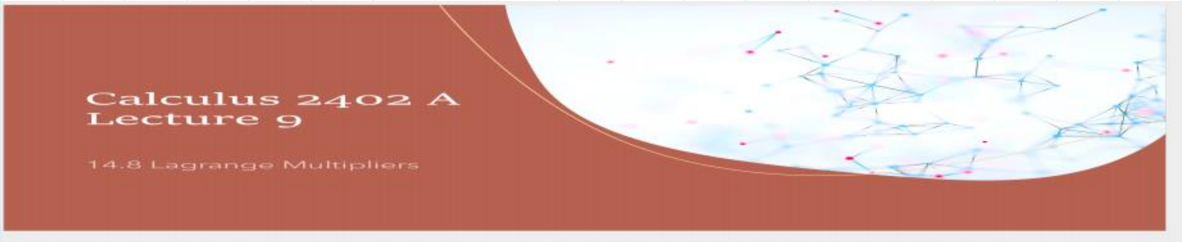
Lecture 9

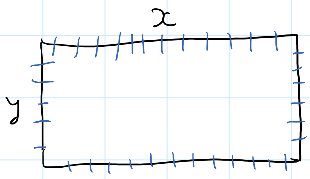
September 28, 2020 10:58 AM



Lagrange Multipliers (sec 14.8)

Consider the following problem.

A farmer has 100 m of fencing and wants to fence off a rectangular field. What are the dimensions of the field that has the largest area?



Let A be the area of the field. Then

$$A = xy$$

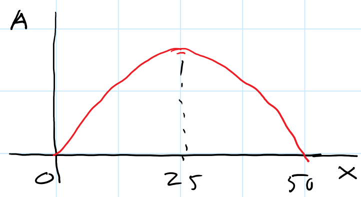
We also has

$$2x + 2y = 100 \Rightarrow x + y = 50$$

$$y = 50 - x$$

$$\therefore \quad A = x(50 - x)$$

A has its maximum value at $x = 25$



$$\Rightarrow \quad y = 50 - x = 50 - 25 = 25$$

$$\therefore \quad A_{max} = (25)(25) = 625 \text{ m}^2 \quad // \text{ Ans.}$$

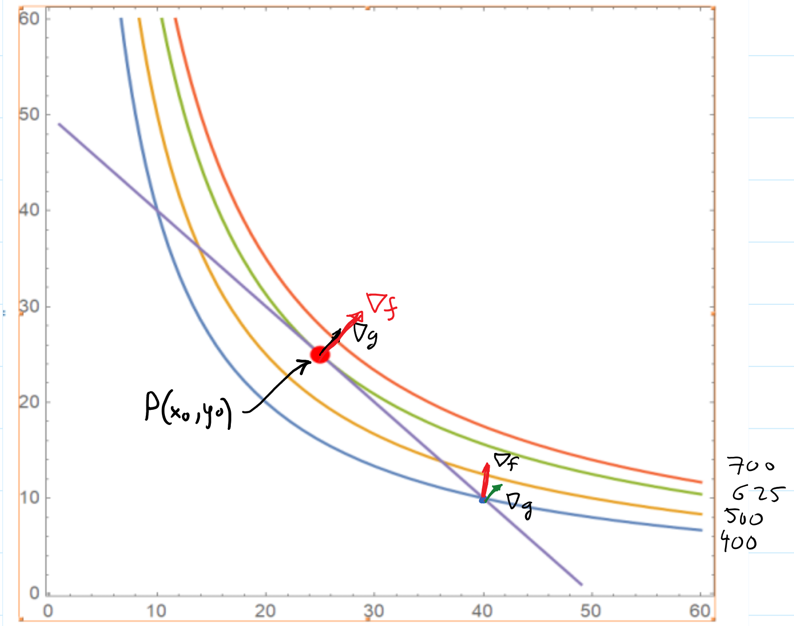
Now, let's look at the above problem in another way. We want to extremize (in this problem, we want to maximize) the function

$$f(x, y) = xy$$

subject to the constraint

$$x + y = 50$$

The level curves of $f(x, y)$ are shown in the below figure



Let $P(x_0, y_0)$ is the point where f has its extreme value. Then at P, both ∇f and ∇g are parallel. Hence,

$$\nabla f = \lambda \nabla g$$

where $f(x,y) = xy$ and $g(x,y) = x + y = 50$

$\nabla f = y\hat{i} + x\hat{j}$ and $\nabla g = \hat{i} + \hat{j}$

$\therefore \quad (y, x) = \lambda (1, 1)$

$$\Rightarrow \begin{cases} y = \lambda \\ x = \lambda \end{cases} \Rightarrow x = y \Rightarrow x = y = 25$$

