UWO CS2214

Tutorial #11

Problem 1 Let A be the set of all ordered pairs of integers for which the second element of the pair is nonzero. Symbolically,

$$A = \mathbf{Z} \times (\mathbf{Z} \setminus \{\mathbf{0}\}).$$

Define a binary relation R on A as follows: For all $(a, b), (c, d) \in A$,

$$(a, b)\ R\ (c, d) \Leftrightarrow ac = bd.$$

1. Is R reflexive?
2. Is R symmetric?
3. Is R anti-symmetric?
4. Is R transitive?
5. Is R an equivalence relation, a partial order, neither, or both?

Solution 1

1. Is R reflexive? No. Indeed, consider $(a, c) \in \mathbf{Z} \times (\mathbf{Z} \setminus \{\mathbf{0}\})$. We have:
$$(a, c)\ R\ (a, c) \Leftrightarrow a^2 = c^2.$$
The statement $a^2 = c^2$ is equivalent to $(a - c)(a + c) = 0$, that is $a = c \vee a = -c$. Therefore, we $(2, 3) \notin R$. Thus, R is not reflexive.
2. Is R symmetric? Yes. Indeed, consider $(a, b), (c, d) in \mathbf{Z} \times (\mathbf{Z} \setminus \{\mathbf{0}\})$. We have:
$$(a, b)\ R\ (c, d) \Leftrightarrow ac = bd,$$
and
$$(c, d)\ R\ (a, b) \Leftrightarrow ca = db,$$
Clearly, we have:
$$ca = db \Leftrightarrow ac = bd,$$
Therefore, we have:
$$(a, b)\ R\ (c, d) \Leftrightarrow (c, d)\ R\ (a, b).$$
3. Is R anti-symmetric? No. Indeed, we have $(6, 10) R (5, 3)$.

4. Is R transitive? No. Indeed, we have $(6,10)R(5,3)$ and $(5,3)R(21,35)$. But we do **not** have $(6,10)R(21,35)$, since $6 \times 21 \neq 10 \times 35$.
5. Is R an equivalence relation, a partial order, neither, or both? Neither. It is not an equivalence relation, since it is not reflexive. It is not a partial order, since it is not anti-symmetric.

Problem 2 1. Show that the relation

$$R = \{(x, y) |\ (x - y) \text{ is an even integer}\}$$

is an equivalence relation on the set $\mathbb{R}$ of real numbers.

2. Show that the relation

$$\boxed{R = \{((x_1, y_1), (x_2, y_2))\ \mid\ (x_1 < x_2)\ \text{ or }\ ((x_1 = x_2)\ \text{ and }\ (y_1 \leq y_2))\}}$$

is a total ordering relation on the set $\mathbb{R} \times \mathbb{R}$.

Solution 2

1. (a) R is reflexive, since for all $x \in \mathbb{R}$, we have $x - x = 0$ which is even, hence for all $x \in \mathbb{R}$, we have $(x, x) \in R$.

   (b) R is symmetric, since for all $x, y \in \mathbb{R}$, if $x - y \equiv 0 \mod 2$ holds then so does $y - x \equiv 0 \mod 2$, that is, if $(x, y) \in R$ holds then so does $(y, x) \in R$.

   (c) R is transitive, since for all $x, y, z \in \mathbb{R}$, if $x - y \equiv 0 \mod 2$ and $y - x \equiv 0 \mod 2$ both hold then so does $x - z = (x - y) + (y - z) \equiv 0 \mod 2$, that is, if $(x, y) \in R$ and $(y, z) \in R$ both hold then so does $(x, z) \in R$.

   Therefore, R is an equivalence relation.

2. (a) R is reflexive, since for all $(x_1, y_1) \in \mathbb{R} \times \mathbb{R}$, we have $((x_1 = x_1)$ and $y_1 \leq y_1$, that is, for all $(x_1, y_1) \in \mathbb{R} \times \mathbb{R}$ we have $((x_1, y_1), (x_1, y_1)) \in R$.

   (b) R is anti-symmetric, since for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R} \times \mathbb{R}$, if $((x_1, y_1), (x_2, y_2)) \in R$ and $((x_2, y_2), (x_1, y_1)) \in R$ both hold then neither $x_1 < x_2$ nor $x_2 < x_1$ holds but both $((x_1 = x_2)$ and $y_1 \leq y_2)$ and $((x_2 = x_1)$ and $y_2 \leq y_1)$ hold, which implies $(x_1, y_1) = (x_2, y_2)$.

