CHAPTER 7
Basic Modal Logic

Modal logic studies arguments whose validity depends on "necessary," "possible," and similar notions. This chapter covers the basics, and the next gets into further modal systems.

7.1 Translations

To help us evaluate modal arguments, we'll construct a little modal language. For now, our language will build on propositional logic, and thus include all the vocabulary, wffs, inference rules, and proofs of the latter. Our language adds two new vocabulary items: "◊" and "□" (diamond and box):

◊A	=	It's possible that A	=	A is true in some possible world.
A	=	It's true that A	=	A is true in the actual world.
□A	=	It's necessary that A	=	A is true in all possible worlds.

Calling something *possible* is a weak claim—weaker than calling it *true*. Calling something *necessary* is a strong claim; it says, not just that the thing is true, but that it *has* to be true—it *couldn't* be false.

"Possible" here means *logically possible (not self-contradictory)*. "I run a mile in two minutes" may be physically impossible; but there's no self-contradiction in the idea, so it's logically possible. Likewise, "necessary" means *logically necessary (self-contradictory to deny)*. "2+2=4" and "All bachelors are unmarried" are examples of **necessary truths;** such truths are based on logic, the meaning of concepts, or necessary connections between properties.

We can rephrase "possible" as *true in some possible world*—and "necessary" as *true in all possible worlds*. A **possible world** is a consistent and complete[1] description of how things might have been or might in fact be. Picture a possible world as a *consistent story* (or novel). The story is *consistent,* in that its statements don't entail self-contradictions; it describes a set of possible situations that are all possible together. The story may or may not be true. The **actual world** is the story that's true—the description of how things in fact are. As before, a grammatically correct formula is called a *wff,* or *well-formed formula*. For now, wffs are strings that we can construct using the propositional rules plus this additional rule:

1. The result of writing "◊" or "□," and then a wff, is a wff.

Don't use parentheses with "◊A" and "□A":

[1] Since we are finite beings, we will in practice only give partial (not "complete") descriptions.

Right:	$\Diamond A$	$\Box A$
Wrong:	$\Diamond(A)$	$\Box(A)$
	$(\Diamond A)$	$(\Box A)$

Parentheses here would serve no purpose.

Now we'll focus on how to translate English sentences into modal logic. Here are some simpler examples:

A is possible (consistent, could be true)	=	$\Diamond A$		
A is necessary (must be true, has to be true)	=	$\Box A$		
A is impossible (self-contradictory)	=	$\sim\Diamond A$	=	A couldn't be true.
	=	$\Box\sim A$	=	A has to be false.

An impossible statement (like "2≠2") is one that's false in every possible world.

These examples are more complicated:

A is consistent (compatible)	=	It's possible that A and B are both true.
with B	=	$\Diamond(A \cdot B)$
A entails B	=	It's necessary that if A then B.
	=	$\Box(A \supset B)$

"Entails" makes a stronger claim than plain "if-then." Compare these two:

"There's rain" entails "There's precipitation"	=	$\Box(R \supset P)$
If it's Saturday, then I don't teach class	=	$(S \supset \sim T)$

The first if-then is logically necessary; every conceivable situation with rain also has precipitation. The second if-then just happens to be true; we can consistently imagine me teaching on Saturday—even if in fact I never do.

These common forms negate the whole wff:

A is inconsistent with B	=	It's not possible that A and B are both true.
	=	$\sim\Diamond(A \cdot B)$
A doesn't entail B	=	It's not necessary that if A then B.
	=	$\sim\Box(A \supset B)$

Here is how we translate "contingent":

A is a contingent statement	=	A is possible and not-A is possible.
	=	$(\Diamond A \cdot \Diamond\sim A)$
A is a contingent truth	=	A is true but could have been false.
	=	$(A \cdot \Diamond\sim A)$

Statements are necessary, impossible, or contingent. But truths are only necessary or contingent (since impossible statements are false).

When translating, it's usually good to mimic the English word order:

necessary not	=	$\Box\sim$	necessary if	=	$\Box($
not necessary	=	$\sim\Box$	if necessary	=	$(\Box$

Use a separate box or diamond for each "necessary" or "possible":

If A is necessary and B is possible, then C is possible=$((\Box A\cdot\Diamond B)\supset\Diamond C)$

The following tricky English forms are ambiguous; translate these into two modal wffs, and say that the English could mean one or the other:

> *"If A is true, then it's necessary (must be) that B"* could mean "$(A\supset\Box B)$" or "$\Box(A\supset B)$."

> *"If A is true, then it's impossible (couldn't be) that B"* could mean "$(A\supset\Box\sim B)$" or "$\Box(A\supset\sim B)$."

So this next sentence could have either of the following two meanings:

"If you're a bachelor, then you must be unmarried."

$(B\supset\Box U)$	=	"If you're a bachelor, then you're *inherently unmarriable* (in no possible world would anyone ever marry you)."
	=	If B, then U (by itself) is necessary.
$\Box(B\supset U)$	=	"It's necessary that *if* you're a bachelor *then* you're unmarried."
	=	It's necessary that if B then U.

