

Binary Relations

Chapter 9

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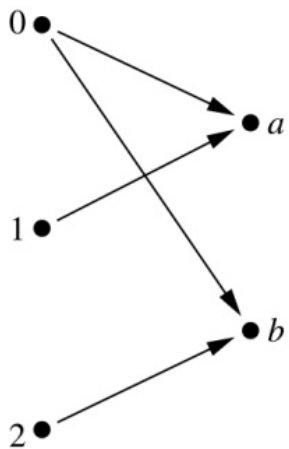
Binary relations

Definition

A *binary relation* R from a set A to a set B is a subset $R \subseteq A \times B$.

Example

- 1 Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$
- 2 $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B .
- 3 We can represent relations from a set A to a set B graphically or using a table:



| R | a | b |
|-----|-----|-----|
| 0 | × | × |
| 1 | × | |
| 2 | | × |

Relations are more general than functions. A function from A to B is a relation where for each element x of A there is exactly one element y of B related to x .

Binary relation on a set

Definition

A binary relation R on a set A is a subset of $A \times A$ or a relation from A to A .

Example

- 1 Suppose that $A = \{a, b, c\}$.
- 2 Then $R = \{(a, a), (a, b), (a, c)\}$ is a binary relation on A .

Example

- 1 Let $A = \{1, 2, 3, 4\}$.
- 2 The ordered pairs in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ are $(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3)$, and $(4, 4)$.

Binary relation on a set

Example

How many relations are there on a finite set A ?

Solution:

- ① Because a relation on A is the same thing as a subset of $A \times A$, we count the subsets of $A \times A$.
- ② Since $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are $2^{|A|^2}$ subsets of $A \times A$.
- ③ Therefore, there are $2^{|A|^2}$ relations on a set A .

Binary relations on a set

Example

Consider these **relations** on the set of integers:

$$R_1 = \{(a, b) \mid a \leq b\}$$

$$R_2 = \{(a, b) \mid a > b\}$$

$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$$

$$R_4 = \{(a, b) \mid a = b\}$$

$$R_5 = \{(a, b) \mid a = b + 1\}$$

$$R_6 = \{(a, b) \mid a + b \leq 3\}$$

Note that these relations are on an infinite set and each of these relations is an infinite set.

Which of these relations contain each of the pairs:

- ① (1,1) ② (1, 2) ③ (2,1) ④ (1, -1) ⑤ (2,2)

Solution: Checking the conditions that define each relation, we see that the pair

① (1,1) is in R_1, R_3, R_4 , and R_6 :

② (1,2) is in R_1 and R_6 :

③ (2,1) is in R_2, R_5 , and R_6 :

④ (1, -1) is in R_2, R_3 , and R_6 :

⑤ (2,2) is in R_1, R_3 , and R_4 .

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Reflexive relations

Definition

The relation R on A is *reflexive* iff $(a, a) \in R$ for every element $a \in A$. That is, R is reflexive if and only if

$$\forall x, \quad x \in A \rightarrow (x, x) \in R$$

Example

The following relations on the integers are reflexive:

$$R_1 = \{(a, b) \mid a \leq b\}$$

$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$$

$$R_4 = \{(a, b) \mid a = b\}$$

holds where $a=b$.

If $A = \emptyset$ then the empty relation is reflexive vacuously. That is, the empty relation on an empty set is reflexive!

The following relations are not reflexive:

$$R_2 = \{(a, b) \mid a > b\} \quad (\text{note that } 3 \not> 3)$$

$$R_5 = \{(a, b) \mid a = b + 1\} \quad (\text{note that } 3 \neq 3 + 1)$$

$$R_6 = \{(a, b) \mid a + b \leq 3\} \quad (\text{note that } 4 + 4 \not\leq 3)$$

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Symmetric relations

Definition

The relation R on A is *symmetric* iff $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$. That is, R is symmetric if and only if

$$\forall x, \forall y, (x, y) \in R \rightarrow (y, x) \in R$$

Example

The following relations on the integers are symmetric:

$$R_3 = \{(a, b) \mid |a| = |b|\}$$

$$R_4 = \{(a, b) \mid a = b\}$$

$$R_6 = \{(a, b) \mid a + b \leq 3\}$$

The following relations are not symmetric:

$$R_1 = \{(a, b) \mid a \leq b\} \quad (\text{note that } 3 \leq 4, \text{ but } 4 \not\leq 3)$$

$$R_2 = \{(a, b) \mid a > b\} \quad (\text{note that } 4 > 3, \text{ but } 3 \not> 4)$$

$$R_5 = \{(a, b) \mid a = b + 1\} \quad (\text{note that } 4 = 3 + 1, \text{ but } 3 \neq 4 + 1)$$

Antisymmetric relations

Definition

The relation R on A is *antisymmetric* if for all $a, b \in A$ if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ holds. That is: R is antisymmetric iff

$$\forall x, \forall y, ((x, y) \in R \wedge (y, x) \in R) \rightarrow x = y$$

Note: if x and y are distinct ($x \neq y$) then R can not have both (x, y) and (y, x) .

Example

The following relations on the integers are antisymmetric:

$$R_1 = \{(a, b) \mid a \leq b\}$$

$$R_2 = \{(a, b) \mid a > b\}$$

$$R_4 = \{(a, b) \mid a = b\}$$

$$R_5 = \{(a, b) \mid a = b + 1\}$$

For any integer, if $a \leq b$ and $b \leq a$ then $a = b$.

The following relations are not antisymmetric:

$$R_3 = \{(a, b) \mid |a| = |b|\} \quad (\text{note that both } (1, -1) \text{ and } (-1, 1) \text{ belong to } R_3)$$

$$R_6 = \{(a, b) \mid a + b \leq 3\} \quad (\text{note that both } (1, 2) \text{ and } (2, 1) \text{ belong to } R_6)$$

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Transitive relations

Definition

The relation R on A is **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$. That is, R is transitive if and only if

$$\forall x \forall y \forall z [(x, y) \in R \wedge (y, z) \in R \rightarrow (x, z) \in R]$$

Example

The following relations on the integers are transitive:

$$R_1 = \{(a, b) \mid a \leq b\}$$

$$R_2 = \{(a, b) \mid a > b\}$$

$$R_3 = \{(a, b) \mid |a| = |b|\}$$

$$R_4 = \{(a, b) \mid a = b\}$$

For any integer, if $a \leq b$ and $b \leq c$ then $a \leq c$.

The following relations are not transitive:

$$R_5 = \{(a, b) \mid a = b + 1\} \quad (\text{note that both } (3, 2) \text{ and } (4, 3) \text{ belong to } R_5, \text{ but not } (4, 2))$$

$$R_6 = \{(a, b) \mid a + b \leq 3\} \quad (\text{note that both } (2, 1) \text{ and } (1, 2) \text{ belong to } R_6, \text{ but not } (2, 2))$$

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Combining relations

Given two relations R_1 and R_2 , we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.

Example

Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relations $R_1 = \{(1,1), (2,2), (3,3)\}$ and $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$ can be combined using basic set operations to form new relations:

- ① $R_1 \cup R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\}$
- ② $R_1 \cap R_2 = \{(1,1)\}$
- ③ $R_1 - R_2 = \{(2,2), (3,3)\}$
- ④ $R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$

