Lecture 1

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Integration By Parts (Sec 7.1)

$$\boxed{\int u\,dv = uv - \int v\,du} \qquad (\mathrm{I})$$

__Proof__ We start with the product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \qquad (1)$$

$$\therefore \quad u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx} \qquad (2)$$

Integrating both sides of (2) with respect to x (w.r.t x)

$$\int u \underbrace{\frac{dv}{dx}dx}_{=dv} = \underbrace{\int \frac{d}{dx}(uv)\,dx}_{=uv} - \int v \underbrace{\frac{du}{dx}dx}_{=du}$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$ which is (I).

__Ex1__: Evaluate $\int x e^{-x}\,dx$.

__Solution__

If we choose $\boxed{u = e^{-x}}$ ✗ then $dv = x\,dx$

$$\Rightarrow du = -e^{-x}\,dx \qquad v = \frac{x^2}{2}$$

Then (I) gives

$$\int x e^{-x}\,dx = \frac{x^2}{2}e^{-x} + \frac{1}{2}\int x^2 e^{-x}\,dx$$

$\Downarrow$ which is harder than the original one

So we should let

$$u = x \qquad dv = e^{-x}\,dx$$

$$\Rightarrow du = dx \qquad v = -e^{-x} + \cancel{C_1}^{\,0}$$

Then (I) gives

$$\int x e^{-x}\,dx = -x e^{-x} + \int e^{-x}\,dx$$

$$\int x e^{-x}\,dx = -x e^{-x} + \underbrace{\int e^{-x}\,dx}_{-e^{-x}}$$

$$= -xe^{-x} - e^{-x} + C \quad // \text{Ans.}$$

Ex 2: Evaluate $\int \ln x \, dx$.

Solution

Let $u = \ln x \qquad dv = dx$

$\Rightarrow du = \frac{1}{x}\,dx \qquad v = x$

Then (I) gives

$$\int \ln x \, dx = x \ln x - \int x \frac{dx}{x}$$

$$= x\ln x - \int dx$$

$$= x\ln x - x + C \quad // \text{Ans.}$$

Ex 3: Evaluate $\int \cos^2 x \, dx$.

Soln

Let $u = \cos x \qquad dv = \cos x\,dx$

$\Rightarrow du = -\sin x\,dx \qquad v = \sin x$

$\therefore$ (I) gives

$$\int \cos^2 x\,dx = \cos x \sin x + \int \underbrace{\sin^2 x}_{1-\cos^2 x}\,dx$$

$$\int \cos^2 x\,dx = \cos x \sin x + \int (1-\cos^2 x)\,dx$$

$$\int \cos^2 x\,dx = \cos x \sin x + \int dx - \int \cos^2 x\,dx$$

$$\therefore \quad 2\int \cos^2 x\,dx = \cos x \sin x + x + C_1$$

$\div 2$, $$\int \cos^2 x\,dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + \underbrace{\frac{C_1}{2}}_{C}$$

$$= \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C \quad // \text{Ans.}$$

Ex 3: Evaluate $\int \sin^8 x\,dx \qquad \int \sin^{20} x\,dx$

Before evaluating the above integral, we will learn a new

Before evaluating the above integral, we will learn a new tool which is called a reduction Formula.

Consider an integral $I_n = \int f(x, n)\, dx$ where n is an arbitrary +ve integer.

By using IBPs, we may express I_n in terms of I_{n-1} or I_{n-2}.

The relationship between I_n and I_{n-1} (or I_{n-2}) is called a reduction formula.

Let find the reduction formula of $I_n = \int \sin^n x\, dx$.

$$= \int \underbrace{\sin^{n-1} x}_{u} \sin x\, dx$$

Using IBPs, let

$$u = \sin^{n-1} x \qquad dv = \sin x\, dx$$

$$\Rightarrow du = (n-1)\sin^{n-2} x (\cos x)\, dx \qquad v = -\cos x$$

Then (I) gives

$$I_n = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos x \underbrace{\cos x}_{1-\sin^2 x}\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x)\, dx$$

$$I_n = -\sin^{n-1} x \cos x + (n-1) \underbrace{\int \sin^{n-2} x\, dx}_{= I_{n-2}} - (n-1) \underbrace{\int \sin^n x\, dx}_{= I_n}$$

$$I_n + (n-1) I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$

$$n I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$

$$\therefore \boxed{I_n = -\frac{\sin^{n-1} x \cos x}{n} + \left(\frac{n-1}{n}\right) I_{n-2}} \qquad (A)$$

which is the reduction formula of $I_n = \int \sin^n x\, dx$ we are looking for.

Applying (A) with $n = 8$,

$$I_8 = \int \sin^8 x\, dx = -\frac{\sin^7 x \cos x}{8} + \frac{7}{8} I_6 \qquad (1)$$

where $I_6 = \int \sin^6 x\, dx$

$$I_6 = \int \sin^6 x\, dx = -\frac{\sin^5 x \cos x}{6} + \frac{5}{6} I_4 \qquad (2)$$

where $I_4 = \int \sin^4 x\, dx$

$$I_4 = -\frac{\sin^3 x \cos x}{4} + \frac{3}{4} I_2 \qquad (3)$$

where $I_4 = \int \sin^4 x \, dx$

$$I_4 = -\frac{\sin^3 x \cos x}{4} + \frac{3}{4} I_2 \qquad (3)$$

where $I_2 = \int \sin^2 x \, dx$

$$I_2 = -\frac{\sin x \cos x}{2} + \frac{1}{2} I_0$$

where $I_0 = \int (\sin x)^0 \, dx = x$

$$= -\frac{\sin x \cos x}{2} + \frac{1}{2} x \qquad (4)$$

$$\therefore \; I_4 = -\frac{\sin^3 x \cos x}{4} + \frac{3}{4}\left(-\frac{\sin x \cos x}{2} + \frac{1}{2} x\right)$$