(c) R is transitive. To prove this consider $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R} \times \mathbb{R}$ such that $((x_1, y_1), (x_2, y_2)) \in R$ and $((x_2, y_2), (x_3, y_3)) \in R$ both hold. We shall prove that $((x_1, y_1), (x_3, y_3)) \in R$ also holds. Four cases must be inspected:

i. $x_1 < x_2$ and $x_2 < x_3$,
ii. $x_1 < x_2$ and $x_2 = x_3$ and $y_2 \leq y_3$,
iii. $x_1 = x_2$ and $y_1 \leq y_2$ and $x_2 < x_3$,
iv. $x_1 = x_2$ and $y_1 \leq y_2$ and $x_2 = x_3$ and $y_2 \leq y_3$,

which respectively imply:

i. $x_1 < x_3$,
ii. $x_1 < x_3$,
iii. $x_1 < x_3$,
iv. $x_1 = x_3$ and $y_1 \leq y_3$,

that is $((x_1, y_1), (x_3, y_3)) \in R$.

3. Therefore, R is an ordering relation on the set $\mathbb{R} \times \mathbb{R}$.

4. R is a total ordering relation on the set $\mathbb{R} \times \mathbb{R}$. Indeed, for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R} \times \mathbb{R}$, we have

(a) either $x_1 < x_2$ (in which case $((x_1, y_1), (x_2, y_2)) \in R$ holds),
(b) or ($x_1 = x_2$ and $y_1 \leq y_2$ (in which case $((x_1, y_1), (x_2, y_2)) \in R$ holds),
(c) or ($x_1 = x_2$ and $y_1 > y_2$ (in which case $((x_2, y_2), (x_1, y_1)) \in R$ holds),
(d) or $x_1 > x_2$ (in which case $((x_2, y_2), (x_1, y_1)) \in R$ holds).

Problem 3 Let R be a binary relation on a set A. We denote by I the *identity relation* on A, that is:

$$I = \{(x, x) \mid x \in A\}.$$

We denote by $r(R)$ the relation given by:

$$r(R) = R \cup I.$$

1. Prove that $r(R)$ is reflexive.

2. Prove that R is reflexive if and only if $r(R) = R$.

Clearly, if R' is a reflexive relation so that $R \subseteq R'$ holds then $r(R) \subseteq R'$ holds as well. For that reason, the relation $r(R)$ can be regarded as the "smallest" reflexive relation containing R and $r(R)$ is called the *reflexive closure* of R.

Solution 3

1. Indeed R reflexive exactly means $I \subseteq R$.

2. From the previous question, if R reflexive, then $I \subseteq R$ holds and thus $r(R) \subseteq R$ holds. Since $R \subseteq r(R)$ clearly holds as well, we have proved the following:

   $$R \text{ reflexive} \quad \to \quad r(R) = R$$

   The converse follow from the previous question.

Problem 4 Let R be a binary relation on a set A. We denote by R^{-1} the *inverse relation* of R, that is, the binary relation on A defined by:

$$R^{-1} = \{(y, x) \ \mid \ (x, y) \in R\}.$$

We denote by $s(R)$ the relation given by:

$$s(R) = R \cup R^{-1}.$$

1. Prove that $s(R)$ is symmetric.

2. Prove that R is symmetric if and only if $s(R) = R$.

3. Prove that if R' is a symmetric relation so that $R \subseteq R'$ holds, then $s(R) \subseteq R'$ holds as well.

From the third question it follows that the relation $s(R)$ can be regarded as the "smallest" symmetric relation containing R. For that reason, $s(R)$ is called the *symmetric closure* of R.

Solution 4

1. Let us prove that $s(R)$ is symmetric, thus let us prove that for all $x, y \in A$, if $(x, y) \in s(R)$, then $(y, x) \in s(R)$ holds as well. So, let $x, y \in A$ and assume that $(x, y) \in s(R)$ holds. Since $s(R) = R \cup R^{-1}$ holds, two cases arise: either $(x, y) \in R$ holds or $(x, y) \in R^{-1}$ holds. Consider the first case. Then, by definition of R^{-1}, we have $(y, x) \in R^{-1}$, thus we have $(y, x) \in s(R)$. Consider now the second case, that is, $(x, y) \in R^{-1}$. Then, by definition of R^{-1}, we have $(y, x) \in R$, thus we have $(y, x) \in s(R)$. Finally, we have shown that $s(R)$ is symmetric.

