A summary

Homogeneous equations

Review

- Matrix (definition, dimensions)

An $m \times n$ matrix A is a rectangular array of entries

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

where m is the number of rows and n is the number of column.

RREF

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The relation between a general matrix and an RREF is:
Theorem A Every matrix can be transformed into a unique RREF by elementary row operations.

Do ask me if you are not sure how to do the row reduction.

Rank of a matrix

Definition The *rank* of a matrix is the number of leading ones in the RREF of A i.e., equals to the non-zero rows in the RREF of A. We can use $r(A)$ to denote the rank of matrix A.

Also, we say that the $m \times n$ matrix A has *full rank* if $r(A) = n$, i.e. if every column of the RREF of A contains the leading one for some row.

SLE and Matrix equations

Consider m linear equations with n variables x_1, x_2 ..., x_n

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

The augmented matrix of an SLE is

$$
\left[A \,\middle|\, \vec{b} \right] = \left[\begin{array}{cccc|c}
a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\
a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn} & b_m
\end{array} \right]
$$

SLE and Matrix equation

We have a matrix equation to represent the SLE.

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{b}$$

where $\vec{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$ and $\vec{b} = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix}^T$.

- The number of rows of A equals to the number of linear equations in SLE.
- The number of columns of A equals to the number of variables of SLE.

Solutions of SLE

For any system of linear equations (or a matrix equation) there are exactly 3 possibilities:

- no solution, or
- a unique solution, or
- infinitely many solutions.

The method to determine which situation of solutions an SLE (or a matrix equation) $A\vec{x} = \vec{b}$ has, is to compare the rank of the coefficient matrix A and the rank of the augmented matrix $\left[A \mid \vec{b}\right]$.

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Explicitly, We have the following theorem

Theorem B Then the SLE $A\vec{x} = \vec{b}$ has:

- no solution if $r(A) < r(\left[A \mid \vec{b}\right])$
- a unique solution if $r(A) = r(\left[A \mid \vec{b}\right]) = n$
- infinitely many solutions if $r(A) = r(\left[A \mid \vec{b}\right])$ and $r(A) < n$.

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For an SLE, we start with its augmented matrix $\left[A \mid \vec{b}\right]$.

Reduce the augmented matrix

$$\left[A \mid \vec{b}\right] \xrightarrow{RREF} \left[R \mid \vec{\beta}\right]$$

where R is the RREF of A and $\left[R \mid \vec{\beta}\right]$ is the RREF of $\left[A \mid \vec{b}\right]$.

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where R is the RREF of A and $\left[R \mid \vec{\beta}\right]$ is the RREF of $\left[A \mid \vec{b}\right]$.

By the definition, the rank of A equals to the number of leading ones in R and the rank of $\left[A \mid \vec{b}\right]$ equals to the number of leading ones in $\left[R \mid \vec{\beta}\right]$.

The case: no solution

- $A\vec{x} = \vec{b}$ has no solution if $r(A) < r(\left[R \mid \vec{\beta}\right])$

Observe the matrix $\left[R \mid \vec{\beta}\right]$.

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In $\left[R \mid \vec{\beta}\right]$, $r(A) < r(\left[R \mid \vec{\beta}\right])$ is equivalent to that there is a row looking like

$$\left[\begin{array}{cccc|c} 0 & 0 & \dots & 0 & 1 \end{array}\right]$$

which corresponds to the equation $0 = 1$.
Hence, in this case, the SLE has no solution.

The case: has solution(s)

If not, it means $r(A) = r(\begin{bmatrix} R & | & \vec{\beta} \end{bmatrix})$. In this case, the SLE has solution(s).

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The number of solutions relies on the difference $n - r(A)$.

- If $n = r(A)$ (i.e., A is full rank), then the SLE has unique solution.
- If $r(A) < n$, then the SLE has an $n - r(A)$ parameters of solutions.

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R has the following properties: the first top n rows are non-zero rows each of which has a leading one. The total number of these leading ones is n which is exactly the number of columns of R. So each colunm of R has a leading one.

