Math 1229A/B

Unit 3:

Lines and Planes

(text reference: Section 1.3)

3 Lines and Planes

Lines in $\Re^2$

You are already familiar with equations of lines. In previous courses you will have written equations of lines in slope-point form, in slope-intercept form, and probably also in standard form for a line in $\Re^2$. Recall that:

$y - y_1 = m(x - x_1)$	is the *slope-point* form equation of the line through point (x_1, y_1) with slope m
$y = mx + b$	is the *slope-intercept* form equation of the line with slope m and y-intercept b
$ax + by = c$	is the *standard form* equation which either of the others can be rearranged to

In this course, we don't use the slope-point or slope-intercept forms of equations of lines. Instead, we use various other, vector-based, forms of equations. But we do still use the standard form.

You already know that given any 2 distinct points, whether in $\Re^2$ or in $\Re^3$, there is exactly one line which passes through both points. Suppose we have 2 points, P and Q. Let ℓ be the line that passes through these two points. Both point P and point Q lie on line ℓ, and so do all the points between them. In fact, the directed line segment $\overrightarrow{PQ}$ lies on line ℓ. When this directed line segment is translated to the origin, the resulting vector most likely doesn't lie on line ℓ (unless the origin happens to lie on line ℓ), but if not, it does lie on a line which is *parallel* to line ℓ. It lies on the line parallel to ℓ which passes through the origin. So this vector does give us some information about the line. (Similar to the information given by knowing the slope of a line in $\Re^2$, although it's not quite the same information.)

If a vector $\vec{v}$ lies on a particular line, or on a line parallel to that line, we say that $\vec{v}$ is *parallel to*, or is *collinear with* that line. And we call $\vec{v}$ a *direction vector* for the line. Not that the line actually has a direction associated with it. It doesn't. It extends in both directions, but has no particular "forwards along the line" or "backwards along the line" associated with it. So don't read too much meaning into the term *direction vector*. If $\vec{v}$ is a direction vector for line ℓ, then so is $-\vec{v}$. And so is every other scalar multiple of $\vec{v}$, except for $0\vec{v}$. Because of course $0\vec{v} = \vec{0}$ which has no direction information. But every other scalar multiple of $\vec{v}$ starts at the origin, and goes either the same or the opposite direction as $\vec{v}$ and therefore also lies on the line parallel to ℓ which passes through the origin. So any such vector would be considered a *direction vector* for line ℓ.

Definition: Any non-zero vector which is parallel to a line ℓ is called a **direction vector** for line ℓ.

For instance, consider the line $x + y = 2$. The points $(1, 1)$, $(2, 0)$, $(0, 2)$, $(-1, 3)$, $(3, -1)$, $(-2, 4)$, $(4, -2)$, etc., all lie on this line. So do $(1/2, 3/2)$ and $(3/2, 1/2)$ and infinitely many other points. Pick any 2 of these points, and find the vector which is the translation to the origin of the directed line segment between them, and you have a direction vector for the line. And we know that for any points P and Q, letting $\vec{p} = \overrightarrow{OP}$ denote the vector from the origin to point P and $\vec{q} = \overrightarrow{OQ}$ denote the vector from the origin to point Q, the vector $\vec{v} = \vec{q} - \vec{p}$ is the translation of directed line segment $\overrightarrow{PQ}$ to the origin. So for instance for points $P(1, 1)$ and $Q(0, 2)$, we have $\vec{p} = (1, 1)$ and $\vec{q} = (0, 2)$, and we see that $\vec{v} = \vec{q} - \vec{p} = (0, 2) - (1, 1) = (-1, 1)$ is a direction vector for the line $x + y = 2$. And other choices of P and Q give other direction vectors which are scalar multiples of this one. (Go ahead, pick some other points, and see what vectors you get.)