2. Let us prove that R is symmetric if and only if $s(R) = R$. First, we assume that R is symmetric and we prove that $s(R) = R$ holds as well. We observe that R symmetric implies that $R^{-1} \subseteq R$ holds and thus we have $s(R) = R$. Conversely, if $s(R) = R$ holds, then $R^{-1} \subseteq R$ holds as well which implies that R is symmetric.

3. Let R' be a symmetric relation so that $R \subseteq R'$ holds. We shall prove that $s(R) \subseteq R'$ holds as well. Since $R \subseteq R'$ holds, it is a routine exercise to prove that $s(R) \subseteq s(R')$ holds as well. Since R' is symmetric, it follows from the second question that $R' = s(R')$. Therefore, we have $s(R) \subseteq R'$, as required.

Problem 5 Let R be a binary relation on a finite set A with cardinality n. We denote by $t(R)$ the *transitive closure* of R, that is, the binary relation on A defined by:

$$t(R) = R \cup R^2 \cup \cdots \cup R^n.$$

1. Let k be an integer such that $2 \leq k \leq n$. Let x, y be in A. We denote by $P(x, y, k)$ the following predicate:

   there exist $(k-1)$ elements $x_2, \ldots, x_k$ of A so that $(x, x_2), (x_2, x_3), \ldots, (x_k, y)$ all belong to R.

   Prove that the following statements are equivalent for all $x, y \in A$:

   (a) $(x, y) \in R^k$,

   (b) $P(x, y, k)$ holds

2. Let k, ℓ be two positive integers, with $k \leq n$ and $\ell \leq n$. Let x, y, z be in A so that $P(x, y, k)$ and $P(y, z, \ell)$ both hold. Prove that $P(x, z, m)$, with $m = \min(n, k + \ell)$, also holds.

3. Prove that $t(R)$ is transitive.

4. Prove that if R transitive, then $R^k \subseteq R$ for all positive integer k.

5. Prove that R transitive if and only if $t(R) = R$.

6. Prove that if R' is a transitive relation so that $R \subseteq R'$ holds, then $t(R) \subseteq R'$ holds as well.

It follows from the last question that the relation $t(R)$ can be regarded as the "smallest" transitive relation containing R. This is the reason why $t(R)$ is called the *transitive closure* of R.

Solution 5

1. We proceed by induction on k, for $1 \leq k \leq n$. We observe that the equivalence $(a) \iff (b)$ is clear for all $x, y, \in A$, when $k = 1$. Indeed, $P(x, y, 1)$ simply means $(x, y) \in R$. Now we assume that for some k, with $1 \leq k < n$, the equivalence $(a) \iff (b)$ holds for all $x, y, \in A$. We shall prove that this equivalence holds for all $x, y, \in A$, with $k+1$ instead of k. So let $x, y, \in A$. Assume first that $(x, y) \in R^{k+1}$ holds and let us prove that $P(x, y, k+1)$ holds as well. By definition of R^{k+1}, we have $R^{k+1} = R \circ R^k$, thus there exists $z \in A$ so that $(x, z) \in R^k$ and $(z, y) \in R$. By induction hypothesis, $(x, z) \in R^k$ is equivalent to $P(x, z, k)$, that is, there exist $(k-1)$ elements $x_2, \ldots, x_k$ of A so that $(x, x_2), (x_2, x_3), \ldots, (x_k, z)$ all belong to R. Putting everything together, we deduce that there exist k elements $x_2, \ldots, x_k, z$ of A so that $(x, x_2), (x_2, x_3), \ldots, (x_k, z), (z, y)$ all belong to R. This latter statement means that $P(x, y, k+1)$ holds, as required. Proving the converse implication (that is, $P(x, y, k+1) \rightarrow (x, y) \in R^{k+1}$) can easily be done using the same arguments as those used for proving the direct implication $(x, y) \in R^{k+1} \rightarrow P(x, y, k+1)$. This completes the proof of this first question.

