MATH 1600 Linear Algebra — Winter 2020

Tutorial 4 - Wednesday

REFs and RREFs

1. Consider the following matrices with real coefficients.

$$M = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right], \quad B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

   (a) Which of the above matrices are in row echelon form? Explain your answer. C, D

   (b) Which of the above matrices are in *reduced* row echelon form? Explain your answer. D.

   $r_2 = r_1 - r_2$.

   (c) Find a sequence of elementary row operations that would turn M into B and a sequence of row operations that would turn C into B. $r_1 = r_1 - r_2$ $r_3 = -r_3$. | $r_1 = r_1 - 2(r_2 - r_3)$.

   (d) Find a sequence of elementary row operations that would turn B into D. | $r_3 = -r_3$ $r_2 = r_2 - r_3$

   (e) Conclude that D is the RREF of M, B and C. ✓

   (f) Suppose that M is an augmented matrix of a linear system, that is $M = (A|\mathbf{b})$ for some 3×3 matrix A and some column vector $\mathbf{b} \in \mathbb{R}^3$. Find A and $\mathbf{b}$, and solve the linear system $A\mathbf{x} = \mathbf{b}$.

$$\begin{aligned} x + 2y &= 0 \\ x + y + z &= 0 \\ y &= 1 \end{aligned} \qquad \begin{cases} x = -2 \\ y = 1 \\ z = 1. \end{cases}$$

Linear Systems

2. Consider the following matrix with real coefficients:

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & -\frac{a+2}{5} & -\frac{1}{5} \\ 0 & 0 & a & 2. \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & \frac{a-3}{5} & -\frac{4}{5} \\ 0 & 1 & -\frac{a+2}{5} & -\frac{1}{5} \\ 0 & 0 & a & 2. \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & a \\ 1 & -4 & 2a+1 \end{bmatrix} \begin{matrix} -1 \\ -1 \\ 2 \end{matrix} \quad \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & -5 & a+2 & 1 \\ 0 & -5 & 2a+2 & 3. \end{bmatrix}$$

where a is a parameter, that is, it is allowed to vary over $\mathbb{R}$.

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 0 & a & 2 \\ 0 & -5 & a+2 & 1 \end{bmatrix}$$

   (a) Find the values of a for which the system $A\mathbf{x} = \mathbf{0}$ is consistent.

   (b) Find the values of a for which the system $A\mathbf{x} = \mathbf{b}$, with $\mathbf{b} = \langle -1, -1, 2 \rangle$, has no solutions, exactly one solution, and infinitely many solutions.

   $a = 0$. $x + y = \frac{1}{3}$.

3. Consider the following linear system with coefficients in $\mathbb{Z}_3$.

$$\begin{aligned} 2x + y + z &= 1 \\ x + 2y &= 0 \\ x + 2y + z &= 0 \end{aligned} \qquad \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right].$$

   (a) Find the coefficient matrix and the augmented matrix system for this system of linear equations.

   (b) Solve this system over $\mathbb{Z}_3$. (Your solution vector(s) must have entries in $\mathbb{Z}_3$ only)

   $x = \frac{2}{3}$, $y = \frac{8}{3}$, $z = 0$

   (c) Let $B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$ be a matrix with entries in $\mathbb{Z}_3$. Find the RREF of B over $\mathbb{Z}_3$ (Your elementary row operations must involve scalars in $\mathbb{Z}_3$ only). Why doesn't the system $B\mathbf{x} = 0$ have infinitely many solutions?

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow[r_2 = r_2 - \frac{1}{2} r_1]{r_1 = \frac{1}{2} r_1} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{3}{2} & 0 \end{bmatrix} \xrightarrow[r_1 = r_1 - \frac{1}{2} r_2]{r_2 = \frac{2}{3} r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

pivots $= r(B) = 2$

Pivots and Free Variables

4. Consider the following $\mathbb{R}$-valued matrix:

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

You are given the information that R is the reduced row echelon form of some 5×7 matrix A.

(a) How many pivots (a.k.a. leading 1's) does R have? 3. How many free variables does R have? 3

(b) How many free variables does A have? Explain your answer. Which variables among $x_1, \ldots, x_7$ are the free ones? 3

because they can be any value.

(c) Without computing, explain why the system $A\mathbf{x} = 0$ has infinitely many solutions.

variable numbers $= 7 > r(A) = 3$