Point-Parallel Form

If we know a direction vector, $\vec{v}$, for a line, and any one point, P, on the line, we can use them to write an equation of the line. Because if we take the vector from the origin to the specified point, $\vec{p} = \overrightarrow{OP}$, and travel from there any non-zero scalar multiple of the direction vector (i.e. form the vector sum of the vector $\vec{p}$ and some scalar multiple of the direction vector), we travel along the line and end up at some other point on the line (i.e. the sum vector goes from the origin to some point on the line). And any point on the line can be reached by doing this. It's just a matter of choosing the right scalar multiple. So let $Q(x, y)$ be any point on the line ℓ which goes through point P and has direction vector $\vec{v}$. Then $\vec{q} = \overrightarrow{OQ} = \vec{p} + t\vec{v}$, for some value of t. And if we write the vectors in component form, it looks like we're adding points together, although of course we're not. If we let $P(x_1, y_1)$ denote the known point on the line, and use $\vec{v} = (v_1, v_2)$ to denote the direction vector, we get $(x, y) = (x_1, y_1) + t(v_1, v_2)$ as an equation which describes all points (x, y) on the line ℓ.

The way we actually write the line is a little different. We use $\vec{x}(t)$ to denote the vector (x, y), i.e. the vector from the origin to an unspecified point on the line. (It is written as $\vec{x}(t)$ to denote that the specific point obtained depends on, i.e. is a function of, the particular t value used.) Since we're writing an equation of the line using a point on the line and a vector which is parallel to the line, we refer to the equation as being in *point-parallel* form.

> **Definition:** The **point-parallel** form equation of the line ℓ which passes through point P and has direction vector $\vec{v}$ is given by
>
> $$\begin{aligned} \vec{x}(t) &= \vec{p} + t\vec{v} \\ \text{i.e. } \vec{x}(t) &= (p_1, p_2) + t(v_1, v_2) \end{aligned}$$

Example 3.1. Write a point-parallel form equation for each of the following lines:
(a) The line through $P(3, 1)$ and $Q(0, 6)$.
(b) The line through $P(1, 2)$ with direction vector $\vec{v} = (2, -1)$.
(c) The line through the origin with direction vector $\vec{v} = (0, 1)$.

Solution:
(a) The directed line segment $\overrightarrow{PQ}$ lies on, and hence is parallel to, the line through P and Q. We have the vector $\vec{p} = (3, 1)$ which goes from the origin to the point P, and the vector $\vec{q} = (0, 6)$ which goes from the origin to the point Q, and so when we translate $\overrightarrow{PQ}$ to the origin, we get

$$\vec{v} = \vec{q} - \vec{p} = (0, 6) - (3, 1) = (-3, 5)$$

as a direction vector for the line. And now, we can use *either* P or Q as the point which we know to be on the line. If we use P, we get the point-parallel form equation:

$$\vec{x}(t) = \vec{p} + t\vec{v} \qquad \Rightarrow \qquad \vec{x}(t) = (3, 1) + t(-3, 5)$$

(b) This time, we don't need to find the direction vector. We have $\vec{p} = (1, 2)$, so we use this and $\vec{v} = (2, -1)$ in the form $\vec{x}(t) = \vec{p} + t\vec{v}$ to get the point-parallel form equation

$$\vec{x}(t) = (1, 2) + t(2, -1)$$

(c) Again, all we need to do is plug the point and the direction vector into the point-parallel form. The point, of course, is the origin, i.e. $(0, 0)$, and the direction vector is $\vec{v} = (0, 1)$. We get:

$$\vec{x}(t) = (0, 0) + t(0, 1)$$

Parametric Equations

There's another form of an equation of a line, which is really more than one equation, that follows directly from the point-parallel form. Remember, the $\vec{x}(t)$ on the left hand side of the point-parallel equation is simply saying that the vector $\vec{x} = (x, y)$, corresponding to any point (x, y) on the line, is determined by the choice of value of the **parameter** t. So as we saw before, the point-parallel equation $\vec{x}(t) = (p_1, p_2) + t(v_1, v_2)$ really says that for any point (x, y) on the line, $(x, y) = (p_1, p_2) + t(v_1, v_2)$. Now, this is a statement about vectors, but (as previously noted) it looks like we're doing arithmetic with points. We're not really, because that wouldn't make any sense, but if we break it down to individual components of vectors, we get statements which are equally true of coordinates of points. Consider the first components of the vectors in the equation. We can express the vector arithmetic being done for that component as $x = p_1 + tv_1$. And if we think about x as the x-coordinate of an unspecified point on the line, and p_1 as the x-coordinate of the point P, and v_1 as the x-coordinate of the endpoint of the direction vector, then the statement $x = p_1 + tv_1$ simply says that you can get the x-coordinate of a point on the line by adding some multiple of the x-coordinate of the endpoint of the direction vector to the x-coordinate of the known point. And then, if you do the same thing with the y-coordinates, using *the same* value of the multiplier, t, the formula $y = p_2 + tv_2$ gives the y-coordinate of the same point on the line. So the point-parallel form equation also gives us *two* equations, which together describe any point on the line. And because it describes the point in terms of the effect of the parameter t, we call these *parametric equations* of the line.

