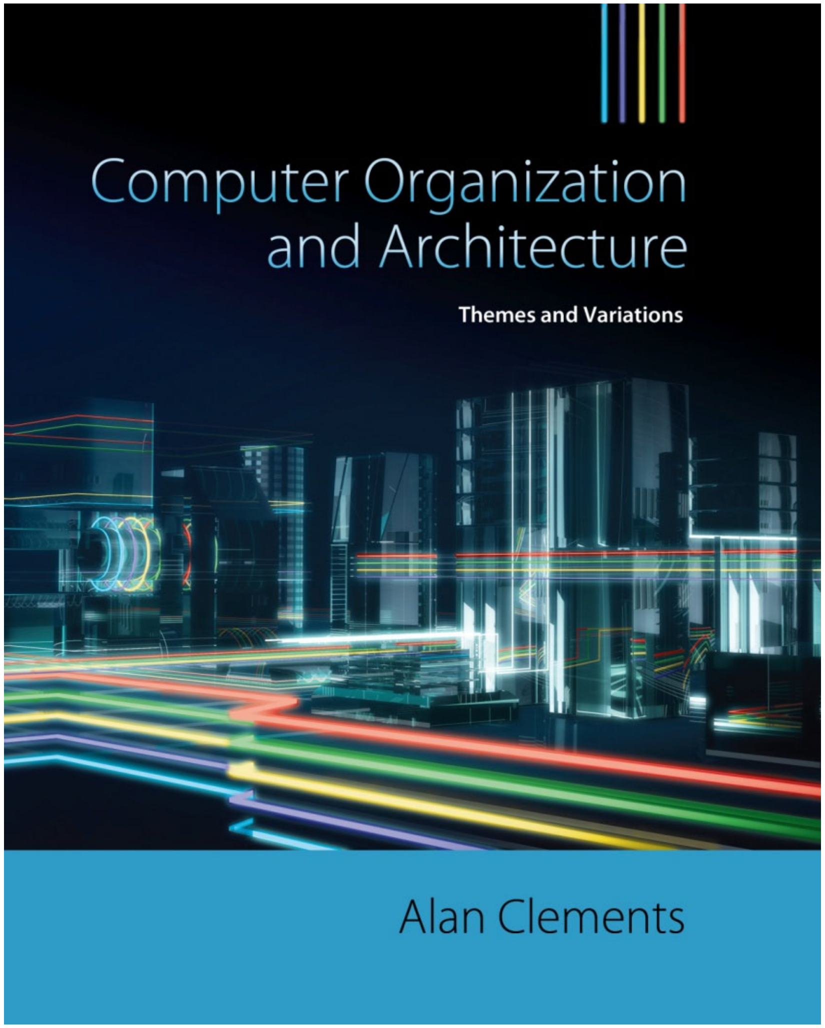
Part 2

Chapter 2

Computer Arithmetic and Digital Logic



1

CENGAGE Learning

Signed Integers

- Signed numbers can be represented in many ways, including
  - ***sign and magnitude***,
  - ***biased representation***, and
  - ***two's complement***.

Sign and Magnitude Representation

- An n-bit word has 2^n possible values
  - for example, an eight-bit word can represent 256 different values.
- One way of representing a negative number is to take the *most-significant bit* and *reserve it* to indicate *the sign of the number*.
  - 0 represents positive numbers and
  - 1 represents negative numbers.
- ***In an eight-bit word, using sign and magnitude representation***
  How many positive numbers can be represented?
  How many negative numbers can be represented?
- The value of a *sign and magnitude number* can be expressed as $(-1)^S \times M$, where S is the sign bit and M is the magnitude of the number.
  - If $S = 0$, $(-1)^0 = +1$ and the number is positive.
  - If $S = 1$, $(-1)^1 = -1$ and the number is negative.
- for example, in 8 bits we can ***interpret*** the numbers 00001101_2 and 10001101_2 as $+13_{10}$ and -13_{10}, respectively.
- Sign and magnitude representation is
  - **not** generally **used in integer arithmetic**,
  - however, it is **used in floating-point arithmetic**.