Combining relations via composition

Definition

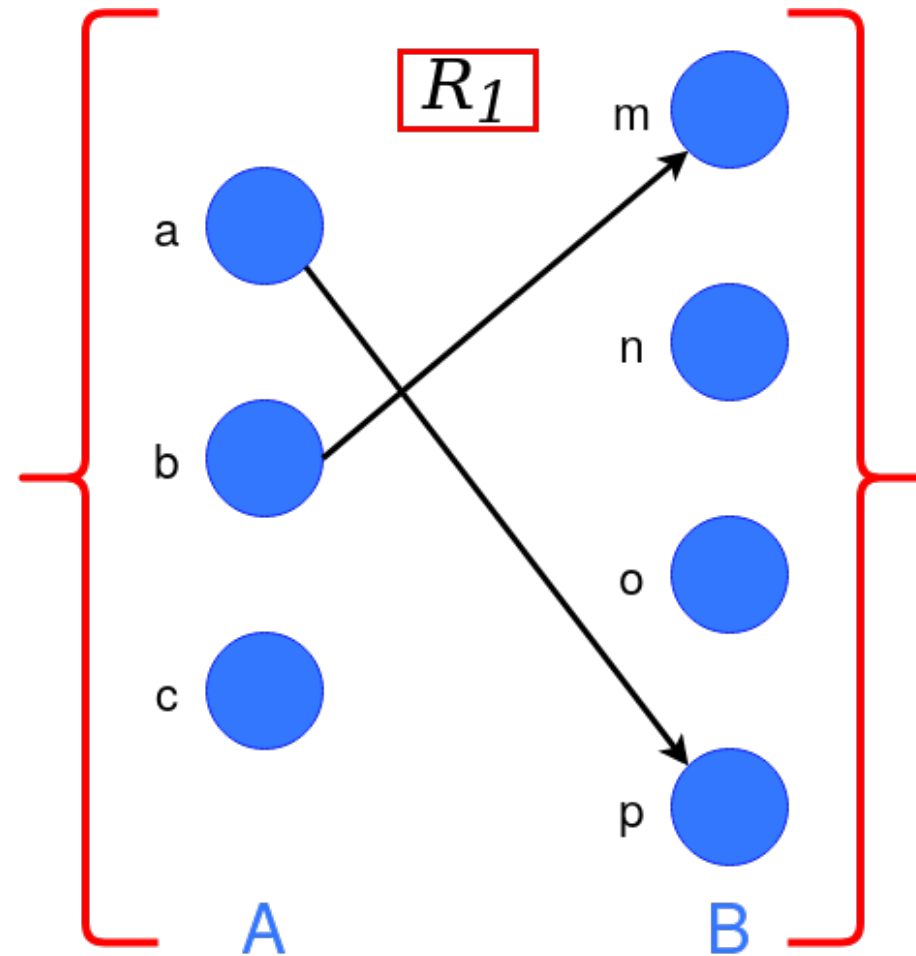
Suppose:

- 1 R_1 is a relation from a set A to a set B .
- 2 R_2 is a relation from B to a set C .

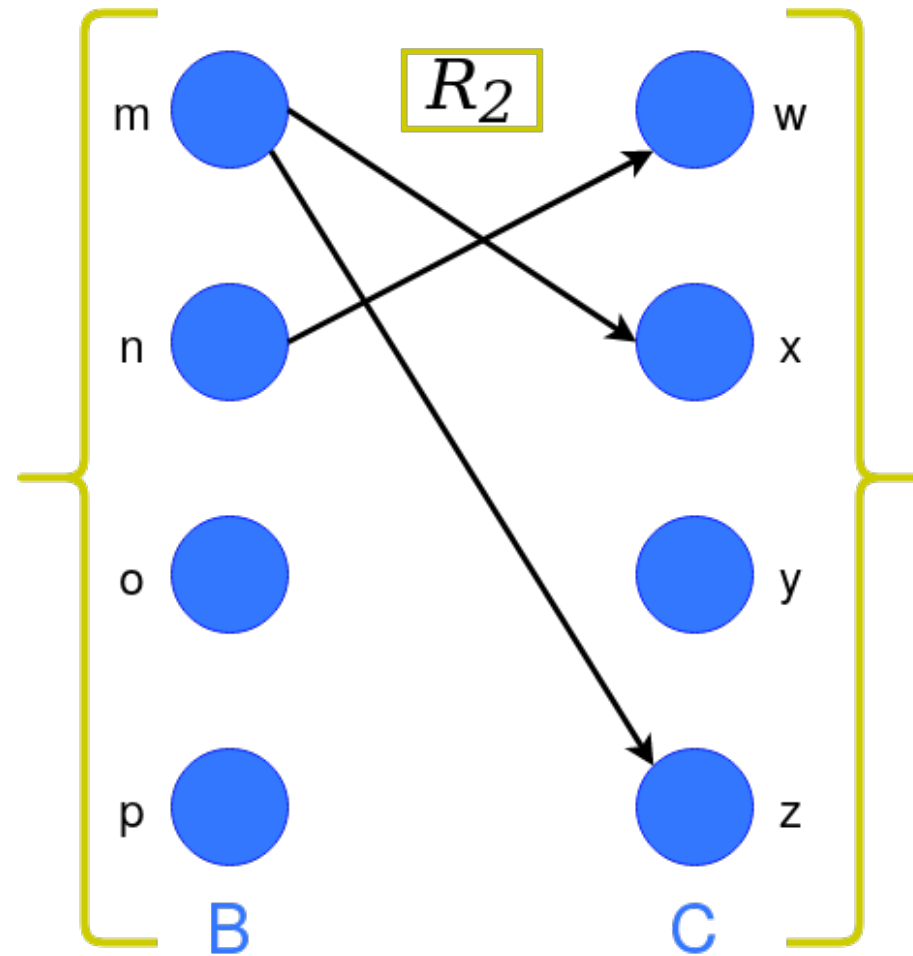
Then the *composition of R_2 with R_1* , is a relation from A to C , denoted $R_2 \circ R_1$, where:

- 1 if (x, y) is a member of R_1 and (y, z) is a member of R_2 then (x, z) is a member of $R_2 \circ R_1$.
- 2 also, if $(x, z) \in R_2 \circ R_1$ then there exists some $y \in B$ such that $(x, y) \in R_1$ and $(y, z) \in R_2$

