Gauss-Jordan Elimination

Elementary row operations

The following operations are the *elementary row operations* which can be performed to a matrix:
1) Multiply any row of the matrix by any non-zero scalar. $R_i \longrightarrow cR_i, \ (c \neq 0)$

2) Interchange the positions of any two rows in the matrix. $R_i \quad \leftrightarrow \quad R_j$

3) Replace any row in the matrix by the sum of that row and a scalar multiple of any other row of the matrix. $R_i \longrightarrow R_i + cR_j$

No other operations are allowed.

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Definition Two matrices are *row equivalent* if one matrix can be transformed into the other by applying a sequence of elementary row operations.

RREF

The following matrices are not RREF

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

RREF

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2} R_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \longleftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 3R_2} \begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

Procedure

1) Always start from the top to the bottom

2) Always read/think from the left to the right

3) If the first non-zero element sitting at a lower row is closer to the left than the first non-zero element sitting at an upper row, then interchange these two rows.

Procedure

1) Always start from the top to the bottom

2) Always read/think from the left to the right

3) If the first non-zero element sitting at a lower row is closer to the left than the first non-zero element sitting at an upper row, then interchange these two rows.

$$\begin{bmatrix} 0 & 0 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 \end{bmatrix} \xrightarrow{R_2 \longrightarrow \frac{1}{2} R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & \frac{3}{2} \end{bmatrix}$$

4) To obtain the leading 1, observe the first non-zero element in each row (from the top to the bottom). If the first non-zero entry is not 1, let this row multiply a non-zero scalar.

5) If the leading 1 is obtained, use it to kill off other non-zero entries in the same column.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & \frac{3}{2} \end{bmatrix} \xrightarrow{R_1 \longrightarrow R_1 - 3R_2} \begin{bmatrix} 1 & 2 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} \end{bmatrix}$$

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6) If a row of entirely zeros is otained, move it to the bottom of the matrix.

7) Repeated the above steps until we have an RREF.

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$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & \frac{3}{2} \end{bmatrix} \xrightarrow{R_1 \longrightarrow R_1 - 3R_2} \begin{bmatrix} 1 & 2 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} \end{bmatrix}$$

6) If a row of entirely zeros is otained, move it to the bottom of the matrix.

7) Repeated the above steps until we have an RREF.

Definition The process of applying elementary row operations to transform a matrix into RREF is called *row-reduction*, or simply *reduction*. We can refer to this as *row-reducing* or *reducing* the matrix.

Examples

Use elementary row operations to find the RREF equivalent to the given matrix.

$$\begin{bmatrix} 2 & 1 & 2 \\ 0 & -4 & 1 \\ 1 & -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 1 & 3 \\ 1 & -1 & -1 & 1 \\ 2 & 0 & 1 & 2 \end{bmatrix}$$

Examples

Use elementary row operations to find the RREF equivalent to the given matrix.

$$\begin{bmatrix} 2 & 1 & 2 \\ 0 & -4 & 1 \\ 1 & -2 & 0 \end{bmatrix} \xrightarrow{R_1 \longrightarrow \frac{1}{2}R_1} \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 0 & -4 & 1 \\ 1 & -2 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 0 & -4 & 1 \\ 0 & -\frac{5}{2} & -1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to -\frac{1}{4}R_2} \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 1 & -\frac{1}{4} \\ 0 & -\frac{5}{2} & -1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - \frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & \frac{9}{8} \\ 0 & 1 & -\frac{1}{4} \\ 0 & -\frac{5}{2} & -1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + \frac{5}{2}R_2} \begin{bmatrix} 1 & 0 & \frac{9}{8} \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & -\frac{13}{8} \end{bmatrix}$$

$$\xrightarrow{R_3 \to -\frac{8}{13}R_3} \begin{bmatrix} 1 & 0 & \frac{9}{8} \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + \frac{1}{4}R_3} \begin{bmatrix} 1 & 0 & \frac{9}{8} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - \frac{9}{8}R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Examples

Use elementary row operations to find the RREF equivalent to the given matrix.

$$\begin{bmatrix} 0 & -1 & 1 & 3 \\ 1 & -1 & -1 & 1 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & -1 & 1 & 3 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & -1 & 1 & 3 \\ 0 & 2 & 3 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to -R_2} \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & -1 & -3 \\ 0 & 2 & 3 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 + R_2} \begin{bmatrix} 1 & 0 & -2 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 2 & 3 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 0 & -2 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 5 & 6 \end{bmatrix} \xrightarrow{R_3 \to \frac{1}{5} R_3} \begin{bmatrix} 1 & 0 & -2 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & \frac{6}{5} \end{bmatrix}$$

$$\xrightarrow[R_2 \to R_2 + R_3]{R_1 \to R_1 + 2R_3} \begin{bmatrix} 1 & 0 & 0 & \frac{2}{5} \\ 0 & 1 & 0 & -\frac{9}{5} \\ 0 & 0 & 1 & \frac{6}{5} \end{bmatrix}$$

Theorem 1) If any sequence of elementary row operations are applied to the augmented matrix for a SLE, a new augmented matrix is obtained and has a corresponding SLE which has exactly the same solution(s) as (i.e. is equivalent to) the original SLE.

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2) Every matrix can be transformed into RREF by applying a finite sequence of elementary row operations.

Theorem 1) If any sequence of elementary row operations are applied to the augmented matrix for a SLE, a new augmented matrix is obtained and has a corresponding SLE which has exactly the same solution(s) as (i.e. is equivalent to) the original SLE.

2) Every matrix can be transformed into RREF by applying a finite sequence of elementary row operations.

3) The row-reduced echelon form of a matrix is unique.

Gauss-Jordan Elimination

Given any system of linear equations:

1. form the augmented matrix for the SLE;

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1. form the augmented matrix for the SLE;

2. use elementary row operations to transform the augmented matrix to one in which the coefficient matrix part is in RREF;

3. determine the set of all solutions to the system corresponding to this final augmented matrix.

The final SLE is equivalent to the original SLE, so these are also the solutions to the original system.

Solve the linear system by the Gauss-Jordan elimination.

$$\begin{aligned} x - 2y + z &= 5 \\ -2x + 3y + z &= 1 \\ x + 3y + 2z &= 2 \end{aligned}$$

(2)

$$\begin{aligned} 3x + 3y + 12z &= 6 \\ x + y + 4z &= 2 \\ 2x + 5y + 20z &= 10 \\ x + 7y + 28z &= 14 \end{aligned}$$

(3)

$$\begin{aligned} x_1 + x_2 - x_3 + x_4 &= 0 \\ x_3 + x_4 + x_5 &= 0 \\ 2x_1 + 2x_2 - x_3 + x_5 &= 0 \\ x_1 + x_2 - 2x_3 - x_5 &= 0 \\ 2x_3 - 4x_4 + 2x_5 &= 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 0 \end{aligned}$$

(4)

$$\begin{aligned}
x_1 + 2x_2 \qquad\qquad\quad &= 1\\
x_1 + 2x_2 + 3x_3 + x_4 &= 0\\
-x_1 - \ x_2 + \ x_3 + \ x_4 &= -2\\
x_2 + \ x_3 + \ x_4 &= -1\\
-x_2 + 2x_3 \qquad &= 0
\end{aligned}$$

(5) Consider the following matrix.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1\\ 0 & 1 & 1 & 0\\ 0 & 1 & k & k \end{array}\right]$$

For which value(s) of k, the SLE corresponding to this matrix has
1) no solutions;
2) a unique solution;