Math 2155, Fall 2021: Homework 9

Instructions for GradeScope:

You may handwrite your answers or type them. Only type solutions if you are able to type the appropriate symbols. If **typed**, use at least a 12pt font, with **one question per page**. You can then directly submit the pdf file to GradeScope.

If **handwritten**:

- Don't erase or cross out more than a word or two.
- Only write on one side of the page. Use dark, large, neat writing.
- If you have a real scanner, great! Most people will scan their solutions using their phone. Use one of the recommended apps: For iOS, use Scannable by Evernote or Genius Scan. For Android, use Genius Scan. Do *not* just take regular photos.
- When scanning, have good lighting and try to avoid shadows.
- It is best to have **one question per scan**. If the solution is short, **fold the page in half** and scan just the half it is on, so there isn't so much blank space. Or, you can scan and then crop the scan. You don't need to scan parts (a), (b), etc. separately or put them on separate pages.
- It works well to scan each question separately and produce one pdf file with one question per page. Most scanning apps will automatically combine your images into one pdf file.
- You must **check the quality** of your scans afterwards, and rescan if needed.

Make sure you are going to `http://gradescope.ca` not `http://gradescope.com`. You can access it through the course OWL site.

You can **resubmit** your work any number of times until the deadline.

See the GradeScope help website for lots of information: `https://help.gradescope.com/`
Select "Student Center" and then either "Scanning Work on a Mobile Device" or "Submitting an Assignment".

Instructions for writing solutions:

- Homework is graded both on **correctness** and on **presentation**.
- Your proofs should be written in **complete sentences**, starting with capitals and ending with periods. They should be in **paragraph form**, similar to the proofs in the textbook. For example, you shouldn't put each sentence on a new line. Paragraphs should be used to break the proof into logical chunks.
- Don't use informal abbreviations such as three dots for "therefore." Logical symbols can be used when they clarify things to the reader, but English words can often be more effective.
- Include all of the steps that are needed to logically justify every claim you make. Do not include unnecessary steps. Try to be **concise and complete**.
- Do not submit any **rough work**. Do your rough work on scratch paper, and only submit a clearly and neatly written answer. Do not cross out or erase more than a word or two. If you write each final solution on a new page, it's easy to start over on a fresh page when you've made a large error.
- You should do the work **on your own**. Read the course syllabus for the rules about scholastic offences, which include sharing solutions with others, uploading material to a website, viewing material of others or on a website (even if you don't use it), etc. The penalty for cheating on homework will be a grade of **0** on the homework set as well as a penalty of **negative 5%** on the overall course grade.
- We may not grade every question.

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Due **Friday, November 19 at 11:59pm**. You can **resubmit** your work any number of times until the deadline.

H9Q1: For each of the following relations on the specified set, either prove that they are partial orders, or explain why they are not. Also prove that they are total orders, or explain why they are not.

(a) $A_1 = \{1, 2, 3\}$ and $R_1 = \{(1, 1), (1, 2), (2, 2), (3, 2), (3, 3)\}$.

(b) $A_2 = \{1, 2, 3\}$ and $R_2 = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3), (1, 3)\}$. reflexive tran ✓ anti ✓.

(c) $A_3 = \mathbb{N}$ and $R_3 = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid a - b \leq 1\}$. $a = b + 1$ ref × trans ✓ anti ✓.

(d) $A_4 = \{1, 2, 3\}$ and $R_4 = \emptyset$. × ×

(e) $A_5 = \emptyset$ and $R_5 = \emptyset$.

Solution: (a) R_1 is a partial order.

Reflexivity: R_1 contains $(1, 1), (2, 2)$ and $(3, 3)$, as required.

Transitivity: We need to check that for all $x, y, z \in A_1$, if xR_1y and yR_1z, then xR_1z. This clearly holds if $x = y$ or $y = z$, so we just need to handle the cases when $x \neq y$ and $y \neq z$. The only related pairs of this form are $(1, 2)$ and $(3, 2)$, and since $2 \neq 3$ and $1 \neq 2$, they don't "compose" in either order, so there is nothing left to check.