Definition: The line ℓ with point-parallel form equation $\vec{x}(t) = (p_1, p_2) + t(v_1, v_2)$ has **parametric equations**

$$\begin{aligned} x &= p_1 + tv_1 \\ y &= p_2 + tv_2 \end{aligned}$$

Notice: Parametric equations of lines in $\Re^2$ **always** come in pairs. You can't have only one parametric equation, telling about just one component/coordinate. That doesn't describe the line. Also, if you use parametric equations to find points on the line, you have to remember to use the *same* value of t in both equations.

Example 3.2. Find parametric equations for each of the lines in Example 3.1.

Solution:
(a) We have the point-parallel form equation $\vec{x}(t) = (3, 1) + t(-3, 5)$. We get the right hand side of the x equation from the first components, and the right hand side of the y equation from the second components. Of course, we don't usually write something like $t(-3)$ or $t(5)$, or even $t5$. We would write this product as $-3t$ or $5t$. And we know that adding a negative is the same as subtracting, so the minus sign in the $-3t$ can replace the plus sign. We get:

$$\begin{aligned} x &= 3 - 3t \\ y &= 1 + 5t \end{aligned}$$

(b) This time we have the point-parallel form equation $\vec{x}(t) = (1, 2) + t(2, -1)$, which gives the parametric equations:

$$\begin{aligned} x &= 1 + 2t \\ y &= 2 - t \end{aligned}$$

(c) And now, we use $\vec{x}(t) = (0, 0) + t(0, 1)$. But unless it's the only thing there, we don't need to write a 0. And we never need to write a 1 multiplier. We get:

$$\begin{array}{rclcrcl} x &=& 0 + 0t & \Rightarrow & x &=& 0 \\ y &=& 0 + 1t & \Rightarrow & y &=& t \end{array}$$

Example 3.3. Write a point-parallel form equation for the line with parametric equations

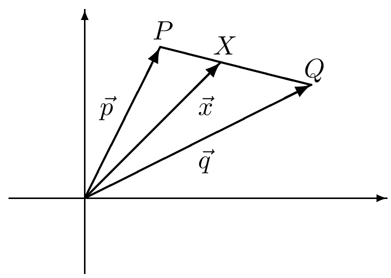
$$\begin{aligned} x &= 1 + 5t \\ y &= 2 \end{aligned}$$

Solution:
We use the x equation to find the first components for our point-parallel equation, and the y equation to find the second components. We need to recognize that in each equation, the number on the right hand side that isn't multiplied by t is the coordinate of the known point, P, and that the number that *is* multiplied by t is the component of the direction vector, $\vec{v}$. So from the first equation, i.e. the x-equation, we see that $p_1 = 1$ and $v_1 = 5$. And from the second equation, since there's no t multiplying it, the 2 must be p_2. So where's the t? It's invisible, which means it must have a 0 multiplier. That is, $v_2 = 0$. So we have the point $P(1, 2)$ and the direction vector $\vec{v} = (5, 0)$, which when we put it in the form $\vec{x}(t) = \vec{p} + t\vec{v}$ gives the point-parallel form equation

$$\vec{x}(t) = (1, 2) + t(5, 0)$$

Two-Point Form

Now, suppose that we have two points, P and Q, and consider the vectors $\vec{p}$ and $\vec{q}$, from the origin to the points. Let X be any point on the line segment joining P and Q. Of course, we can consider X to be a point on the *directed* line segment $\overrightarrow{PQ}$. How can we describe the vector $\vec{x}$, from the origin to point X? Let's look at a picture.