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$$R = \begin{bmatrix} I_n \\ \mathbf{0} \end{bmatrix}$$

where $\mathbf{0}$ is an $(m-n) \times n$ zero matrix.

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How about $\vec{\beta}$? The components at $n+1, \ldots, m$ of $\vec{\beta}$ must be zero. Otherwise, any nonzero element at one of these components will contribute a leading one, which will induce the rank of $\left[R \mid \vec{\beta}\right])$ greater than the rank of A. Hence $\vec{\beta} = [\beta_1 \ \ldots \ \beta_n \ 0 \ \ldots \ 0]^T$.

The case: unique solution

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Reason:

Now we know $R = \begin{bmatrix} I_n \\ \mathbf{0} \end{bmatrix}$ and $\vec{\beta} = [\beta_1 \ \ldots \ \beta_n \ 0 \ \ldots \ 0]^T$, where $\mathbf{0}$ is an $(m-n) \times n$ zero matrix.

The augmented matrix $\left[R \mid \vec{\beta}\right]$ represents $R\vec{x} = \vec{\beta}$.

By a substitution R and $\vec{\beta}$, we have $R\vec{x} = \begin{bmatrix} I_n \\ \mathbf{0} \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{x} \\ \vec{0} \end{bmatrix} = \vec{\beta}$ and the unique solution is $\vec{x} = (\beta_1, \ldots, \beta_n)$.

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- If $r(A) = r(\left[R \mid \vec{\beta}\right]) < n$, then the SLE has infinitely many solutions.

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  The number $r = r(A)$ is the number of leading ones in R. Each leading one exists at a certain column of R.
  The variables corresponding to a column which does not have a leading one of R will become parameters for the solutions. Hence, there are $n - r(A)$ parameters.

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The variables corresponding to a column which does not have a leading one of R will become parameters for the solutions. Hence, there are $n - r(A)$ parameters.
Then the variables corresponding to a column which contains a leading one of R can be expressed in a way of linear sums of parameters due to the the linear equations.

Example

For example, the following RREF

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & 5 & 1 \\ 0 & 1 & 0 & 5 & -12 & 3 \\ 0 & 0 & 1 & -3 & 13 & 7 \end{array}\right]$$

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corresponds to

$$\begin{aligned} x_1 \qquad\quad + 3x_4 + 5x_5 &= 1 \\ x_2 \quad + 5x_4 - 12x_5 &= 3 \\ x_3 - 3x_4 + 13x_5 &= 7 \end{aligned}$$

Let $x_4 = s$ and $x_5 = t$ (two parameters). The SLE has solutions

$$(1 - 3s - 5t, 3 - 5s + 12t, 7 + 3s - 13t, s, t).$$

To summarise,
Theorem The SLE $A\vec{x} = \vec{b}$, where A is an $m \times n$ matrix, has:

- no solution if $r(A) < r(\left[A \mid \vec{b}\right])$
- a unique solution if $r(A) = r(\left[A \mid \vec{b}\right]) = n$
- $n - r(A)$ parameters of solutions if $r(A) = r(\left[A \mid \vec{b}\right])$ and $r(A) < n$.

Example

Suppose that the augmented matrix of a linear system is given by

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & x & y \end{array}\right]$$

For what values of x and y is there

(a) No solution?

(b) Exactly one solution?

(c) Infinitly many solutions?

Example

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For what values of x and y is there

(a) No solution? $x = 0$ and $y \neq 0$

(b) Exactly one solution? $x \neq 0$ and any values of y

(c) Infinitly many solutions? $x = 0$ and $y = 0$

Homogeneous equations

Definition The SLE $A\vec{x} = \vec{b}$ is *homogeneous* if $\vec{b} = \vec{0}$, i.e., all the constants on the right hand side are zero.

