'q:

- Exam pick-up : Thursday, Oct 31, 3:30 – 5:00 pm

- Quiz after reading break : Topics include [redacted], Chapter 4: 4.1, 4.3, [redacted], 4.5, 4.7, 4.4.

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call:

Ch. 4.1,
4.3,
4.5,
4.7

Basic principles: $f'(x) > 0 \Rightarrow$ increasing

① $\underbrace{f'(x) < 0 \Rightarrow \text{decreasing}}$

$\Downarrow$

local min/max can only occur at $\overbrace{f'(x) = 0}^{\text{critical points.}}$

② $f''(x) > 0 \Rightarrow$ concave up (think $\boxed{x^2}$ )

$\underbrace{f''(x) < 0 \Rightarrow \text{concave down (think } \boxed{-x^2}\text{)}}$ 

$\Downarrow$

at local min $\rightarrow$ concave up

at local max $\rightarrow$ concave down

[redacted] ③ $\underbrace{\text{Endpoints}}_{\text{if they exist}}$ can be maxima or minima and need to be checked seperately.

4.7: word problems

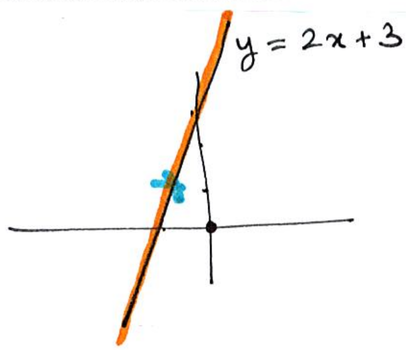
eg: Q. Find point on the line $y = 2x+3$ that is closest to the origin.

Method: a) Find the "quanitity" that you are trying to minimize / maximize

b) Express it ~~maticallly~~ mathematically

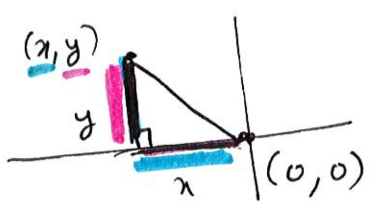
c) Find endpoints if any

$\longrightarrow$ You have reduced the problem to finding absolute max / min.



a) distance between a point ont the line and the origin

b) Let (x, y) be a point on the line.



$d = \sqrt{x^2 + y^2}$ by Pythagoras

$$\boxed{= \sqrt{x^2 + (2x+3)^2}}$$

Becawe we are on the line $y = 2x+3$

*. Free variable $= x$.

ⓒ Endpoints: (x, y) is a point on this line $y = 2x+3$

x has no bounds

$\Rightarrow$ no endpoints.

Q. Find absolute min of

$$f(x) = \sqrt{x^2 + (2x+3)^2}$$

$g(x) = x^2 + (2x+3)^2$

$g'(x) = 2x + 2(2x+3) \cdot 2$

$= 2x + 8x + 12 = 0$

$x = -\frac{6}{5}$.

Ans:

$$f'(x) = \left(\sqrt{x^2 + (2x+3)^2}\right)'$$

$$= \left(\left(x^2 + (2x+3)^2\right)^{1/2}\right)'$$

$$= \frac{1}{2} \cdot \left(x^2 + (2x+3)^2\right)^{-1/2} \cdot \left(x^2 + (2x+3)^2\right)'$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{x^2 + (2x+3)^2}} \cdot \left(2x + 2 \cdot (2x+3) \cdot (2x)'\right)$$

$$= \frac{1}{2} \frac{1}{\sqrt{x^2 + (2x+3)^2}} \cdot \left(2x + 2 \cdot (2x+3) \cdot 2\right)$$

$$= \frac{5x+6}{\sqrt{x^2 + (2x+3)^2}}$$

For ~~maximum~~ min: $f'(x) = 0$ Critical points

$$\Rightarrow \frac{5x+6}{\sqrt{x^2 + (2x+3)^2}} = 0$$

$$\Rightarrow 5x + 6 = 0$$

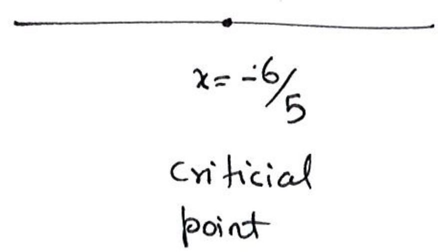
$$\Rightarrow \boxed{x = -\frac{6}{5}}$$

. To find min / max we need to find concavity.