$\therefore \quad f_{max} = (25)(25) = 625$ //

which leads us to

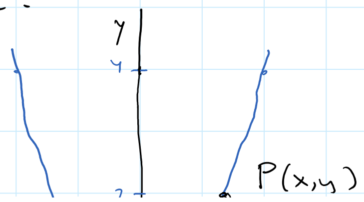
Method of Lagrange Multipliers: To find the maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$. Let (x_0, y_0, z_0) be the point where f has its extreme value then

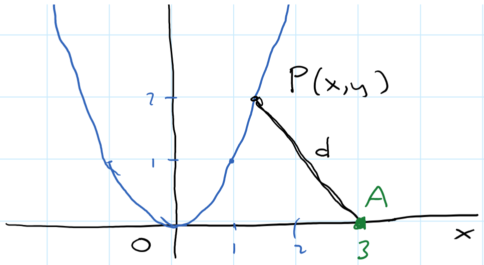
$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

The parameter λ is called a Lagrange multiplier.

<u>Ex1</u>: Find the shortest distance from the point $(3, 0)$ to the parabola $y = x^2$.

Solution





$$d = AP = \sqrt{(x_P - x_A)^2 + (y_P - y_A)^2} = \sqrt{(x-3)^2 + (y-0)^2}$$
$$= \sqrt{(x-3)^2 + y^2}$$

It is better if we work with d^2 instead of d so we let

$$f(x,y) = (x-3)^2 + y^2$$

we want to minimize f subject to the constraint

$$y = x^2 \quad \text{or} \quad g(x,y) = x^2 - y = 0$$

At the extreme values of f

$$\nabla f = \lambda \nabla g$$
$$2(x-3)\hat{i} + 2y\hat{j} = \lambda\left[2x\hat{i} - \hat{j}\right]$$

Equating the components

$$\begin{cases} \cancel{2}(x-3) = \cancel{2}x\lambda \Rightarrow x-3 = x\lambda \\ 2y = -\lambda \Rightarrow \lambda = -2y \end{cases}$$

$$\therefore \quad x-3 = x(-2y) = x(-2x^2)$$
$$x-3 = -2x^3$$

$$\therefore \quad 2x^3 + x - 3 = 0 \qquad (*)$$

We note that this equation has a root as 1. Hence,

$$(x-1)\underbrace{(2x^2+2x+3)}_{\text{has complex roots}} = 0$$

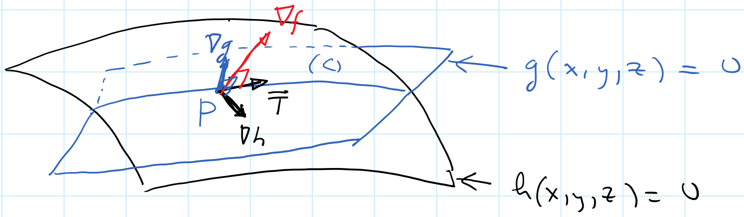
$$\therefore \quad x = 1$$

$$d_{min} = \sqrt{(1-3)^2 + (1)^2} = \sqrt{(-2)^2+1} = \sqrt{5} \quad // \text{Ans.}$$

Problems with more than one constraint

Suppose we want to extremize $f(x,y,z)$ subject to two constraints of the form $g(x,y,z) = 0$ and $h(x,y,z) = 0$.

Let (C) be the intersection curve of the surfaces $g(x,y,z)=0$ and $h(x,y,z)=0$. Assume f has extreme value at a point $P(x_0,y_0,z_0)$ on (C).



Let $\vec{T}$ be the tangent to (C) at P. Then

$$\vec{T} = \nabla g \times \nabla h \quad \Rightarrow \quad \vec{T} \perp \nabla h \text{ and } \vec{T} \perp \nabla g$$

Since f has extreme value on (C) at $P(x_0, y_0, z_0)$, $\nabla f \perp \vec{T}$. Hence,

∇f has to be in the plane formed by ∇g and ∇h, ie,

$$\nabla f = \lambda \nabla g + \mu \nabla h \quad \text{at} \quad P(x_0, y_0, z_0)$$

where λ, μ are Lagrange multipliers.

