11.8 Power Series.

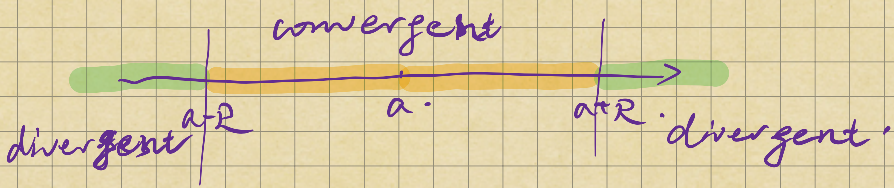
A series of the form.

$$c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n = \sum_{n=0}^{n} c_n(x-a)^n.$$

is called a power series in $(x-a)$ about the point $x-a$.

$c_0, c_1 \ldots\ c_n$ are called coefficient of the series.

the point $x=a$ is called the centre of the series.



In general, the series converges at $|x-a|<R$ and diverges at $|x-a|>R$ where R is a number called the radius of converges' of the series.

At the endpoints where $|x-a|=R$, the series may or may not be convergent.

The interval $(x-R, x+R)$ is called the interval of convergent and the point $x=a$ is called the centre of the convergence.

e.g. 1. $\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{x+2}{2}\right)^n.$ center, radius and interval.

$$\frac{1}{n}\left(\frac{x+2}{2}\right)^n = \frac{1}{n} \cdot \frac{(x+2)^n}{2^n} = c_n(x-a)^n.$$

$$c_n = \frac{1}{n} \cdot \frac{1}{2^n} \qquad a = -2.$$

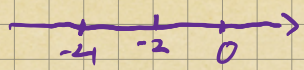
To obtain the radius of convergence, we use ratio test.

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{n+1} \cdot \frac{1}{2^{n+1}} (x+2)^{n+1}}{\frac{1}{n} \cdot \frac{1}{2^n} \cdot (x+2)^n} = \frac{n(x+2)}{(n+1)\times 2}.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x+2}{2}\right| \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = \left|\frac{x+2}{2}\right| \lim_{n\to\infty}\left|\frac{1}{1+\frac{1}{n}}\right| = \left|1+\frac{x}{2}\right|$$

if the series is convergents, then $|1+\frac{x}{2}|<1 \Rightarrow \underline{|2+x|<2}.$

$\Rightarrow$ 2 is radius of convergent. $|x-a|<R.$



the series converges in $(-4, 0)$.

at $x=-4$. the power series: $\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{-4+2}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n}(-1)^n.$

which is convergent by the AST. (last class)

at $x=0$, the power series: $\sum_{n=1}^{\infty} \frac{1}{n} \cdot 1$

which is diverses because it is a harmonic series.

$\Rightarrow$ the series converges at the interval $[-4, 0)$

diverges at the interval $(-\infty, -4) \cup [0, +\infty)$.

e.g. 2. $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \cdot (x-0)^n.$

$C_n = \frac{1}{\sqrt{n}}$ Centre: $x=0$.

ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{x^{n+1} \cdot \sqrt{n}}{x^n \sqrt{n+1}} = |x| \frac{\sqrt{n}}{\sqrt{n+1}} = |x| \cdot \frac{1}{\sqrt{1+\frac{1}{n}}}$

$\lim_{n\to\infty} |x| \frac{1}{\sqrt{1+\frac{1}{n}}} = |x|.$

if the series is convergent, $|x|<1$

$\Rightarrow$ the series converges in the interval $(-1, 1)$.

at the end points:

$x=-1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (-1)^n$

$n < n+1.$

$\sqrt{n} < \sqrt{n+1}.$

$\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}.$

$a_n > a_{n+1}.$

the series is decreasing

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

$\therefore$ the series convesis by AST.

$x=1 \Rightarrow \sum_{x=1}^{n} \frac{1}{\sqrt{n}} = \sum_{n=1}^{n} \frac{1}{n^{\frac{1}{2}}} \Leftarrow$ p.series.

$\because p = \frac{1}{2} < 1 \quad \therefore$ diverses.

$\therefore$ converges in $[-1, 1)$.

e.g. 3. $\sum_{n=1}^{\infty} (-1)^n \cdot n^2 \cdot \frac{x^n}{2^n}.$

$= \sum_{n=1}^{\infty} (-1)^n \cdot \frac{n^3}{2^n} x^n.$

$c_n = (-1)^n \frac{n^3}{2^n}$

$a = 0$

$a_n = (-1)^n \cdot \frac{n^2}{2^n} x^n.$

$\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{n+1}{n}\right)^2 \cdot \frac{x}{2}. \quad \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{x}{2} = \frac{x}{2}$

$\left|\frac{x}{2}\right| < 1 \quad -2 < x < 2 \quad R = 2.$

at the endpoint:

$x=-2: \sum_{n=1}^{\infty} (-1)^n \frac{n^2}{2^n} \cdot (-2)^n = \sum_{n=1}^{\infty} n^2 = \sum_{n=1}^{\infty} \frac{1}{n^{-2}}.$

p series. $\because p = -2 < 1 \quad \therefore$ diverges.

$x = 2 \quad \sum_{n=1}^{\infty} (-1)^n \frac{n^3}{2^n} \cdot 2^n = (-1)^n \cdot n^2.$

$\lim_{n\to\infty} n^2 = \infty$ is not fine $\Rightarrow$ diverges.

$\therefore$ the series converges at $x \in (-2, 2)$.

If the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius converges R.

then the function f define by.

$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$.

is differentiable at $(a-R, a+R)$ and.

1) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$.

$$= \sum_{n=0}^{\infty} nc_n(x-a)^{n-1} = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

2) $\int f(x)dx = C + c_0x + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$

$$= C + \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1}.$$

The radius of convergent of the power series in (i) & (ii) are R.

e.g. Find power series of $\frac{1}{(1-x)^2}$.

Let's consider the geomatrics series

$1 + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$.

if $|x|<1$, then $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$.

$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ $\quad \sum_{n=0}^{\infty} \frac{d}{dx}(x^n) = \frac{-(-1)}{(1-x)^2}$

$$\sum_{n=0}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

That is,

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n \quad |x|<1.$$

e.g.2. Find the power series of $\ln(1-x)$.

$$\int \frac{1}{1-x}\,dx = -\ln|1-x| + C.$$

$$\frac{1}{1-x} = 1 + x + \cdots + x^n = \sum_{n=0}^{\infty} x^n \text{ for } |x|^n.$$

$$-\ln|1-x| + C = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n}. = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

If $x \in (-\infty, 1)$, $|x-1| = x-1.$

$$\ln|1-x| = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}. \quad |x|<1.$$