Tutorial #3

Problem 1 Professor Cuthbert Calculus has designed a machine which consists of three components A, B, C which are either running or stopped. The constraints on those components are the following:

- 1. if A is running, then at least one of the components B or C is stopped,
- 2. if B is stopped, then at least one of the components A or C is running,
- 3. if C is running, then B is running as well.

Can the machine of Professor Cuthbert Calculus be built, that is, is the conjunction of the above three statements satisfiable. Justify your answer.

Solution 1 Let us denote by A, B, C Boolean variables stating that the respective components A, B, C are running. Then the 3 constraints can be rephrased as follows in propositional logic:

- 1. $A \rightarrow (\neg B \vee \neg C),$
- $2. \neg B \rightarrow (A \lor C),$
- 3. $C \rightarrow B$.

Because of the third constraint, namely $C \to B$, it is natural to test whether the conjunction of the three constraints is satisfiable with B = C = true. Then, since $\neg B \lor \neg C =$ false, to satisfy the first constraint, we must have A = false. With those values of the Boolean variables A, B, C, the second constraint is satisfied. Therefore, the machine of Professor Cuthbert Calculus be built.

Problem 2 Prove that $\log_2(9)$ is irrational.

Solution 2 if $\log_2(9)$ were equal to $\frac{m}{n}$, with m, n positive integers, without common factors, then, by the definition of logarithms, we would have

$$2^{\frac{m}{n}} = 9$$
.

Raising both sides to the n-th power, we obtain:

$$2^m = 9^n$$
.

Since n and m are non-zero, the numbers 9^n and 2^m are greater or equal to 9 and 2, respectively. Moreover, the numbers 9^n and 2^m are odd and even, respectively. Since a number cannot be both even and odd, the numbers 9^n and 2^m cannot be equal and we have reached a contradiction. Therefore, the number $\log_2(9)$ is irrational.

Problem 3 Let p, q, r, s be Boolean variables. For each of the following propositions, determine whether it is satisfiable or not:

- 1. $(p \lor (q \land (q \lor s)) \land (\neg p \lor (\neg q \land (\neg q \lor r)) \land (p \lor s) \land (\neg p \lor r).$
- 2. $(p \lor (q \land (q \lor s)) \land (\neg p \lor (\neg q \land (\neg q \lor r)) \land (p \lor \neg q) \land (\neg p \lor q)$

Solution 3

1. Using the absorption laws, the sub-expression $(q \land (q \lor s))$ can simply be rewritten as q and the sub-expression $(\neg q \land (\neg q \lor r))$ can simply be rewritten as $\neg q$. Therefore, the entire proposition becomes

$$(p \lor q) \land (\neg p \lor \neg q) \land (p \lor s) \land (\neg p \lor r).$$

Let us look first at $(p \lor q) \land (\neg p \lor \neg q)$. Both $(p \lor q)$ and $(\neg p \lor \neg q)$ are true if and only if p and q have opposite truth values. (This can be verified with a truth table.) Assume we choose $p = \mathsf{true}$ and $q = \mathsf{false}$. Then $(p \lor s)$ is true whatever is the truth value of s, meanwhile satisfying $(\neg p \lor r)$ requires to set $r = \mathsf{true}$. Finally, we can conclude that the entire proposition is satisfied with $p = \mathsf{true}$, $q = \mathsf{false}$ and $r = \mathsf{true}$.

2. Here again, $(q \land (q \lor s))$ can simply be rewritten as q and $(\neg q \land (\neg q \lor r))$ can simply be rewritten as $\neg q$. And the entire proposition becomes

$$(p \vee q) \ \land \ (\neg p \vee \neg q) \ \land \ (p \vee \neg q) \ \land \ (\neg p \vee q).$$

Remember that $(p \lor q) \land (\neg p \lor \neg q)$ means $p \leftrightarrow \neg q$, that is, p and q have **opposite** truth values. Similarly, the sub-expression $(p \lor \neg q) \land (\neg p \lor q)$ means that p and q have the same truth values, that is, $p \leftrightarrow q$. Therefore, the entire proposition becomes

$$(p \leftrightarrow \neg q) \land (p \leftrightarrow q),$$

which is clearly false. Finally, we can conclude that the entire proposition cannot be satisfied.