Consider the directed line segment $\overrightarrow{PX}$. Suppose that we travel along the vector $\vec{p}$ and then along the directed line segment $\overrightarrow{PX}$. Then we started at the origin and ended up at the point X, the same as if we travelled along the vector $\vec{x}$. In terms of vector sums, we travelled $\vec{p} + \vec{u}$ where $\vec{u}$ is the translation of $\overrightarrow{PX}$ to the origin. So we have $\vec{x} = \vec{p} + \vec{u}$.

But let's think more about $\overrightarrow{PX}$ and $\vec{u}$. $\overrightarrow{PX}$ is a piece of the directed line segment $\overrightarrow{PQ}$. For instance, if X was exactly one-third of the way along $\overrightarrow{PQ}$, then $\overrightarrow{PX}$ would be equivalent to one-third of $\overrightarrow{PQ}$. And we know that $\overrightarrow{PQ} = \vec{q} - \vec{p}$. So we could say (if X happened to be exactly one-third of the way along $\overrightarrow{PQ}$) that $\vec{u} = (1/3)(\vec{q} - \vec{p})$. Now, we don't necessarily have X being one-third of the way along. We're considering *any* point X on $\overrightarrow{PQ}$. But then we do know that X is $100t\%$ of the way along $\overrightarrow{PQ}$, for some value t between 0 and 1. (For instance, if X is 20% of the way along, then $t = .2$.) And then this means that we have $\overrightarrow{PX} = t\overrightarrow{PQ}$ so that $\vec{u} = t(\vec{q} - \vec{p})$. Therefore we also have

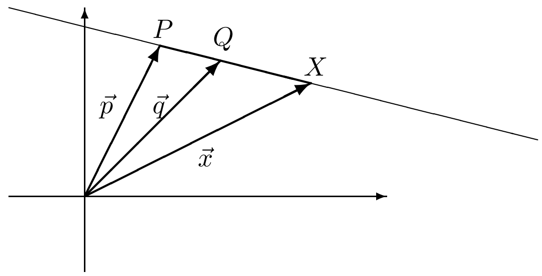
$$\vec{p} + \vec{u} = \vec{p} + t(\vec{q} - \vec{p}) = \vec{p} + t\vec{q} - t\vec{p} = (1 - t)\vec{p} + t\vec{q}$$

That is, for any point X along the line segment joining point P and point Q, we have

$$\vec{x} = (1 - t)\vec{p} + t\vec{q}$$

where t is some value between 0 and 1.

But nothing that we did here really required being in between P and Q. We could do something similar for *any point on the line containing* P *and* Q. The only difference is that t would no longer necessarily be between 0 and 1. That is, for any point X on the line containing the points P and Q, we could travel from the origin to point P, and then travel some scalar multiple of the vector $\vec{u}$ to end up at the point X. For instance, consider the diagram shown below. As before, we have $\vec{x} = \vec{p} + t\vec{u}$, but now t is bigger than 1. Or if we needed to go the other direction along the line, from P, then t would be negative.



And so for any point X on the line, we have

$$\vec{x} = \vec{p} + t\vec{u} = \vec{p} + t(\vec{q} - \vec{p}) = \vec{p} + t\vec{q} - t\vec{p} = (1-t)\vec{p} + t\vec{q}$$

for some value t. That is, we can express the line containing two points P and Q as the line containing all points $X(x, y)$ such that $\vec{x} = (1-t)\vec{p} + t\vec{q}$ for some value t. And so we have another form of equation for the line. We call this the *two-point* form, and as before, we write $\vec{x}(t)$ instead of just $\vec{x}$.

Definition: The **two-point** form of equation for the line through points P and Q is:

$$\vec{x}(t) = (1-t)\vec{p} + t\vec{q}$$

Example 3.4. Write equations in two-point form for each of the lines in Example 3.1.

Solution:
(a) The line here is the line through the points $P(3, 1)$ and $Q(0, 6)$, so we have $\vec{p} = (3, 1)$ and $\vec{q} = (0, 6)$ and we get the two-point form