But $f''(x)$ is too complicated.

So instead we use the following trick

So far : $f'(x) = \dfrac{5x+6}{\sqrt{x^2+(2x+3)^2}}$



To avoid finding $f''(x)$.

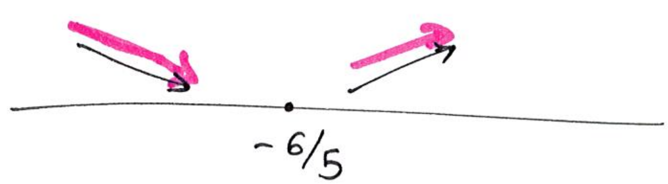
Check signs of $f'(x)$ before and after $x = -6/5$

$x < -\frac{6}{5}$ $\quad f'(x) = \dfrac{5x+6}{\sqrt{}} < 0$

$\Rightarrow$ f decreases to left of $x = -6/5$

$x > -6/5$ $\quad f'(x) = \dfrac{5x+6}{\sqrt{}} > 0$

$\Rightarrow$ f increases to right of $x = -6/5$



$\Rightarrow$ $\boxed{x = -\frac{6}{5} \text{ is a minima}}$

Back to the original question, $\boxed{y = 2x+3 = 2\cdot\left(\frac{-6}{5}\right)+3}$

Q. Find the rectangle with the smallest perimeter whose area is A. (A is some constant).

A. a) quantity to minimize = perimeter

b) $= 2l + 2b$

Given Area $= A$

$\Rightarrow \boxed{l \cdot b = A}$ —— $(*)$

quantity to ~~optimize~~ minimize : $\boxed{\dfrac{2A}{b} + 2b}$ using $(*)$ $l = \dfrac{A}{b}$

c) endpoints : $l \geq 0$, $\boxed{b \geq 0}$ (as l, b are dimensions)

we should see what happens when $\lim_{b \to 0}$.

Q. Find minima for

$$f(b) = \frac{2A}{b} + 2b$$

A: For minima : $f'(b) = 0$

$$\Rightarrow \left(\frac{2A}{b}\right)' + (2b)' = (2A \cdot b^{-1})' + (2b)'$$

$$= 2A \cdot (b^{-1})' + 2(b')$$

$$= 2A \cdot -(b^{-2}) + 2 = 0$$

$$\Rightarrow \frac{-2A}{b^2} + 2 = 0$$

$$\Rightarrow \quad -\frac{2A}{b^2} = -2$$

$$\Rightarrow \quad \frac{A}{b^2} = 1$$

$$\Rightarrow \quad A = b^2$$

$$\Rightarrow \quad b = \pm\sqrt{A}$$ but b = breadth and hence cannot be negative

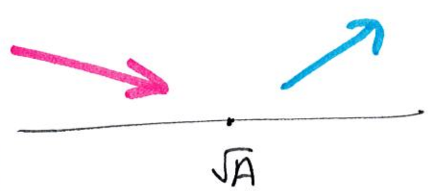
$$\Rightarrow \quad \boxed{b = +\sqrt{A}}$$

$$f'(b) = \frac{-2A}{b^2} + 2$$

$b = \sqrt{A}$ critical point

- to left of $b = \sqrt{A}$ $\quad f'(b) < 0$

to right of $b = \sqrt{A}$ $\quad f'(b) > 0$



$$\Rightarrow \quad \boxed{\begin{array}{l} b = \sqrt{A} \text{ is a local minima} \\ l = \dfrac{A}{b} = \dfrac{A}{\sqrt{A}} = \sqrt{A} \qquad \boxed{\text{Square!}} \end{array}}$$

Aside:

Not needed

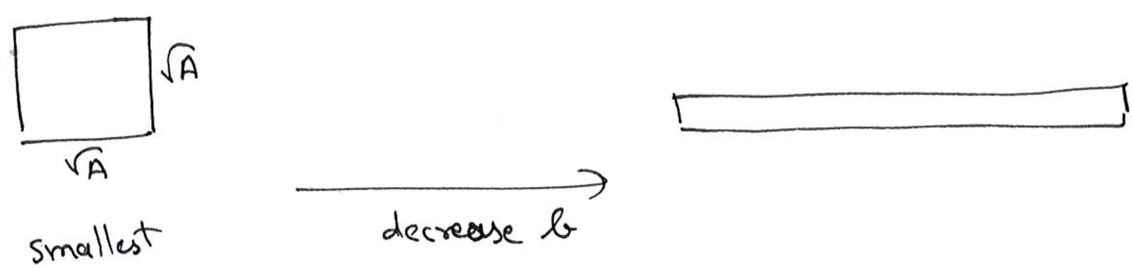
as $\lim_{b \to 0^+}$

perimeter $= \dfrac{2A}{b} + 2b$

$\lim_{b \to 0^+} = \infty$ no minima at the boundary

Q: Find the rectangle with the largest perimeter whose area is A. (A is some constant).

A: There is <u>no maxima</u>. i.e. no such rectangle exists



smallest perimeter

4.4 l'Hospital's Rule :

method for finding limits of indeterminate forms $\frac{0}{0}, \pm\frac{\infty}{\infty}$. **only**

Very important
Do not apply to other situations.

$\boxed{\text{Case } \frac{0}{0}}$ (similar proof works for $\frac{\infty}{\infty}$)

Suppose $f(a) = 0 = g(a)$

Proof: Derivation
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)}$$

$$= \lim_{x\to a} \frac{(f(x)-f(a))/(x-a)}{(g(x)-g(a))/(x-a)}$$

Divide numerator, denominator by $(x-a)$.

$$= \frac{\displaystyle\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\displaystyle\lim_{x\to a} \frac{g(x)-g(a)}{x-a}}$$

L'Hospital's Rule $\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}}$ if $\frac{f(x)}{g(x)} \to \frac{0}{0}$ or $\pm\frac{\infty}{\infty}$

eg: $\lim_{h \to 0} \frac{\sin h}{h}$ ~~$\cos h$~~

More commonly, $\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \quad \text{when } \frac{f(x)}{g(x)} \to \frac{0}{0} \text{ or } \frac{\infty}{\infty}}$

eg • $\lim_{h \to 0} \frac{\sin h}{h} = \lim_{h \to 0} \frac{(\sin h)'}{h'}$ by L'Hospital's rule

(Plug in we get $\frac{0}{0}$)

$$= \lim_{h \to 0} \frac{\cos h}{1}$$

$$= \frac{\cos 0}{1}$$

$$= 1$$

• $\lim_{h \to 0} \frac{e^h - 1}{h} = \lim_{h \to 0} \frac{(e^h - 1)'}{h'}$

Plug in $\frac{e^0 - 1}{0} = \frac{0}{0}$

use L'Hospital's Rule

$$= \lim_{h \to 0} \frac{e^h}{1}$$

$$= \frac{e^0}{1}$$

$$= 1$$

$$\lim_{x \to 0} \frac{\cos x - 1}{x^2} = \lim_{x \to 0} \frac{(\cos x - 1)'}{(x^2)'}$$

Plug in $\frac{\cos 0 - 1}{0} = \frac{0}{0}$

Use L'Hospital's Rule

$$= \lim_{x \to 0} \frac{-\sin x}{2x}$$

$\frac{0}{0}$ again,

Do one more iteration

$$= \lim_{x \to 0} \frac{(-\sin x)'}{(2x)'}$$

$$= \lim_{x \to 0} \frac{-\cos x}{2}$$

$$\Rightarrow \boxed{\lim_{x \to 0} \frac{\cos x - 1}{x^2} = \frac{-\cos 0}{2} = -\frac{1}{2}}$$

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eg: $\lim_{x \to \frac{\pi}{2}} \frac{\sin x}{x}$

