The tangent plane to surface $z = f(x, y)$ at the point

$P(a, b, f(a, b))$

is : $z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$.

Replace z by $f(x,y)$: $f(x,y) - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$,

when x, y is close to a, b, then $f(x,y)$ is close to $L(x,y)$

where $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$.

$L(x,y)$ is called a linear approximation of f at (a,b).

e.g. 1 Find the tangent plane to the elliptic paraboloid line

$z = f(x,y) = 2x^2 + y^2$ at the point $(1, 1, 3)$.

$f_x = 4x \quad f_x(1,1) = 4$

$f_y = 2y \quad f_y(1,1) = 2$.

$\Rightarrow z = 4(x-1) + 2(y-1) + 3$

$= 4x + 2y - 3$.

sketch the level curves of $f(x,y) = 2x^2 + y^2$:

$L(x,y) = 2x + y$

$f(x,y) = 2x^2 + y^2$.

$\Rightarrow$ at a very small $\Delta x, \Delta y$,

$L(x,y) \simeq f(x,y)$.

where $dz = f_x(x,y)\Delta x + f_y(x,y)\Delta y$.

$= f_x(x,y)\,dx + f_y(x,y)\,dy$

dz is called the total differential

e.g. 2. Find the linear approximation of the function

$f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$ at $(3, 2, 6)$ and use it to

approximate the number $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$

$L(x,y,z) = f(3,2,6) + f_x(3,2,6)\Delta x + f_y(3,2,6)\Delta y + f_z(3,2,6)\Delta z$

$= \sqrt{49} \quad +$

$f_x(3,2,6)\Delta x = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}} = \frac{3}{7}$

$f_y(3,2,6)\Delta y = \qquad = \frac{2}{7}$

$f_z(3,2,6)\Delta z = \qquad = \frac{6}{7}.$

$\therefore L(x,y,z) = 7 + \frac{3}{7}(x-3) + \frac{2}{7}(y-2) + \frac{6}{7}(z-6).$

$\therefore \sqrt{(\quad)^2+(\quad)^2+(\quad)^2} \simeq (3.02)\cdot\frac{3}{7} + (1.97)\frac{2}{7} + (5.99)\frac{6}{7}$

$\simeq 6.991428$

the exact number is 6.991523

the percentage relative error is

$\left|\frac{6.991428-6.991523}{6.991523}\right| \times 100\% = 0.0013$, a very good approximation

Defination: if $z(x,y)$ is differentiable at point (a,b), then $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$ can be express in the form $\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$ where $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$.

e.g. 3. show $f(x,y) = xy - 5y^2$. is differentiable by finding $\varepsilon_1, \varepsilon_2$.

$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y).$

$= (x+\Delta x)(y+\Delta y) - 5(y+\Delta y)^2 - xy + 5y^2.$

$= \cancel{xy} + x\Delta y + \Delta x y + \Delta x\Delta y - \cancel{5y^2} - 10y\Delta y - 5\Delta y^2 - \cancel{xy} + \cancel{5y^2}$

$= x\Delta y + \Delta x y + \Delta x\Delta y - 10y\Delta y - 5\Delta y^2.$

$= \underbrace{y\Delta x}_{f_x} + \underbrace{(x-10y)\Delta y}_{f_y} + \underbrace{\Delta x\Delta y}_{\varepsilon_1\Delta x} - \underbrace{5\Delta y^2}_{\varepsilon_2\Delta y}.$

$\varepsilon_1 = \Delta y, \quad \varepsilon_2 = -5\Delta y$

$\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

Therom: If the partial