UWO CS2214

# Tutorial #3

**Problem 1** Professor Cuthbert Calculus has designed a machine which consists of three components $A$, $B$, $C$ which are either running or stopped. The constraints on those components are the following:

1. if $A$ is running, then at least one of the components $B$ or $C$ is stopped,
2. if $B$ is stopped, then at least one of the components $A$ or $C$ is running,
3. if $C$ is running, then $B$ is running as well.

Can the machine of Professor Cuthbert Calculus be built, that is, is the conjunction of the above three statements satisfiable. Justify your answer.

**Solution 1** Let us denote by $A$, $B$, $C$ Boolean variables stating that the respective components $A$, $B$, $C$ are running. Then the 3 constraints can be rephrased as follows in propositional logic:

1. $A \quad \rightarrow \quad (\neg B \vee \neg C)$,
2. $\neg B \quad \rightarrow \quad (A \vee C)$,
3. $C \quad \rightarrow \quad B$.

Because of the third constraint, namely $C \ \rightarrow \ B$, it is natural to test whether the conjunction of the three constraints is satisfiable with $B = C = \mathsf{true}$. Then, since $\neg B \vee \neg C = \mathsf{false}$, to satisfy the first constraint, we must have $A = \mathsf{false}$. With those values of the Boolean variables $A$, $B$, $C$, the second constraint is satisfied. Therefore, the the machine of Professor Cuthbert Calculus be built.

**Problem 2** Prove that $\log_2(9)$ is irrational.

**Solution 2** if $\log_2(9)$ were equal to $\frac{m}{n}$, with $m, n$ positive integers, without common factors, then, by the definition of logarithms, we would have

$$2^{\frac{m}{n}} = 9.$$

Raising both sides to the $n$-th power, we obtain:

$$2^m = 9^n.$$

Since $n$ and $m$ are non-zero, the numbers $9^n$ and $2^m$ are greater or equal to 9 and 2, respectively. Moreover, the numbers $9^n$ and $2^m$ are odd and even, respectively. Since a number cannot be both even and odd, the numbers $9^n$ and $2^m$ cannot be equal and we have reached a contradiction. Therefore, the number $\log_2(9)$ is irrational.

**Problem 3** Let $p, q, r, s$ be Boolean variables. For each of the following propositions, determine whether it is satisfiable or not :

1. $(p \vee (q \wedge (q \vee s)) \;\wedge\; (\neg p \vee (\neg q \wedge (\neg q \vee r)) \;\wedge\; (p \vee s) \;\wedge\; (\neg p \vee r)$.
2. $(p \vee (q \wedge (q \vee s)) \;\wedge\; (\neg p \vee (\neg q \wedge (\neg q \vee r)) \;\wedge\; (p \vee \neg q) \;\wedge\; (\neg p \vee q)$

**Solution 3**

1. Using the absorption laws, the sub-expression $(q \wedge (q \vee s))$ can simply be rewritten as $q$ and the sub-expression $(\neg q \wedge (\neg q \vee r))$ can simply be rewritten as $\neg q$. Therefore, the entire proposition becomes
$$(p \vee q) \;\wedge\; (\neg p \vee \neg q) \;\wedge\; (p \vee s) \;\wedge\; (\neg p \vee r).$$
Let us look first at $(p \vee q) \wedge (\neg p \vee \neg q)$. Both $(p \vee q)$ and $(\neg p \vee \neg q)$ are true if and only if $p$ and $q$ have opposite truth values. (This can be verified with a truth table.) Assume we choose $p = \mathsf{true}$ and $q = \mathsf{false}$. Then $(p \vee s)$ is true whatever is the truth value of $s$, meanwhile satisfying $(\neg p \vee r)$ requires to set $r = \mathsf{true}$. Finally, we can conclude that the entire proposition is satisfied with $p = \mathsf{true}$, $q = \mathsf{false}$ and $r = \mathsf{true}$.
2. Here again, $(q \wedge (q \vee s))$ can simply be rewritten as $q$ and $(\neg q \wedge (\neg q \vee r))$ can simply be rewritten as $\neg q$. And the entire proposition becomes
$$(p \vee q) \;\wedge\; (\neg p \vee \neg q) \;\wedge\; (p \vee \neg q) \;\wedge\; (\neg p \vee q).$$
Remember that $(p \vee q) \wedge (\neg p \vee \neg q)$ means $p \leftrightarrow \neg q$, that is, $p$ and $q$ have **opposite** truth values. Similarly, the sub-expression $(p \vee \neg q) \;\wedge (\neg p \vee q)$ means that $p$ and $q$ have the same truth values, that is, $p \leftrightarrow q$. Therefore, the entire proposition becomes
$$(p \leftrightarrow \neg q) \;\wedge\; (p \leftrightarrow q),$$
which is clearly false. Finally, we can conclude that the entire proposition cannot be satisfied.