Plug in

$$\frac{\sin \pi/2}{\pi/2}$$

$$= \frac{1}{\pi/2}$$

$$= \frac{2}{\pi} \leftarrow \text{Ans}$$

If ~~we~~ we use L'Hospital's rule

$$= \lim_{x \to \pi/2} \frac{\cos x}{1}$$

$$= \frac{\cos \pi/2}{1}$$

$$= 0.$$

Wrong

eg: $\lim_{x \to 0^+} \frac{\cos x - 1}{x} = \lim_{x \to 0^+} \frac{-\sin x}{1}$

$= \frac{\sin 0}{1}$

$= 0.$

Plug in $\frac{\cos 0 - 1}{0} = \frac{0}{0}$

Use L'Hospital's rule

$\frac{0}{0}, \frac{\infty}{\infty}$ → Use L'Hospitals

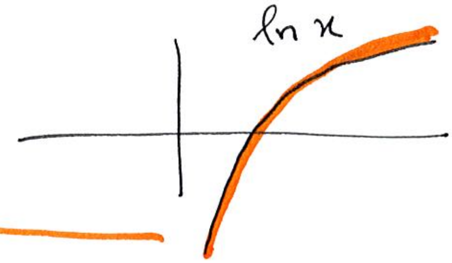
Other indeterminate forms:

$0 \cdot \infty$ → "change 0 to $1/\infty$ or ∞ to $1/0$"

0^0
∞^0 → do logarithmic differentiation
1^∞

$\infty - \infty$ → do algebraic simplification

eg: Find $\lim_{x \to 0^+} \underbrace{x}_{\to 0} \cdot \underbrace{\ln x}_{\to -\infty}$.



[Form: $0 \cdot \infty$ → Change to $\frac{0}{0}$ or $\frac{\infty}{\infty}$]

$\lim_{x \to 0^+} x \cdot \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x}$ or $\lim_{x \to 0^+} \frac{x}{1/\ln x}$

- Can use either
- pick the one whose derivatives are simple

$$\lim_{x\to 0^+} x\cdot \ln x = \lim_{x\to 0^+} \frac{\ln x}{(1/x)} \rightsquigarrow \frac{\infty}{\infty}$$

Can use L'Hospital's Rule

$$= \lim_{x\to 0^+} \frac{(\ln x)'}{(1/x)'}$$

$$= \lim_{x\to 0^+} \frac{1/x}{-1/x^2}$$

$$= \lim_{x\to 0^+} - x^2\cdot\left(\frac{1}{x}\right)$$

$$= \lim_{x\to 0^+} -x$$

$$\lim_{x\to 0^+} \underbrace{x}_{\to\, 0}\cdot\underbrace{\ln x}_{\to\, -\infty} = 0$$

$$\boxed{0\cdot\infty \xrightarrow[\text{to}]{\text{change}} \frac{0}{0} \text{ or } \frac{\infty}{\infty}}$$

- 0^0 or ∞^∞ or $(-)^{(-)}$ $\rightsquigarrow$ logarithmic differentiation

eg: $\lim_{x\to 0^+} x^x$

- Find ① $\lim_{x\to 0^+} \ln(x^x)$

② Exponentiate this answer to get back $\lim_{x\to 0} x^x$.

Ans: ① $\lim_{x\to 0^+} \ln(x^x) = \lim_{x\to 0^+} x \cdot \ln x$

we just did this

$0\cdot\infty \to$ change to $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$$= \lim_{x\to 0^+} \frac{\ln x}{1/x}$$

$\vdots$ L'Hospital's rule

$$= 0$$

② $\lim_{x\to 0^+} x^x = e^0 = 1$

Q. $\lim_{x\to 0^+} (1+\sin x)^{\cot x}$

① Find $\lim_{x\to 0^+} \ln\left((1+\sin x)^{\cot x}\right)$

$$= \lim_{x\to 0^+} \cot x \cdot \ln(1+\sin x)$$

$$= \lim_{x\to 0^+} \frac{\cos x}{\sin x} \cdot \ln(1+\sin x)$$

Plug in $x=0$

$\frac{0}{0}$, L'Hospital's rule

$\vdots$ ← Exercise (next page)

$$= 1$$

② original limit $= e^1 = e$.

Exercise :

$$= \lim_{x \to 0^+} \cos x \cdot \lim_{x \to 0^+} \frac{\ln(1+\sin x)}{\sin x}$$

$\frac{0}{0}$, L'Hospital's Rule

$$= 1 \cdot \lim_{x \to 0^+} \frac{(\ln(1+\sin x))'}{(\sin x)'}$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{(1+\sin x)} \cdot \cancel{\cos x}}{\cancel{\cos x}}$$

$$= \lim_{x \to 0^+} \frac{1}{1+\sin x}$$

$$= 1$$