Counting

Chapter 6

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Plan for Chapter 6

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Basic counting principles: the product rule

The Product Rule : Assume that some procedure can be broken down into a sequence of two (or more) consecutive and independent tasks. Assume also that there are n_1 ways to do the first task and n_2 ways to do the second task. Then there are $n_1 \cdot n_2$ ways to do the procedure.

Example

How many bit strings of length seven are there?

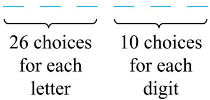
Solution: Since each of the seven bits is either a 0 or a 1, the answer is $2^7 = 128$.

The product rule

Example

How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

Solution: By the product rule, there are $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000$ different possible license plates.



Counting functions

Example (Counting Functions)

How many functions are there from a set with m elements to a set with n elements?

Since a function represents a choice of one of the n elements of the codomain for each of the m elements in the domain, the product rule tells us that the number of such functions is:

$$n \cdot n \cdots n = n^m.$$

Example (Counting One-to-One Functions)

How many one-to-one functions are there from a set with m elements to one with n elements? Suppose the elements in the domain are $a_1, a_2, \ldots, a_m$. There are n ways to choose the value of a_1 and $n - 1$ ways to choose a_2, etc. The product rule yields:

$$n(n-1)(n-2)\cdots(n-m+1).$$

Telephone numbering plan

Example

The *North American numbering plan* (*NANP*) specifies that a telephone number consists of 10 digits, consisting of a three-digit area code, a three-digit office code, and a four-digit station code. There are some restrictions on the digits.

1. Let X denote a digit from 0 through 9.
2. Let N denote a digit from 2 through 9.
3. Let Y denote a digit that is 0 or 1.
4. In the old plan (in use in the 1960s) the format was $NYX - NNX - XXXX$.
5. In the new plan, the format is $NXX - NXX - XXXX$.

How many different telephone numbers are possible under the old plan and the new plan?

Solution: Use the Product Rule.

1. There are $8 \cdot 2 \cdot 10 = 160$ area codes with the format NYX.
2. There are $8 \cdot 10 \cdot 10 = 800$ area codes with the format NXX.
3. There are $8 \cdot 8 \cdot 10 = 640$ office codes with the format NNX.
4. There are $10 \cdot 10 \cdot 10 \cdot 10 = 10,000$ station codes with the format $XXXX$.

Number of old plan telephone numbers: $160 \cdot 640 \cdot 10,000 = 1,024,000,000$.
Number of new plan telephone numbers: $800 \cdot 800 \cdot 10,000 = 6,400,000,000$.

Counting subsets of a finite set

Example (Counting Subsets of a Finite Set)

Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$.

Solution:

1. When the elements of S are listed in an arbitrary order, there is a one-to-one correspondence between subsets of S and bit strings of length $\mid S \mid$.
2. When the $i-$th element is in the subset, the bit string has a 1 in the $i-$th position and a 0 otherwise.
3. By the product rule, there are $2^{|S|}$ such bit strings, and therefore $2^{|S|}$ subsets.

Product rule in terms of sets

Definition (Product Rule of Sets)

If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set. That is:

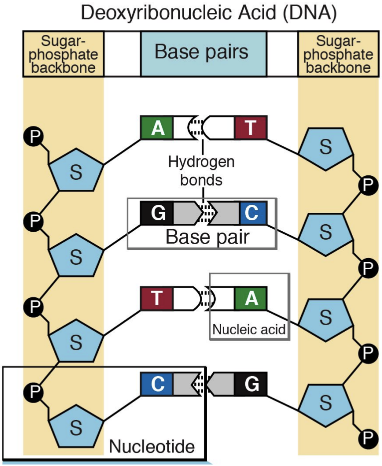
$$|A_1 \times A_2 \times \cdots \times A_m| \;=\; |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|.$$

Indeed:

1. The task of choosing an element in the Cartesian product of $A_1 \times A_2 \times \cdots \times A_m$ is done by choosing an element in A_1, an element in A_2, ..., and an element in A_m.
2. By the product rule, it follows that:

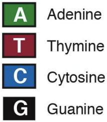
$$|\, A_1 \times A_2 \times \cdots \times A_m \,| \quad = \quad |\, A_1 \,| \cdot |\, A_2 \,| \cdot \cdots \cdot |\, A_m \,|$$

DNA and genomes



A gene (DNA) can be abstractly represented as a string with elements from the alphabet $\Sigma = \{A, T, C, G\}$ e.g.

