COUNTING

OUTLINE:

1) Product and sum rules
2) Pigeonhole principle
3) Permutations and combinations
4) Binomial coefficients

1. PRODUCT AND SUM RULES

The product rule

- Assume that some procedure can be broken down into a sequence of k consecutive and independent actions. Assume also that there are n_1 choices to perform the first action, n_2 choices to perform the second action, and so on. Then there are $n_1 \cdot n_2 \cdot \ldots \cdot n_k$ ways to perform the whole procedure.

The product rule

- EX: How many Canadian postal codes can there be?
- A Canadian postal code is a sequence of 6 characters of the form ‘xnx nxn’, where x is an uppercase english letter and n is a decimal digit.
- There are (theoretically) 26 ways of choosing the 1st character of the code, 10 ways of choosing the 2nd, 26 ways for the 3rd, 10 for the 4th, 26 for the 5th and 10 for the 6th
- By the product rule, there can be (theoretically) $26 \cdot 10 \cdot 26 \cdot 10 \cdot 26 \cdot 10 = 17576000$ Canadian postal codes.

The product rule

- EX: How many Canadian postal codes can there be?
- Actually, Canadian postal codes do not include the letters D,F,I,O,Q,U; in addition, the 1st character cannot be W or Z
- With these limitations, there are 18 ways of choosing the 1st character of the code, 10 ways of choosing the 2nd, 20 ways for the 3rd, 10 for the 4th, 20 for the 5th and 10 for the 6th
- By the product rule, there can be $18 \cdot 10 \cdot 20 \cdot 10 \cdot 20 \cdot 10 = 7200000$ Canadian postal codes.

The product rule

- EX: How many distinct functions can there be from a domain of cardinality k to a codomain of cardinality n?
- Each element of the domain must have one and only one image in the codomain.
- Create an arbitrary ordering of the elements of the domain
- The image of the 1st element of the domain can be chosen among n possible elements of the codomain; the same holds for the 2nd element of the domain and so on for all k elements of the domain.
- By the product rule, there are n^k possible functions form a domain of cardinality k to a codomain of cardinality n.

The extended product rule

- Slight generalization: if the procedure at hands consists of k actions $A_1,\ldots,A_k$ in sequence, and each action A_i can be performed in n_i ways regardless of how the previous actions were performed (i.e., the actual available choices for action A_i may depend on the previous actions, but the number of available choices for A_i does not depend on the previous actions). Then there are $n_1 \cdot n_2 \cdot \ldots \cdot n_k$ ways to perform the whole procedure.

The extended product rule

- EX: How many distinct injective functions can there be from a domain of cardinality k to a codomain of cardinality n (with $k \leq n$)?
- Each element of the domain must have one and only one image in the codomain, and distinct elements of the domain cannot have the same image.
- Create an arbitrary ordering of the elements of the domain: $a_1,a_2,...,a_k$.
- The image of a_1 can be chosen among n possible elements of the codomain; the image of a_2 cam be chosen among n-1 elements of the codomain (because 1 element is already reserved as the image of a_1), and so on, up to the image of a_k, which can be chosen among n-k+1 elements of the codomain. Which exact elements of the codomain are available as images for a_i depends on which elements have been already chosen as images of $a_1,a_2,...,a_{i-1}$, but the number of available elements does not depend on that.
- By the extended product rule, there are $n \cdot (n-1) \cdot ... \cdot (n-k+1)$ possible injective functions from a domain of cardinality k to a codomain of cardinality $n \geq k$.

The product rule in set theory

- More formally, the (basic or extended) product rule is the set-theoretic principle stating that the cardinality of a cartesian product of sets is the product of the cardinalities of the sets:

$$|A_1 \times A_2 \times \ldots \times A_k| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_k|$$

- The cartesian product represents the whole procedure, while the single factors A_i are the single independent actions involved. Using the cartesian product ensures that the single actions are independent: given any choice for the 1st action, all the choices for the 2nd action are available, and so on (because the cartesian product is the set of all possible tuples with entries in the factor sets).

