MATH 1600 Linear Algebra — Winter 2020

Tutorial 8 - Wednesday

Dimension and basis of subspaces

1. Which of the following sets are subspaces of $\mathbb{R}^n$? Find a basis for each subspace.

   (a) $S_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;:\; y = |x| \right\}$ ✓ (1,0), (0,1),

   (b) $S_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;:\; x + y - 2z = 0 \right\}$ ✗.

   (c) $S_3 = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \;:\; xw = yz \right\}$ ✗

   (d) $S_4 = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \;:\; x = -y,\, z = -w \right\}$ ✗

   (e) How many vectors do we need to add to the basis of S_2 to create a basis of $\mathbb{R}^3$. Find such vectors. How about S_4?

2. Find the dimension of the following subspaces:

   (a) $S_1 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^2$ 2

   (b) $S_2 = \operatorname{span}\left\{ \begin{pmatrix} 4 \\ -3 \end{pmatrix}, \begin{pmatrix} -20 \\ 15 \end{pmatrix} \right\} \subseteq \mathbb{R}^2$ 1

   (c) $S_3 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ -9 \end{pmatrix} \right\} \subseteq \mathbb{R}^3$

3. Let A be the following matrix:

$$A = \begin{pmatrix} 1 & 1 & 0 & -1 \\ -1 & 2 & -3 & 1 \\ 1 & 4 & -3 & 0 \end{pmatrix}$$

$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ [1 0 1 −1] [0 1 −1 0] [0 0 0 1]

$\begin{matrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 1 \end{matrix}$ $\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ r.

   (a) Find a basis for the row space row(A) and the rank of A.

   (b) Find a basis for the column space col(A).

   (c) Find the nullity of A and a basis for the null space null(A). [−1, −1, 1, −1].

4. Either prove or provide a counterexample for each of the following statements.

   (i) There is a subspace of $\mathbb{R}^5$ spanned by 6 linearly independent vectors.

   (ii) If A has nullity equal to the number of its columns, then A is the zero matrix.

   (iii) If a matrix A has odd number of columns then $\operatorname{rank}(A) \neq \operatorname{nullity}(A)$.

$= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} x_3$

$X_1 =$ $\quad x_1 = -x_2 - x_3$

$A = \left[\begin{array}{c|c|c|c} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \end{array}\right].$

$x_1 + x_2 + x_3 = 0.$

5. Let A be a 3×4 matrix.

   (a) If $\text{rank}(A) = 3$ and $X = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}$ is a solution of $AX = 0$, then find a basis for the null space $\text{null}(A)$.

   $\begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix}.$

   $\begin{bmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

   (b) If $\text{nullity}(A) = 3$, then prove that any non-zero row of A forms a basis for $\text{row}(A)$.

6. Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ form a basis.

   (a) Prove that $\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_3 + \mathbf{v}_1$ form a basis for $\mathbb{R}^3$ too.

   (b) Do the vectors $\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_3, \mathbf{v}_1 - \mathbf{v}_3$ form a basis? Explain.

7. Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

   (a) Prove that the set $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis of $\mathbb{R}^3$.

   (b) Find the coordinates of $\mathbf{v}$ and $\mathbf{w}$ with respect to the basis $\mathcal{B}$.

$\begin{bmatrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$

RREF $\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$