AGTCTCCATGAAGCACGTTTAC. . .



DNA and genomes

1. A *gene* is a segment of a DNA molecule that encodes a particular protein. The entirety of genetic information of an organism is called its *genome*.
2. The DNA of bacteria has between 10^5 and 10^7 *nucleotides* (characters in a string over the alphabet $\{A, T, C, G\}$). Mammals have between 10^8 and 10^{10} nucleotides. So, by the product rule there are at least 4^{10^5} different sequences of bases in the DNA of bacteria and 4^{10^8} different sequences of bases in the DNA of mammals.
3. The human genome includes approximately 23,000 genes, each with 1,000 or more nucleotides.
4. Biologists, mathematicians, and computer scientists all work on determining the DNA sequence (genome) of different organisms.

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Basic counting principles: the sum rule

Definition (The Sum Rule)

If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the ways of the set of n_1 ways is the same as any of the ways of the set with n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example

The mathematics department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student.

Solution: By the sum rule it follows that there are $37 + 83 = 120$ possible ways to pick a representative.

The sum rule in terms of sets.

Definition (Sum Rule in Terms of Sets)

$| A \cup B | = | A | + | B |$ as long as A and B are disjoint sets. Or more generally:

$$| A_1 \cup A_2 \cup \cdots \cup A_m | = | A_1 | + | A_2 | + \cdots + | A_m |$$

when $A_i \cap A_j = \varnothing$ for all i, j.

The case where the sets have elements in common will be discussed when we consider the subtraction rule.

Combining the sum and product rule

Example

Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.

Solution:

1. Use the sum and product rules.
2. $26 + 26 \cdot 10 = 286$

Counting passwords

Combining the sum and product rule allows us to solve more complex problems.

Example

Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution:

1. Let P be the total number of passwords, and let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.
2. By the sum rule $P = P_6 + P_7 + P_8$.
3. To find each of P_6, P_7, and P_8, we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters. We find that:

$$\begin{aligned}
P_6 &= 36^6 - 26^6 &&= 2,176,782,336 - 308,915,776 &&= 1,867,866,560.\\
P_7 &= 36^7 - 26^7 &&= 78,364,164,096 - 8,031,810,176 &&= 70,332,353,920\\
P_8 &= 36^8 - 26^8 &&= 2,821,109,907,456 - 208,827,064,576 &&= 2,612,282,842,
\end{aligned}$$

4. Consequently, $P = P_6 + P_7 + P_8 = 2,684,483,063,360.$

Internet addresses

1. Version 4 of the Internet Protocol (IPv4) uses 32 bits.

Bit Number	0	1	2	3	4	8	16	24	31
Class A	0	netid				hostid			
Class B	1	0	netid				hostid		
Class C	1	1	0	netid				hostid	
Class D	1	1	1	0	Multicast Address				
Class E	1	1	1	1	0	Address			

2. **Class A Addresses** : used for the largest networks, a 0, followed by a 7-bit netid and a 24-bit hostid.
3. **Class B Addresses** : used for the medium-sized networks, a 10, followed by a 14-bit netid and a 16-bit hostid.
4. **Class C Addresses** : used for the smallest networks, a 110, followed by a 21-bit netid and a 8-bit hostid.
   a. Neither Class D nor Class E addresses are assigned as the address of a computer on the internet. Only Classes A, B, and C are available.
   b. 1111111 is not available as the netid of a Class A network.
   c. Hostids consisting of all 0s and all 1s are not available in any network.

Counting internet addresses

Example

How many different IPv4 addresses are available for computers on the internet?