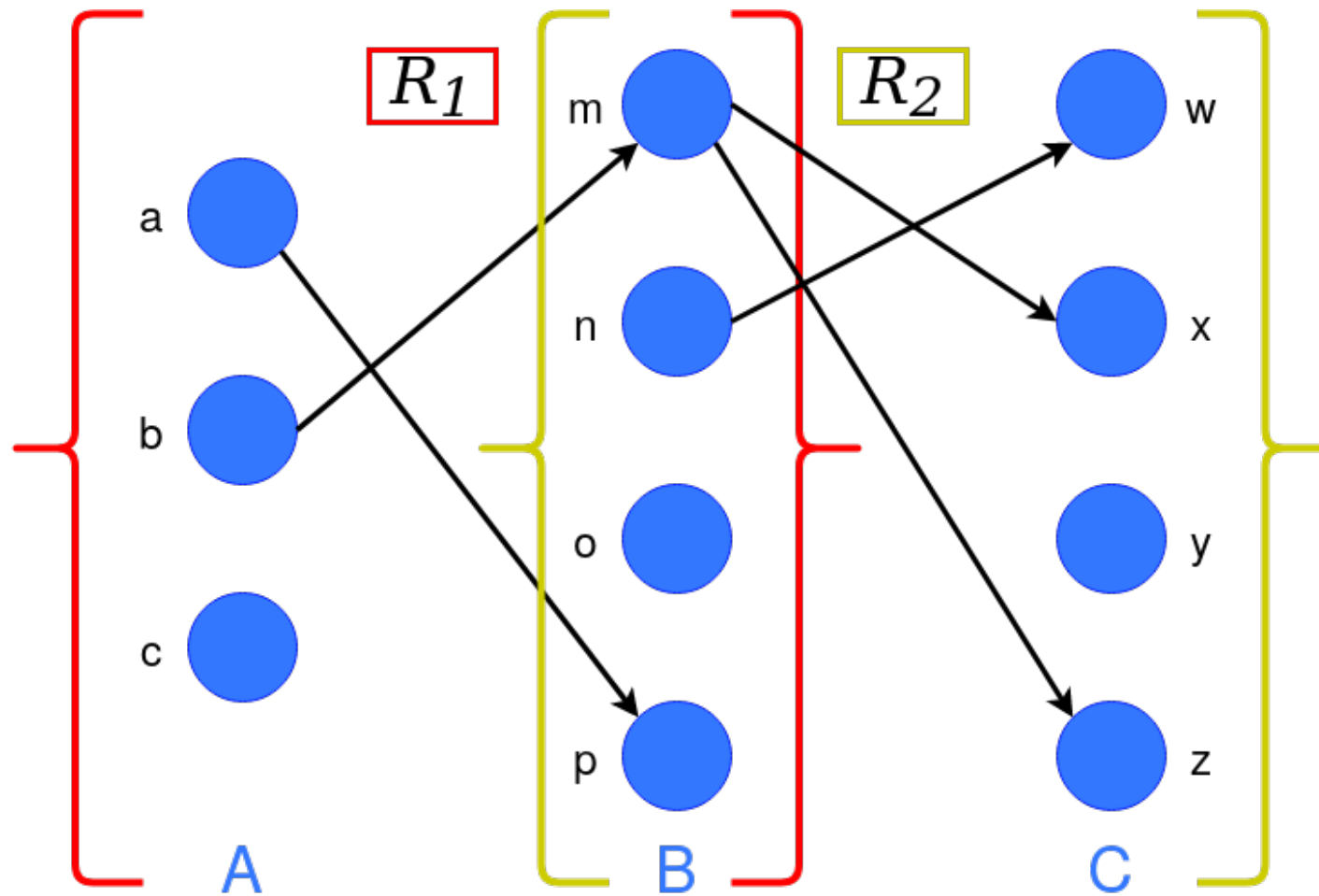
Representing a composition



Representing a composition

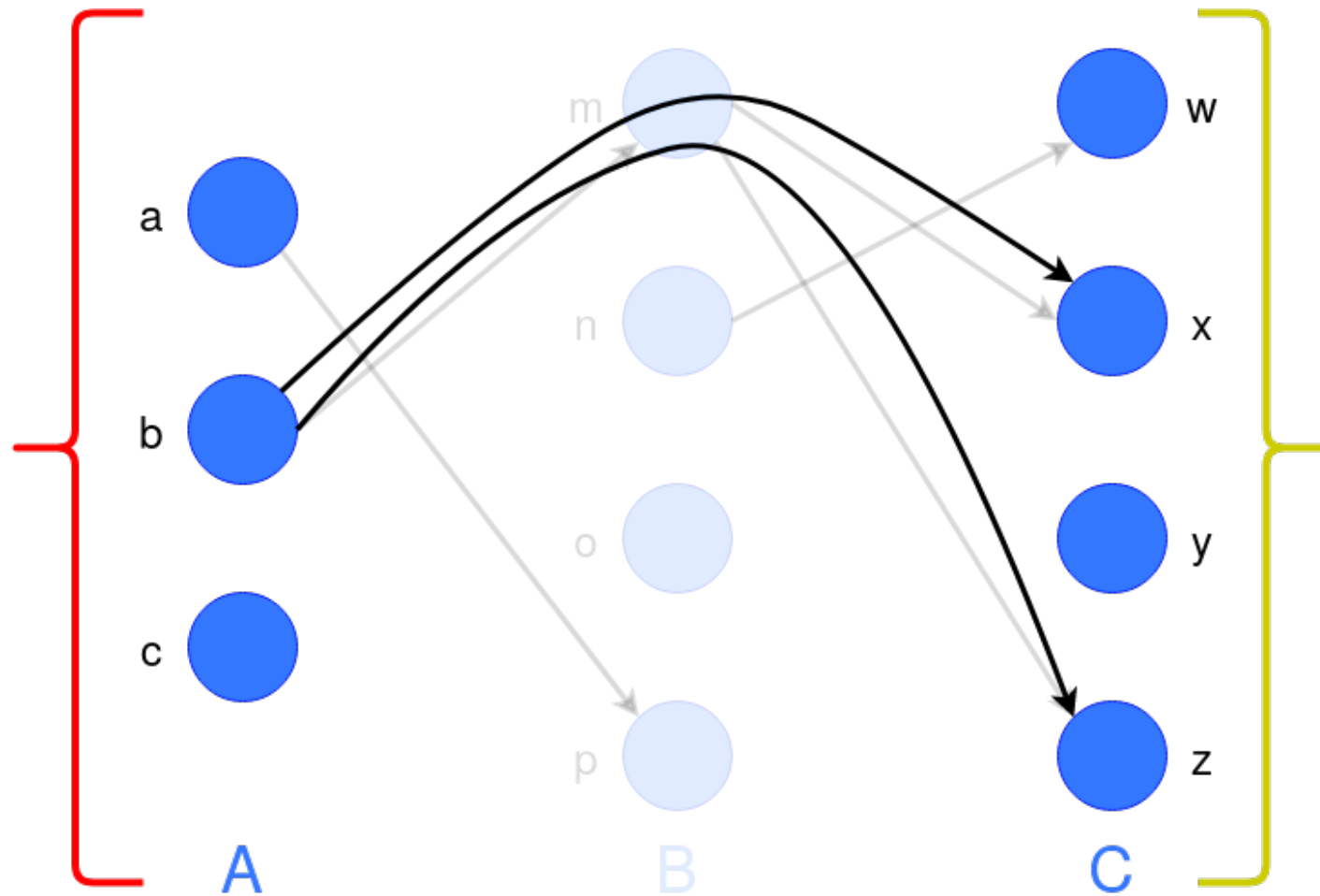


Representing a composition



$$R_2 \circ R_1 = \{(b, z), (b, x)\}$$

Representing a composition



$$R_2 \circ R_1 = \{(b, z), (b, x)\}$$

Composition of a relation with itself

Definition

Let R be a binary relation on a set A . Then the composition of R with R , denoted $R \circ R$, is a relation on A where:

- ① if (x, y) is a member of R
- ② and (y, z) is a member of R
- ③ then (x, z) is a member of $R \circ R$.

Example

Let R be a relation on the set of all people such that (a, b) is in R if person a is parent of person b . Then (a, c) is in $R \circ R$ iff there is a person b such that (a, b) is in R and (b, c) is in R . In other words, (a, c) is in $R \circ R$ iff a is a grandparent of c .

Powers of a relation

Definition

Let R be a binary relation on A . Then the powers R^n of the relation R can be defined inductively by:

- ① Basis Step: $R^1 = R$
- ② Inductive Step: $R^{n+1} = R^n \circ R$

The powers of a transitive relation are subsets of the relation.

This is established by the following theorem:

Theorem

The relation R on a set A is transitive iff $R^n \subseteq R$ for all positive integers n .

(see Tutorial 11.)

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Representing relations using matrices

- ① A relation between finite sets can be represented using a **zero-one matrix**.
- ② Suppose R is a relation from $A = \{a_1, a_2, \dots, a_m\}$ to $B = \{b_1, b_2, \dots, b_n\}$.
 - Ⓐ The elements of the two sets can be listed in any particular arbitrary order. When $A = B$, we use the same ordering.
- ③ The relation R is represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

- ④ The matrix representing R has a 1 as its (i, j) entry, when a_i is related to b_j , and, a 0 if a_i is not related to b_j .

Examples of matrices representing relations

Example

Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B such that:

$$R = \{(a, b) \mid a \in A, b \in B, a > b\}$$

What is the matrix representing R (assuming the ordering of elements is the same as the increasing numerical order) ?

Solution:

Because $R = \{(2,1), (3,1),(3,2)\}$, the matrix is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Examples of matrices representing relations

Example

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

Solution:

Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that:

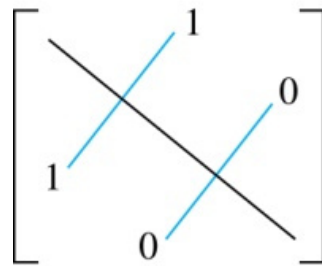
$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), \{(a_3, b_3), (a_3, b_5)\}\}$$

Matrices of relations on sets

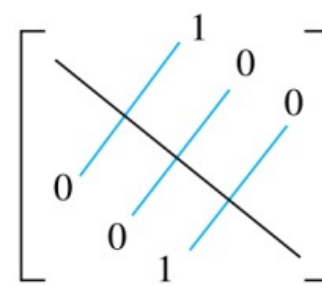
- ① If R is a **reflexive** relation, all the elements on the main diagonal of M_R are equal to 1.

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

- ② R is a **symmetric** relation, if and only if $m_{ij} = 1$ whenever $m_{ji} = 1$.