The sum rule

- Assume that some procedure can be performed by choosing among k mutually exclusive actions (i.e., we choose one and only one). Assume also that there are n_1 choices to perform the first action, n_2 choices to perform the second action, and so on. Finally, assume that the choices available for each action are not available choices for any other action. Then there are $n_1+n_2+...+n_k$ ways to perform the whole procedure.

The sum rule

- EX: For the next take-out night with your friends you can try either “Federico’s Pizzeria”, which offers 16 choices of real Italian pizzas, or “Freddy Spaghetti”, which is famous for its 12 varieties of pasta, or “Little Italy”, wits its freshly made 5 risottos of the day.
- In total, you have 16+12+5 = 33 choices for your dinner.

The sum rule in set theory

- More formally, the sum rule is the set-theoretic principle stating that the cardinality of the union of mutually disjoint sets is the sum of the cardinalities of the sets:

$$|A_1 \cup A_2 \cup \ldots \cup A_k| = |A_1| + |A_2| + \ldots + |A_k|$$

$$\textit{provided } \forall\, i \neq j \; \left(A_i \cap A_j = \emptyset \right)$$

The extended sum rule (aka the inclusion-exclusion principle)

- Assume that some procedure can be performed by choosing among 2 mutually exclusive actions (i.e., we choose one and only one). Assume also that there are n_1 choices to perform the first action and n_2 choices to perform the second action, but we do not ask the available choices for each action to be unavailable for the other action. Then the number of ways to perform the whole procedure is:

  n_1+n_2-(number of choices that are common to the 2 different actions).

- The textbook calls this rule the "subtraction rule"
- There are more general versions, using an arbitrary number of choices, which you have already seen in the tutorial on set theory

The inclusion-exclusion principle

- EX: In a (very little) portable memory, cells are identified with a binary strings of length 6 (the address). The cells whose address starts with 0 or ends with 10 are reserved for the file system and for backup purposes. How many available cells does the memory have?
- Let us first count the reserved cells:
  - The cells whose address starts with 0 are $2^5 = 32$ (product rule on the other 5 bits of the address)
  - The cells whose address ends with 10 are $2^4 = 16$ (product rule on the other 4 bits of the address)
  - The cells whose address both starts with 0 and ends with 10 are $2^3 = 8$ (product rule on the other 3 bits of the address)
  - By the inclusion-exclusion principle, the reserved cells are $32+16-8 = 40$
- Therefore, the available cells are (total cells)-(reserved cells) = $2^6-40 = 24$

2. PIGEONHOLE PRINCIPLE

The pigeonhole principle

- Theorem (Pigeonhole Principle): Let n>0 be a positive integer. If n+1 objects are placed into n “boxes” (concrete or abstract), then at least one box contains at least 2 objects.
- Proof: Assume by contradiction that no box contains more than 1 object. Then the total number of objects would be at most equal to the number of boxes, that is n. This contradicts the fact that there are n+1 objects.
- Statement of the principle in terms of functions: for any positive integer n>0, there is no injective function from a set of cardinality n+1 to a set of cardinality n.

The pigeonhole principle

- EX: Suppose you have only black, blue and red socks, all mixed together in your drawer.
- Suppose also (don’t ask me why, that’s your problem) that you pull a number of socks from the drawer without looking.
- What is the minimum number of socks you have to pull from the drawer to be certain to get a pair of socks of the same colour?
- By the pigeonhole principle, you need to pick 4 socks. If you pick 2 or 3 socks, you may get 2 of the same colour. However, if you pick 4 or more, you are guaranteed to have at least 2 of the same colour because there are only 3 possible colours.

The generalized pigeonhole principle

- Theorem (Pigeonhole Principle): Let n and k be positive integers. If n objects are placed into k “boxes”, then at least one box contains at least $\lceil n/k \rceil$ objects
- Proof: Assume by contradiction that no box contains more than $\lceil n/k \rceil - 1$ objects. Then the total number of objects would be at most $k(\lceil n/k \rceil - 1)$ objects. But, since $\lceil n/k \rceil < n/k+1$, we have $k(\lceil n/k \rceil - 1) < k((n/k+1)-1) = n$. This contradicts the fact that there are n objects.