Anti-symmetry: Again, it is enough to check that if xR_1y and $x \neq y$, then yR_1x does not hold. The only pairs to check are $(1, 2)$ and $(3, 2)$, and neither of $(2, 1)$ and $(2, 3)$ are in R_1, so we are done.

R_1 is not a total order, since neither of $(1, 3)$ nor $(3, 1)$ is in R_1.

(b) R_2 is a partial order and a total order. Both are straightforward to check.

(c) R_3 is not a partial order (and therefore not a total order either). It fails to be anti-symmetric, since both $(1, 2)$ and $(2, 1)$ are in R_3. (It also fails to be transitive, since $(3, 2)$ and $(2, 1)$ are in R_3, but $(3, 1)$ is not in R_3.)

(d) R_4 is not a partial order (and therefore not a total order either). It fails to be reflexive, since $(1, 1)$ is not in R_4. (The other properties hold.)

(e) R_5 is a partial order and a total order. All of the conditions start with "for all x in A, ..." and so they hold vacuously when A is empty.

H9Q2: The following are both partial orders on the specified set (you don't need to prove this). For both, state what the R-smallest element is, or explain why it doesn't exist. Also, give the set of R-minimal elements. Explain all of your work.

(a) $A = \{\text{cat}, \text{dog}, \text{pill}, \text{caterpillar}\}$ and for words u and v in A, we have uRv iff u appears in v. (So cat$R$cat and cat$R$caterpillar, but dog is not related to caterpillar.)

caterpillar

cat pill dog.

(b) $A = \{B \subseteq \mathbb{N} \mid B \text{ has a least 2 elements}\}$ and R is the $\subseteq$ relation.

Solution: (a) There is no R-smallest element. If u was R-smallest, then we would have uRdog and uRcat, but there is no word $u \in A$ that appears in both dog and cat.

The R-minimal elements are cat, dog and pill. In all three cases, there is no different word $u \in A$ that appears in the given word. On the other hand, caterpillar is not R-minimal, since cat$R$caterpillar.

(Note that if you find the R-minimal elements first, then it *follows* that there is no R-smallest element, by Theorem 4.4.6 in the text.)

(b) There is no $\subseteq$-smallest element. If B was $\subseteq$-smallest, then it would have to be a subset of both $\{1, 2\}$ and $\{3, 4\}$, but the only subset of both is the empty set, and it does not have at least two elements.

The set of $\subseteq$-minimal elements is $\{B \subseteq \mathbb{N} \mid B \text{ has exactly 2 elements}\}$. These are all minimal, since any subset of a set B with 2 elements is either equal to B or has fewer than 2 elements, so is not in A. On the other hand, if a set B has more than 2 elements, then we can choose a 2 element subset C of B and get that $C \subseteq B$ and $C \neq B$, showing that B is not $\subseteq$-minimal.

(Again, it follows from the second paragraph that there is no R-smallest element.)

H9Q3: Let R and S be partial orders on a set A. Consider the following statements:

Statement 1: If $R \subseteq S$ and a is an S-smallest element of A, then a is an R-smallest element of A.

Statement 2: If $R \subseteq S$ and a is an S-minimal element of A, then a is an R-minimal element of A.

One of these is true and one is false. Prove the correct one and give a counterexample for the other one.

Solution: Statement 1 is not correct. For example, let $A = \{1, 2\}$, let $S = \{(1, 1), (1, 2), (2, 2)\}$ and let $R = i_A$. Then R and S are partial orders on A, with $R \subseteq S$. The element 1 is S-smallest, since $1S1$ and $1S2$, but it is not R-smallest, since $1R2$ does not hold.

Proof of Statement 2. Assume that $R \subseteq S$ and that a is an S-minimal element of A. We need to prove that if bRa, then $b = a$. So let $b \in A$ be arbitrary, and assume bRa. Since $R \subseteq S$, we also have bSa. And since a is S-minimal, this implies that $b = a$, as required. So a is R-minimal. $\square$