Biased Representation

- Assume that we have values ranging from –7 to +7
- To go around the negative sign, we can *shift the scale* (*by adding a bias = 7 to all values*) to have *only non-negative values*.
  - The original number = –7 ➔ The biased number = 0
  - The original number = –6 ➔ The biased number = 1
  - ....
  - The original number = 0 ➔ The biased number = 7
  - ....
  - The original number = +6 ➔ The biased number = 13
  - The original number = +7 ➔ The biased number = 14

  Excess-7 code
- To convert a biased number to its original value, you need to *shift back the scale* (*by subtracting the bias from the biased number*).
  - The biased number = 0 ➔ The original number = –7
  - The biased number = 1 ➔ The original number = –6
  - ....
  - The biased number = 7 ➔ The original number = 0
  - ....
  - The biased number = 13 ➔ The original number = +6
  - The biased number = 14 ➔ The original number = +7
- In this representation, ***biased numbers are unsigned integers***; yet, they ***represent both positive and negative values***.
- The ***biased representation*** is also called ***excess-K***, where ***K*** is a pre-specified biasing value

Complementary Arithmetic

- A number and its ***complement*** add up to a constant;
  - In *n-digit* ***decimal*** *ten's complement* (*radix complement*) arithmetic, if P is a number, then its *ten's* complement is Q, where P + Q = 10^n.
    - the *ten's* complement of 12 is 88 because 12 + 88 = 100.
    - the *ten's* complement of 88 is 12 because 88 + 12 = 100.
  - In *n-bit* ***binary*** *two's complement* (*radix complement*) arithmetic, if P is a number, then its *two's* complement is Q, where P + Q = 2^n.
    - the *two's* complement of 101_2 is 011_2 because $101_2 + 011_2 = 1000_2$.
    - the *two's* complement of 011_2 is 101_2 because $011_2 + 101_2 = 1000_2$.

Two's Complement Arithmetic

- In binary arithmetic, the *two's complement* of a number is formed by
  - *Subtracting the number from 2^n*.

    The *two's* complement of 01100101_2 is
    $100000000_2 - 01100101_2 = 10011011_2$
  - *Flipping (inverting) all the bits of the number* and *adding 1*.

    The *two's* complement of 01100101_2 is
    $10011010_2 + 1_2 = 10011011_2$.
  - *Processing all the bits of the number from the least significant bit (LSB) towards the most significant bit (MSB)*
    - copying all the zeros until the first 1 is reached,
    - copying that 1,
    - flipping (inverting) all the remaining bits.

    The *two's* complement of 01100100_2 is 10011100_2.
    The *two's* complement of 01100101_2 is 10011011_2.

Two's Complement Arithmetic

- The *two's* complement of a *positive* number is the corresponding *negative* value of the *positive* number when it is represented in *two's* complement.
- The *two's* complement of a *negative* number is the corresponding *positive* value of the *negative* number when it is represented in *two's* complement.

- We are interested in ***complementary*** arithmetic because *subtracting* a number is the same as *adding its complement*.
  - To subtract 001100101_2 from a binary number, we just need to add its *two's* complement (i.e., 110011011_2) to the number.

Two's Complement Arithmetic

- The *two's* complement of an *n*-bit binary value, N, is defined as $2^n - N$.
- If $N = 5_{10} = 00000101_2$ (8-bit arithmetic), the *two's* complement of N is given by $2^8 - 00000101_2 = 100000000_2 - 00000101_2 = 11111011_2$.
- 11111011_2 represents -00000101_2 (i.e., -5_{10}) or -123_{10} depending only on whether we interpret 11111011_2 as a *two's complement integer* or as a *sign-magnitude integer*, respectively.

This is -123 not +123

This is sign-magnitude not unsigned

Two's Complement Arithmetic

- This example demonstrates 8-bit *two's* complement arithmetic. We begin by writing down the representations of +5, –5, +7 and –7.