The box-inside "$(B\supset\Box U)$" posits an *inherent necessity,* given that the antecedent is true, "You're unmarried" is inherently necessary. This version is insulting and presumably false. The box-outside "$\Box(B\supset U)$" posits a *relative necessity,* what is necessary is, not "You're a bachelor" or "You're unmarried" by itself, but only the connection between the two. This version is trivially true because "bachelor" means *unmarried man.*

The medievals called the box-inside form the "necessity of the *consequent*" (the second part being necessary); they called the box-outside form the "necessity of the *consequence*" (the whole if-then being necessary). The ambiguity is important philosophically; several intriguing but fallacious philosophical arguments depend on the ambiguity for their plausibility.

It's not ambiguous if you say the second part "by itself" is necessary or impossible—or if you use "entails" or start with "necessary." These forms aren't ambiguous:

If A, then B (by itself) is necessary	=	$(A\supset\Box B)$
A entails B	=	$\Box(A\supset B)$
Necessarily, if A then B	=	$\Box(A\supset B)$
It's necessary that if A then B	=	$\Box(A\supset B)$
"If A then B" is a necessary truth	=	$\Box(A\supset B)$

The ambiguous forms have if-then with a strong modal term (like "necessary," "must," "impossible," or "can't") in the then-part, like these:[1]

If A, then it's necessary that B.	If A, then it's impossible that B.
If A, then it must be that B.	If A, then it can't be that B.

When you translate an ambiguous English sentence, say that it's ambiguous and give both translations. When you do an English argument with an ambiguous statement, give both translations and work out both arguments.

7.1a Exercise—LogiCola J (BM & BT)

Using these equivalences, translate these English sentences into wffs. Be sure to translate ambiguous forms both ways.

G	=	There s a God (God exists)	R	=	There s rain
E	=	There's evil (Evil exists)	P	=	There's precipitation
M	=	There's matter (Matter exists)			

"God exists and evil doesn't exist" entails "There's no matter."	$\Box((G\cdot{\sim}E)\supset{\sim}M)$

1. It's necessary that God exists.
2. "There's a God" is self-contradictory.
3. It isn't necessary that there's matter.
4. It's necessary that there's no matter.
5. "There's rain" entails "There's precipitation."
6. "There's precipitation" doesn't entail "There's rain."
7. "There's no precipitation" entails "There's no rain."
8. If rain is possible, then precipitation is possible.
9. God exists.
10. If there's rain, then there must be rain.
11. It isn't possible that there's evil.
12. It's possible that there's no evil.
13. If there's rain, then it's possible that there's rain.
14. "There's matter" is compatible with "There's evil."
15. "There's a God" is inconsistent with "There's evil."
16. Necessarily, if there's a God then there's no evil.
17. If there's a God, then there can't be evil.
18. If there must be matter, then there's evil.
19. Necessarily, if there's a God then "There's evil" (by itself) is self-contradictory.

[1] There's an exception to this rule: if the if-part is a claim about necessity or possibility, then just use the box-inside form. So "If A is necessary then B is necessary" is just "$(\Box A\supset{\sim}\Box B)$"—and "If A is possible then B is impossible" is just "$(\Diamond A\supset{\sim}\Diamond B)$."

20. It's necessary that either there's a God or there's matter.
21. Either it's necessary that there's a God or it's necessary that there's matter.
22. "There's rain" is a contingent statement.
23. "There's rain" is a contingent truth.
24. "If there's rain, then there's evil" is a necessary truth.
25. If there's rain, then "There's evil" (by itself) is logically necessary.
26. If there's rain, then it's necessary that there's evil.
27. It's necessary that it's possible that there's matter.
28. "There's a God" isn't a contingent truth.
29. If there's a God, then it must be that there's a God.
30. It's necessary that if there's a God then "There's a God" (by itself) is necessary.

7.2 Proofs

Modal proofs work much like propositional proofs; but we need to add possible worlds and four new inference rules.

A **world prefix** is a string of zero or more instances of "W." So "" (zero instances), "W," "WW," and so on are world prefixes; these represent possible worlds, with the blank world prefix ("") representing the actual world. A *derived step* is now a line consisting of a world prefix and then "∴" and then a wff. And an *assumption* is now a line consisting of a world prefix and then "asm:" and then a wff. Here are examples of derived steps and assumptions:

∴ A	(So A is true in the actual world.)	asm: A	(Assume A is true in the actual world.)
W ∴ A	(So A is true in world W.)	W asm: A	(Assume A is true in world W.)
WW ∴ A	(So A is true in world WW.)	WW asm: A	(Assume A is true in world WW.)

Seldom do we need to assume something in another world.

We'll still use the S- and I-rules and RAA in modal proofs. Unless otherwise specified, we can use an inference rule only within a given world; so if we have "(A⊃B)" and "A" in the same world, then we can infer "B" in this same world. RAA needs additional wording *(italicized below)* for world prefixes:

> RAA: Suppose that some pair of not-blocked-off lines *using the same world prefix* have contradictory wffs. Then block off all the lines from the last not-blocked-off assumption on down and infer a step consisting in *this assumption's world prefix followed by* "∴" followed by a contradictory of that assumption.

To apply RAA, lines with the same world prefix must have contradictory wffs. Having "W ∴ A" and "WW ∴ ~A" isn't enough; "A" may well be true in one world but false in another. But "WW ∴ A" and "WW ∴ ~A" provide a genuine contradiction. The line derived using RAA must have the same world prefix as the assumption; if "W asm: A" leads to a contradiction in any world, then RAA lets us derive "W ∴ ~A."

