Algorithms and Decision Procedures for Context-Free Languages

Chapter 14

Decision Procedures for CFLs

Membership: Given a language *L* and a string *w*, is *w* in *L*?

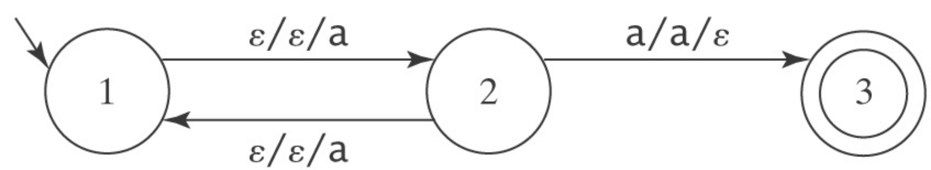
Two approaches:

- If *L* is context-free, then there exists some context-free grammar *G* that generates it. Try derivations in *G* and see whether any of them generates *w*.

Problem: $S \rightarrow S\ T \mid a$ Try to derive aaa

- If *L* is context-free, then there exists some PDA *M* that accepts it. Run *M* on *w*.

Problem:



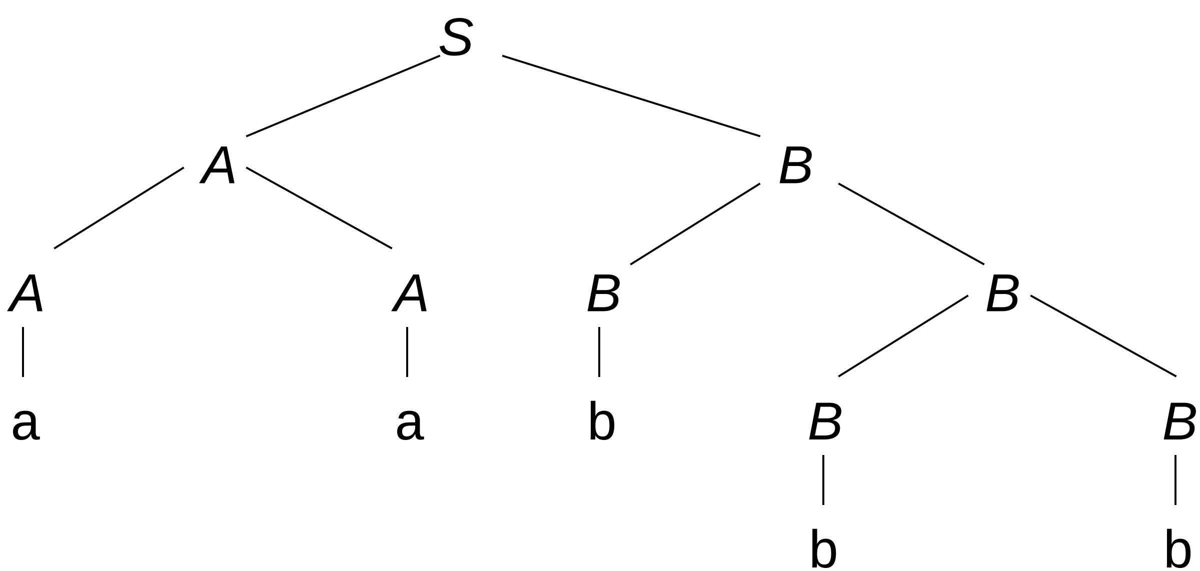
Normal Forms for Grammars

Chomsky Normal Form, in which all rules are of one of the following two forms:

- $X \rightarrow a$, where $a \in \Sigma$, or
- $X \rightarrow BC$, where B and C are elements of $V - \Sigma$.

Advantages:

- Parsers can use binary trees.
- Exact length of derivations is known:



Conversion to Chomsky Normal Form

For any context-free grammar G, there exists a context-free grammar G' in Chomsky normal form such that $L(G') = L(G) - \{\varepsilon\}$.

Algorithm:

1. Remove all ε-rules, using the algorithm *removeEps*. (We have seen this before.)

2. Remove all unit productions (rules of the form $A \rightarrow B$).

3. Remove all rules whose right hand sides have length greater than 1 and include a terminal:
(e.g., $A \rightarrow \mathrm{a}B$ or $A \rightarrow B\mathrm{a}C$)

4. Remove all rules whose right hand sides have length greater than 2:
(e.g., $A \rightarrow BCDE$)

Removing ε-Productions

Definition: a rule is ***modifiable*** iff it is of the form:

$$P \rightarrow \alpha R \beta, \text{ for some nullable } R,\ P \neq \alpha\beta \neq \varepsilon$$

removeEps(*G*: cfg) =

1. Let $G' = G$.
2. Find the set N of nullable variables in G'.
3. For each modifiable rule $P \rightarrow \alpha R \beta$ of G do

   Add the rule $P \rightarrow \alpha\beta$.
4. Delete from G' all rules of the form $X \rightarrow \varepsilon$.
5. Return G'.

