in Figure 8).

If we decrease the step size from 0.5 to 0.25, we get the better Euler approximation shown in Figure 9.



Euler approximation with step size 0.25

Figure 9

In general, Euler's method says to start at the point given by the initial value and proceed in the direction indicated by the direction field.

Stop after a short time, look at the slope at the new location, and proceed in that direction.

Keep stopping and changing direction according to the direction field.

Euler's method does not produce the exact solution to an initial-value problem—it gives approximations.

But by decreasing the step size (and therefore increasing the number of midcourse corrections), we obtain successively better approximations to the exact solution. (Compare Figures 7, 8, and 9.)

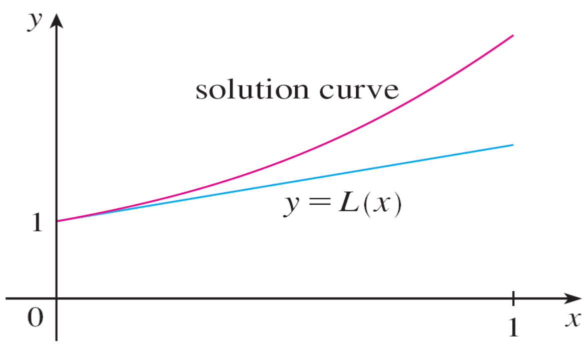


Figure 7

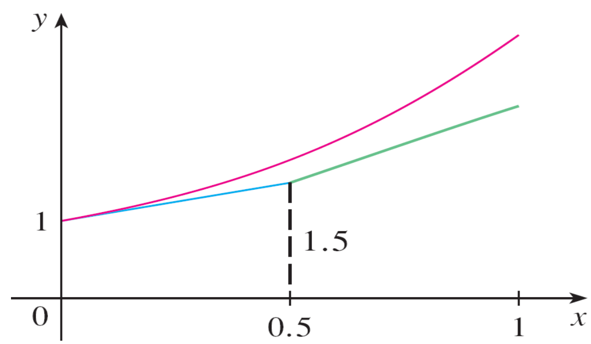


Figure 8

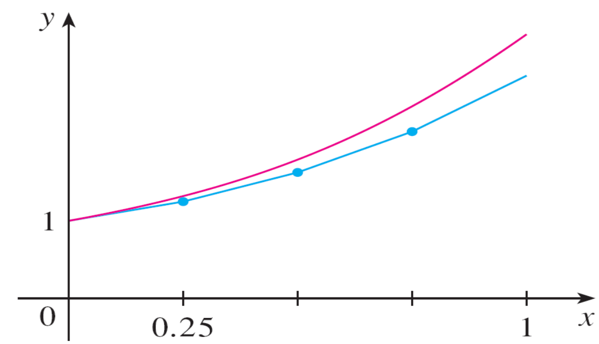


Figure 9

For the general first-order initial-value problem $y' = F(x, y)$, $y(x_0) = y_0$, our aim is to find approximate values for the solution at equally spaced numbers x_0, $x_1 = x_0 + h$, $x_2 = x_1 + h, \ldots$, where h is the step size.

The differential equation tells us that the slope at (x_0, y_0) is $y' = F(x_0, y_0)$, so Figure 10 shows that the approximate value of the solution when $x = x_1$ is