Modal proofs use four new inference rules. These two reverse-squiggle (RS) rules hold regardless of what pair of contradictory wffs replaces "A"/"~A" (here "→" means we can infer whole lines from left to right):

Reverse squiggle	
	~☐A → ◇~A
	~◇A → ☐~A

These let us go from "not necessary" to "possibly false"—and from "not possible" to "necessarily false." Use these rules only within the same world. We can reverse squiggles on complicated formulas, so long as the whole formula begins with "~☐" or "~◇":

Right:	*Right:*	*Wrong:*
~◇~B	~☐(C · ~D)	(P ⊃ ~☐Q)
∴ ☐~~B	∴ ◇~(C · ~D)	∴ (P ⊃ ◇~Q)

In the first example, it would be simpler to conclude "☐B" (eliminating the double negation). Reverse squiggles whenever you have a wff that begins with "~" and then a modal operator; reversing a squiggle moves the modal operator to the beginning of the formula, so we can later drop it.

We drop modal operators using the next two rules (which hold regardless of what wff replaces "A"). Here's the drop-diamond (DD) rule:

Drop diamond	
	◇A → W ∴ A,
	use a *new* string of W's

Here the line with "◇A" can use any world prefix—and the line with "∴ A" must use a *new* string (one not occurring in earlier lines) of one or more W's. If "A" is possible, then "A" is thereby true in *some* possible world; we can give this world a name—but a *new* name, since "A" needn't be true in any of the worlds used in the proof so far. In proofs, we'll use "W" for the first diamond we drop, "WW" for the second, and so forth. So if we drop two diamonds, then we must introduce two worlds:

◇H	Heads is possible, tails is possible; call an imagined
◇T	world with heads "W," and one with tails "WW."
W ∴ H ←	"W" is OK because it occurs in no earlier line.
WW ∴ T ←	Since "W" has now occurred, we use "WW."

We can drop diamonds from complicated formulas, so long as the diamond *begins* the wff:

Right:	*Wrong:*	*Wrong:*
$\Diamond(A \cdot B)$	$(\Diamond A \supset B)$	$(\Diamond A \cdot \Diamond B)$
$W \therefore (A \cdot B)$	$W \therefore (A \supset B)$	$W \therefore (A \cdot B)$

The last two formulas don't *begin* with a diamond; instead, they begin with "(." Drop only *initial* diamonds—and introduce a new and different world prefix whenever you drop a diamond.

Here's the drop-box (DB) rule:

Drop box	$\Box A \rightarrow \overset{\downarrow}{W} \therefore A$, use any world prefix

Here the line with "□A" can use any world prefix—and the line with "∴ A" can use any world prefix too (including the blank one). If "A" is necessary, then "A" is true in *all* possible worlds, and so we can put "A" in any world we like. However, it's bad strategy to drop a box using a new world; instead, stay in old worlds. As before, we can drop boxes from complicated formulas, so long as the box *begins* the wff:

Right:	*Wrong:*	*Wrong:*
$\Box(A \supset B)$	$(\Box A \supset B)$	$(\Box A \supset \Box B)$
$W \therefore (A \supset B)$	$W \therefore (A \supset B)$	$W \therefore (A \supset B)$

The last two formulas begin, not with a box, but with a left-hand parenthesis. "(□A⊃B)" and "(□A⊃□B)" are if-then forms and follow the if-then rules: if we have the first part true, we can get the second true; if we have the second part false, we can get the first false; and if we get stuck, we need to make another assumption.

Here's an English version of a modal proof:

```
1      Necessarily, if there's rain then there's precipitation.
2      It's possible that there's rain.
    [∴ It's possible that there's precipitation.
3   ┌ Assume: It's not possible that there's precipitation.
4   │ ∴ It's necessary that there's no precipitation.   {from 3, reverse squiggle}
5   │ ∴ In world W: There's rain.   {from 2, drop diamond, call the world "W"}
6   │ ∴ In world W: If there's rain then there's precipitation.   {from 1, drop box}
7   │ ∴ In world W: There's precipitation.   {from 5 and 6}
8   └ ∴ In world W: There's no precipitation.   {from 4, drop box}
9    ∴ It's possible that there's precipitation.   {from 3; 7 contradicts 8}
```

The steps here should make sense. Our proof strategy is: first reverse squiggles, then drop diamonds into new worlds, and finally drop boxes into the same old worlds. Here's the proof in symbols:

```
     1    □(R ⊃ P)   Valid
  *  2    ◇R
         [∴ ◇P
  *  3   ┌ asm: ~◇P
     4   │ ∴ □~P   {from 3}
     5   │ W ∴ R   {from 2}
  *  6   │ W ∴ (R ⊃ P)   {from 1}
     7   │ W ∴ P   {from 5 and 6}
     8   └ W ∴ ~P   {from 4}
     9   ∴ ◇P   {from 3; 7 contradicts 8}
```

After making the assumption (line 3), we reverse the squiggle to move the modal operator to the outside (line 4). We drop the diamond using a new world (line 5). We drop the box in line 1 using this same world—and use an I-rule to get "P" in world W (lines 6 and 7). Then we drop the box in line 4 using this same world, to get "~P" in world W (line 8). Since we have a contradiction, we use RAA to give us the original conclusion.

We starred lines 2, 3, and 6; as before, starred lines largely can be ignored in deriving further steps. Here are two new starring rules—with examples:

Star any wff on which you reverse squiggles.	Star any wff from which you drop a diamond.
* ~□A ――――― ∴ ◇~A	* ◇A ――――― W ∴ A

When we reverse squiggles or drop diamonds, the new line has the same information. Don't star when dropping a box; we can never exhaust a "necessary" statement—and we may have to use it again later in the proof.

