Math 2155, Fall 2021: Homework 7

Instructions for GradeScope:

You may handwrite your answers or type them. Only type solutions if you are able to type the appropriate symbols. If **typed**, use at least a 12pt font, with **one question per page**. You can then directly submit the pdf file to GradeScope.

If **handwritten**:

- Don't erase or cross out more than a word or two.
- Only write on one side of the page. Use dark, large, neat writing.
- If you have a real scanner, great! Most people will scan their solutions using their phone. Use one of the recommended apps: For iOS, use Scannable by Evernote or Genius Scan. For Android, use Genius Scan. Do *not* just take regular photos.
- When scanning, have good lighting and try to avoid shadows.
- It is best to have **one question per scan**. If the solution is short, **fold the page in half** and scan just the half it is on, so there isn't so much blank space. Or, you can scan and then crop the scan. You don't need to scan parts (a), (b), etc. separately or put them on separate pages.
- It works well to scan each question separately and produce one pdf file with one question per page. Most scanning apps will automatically combine your images into one pdf file.
- You must **check the quality** of your scans afterwards, and rescan if needed.

Make sure you are going to `http://gradescope.ca` not `http://gradescope.com`. You can access it through the course OWL site.

You can **resubmit** your work any number of times until the deadline.

See the GradeScope help website for lots of information: `https://help.gradescope.com/`
Select "Student Center" and then either "Scanning Work on a Mobile Device" or "Submitting an Assignment".

Instructions for writing solutions:

- Homework is graded both on **correctness** and on **presentation**.
- Your proofs should be written in **complete sentences**, starting with capitals and ending with periods. They should be in **paragraph form**, similar to the proofs in the textbook. For example, you shouldn't put each sentence on a new line. Paragraphs should be used to break the proof into logical chunks.
- Don't use informal abbreviations such as three dots for "therefore." Logical symbols can be used when they clarify things to the reader, but English words can often be more effective.
- Include all of the steps that are needed to logically justify every claim you make. Do not include unnecessary steps. Try to be **concise and complete**.
- Do not submit any **rough work**. Do your rough work on scratch paper, and only submit a clearly and neatly written answer. Do not cross out or erase more than a word or two. If you write each final solution on a new page, it's easy to start over on a fresh page when you've made a large error.
- You should do the work **on your own**. Read the course syllabus for the rules about scholastic offences, which include sharing solutions with others, uploading material to a website, viewing material of others or on a website (even if you don't use it), etc. The penalty for cheating on homework will be a grade of **0** on the homework set as well as a penalty of **negative 5%** on the overall course grade.
- We may not grade every question.

$(x+a)(x+b) = x^2 + 4x + c.$

$x^2+(a+b)x+ab = x^2+4x+c$.

$a+b=4$ $\quad a=4-b$. $\quad b=2$

$ab=c$. $\quad b(4-b)=c$. $\quad b=1$

$b=3$. False.

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Due **Friday, October 29 at 11:59pm**. You can **resubmit** your work any number of times until the deadline.

H7Q1: Consider the following statement. If it is true, prove it. If not, give a counterexample.

Statement 1: There exists a unique $c \in \mathbb{Z}$ such that $x^2 + 4x + c$ factors over $\mathbb{Z}$. (That is, it factors as $(x + a)(x + b)$ for $a, b \in \mathbb{Z}$.)

Solution: Statement 1 is not correct because there are at least two values of c such that $x^2 + 4x + c$ factors. When $c = 4$, we have $x^2 + 4x + 4 = (x + 2)^2$. And when $c = 3$, we have $x^2 + 4x + 3 = (x + 1)(x + 3)$.

$\frac{y}{1+x} = y - x$. $\quad \boxed{x = y-1}$.

$y = (y-x)(1+x)$

H7Q2: Prove the following:

$y = -x^2 + (y-1)x + y$.

Theorem 1: For every real number x, if $x \neq -1$ and $x \neq 0$, there is a unique real number y such that $\frac{y}{1+x} = y - x$.

$x^2y - 2y + 2x^2 - 4 = 0$. $\quad x = \pm\sqrt{2}$: y can be any num

$y(x^2-2) = -2(x^2-2)$. $\quad$ else: $y = -2$.

Theorem 2: There exists a unique real number y such that for every real number x, $2y - 2x^2 = x^2y - 4$.

Solution:

Proof of Theorem 1. Let $x \in \mathbb{R}$ be arbitrary, and assume that $x \neq -1$ and $x \neq 0$.

