Dot products and cross products

Xin Fu

Western University

xfu82@uwo.ca

Other notations

Recall

$$c\vec{v} = \begin{cases} (cv_1, cv_2) & \text{if } \vec{v} = (v_1, v_2) \text{ in } \mathbb{R}^2 \\ (cv_1, cv_2, cv_3) & \text{if } \vec{v} = (v_1, v_2, v_3) \text{ in } \mathbb{R}^3. \end{cases}$$

$$\vec{u}+\vec{v} = \begin{cases} (u_1 + v_1, u_2 + v_2) & \text{if } \vec{u} = (u_1, u_2) \text{ and } \vec{v} = (v_1, v_2) \text{ in } \mathbb{R}^2 \\ (u_1 + v_1, u_2 + v_2, u_3 + v_3) & \text{if } \vec{u} = (u_1, u_2, u_3) \text{ and } \vec{v} = (v_1, v_2, v_3) \text{ in } \mathbb{R}^3. \end{cases}$$

Other notations

Recall

$$c\vec{v} = \begin{cases} (cv_1, cv_2) & \text{if } \vec{v} = (v_1, v_2) \text{ in } \mathbb{R}^2 \\ (cv_1, cv_2, cv_3) & \text{if } \vec{v} = (v_1, v_2, v_3) \text{ in } \mathbb{R}^3. \end{cases}$$

$$\vec{u}+\vec{v} = \begin{cases} (u_1 + v_1, u_2 + v_2) & \text{if } \vec{u} = (u_1, u_2) \text{ and } \vec{v} = (v_1, v_2) \text{ in } \mathbb{R}^2 \\ (u_1 + v_1, u_2 + v_2, u_3 + v_3) & \text{if } \vec{u} = (u_1, u_2, u_3) \text{ and } \vec{v} = (v_1, v_2, v_3) \text{ in } \mathbb{R}^3. \end{cases}$$

By this way, we can write

$$(v_1, v_2) = v_1(1, 0) + v_2(0, 1)$$
$$(v_1, v_2, v_3) = v_1(1, 0, 0) + v_2(0, 1, 0) + v_3(0, 0, 1).$$

Recall

$$c\vec{v} = \begin{cases} (cv_1, cv_2) & \text{if } \vec{v} = (v_1, v_2) \text{ in } \mathbb{R}^2 \\ (cv_1, cv_2, cv_3) & \text{if } \vec{v} = (v_1, v_2, v_3) \text{ in } \mathbb{R}^3. \end{cases}$$

$$\vec{u}+\vec{v} = \begin{cases} (u_1 + v_1, u_2 + v_2) & \text{if } \vec{u} = (u_1, u_2) \text{ and } \vec{v} = (v_1, v_2) \text{ in } \mathbb{R}^2 \\ (u_1 + v_1, u_2 + v_2, u_3 + v_3) & \text{if } \vec{u} = (u_1, u_2, u_3) \text{ and } \vec{v} = (v_1, v_2, v_3) \text{ in } \mathbb{R}^3. \end{cases}$$

By this way, we can write

$$(v_1, v_2) = v_1(1, 0) + v_2(0, 1)$$
$$(v_1, v_2, v_3) = v_1(1, 0, 0) + v_2(0, 1, 0) + v_3(0, 0, 1).$$

Moreover, let $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$ in $\mathbb{R}^2$ or let $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$ and $\vec{k} = (0, 0, 1)$ in $\mathbb{R}^3$. Then we can rewrite vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ as

$$(v_1, v_2) = v_1\vec{i} + v_2\vec{j} \quad \text{in } R^2$$
$$(v_1, v_2, v_3) = v_1\vec{i} + v_2\vec{j} + v_3\vec{k} \quad \text{in } R^3.$$

The vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ are written as

$$(v_1, v_2) = v_1\vec{i} + v_2\vec{j} \text{ in } R^2$$
$$(v_1, v_2, v_3) = v_1\vec{i} + v_2\vec{j} + v_3\vec{k} \text{ in } R^3.$$

Examples

1. Express $(1, 2, 3)$ in terms of $\vec{i}, \vec{j}, \vec{k}$.
2. Express $2\vec{i} - \vec{j} + 4\vec{k}$ as an ordered triple.
3. Find the length of $\vec{i} - \vec{j} + 2\vec{k}$.
4. Simplify $2(3\vec{i} + 2\vec{k}) - 2(-\vec{i} + 4\vec{j} - 2\vec{k})$.

The vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ are written as

$$(v_1, v_2) = v_1\vec{i} + v_2\vec{j} \text{ in } R^2$$
$$(v_1, v_2, v_3) = v_1\vec{i} + v_2\vec{j} + v_3\vec{k} \text{ in } R^3.$$

Examples

1. Express $(1, 2, 3)$ in terms of $\vec{i}, \vec{j}, \vec{k}$. $\vec{i} + 2\vec{j} + 3\vec{k}$
2. Express $2\vec{i} - \vec{j} + 4\vec{k}$ as an ordered triple. $(2, -1, 4)$
3. Find the length of $\vec{i} - \vec{j} + 2\vec{k}$.
$\left\|\vec{i} - \vec{j} + 2\vec{k}\right\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$
4. Simplify $2(3\vec{i} + 2\vec{k}) - 2(-\vec{i} + 4\vec{j} - 2\vec{k})$.
$2(3\vec{i} + 2\vec{k}) - 2(-\vec{i} + 4\vec{j} - 2\vec{k}) = 6\vec{i} + 4\vec{k} + 2\vec{i} - 8\vec{j} + 4\vec{k} = 8\vec{i} - 8\vec{j} + 8\vec{k}$

Dot products

Definition The *dot product* of $\vec{u}$ and $\vec{v}$ is denoted by $\vec{u} \cdot \vec{v}$ and is defined by

$$\vec{u} \cdot \vec{v} = \begin{cases} u_1 v_1 + u_2 v_2 & \text{if } \vec{u}, \vec{v} \text{ in } \mathbb{R}^2 \\ u_1 v_1 + u_2 v_2 + u_3 v_3 & \text{if } \vec{u}, \vec{v} \text{ in } \mathbb{R}^3. \end{cases}$$

Definition The *dot product* of $\vec{u}$ and $\vec{v}$ is denoted by $\vec{u} \cdot \vec{v}$ and is defined by

$$\vec{u} \cdot \vec{v} = \begin{cases} u_1 v_1 + u_2 v_2 & \text{if } \vec{u}, \vec{v} \text{ in } \mathbb{R}^2 \\ u_1 v_1 + u_2 v_2 + u_3 v_3 & \text{if } \vec{u}, \vec{v} \text{ in } \mathbb{R}^3. \end{cases}$$

Note that the dot product of vectors is a number!

Definition The *dot product* of $\vec{u}$ and $\vec{v}$ is denoted by $\vec{u} \cdot \vec{v}$ and is defined by

$$\vec{u} \cdot \vec{v} = \begin{cases} u_1 v_1 + u_2 v_2 & \text{if } \vec{u}, \vec{v} \text{ in } \mathbb{R}^2 \\ u_1 v_1 + u_2 v_2 + u_3 v_3 & \text{if } \vec{u}, \vec{v} \text{ in } \mathbb{R}^3. \end{cases}$$

Note that the dot product of vectors is a number!

Examples

Compute the dot product $\vec{u} \cdot \vec{v}$.

1. $\vec{u} = (2, -1)$ and $\vec{v} = (5, 3)$.
2. $\vec{u} = (1, -1, 4)$ and $\vec{v} = (-2, 3, 1)$.
3. $\vec{u} = (2, 1, -3)$ and $\vec{v} = (1, 1, 1)$.
4. $\vec{u} \cdot \vec{u}$.

Definition The *dot product* of $\vec{u}$ and $\vec{v}$ is denoted by $\vec{u} \cdot \vec{v}$ and is defined by

$$\vec{u} \cdot \vec{v} = \begin{cases} u_1 v_1 + u_2 v_2 & \text{if } \vec{u}, \vec{v} \text{ in } \mathbb{R}^2 \\ u_1 v_1 + u_2 v_2 + u_3 v_3 & \text{if } \vec{u}, \vec{v} \text{ in } \mathbb{R}^3. \end{cases}$$

Note that the dot product of vectors is a number!

Examples

Compute the dot product $\vec{u} \cdot \vec{v}$.

1. $\vec{u} = (2, -1)$ and $\vec{v} = (5, 3)$. $\vec{u} \cdot \vec{v} = 2 \times 5 + (-1) \times 3 = 7$
2. $\vec{u} = (1, -1, 4)$ and $\vec{v} = (-2, 3, 1)$.
   $\vec{u} \cdot \vec{v} = 1 \times (-2) + (-1) \times 3 + 4 \times 1 = -1$
3. $\vec{u} = (2, 1, -3)$ and $\vec{v} = (1, 1, 1)$. $\vec{u} \cdot \vec{v} = 2 \times 1 + 1 \times 1 + (-3) \times 1 = 0$
4. $\vec{u} \cdot \vec{u}$. $\|\vec{u}\|^2$

Theorem Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let c be a scalar. Then

$$\begin{aligned}
\vec{u}\cdot\vec{v} &= \vec{v}\cdot\vec{u} && \text{(Commutativity)}\\
c(\vec{u}\cdot\vec{v}) &= (c\vec{u})\cdot\vec{v} = \vec{u}\cdot(c\vec{v}) && \text{(Scarlars factor out)}\\
\vec{u}\cdot(\vec{v}+\vec{w}) &= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{w} && \text{(Distributive law)}\\
\vec{u}\cdot\vec{0} &= 0 &&\\
\vec{u}\cdot\vec{u} &= \|\vec{u}\|^2 &&
\end{aligned}$$