Here's another modal proof:

```
     1    □(A · B)
         [∴ □A
  *  2   ┌ asm: ~□A
  *  3   │ ∴ ◇~A          First reverse squiggles
     4   │ W ∴ ~A          and drop diamonds.
     5   │ W ∴ (A · B)     Lastly drop boxes.
     6   └ W ∴ A
     7   ∴ □A   {from 2; 4 contradicts 6}
```

Here it would be useless and bad strategy to drop the box into the actual world—to go from "□(A·B)" in line 1 to "∴ (A·B)" with no initial W's.

Drop a box into the actual world in only two cases:

> Drop a box into the actual world just if:
>
> - the original premises or conclusion have an unmodalized instance of a letter, or
> - you've done everything else possible (including further assumptions if needed) and still have no other worlds.

These examples illustrate the two cases:

1	□(A · B)		1	□~A	
	[∴ A	*unmodalized*		[∴ ~□A	*no other*
2	┌ asm: ~A		2	┌ asm: □A	*worlds*
3	│ ∴ (A · B)	{from 1}	3	│ ∴ A	{from 2}
4	└ ∴ A	{from 3}	4	└ ∴ ~A	{from 1}
5	∴ A	{from 2; 2 contradicts 4}	5	∴ ~□A	{from 2; 3 contradicts 4}

In the first argument, "A" in the conclusion is *unmodalized*—which means that it doesn't occur as part of a larger wff beginning with a box or diamond. Whenever our original premises or conclusion have an unmodalized instance of a letter, our standard strategy will be to drop all boxes into the actual world.[1]

In the second argument, we drop boxes into the actual world because there are no other old worlds to use (because there was no "◊" to drop) and we've done everything else that we can do. This second case is fairly unusual—while the first is quite common. Our proof strategy works much like before. We first assume the opposite of the conclusion; then we use our four new rules plus the S- and I-rules to derive whatever we can. If we find a contradiction, we apply RAA. If we get stuck and need to break up a wff of the form "~(A·B)" or "(A∨B)" or "(A⊃B)," then we make another assumption. If we get no contradiction and yet can't do anything further, then we try to refute the argument.

Reverse squiggles and drop diamonds first, drop boxes last:

1. FIRST REVERSE SQUIGGLES: For each unstarred, not-blocked-off step that begins with "~" and then a box or diamond, derive a step using the reverse-squiggle rules. Star the original step.

2. AND DROP DIAMONDS: For each unstarred, not-blocked-off step that begins with a diamond, derive an instance using the next available *new* world prefix (unless some such instance already occurs in previous not-blocked-off steps). Star the original step.

   Note: Don't drop a diamond if you already have a not-blocked-off instance in previous steps—there's no point in deriving a second instance. For example, don't drop "◊A" if you already have "W ∴ A."

[1] In "(A·◊B)," the first letter is unmodalized. If this formula was a premise or conclusion, then our standard strategy would say to drop all boxes into the actual world.

3. LASTLY DROP BOXES: For each not-blocked-off step that begins with a box, derive instances using each *old* world. Don't star the original step; you might have to use it again.

   Note: Drop a box into the actual world just if (a) the premises or conclusion have an unmodalized instance of a letter, or (b) you've done everything else (including further assumptions if needed) and still have no other worlds.

Be sure to drop diamonds before boxes. Introduce a new world each time you drop a diamond, and use the same old worlds when you drop a box. And drop only *initial* diamonds and boxes.

We won't see invalid modal arguments until later.

7.2a Exercise—LogiCola KV

Prove each of these arguments to be valid (all are valid).

```
    □(A ⊃ B)
    ◇~B
  ∴ ◇~A
```

```
    1      □(A ⊃ B)   Valid
*   2      ◇~B
         [ ∴ ◇~A
*   3    ┌ asm: ~◇~A
    4    │ ∴ □A   {from 3}
    5    │ W ∴ ~B   {from 2}
*   6    │ W ∴ (A ⊃ B)   {from 1}
    7    │ W ∴ A   {from 4}
    8    └ W ∴ B   {from 6 and 7}
    9    ∴ ◇~A   {from 3; 5 contradicts 8}
```

1. ◇(A · B)
∴ ◇A

2. A
∴ ◇A

3. ~◇(A · ~B)
∴ □(A ⊃ B)

4. □(A ∨ ~B)
~□A
∴ ◇~B

5. (◇A ∨ ◇B)
∴ ◇(A ∨ B)

6. (A ⊃ □B)
◇~B
∴ ◇~A

7. ~◇(A · B)
◇A
∴ ~□B

8. □A
∴ ◇A

9. □A
~□B
∴ ~□(A ⊃ B)

10. □(A ⊃ B)
∴ (□A ⊃ □B)

7.2b Exercises—LogiCola KV

First appraise intuitively. Then translate into logic (using the letters given) and prove to be valid (all are valid).

1. “You knowingly testify falsely because of threats to your life” entails “You lie.”
   It’s possible that you knowingly testify falsely because of threats to your life but don’t intend to deceive. (Maybe you hope no one will believe you.)
   ∴ “You lie” is consistent with “You don’t intend to deceive.” [Use T, L, and I. This argument is from Tom Carson, who writes on the morality of lying.]

