

- Announcements :
- 1) Tentatively, you can collect your midterms on Thursday 3:30 - 5:00pm, from our TAs (James & Eli).
 - 2) HW05 is due this Friday.
 - 3) There will be Quiz 03 on the Monday after the Reading week.

Recall: For absolute min/max in an interval:

4.1

- ① Find critical points
- ② Compare the y-values ~~at~~ of critical points and end points.

For local min/max:

- ① Find critical points.
- ② At each critical point $x=a$, find the second derivative $f''(a)$.
- ③ If $f''(a) < 0 \Rightarrow a$ is local max
 $f''(a) > 0 \Rightarrow a$ is local min.

A15)

 $f(x) = \sin(2x) + 3$, g is a function.

$$g'(0) = -4$$

$$(fg)''(0) = 4$$

 \leftarrow Product.Q. Find $g'(0)$.Ans. we need to use product rule

$$(fg)' = f'g + g'f$$

 \leftarrow Product rule

$$(fg)'' = (f'g + g'f)'$$

$$= (f'g)' + (g'f)'$$

$$= f''g + f'g'$$

 \leftarrow Product Rule on each term

$$+ g''f + g'f'$$

at $x=0$

$$\left[(fg)''(0) = \underbrace{f''(0)}_{-4} \cdot g(0) + \overset{\times}{f'(0)} \cdot g'(0) + \underbrace{g''(0)}_{-4} \cdot f(0) + g'(0) \cdot \overset{\times}{f'(0)} \right] - (\times)$$

$$f(x) = \sin(2x) + 3$$

at $x=0$

$$f(0) = 3$$

$$f'(x) = 2 \cos(2x)$$

 \longrightarrow

$$f'(0) = 2$$

$$f''(x) = -2 \cdot 2 \cdot \sin(2x)$$

$$\underline{f''(0) = 0}$$

Plugging everything back in (*)

$$4 = 0 \cdot g'(0) + \underbrace{2 \cdot g'(0)} + (-4) \cdot 3 + \underbrace{g'(0) \cdot 2}$$

$$\Rightarrow 4 = 4 \cdot g'(0) - 12$$

$$\Rightarrow 16 = 4 \cdot g'(0)$$

$$\Rightarrow \boxed{4 = g'(0)}$$

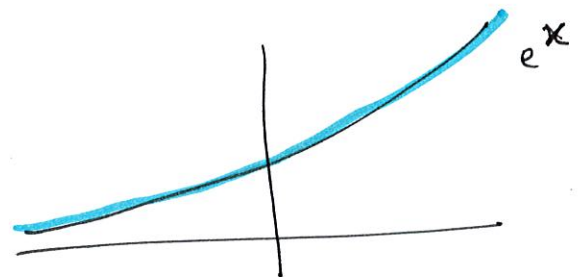
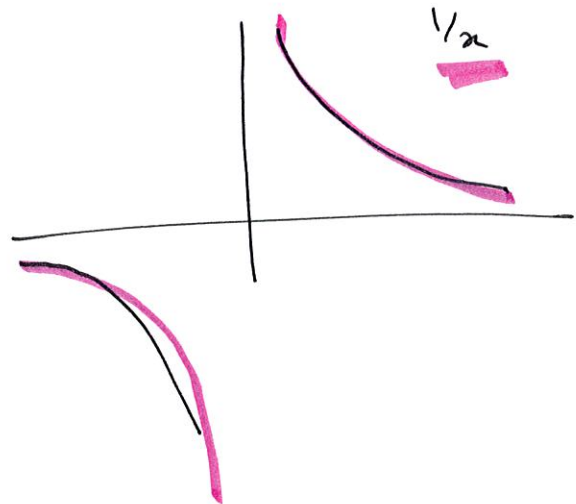
$$\lim_{x \rightarrow 0^-} e^{1/x} = 0$$

Reason: as $x \rightarrow 0^-$,

$$\frac{1}{x} \rightarrow -\infty$$

$$\text{as } \frac{1}{x} \rightarrow -\infty,$$

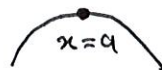
$$e^{1/x} \rightarrow 0.$$



Definitions:

03

- Local max : $x=a$ is local max for $f(x)$ if for all points near a , $f(x) \leq f(a)$.