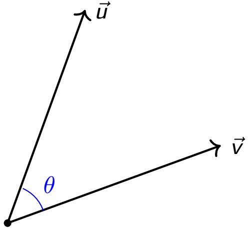
Theorem Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let c be a scalar. Then

$$\begin{aligned} \vec{u}\cdot\vec{v} &= \vec{v}\cdot\vec{u} && \text{(Commutativity)} \\ c(\vec{u}\cdot\vec{v}) &= (c\vec{u})\cdot\vec{v} = \vec{u}\cdot(c\vec{v}) && \text{(Scarlars factor out)} \\ \vec{u}\cdot(\vec{v}+\vec{w}) &= \vec{u}\cdot\vec{u}+\vec{u}\cdot\vec{w} && \text{(Distributive law)} \\ \vec{u}\cdot\vec{0} &= 0 \\ \vec{u}\cdot\vec{u} &= \|\vec{u}\|^2 \end{aligned}$$

Examples Compute the dot product of each pair of vectors.

1. $2\vec{i}-\vec{j}$ and $\vec{i}+\vec{k}$.
2. If $\vec{v}=-\frac{2}{3}\vec{u}$ and $\vec{u}\cdot\vec{v}=-6$. Find $\|\vec{v}\|$.

Given two nonzero vectors, the dot product can be used to find the angle formed by these two nonzero vectors.



Theorem 1 Let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. Let θ be the angle between them. Then

$$\vec{u} \cdot \vec{v} = \|u\| \, \|v\| \, cos\theta.$$

Alternatively, for nonzero $\vec{u}$ and $\vec{v}$

$$cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|u\| \, \|v\|}.$$

Theorem 1 Let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. Let θ be the angle between them. Then

$$\vec{u} \cdot \vec{v} = \|u\| \, \|v\| \, cos\theta.$$

Alternatively, for nonzero $\vec{u}$ and $\vec{v}$

$$cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|u\| \, \|v\|}.$$

Examples Find the cosine of the angle between each pair of vectors.

1. $\vec{u} = (2, -1)$ and $\vec{v} = (5, 3)$.
2. $\vec{u} = (2, 1, -3)$ and $\vec{v} = (1, 1, 1)$.
3. $\vec{u} = (1, -1, 4)$ and $\vec{v} = (-2, 3, 1)$.

Definition Vectors $\vec{u}$ and $\vec{v}$ are *orthogonal* if $\vec{u} \cdot \vec{v} = 0$.

Theorem 2 Let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. Let θ be the angle between them. Then θ is

1. an acute angle if $\vec{u} \cdot \vec{v} > 0$;
2. a right angle if $\vec{u} \cdot \vec{u} = 0$;
3. an obtuse angle if $\vec{u} \cdot \vec{v} < 0$.

Cross product

Given two nonzero vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$, the *cross product* $\vec{u} \times \vec{v}$ is a vector in $\mathbb{R}^3$ which is both orthogonal to $\vec{u}$ and $\vec{v}$. Namely, we have $\vec{u} \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot (\vec{u} \times \vec{v}) = 0$.

Given two nonzero vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$, the *cross product* $\vec{u} \times \vec{v}$ is a vector in $\mathbb{R}^3$ which is both orthogonal to $\vec{u}$ and $\vec{v}$. Namely, we have $\vec{u} \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot (\vec{u} \times \vec{v}) = 0$.

Definition Let $\vec{u} = (u_1, u_2, u_3)$ and $v_1, \vec{v_2}, v_3$ in $\mathbb{R}^3$. The *cross product* of $\vec{u}$ and $\vec{v}$ is the vector

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).$$

Given two nonzero vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$, the *cross product* $\vec{u} \times \vec{v}$ is a vector in $\mathbb{R}^3$ which is both orthogonal to $\vec{u}$ and $\vec{v}$. Namely, we have $\vec{u} \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot (\vec{u} \times \vec{v}) = 0$.

Definition Let $\vec{u} = (u_1, u_2, u_3)$ and $v_1, \vec{v_2}, v_3$ in $\mathbb{R}^3$. The *cross product* of $\vec{u}$ and $\vec{v}$ is the vector

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).$$

Example Find a vector that is both orthogonal to $\vec{u} = (1, 2, 1)$ and $\vec{v} = (2, 0, -3)$.

Given two nonzero vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$, the *cross product* $\vec{u} \times \vec{v}$ is a vector in $\mathbb{R}^3$ which is both orthogonal to $\vec{u}$ and $\vec{v}$. Namely, we have $\vec{u} \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot (\vec{u} \times \vec{v}) = 0$.

Definition Let $\vec{u} = (u_1, u_2, u_3)$ and $v_1, \vec{v_2}, v_3$ in $\mathbb{R}^3$. The *cross product* of $\vec{u}$ and $\vec{v}$ is the vector

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).$$

Example Find a vector that is both orthogonal to $\vec{u} = (1, 2, 1)$ and $\vec{v} = (2, 0, -3)$.

$$\begin{aligned}\vec{u} \times \vec{v} &= (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1) \\ &= (2 \times (-3) - 1 \times 0, 1 \times 2 - 1 \times (-3), 1 \times 0 - 2 \times 2) \\ &= (-6, 5, -4).\end{aligned}$$