Lecture 22

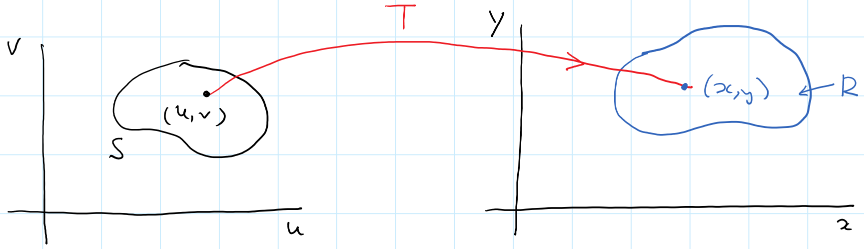
October 30, 2020 11:15 AM

Change of variables in Multiple Integrals (sec 15.9)

Consider a change of variables that given by a transformation T from the uv-plane to the xy-plane

$$T(u,v) = (x,y)$$

where $x = x(u,v)$ and $y = y(u,v)$

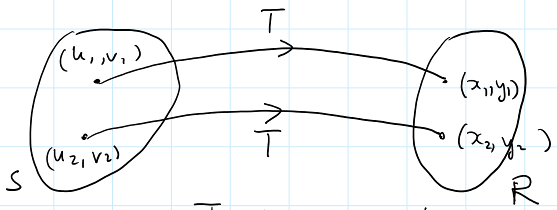


We assume that T is a C^1 transformation, ie, both $x(u,v)$ and $y(u,v)$ have continuous first partial derivatives.

If $T(u_1,v_1) = (x_1,y_1)$ then (x_1,y_1) is called the image of (u_1,v_1).

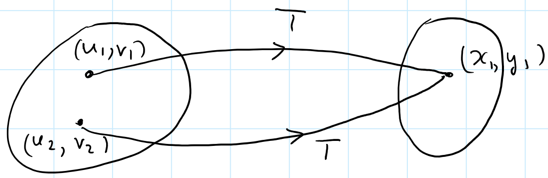
If $(u_1,v_1) \neq (u_2,v_2)$ and $T(u_1,v_1) \neq T(u_2,v_2)$, ie $(x_1,y_1) \neq (x_2,y_2)$ then T is called a one-to-one transformation.

(i)



T is one-to-one

(ii)



T is NOT one-to-one

T transforms a region S in the uv-plane to a region R in the xy-plane. R is called the image of S under T.

If T is one-to-one, the inverse of T, denoted as T^{-1}, exists. Therefore, if

$$T: \quad x = x(u,v) \quad , \quad y = y(u,v)$$

then $\quad T^{-1}: \quad u = u(x,y) \quad , \quad v = v(x,y)$

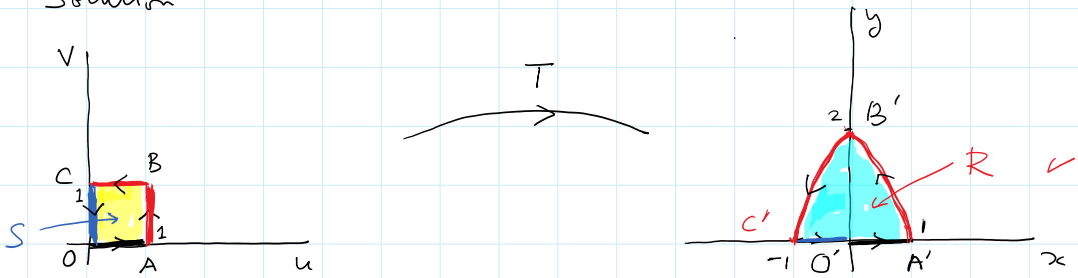
Ex 1: A transformation is defined by

$$x = x(u,v) = u^2 - v^2 \quad , \quad y = y(u,v) = 2xy$$

Find the image of the square

$$S = \{ (u,v) \mid 0 \leq u \leq 1 \quad , \quad 0 \leq v \leq 1 \}$$

Solution



$OA \; (v = 0 \; , 0 \leq u \leq 1) \xrightarrow{T} O'A' \quad x = u^2 - v^2 = u^2 \Rightarrow 0 \leq x \leq 1$

$\qquad y = 2uv = 0$

$\therefore$ $O'A'$ is the segment on the x-axis for $0 \leq x \leq 1$

$AB \; (u = 1 \; , \; 0 \leq v \leq 1) \xrightarrow{T} A'B' \quad x = u^2 - v^2 = 1 - v^2 = 1 - \frac{y^2}{4} \Rightarrow 4x = 4 - y^2$