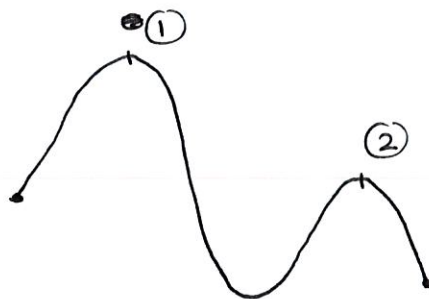


- Similarly, local min



- Absolute max/min : $x=a$ is absolute max in an interval $[c,d]$ if $f(a) \geq f(x)$ for all $x \in [c,d]$.

eg:

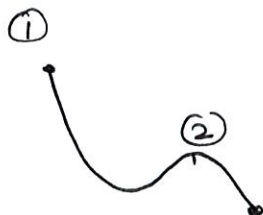


①, ② are both local max

but only

① is the absolute max.

- absolute max/min can also occur at the endpoints



① is the absolute max.

• Similarly, there is absolute min.

• ① $x=a$ is critical point of $f(x)$ if $f'(a)=0$

② $f'(a) > 0 \Rightarrow f(x)$ increasing near a .

$f'(a) < 0 \Rightarrow f(x)$ decreasing near a .

③ ~~$x=a$~~ is inflection point of $f(x)$ if $f''(a)=0$.

④ $f''(a) > 0 \Rightarrow f(x)$ is concave ~~downward~~ upward near a .

$f''(a) < 0 \Rightarrow f(x)$ is concave downward near a .

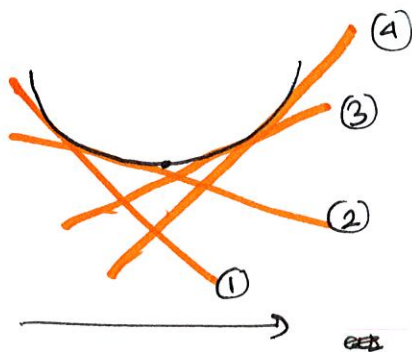
• at a local min/max $f(x)$ cannot be increasing nor can it be decreasing.
 \Rightarrow local min/max are critical points.

• How to tell if a critical point is a min/max?
 \parallel
 $f'(a)=0$.

Ans: • $f'(x)$ is the rate of change of $f(x)$. ~~at~~

• $f''(x)$ is the rate of change of $f'(x)$.

At local min:



as we increase x
the slope of
the tangent increases

\Rightarrow as we increase x , $f'(x)$ increases
(at a local min)

$\Rightarrow (f'(x))' = f''(x)$ is positive at a

i.e. at local min
 $x=a$, $f''(a) > 0$



\leftarrow This shape is called concave upward.

Basic example: $f(x) = x^2$

$$f'(x) = 2x$$

$$f''(x) = 2 > 0 \quad \text{for all } x$$



Concave upward
for all x

at local max,
 $x=a$

$$f''(a) < 0$$



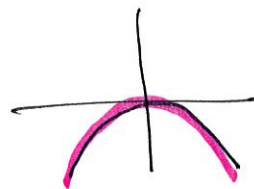
\leftarrow This shape is called concave downward.

Basic example:

$$f(x) = -x^2$$

$$f'(x) = -2x$$

$$f''(x) = -2 < 0$$



\Rightarrow for all x
concave downward

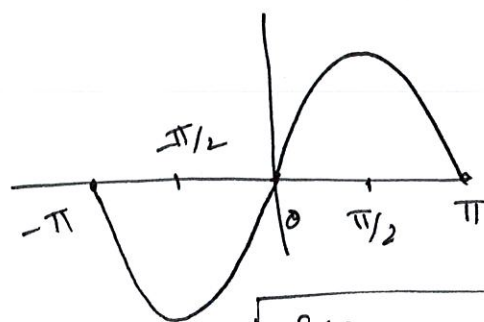
(06)

$f''(a) = 0 \Rightarrow$ neither concave upward nor concave downward.

Inflection point \equiv the curve is changing its concavity at the inflection point.

eg: $f(x) = \sin(x) \quad x \in (-\pi, \pi)$

Find ~~the~~ local min/max, concavity, inflection point etc.



inflection point.