Problem 4 For any real number x, the *absolute value* of x, denoted by |x|, is defined as follows:

$$|x| = \begin{cases} x, & \text{if } x \ge 0 \\ -x, & \text{if } x < 0. \end{cases}$$

Prove that for all real numbers a, b, the following properties hold:

- 1. $|a+b| \leq |a| + |b|$ (called the triangular inequality),
- 2. |a-b| > |a| |b| (called the reverse triangular inequality),
- 3. if b is non-negative then we have: $|a| \le b \iff -b \le a \le b$.

Solution 4

- 1. Let a, b be two real numbers. We consider 4 cases
 - Case 1: $a \ge 0$ and $b \ge 0$. Then $a + b \ge 0$ and we have:

$$|a + b| = a + b = |a| + |b|.$$

• Case 2: a < 0 and $b \ge 0$. In this case a + b = -|a| + b and thus the sign of a + b depends on whether $|a| \le b$ or |a| > b holds. If $|a| \le b$ holds, then $a + b \ge 0$ and we have:

$$|a+b| = -|a| + b \le |a| + b = |a| + |b|.$$

If |a| > b holds, then a + b < 0 and we have:

$$|a+b| = |a| - b \le |a| + b = |a| + |b|.$$

- Case 3: $a \ge 0$ and b < 0. This case is simply deduced from the previous one by exchanging the role of a and b.
- Case 4: a < 0 and b < 0. Then a + b < 0 and we have:

$$|a + b| = -(a + b) = -|a| - |b| \le |a| + |b|.$$

QED. It should be noted, as pointed by one student in class, that other formulas about absolute values can be used to avoid the case discussion. These formulas are

$$\sqrt{a^2} = |a|$$
 and $|a \times b| = |a| \times |b|$.

Since $ab \leq |a \times b|$ and $|a \times b| = |a| \times |b|$ both hold, we deduce:

$$2ab \le 2|a| \times |b|,$$

and thus

$$a^{2} + 2ab + b^{2} \le |a|^{2} + 2|a| \times |b| + |b|^{2},$$

leading to

$$(a+b)^2 \le (|a|+|b|)^2$$
.

Taking the square-root of each side yields:

$$\sqrt{(a+b)^2} \le \sqrt{(|a|+|b|)^2},$$

that is,

$$|a+b| \le ||a|+|b|| = |a|+|b|.$$

2. Let a, b be two real numbers. One could proceed again by case inspection, discussing whether a - b is non-negative or not, and discussing whether |a| - |b| is non-negative or not. But there is a faster way, by applying the triangular inequality twice:

• From a = (a - b) + b, we deduce

$$|a| \le |a - b| + |b|,$$

and thus

$$|a| - |b| \le |a - b|.$$

• From -b = (a - b) + (-a) and |a| = |-a| and |b| = |-b|, we deduce

$$|b| \le |a - b| + |a|,$$

and thus

$$|b| - |a| \le |a - b|.$$

From $|a| - |b| \le |a - b|$ and $|b| - |a| \le |a - b|$, we deduce

$$||a| - |b|| \le |a - b|$$

Indeed, |a| - |b| is equal to either |a| - |b| or |b| - |a|. QED.

3. Let a, b be two real numbers. We have the following equivalences:

$$|a| \le b \iff (a \ge 0 \land |a| \le b) \lor (a < 0 \land |a| \le b)$$

$$\iff (a \ge 0 \land a \le b) \lor (a < 0 \land -a \le b)$$

$$\iff (a \ge 0 \land a \le b) \lor (a < 0 \land -b \le a)$$

$$\iff (a \ge 0 \land -b \le a \le b) \lor (a < 0 \land -b \le a \le b)$$

$$\iff -b \le a \le b$$

Indeed, for the second last equivalence, we can replace $(a \ge 0 \land a \le b)$ with $(a \ge 0 \land -b \le a \le b)$ since we know that $b \ge 0$ holds anyway. Similarly, we can replace $(a < 0 \land -b \le a)$ with $(a < 0 \land -b \le a \le b)$ for the same reason. QED. Of course, we can also prove the property

$$|a| < b \iff -b < a < b$$

by case inspection, discussing $a \ge 0$ or a < 0.