The generalized pigeonhole principle

- EX: This term, there are 224 students taking CS2214. Therefore, there are at least $\lceil 224/12 \rceil = 19$ students who were born in the same month.
- EX: How many times do you have to roll a d6 die to be sure to get at least 3 equal outcomes (no matter which)? Let n be the number of rolls. We have $k = 6$ possible outcomes (the boxes). We get at least 3 equal outcomes when $\lceil n/6 \rceil \geq 3$. The smallest n satisfying the inequality $\lceil n/6 \rceil \geq 3$ is $n = 2 \cdot 6+1 = 13$.

3. PERMUTATIONS AND COMBINATIONS

Permutations

- A permutation of a set S is an ordered list of the elements of the set.
- EX: The permutations of the set S = {1,2,3} are
  - 1,2,3
  - 1,3,2
  - 2,1,3
  - 2,3,1
  - 3,1,2
  - 3,2,1

Permutations

- If $|S| = n$, and $k \le n$, a k-permutation of elements of S is an ordered list of k elements of the set. The number of k-permutations on a set of cardinality n is denoted $P(n,k)$. This number depends only on n and k, not on the actual elements of S.
- In set-theoretic terms, k-permutations are k-tuples of elements of S with distinct entries
- EX: The 2-permutations of the set $S = \{1,2,3\}$ are

  (1,2) (2,1) (1,3) (3,1) (2,3) (3,2)

  Therefore, $P(3,2) = 6$
- EX: The 2-permutations of the set $T = \{a,b,c\}$ are

  (a,b) (b,a) (a,c) (c,a) (b,c) (c,b)

  As expected, we get $P(3,2) = 6$ again.
- If $|S| = n$, the n-permutations of S are just the permutations of S

The number of permutations

- Theorem: For any positive integers n and k, with k≤n, the number of k-permutations of a set with cardinality n is

$$P(n,k)=n\cdot(n-1)\cdot\ldots\cdot(n-k+1)=\frac{n!}{(n-k)!}$$

- Proof: by the extended product rule (see also the number of injective functions from a k-element domain to a n-element codomain):
- The 1st element of the permutation can be chosen among n objects, the 2nd among n-1 objects (one has already been taken), and so on, up to the kth element, which can be chosen among n-k+1 objects.
- For the sake of completeness, we also define P(n,0) = 1.

The number of permutations

- EX: For each L = 1,...,10, count the strings of length L made of distinct decimal digits
  - L=1: P(10,1) = 10 (just pick a digit between 0 and 9)
  - L=2: P(10,2) = 10·9 = 90 (10 ways to pick the 1st digit, 9 ways to pick the 2nd)
  - L=3: P(10,3) = 10·9·8 = 720
  - L=4: P(10,4) = 10·9·8·7 = 5040
  - L=5: P(10,5) = 10·9·8·7·6 = 30240
  - L=6: P(10,6) = 10·9·8·7·6·5 = 151200
  - L=7: P(10,7) = 10·9·8·7·6·5·4 = 604800
  - L=8: P(10,8) = 10·9·8·7·6·5·4·3 = 1814400
  - L=9: P(10,9) = 10·9·8·7·6·5·4·3·2 = 10! = 3631101
  - L=10: P(10,10) = 10! = 3631101

The number of permutations

- EX: how many permutations of the letters A,B,C,D,E,F contain the string ‘ACE’?
- We treat ‘ACE’ as a single object, and each remaining letter as an object on its own
- So we have 4 objects: ‘ACE’, ‘B’, ‘D’, ‘F’
- The permutations of 4 objects are $P(4,4) = 4! = 24$

