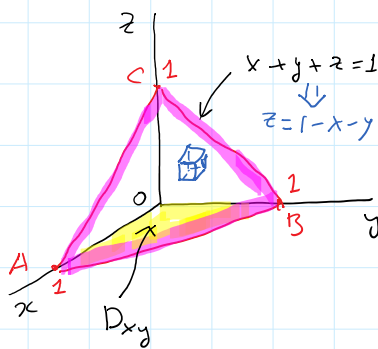CALCULUS 2402A
LECTURE 17

15.6 Triple Integrals (Part 2)

Ex 2: Find the mass of the tetrahedron bounded by the planes
$x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$ and the mass density is
$f(x, y, z) = y$.

Solution

In this case, D is the tetrahedron
OABC. We can consider D as a
type I region so we integrate wrt z
first.

Consider a typical volume element dV.
It mass dm is

$$dm = (\text{mass density}) \text{ volume}$$
$$= f(x,y,z) \, dV$$
$$= y \, dV$$

$$\therefore \quad m = \iiint_D dm = \iiint_D y \, dV$$

$$= \iint_{D_{xy}} \left(\int_{z=0 \,(\text{the } xy\text{-plane})}^{z=1-x-y \,(\text{the plane } ABC)} y \, dz \right) dA$$

$$= \int_0^1 \int_{y=0}^{1-x} \int_0^{1-x-y} y \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} y(1-x-y) \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \left[(1-x)y - y^2 \right] dy \, dx$$

$$= \int_0^1 \left((1-x)\frac{y^2}{2} - \frac{y^3}{3} \right) \Big|_{y=0}^{1-x} dx$$

$$= \int_0^1 \left[(1-x)\frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} \right] dx$$
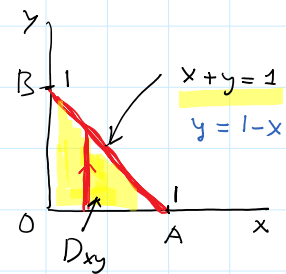
$$= \int_0^1 \left[(1-x)\frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} \right] dx$$

$$= \int_0^1 \left[\frac{(1-x)^3}{2} - \frac{(1-x)^3}{3} \right] dx$$

$$= \int_0^1 \frac{(1-x)^3}{6} dx \qquad \text{Let } \boxed{u = 1-x}$$
$$\hspace{10cm} du = -dx$$

$$= \int_1^0 \frac{u^3}{6} (-du)$$

$$\underset{\underset{\text{Switching the limits}}{\uparrow}}{=} \frac{1}{6} \int_0^1 u^3 \, du = \frac{1}{6} \left(\frac{u^4}{4} \right) \Big|_0^1 = \frac{1}{6} \left(\frac{1}{4} \right) = \frac{1}{24} \quad /\!/ \text{Ans.}$$

In double integrals, there are two ways of integration. They are $dy\,dx$ and $dx\,dy$. However, for triple integrals, there are $3! = 6$ ways of integration. They are

$$\begin{array}{ccc} dx\,dy\,dz & dy\,dz\,dx & dz\,dx\,dy \\ dx\,dz\,dy & dy\,dx\,dz & dz\,dy\,dx \end{array}$$

Ex 3: Consider the iterated integral

$$I = \int_0^1 \int_0^{x^2} \int_0^y f(x,y,z) \, dz \, dy \, dx \qquad \Longleftarrow \quad (\bigstar)$$

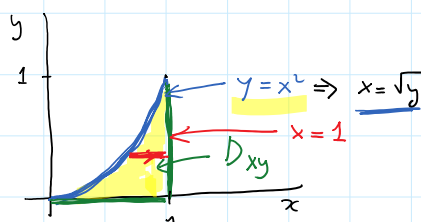Change the order of integration to $dz\,dx\,dy$, $dx\,dy\,dz$, $dx\,dz\,dy$, $dy\,dz\,dx$ and $dy\,dx\,dz$.

Solution

$(\bigstar)$ can be written as

$$I = \iint_{D_{xy}} \left(\int_0^y f(x,y,z) \, dz \right) dA$$

where D_{xy} is the projection of D onto the xy-plane.

$$D_{xy} = \left\{ (x,y) \mid 0 \le x \le 1, \ 0 \le y \le x^2 \right\}$$

$$I = \int_0^1 \int_{x=\sqrt{y}}^{x=1} \int_0^y f(x,y,z) \, dz \, dx \, dy$$

$$\int_0^{} \int_{x=\sqrt{y}}^{} \int_0^{} f(x,y,z)\, dz\, dx\, dy$$
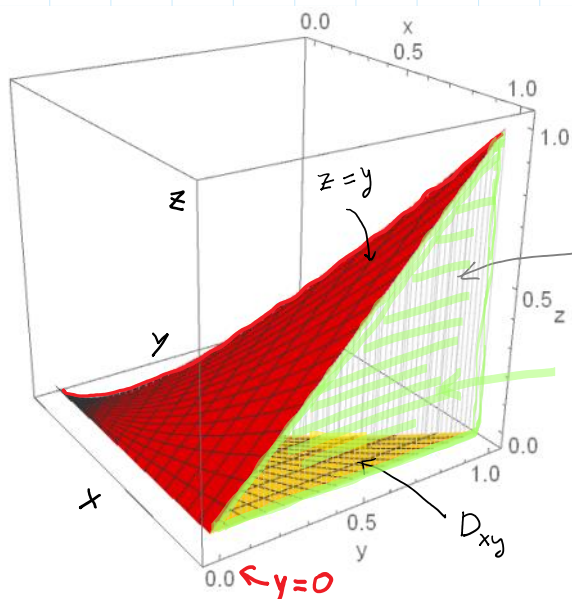
$$= \int_0^1 \int_{\sqrt{y}}^1 \int_0^y f(x,y,z)\, dz\, dx\, dy \qquad // Ans.$$
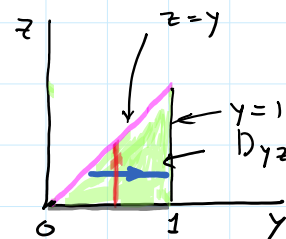
If we integrate w.r.t x first, then