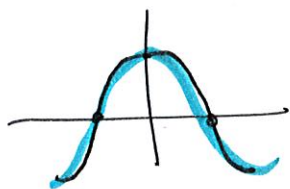
Ans: ~~local min~~

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= +\cos x \\ f''(x) &= -\sin x \end{aligned}$$

① Critical points:

$$f'(x) = 0$$

$$\Rightarrow \cos x = 0 \text{ in } (-\pi, \pi)$$



\Rightarrow

$$x = -\frac{\pi}{2}, \frac{\pi}{2}$$

② At the critical points

$$x = -\frac{\pi}{2}$$

$$f''(-\frac{\pi}{2}) = -\sin(-\frac{\pi}{2})$$

$$= -(-1)$$

$$= 1 > 0$$

\Rightarrow concave up

\Rightarrow local min.

$$x = \frac{\pi}{2}$$

$$f''(\frac{\pi}{2}) = -\sin(\frac{\pi}{2}) = -1 < 0$$

\Rightarrow concave down

\Rightarrow local max.

(3) Point of inflection:

$$f''(x) = 0$$

$$\Rightarrow -\sin x = 0 \quad \text{in } (-\pi, \pi)$$

\Rightarrow $x = 0$ \leftarrow at $x = 0$ $\sin x$ goes from being concave upward to concave downward.

Ch 4.

4.1, 4.3, 4.5, 4.7

critical points
local min/max
concavity

- ✓ 4.1 absolute min/max
- ✓ 4.3 local min/max
- 4.5 sketching graphs
- 4.7 word problems

* Q Sketch $f(x) = x + 2\sin x$ in $[0, 2\pi]$.

- (Find critical points, check up/down concavity inflection at the critical point.)

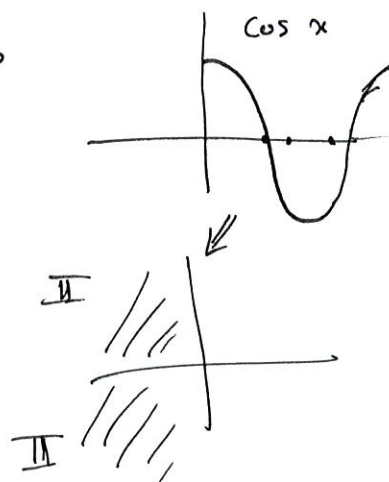
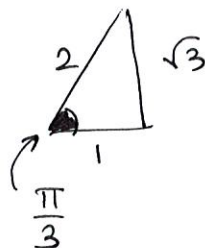
① Critical points:

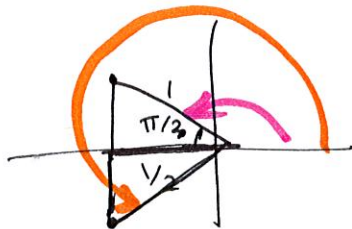
$$f'(x) = 0$$

$$\Rightarrow 1 + 2\cos x = 0$$

$$\Rightarrow 2\cos x = -1$$

$$\Rightarrow \boxed{\cos x = -\frac{1}{2}}$$





$$\pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\pi + \frac{\pi}{3} = \frac{4\pi}{3}$$

The critical points are

$$x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

(2) ~~f'(x)~~ $f(x) = x + 2\sin x$

$$f'(x) = 1 + 2\cos x$$

$$f''(x) = -2\sin x$$

i) at $x = \frac{2\pi}{3}$

$$f''\left(\frac{2\pi}{3}\right) = -2\sin\left(\frac{2\pi}{3}\right) < 0$$

as \sin is positive
in second quadrant

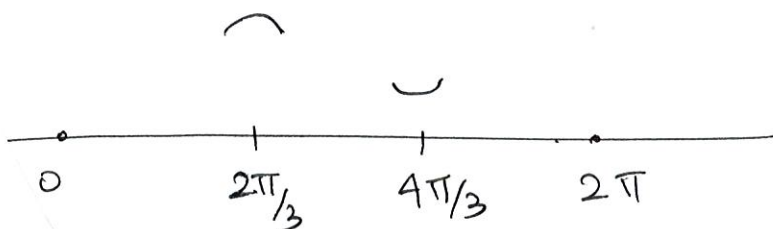
$\Rightarrow x = \frac{2\pi}{3}$ is local max i.e. $f(x)$ is concave
downward at $\frac{2\pi}{3}$

ii) at $x = \frac{4\pi}{3}$

$$f''\left(\frac{4\pi}{3}\right) = -2\sin\left(\frac{4\pi}{3}\right) > 0$$

as \sin is negative
in third quadrant

$\Rightarrow x = \frac{4\pi}{3}$ is local min i.e. $f(x)$ is concave
upward at $\frac{4\pi}{3}$