The number of permutations

- EX: how many permutations of the letters A,B,C,D,E,F have the vowels in consecutive positions?
- The permutations we want feature either the string ‘AE’ or the string ‘EA’ (but not both, since letters are not repeated), so we are going to apply the sum rule, with strings containing ‘AE’ as 1st “action” and strings containing ‘EA’ as 2nd “action”.
- We first compute the permutations of the 5 objects ‘AE’, ‘B’, ‘C’, ‘D’, ‘F’: n_1 = P(5,5) = 5! = 120
- The permutations of the 5 objects ‘EA’, ‘B’, ‘C’, ‘D’, ‘F’ are also n_2 = P(5,5) = 5! = 120
- By the sum rule, the number of the permutations under consideration is n_1+n_2 = P(5,5)+P(5,5) = 240

Combinations

- If $|S| = n$, and $k \leq n$, a k-combination of elements of S is an unordered selection of k elements of the set. The number of k-combinations on a set of cardinality n is denoted $C(n,k)$. This number depends only on n and k, not on the actual elements of S.
- In set-theoretic terms, a k-combination of elements of S is a cardinality k subset of S
- EX: The 2-combinations of the set S = {1,2,3} are

  {1,2} {1,3} {2,3}

  Therefore, C(3,2) = 3
- EX: The 2-combinations of the set T = {a,b,c} are

  {a,b} {a,c} {b,c}

  As expected, we get C(3,2) = 3 again.
- If $|S| = n$, the n-combinations of S are called just permutations of S.

The number of combinations

- Theorem: For any positive integers n and k, with k≤n, the number of k-combinations of a set with cardinality n is

$$C(n,k)=\frac{P(n,k)}{P(k,k)}=\frac{n!}{k!(n-k)!}$$

- Proof: look at the map from the set of k-permutations on n elements to the set of k-combinations on the same n elements that sends a given permutation to the set of its entries. This map is not injective, because permutations made of the same entries, but in a different order, have the same combination as image. The preimages of each k-combination are all the possible orderings of its k elements. But, by definition, the number of such orderings is the number of k-permutations of its k elements, i.e. P(k,k). So each k-combination corresponds to P(k,k) k-permutations. In symbols,

$$P(n,k)=C(n,k)\cdot P(k,k)$$

The number of combinations

- EX: How many ways are there to randomly select 3 members out of an assembly of 40 people?
- Since clearly the 3 members are distinct, and since the order of selection does not matter, we are interested in the number of possible 3-combinations of 40 objects, that is,

$$C(40,3)=\frac{40!}{3!\,(40-3)!}=\frac{40!}{3!\,(37)!}=\frac{40\cdot 39\cdot 38\cdot (37!)}{3!\,37!}=\frac{40\cdot 39\cdot 38}{3\cdot 2}$$

$$=40\cdot 13\cdot 19=9880$$

The number of combinations

- Fun fact: C(n,k) = C(n,n-k)
- EX: How many ways are there to randomly select 37 = 40-3 members out of an assembly of 40 people?
- We are interested in the number of possible 37-combinations of 40 objects, that is,

$$C(40,37) = \frac{40!}{37!\,(40-37)!} = \frac{40!}{37!\,3!} = \frac{40\cdot 39\cdot 38\cdot (37!)}{37!\,3!} = \frac{40\cdot 39\cdot 38}{3\cdot 2}$$

$$= 40\cdot 13\cdot 19 = 9880 = C(40,3)$$

4. BINOMIAL COEFFICIENTS

Binomial expansions

- The number $C(n,k) = \frac{n!}{n!(n-k)!}$ ¿

  is called binomial coefficient, and is denoted $\binom{n}{k}$

- The name comes from the fact that $\binom{n}{k}$ is the coefficient of the term $x^{n-k}y^k$ in the expansion of $(x+y)^n$.

- EX: $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$

Binomial expansions

- Corollary: for any n$\in$**N**, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$
- Proof: in the binomial expansion of $(x+y)^n$, i.e.,

  $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$, set x = y = 1
- We get

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k}$$

Pascal’s identity

- Theorem: for any natural numbers n,k, with k≤n,

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

- Proof: boring calculations

Pascal’s (or Tartaglia’s) triangle

- The binomial coefficients for varying n and k can be stored in a triangle:

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
\quad \Longrightarrow \quad
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

- By Pascal’s identity, each entry is the sum of the 2 entries above it (no entry counts as 0)