$+5_{10} = 00000101_2$ $-5_{10} = 11111011_2$ $+7_{10} = 00000111_2$ $-7_{10} = 11111001_2$

- Now $7_{10} - 5_{10}$ can be calculated as $7_{10} + (-5_{10})$ by adding the binary value for 7_{10} to the *two's* complement of 5_{10}.

```
11111111
 00000111        7
+11111011       -5
100000010        2
```

The result is correct if the left-hand carry-out is ignored.

Two's Complement Arithmetic

- Now consider the addition of -7_{10} to $+5_{10}$, i.e., $+5_{10} + (-7_{10})$

```
        1
  00000101        5
 +11111001       -7
  --------       --
  11111110       -2
```

The result is 11111110_2 (the carry-out bit is 0).

The expected answer is -2_{10};
that is, $2^8 - 2_{10} = 100000000_2 - 00000010_2 = 11111110_2$.

Two's Complement Arithmetic

- *Two's* complement arithmetic is not magic.
- Consider the calculation $Z = X - Y$ in n-bit arithmetic
  - we do it by *adding* the *two's* complement of Y to X.
- The *two's* complement of Y is defined as $2^n - Y$.

  We get

$$Z = X - Y = X + (2^n - Y) = 2^n + (X - Y).$$

- This is
  - *the desired result*, $X - Y$
  - If the result of $X - Y$ is *negative*, the *two's* complement will be automatically calculated, i.e., $2^n + (X - Y)$, where $(X - Y)$ is *negative*.
  - Otherwise, we discard the *unwanted carry-out digit* (i.e., 2^n) at the leftmost position

Two's Complement Arithmetic

Time examples using radix complement

In the 24H system *(i.e., radix = 24)*, assume that

- the time right now is 2 am and we want to subtract 5 hours from 2 am
  - The radix complement of 5 is 24 – 5 = 19
  - To subtract 5 from 2 in the 24H system, it is equivalent to adding the 2 to the complement of 5, i.e., 2 + 19 = 21 *(correct answer)*
  - The 21 can also be read as –3 in the radix complement system
- the time right now is 5 am and we want to subtract 2 hours from 5 am
  - The radix complement of 2 is 24 – 2 = 22
  - To subtract 2 from 5 in the 24H system, it is equivalent to adding the 5 to the complement of 2, i.e., 5 + 22 = 27
  - As all the time values must be between 0 and 23, we need to subtract a full 24H from the result, i.e., 27 – 24 = 3 *(correct answer)*

Two's Complement Arithmetic

Let $X = 9_{10} = 00001001_2$ and $Y = 6_{10} = 00000110_2$

$-X = -9_{10} = 100000000_2 - 00001001_2 = 11110111_2$
$-Y = -6_{10} = 100000000_2 - 00000110_2 = 11111010_2$

```
1.  +X   +9      00001001
    +Y   +6     +00000110
         +15     00001111
```

```
                 11111
2.  +X   +9      00001001
    -Y   -6     +11111010
         +3     100000011
```

```
                      11
3.  -X   -9      11110111
    +Y   +6     +00000110
         -3      11111101
```

```
                 1111111
4.  -X   -9      11110111
    -Y   -6     +11111010
         -15    111110001
```

11111101 is a negative *two's* complement number
The two's complement of **11111101** is **00000011**$_2$
Hence, **11111101** = **–00000011**$_2$ = -3_{10}

11110001 is a negative *two's* complement number
The two's complement of **11110001** is **00001111**$_2$
Hence, **11110001** = **–00001111**$_2$ = -15_{10}

All four examples give the result we would expect when the result is interpreted as a *two's* complement number.

Properties of Two's complement Numbers

- The *two's* complement system is a true complement system in that $+X + (-X) = 0$.
- There is *one* unique zero 00…0.
- This is *not the case with the signed magnitude* number system
- The *most-significant bit* of a *two's* complement number is a *sign bit*.
  - The number is *positive* if the most-significant bit is *0*, and
  - *negative* if it is *1*.