2. Necessarily, if you don’t decide then you decide not to decide.
   Necessarily, if you decide not to decide then you decide.
   ∴ Necessarily, if you don’t decide then you decide. [Use D for “You decide” and N for “You decide not to decide.” This is adapted from Jean-Paul Sartre.]

3. If truth is a correspondence with the mind, then “There are truths” entails “There are minds.”
   “There are minds” isn’t logically necessary.
   Necessarily, if there are no truths then it is not true that there are no truths.
   ∴ Truth isn’t a correspondence with the mind. [Use C, T, and M.]

4. There’s a perfect God.
   There’s evil in the world.
   ∴ “There’s a perfect God” is logically compatible with “There’s evil in the world.” [Use G and E. Most who doubt the conclusion would also doubt premise 1.]

5. “There’s a perfect God” is logically compatible with T.
   T logically entails “There’s evil in the world.”
   ∴ “There’s a perfect God” is logically compatible with “There’s evil in the world.” [Use G, T, and E. Here T (for “theodicy”) is a possible explanation of why God permits evil that’s consistent with God’s perfection and entails the existence of evil. T might say: “The world has evil because God, who is perfect, wants us to make significant free choices to struggle to bring a half-completed world toward its fulfillment; moral evil comes from the abuse of human freedom and physical evil from the half-completed state of the world.” The basic argument (but not the specific T) is from Alvin Plantinga.]

6. “There’s a perfect God and there’s evil in the world and God has some reason for permitting the evil” is logically consistent.
   ∴ “There’s a perfect God and there’s evil in the world” is logically consistent. [Use G, E, and R. This is Ravi Zacharias’s version of Plantinga’s argument.]

7. God is omnipotent.
   “You freely always do the right thing” is logically possible.
   If “You freely always do the right thing” is logically possible and God is omnipotent, then it’s possible for God to bring it about that you freely always do the right thing.
   ∴ It’s possible for God to bring it about that you freely always do the right thing. [Use O, F, and B. This argument is from J.L.Mackie. He thought God had a third option

besides making robots who always act rightly and free beings who sometimes act wrongly: he could make free beings who always act rightly.]

8. "God brings it about that you do A" is inconsistent with "You freely do A."
"God brings it about that you freely do A" entails "God brings it about that you do A."
"God brings it about that you freely do A" entails "You freely do A."
∴ It's impossible for God to bring it about that you freely do A. [Use B, F, and G. This attacks the conclusion of the previous argument.]

9. "This is a square" entails "This is composed of straight lines."
"This is a circle" entails "This isn't composed of straight lines."
∴ "This is a square and also a circle" is self-contradictory. [Use S, L, and C.]

10. "This is red and there's a blue light that makes red things look violet to normal observers" entails "Normal observers won't sense redness."
"This is red and there's a blue light that makes red things look violet to normal observers" is logically consistent.
∴ "This is red" doesn't entail "Normal observers will sense redness." [Use R, B, and N. This argument is from Roderick Chisholm.]

11. "All brown dogs are brown" is a necessary truth.
"Some dog is brown" isn't a necessary truth.
"Some brown dog is brown" entails "Some dog is brown."
∴ "All brown dogs are brown" doesn't entail "Some brown dog is brown." [Use A for "All brown dogs are brown," X for "Some dog is brown," and S for "Some brown dog is brown." This attacks a doctrine of traditional logic (see Section 2.8), that "all A is B" entails "some A is B."]

12. It's necessary that, if God exists as a possibility and not in reality, then there could be a being greater than God (namely, a similar being that also exists in reality).
"There could be a being greater than God" is self-contradictory (since "God" is defined as "a being than which no greater could be").
It's necessary that God exists as a possibility.
∴ It's necessary that God exists in reality. [Use P for "God exists as a possibility," R for "God exists in reality," and C for "There could be a being greater than God." This is a modal version of St Anselm's ontological argument.]

13. If "X is good" and "I like X" are interchangeable, then "I like hurting people" logically entails "Hurting people is good."
"I like hurting people but hurting people isn't good" is consistent.
∴ "X is good" and "I like X" aren't interchangeable. [Use I, L, and G. This argument attacks subjectivism.]

14. "You sin" entails "You know what you ought to do and you're able to do it and you don't do it."

It's necessary that if you know what you ought to do then you want to do it.
It's necessary that if you want to do it and you're able to do it then you do it.
∴ It's impossible for you to sin. [Use S, K, A, D, and W.]

15. Necessarily, if it is true that there are no truths then there are truths.
∴ It is necessary that there are truths. [Use T for "There are truths."]

7.3 Refutations

Applying our proof strategy to an invalid argument leads to a refutation:

It's possible that it's not heads.
It's necessary that it's heads or tails.
∴ It's necessary that it's tails.