(a) **Symmetric**



(b) **Antisymmetric**

- ③ R is an **antisymmetric** relation, if and only if $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.

Example of a relation on a set

Example

Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution:

- 1 Because all the diagonal elements are equal to 1, R is reflexive.
- 2 Because M_R is symmetric, R is symmetric and not antisymmetric because both $m_{1,2}$ and $m_{2,1}$ are 1.

Matrices for combinations of relations

- 1 The matrix of the **union of two relations** is the **join** (Boolean OR) between the matrices of the component relations:

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$$

- 2 The matrix of the **intersection of two relations** is the **meet** (Boolean AND) between the matrices of the component relations:

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$$

- 3 The matrix of the **composite relation** $R_1 \circ R_2$ is the **Boolean product** of the matrices of the component relations:

$$M_{R_1 \circ R_2} = M_{R_1} \odot M_{R_2}$$

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Representing relations using directed graphs (a.k.a. *digraphs*)

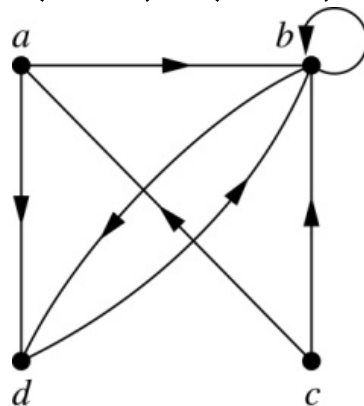
Definition

A *directed graph*, or *digraph*, consists of a set V of *vertices* or *nodes* together with a set E of ordered pairs of elements of V called (directed) *edges* or *arcs*.

- 1 The vertex a is called the *initial vertex* of the edge (a, b) , and the vertex b is called the *terminal vertex* of this edge.
- 2 An edge of the form (a, a) is called a *loop*.

Example

The directed graph with vertices a, b, c , and d , and edges (a, b) , (a, d) , (b, b) , (b, d) , (c, a) , (c, b) , and (d, b) is shown here:



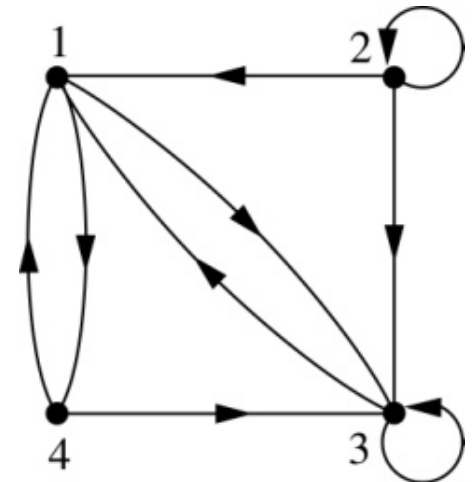
Examples of digraphs representing relations

Example

What are the ordered pairs in the relation represented by this directed graph?

Solution:

The ordered pairs in the relation are:



① (1, 3)

② (1, 4)

③ (2, 1)

④ (2, 2)

⑤ (2, 3)

⑥ (3, 1)

⑦ (3, 3)

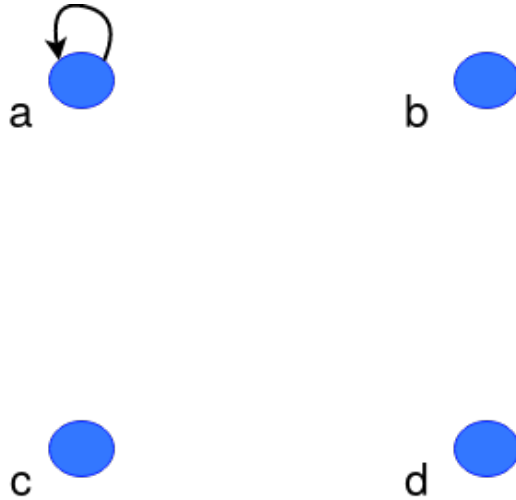
⑧ (4, 1)

⑨ (4, 3)

Determining which properties a relation has from its digraph

- ① *Reflexivity*: A loop must be present at all vertices.
- ② *Symmetry*: If (x, y) is an edge, then so is (y, x) .
- ③ *Antisymmetry*: If (x, y) with $x \neq y$ is an edge, then (y, x) is not an edge.
- ④ *Transitivity*: If (x, y) and (y, z) are edges, then so is (x, z) .

Determining which properties a relation has from its digraph – Example 1



1 *Reflexive?*

No, not every vertex has a loop

2 *Symmetric?*

Yes (trivially), there is no edge from one vertex to another

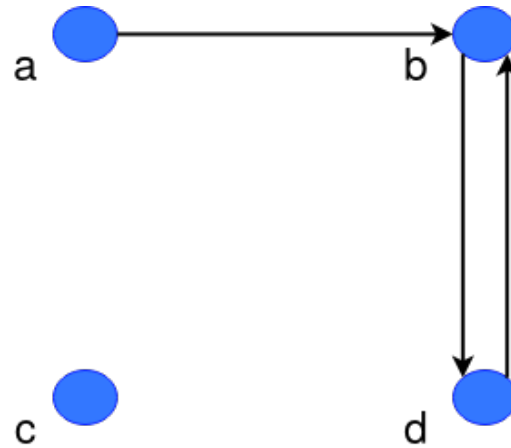
3 *Antisymmetric?*

Yes (trivially), there is no edge from one vertex to another

4 *Transitive?*

Yes, (trivially) since there is no edge from one vertex to another

Determining which properties a relation has from its digraph – Example 2



1 Reflexive? ✗

No, there are no loops

2 Symmetric? ✗

No, there is an edge from a to b , but not from b to a

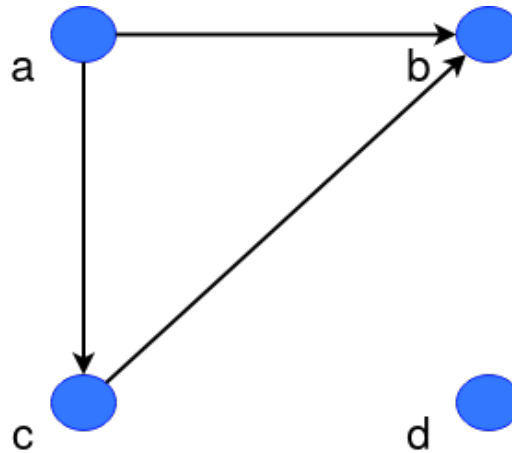
3 Antisymmetric? ✗

No, there is an edge from d to b and b to d

4 Transitive? ✗

No, there are edges from a to b and from b to d , but there is no edge from a to d

Determining which properties a Relation has from its digraph – Example 3



① *Reflexive?*

No, there are no loops

② *Symmetric?*

No, for example, there is no edge from c to a

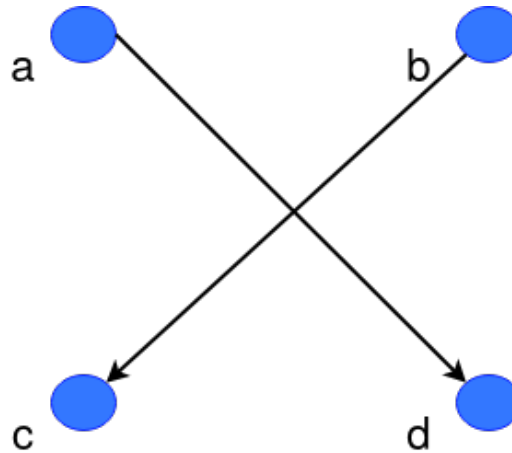
③ *Antisymmetric?*

Yes, whenever there is an edge from one vertex to another, there is not one going back