Solution:

1. Use both the sum and the product rule. Let x be the number of available addresses, and let x_A, x_B, and x_C denote the number of addresses for the respective classes.
2. To find, $x_A : 2^7 - 1 = 127$ netids. $2^{24} - 2 = 16,777,214$ hostids.
   $x_A = 127 \cdot 16,777,214 = 2,130,706,178$.
3. To find, $x_B : 2^{14} = 16,384$ netids. $2^{16} - 2 = 16,534$ hostids.
   $x_B = 16,384 \cdot 16,534 = 1,073,709,056$.
4. To find, $x_C : 2^{21} = 2,097,152$ netids. $2^8 - 2 = 254$ hostids.
   $x_C = 2,097,152 \cdot 254 = 532,676,608$.
5. Hence, the total number of available IPv4 addresses is

$$\begin{aligned} x &= x_A + x_B + x_C \\ &= 2,130,706,178 + 1,073,709,056 + 532,676,608 \\ &= 3,737,091,842 \end{aligned}$$

Not enough today: the newer IPv6 protocol solves the problem of too few addresses.

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Basic counting principles: subtraction rule

Definition (Subtraction Rule)

If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

Also known as, the *principle of inclusion-exclusion*:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Counting bit strings

Example

How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution:

1. Use the subtraction rule.
2. Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$
3. Number of bit strings of length eight that end with bits 00: $2^6 = 64$
4. Number of bit strings of length eight that start with a 1 bit and end with bits 00 : $2^5 = 32$
5. Hence, the number is $128 + 64 - 32 = 160$.

$$\underbrace{\underline{1}\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }}_{2^7=128 \text{ ways}}$$

$$\underbrace{\underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{0}\ \underline{0}}_{2^6=64 \text{ ways}}$$

$$\underbrace{\underline{1}\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{0}\ \underline{0}}_{2^5=32 \text{ ways}}$$

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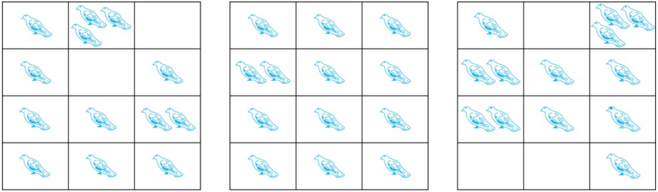
The pigeonhole principle

If a flock of 13 pigeons roosts in a set of 12 pigeonholes, one of the pigeonholes must have more than 1 pigeon.



(a) (b) (c)

Theorem (Pigeonhole Principle)

If $k+1$ objects (for $k>0$) are placed into k boxes, then at least one box contains two or more objects.

Proof.

We use a proof by contraposition.

1. Suppose none of the k boxes has more than one object.
2. Then the total number of objects would be at most k.
3. This contradicts the statement that we have $k+1$ objects.

■

The pigeonhole principle

Corollary

A function f from a set with $k+1$ elements to a set with k elements is not one-to-one.

Proof.

Use the pigeonhole principle.

1. Create a box for each element y in the *codomain* of f.
2. Put in these boxes all of the elements x from the domain such that $f(x) = y$.
3. Because there are $k+1$ elements and only k boxes, at least one box has two or more elements.
4. Hence, f can't be one-to-one.

■

Pigeonhole principle

Example

Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Example

Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.

Solution:

1. Let n be a positive integer. Consider the $n + 1$ integers $1, 11, 111, \ldots, 11\ldots1$ (where the last has $n + 1$ digits).
2. There are n possible remainders when an integer is divided by n .
3. By the pigeonhole principle, when each of the $n + 1$ integers is divided by n, at least two must have the same remainder.
4. Subtract the smaller from the larger and the result is a multiple of n that has only 0s and 1s in its decimal expansion.

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The generalized pigeonhole principle

Theorem (The Generalized Pigeonhole Principle)

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil \frac{N}{k} \rceil$ objects.

Proof.

We use a proof by contraposition.

1. Suppose that none of the boxes contains more than $\lceil \frac{N}{k} \rceil - 1$ objects.
2. Then the total number of objects is at most $k(\lceil \frac{N}{k} \rceil - 1) < k((\frac{N}{k} + 1) - 1) = k\frac{N}{k} = N$, where the inequality $\lceil \frac{N}{k} \rceil < \frac{N}{k} + 1$ has been used.
3. This is a contradiction because there are a total of N objects.

