1. For each of the following statements, circle **T** if the statement is always true and **F** if it can be false. Give a one-sentence justification for your answer.

[2] (a) Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be non-zero vectors in $\mathbb{R}^3$. If $\vec{u}$ and $\vec{v}$ are both orthogonal to $\vec{w}$, then $\vec{u}$ is parallel to $\vec{v}$.

Solution: False. For example, $\vec{u} = [1, 0, 0]$, $\vec{v} = [0, 1, 0]$ and $\vec{w} = [0, 0, 1]$ are all orthogonal.

[2] (b) Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$. Then $\|\vec{u} - \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$.

Solution: True. This follows from the triangle inequality. A picture would be a sufficient explanation.

[2] (c) The planes $2x - 3y + z = 4$ and $-4x + 6y - 2z = 1$ in $\mathbb{R}^3$ are parallel.

Solution: True. The normal vector of the second one is $[-4, 6, -2]$, which is twice the normal vector of ~~the first one~~.

[2] (d) Let A denote the coefficient matrix of a system of 4 linear equations in 4 unknowns. If the rank of A is 3, then this system has infinitely many solutions.

Solution: False. The system may not be consistent.

[2] 2. Given that $\vec{u} \cdot \vec{v} = 0$, $\vec{u} \cdot \vec{w} = 1$, $\vec{v} \cdot \vec{w} = 2$ and $\|v\| = 1$, compute $(2\vec{u} + \vec{v}) \cdot (2\vec{v} + 3\vec{w})$.

Solution:

$$\begin{aligned}(2\vec{u} + \vec{v}) \cdot (2\vec{v} + 3\vec{w}) &= 2\vec{u} \cdot (2\vec{v} + 3\vec{w}) + \vec{v} \cdot (2\vec{v} + 3\vec{w}) \\ &= 4\vec{u} \cdot \vec{v} + 6\vec{u} \cdot \vec{w} + 2\vec{v} \cdot \vec{v} + 3\vec{v} \cdot \vec{w} \\ &= 4(0) + 6(1) + 2\|\vec{v}\|^2 + 3(2) = 14.\end{aligned}$$

3. Let $\vec{u} = [1, \sqrt{2}, 1]$ and $\vec{v} = [0, 0, 1]$ be vectors in $\mathbb{R}^3$.

[2] (a) Find the unit vector in the same direction as $\vec{u}$.

Solution:

$$\|\vec{u}\| = \sqrt{1^2 + \sqrt{2}^2 + 1^2} = \sqrt{4} = 2$$

so the unit vector in the same direction as $\vec{u}$ is

$$\frac{1}{2}\vec{u} = \frac{1}{2}[1, \sqrt{2}, 1] = \left[\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}\right].$$

[2] (b) Compute the angle between $\vec{u}$ and $\vec{v}$.

Solution: The cosine of the angle θ between $\vec{u}$ and $\vec{v}$ is

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{1}{2 \cdot 1} = 1/2.$$

So the angle is $60°$.

4. Let ℓ be the line through the points $P = (1, 2)$ and $Q = (5, 5)$.

[2] (a) Find a direction vector for the line ℓ and write parametric equations of the line ℓ.

Solution: A direction vector for ℓ is $\vec{d} = \vec{PQ} = [4, 3]$, and the position vector for P is $\vec{p} = [1, 2]$. So the parametric equations are

$$\begin{aligned} x &= 1 + 4t \\ y &= 2 + 3t. \end{aligned}$$

[4] (b) Find the distance from the point $R = (6, 12)$ to the line ℓ.

Solution: The distance is

$$d(R, \ell) = \|\vec{v} - \mathrm{proj}_{\vec{d}}(\vec{v})\|,$$

where $\vec{v} = \vec{PR} = [5, 10]$. We compute

$$\mathrm{proj}_{\vec{d}}(\vec{v}) = \frac{\vec{d} \cdot \vec{v}}{\vec{d} \cdot \vec{d}}\vec{d} = \frac{50}{25}[4, 3] = [8, 6]$$

and

$$\vec{v} - \mathrm{proj}_{\vec{d}}(\vec{v}) = [5, 10] - [8, 6] = [-3, 4],$$

which has length $\sqrt{(-3)^2 + 4^2} = 5$.