2. Let k, ℓ be two positive integers, with $k \leq n$ and $\ell \leq n$. Let x, y, z be in A so that $P(x, y, k)$ and $P(y, z, \ell)$ both hold. We shall prove that $P(x, z, m)$, with $m = \min(n, k+\ell)$, holds as well. Recall first that $P(x, y, k)$ means that there exist $(k-1)$ elements $x_2, \ldots, x_k$ of A so that $(x, x_2), (x_2, x_3), \ldots, (x_k, y)$ all belong to R. Similarly, $P(y, z, \ell)$ means that there exist $(\ell - 1)$ elements $x_{k+2}, \ldots x_{\ell+k}$ so that $(y, x_{k+2}), \ldots, (x_{\ell+k}, z)$ all belong to R. It follows that there exist $x_2, \ldots, x_k, x_{k+1}, x_{k+2}, \ldots x_{\ell+k} \in A$ with $y = x_{k+1}$, so that

$$(x, x_2), (x_2, x_3), \ldots, (x_k, x_{k+1}), (x_{k+1}, x_{k+2}), \ldots, (x_{\ell+k}, z)$$

all belong to R. The number of these "intermediate points"

$$x_2, \ldots, x_k, x_{k+1}, x_{k+2}, \ldots x_{\ell+k}$$

is $\ell - k - 1$. But if $\ell - k - 1$ exceeds $n - 1$ then there is necessarily some repetitions among those points and thus some arcs can be removed.

Indeed, since the set A counts n elements, the number of these "intermediate points" (excluding x and z) is at most $n-1$ if $x = z$ holds and $n-2$ otherwise. Therefore, the number of intermediate points is $m-1$ with $m = \min(n, k+\ell)$. Therefore, we have $P(x, z, m)$, as required.

3. Let us prove that $t(R)$ is transitive. Let x, y, z be in A so that $(x, y) \in t(R)$ and $(y, z) \in t(R)$ both hold. Let us prove that $(x, z) \in t(R)$ as well. Recall that, by definition of $t(R)$, we have:

$$t(R) = R \cup R^2 \cup \cdots \cup R^n.$$

Therefore, the statement $(x, y) \in t(R)$ means that there exists a positive integer $k \leq n$ so that $(x, y) \in R^k$. Similarly, the statement $(y, z) \in t(R)$ means that there exists a positive integer $\ell \leq n$ so that $(y, z) \in R^\ell$. From the first question, we deduce that $P(x, y, k)$ and $P(y, z, \ell)$ both hold. Then, from the second question, we deduce that $P(x, z, m)$, with $m = \min(n, k+\ell)$, also holds. This implies, using the first question again that $(x, z) \in R^m$. Since $m \leq n$ holds, it follows that (x, z) belongs to one of $R, R^2, \ldots, R^n$. In other words, (x, z) belongs to $t(R)$, as required. This completes the proof that $t(R)$ is transitive.

4. The proof is by induction $k \geq 1$. The *base step* $k = 1$ is clear since we obviously have $R \subseteq R$. We now prove the *inductive step*. We assume that $R^k \subseteq R$ holds for some $k \geq 1$. We shall prove that $R^{k+1} \subseteq R$ holds as well. Recall that we have $R^{k+1} = R \circ R^k$. Since $R^k \subseteq R$ holds (by induction hypothesis) a routine proof yields

$$R \circ R^k \quad \subseteq \quad R \circ R.$$

Since R is transitive, it follows directly from the definition of the composition of two relations that $R \circ R \ \subseteq \ R$ holds. Therefore, we have $R^{k+1} \ \subseteq \ R$, which completes the proof of the inductive step and thus the proof of the fact that if R transitive, then $R^k \subseteq R$ for all positive integer k.

5. We prove the equivalence:

$$R \text{ transitive} \quad \Longleftrightarrow \quad t(R) = R.$$

We first assume that R is transitive. Recall that we have:

$$t(R) = R \cup R^2 \cup \cdots \cup R^n.$$

From the previous question, we have $R^k \subseteq R$, for all positive integer $k \geq 1$. This clearly implies $t(R) = R$. Conversely, if $t(R) = R$ holds, then from the third question, we deduce that R is transitive, as required.

6. Let R' be a transitive relation so that $R \subseteq R'$ holds. We prove that $t(R) \subseteq R'$ holds as well. From $R \subseteq R'$, an easy routine proof (similar to the proof of the fourth question) yields $t(R) \subseteq t(R')$. Since R' is transitive, the fifth question yields $t(R') = R'$. Therefore, we have $t(R) \subseteq R'$, as required.