| Question: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Total |
|---|---|---|---|---|---|---|---|---|---|
| Points: | 8 | 5 | 5 | 6 | 7 | 7 | 9 | 8 | 55 |
| Score: | | | | | | | | | |

Name (print): _____

Signature: _____

UWO ID number: _____

The UNIVERSITY of WESTERN ONTARIO
DEPARTMENT of COMPUTER SCIENCE

CS 2214B, Winter 2022 - FINAL EXAM

24 April 2022

# VERSION B

INSTRUCTIONS:

1. This exam is 8 pages long. It is printed single-sided. There are 8 questions.
2. All questions must be answered in the space provided. Write your answer clearly.
3. You can use the back of each page as scrap paper for your calculations. Only what you write on the **front** pages is going to be marked.
4. Show all your of your work and **justify your answers fully**. Unjustified, irrelevant or illegible answers will receive little or no credit.
5. Do not unstaple the exam booklet.
6. **No aids are permitted. In particular, calculators, phones and other electronics are not allowed and may be confiscated.**

# QUESTIONS

1. (a) (4 points) Write the truth table of the formula $A = ((a \to b) \to \neg a) \to a$.

(b) (1 point) Is the formula a tautology, a contingency, or a contradiction?

(c) (1 point) Is the formula satisfiable? If so, list the interpretations which make the formula true.

(d) (2 points) Find a formula which is logically equivalent to $A$ but is in CNF (conjunctive normal form), and prove the logical equivalence with a method of your choice.

2. (5 points) Let

$$M = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}.$$

Find a formula for the powers $M^n$ ($n \in \mathbb{N}$) and prove it by mathematical induction.

3. Write in roster notation the sets

(a) (1 point) $A = \{x | x \in \mathbb{Z} \wedge x^2 < 5\}$

(b) (1 point) $B = \{x | x$ is a vowel of the expression "epic failure in the final exam"$\}$

(c) (1 point) $A \times B \times (A \cap B)$.

(d) (1 point) $P(B \setminus A)$.

(e) (1 point) $A \cup B$

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4. Let $A = \{-2, -1, 0, 1, 2\}$.

   (a) (2 points) Represent the relation $R = \{(0,0), (0,1), (0,-1), (2,2), (2,-2)\}$ with a directed graph.

   (b) (2 points) Represent the symmetric closure $S_R$ of $R$ (i.e., the smallest symmetric relation on $A$ containing $R$) in roster notation.

   (c) (2 points) Is $R$ a function? (Remember to justify your answer)

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5. Let $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = |x|$, $\quad g : \mathbb{Z} \to \mathbb{Q}$, $g(x) = 1/(x^2 + 1)$.

   (a) (3 points) Determine the properties of $f$ and $g$ (that is, if they are injective, surjective, bijective)

   (b) (1 point) Determine, if it exists, the function $f^{-1}$

   (c) (1 point) Determine, if it exists, the function $g^{-1}$

   (d) (2 points) Determine, if they exist, the functions $f \circ g$ and $g \circ f$

6. Let $G = (V, E)$ be the graph with vertex set $V = \{0, 1, 2, 3, 4\}$ and incidence matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}$$

(a) (2 points) Draw $G$.

(b) (2 points) What are the strong and weak connected components of $G$? Is $G$ strongly or weakly connected?

(c) (2 points) Does G have any Hamilton circuits? If so, show one; if not, explain why.

(d) (1 point) Write the sequence of in-degrees and the sequence of out-degrees of $G$.

7. (a) (5 points) Solve the modular congruence $20(21x - 20) \equiv -376(739x + 360) \mod 369$.

(b) (1 point) What is the $lcm(58, 369)$?

(c) (3 points) With respect to a crazy long alphabet of 369 characters, would $\mathbf{e} : \mathbb{Z}_{369} \to \mathbb{Z}_{369}$, $\mathbf{e}(x) = 58x + 275 \mod 369$ be a well-defined encryption function for an affine cipher? If so, find the corresponding decryption function; if not, explain why.

8. An urn contains 10 red balls, numbered from 0 to 9, and 9 blue balls, marked with letters from $a$ to $i$. The balls are otherwise identical.

   (a) (1 point) How many combinations of 2 letters and 3 digits (in any order) can be formed with the balls in the urn?

   (b) (1 point) How many combinations of 2 letters followed by 3 digits can be formed with the balls in the urn?

   (c) (1 point) If we randomly extract 6 balls at once, how many ways are there to arrange the corresponding characters into a passcode for a bank vault?

   (d) (1 point) What is the probability that a randomly picked ball features a prime number?

(e) (1 point) For 12 times, we extract a ball from the urn, we record its features and we put it back in the urn. What is the probability that at most 2 balls are red?

(f) (3 points) We pick a ball from the urn, we record its features, we discard it and then we pick another ball. Are the events E = "the 1st ball is blue" and F = "the 2nd ball has an even number" dependent or independent? What is the probability that the 2nd ball has an even number, knowing that the 1st ball is blue?