5. Let $\mathcal{P}$ be the plane in $\mathbb{R}^3$ given by the parametric equations

$$\begin{aligned} x &= -5 + s \\ y &= - 2s + t \\ z &= 1 + 6s - 3t \end{aligned}$$

[3] (a) Find a normal vector to the plane $\mathcal{P}$.

Solution: Directions vectors for $\mathcal{P}$ are $\vec{u} = [1, -2, 6]$ and $\vec{v} = [0, 1, -3]$. One way to get a vector orthogonal to both of these is to use the cross product:

$$\vec{n} = \vec{u} \times \vec{v} = [-2(-3) - 6(1), 6(0) - 1(-3), 1(1) - (-2)(0)] = [0, 3, 1].$$

(Note that this can be easily checked!)

[2] (b) Find a general equation for the plane $\mathcal{P}$.

Solution: We compute $\vec{n} \cdot \vec{x} = [0, 3, 1] \cdot [x, y, z] = 3y + z$, so the equation is of the form $3y + z = d$. Since $(-5, 0, 1)$ is a point on $\mathcal{P}$ (taking $s = t = 0$ in the parametric equations), we determine that $d = 3(0) + 1(1) = 1$, so the answer is $3y + z = 1$.

[3] (c) Give the general equation for a plane $\mathcal{P}'$ that intersects $\mathcal{P}$ in a line, and explain how you know that the intersection is exactly a line.

Solution: We can choose *any* plane whose normal vector is not parallel to $\vec{n}$, For example, $x = 0$ will work, or $x + y + z = 17$, or many others. If the normal vectors are not parallel, then the planes are not parallel, so they must intersect in a line.

6. Recall that the Universal Product Code (UPC) uses code words in $\mathbb{Z}_{10}^{12}$ and has check vector $\vec{c} = [3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1]$.

[3] (a) Find the missing digit y in the UPC $[0, 4, 3, 7, 0, 6, 5, y, 9, 1, 2, 1]$.

Solution: Writing $\vec{v}$ for $[0, 4, 3, 7, 0, 6, 5, y, 9, 1, 2, 1]$, we compute that

$$\vec{c} \cdot \vec{v} = 0 + 4 + 9 + 7 + 0 + 6 + 15 + y + 27 + 1 + 6 + 1 = 6 + y \pmod{10}$$

So to get 0 modulo 10, we need to take $y = 4$.

[2] (b) Find a valid UPC code with only one non-zero digit, or explain why this is not possible.

Solution: This is not possible. If there is only one non-zero digit y, then $\vec{c} \cdot \vec{v}$ would equal either y or $3y$, and we would need this to be a multiple of 10. But for $y = 1, 2, \ldots, 9$, neither y nor $3y$ is a multiple of 10.

7. Consider the system of linear equations

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ 2x + y + z &= 5 \\ x + 4y - 3z &= -1 \end{aligned}$$

[1] (a) Write down the augmented matrix of this linear system.

Solution:

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 2 & 1 & 1 & 5 \\ 1 & 4 & -3 & -1 \end{array}\right]$$

[3] (b) Compute the reduced row-echelon form of the augmented matrix above. Indicate all elementary row operations that you are performing.

Solution: Row reduction leads to:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The details must be shown. Common mistakes:
1) Getting to row echelon form, but not **reduced** row echelon form.
2) Arithmetic errors. If you find messy fractions, this is a hint that you made a mistake. Be careful!
3) Disorganized approach. Follow the guidelines when doing row reduction, clearing one column at a time.
4) Using row operations that are not one of the elementary row operations given in the text, e.g., $3R_1 + 4R_2$.

[2] (c) Use the result of the previous part to find all solutions of the linear system.

Solution: x and y are leading variables, and z is a free variable, so we get:

$$\begin{aligned} x &= 3 - t \\ y &= -1 + t \\ z &= t \end{aligned}$$

[1] (d) What is the rank of the augmented matrix you found in part (a)?

Solution: It has rank 2, because there are two nonzero rows in the reduced row echelon form.