$$= -\frac{\sin^3 x \cos x}{4} - \frac{3}{8} \sin x \cos x + \frac{3}{8} x$$

$$I_6 = -\frac{\sin^5 x \cos x}{6} + \frac{5}{6}\left(-\frac{\sin^3 x \cos x}{4} - \frac{3}{8} \sin x \cos x + \frac{3}{8} x\right)$$

$$\therefore \; I_8 = -\frac{\sin^7 x \cos x}{8} + \frac{7}{8}\left[-\frac{\sin^5 x \cos x}{6} - \frac{5}{24} \sin^3 x \cos x - \frac{5}{16} \sin x \cos x + \frac{5}{16} x\right] + C \,//$$

Ex 4: Evaluate $\int x^3 e^{-x} \, dx$.

Using the adjacent table, we obtain

$$\int x^3 e^{-x} dx = -x^3 e^{-x} - 3x^2 e^{-x} - 6x e^{-x} - 6 e^{-x} + C$$

// Ans.

$\frac{d}{dx}$ u	dv $\int dx$	
x^3	e^{-x}	
$3x^2$	$-e^{-x}$	$+$
$6x$	e^{-x}	$-$
6	$-e^{-x}$	$+$
0	e^{-x}	$-$

Trigonometric Integrals (Sec 7.2)

We need addition formulas

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad (i)$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b \qquad (ii)$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b \qquad (iii)$$

$$\cos(a-b) = \cos a \cos b + \sin a \sin b \qquad (iv)$$

Let $a = b = x$. Then (iii) becomes

$$\cos(2x) = \cos x \cos x - \sin x \sin x$$

$$\boxed{\cos(2x) = \cos^2 x - \sin^2 x} \qquad \text{(Double Angle Formula 1)}$$

$$\boxed{\cos(2x) = \cos^2 x - \sin^2 x}$$ (Double Angle Formula 1)

If we replace $\sin^2 x$ by $1-\cos^2 x$, then DAF1 becomes

$$\cos(2x) = \cos^2 x - (1-\cos^2 x)$$

$$\boxed{\cos(2x) = 2\cos^2 x - 1}$$ (DAF 2) $\leftarrow$

If we replace $\cos^2 x$ by $1-\sin^2 x$, then DAF 1 becomes

$$\cos(2x) = 1-\sin^2 x - \sin^2 x$$

$$\boxed{\cos(2x) = 1 - 2\sin^2 x}$$ (DAF 3)

<u>Ex1</u>: Evaluate $\int \cos^2 x \, dx$.

<u>Soln</u> Using DAF 2

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\therefore \int \cos^2 x \, dx = \frac{1}{2}\int (1+\cos 2x)\, dx$$

$$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right] + C$$

$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C \quad // \text{Ans.}$$

$2\sin x \cos x$

<u>N.B</u>: In 7.1, we got $\int \cos^2 x \, dx = \frac{1}{2}x + \frac{1}{2}\sin x \cos x + C$

If we set $a = b = x$ then (i) gives

$$\boxed{\sin 2x = 2\sin x \cos x}$$ (DAF 4)

then both answers are the same!

<u>Ex2</u>: Evaluate $\int \cos^4 x \, dx$

<u>Soln</u>

$$\cos^4 x = (\cos^2 x)^2 = \left[\frac{1}{2}(1+\cos 2x)\right]^2 = \frac{1}{4}(1+\cos 2x)^2$$

$$= \frac{1}{4}\left(1 + 2\cos 2x + \cos^2 2x\right)$$

$$= \frac{1}{4}\left(1 + 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right)$$

$$= \frac{1}{4}\left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right)$$

$$\therefore \int \cos^4 x \, dx = \frac{1}{4} \int \left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x \right) dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x + \sin 2x + \frac{1}{8}\sin 4x \right] + C \quad // \text{Ans.}$$

Ex3: Evaluate $\int \cos^5 x \, dx$.

Convert this factor to sine terms by using $\cos^2 x + \sin^2 x = 1$

Soln

$$\int \cos^5 x \, dx = \int \boxed{\cos^4 x} \cos x \, dx$$

$$= \int (\cos^2 x)^2 \cos x \, dx$$

$$= \int (1 - \sin^2 x)^2 \cos x \, dx$$

$$= \int (1 - 2\sin^2 x + \sin^4 x) \cos x \, dx$$

$$= \int \cos x \, dx - 2 \int \sin^2 x \cos x \, dx + \int \sin^4 x \cos x \, dx$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C \quad // \text{Ans.}$$

Ex4: Evaluate $\int \sec x \, dx$

Soln

$$\int \sec x \, dx = \int \sec x \,(1)\, dx$$

$$= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx = \int \frac{du}{u}$$

Let $u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x)\, dx$

$$= \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

// Ans.

Remarks

$$\int \sin^2 x \underbrace{\cos x \, dx} = \int u^2 \, du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3 x + C \quad //$$

Let $\boxed{u = \sin x}$

$du = \cos x \, dx$

$$\cos^2(2x) = \frac{1}{2}(1 + \cos(4x)) = \frac{1}{2} + \frac{1}{2}\cos 4x$$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta) \qquad \theta = 2x$$