Existence: Let $y = 1 + x$. Then $\frac{y}{1+x} = \frac{1+x}{1+x} = 1 = 1 + x - x = y - x$, as required.

Uniqueness: Suppose that $z \in \mathbb{R}$ satisfies $\frac{z}{1+x} = z - x$. Then

$$z = (1 + x)(z - x) = (1 + x)z - x(1 + x).$$

Therefore, $-xz = -x(1 + x)$. Since $x \neq 0$, it follows that $z = 1 + x$. $\square$

Proof of Theorem 2. Existence: Let $y = -2$. Then, for any real number x, we have that

$$2y - 2x^2 = -4 - 2x^2 = x^2y - 4,$$

as required.

Uniqueness: Let z be a real number with the property that $2z - 2x^2 = x^2z - 4$, holds for every real number x. In particular, this must hold when $x = 0$, which gives that $2z = -4$. Therefore, we must have $z = -2$, as required. $\square$

H7Q3: Let A and B be sets.

(a) Prove that there exists a set C such that:

   (1) $A \subseteq C$ and $B \subseteq C$, and

   (2) for every set D such that $A \subseteq D$ and $B \subseteq D$, we have $C \subseteq D$.

(b) Is the set C the unique set with these properties? If so, prove this; otherwise, give an example of sets A, B and two choices of C that both satisfy (1) and (2).

Solution:

Proof of (a). We will show that $C = A \cup B$ has the two desired properties.

(1) says that $A \subseteq A \cup B$ and $B \subseteq A \cup B$, and these are clear. For example, every element $a \in A$ is in $A \cup B$.

To prove (2), let D be such that $A \subseteq D$ and $B \subseteq D$. Let us now prove that $C \subseteq D$. Let $x \in C$. Then since $C = A \cup B$, we have that $x \in A$ or $x \in B$. Suppose that $x \in A$. We have that $A \subseteq D$, by our assumption on D, and therefore $x \in D$. Now suppose that $x \in B$. We again have by assumption on D that $B \subseteq D$ and therefore $x \in D$. Both cases give us that $x \in D$, therefore we can conclude that $\forall x \in C, x \in D$ and therefore $C \subset D$. □

Proof of (b). The set C is unique. In order to prove it, we assume that arbitrary sets C and C' both satisfy properties (1) and (2) and we prove that $C = C'$.

We first prove that $C \subseteq C'$. We have that $A \subseteq C'$ and $B \subseteq C'$ by property (1) for C', so by applying property (2) for C with $D = C'$, we get that $C \subseteq C'$.

The reverse inclusion $C' \subseteq C$ follows from a similar argument with C and C' reversed. We conclude that $C = C'$, which finishes our proof. □

H7Q4: Let U be a set. Prove that for all $A, B \in \mathcal{P}(U)$, the following are equivalent:

(a) $B \subseteq A$

(b) $(U \setminus A) \cap B = \emptyset$

(c) $(U \setminus B) \cup A = U$.

Solution:

Proof. Let $A, B \in \mathcal{P}(U)$ be arbitrary. We will prove $(a) \implies (b) \implies (c) \implies (a)$.

$(a) \implies (b)$: Assume $B \subseteq A$. Suppose that $x \in (U \setminus A) \cap B$. Then $x \in U$, $x \notin A$ and $x \in B$. But this contradicts the assumption that $B \subseteq A$. So $(U \setminus A) \cap B$ must be empty.

$(b) \implies (c)$: Assume that $(U \setminus A) \cap B = \emptyset$. Since $U \setminus B \subseteq U$ and $A \subseteq U$, it follows that $(U \setminus B) \cup A \subseteq U$. We must show the reverse inclusion. So let x in U.

Case 1: $x \in A$. Then clearly $x \in (U \setminus B) \cup A$, so we are done.

Case 2: $x \notin A$. Then $x \in U \setminus A$. Since $U \setminus A$ and B are disjoint, we must have $x \notin B$. Therefore, $x \in U \setminus B$. So $x \in (U \setminus B) \cup A$.

In either case, $x \in (U \setminus B) \cup A$, so we have shown that $U \subseteq (U \setminus B) \cup A$.

$(c) \implies (a)$: Assume that $(U \setminus B) \cup A = U$. Let $x \in B$. Since $B \subseteq U$, we also have $x \in U$. Therefore, $x \in (U \setminus B) \cup A$. But since $x \in B$, we have that $x \notin U \setminus B$. Therefore we must have $x \in A$. We have shown that $B \subseteq A$. $\square$