$$\vec{x}(t) = (1-t)(3, 1) + t(0, 6)$$

(b) This time, we have the line through $P(1, 2)$ with direction vector $\vec{v} = (2, -1)$. We don't know two points on the line, so we need to find a second point. We saw in Example 3.1(b) that a point-parallel form equation for this line is $\vec{x}(t) = (1, 2) + t(2, -1)$. We can choose any value of t other than 0 to get another point on the line. (*Notice:* We don't want to use $t = 0$, because that will just give us the point we already know. But any other value of t will do.) For instance, using $t = 1$ we have $\vec{x}(1) = (1, 2) + 1(2, -1) = (1, 2) + (2, -1) = (3, 1)$, so we see that $Q(3, 1)$ is another point on the same line. Now that we know two points on the line, we can find a two-point form equation. Notice, though, that since we have already been using t as the parameter for the point-parallel form equation, we should use a different name for the parameter in the two-point form equation. (Especially since we gave t a specific value. We wouldn't want to get confused and think the parameter in the two-point form equation was supposed to have that value too.) Notice also that it doesn't matter in the least what letter we use to represent the parameter (which is just a scalar multiplier). So we can use s instead. We get:

$$\vec{x}(s) = (1-s)(1, 2) + s(3, 1)$$

(c) This time, we have the line through the origin with direction vector $(0,1)$. We know that the point $(0,0)$ is on the line, and clearly the point $(0,1)$ is also on the line (because the vector $(0,1)$ is on the line, since the line does pass through the origin). So a two-point form equation for this line is

$$\vec{x}(t) = (1-t)(0,0) + t(0,1)$$

Point-Normal Form

When two lines meet at right angles, we call them **perpendicular**. (You knew that.) And we have already learnt that when two vectors are perpendicular, there's another word we use. Instead of saying they're perpendicular, we say they are **orthogonal**. When we're talking about a vector and a line, there's yet another word that we use. (This was mentioned earlier, but is now defined.)

> **Definition:** A vector which is perpendicular to a particular line in $\Re^2$ is said to be **normal** to the line and is called a **normal** for that line, or a **normal vector** for the line.

So *orthogonal* and *normal* really just mean *perpendicular*, but the three words are used in different contexts.

If $\vec{n}$ is a normal vector for a particular line, then it is orthogonal to any direction vector for the line. We have already seen that if we know a direction vector for a line, and one point on the line, we can write a vector equation for the line, in *point-parallel form*. Similarly, if we know a normal vector for a line in $\Re^2$, and one point on the line, we can write a vector equation for the line. We call it the **point-normal form** of the line. The equation comes from the fact that the dot product of two orthogonal vectors is 0.

Suppose $\vec{n}$ is a normal vector for a particular line, and P is a point on that line. Let X be any other point on the line. Then the directed line segment $\overrightarrow{PX}$ lies on the line, and so the vector $\vec{x} - \vec{p}$, which is equivalent to $\overrightarrow{PX}$, is parallel to the line. But then $\vec{n}$ is orthogonal to $\vec{x} - \vec{p}$, and so $\vec{n} \bullet (\vec{x} - \vec{p}) = 0$. So if $\vec{n} = (n_1, n_2)$ and the point is $P(p_1, p_2)$, then we have $(n_1, n_2) \bullet (\vec{x} - (p_1, p_2)) = 0$. This equation is the form we call point-normal. As always, it describes all the points $X(x, y)$ which lie on the line.

> **Definition:** Let ℓ be any line in $\Re^2$. If $\vec{n} = (n_1, n_2)$ is a normal for line ℓ, and $P(p_1, p_2)$ is a point on line ℓ, then an equation for line ℓ in **point-normal form** is:
>
> $$(n_1, n_2) \bullet (\vec{x} - (p_1, p_2)) = 0$$

Example 3.5. Write an equation in point-normal form for the line through $P(1,2)$ with normal $\vec{n} = (-1, 1)$.

Solution:
We get the equation:

$$(-1, 1) \bullet (\vec{x} - (1, 2)) = 0$$

Recall that in $\Re^2$, the vector $(b, -a)$ is orthogonal to the vector (a, b), because $(b, -a) \bullet (a, b) = 0$. This means that whenever we know a direction vector for a line in $\Re^2$ (i.e. a vector which is parallel to the line) then we can easily find a normal for the line (i.e. a vector which is perpendicular to the line). And vice versa. So it's easy to find the point-parallel form of a line from the point-normal form, and also to find the point-normal form from the point-parallel form.