■

Example

Among 200 students in CS2214 there are at least $\lceil \frac{200}{12} \rceil = 17$ who were born in the same month.

The generalized pigeonhole principle

Example

How many cards (N) must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

Solution:

1. We assume four boxes; one for each suit. Using the generalized pigeonhole principle, at least one box contains at least $\lceil \frac{N}{4} \rceil$ cards.
2. At least three cards of one suit are selected if $\lceil \frac{N}{4} \rceil \geq 3$.
3. The smallest integer N such that $\lceil \frac{N}{4} \rceil \geq 3$ is $N = 2 \cdot 4 + 1 = 9$.

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Permutations

Definition

A *permutation* of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an r-permutation.

Example

Let $S = \{1,2,3\}$.

1. The ordered arrangement 3,1,2 is a permutation of S.
2. The ordered arrangement 3,2 is a 2-permutation of S.

The number of r -permutations of a set with n elements is denoted by $P(n,r)$.

3. The 2-permutations of $S = \{1,2,3\}$ are 1,2; 1,3; 2,1; 2,3; 3,1; 3,2. Hence, $P(3,2) = 6$.

A formula for the number of permutations

Theorem

If n is a positive integer and r is an integer, with $1 \leq r \leq n$, then there are $P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$ r-permutations of a set with n distinct elements.

Proof.

Use the product rule.

1. The first element can be chosen in n ways.
2. The second in $n-1$ ways, and so on until there are $(n-(r-1))$ ways to choose the last element.

∎

Note that $P(n, 0) = 1$ as there is only one way to order zero elements.

Corollary

If n and r are integers, with $1 \leq r \leq n$, then

$$P(n, r) = \frac{n!}{(n-r)!}$$

Solving counting problems by counting permutations

Example

How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution:

1. $P(100,3) = 100 \cdot 99 \cdot 98 = 970,200$

Solving counting problems by counting permutations (*continued*)

Example

Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution:

1. The first city is chosen, and the rest are ordered arbitrarily.
2. Hence the orders are:
3. $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$
4. If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!

Solving counting problems by counting permutations (*continued*)

Example

How many permutations of the letters $ABCDEFGH$ contain the string ABC?

Solution:

1. We solve this problem by counting the permutations of six objects, ABC, D, E, F, G, and H.
2. $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

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Combinations

Definition

An r-combination of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.

1. The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$. The notation $\binom{n}{r}$ is also used and is called a *binomial coefficient*.

Example

Let S be the set $\{a, b, c, d\}$. Then $\{a, c\}$ is a 2-combination from S. It is the same as $\{c, a\}$ since the order does not matter.

1. $C(4, 2) = 6$ because the 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.

Combinations

Theorem

The number of r – combinations of a set with n elements, where $n \geq r \geq 0$, equals

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

Proof.

By the product rule:

$$P(n,r) = C(n,r) \cdot P(r,r)$$

1. **goal:** get ordered arrangement of r elements from a set of n
2. **task 1:** get unordered selection of r elements from a set of n
3. **task 2:** get ordered arrangement of r elements from a set of r

Therefore,

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{\frac{n!}{(n-r)!}}{\frac{r!}{(r-r)!}} = \frac{n!}{(n-r)!r!}$$

∎

Combinations

Example

How many poker hands of five cards can be dealt from a standard deck of 52 cards?

Solution:

1. Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$C(52,5) = \frac{52!}{5!47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960$$

Example

How many different ways are there to select 47 cards from a standard deck of 52 cards?

Solution:

1. $C(52,47) = \frac{52!}{47!5!} = C(52,5) = 2,598,960$

This is a special case of a general result. →

Combinations

Corollary

Let n and r be non-negative integers with $r \leq n$. Then $C(n,r) = C(n,n-r)$.

Proof.

1. From Theorem 2, it follows that
$$C(n,r) = \frac{n!}{(n-r)!r!}$$
2. and
$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!}$$
3. Hence, $C(n,r) = C(n,n-r)$.

∎

Combinations

Example

How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.

