Elementary operations

Definition Two SLEs are *equivalent* if they have the same solutions.

For example

$$\begin{aligned} x + 2y \quad\quad &= 2 \\ x - \ y \ + 2z &= 1 \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}x + y \quad\quad &= 1 \\ 3y - 2z &= 1 \end{aligned}$$

Remark. To solve a SLE, we mean to find its solutions to it.

Solve the SLE

$$
\begin{aligned}
x + y &= 2 \\
x - y &= 0
\end{aligned}
$$

Method 1. Substituting $x = y$ to the equation $x + y = 2$, we have $2x = 2$. So $x = 1$ and $y = 1$.

Solve the SLE

$$x + y = 2$$
$$x - y = 0$$

Method 1. Substituting $x = y$ to the equation $x + y = 2$, we have $2x = 2$. So $x = 1$ and $y = 1$.

Method 2. The left-hand side (LHS) of the first equation minus the left-hand side of the second equation is

$$(x + y) - (x - y) = 2y.$$

To make sure we will get an equality, the right-hand side (LHS) of the first equation should substract the right-hand side of the second equation, which is 2.
So $2y = 2$ and $y = 1$, $x = 1$.

Definition The following operations are the *elementary operations* which are allowed in solving a SLE:

- Multiply an equation by a non-zero scalar.

- Interchange the positions of two equations in the system.

- Replace one of the equations by the sum of that equation and a scalar multiple of another one of the equations in the system.

No other operations are allowed.

Definition The following operations are the *elementary operations* which are allowed in solving a SLE:

- Multiply an equation by a non-zero scalar.

- Interchange the positions of two equations in the system.

- Replace one of the equations by the sum of that equation and a scalar multiple of another one of the equations in the system.

No other operations are allowed.

Theorem. Performing one of the elementary operations to transform a system of linear equations always results in a SLE which is equivalent to the original system.

1. Find all solutions to the system of linear equations

$$\begin{aligned} x + y + z &= 5 \\ 3x + 2y + z &= 15 \\ y + 2z &= 0 \end{aligned}$$

2. Solve the SLE: $2x = 2 - 2y$ and $3y - 6 = -3x$.
3. Solve the SLE

$$\begin{aligned} -x + y + z &= 0 \\ 2x - 2y + z &= 3 \\ y - 3z &= 0 \end{aligned}$$

1. Find all solutions to the system of linear equations

$$\begin{aligned} x + y + z &= 5 \\ 3x + 2y + z &= 15 \\ y + 2z &= 0 \end{aligned}$$

2. Solve the SLE: $2x = 2 - 2y$ and $3y - 6 = -3x$.
3. Solve the SLE

$$\begin{aligned} -x + y + z &= 0 \\ 2x - 2y + z &= 3 \\ y - 3z &= 0 \end{aligned}$$

We had examples of the only 3 things that can happen when we solve a system of linear equation:
The SLE has a unique solution, does not have a solution or has an r-parametric family of solutions.

Matrices

Definition. A *matrix* is a rectangular array of numbers, each of which is called an *entry* of the matrix.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

A matrix with m rows and n columns is called an $m \times n$ matrix (pronounced "m by n").

Definition. A *matrix* is a rectangular array of numbers, each of which is called an *entry* of the matrix.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

A matrix with m rows and n columns is called an $m \times n$ matrix (pronounced "m by n").

For instance,

$$\begin{bmatrix} 2 & -3 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & -3 \\ 1 & \frac{2}{3} \\ \sqrt{3} & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}.$$

Definition. Consider a SLE with m linear equations in n variables, written in standard form. The *coefficient matrix* of the SLE is the $m \times n$ matrix in which the (i, j)-entry is the coefficient, in the i-th equation, of the j-th variable.

Definition. Consider a SLE with m linear equations in n variables, written in standard form. The *coefficient matrix* of the SLE is the $m \times n$ matrix in which the (i, j)-entry is the coefficient, in the i-th equation, of the j-th variable.

For instance,

$$\begin{aligned} -x + y + z &= 0 \\ 2x - 2y + z &= 3 \\ y - 3z &= 0 \end{aligned}$$

Definition. Consider a SLE with m linear equations in n variables, written in standard form. The *coefficient matrix* of the SLE is the $m \times n$ matrix in which the (i, j)-entry is the coefficient, in the i-th equation, of the j-th variable.

For instance,

$$\begin{aligned} -x + y + z &= 0 \\ 2x - 2y + z &= 3 \\ y - 3z &= 0 \end{aligned}$$

Its coefficient matrix is

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & -2 & 1 \\ 0 & 1 & -3 \end{bmatrix}$$

Augmented matrix for a SLE

Definition. The *augmented matrix* for a SLE is obtained by adding the m right hand side values of the equations to the coefficient matrix, as an extra column in the matrix.

We always delineate this extra column in an augmented matrix by placing a vertical line before it.

Augmented matrix for a SLE

Definition. The *augmented matrix* for a SLE is obtained by adding the m right hand side values of the equations to the coefficient matrix, as an extra column in the matrix.

We always delineate this extra column in an augmented matrix by placing a vertical line before it.
For example,

$$\begin{aligned} x + y + z &= 5 \\ 3x + 2y + z &= 15 \\ y + 2z &= 0 \end{aligned}$$

Definition. The *augmented matrix* for a SLE is obtained by adding the m right hand side values of the equations to the coefficient matrix, as an extra column in the matrix.

We always delineate this extra column in an augmented matrix by placing a vertical line before it.
For example,

$$\begin{aligned} x + y + z &= 5 \\ 3x + 2y + z &= 15 \\ y + 2z &= 0 \end{aligned}$$

Its augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 3 & 2 & 1 & 15 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

Find the coefficient matrices and the augmented matrices for the following linear systems.

1.

$$\begin{aligned} x - 3y + z &= 1 \\ -2x \qquad + z &= 0 \\ y + 2z &= -5 \end{aligned}$$

2.

$$\begin{aligned} x_1 - 3x_2 + 1 &= x_4 + x_6 \\ -x_2 + x_3 - x_5 &= 2 \\ x_1 + x_3 - x_4 + 3 - x_6 &= 0 \end{aligned}$$

3. If the augmented matrix for a particular system of linear equations is

$$\left[\begin{array}{cc|c} -1 & 7 & 3 \\ 0 & 5 & 6 \end{array}\right],$$

write the SLE.