**Problem 4** For any real number $x$, the *absolute value* of $x$, denoted by $|x|$, is defined as follows:
$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$
Prove that for all real numbers $a, b$, the following properties hold:

1. $|a + b| \leq |a| + |b|$ (called the *triangular inequality*),
2. $|a - b| \geq \big| |a| - |b| \big|$ (called the *reverse triangular inequality*),
3. if $b$ is non-negative then we have: $|a| \leq b \iff -b \leq a \leq b$.

**Solution 4**

1. Let $a, b$ be two real numbers. We consider 4 cases
   - **Case 1:** $a \geq 0$ and $b \geq 0$. Then $a + b \geq 0$ and we have:
   $$|a + b| = a + b = |a| + |b|.$$
   - **Case 2:** $a < 0$ and $b \geq 0$. In this case $a + b = -|a| + b$ and thus the sign of $a + b$ depends on whether $|a| \leq b$ or $|a| > b$ holds. If $|a| \leq b$ holds, then $a + b \geq 0$ and we have:
   $$|a + b| = -|a| + b \leq |a| + b = |a| + |b|.$$
   If $|a| > b$ holds, then $a + b < 0$ and we have:
   $$|a + b| = |a| - b \leq |a| + b = |a| + |b|.$$
   - **Case 3:** $a \geq 0$ and $b < 0$. This case is simply deduced from the previous one by exchanging the role of $a$ and $b$.
   - **Case 4:** $a < 0$ and $b < 0$. Then $a + b < 0$ and we have:
   $$|a + b| = -(a + b) = -|a| - |b| \leq |a| + |b|.$$

   QED. It should be noted, as pointed by one student in class, that other formulas about absolute values can be used to avoid the case discussion. These formulas are
   $$\sqrt{a^2} = |a| \quad \text{and} \quad |a \times b| = |a| \times |b|.$$
   Since $ab \leq |a \times b|$ and $|a \times b| = |a| \times |b|$ both hold, we deduce:
   $$2ab \leq 2|a| \times |b|,$$
   and thus
   $$a^2 + 2ab + b^2 \leq |a|^2 + 2|a| \times |b| + |b|^2,$$
   leading to
   $$(a + b)^2 \leq (|a| + |b|)^2.$$
   Taking the square-root of each side yields:
   $$\sqrt{(a + b)^2} \leq \sqrt{(|a| + |b|)^2},$$
   that is,
   $$|a + b| \leq ||a| + |b|| = |a| + |b|.$$

2. Let $a, b$ be two real numbers. One could proceed again by case inspection, discussing whether $a - b$ is non-negative or not, and discussing whether $|a| - |b|$ is non-negative or not. But there is a faster way, by applying the triangular inequality twice:

- From $a = (a-b) + b$, we deduce

$$|a| \leq |a-b| + |b|,$$

and thus

$$|a| - |b| \leq |a-b|.$$

- From $-b = (a-b) + (-a)$ and $|a| = |-a|$ and $|b| = |-b|$, we deduce

$$|b| \leq |a-b| + |a|,$$

and thus

$$|b| - |a| \leq |a-b|.$$

From $|a| - |b| \leq |a-b|$ and $|b| - |a| \leq |a-b|$, we deduce

$$\big|\, |a| - |b| \,\big| \leq |a-b|$$

Indeed, $\big|\, |a| - |b| \,\big|$ is equal to either $|a| - |b|$ or $|b| - |a|$. QED.

3. Let $a, b$ be two real numbers. We have the following equivalences:

$$\begin{array}{lcl} |a| \leq b & \iff & (a \geq 0 \wedge |a| \leq b) \;\vee\; (a < 0 \wedge |a| \leq b) \\ & \iff & (a \geq 0 \wedge a \leq b) \;\vee\; (a < 0 \wedge -a \leq b) \\ & \iff & (a \geq 0 \wedge a \leq b) \;\vee\; (a < 0 \wedge -b \leq a) \\ & \iff & (a \geq 0 \wedge -b \leq a \leq b) \;\vee\; (a < 0 \wedge -b \leq a \leq b) \\ & \iff & -b \leq a \leq b \end{array}$$

Indeed, for the second last equivalence, we can replace $(a \geq 0 \wedge a \leq b)$ with $(a \geq 0 \wedge -b \leq a \leq b)$ since we know that $b \geq 0$ holds anyway. Similarly, we can replace $(a < 0 \wedge -b \leq a)$ with $(a < 0 \wedge -b \leq a \leq b)$ for the same reason. QED. Of course, we can also prove the property

$$|a| \leq b \quad \iff \quad -b \leq a \leq b$$

by case inspection, discussing $a \geq 0$ or $a < 0$.