Solution:

1. By Theorem 2, the number of combinations is
$$C(10,5) = \frac{10!}{5!5!} = 252$$

Example

A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission?

Solution:

1. By Theorem 2, the number of possible crews is
$$C(30,6) = \frac{30!}{6!24!} = \frac{30\cdot 29\cdot 28\cdot 27\cdot 26\cdot 25}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 593,775$$

Plan for Chapter 6

1. The Basics of Counting
1.1 The Product Rule
1.2 The Sum Rule
1.3 The Subtraction Rule

2. The Pigeonhole Principle
2.1 The Pigeonhole Principle
2.2 The Generalized Pigeonhole Principle

3. Permutations and Combinations
3.1 Permutations
3.2 Combinations

4. Binomial Coefficients and Identities
4.1 The Binomial Theorem
4.2 Pascal's Identity and Triangle

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Powers of binomial expressions

A *binomial* expression is the sum of two terms, such as $x + y$. More generally, these terms can be products of constants and variables.

1. We can use counting principles to find the coefficients in the expansion of $(x + y)^n$ where n is a positive integer.
2. To illustrate this idea, we first look at the process of expanding $(x + y)^3$.
3. $(x + y)(x + y)(x + y)$ expands into a sum of terms that are the product of a term from each of the three sums.
4. Terms of the form x^3, x^2y, xy^2, y^3 arise. The question is what are the coefficients of those terms?
   a. To obtain x^3, an x must be chosen from each of the sums. There is only one way to do this. So, the coefficient ofx^3 is 1.
   b. To obtain x^2y, an x must be chosen from two of the sums and a y from the other. There are $\binom{3}{2}$ ways to do this and so the coefficient of x^2y is 3.
   c. To obtain xy^2, an x must be chosen from of the sums and a y from the other two . There are $\binom{3}{1}$ ways to do this and so the coefficient of xy^2 is 3.
   d. To obtain y^3, a y must be chosen from each of the sums. There is only one way to do this. So, the coefficient of y^3 is 1.
5. We have used a counting argument to show that $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.
6. Next we present the binomial theorem gives the coefficients of the terms in the expansion of $(x + y)^n$.

Binomial theorem

Theorem (Binomial Theorem)

Let x and y be variables, and n a non-negative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

Proof.

We use combinatorial reasoning.

1. All terms in the expansion of $(x+y)^n$ are of the form $x^{n-j}y^j$ for $j = 0, 1, 2, \ldots, n$.
2. To form the term $x^{n-j}y^j$, it is necessary to choose $n-j$ x's from the n sums.
3. Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{j}$.

∎

Using the binomial theorem

Example

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution:

1. We view the expression as $(2x + (-3y))^{25}$.
2. By the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j$$

3. Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$.

$$\binom{25}{13} = \frac{25!}{13!12!} = 5200300.$$

A useful identity

Corollary

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \textit{ with } n \geq 0$$

Proof.

(*using binomial theorem*): With $x = 1$ and $y = 1$, from the binomial theorem we see that:

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$$

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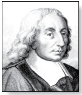
Pascal's Identity



Blaise Pascal (1623 - 1662)

Definition (Pascal's Identity)

If n and k are integers with $n > k \geq 0$, then

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

Proof.

1. Consider a set $S = \{s_1, s_2, \ldots, s_n, s_{n+1}\}$ with $n+1$ elements.
2. A subset $X \subseteq S$ with $k+1$ elements is obtained
   a. either by choosing s_{n+1} and k elements from $\{s_1, s_2, \ldots, s_n\}$, which can be done in $\binom{n}{k}$ ways,
   b. or by not choosing s_{n+1}, thus choosing $k+1$ elements from $\{s_1, s_2, \ldots, s_n\}$, which can be done in $\binom{n}{k+1}$ ways.
3. This yields the result.

■

Pascal's triangle

The n-th row in the triangle consists of the binomial coefficients $\binom{n}{k}$, $k = 0, 1, \ldots, n$.

(a) (b)

By Pascal's identity, adding two adjacent binomial coefficients results is the binomial coefficient in the next row between these two coefficients.