④ *Transitive?*

Yes

Determining which properties a relation has from its digraph – Example 4



① *Reflexive?*

No, there are no loops

② *Symmetric?*

No, for example, there is no edge from d to a

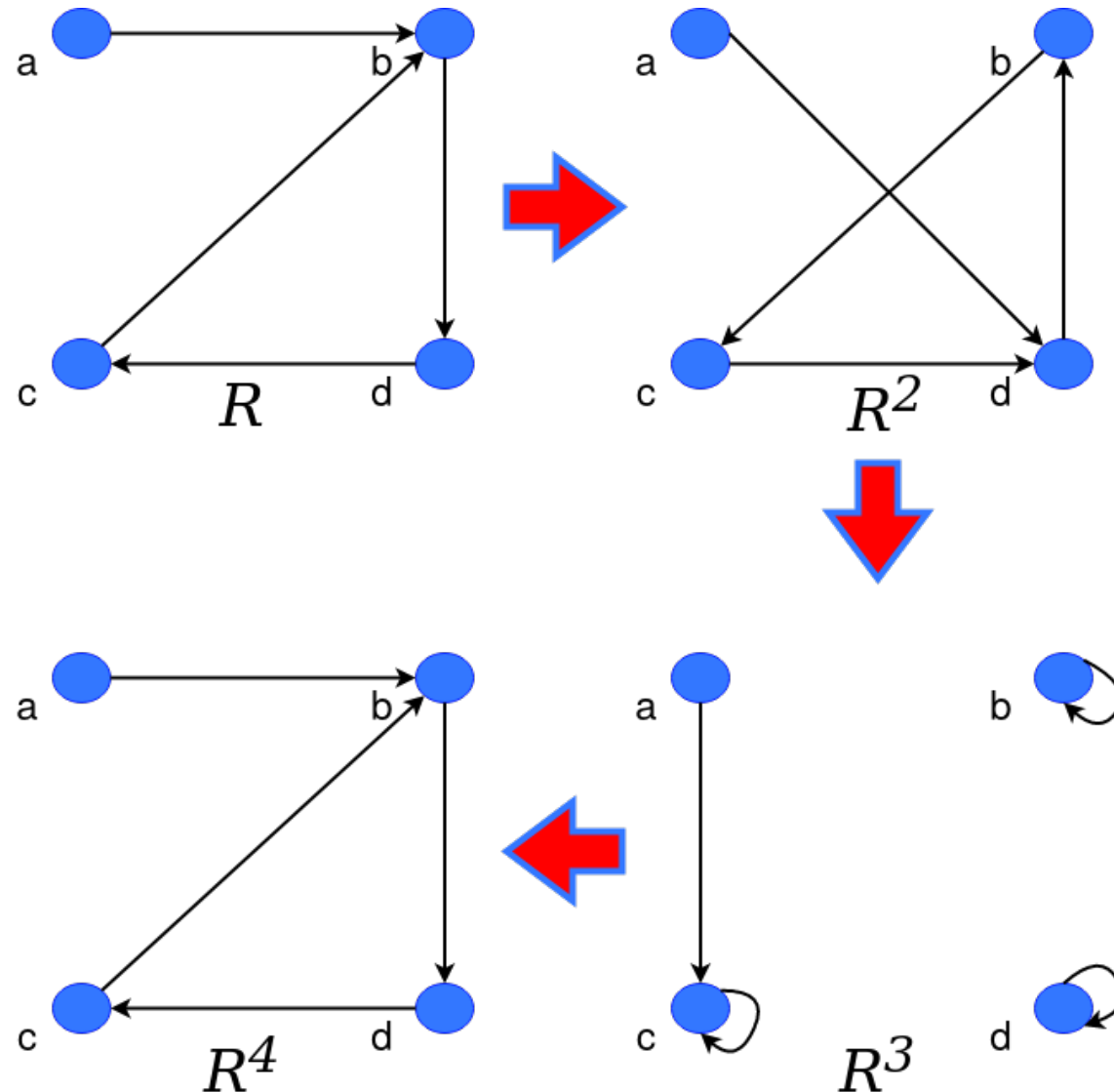
③ *Antisymmetric?*

Yes, whenever there is an edge from one vertex to another, there is not one going back

④ *Transitive?*

Yes (trivially), there are no two edges where the first edge ends at the vertex where the second edge begins

Example of the powers of a relation



The pair (x, y) is in R^n if there is a path of length n from x to y in R (following the direction of the arrows).

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Equivalence relations

Definition

A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Definition

Two elements a and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Example

Assume C is the set of all cars and a relation R on C such that $R = \{(a, b) \mid a \in C, b \in C \text{ and, } a \text{ and } b \text{ have the same color}\}$.

R is an *equivalence relation* on C .

Strings

Example

Suppose that R is the relation on the set of strings of English letters such that $(a, b) \in R$ if and only if $\ell(a) = \ell(b)$, where $\ell(x)$ is the length of the string x . Is R an equivalence relation?

Solution: Show that all of the properties of an equivalence relation hold:

- ① *Reflexivity*: Because $\ell(a) = \ell(a)$, it follows that $(a, a) \in R$ for all strings a .
- ② *Symmetry*: Assume $(a, b) \in R$. Since $\ell(a) = \ell(b)$, then $\ell(b) = \ell(a)$ also holds and we have $(b, a) \in R$.
- ③ *Transitivity*: Suppose that $(a, b) \in R$ and $(b, c) \in R$. Since $\ell(a) = \ell(b)$ and $\ell(b) = \ell(c)$ both hold, then $\ell(a) = \ell(c)$ also holds and we have $(a, c) \in R$.

Yes, R is an equivalence relation.

Congruence modulo m

Example

Let m be an integer with $m > 1$. Show that the relation

$$R = \{(a, b) \mid a \equiv b \pmod{m}\}$$

is an equivalence relation on the set of integers.

Solution: Recall that $a \equiv b \pmod{m}$ if and only if m divides $a - b$.

① *Reflexivity:*

$a \equiv a \pmod{m}$ since $a - a = 0$ is divisible by m since $0 = 0 \cdot m$.

② *Symmetry:*

Ⓐ Suppose that $a \equiv b \pmod{m}$.

Ⓑ Then $a - b$ is divisible by m , and so $a - b = km$, where k is an integer.

Ⓒ It follows that $b - a = (-k)m$, so $b \equiv a \pmod{m}$.

③ *Transitivity:*

Ⓐ Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$.

Ⓑ Then m divides both $a - b$ and $b - c$.

Ⓒ Hence, there are integers k and ℓ with $a - b = km$ and $b - c = \ell m$.

Ⓓ We obtain by adding the equations:

$$a - c = (a - b) + (b - c) = km + \ell m = (k + \ell)m.$$

Ⓔ Therefore, $a \equiv c \pmod{m}$.

Divides

Example

Show that the “divides” relation on the set of positive integers is not an equivalence relation.

Solution:

① *Reflexive*:

$a \mid a$ for all a .

② *Not symmetric*:

For example, $2 \mid 4$, but $4 \nmid 2$. Hence, the relation is not symmetric.

③ *Transitive*:

Ⓐ Suppose that a divides b and b divides c .

Ⓑ Then there are positive integers k and ℓ such that $b = ak$ and $c = b\ell$.

Ⓒ Hence, we have $c = a(k\ell)$. So, we have: a divides c .

Ⓓ Therefore, the relation is transitive.

The properties of reflexivity, and transitivity do hold, but the relation is not symmetric. Hence, “divides” is not an equivalence relation.

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Equivalence classes

Definition

Let R be an equivalence relation on a set A . The set of all elements that are related to an element a of A is called the *equivalence class* of a , denoted by $[a]_R$.

$$[a]_R := \{s \in A \mid (a, s) \in R\} \equiv \{s \in A \mid s \sim a\}$$

When only one relation is under consideration, we can write $[a]$, without the subscript R , for this equivalence class.

① If $b \in [a]_R$, then b is a *representative* of this equivalence class. Any element of a class can be used as representative.

② The equivalence classes of the relation “congruence modulo m ” are called the *congruence classes modulo m* . The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\dots, a - 2m, a - m, a + m, a + 2m, \dots\}$.

$$[0]_4 = \{\dots, -8, -4, 0, 4, 8, \dots\} \quad [1]_4 = \{\dots, -7, -3, 1, 5, 9, \dots\}$$

$$[2]_4 = \{\dots, -6, -2, 2, 6, 10, \dots\} \quad [3]_4 = \{\dots, -5, -1, 3, 7, 11, \dots\}$$

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Equivalence classes and partitions

Theorem

Let R be an equivalence relation on a set A . These statements for elements a and b of A are equivalent:

- ❶ $(a, b) \in R$,
- ❷ $[a] = [b]$,
- ❸ $[a] \cap [b] \neq \emptyset$.

