Row-reduced echelon form

Recap

System of linear equations (SLE)

- a linear equation is
$$a_1x_1 + \ldots + a_nx_n = b$$

where a_i are coefficients and b is a constant number.

- an SLE is a finite number of linear equations
- standard form for an SLE
- elementary operations to solve an SLE

Matrix (*pl.* matrices)

- a matrix is a rectangular array of numbers.
- coefficient matrices for SLEs and augmented matrices for SLEs

For example,

$$
\begin{aligned}
x + y + z &= 5 \\
3x + 2y + z &= 15 \\
y + 2z &= 0
\end{aligned}
$$

Its augmented matrix is

$$
\left[\begin{array}{ccc|c}
1 & 1 & 1 & 5 \\
3 & 2 & 1 & 15 \\
0 & 1 & 2 & 0
\end{array}\right]
$$

With an augmented matrix, we can also write an SLE in a standard form.

Elementary operations to SLE

The following operations are the elementary operations to SLE
• Multiply an equation by a non-zero scalar.

• Interchange the positions of two equations in the system.

• Replace one of the equations by the sum of that equation and a scalar multiple of another one of the equations in the system.

From the correspondence of the SLE and the augmented matrix, we can perform operations to a matrix.

Elementary row operations

The following operations are the *elementary row operations* (abbreviated ero's) which can be performed to a matrix:
1) Multiply any row of the matrix by any non-zero scalar.

2) Interchange the positions of any two rows in the matrix.

3) Replace any row in the matrix by the sum of that row and a scalar multiple of any other row of the matrix.

No other operations are allowed.

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The **goal** is to transform a matrix to an RREF by performing elementary row operations.

Definition A matrix A is said to be in *row-reduced echelon form*, abbreviated as RREF, if each of the following four conditions is met:
(i) Each row of A that does not contain entirely zeros has a 1 as its first (from left to right) non-zero entry (We call this 1 as the leading 1).

(ii) Each column of A which contains a leading 1 for some row contains no other nonzero entries (i.e. all other entries are 0's).

(iii) In any two rows of A which each contain some non-zero entries, the leading 1 from the lower row must occur farther to the right than the leading 1 from the upper row.

(iv) All rows of A which consist entirely of zeros are placed at the "bottom" of the matrix.

Examples

(i) Each row of A that does not contain entirely zeros has a 1 as its first (from left to right) non-zero entry (We call this 1 as the leading 1).

Example

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Non-example

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

(ii) Each column of A which contains a leading 1 for some row contains no other nonzero entries (i.e. all other entries are 0's).

Example

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Non-example Example

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(iii) In any two rows of A which each contain some non-zero entries, the leading 1 from the lower row must occur farther to the right than the leading 1 from the upper row.

Example

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Non-example Example

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

(iv) All rows of A which consist entirely of zeros are placed at the "bottom" of the matrix.
Example

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Non-example Example

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Examples

Which of the following are RREFs?

$$\left[\begin{array}{cccc|c} 1 & 0 & 7 & 2 & 3 \\ 0 & 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Suppose that we have an RREF augmented matrix. We know it corresponding to an SLE. We can “see” solutions from an RREF augmented matrix.

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$$\begin{aligned} x + y &= 1 \\ x - y &= 0 \end{aligned}$$

and its augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 0 \end{array}\right]$$

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The RREF augmented matrix

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{array}\right]$$

corresponds to an SLE equivalent to the given one.

$$\begin{aligned} x + 0y &= \tfrac{1}{2} \\ 0x + y &= \tfrac{1}{2} \end{aligned}$$

Examples

For each of the following, find all solutions to the SLE corresponding to the given RREF augmented matrix, where the variables are as stated.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \text{with variables } x, y, z$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 7 & 2 & 3 \\ 0 & 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \text{with variables } x_1, x_2, x_3, x_4$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \quad \text{with variables } w, x, y \text{ and } z.$$