Example 3.6.
(a) Find an equation in point-normal form for the line $\vec{x}(t) = (0,1) + t(2,-1)$.
(b) Write an equation in point-parallel form for the line from Example 3.5.
(c) Write a point-normal form equation for the line with parametric equations

$$x = 3 + t \qquad \text{and} \qquad y = 2t - 4$$

Solution:
(a) We have $\vec{x}(t) = (0,1) + t(2,-1)$, which we recognize as a point-parallel form equation for the line through point $(0,1)$ parallel to the vector $(2,-1)$. Since the vector $(2,-1)$ is parallel to the line, then the vector $(1,2)$, obtained by switching the components and changing one of the signs, is perpendicular to the line. That is, $\vec{n} = (1,2)$ is a normal for this line. So a point-normal form equation for the line is

$$(1,2) \bullet (\vec{x} - (0,1)) = 0$$

(b) In Example 3.5 we found the point-normal form equation $(-1,1) \bullet (\vec{x} - (1,2)) = 0$ for a particular line. Since $(-1,1)$ is a normal for this line, and the vector $(1,1)$ is orthogonal to $(-1,1)$, then the vector $(1,1)$ is parallel to the line, i.e. is a direction vector for the line. And of course $(1,2)$ is a point on the line. So a point-parallel form equation for this line is

$$\vec{x}(t) = (1,2) + t(1,1)$$

(c) From the parametric equations of the line we can identify both a point on the line and a direction vector for the line. Remember, the multiplier on t is the component of the direction vector, while the number without a t is the coordinate of the known point. Keeping this in mind allows us to correctly identify both the known point and the direction vector from the parametric equations, even when they look a bit different than we expect.

Here, the parametric equations are given as

$$\begin{aligned} x &= 3 + t \\ y &= 2t - 4 \end{aligned}$$

We're more accustomed to seeing the form we have in the x equation. The form in the y equation, with the t term coming before the non-t term, is different. This is just done to avoid having a "leading negative". Equations look less tidy when the first thing on one side of the equation is a negative sign, so mathematicians often avoid writing things that way. That is, the given parametric equations are just a tidier form of

$$\begin{aligned} x &= 3 + t \\ y &= -4 + 2t \end{aligned}$$

In this form we see that the corresponding point-parallel form equation is $\vec{x}(t) = (3,-4) + t(1,2)$. So $(1,2)$ is a direction vector for the line and therefore $(2,-1)$ is a normal for the line. Thus we can write a point-normal form equation as

$$(2,-1) \bullet (\vec{x} - (3,-4)) = 0$$

Standard Form

Using the point-normal form of a line, we can get another form of equation for a line in $\Re^2$ – one which is already familiar to you. We get it by writing the vector $\vec{x}$ as $\vec{x} = (x, y)$ and distributing the dot product over the bracket in the point-normal form equation. (The vector (x, y), as always, just represents any vector whose endpoint (x, y) is a point on the line. That is, (x, y) is any unspecified point on the line.) For any line with normal vector (n_1, n_2) containing point (p_1, p_2) we get

$$\begin{aligned}(n_1, n_2) \bullet (\vec{x} - (p_1, p_2)) = 0 \quad &\Rightarrow \quad (n_1, n_2) \bullet ((x, y) - (p_1, p_2)) = 0 \\ &\Rightarrow \quad [(n_1, n_2) \bullet (x, y)] - [(n_1, n_2) \bullet (p_1, p_2)] = 0 \\ &\Rightarrow \quad n_1 x + n_2 y = (n_1, n_2) \bullet (p_1, p_2)\end{aligned}$$

But if (n_1, n_2) is a known normal vector to the line, and (p_1, p_2) is a known point on the line (so that n_1, n_2, p_1 and p_2 are all just numbers and we know which numbers they are), then $(n_1, n_2) \bullet (p_1, p_2)$ is just a number, i.e. is a known scalar, which we could call c. So we have $n_1 x + n_2 y = c$. Or, to make this general form look more familiar to you, we could use a and b in place of n_1 and n_2 and write $ax + by = c$. Ah. You've seen that before, haven't you? That's the standard form of an equation of a line.

Definition: The equation $ax + by = c$ is called the **standard form** equation of a line.

And what we've seen is that in this kind of equation, the coefficients a and b are the components of a normal vector for the line. And we also saw how to find the constant c.