We show $(i) \rightarrow (ii)$. All other implications are proved similarly.

- ❶ Assume that $(a, b) \in R$.
- ❷ Now suppose that $c \in [a]$.
- ❸ Then $(a, c) \in R$.
- ❹ Because $(a, b) \in R$ and R is symmetric, $(b, a) \in R$.
- ❺ Because R is transitive and $(b, a) \in R$ and $(a, c) \in R$, it follows that $(b, c) \in R$.
- ❻ Hence, $c \in [b]$. Therefore, $[a] \subseteq [b]$.

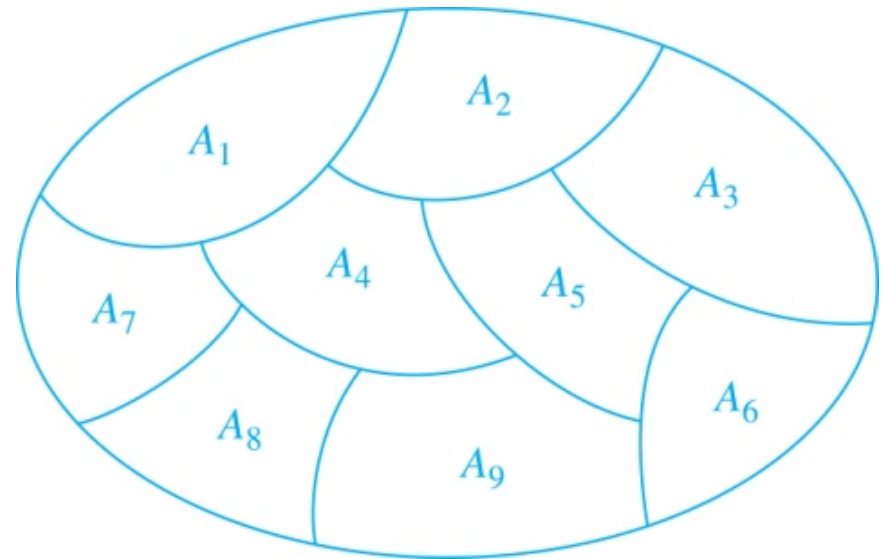
A similar argument proved $[b] \subseteq [a]$. Hence, we have $[b] = [a]$.

Partition of a set

Definition

A *partition* of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i , where $i \in I$ (where I is an index set), forms a partition of S if and only if

- ① $A_i \neq \emptyset$ for $i \in I$,
- ② $A_i \cap A_j = \emptyset$ when $i \neq j$,
- ③ and $\bigcup_{i \in I} A_i = S$



A partition of a set

An equivalence relation partitions a set

- 1 Let R be an equivalence relation on a set A . The union of all the equivalence classes of R is all of A , since an element a of A is in its own equivalence class $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_R = A$$

- 2 From the previous theorem, it follows that these **equivalence classes are either equal or disjoint**, so $[a]_R \cap [b]_R = \emptyset$ when $[a]_R \neq [b]_R$.
- 3 Therefore, the equivalence classes form a partition of A , because they split A into disjoint subsets.

An equivalence relation partitions a set

Theorem

Let R be an equivalence relation on a set S . Then the equivalence classes of R form a partition of S . Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S , there is an equivalence relation R that has the sets $A_i, i \in I$, as its equivalence classes.

Proof.

We have already shown the first part of the theorem. For the second part, assume that $\{A_i \mid i \in I\}$ is a partition of S . Let R be the relation on S consisting of the pairs (x, y) where x and y belong to the same subset A_i in the partition. We must show that R satisfies the properties of an equivalence relation.

① Reflexivity:

For every $a \in S$, $(a, a) \in R$, because a is in the same subset as itself.

② Symmetry:

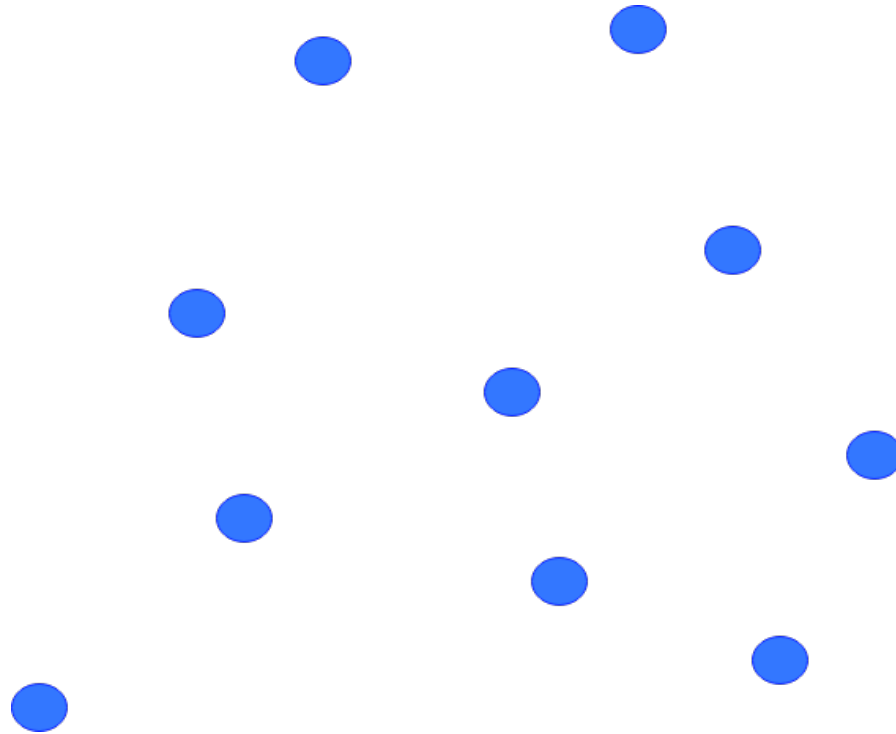
If $(a, b) \in R$, then b and a are in the same subset of the partition, so $(b, a) \in R$.

③ Transitivity:

If $(a, b) \in R$ and $(b, c) \in R$, then a and b are in the same subset of the partition, as are b and c . Since the subsets are disjoint and b belongs to both, the two subsets of the partition must be identical. Therefore, $(a, c) \in R$ since a and c belong to the same subset of the partition.