*	1	$\Diamond \sim H$	
	2	$\Box(H \vee T)$	
		[∴ $\Box T$	
*	3	asm: $\sim\Box T$	
*	4	∴ $\Diamond \sim T$	{from 3}
	5	W ∴ $\sim T$	{from 4}
	6	WW ∴ $\sim H$	{from 1}
*	7	W ∴ $(H \vee T)$	{from 2}
*	8	WW ∴ $(H \vee T)$	{from 2}
	9	W ∴ H	{from 5 and 7}
	10	WW ∴ T	{from 6 and 8}

Invalid

W	H, ~T
WW	T, ~H

After making the assumption (line 3), we reverse a squiggle to move the modal operator to the outside (line 4). Then we drop the two diamonds, using a new and different world each time (lines 5 and 6). We drop the box twice, using first world W and then world WW (lines 7 and 8). Since we reach no contradiction, we gather the simple pieces to give a refutation. Here our refutation has two possible worlds: one with heads-and-not-tails and another with tails-and-not-heads. Presumably, our refutation will make premises true and conclusion false, and thus show the argument to be invalid. But how are we to calculate the truth value of the premises and conclusion?

In evaluating the premises and conclusion, evaluate each wff (or part of a wff) that starts with a diamond or box according to these rules:

"◊A" is true if and only if *at least one world* has "A" true.	"□A" is true if and only if *all worlds* have "A" true.

Recall the galaxy of possible worlds that we reached for our last argument:

W	H, ~T
WW	T, ~H

This galaxy makes our premises true and conclusion false. Here's how we'd evaluate each wff:

◊~H ← First premise: by our rule, this is true if and only if *at least one world* has "~H" true.
But world WW has "~H" true.
1 ← So "◊~H" is true.
□(H∨T) ← Second premise: by our rule, this is true if and only if *all worlds* have "(H∨T)" true.
In world W: (H∨T)=(1∨0)=1.
In world WW: (H∨T)=(0∨1)=1.
1 ← So "□(H∨T)" is true.
□T ← Conclusion: by our rule, this is true if and only if *all worlds* have T true.
But world W has T false.
0 ← So "□T" is false.

So this galaxy of possible worlds shows our argument to be invalid.

If our refutation had neither "H" nor "~H" in world W, then the value of H in world W would be "?" (unknown). This would happen in our example if we neglected to go from "□(H∨T)" to "W ∴ (H∨T)."

As before, it's important to check that our refutation works. If we don't get premises all true and conclusion false, then we did something wrong—and we should check what we did with the formula that didn't come out right.

So far we've evaluated formulas that begin with a diamond or box, like "◊~H" or "□(H∨T)." But some formulas, like "(◊H⊃□T)," have the diamond or box further inside. In these cases, we'd first evaluate parts of the wff that begin with a diamond or box, and then substitute "1" or "0" for these parts. With "(◊H⊃□T)," we'd first evaluate "◊H" and "□T" to see whether these are "1" or "0"; then we'd substitute "1" or "0" for these parts and determine the truth value of the whole formula.

I'll give examples to show how this works. Let's assume the same galaxy:

W	H, ~T
WW	T, ~H

Now let's evaluate three sample wffs:

~□H ← Here we'd first evaluate "□H." This is true if and only if *all worlds* have H true.
Since world WW has H false, "□H" is false.
~0 ← So we substitute "0" for "□H."
1 ← So "~□H" is true.

(◊H⊃□T)	←	Here we'd first evaluate "◊H" and "□T." "◊H" is true if and only if *at least one world* has H true. Since world W has H true, "◊H" is true. "□T" is true if and only if *all worlds* have T true. Since world W has T false, "□T" is false.
(1⊃0)	←	So we substitute "1" for "◊H" and "0" for "□T."
0	←	So "(◊H⊃□T)" is false.
~□(H⊃~T)	←	Here we'd first evaluate "□(H⊃~T)." This is true if and only if *all worlds* have "(H⊃~T)" true. In world W: (H⊃~T)=(1⊃~0)=(1⊃1)=1. In world WW: (H⊃~T)=(0⊃~1)=(0⊃0)=1.
~1	←	So "□(H⊃~T)" is true; so we substitute "1" for it.
0	←	So "~□(H⊃~T)" is false.

The key thing is to evaluate each part starting with a diamond or box, and then substitute "1" or "0" for it.

In working out English modal arguments, you'll sometimes find an ambiguous premise. Premise 1 is ambiguous in the following argument:

> If you're a bachelor, then you must be unmarried.
> You're a bachelor.
> ∴ It's logically necessary that you're unmarried.

Premise 1 could have either of these two meanings:

(B⊃□U)	"If you're a bachelor, then you're *inherently unmarriable*—in no possible world would anyone ever marry you." (We hope this is false.)
□(B⊃U)	"It's necessary that *if* you're a bachelor *then* you're unmarried." (This is trivially true because "bachelor" means *unmarried man*.)

In such cases, say that the argument is ambiguous and work out both versions:

Box-inside version:

```
* 1   (B ⊃ □U)   Valid
  2   B
     [ ∴ □U
  3  ┌ asm: ~□U
  4  └ ∴ □U  {from 1 and 2}
  5   ∴ □U  {from 3; 3 contradicts 4}

     (While this is valid,
      premise 1 is false.)
```

Box-outside version:

```
  1   □(B ⊃ U)                    Invalid
  2   B                          ┌────────┐
     [ ∴ □U                      │  B, U  │
* 3   asm: ~□U                 W ├────────┤
* 4   ∴ ◇~U   {from 3}           │ ~B, ~U │
  5   W ∴ ~U   {from 4}          └────────┘
* 6   W ∴ (B ⊃ U)   {from 1}
* 7   ∴ (B ⊃ U)   {from 1}
  8   W ∴ ~B   {from 5 and 6}
  9   ∴ U   {from 2 and 7}
```

Both versions are flawed; the first has a false premise, while the second is invalid. So the proof that you're inherently unmarriable ("□U"—"It's logically necessary that you're unmarried") fails.