$L(G') = L(G) - \{\varepsilon\}$

Example

Example:

$S \rightarrow aACa$
$A \rightarrow B \mid a$
$B \rightarrow C \mid c$
$C \rightarrow cC \mid \varepsilon$

Nullable: A, B, C

$S \rightarrow aACa \mid aAa \mid aCa \mid aa$
$A \rightarrow B \mid a$
$B \rightarrow C \mid c$
$C \rightarrow cC \mid c$

Unit Productions

A ***unit production*** is a rule $A \rightarrow B$ (right-hand side consists of a single nonterminal symbol)

removeUnits(G)

1. Let $G' = G$.
2. Until no unit productions remain in G' do:
   - 2.1 Choose some unit production $X \rightarrow Y$.
   - 2.2 Remove it from G'.
   - 2.3 Consider only rules that still remain.
     For every rule $Y \rightarrow \beta$, where $\beta \in V^*$, do:
     Add to G' the rule $X \rightarrow \beta$ unless it is a rule that has already been removed once.
3. Return G'.

Example

$$S \to aACa \mid aAa \mid aCa \mid aa$$
$$A \to B \mid a$$
$$B \to C \mid c$$
$$C \to cC \mid c$$

Remove $A \to B$. Add $A \to C \mid c$.
Remove $B \to C$. Add $B \to cC$ ($B \to c$, already there).
Remove $A \to C$. Add $A \to cC$ ($A \to c$, already there).

So *removeUnits* returns:

$$S \to aACa \mid aAa \mid aCa \mid aa$$
$$A \to a \mid c \mid cC$$
$$B \to c \mid cC$$
$$C \to cC \mid c$$

Mixed Rules

removeMixed(G) =

1. Let G' = G.
2. Create a new nonterminal T_a for each terminal a in Σ.
3. Modify each rule whose right-hand side has length greater than 1 and that contains a terminal symbol by substituting T_a for each occurrence of the terminal a.
4. Add to G, for each T_a, the rule $T_a \rightarrow a$.
5. Return G'.

Example

$$S \to aACa \mid aAa \mid aCa \mid aa$$
$$A \to a \mid c \mid cC$$
$$B \to c \mid cC$$
$$C \to cC \mid c$$

removeMixed returns:

$$S \to T_aACT_a \mid T_aAT_a \mid T_aCT_a \mid T_aT_a$$
$$A \to a \mid c \mid T_cC$$
$$B \to c \mid T_cC$$
$$C \to T_cC \mid c$$
$$T_a \to a$$
$$T_c \to c$$

Long Rules

removeLong(G) =

1. Let $G' = G$.
2. For each rule r of the form:

$$A \rightarrow N_1N_2N_3N_4 \ldots N_n,\ n > 2$$

create new nonterminals $M_2, M_3, \ldots M_{n-1}$.

3. Replace r with the rule $A \rightarrow N_1M_2$.
4. Add the rules:

$$M_2 \rightarrow N_2M_3,$$
$$M_3 \rightarrow N_3M_4,$$
$$\ldots$$
$$M_{n-1} \rightarrow N_{n-1}N_n.$$

5. Return G'.

Example

$S \rightarrow T_a A C T_a \mid T_a A T_a \mid T_a C T_a \mid T_a T_a$

$A \rightarrow a \mid c \mid T_c C$

$B \rightarrow c \mid T_c C$

$C \rightarrow T_c C \mid c$

$T_a \rightarrow a$

$T_c \rightarrow c$

removeLong returns:

$S \rightarrow T_a S_1$ $\quad S \rightarrow T_a S_3$ $\quad S \rightarrow T_a S_4$ $\quad S \rightarrow T_a T_a$

$S_1 \rightarrow A S_2$ $\quad S_3 \rightarrow A T_a$ $\quad S_4 \rightarrow C T_a$

$S_2 \rightarrow C T_a$

$A \rightarrow a \mid c \mid T_c C$

$B \rightarrow c \mid T_c C$

$C \rightarrow T_c C \mid c$

$T_a \rightarrow a$

Using a Grammar

decideCFLusingGrammar(L: CFL, w: string) =

1. If given a PDA, build G so that $L(G) = L(M)$.

2. If $w = \varepsilon$ then if S_G is nullable then accept, else reject.

3. If $w \neq \varepsilon$ then:

    3.1 Construct G' in Chomsky normal form such that $L(G') = L(G) - \{\varepsilon\}$.

    3.2 If G derives w, it does so in $2 \cdot |w| - 1$ steps. Try all derivations in G of $2 \cdot |w| - 1$ steps. If one of them derives w, accept. Otherwise reject.

Emptiness

Given a context-free language L, is $L = \varnothing$?

decideCFLempty(G: context-free grammar) =

1. Let G' = *removeunproductive*(G).

2. If S is not present in G' then return *True*
   else return *False*.

Finiteness

Given a context-free language *L*, is *L* infinite?

decideCFLinfinite(*G*: context-free grammar) =

1. Lexicographically enumerate all strings in Σ^* of length greater than *k* and less than or equal to 2*k*.

2. If, for any such string *w*, *decideCFL*(*L, w*) returns *True* then return *True*. *L* is infinite.

3. If, for all such strings *w*, *decideCFL*(*L, w*) returns *False* then return *False*. *L* is not infinite.

decideCFLempty2(*G*: context-free grammar)

1. Check for strings of length up to *k.*

The Undecidable Questions about CFLs

- Is $L = \Sigma^*$?
- Is the complement of L context-free?
- Is L regular?
- Is $L_1 = L_2$?
- Is $L_1 \subseteq L_2$?
- Is $L_1 \cap L_2 = \varnothing$?
- Is L inherently ambiguous?
- Is G ambiguous?