$\qquad y = 2uv = 2v \Rightarrow v = \frac{y}{2}$

$\qquad (0 \leq y \leq 2) \qquad 4x = 4 - y^2$

a parabola

$BC \; (v = 1 \; , \; 1 \geq u \geq 0) \xrightarrow{T} B'C' \quad x = u^2 - v^2 = u^2 - 1 = \frac{y^2}{4} - 1$

$\qquad y = 2uv = 2u \Rightarrow u = \frac{y}{2}$

$\qquad 2 \geq y \geq 0 \qquad 4x = y^2 - 4$

a parabolic arc with $2 \geq y \geq 0$

$CO \; (u = 0 \; , \; 1 \geq v \geq 0) \xrightarrow{T} C'O' \quad x = u^2 - v^2 = -v^2 \Rightarrow -1 \leq x \leq 0$

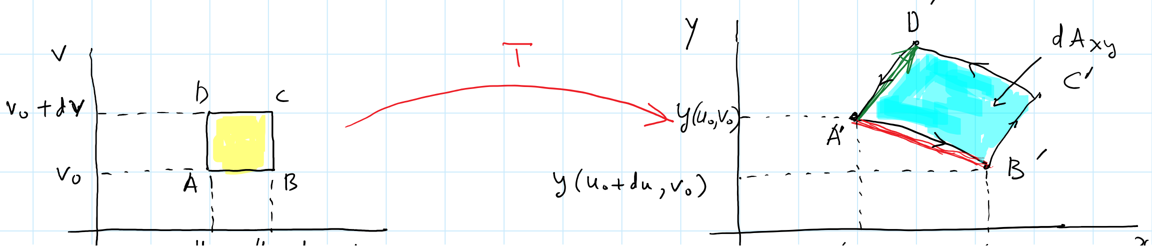
$\qquad y = 2uv = 0 \Rightarrow$ (the x-axis)

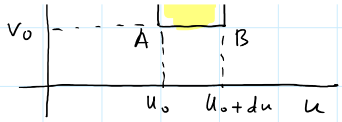
$C'O'$ is the segment on the x-axis where $-1 \leq x \leq 0$.

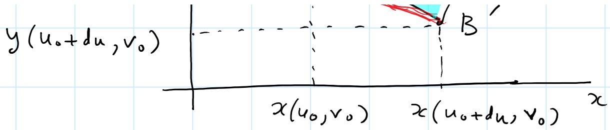
The region R in this example has the same orientation of S.

// Ans.

The area element dA







$dA_{uv} = du\,dv$

Under the transformation T

$$x = x(u,v) \quad \text{and} \quad y = y(u,v)$$

$$A(u_0,v_0) \xrightarrow{T} A'(x(u_0,v_0), y(u_0,v_0))$$

$$B(u_0+du, v_0) \xrightarrow{T} B'(x(u_0+du, v_0), y(u_0+du, v_0))$$

$$C(u_0+du, v_0+dv) \xrightarrow{T} C'(x(u_0+du, v_0+dv), y(u_0+du, v_0+dv))$$

$$D(u_0, v_0+dv) \xrightarrow{T} D'(x(u_0, v_0+dv), y(u_0, v_0+dv))$$

then

$$\overrightarrow{A'B'} = \Big(\underbrace{x(u_0+du, v_0) - x(u_0,v_0)}_{\frac{\partial x}{\partial u}du},\ \underbrace{y(u_0+du, v_0) - y(u_0,v_0)}_{\frac{\partial y}{\partial u}du}\Big)$$

$$\overrightarrow{A'B'} = \left(\frac{\partial x}{\partial u}du,\ \frac{\partial y}{\partial u}dv\right)$$

$$\overrightarrow{A'D'} = \Big(\underbrace{x(u_0, v_0+dv) - x(u_0,v_0)}_{\frac{\partial x}{\partial v}dv},\ \underbrace{y(u_0, v_0+dv) - y(u_0,v_0)}_{\frac{\partial y}{\partial v}dv}\Big)$$

$$\overrightarrow{A'D'} = \left(\frac{\partial x}{\partial v}dv,\ \frac{\partial y}{\partial v}dv\right)$$