at $x=0, f(0)=0$

$f(x) = x + 2\sin x$

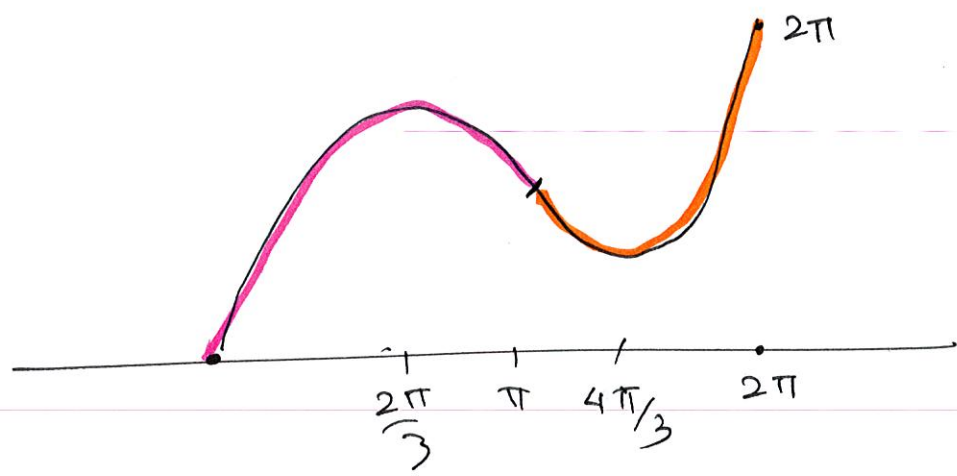
local
max

$$\begin{aligned} x = \frac{2\pi}{3}, \quad f(x) &= f\left(\frac{2\pi}{3}\right) \\ &= \frac{2\pi}{3} + 2 \cdot \sin\left(\frac{2\pi}{3}\right) \\ &= \frac{2\pi}{3} + 2 \cdot \frac{\sqrt{3}}{2} \end{aligned}$$

local
min

$x = \frac{4\pi}{3}, f\left(\frac{4\pi}{3}\right) = \text{something}$

$$\begin{aligned} x = 2\pi, \quad f(2\pi) &= 2\pi + 2\sin(2\pi) \\ &= 2\pi \end{aligned}$$



Point of
inflection :

$$\begin{aligned} f(x) &= x + 2\sin x \\ f'(x) &= 1 + 2\cos x \end{aligned}$$

$$\boxed{f''(x) = -2\sin x}$$

$$f''(x) = 0$$

$$\Rightarrow \sin x = 0$$

$$\Rightarrow \boxed{x = \pi}$$

← curve goes
from \uparrow
concave down
to concave up

Q. Sketch

$$f'(x) = 4x^2 - 12x = 0$$
$$f(x) = x^4 - 4x^3$$
$$4x^2(x-3) = 0$$
$$x=0/x=3$$

$$f''(x) = 12x^2 - 24x = 0$$
$$x=0 \text{ inflection}$$
$$x=3 \Rightarrow f''(x) > 0$$
$$\sim ?$$

Ans:

$$f(x) = x^4 - 4x^3$$

$$f'(x) = 4x^3 - 12x^2$$

$$f''(x) = 12x^2 - 24x$$

Critical points:

$$f'(x) = 0$$

$$\Rightarrow 4x^3 - 12x^2 = 0$$

$$\Rightarrow 4x^2(x-3) = 0$$

$$\Rightarrow x^2 = 0, \quad x-3 = 0$$

$$\Rightarrow \boxed{x=0}, \quad \boxed{x=3}$$

① $x=0$, check concavity:

$$f''(0) = 12 \cdot 0 - 24 \cdot 0$$

$$= 0$$

$\Rightarrow x=0$ is a point of inflection

$f(x)$ is changing concavity at $x=0$.

② $x=3$, check concavity:

$$f''(3) = 12 \cdot (3)^2 - 24 \cdot 3$$

$$= 12 \cdot 3 \cdot (3 - 2 \cdot 1)$$

$$= 12 \cdot 3 \cdot 1 > 0$$

\Rightarrow concave up

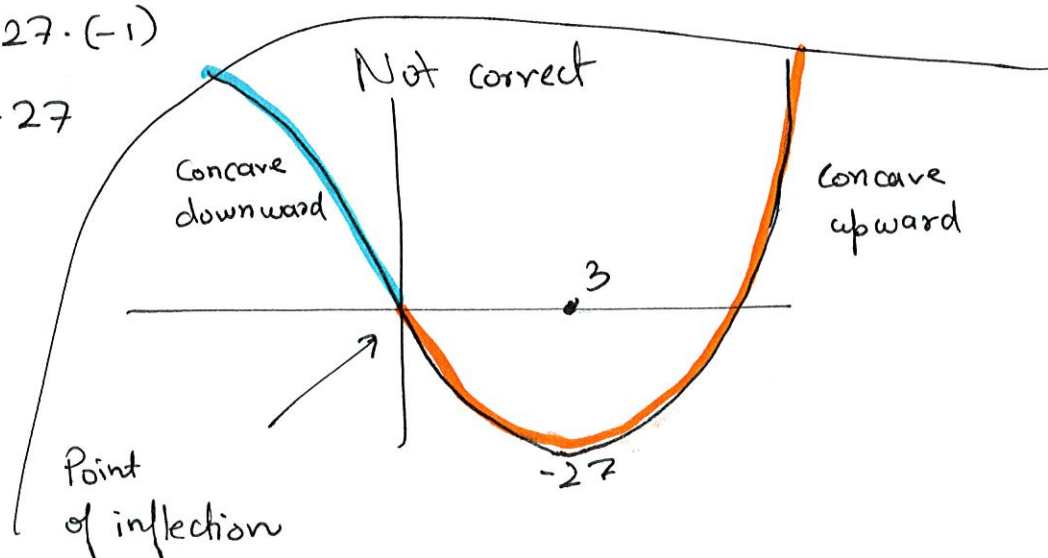
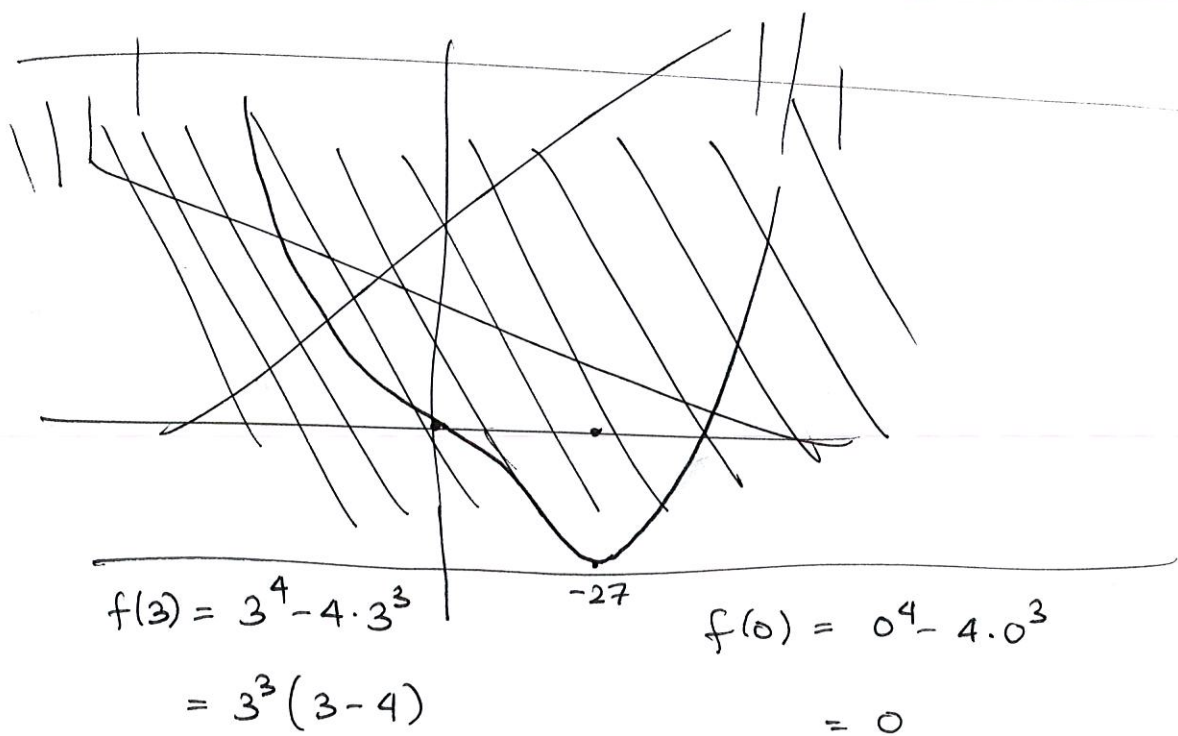
$\Rightarrow x=3$ is local min