  This is the same as the signed magnitude number system
- The range of an *n*-bit *two's* complement number is from -2^{n-1} to $+2^{n-1} - 1$.
  - For $n = 4$, the range is –8 to +7.
    The total number of different values is $2^n = 16$
    - 8 negative,
    - zero and
    - 7 positive.
  - For $n = 8$, the range is –128 to +127.
    The total number of different values is $2^n = 256$
    - 128 negative,
    - zero and
    - 127 positive.
- ***How about the signed magnitude number system?***

000001 => 011111
000000
these two are same number.
011111 + 100001 = 0
100000.

for 2's complement => zero 's zero.

2's complement: 2^{n-1} negative 1 0
$2^{n-1}-1$ positive.

Arithmetic Overflow

- The *range* of *two's* complement numbers in n-bits is from -2^{n-1} to $+2^{n-1} - 1$.
- What happens if we ***violate*** this rule by carrying out an operation whose ***result falls outside the range*** of values that can be represented by *two's* complement numbers?
  - In a *five-bit* representation, the range of *valid two's complement numbers* is *–16* to *+15*.

Case 1

```
  5  =  00101
 +7  =  00111
 12     01100  =  12₁₀
```

Case 2

```
 12  =  01100
+13  =  01101
 25     11001  =  -7₁₀
```

(as a *two's* complement value)

In *Case 1* we get the *expected* answer of $+12_{10}$, but

In *Case 2* we get a negative result because the *sign bit is 1*.

- If the answer were regarded as an *unsigned binary number,* it would be +25, which is, of course, the correct answer.
- However, as the *two's* complement system has been chosen to represent signed numbers, all answers must be interpreted in this light.

Arithmetic Overflow

- If we add together two negative numbers whose total is less than –16, we will go out of range.

Case 3

```
 –9 =   10111
–12 =  +10100
–21    101011 = +11₁₀
```

(as a *two's* complement value)

In *Case 3* we get a positive result because the *sign bit is 0*.

Arithmetic Overflow

- The last two examples demonstrate *arithmetic overflow*
- *Arithmetic overflow* occurs during a *two's complement addition* when
  - the result of *adding two positive* numbers *yields a negative* result, or
  - the result of *adding two negative* numbers *yields a positive* result.

- ***If the sign bits of A and B are the same but the sign bit of the result is different, it means arithmetic overflow has occurred.***

- ***Overflow never occurs when adding two numbers with opposite signs***

- If a_{n-1} is the sign bit of *A*,
  b_{n-1} is the sign bit of *B*, and
  s_{n-1} is the sign bit of the sum of *A* and *B*,

  then

  overflow is defined by the *logical expression* as follow:

  $$V = (a_{n-1})^c \cdot (b_{n-1})^c \cdot s_{n-1} + a_{n-1} \cdot b_{n-1} \cdot (s_{n-1})^c$$

  where $(\ )^c$ is the complement binary operation

Arithmetic Overflow

- In practice, the overflow can be detected from the *carry bits in to* (i.e., C_{in}) and *out of* (i.e., C_{out}) the *most-significant bit* of an adder;

overflow occurs when $C_{in} \neq C_{out}$

8-bit examples in slide#30

```
                 Cout Cin
                  00000000
+X    +9          00001001
+Y    +6         +00000110
      +15         00001111
```

```
                 Cout Cin
                  11111000
+X    +9          00001001
-Y    -6         +11111010
      +3         100000011
```

```
                 Cout Cin
                  00000110
-X    -9          11110111
+Y    +6         +00000110
      -3          11111101
```

```
                 Cout Cin
                  11111110
-X    -9          11110111
-Y    -6         +11111010
      -15        111110001
```

5-bit examples in slides#32,33

```
       Cout Cin
        00111
 5  =   00101
+7  =   00111
12      01100  = +12₁₀
```

```
       Cout Cin
        01100
 12  =  01100
+13  =  01101
 25     11001  = -7₁₀
```

Overflow occurred

```
       Cout Cin
        10100
 -9  =   10111
-12  =  +10100
-21     101011  = +11₁₀
```

Overflow occurred

Shifting Operations

- In a shift operation, the *bits of a word* are *shifted* one or more places to the *left* or *right*.
- The ***arithmetic*** shift is just one type of shift operation.
- In Chapter 3, we will cover the other types of shift operations.