In the second version, the refutation uses an actual world and a possible world W. An unmodalized instance of a letter, like B in premise 2, should be evaluated according to the actual world; so here B is true. Premise 1 "$\Box(B \supset U)$" also comes out true, since "$(B \supset U)$" is true in both worlds:

In the actual world:	$(B \supset U)=(1 \supset 1)=1$
In world W:	$(B \supset U)=(0 \supset 0)=1$

And conclusion "$\Box U$" is false, since "U" is false in world W. So the galaxy makes the premises all true and conclusion false, establishing invalidity.

Arguments with a modal ambiguity, like this one, often have one interpretation that has a false premise and another that is invalid. Such arguments often seem sound until we focus on the ambiguity.

7.3a Exercise—LogiCola KI

Prove each of these arguments to be invalid (all are invalid).

$\Box(A \supset B)$
$\Diamond A$
$\therefore \Box B$

	1	$\Box(A \supset B)$	
*	2	$\Diamond A$	
		$[\therefore \Box B$	
*	3	asm: $\sim\Box B$	
*	4	$\therefore \Diamond\sim B$	{from 3}
	5	$W \therefore \sim B$	{from 4}
	6	$WW \therefore A$	{from 2}
*	7	$W \therefore (A \supset B)$	{from 1}
*	8	$WW \therefore (A \supset B)$	{from 1}
	9	$W \therefore \sim A$	{from 5 and 7}
	10	$WW \therefore B$	{from 6 and 8}

Invalid

W	$\sim A, \sim B$
WW	A, B

1. $\Diamond A$
   $\therefore \Box A$

2. A
   $\therefore \Box A$

3. $\Diamond A$
   $\Diamond B$
   $\therefore \Diamond(A \cdot B)$

4. $\Box(A \supset \sim B)$
   B
   $\therefore \Box\sim A$

5. $(\Box A \supset \Box B)$
   $\therefore \Box(A \supset B)$

6. $\Diamond A$
   $\sim\Box B$
   $\therefore \sim\Box(A \supset B)$

7. $\Box(C \supset (A \vee B))$
   $(\sim A \cdot \Diamond\sim B)$
   $\therefore \Diamond\sim C$

8. $\Diamond A$
   $\sim\Box B$
   $\therefore \sim\Box(A \supset B)$

9. $\Box((A \cdot B) \supset C)$
   $\Diamond A$
   $\Diamond B$
   $\therefore \Diamond C$

10. $\sim\Box A$
    $\Box(B \equiv A)$
    $\therefore \sim\Diamond B$

7.3b Exercise—LogiCola KC

First appraise intuitively. Then translate into logic (using the letters given) and say whether valid (and give a proof) or invalid (and give a refutation). Translate ambiguous English arguments both ways and work out each symbolization separately.

1. If the pragmatist view of truth is right, then "A is true" entails "A is useful to believe."
   "A is true but not useful to believe" is consistent.
   ∴ The pragmatist view of truth isn't right. [Use P, T, and B.]

2. You know.
   "You're mistaken" is logically possible.
   ∴ "You know and are mistaken" is logically possible. [Use K and M.]

3. Necessarily, if this will be then this will be.
   ∴ If this will be, then it's necessary (in itself) that this will be. [Use B. This illustrates two senses of "Que será será"—"Whatever will be will be." The first sense is a truth of logic while the second is a form of fatalism.]

4. If I'm still, then it's necessary that I'm not moving.
   I'm still.
   ∴ "I'm not moving" is a logically necessary truth. (In other words, I'm logically incapable of moving.) [Use S and M. This is adapted from the medieval thinker Boethius, who used a similar example to explain the box-inside/box-outside distinction.]

5. It's necessarily true that if you're morally responsible for your actions then you're free.
   It's necessarily true that if your actions are uncaused then you aren't morally responsible for your actions.
   ∴ "You're free" doesn't entail "Your actions are uncaused." [Use R, F, and U. This argument is from A.J.Ayer.]

6. If "One's conscious life won't continue forever" entails "Life is meaningless," then a finite span of life is meaningless.
   If a finite span of life is meaningless, then an infinite span of life is meaningless.
   If an infinite span of life is meaningless, then "One's conscious life will continue forever" entails "Life is meaningless."
   ∴ If it's possible that life is not meaningless, then "One's conscious life won't continue forever" doesn't entail "Life is meaningless." [Use C, L, F, and I.]

7. If you have money, then you couldn't be broke.
   You could be broke.
   ∴ You don't have money. [Use M and B. Is this argument just a valid instance of *modus tollens:* "(P⊃Q), ~Q ∴ ~P"?]

8. If you know, then you couldn't be mistaken.
   You could be mistaken.

∴ You don't know. [Use K and M. Since we could repeat this reasoning for any alleged item of knowledge, the argument seems to show that genuine knowledge is impossible.]

9. It's necessary that if there's a necessary being then "There's a necessary being" (by itself) is necessary.
"There's a necessary being" is logically possible.
∴ "There's a necessary being" is logically necessary. [Use N for "There's a necessary being" or "There's a being that exists of logical necessity"; this being is often identified with God. This argument is from Charles Hartshorne and St Anselm; it's sometimes called "Anselm's second ontological argument." The proof raises logical issues that we'll deal with in the next chapter.]