$$y_1 = y_0 + hF(x_0, y_0)$$

Similarly,

$$y_2 = y_1 + hF(x_1, y_1)$$

In general,

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$$

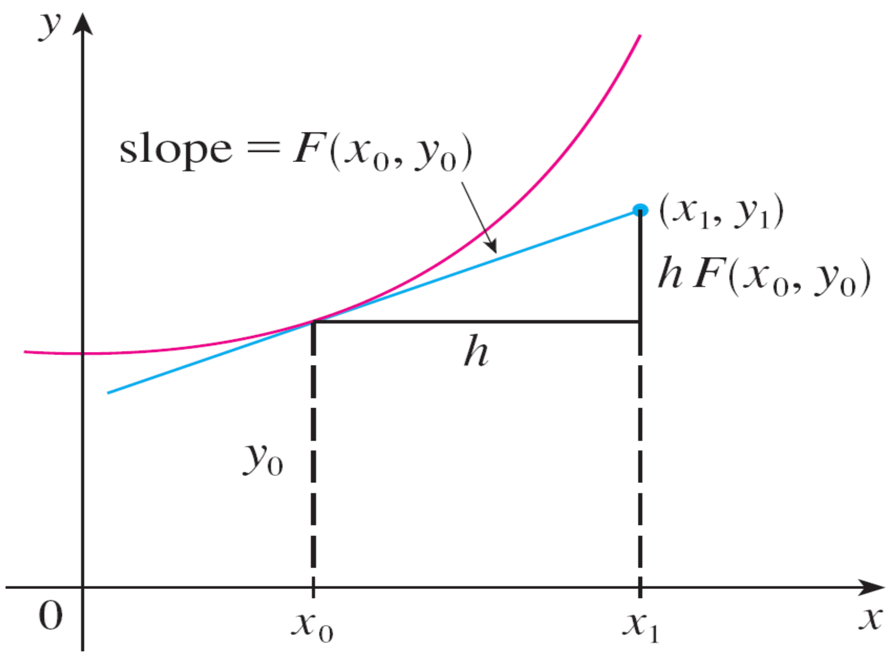


Figure 10

Euler's Method

Euler's Method Approximate values for the solution of the initial-value problem $y' = F(x, y)$, $y(x_0) = y_0$, with step size h, at $x_n = x_{n-1} + h$, are

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}) \qquad n = 1, 2, 3, \cdots$$

Example 3

Use Euler's method with step size 0.1 to construct a table of approximate values for the solution of the initial-value problem

$$y' = x + y \qquad\qquad y(0) = 1$$

Solution:

We are given that $h = 0.1$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = x + y$. So we have

$$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1(0 + 1) = 1.1$$

$$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1(0.1 + 1.1) = 1.22$$

$$y_3 = y_2 + hF(x_2, y_2) = 1.22 + 0.1(0.2 + 1.22) = 1.362$$

This means that if $y(x)$ is the exact solution, then $y(0.3) \approx 1.362$.

Example 3 – *Solution*

Proceeding with similar calculations, we get the values in the table:

n	x_n	y_n	n	x_n	y_n
1	0.1	1.100000	6	0.6	1.943122
2	0.2	1.220000	7	0.7	2.197434
3	0.3	1.362000	8	0.8	2.487178
4	0.4	1.528200	9	0.9	2.815895
5	0.5	1.721020	10	1.0	3.187485

For a more accurate table of values in Example 3 we could decrease the step size.

But for a large number of small steps the amount of computation is considerable and so we need to program a calculator or computer to carry out these calculations.

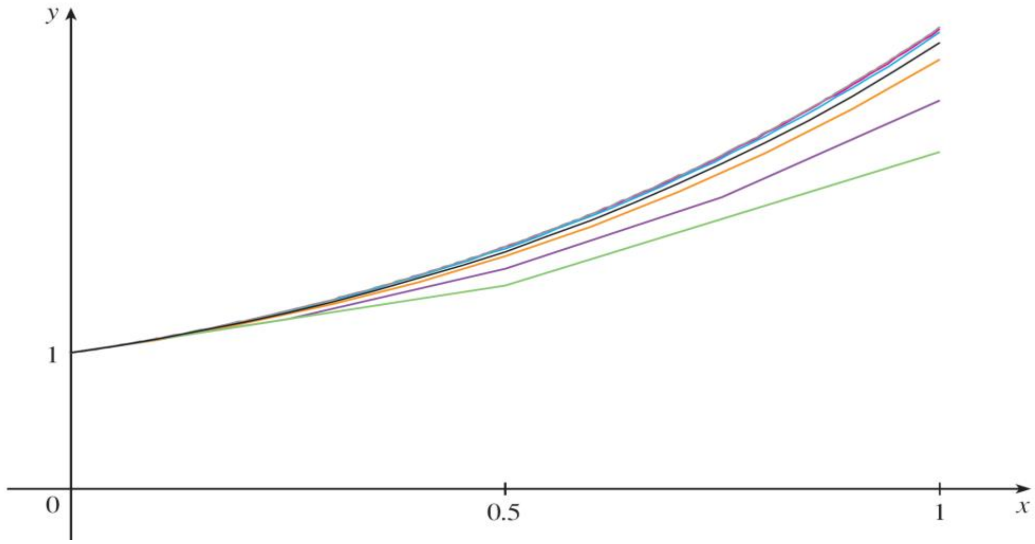
The following table shows the results of applying Euler's method with decreasing step size to the initial-value problem of Example 3.

Step size	Euler estimate of $y(0.5)$	Euler estimate of $y(1)$
0.500	1.500000	2.500000
0.250	1.625000	2.882813
0.100	1.721020	3.187485
0.050	1.757789	3.306595
0.020	1.781212	3.383176
0.010	1.789264	3.409628
0.005	1.793337	3.423034
0.001	1.796619	3.433848

Notice that the Euler estimates in the table seem to be approaching limits, namely, the true values of $y(0.5)$ and $y(1)$.

Figure 11 shows graphs of the Euler approximations with step sizes 0.5, 0.25, 0.1, 0.05, 0.02, 0.01, and 0.005.

They are approaching the exact solution curve as the step size h approaches 0.



Euler approximations approaching the exact solution

Figure 11