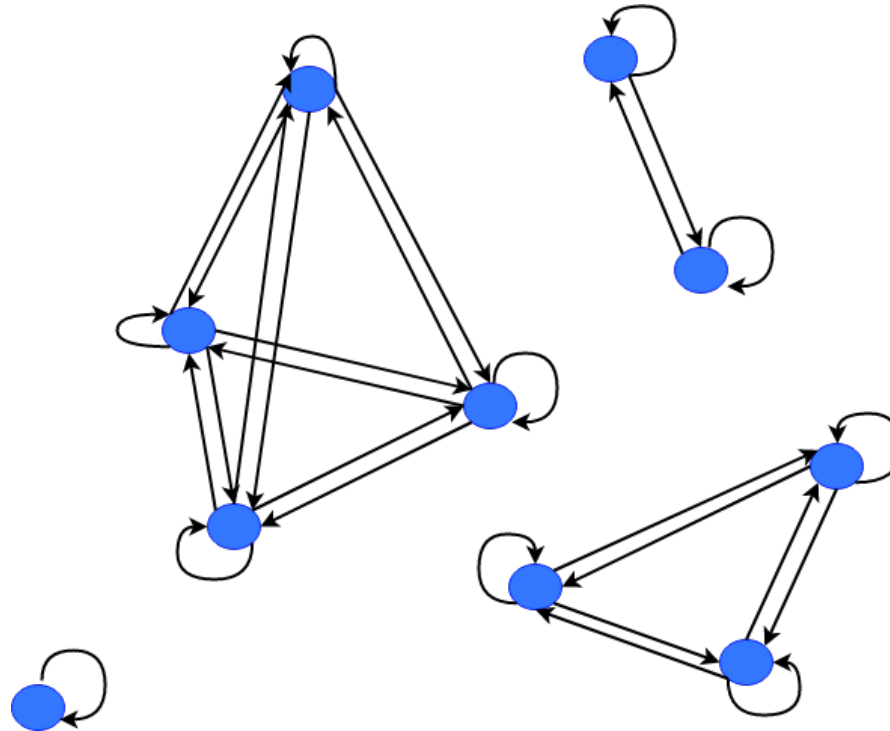


An equivalence relation digraph representation



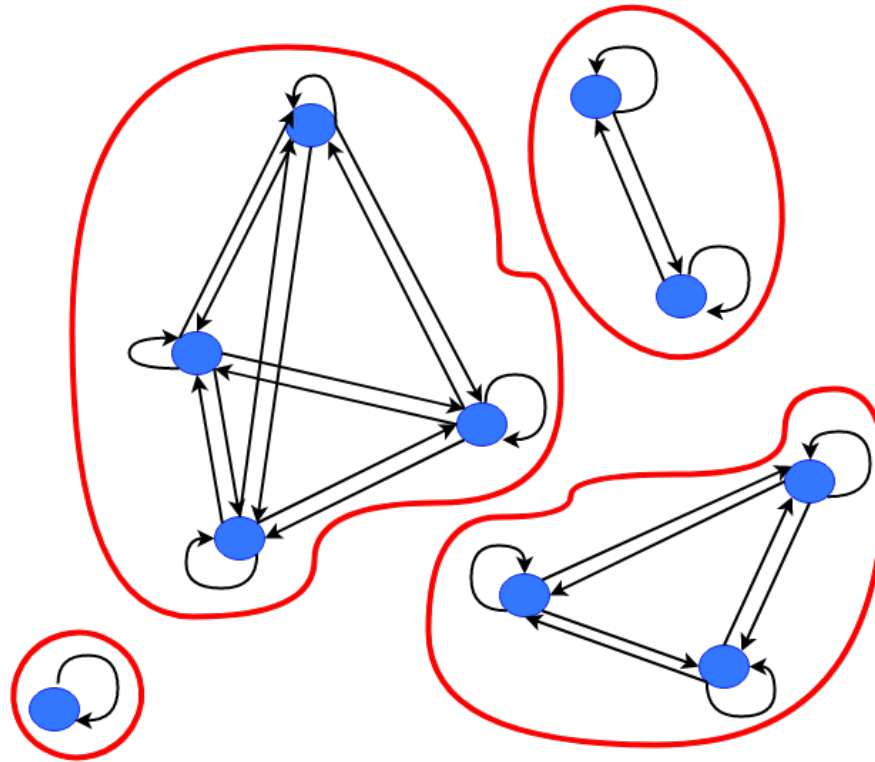
Set A.

An Equivalence Relation digraph representation



Equivalence relation R on set A .

An Equivalence Relation digraph representation



Digraph for equivalence relation R on finite set A is a union of **disjoint sub-graphs** (representing **disjoint equivalent classes**). Nodes in each distinct subgraph (equivalence class) are fully interconnected.

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Partial orderings

Definition

A relation R on a set S is called a *partial ordering*, or *partial order*, if it is *reflexive*, *antisymmetric*, and *transitive*.

A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R) . Members of S are called *elements* of the poset.

Partial orderings

Example

Show that the “greater than or equal” relation (\geq) is a partial ordering on the set of integers.

Solution:

① *Reflexivity*:

$a \geq a$ for every integer a .

② *Antisymmetry*:

If $a \geq b$ and $b \geq a$, then $a = b$.

③ *Transitivity*:

If $a \geq b$ and $b \geq c$, then $a \geq c$.

These properties all follow from the order axioms for the integers.)

Partial orderings

Example

Show that the **divisibility** relation is a partial ordering on the set of positive integers.

Solution:

① *Reflexivity*:

$a \mid a$ for all integers a .

② *Antisymmetry*:

If a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$.

③ *Transitivity*:

Ⓐ Suppose that a divides b and that b divides c .

Ⓑ Then, there are positive integers k and ℓ such that $b = ak$ and $c = b\ell$.

Ⓒ Hence, $c = a(k\ell)$, so that a divides c .

Ⓓ Therefore, the relation is transitive.

(\mathbb{Z}^+, \mid) is a poset.

Partial orderings

Example

Show that the **inclusion** relation (\subseteq) is a partial ordering on the power set of a set S .

Solution:

① *Reflexivity*:

$A \subseteq A$ whenever A is a subset of S .

② *Antisymmetry*:

If A and B are subsets of S , with $A \subseteq B$ and $B \subseteq A$, then $A = B$.

③ *Transitivity*:

If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

The properties all follow from the definition of set inclusion.

Comparability

Definition

The elements a and b of a poset (S, \leq) are *comparable* if either $a \leq b$ or $b \leq a$ holds. When a and b are elements of S so that neither $a \leq b$ nor $b \leq a$ holds, then a and b are called *incomparable*.

The symbol \leq is used to denote the relation in any poset.

Definition

If (S, \leq) is a poset and **any two elements of S are comparable**, then S is called a ***totally ordered*** or *linearly ordered set*, and \leq is called a *total order* or a *linear order*. (A totally ordered set is also called a *chain*.)

Definition

(S, \leq) is a **well-ordered set** if it is a poset such that \leq is a **total ordering** and **every nonempty subset of S has a least element**.

Example

- ① (\mathbb{Z}, \leq) is a totally ordered set
- ② $(\mathbb{Z}, |)$ is a partially ordered but not totally ordered set
- ③ (\mathbb{N}, \leq) is a well-ordered set

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Lexicographic order

Definition

Given two **partially ordered sets** (A_1, \leq_1) and (A_2, \leq_2) , the **lexicographic ordering** on $A_1 \times A_2$ is defined by specifying when (a_1, a_2) is less than (b_1, b_2) , written, $(a_1, a_2) < (b_1, b_2)$, which holds either if $a_1 <_1 b_1$ or if $a_1 = b_1$ and $a_2 <_2 b_2$ holds.

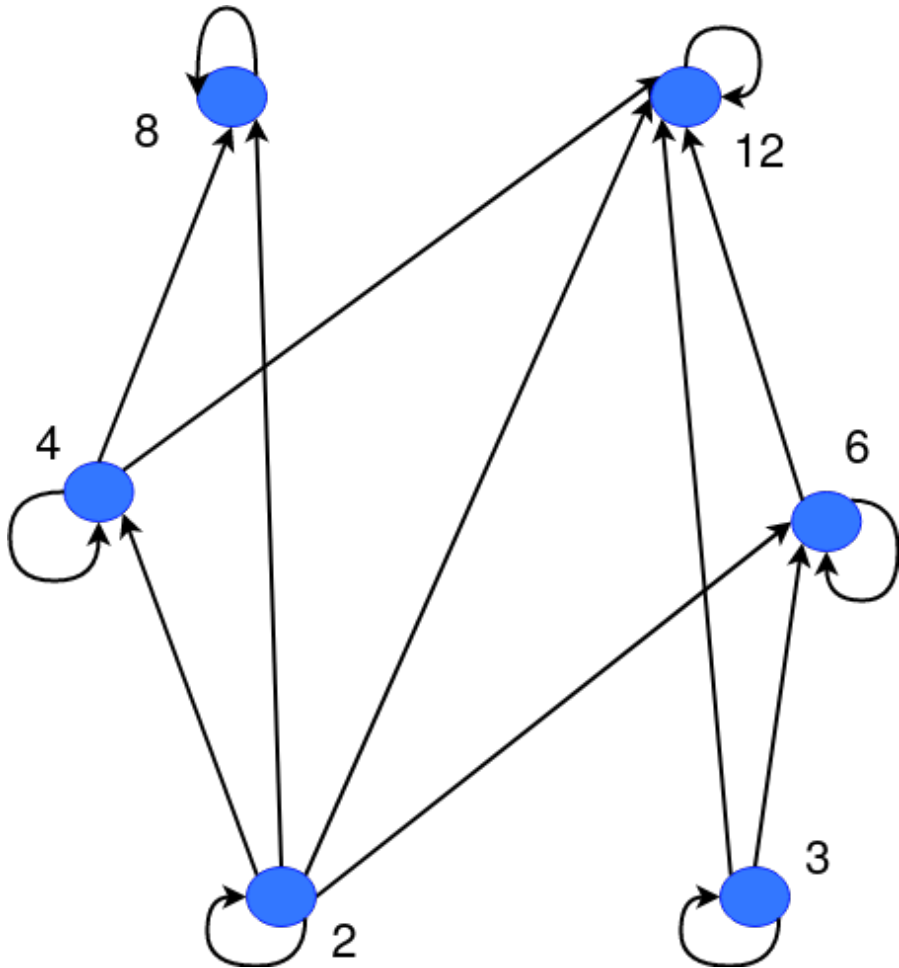
This definition can be easily extended to a lexicographic ordering on strings.