Example 2: Find the maximum and minimum of $f(x,y,z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

Solution

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

where $f(x,y,z) = x + 2y + 3z$

$g(x,y,z) = x - y + z = 1$

and $h(x,y,z) = x^2 + y^2 = 1$

$\Rightarrow \quad \nabla f = (1, 2, 3)$

$\nabla g = (1, -1, 1)$

$\nabla h = (2x, 2y, 0)$

then $(1, 2, 3) = \lambda (1, -1, 1) + \mu (2x, 2y, 0)$

Equating the components

$1 = \lambda + 2\mu x \Rightarrow \cancel{2}\mu x = -\cancel{2} \Rightarrow \mu x = -1 \Rightarrow \mu = -\frac{1}{x}$

$2 = -\lambda + 2\mu y \Rightarrow 2\mu y = 5 \Rightarrow 2\left(-\frac{1}{x}\right) y = 5$

$\boxed{3 = \lambda}$ $\qquad \Rightarrow \boxed{y = -\frac{5}{2}x}$

Subst this into $x^2 + y^2 = 1$

$$x^2 + \left(-\frac{5}{2}x\right)^2 = 1$$

$$x^2 + \frac{25}{4}x^2 = 1 \Rightarrow \frac{29}{4}x^2 = 1$$

$$x = \pm \frac{2}{\sqrt{29}} \Rightarrow y = \mp \frac{5}{\sqrt{29}}$$

$$x = \pm \frac{2}{\sqrt{29}} \Rightarrow y = \mp \frac{5}{\sqrt{29}}$$

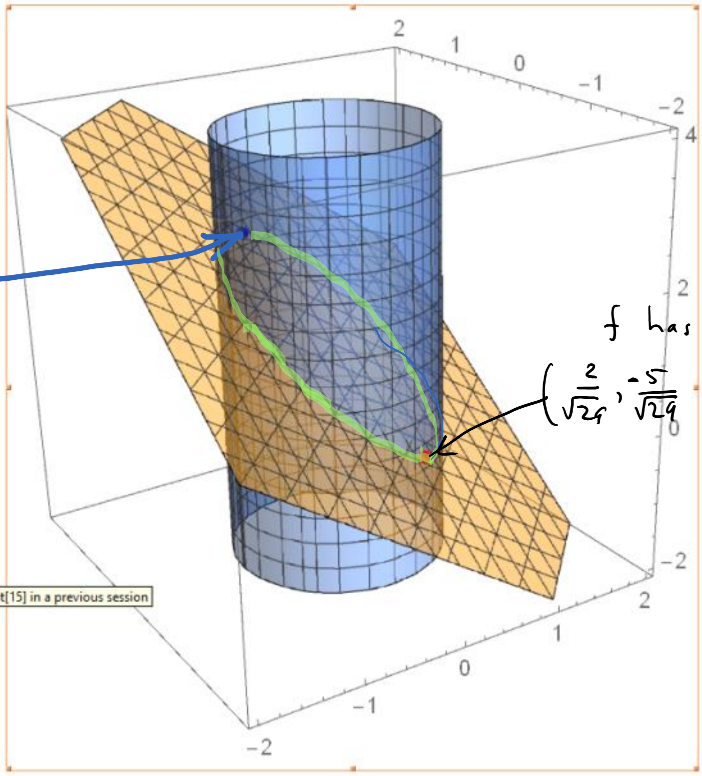
$$\therefore\ z = 1 - x + y = 1 - \left(\frac{2}{\sqrt{29}}\right) - \frac{5}{\sqrt{29}} = 1 - \frac{7}{\sqrt{29}}$$

and $$z = 1 - x + y = 1 - \left(-\frac{2}{\sqrt{29}}\right) + \frac{5}{\sqrt{29}} = 1 + \frac{7}{\sqrt{29}}$$

$$\therefore\ f\left(\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}, 1 - \frac{7}{\sqrt{29}}\right) = x + 2y + 3z$$
$$= \frac{2}{\sqrt{29}} - \frac{10}{\sqrt{29}} + 3\left(1 - \frac{7}{\sqrt{29}}\right)$$
$$= 3 - \sqrt{29} \Leftarrow \text{Min value}$$

$$f\left(-\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1 + \frac{7}{\sqrt{29}}\right) = -\frac{2}{\sqrt{29}} + \frac{10}{\sqrt{29}} + 3\left(1 + \frac{7}{\sqrt{29}}\right) = 3 + \sqrt{29} \Leftarrow \text{Max value}$$

f has its max value @ $\left(-\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1 + \frac{7}{\sqrt{29}}\right)$



f has min value @ $\left(\frac{2}{\sqrt{29}}, \frac{-5}{\sqrt{29}}, 1 - \frac{7}{\sqrt{29}}\right)$

See you on Wednesday!