④ What happens at $+\infty$, $-\infty$?

$$\lim_{x \rightarrow \infty} x^4 - 4x^3 = \lim_{x \rightarrow \infty} \underbrace{x^3}_{\rightarrow \infty} \underbrace{(x-4)}_{\rightarrow \infty} = \infty$$

$$\lim_{x \rightarrow -\infty} x^4 - 4x^3 = \lim_{x \rightarrow -\infty} \underbrace{x^3}_{-\infty} \underbrace{(x-4)}_{-\infty} = +\infty$$

- at both $\infty, -\infty$, $\lim f(x) \approx +\infty$
- at $x=3$ local min
- at $x=0$, both critical point, inflection point.



Step ⑤ Points of inflection:

$$f''(x) = 0$$

$$\Rightarrow 12x^2 - 24x = 0$$

$$\Rightarrow 12x(x-2) = 0$$

$$\Rightarrow x = 0, \quad x = 2 = 0$$

$$\Rightarrow x = 0, \quad x = 2$$

Summary:

$x = 0$

critical point & point of inflection

$$f(2) = 2^4 - 4 \cdot 2^3$$

$$= 2^3(2-4)$$

$$= 2^3 \cdot (-2)$$

$$= -16$$

$x = 2$

only point of inflection

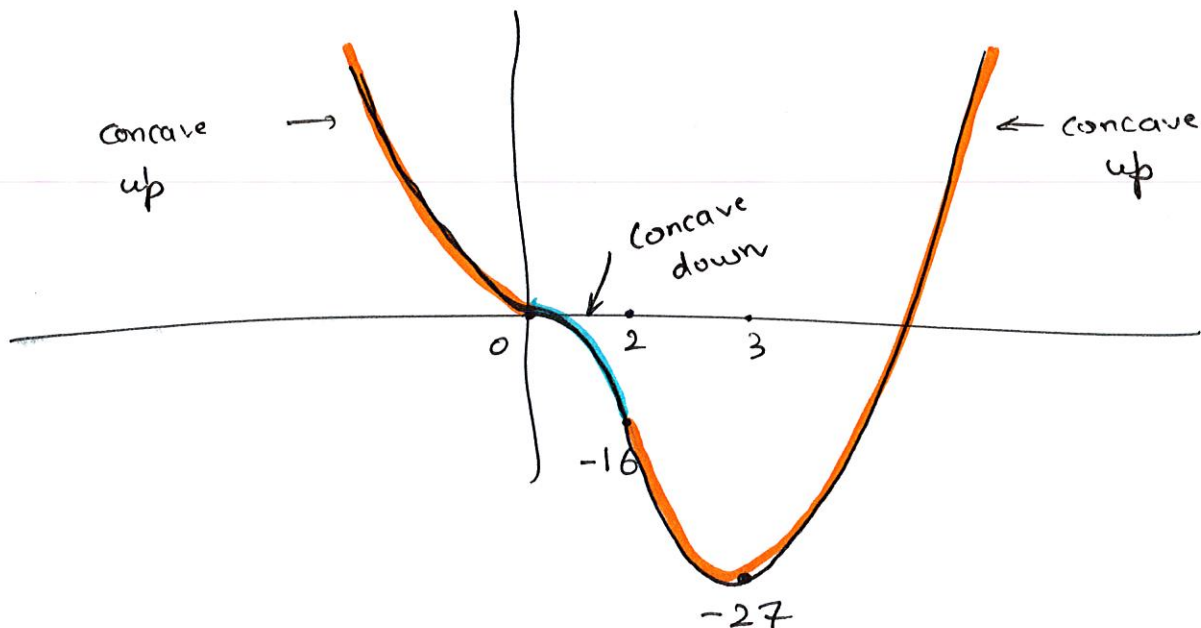
$x = 3$

only critical point, local min

$\lim_{x \rightarrow \infty}$

$\lim_{x \rightarrow -\infty}$

are both $+\infty$



~~At a critical po~~



• To sketch functions

• find critical points, find concavity at these critical points

• find inflection points,

• find "y-values" at all of the above points.

• If we have endpoints find y-values at them.

else find $\lim_{x \rightarrow \infty} f(x)$, $\lim_{x \rightarrow -\infty} f(x)$

• find points of discontinuities (vertical asymptotes).

''

how to prove a point
where it is discontinuous?

$\lim \neq \lim$.