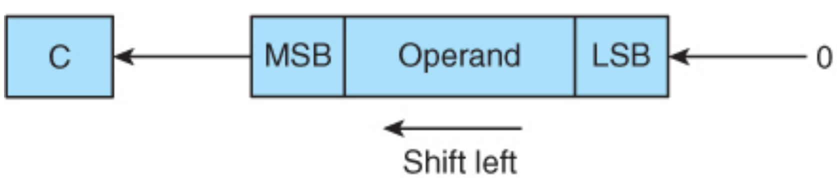
Shifting Operations (left)

- Figure 2.2(a) describes the arithmetic *shift left*.
  - A *zero* enters into the vacated *least-significant bit* position and
  - the bit *shifted out* of the *most-significant bit* position is recorded in the computer's *carry flag*.

  - In *two's* complement and unsigned numbers:
    - Arithmetic *shift left* means *multiplying* by 2.

- If 00100111_2 $(39)_{10}$ is shifted one place left it becomes 01001110_2 $(78)_{10}$.
- If 11100011_2 $(-29)_{10}$ is shifted one place left it becomes 11000110_2 $(-58)_{10}$.

FIGURE 2.2 Arithmetic shift operations



(a) Arithmetic shift left:

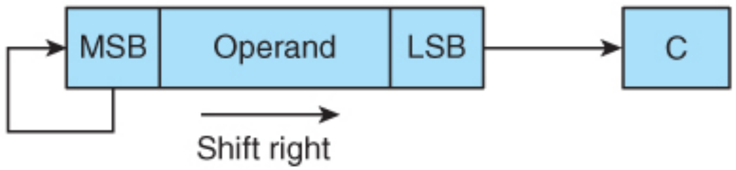
A zero enters the least-significant bit position and the most-significant bit is copied into the carry flag. For example: 11000101 becomes 10001010 after shifting one place left.

Shifting Operations (right)

- Figure 2.2(b) describes the arithmetic *shift right*.
  - the sign bit (the MSB) is *replicated* into the vacated *most-significant bit* position and
  - the bit *shifted out* of the *least-significant bit* position is recorded in the computer's *carry flag*.

  - In *two's* complement and unsigned numbers:
    - Arithmetic *shift right* means *dividing* by 2.
- Shifting 00001100_2 $(12)_{10}$ one place right produces 00000110_2 $(6)_{10}$.
- Shifting 00001101_2 $(13)_{10}$ one place right produces 00000110_2 $(6)_{10}$ as well.
- Shifting 11100010_2 $(-30)_{10}$ one place right produces 11110001_2 $(-15)_{10}$.
- Shifting 11100011_2 (-29) one place right produces 11110001_2 (-15) as well.

FIGURE 2.2 Arithmetic shift operations



(b) Arithmetic shift right:

A copy of the most-significant bit enters the most-significant bit position. All other bits are shifted one place right. The least-significant bit is copied into the carry flag.

For example; 00100101 becomes 00010010 after shifting one place right, and 11100101 becomes 11110010 after shifting one place right.

Binary Multiplication

- Multiplication can be implemented using arithmetic *shift left* and addition operations to add up the partial products as they are formed.

Multiplicand	Multiplier	Step	Partial products														
01101001	0100100**1**	1								0	1	1	0	1	0	0	1
01101001	010010**0**1	2							0	0	0	0	0	0	0	0	
01101001	01001**0**01	3						0	0	0	0	0	0	0	0		
01101001	0100**1**001	4					0	1	1	0	1	0	0	1			
01101001	010**0**1001	5				0	0	0	0	0	0	0	0				
01101001	01**0**01001	6			0	0	0	0	0	0	0	0					
01101001	0**1**001001	7		0	1	1	0	1	0	0	1						
01101001	**0**1001001	8	0	0	0	0	0	0	0	0							
		Result	**0**	**0**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**0**	**1**