$$I = \iint_{D_{yz}\ ?} \left(\int_{\sqrt{y}}^1 f(x,y,z)\, dx \right) dA = \int_0^1 \int_{y=z}^{y=1} \int_{\sqrt{y}}^1 f(x,y,z)\, dx\, dy\, dz$$

$$= \int_0^1 \int_z^1 \int_{\sqrt{y}}^1 f(x,y,z)\, dx\, dy\, dz \qquad // Ans.$$

$$I = \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\, dx\, dz\, dy \qquad // Ans.$$
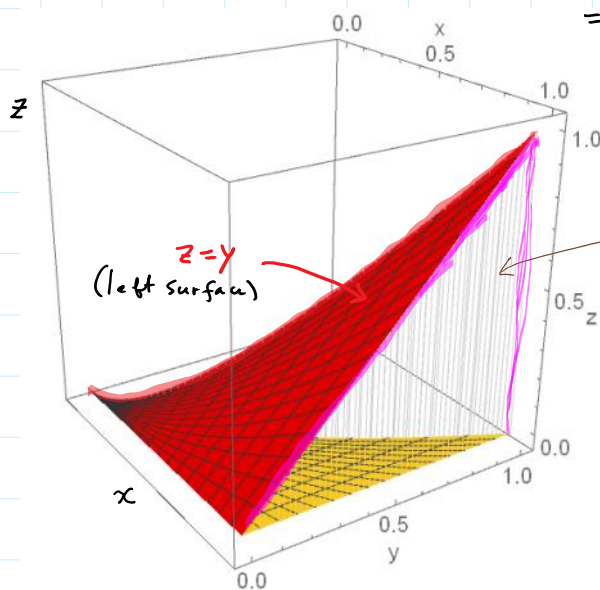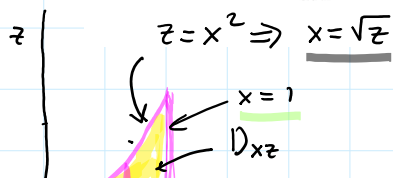
$$y = x^2 \implies x = \sqrt{y}$$

$$X = 1$$

$$I = \iint_{D_{xz}\ ?} \left(\int_z^{x^2} f(x,y,z)\, dy \right) dA = \int_0^1 \int_{z=0}^{z=x^2} \int_z^{x^2} f(x,y,z)\, dy\, dz\, dx$$

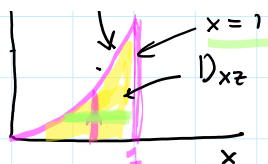$$= \int_0^1 \int_0^{x^2} \int_z^{x^2} f(x,y,z)\, dy\, dz\, dx \qquad // Ans.$$

$$y = x^2 \ (\text{a cylinder})$$
The right surface

$$LEFT \longrightarrow RIGHT$$

The curve of intersection of the surfaces $z = y$ and $y = x^2$ is $z = x^2$

$$z = x^2 \implies x = \sqrt{z}$$

$$x = 1$$

$$D_{xz}$$

$$I = \int_0^1 \int_{}^{x=1} \int_{}^{x^2} f(x,y,z)\, dy\, dx\, dz$$

$$I = \int_0^1 \int_{x=\sqrt{z}}^{x=1} \int_z^{x^2} f(x,y,z)\, dy\, dx\, dz$$

$$= \int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} f(x,y,z)\, dy\, dx\, dz \qquad /\!/ \text{ Ans.}$$

<u>N.B</u>: If $f(x,y,z) \equiv 1$ throughout the region D then the triple integral becomes $\iiint_D dV$ which is the volume V of D.

$$V = \int_0^1 \int_0^{x^2} \int_0^y dz\, dy\, dx = \int_0^1 \int_0^{x^2} y\, dy\, dx$$

$$= \int_0^1 \left(\frac{y^2}{2}\right)\Big|_{y=0}^{x^2} dx = \int_0^1 \frac{x^4}{2}\, dx = \frac{1}{2}\frac{x^5}{5}\Big|_0^1 = \frac{1}{10} \quad /\!/ \text{ Ans.}$$

OR

$$V = \int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} dy\, dx\, dz = \int_0^1 \int_{\sqrt{z}}^1 (x^2 - z)\, dx\, dz$$

$$\sqrt{z} = z^{\frac{1}{2}}$$

$$= \int_0^1 \left(\frac{x^3}{3} - zx\right)\Big|_{x=\sqrt{z}}^1 dz = \int_0^1 \left[\frac{1}{3} - z - \left(\frac{z^{3/2}}{3} - z\, z^{\frac{1}{2}}\right)\right] dz$$

$$\underbrace{\qquad}_{-\frac{2}{3}z^{3/2}}$$

$$= \int_0^1 \left(\frac{1}{3} - z + \frac{2}{3}z^{3/2}\right) dz = \left(\frac{1}{3}z - \frac{z^2}{2} + \frac{2}{3}\frac{2}{5}z^{5/2}\right)\Big|_0^1$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{4}{15} = \frac{10 - 15 + 8}{30} = \frac{3}{30} = \frac{1}{10} \quad /\!/ \text{ Ans.}$$

the same!

See you on Wednesday!