We note that

$$dA_{xy} \approx \text{area of the parallelogram } A'B'C'D'$$
$$\approx \|\overrightarrow{A'B'} \times \overrightarrow{A'D'}\|$$

$$\overrightarrow{A'B'} \times \overrightarrow{A'D'} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial x}{\partial u}du & \frac{\partial y}{\partial u}du & 0 \\ \frac{\partial x}{\partial v}dv & \frac{\partial y}{\partial v}dv & 0 \end{vmatrix}$$

$$= \hat{k}\begin{vmatrix} \frac{\partial x}{\partial u}du & \frac{\partial y}{\partial u}du \\ \frac{\partial x}{\partial v}dv & \frac{\partial y}{\partial v}dv \end{vmatrix}$$

$$= \hat{k}\begin{vmatrix} \frac{\partial x}{\partial u}du & \frac{\partial x}{\partial v}dv \\ \frac{\partial y}{\partial u}du & \frac{\partial y}{\partial v}dv \end{vmatrix} = \hat{k}\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv$$

$$\therefore\ dA_{xy} = \|\overrightarrow{A'B'} \times \overrightarrow{A'D'}\| = \left\|\hat{k}\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv\right\|$$

$du, dv > 0$

$du\,dv > 0$

$$= \left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv\right| \underbrace{\|\hat{k}\|}_{1}$$

$$= \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right| \underbrace{du\,dv}_{dA_{uv}}$$

$$\boxed{dA_{xy} = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv}$$

where
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

which is called the Jacobian of the transformation T.

$\therefore$ We have the theorem of change of variables for a double integral

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

where R is the image of S under the transformation T.

Ex2: Consider the transformation

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

then
$$dx\,dy = \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| dr\,d\theta \qquad (*)$$

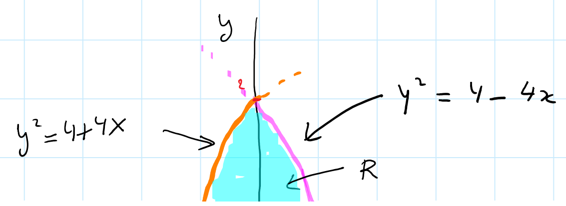
where
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta - (-r\sin^2\theta)$$
$$= r\cos^2\theta + r\sin^2\theta$$
$$= r\,\underbrace{(\cos^2\theta + \sin^2\theta)}_{1}$$
$$= r \qquad (r > 0)$$
$$|r| = r$$

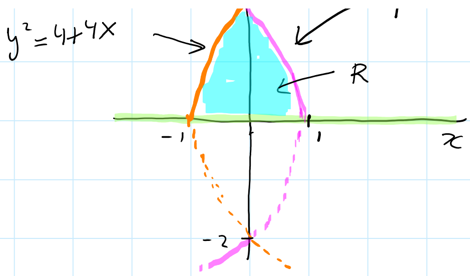
$(*)$ becomes
$$dA = |r|\,dr\,d\theta$$
$$\boxed{dA = r\,dr\,d\theta}$$

which is the area element in polar coordinates. //

Ex3: Evaluate $I = \iint_R y\,dA$ where R is the region bounded by the x-axis and parabolas $y^2 = 4 - 4x$, $y^2 = 4 + 4x$, $y \geq 0$.

Solution



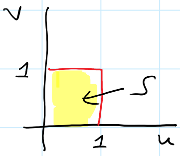


How do we find a transformation which transforms R into a simpler region S.

However, if we recall Ex 1, the transformation

$$x = u^2 - v^2 \quad , \quad y = 2uv$$

transforms the Square S, $0 \le u \le 1$, $0 \le v \le 1$ to the above region R. Hence,



$$I = \iint_R y\,dA = \iint_S (2uv)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = (2u)(2u) - (2v)(-2v) = \underline{4u^2 + 4v^2}$$

$$\therefore \quad I = \int_0^1 \int_0^1 (2uv)\, 4(u^2+v^2)\, du\, dv$$

$$= 8 \int_0^1 \int_0^1 (u^3 v + uv^3)\, du\, dv$$

$$= 8 \int_0^1 \left(\frac{u^4}{4}v + \frac{u^2}{2}v^3\right)\Bigg|_{u=0}^{1} dv$$

$$= 8 \int_0^1 \left(\frac{v}{4} + \frac{v^3}{2}\right) dv$$

$$= 8\left(\frac{v^2}{8} + \frac{v^4}{8}\right)\Bigg|_0^1 = \cancel{8}\left(\frac{v^2+v^4}{\cancel{8}}\right)\Bigg|_0^1 = (v^2+v^4)\Big|_0^1$$

$$= (1)^2 + (1)^4 = 1 + 1 = 2 \quad // \text{Ans.}$$

✕

See you after the reading week!