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For example

$$\begin{aligned} 2x + y - z &= 0 \\ y + z &= 0 \\ 3x + 2z &= 0 \end{aligned}$$

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For example

$$\begin{aligned} 2x + y - \ z &= 0 \\ y + \ z &= 0 \\ 3x \qquad + 2z &= 0 \end{aligned}$$

is homogeneous,but

$$\begin{aligned} 2x + y - \ z &= 0 \\ y + \ z &= 0 \\ 3x \qquad + 2z &= 1 \end{aligned}$$

If $3x + 2z = 0$

✓

is NOT homogeneous.

Solutions of homogeneous equations

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Indeed, without using the Theorem of rank and solutions of SLE, it is not difficult to see that the zero vector $\vec{0}$ is always a solution because $A\vec{0} = \vec{0}$.

Solutions of homogeneous equations

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Indeed, without using the Theorem of rank and solutions of SLE, it is not difficult to see that the zero vector $\vec{0}$ is always a solution because $A\vec{0} = \vec{0}$.
Definition The zero vector $\vec{0}$ is called *trivial solution* of the homogeneous system $A\vec{x} = \vec{0}$.

Solutions of homogeneous equations

The augmented matrix of a homogeneous equation has a form

$$\left[A \,\middle|\, \vec{0}\right].$$

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$$[A \mid \vec{0}] \,.$$

If reduce it, we have

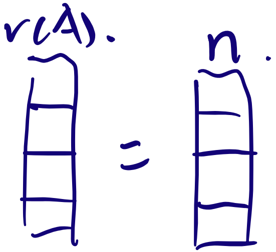
the RREF form of A.

$$[A \mid \vec{0}] \xrightarrow{RREF} [R \mid \vec{0}]$$

Hence $r(A) = r([A \mid \vec{0}])$.
Therefore, any homogenenous equation $A\vec{x} = \vec{0}$ always has solution(s).

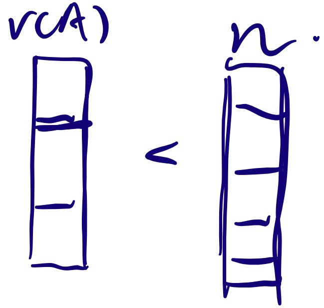
Theorem Every homogeneous equation (or a homogeneous linear system) $A\vec{x} = 0$, where A is an $m \times n$ matrix, has

- either exactly one solution (only the trivial solution) if $r(A) = n$
- or infinitly many solutions if $r(A) < n$.



Because of the property $r(A) \leq m$ and $r(A) \leq n$,

Corollary Any homogeneous SLE in which where the number of variables (i.e. equals to n) is larger than the number of equations (i.e. equals to m) has infinitely many solutions.



If possible, determine how many solutions each of the following SLE's has just from looking at it.

(a)
$$\begin{aligned} 2x + 2y - 5z &= 0 \\ 23x + 14y - z &= 0 \end{aligned}$$

inf

(b)
$$\begin{aligned} 2x + 2y + 5z &= 0 \\ 23x + 14y - z &= 1 \end{aligned}$$

inf.

(d)
$$\begin{aligned} x_1 \qquad\quad + 3x_4 + 5x_5 + 2x_6 &= 1 \\ x_2 \quad + 5x_4 - 3x_5 + 21x_6 &= 3 \\ x_3 - 3x_4 + 7x_5 \qquad\qquad &= 7 \end{aligned}$$

inf.

In the end, we collect all info together and consider a homogeneous SLE $A\vec{x} = \vec{b}$ with n equations and n variables.

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Theorem If A is an $n \times n$ matrix, then the following conditions are equivalent:

1. A is invertible (i.e., nonsingular).
2. $r(A) = n$ (i.e., A has full rank).
3. The RREF of A is I (i.e., A is row-equivalent to the identity matrix I).
4. The system $A\vec{x} = \vec{b}$ has an unique solution (for all $n \times 1$ column vectors $\vec{b}$).
5. The homogeneous system $A\vec{x} = \vec{0}$ has only the trivial solution.