10. It's necessary that either I'll do it or I won't do it.
∴ Either it's necessary that I'll do it, or it's necessary that I won't do it. [Use D for "I'll do it." Aristotle and the stoic Chrysippus discussed this argument for fatalism (which claims that every event happens of inherent necessity). Chrysippus thought this argument was fallacious and was like arguing "Everything is either A or non-A; hence either everything is A or everything is non-A."]

11. "This agent's actions were all determined" is consistent with "I describe this agent's character in an approving way."
"I describe this agent's character in an approving way" is consistent with "I praise this agent."
∴ "This agent's actions were all determined" is consistent with "I praise this agent." [Use D, A, and P.]

12. If thinking is just a chemical brain process, then "I think" entails "There's a chemical process in my brain."
"There's a chemical process in my brain" entails "I have a body."
"I think but I don't have a body" is logically consistent.
∴ Thinking isn't just a chemical brain process. [Use J, T, C, and B. This argument attacks a form of materialism.]

13. If "I did that on purpose" entails "I made a prior purposeful decision to do that," then there's an infinite chain of previous decisions to decide.
It's impossible for there to be an infinite chain of previous decisions to decide.
∴ "I did that on purpose" is consistent with "I didn't make a prior purposeful decision to do that." [Use D, P, and I. This argument is from Gilbert Ryle.]

14. God knew that you'd do it.
If God knew that you'd do it, then it was necessary that you'd do it.
If it was necessary that you'd do it, then you weren't free.
∴ You weren't free. [Use K, D, and F. This argument is the focus of an ancient controversy. Would divine foreknowledge preclude human freedom? If it would,

then should we reject human freedom (as did Luther) or divine foreknowledge (as did Charles Hartshorne)? Or perhaps (as the medieval thinkers Boethius, Aquinas, and Ockham claimed) is the argument that divine foreknowledge precludes human freedom itself fallacious?]

15. If "good" means "socially approved," then "Racism is socially approved" logically entails "Racism is good."
"Racism is socially approved but not good" is consistent.
∴ "Good" doesn't mean "socially approved." [Use M, S, and G. This argument attacks cultural relativism.]

16. Necessarily, if God brings it about that A is true, then A is true.
A is a self-contradiction.
∴ It's impossible for God to bring it about that A is true. [Use B and A, where B is for "God brings it about that A is true."]

17. If this is experienced, then this must be thought about.
"This is thought about" entails "This is put into the categories of judgments."
∴ If it's possible for this to be experienced, then it's possible for this to be put into the categories of judgments. [Use E, T, and C. This argument is from Immanuel Kant, who argued that our mental categories apply, not necessarily to everything that exists, but rather to everything that we could experience.]

18. Necessarily, if formula B has an all-1 truth table then B is true.
∴ If formula B has an all-1 truth table, then B (taken by itself) is necessary. [Use A and B. This illustrates the box-outside versus box-inside distinction.]

19. Necessarily, if you mistakenly think that you exist then you don't exist.
Necessarily, if you mistakenly think that you exist then you exist.
∴ "You mistakenly think that you exist" is impossible. [Use M and E. This relates to Descartes's "I think, therefore I am" *("Cogito ergo sum")*.]

20. If "good" means "desired by God," then "This is good" entails "There's a God."
"There's no God, but this is good" is consistent.
∴ "Good" doesn't mean "desired by God." [Use M, A, and B. This attacks one form of the divine command theory of ethics. Some (see problem 9 of this section and problem 14 of Section 7.2b) would dispute premise 2 and say that "There's no God" is logically impossible.]

21. If Plato is right, then it's necessary that ideas are superior to material things.
It's possible that ideas aren't superior to material things.
∴ Plato isn't right. [Use P and S.]

22. "I seem to see a chair" doesn't entail "There's some actual chair that I seem to see."

If we directly perceive material objects, then "I seem to see a chair and there's some actual chair that I seem to see" is consistent.
∴ We don't directly perceive material objects. [Use S, A, and D.]

23. "There's a God" is logically incompatible with "There's evil in the world."
There's evil in the world.
∴ "There's a God" is self-contradictory. [Use G and E.]

24. If you do all your homework right, then it's impossible that you get this problem wrong.
It's possible that you get this problem wrong.
∴ You don't do all your homework right. [Use R and W.]

25. "You do what you want" is compatible with "Your act is determined."
"You do what you want" entails "Your act is free."
∴ "Your act is free" is compatible with "Your act is determined." [Use W, D, and F.]

26. It's necessarily true that if God doesn't exist in reality then there's a being greater than God.
It's not possible that there's a being greater than God (since "God" is defined as "a being than which no being could be greater").
∴ It's necessary that God exists in reality. [Use R and B. This is a simplified modal form of St Anselm's ontological argument.]

27. It was always true that you'd do it.
If it was always true that you'd do it, then it was necessary that you'd do it.
If it was necessary that you'd do it, then you weren't free.
∴ You weren't free. [Use A (for "It was always true that you'd do it"—don't use a box here), D, and F. This argument is much like problem 14. Are statements about future contingencies (for example, "I'll brush my teeth tomorrow") true or false before they happen? Should we do truth tables for such statements in the normal way, assigning them "1" or "0"? Does this preclude human freedom? If so, should we then reject human freedom? Or should we claim that statements about future contingencies aren't "1" or "0" but must instead have some third truth value (maybe "1/2")? Or is the argument fallacious?]