Example

Consider strings of lowercase English letters. A **lexicographic ordering** can be defined using the ordering of the letters in the alphabet. **This is the same ordering as that used in dictionaries.**

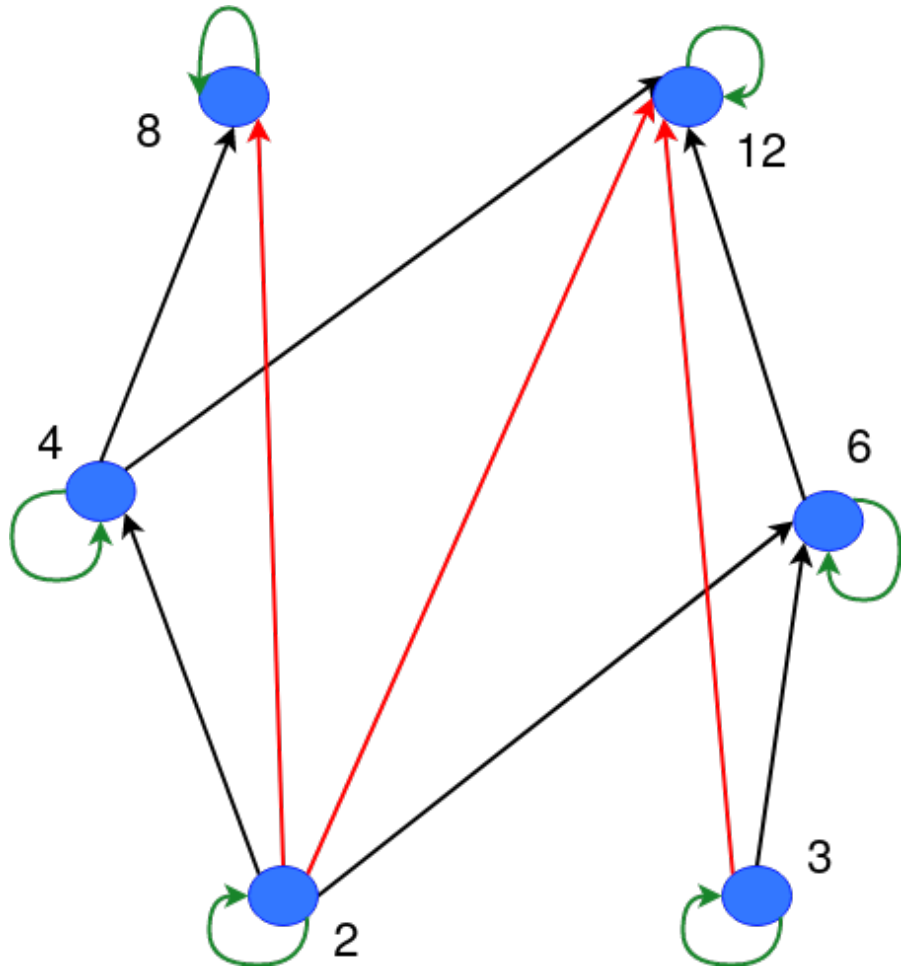
- ① *discreet* < *discrete*, because these strings differ in the seventh position and $e < t$.
- ② *discreet* < *discreetness*, because the first eight letters agree, but the second string is longer.

Partial ordering relation digraph representation



poset $R = (X, |)$ for
divisibility $|$ on set
 $X = \{2, 3, 4, 6, 8, 12\}$

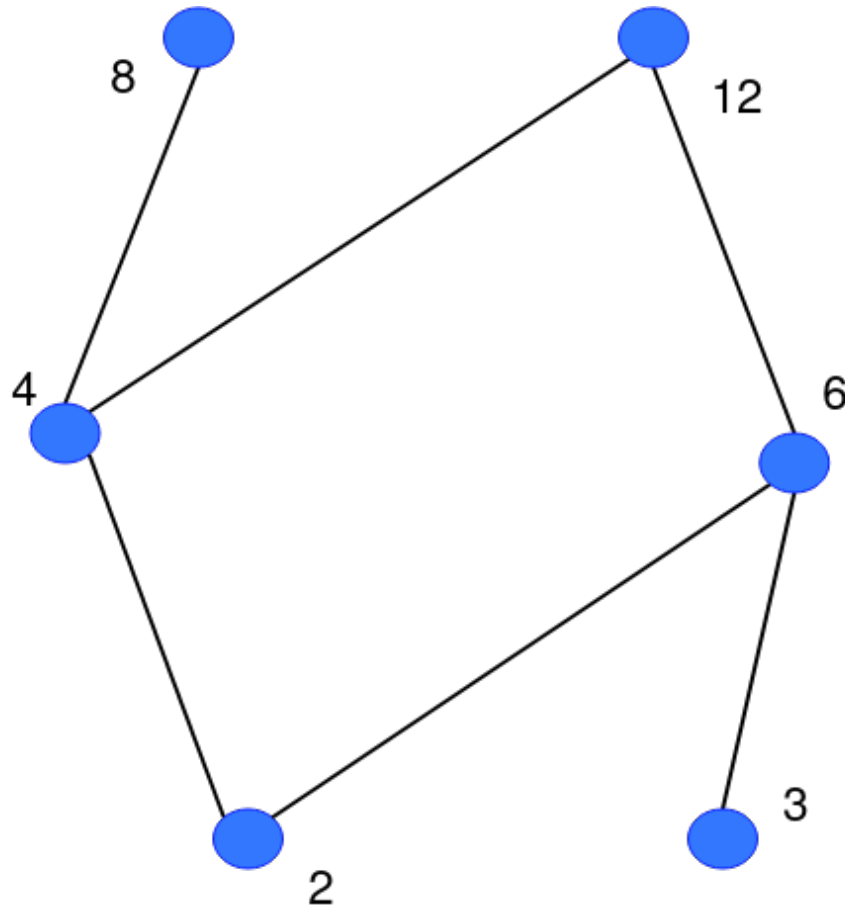
Partial ordering relation Hasse diagram



poset $R = (X, |)$ for
divisibility $|$ on set
 $X = \{2, 3, 4, 6, 8, 12\}$

- 1 Leave out all edges that are implied by **reflexivity** (loop)
- 2 Leave out all edges that are implied by **transitivity**

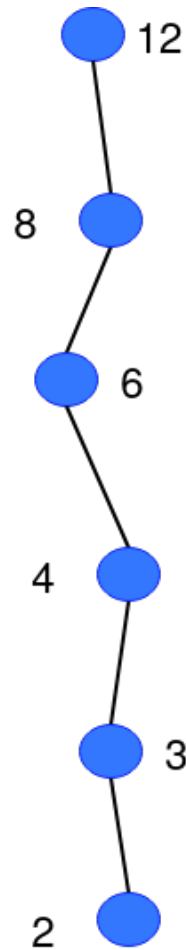
Partial ordering relation Hasse diagram



poset $R = (X, |)$ for
divisibility $|$ on set
 $X = \{2, 3, 4, 6, 8, 12\}$

Can also drop “direction” assuming that (partial) order is **upward**

Partial ordering relation Hasse diagram



poset $R = (X, \leq)$ for
“less than or equal” on set
 $X = \{2, 3, 4, 6, 8, 12\}$

Totally ordered sets are also called “chains”