Theorem 3.1. *If $ax + by = c$ is a standard form equation for line ℓ, then $\vec{n} = (a, b)$ is a normal vector for line ℓ. Also, if $P(p_1, p_2)$ is a point on the line, then $c = \vec{n} \bullet \vec{p}$.*

So if we know a point-normal equation for a line, i.e. if we know a normal for the line and we know a point on the line, then it's easy to find a standard form equation for the line. We simply use the components of the normal as the coefficients of x and y, and use the normal vector and the point to find the right hand side value, c. Likewise, if we have an equation of a line in standard form, we can easily find a normal to the line, because the coefficients of x and y are the components of a normal vector to the line. And then we just need to find any point on the line, to write a point-normal form equation of the line.

Example 3.7. Write a standard form equation for the line in Example 3.5.

Solution:
In Example 3.5 we had the line through $P(1, 2)$ with normal vector $\vec{n} = (-1, 1)$. We use the components of the normal vector as the coefficients of x and y in the standard form equation, so we have $(-1)x + (1)y = c$, or $-x + y = c$. We find the value of c using $c = \vec{n} \bullet \vec{p}$. We get

$$c = (-1, 1) \bullet (1, 2) = (-1)(1) + (1)(2) = -1 + 2 = 1$$

So the standard form equation is $-x + y = 1$. However, we don't usually write something like this with a leading negative. So we multiply the whole equation (i.e. both sides of the equation) by -1 to get rid of it. We get $x - y = -1$. (*Notice:* $(1, -1) = -(-1, 1) = -\vec{n}$ is parallel to (collinear with) $\vec{n}$, and is therefore another normal vector for this line.)

Example 3.8. Write a point-normal form equation for the line $x - 2y = 5$.

Solution:
We use the coefficients of x and y as the components of a normal vector for the line. Of course, $x - 2y = 1x + (-2)y$, so the coefficients are 1 and -2. That is, we get $\vec{n} = (1, -2)$ as a normal vector for the line. Now, we just need to find any point on the line. We plug in any convenient x-value and solve for y. Or we plug in any convenient y-value and solve for x. For instance, when $y = 0$ we have $x - 2(0) = 5$, so $x - 0 = 5$. That is, we see that when $y = 0$ we must have $x = 5$. So $P(5, 0)$ is a point on the line. Now we can write the point-normal form equation:

$$(1, -2) \bullet (\vec{x} - (5, 0)) = 0$$

Example 3.9. Write a standard form equation of the line $\vec{x}(t) = (3, 2) + t(2, 7)$.

Solution:
From the given point-parallel form equation, we see that $P(3, 2)$ is a point on the line and $\vec{v} = (2, 7)$ is a direction vector for the line, i.e. is parallel to the line. And so $\vec{n} = (7, -2)$ is a normal vector to the line, so the standard form equation has $7x - 2y = c$ for some value c. And we can find c using

$$c = (7, -2) \bullet (3, 2) = 7(3) + (-2)(2) = 21 - 4 = 17$$

Therefore the standard form equation is $7x - 2y = 17$.

Lines in $\Re^3$

Of course, we can have lines in 3-space, as well as in the plane. And there's a lot that's the same in $\Re^3$ as it was in $\Re^2$, so we use the same terminology and notation.

For instance, when we move from 2 dimensions to 3, it's still true that given any 2 points, there is exactly one line that passes through both those points. And the vector equivalent to the directed line segment between those points is parallel to that line, so we still call it a direction vector for the line. That is, we define the term direction vector the same way in $\Re^3$ as we did in $\Re^2$.

Definition: If $\vec{v} \in \Re^3$ is parallel to some line ℓ in $\Re^3$, we say that $\vec{v}$ is a **direction vector** for ℓ.

As in $\Re^2$, we can use a direction vector for a line (i.e. a vector parallel to the line) and any one point on the line to write a point-parallel equation for the line. And from that we can write parametric equations. Or we could write a 2-point form equation, instead.

The only difference is that now the points have 3 coordinates and the vectors have 3 components. Of course, for parametric equations this means that we have a third equation, corresponding to the z components of the vectors.

These observations are summarized in the following definitions.

Definition: Let $P(p_1, p_2, p_3)$ and $Q(q_1, q_2, q_3)$ be any points in $\Re^3$ and let $\vec{v} = (v_1, v_2, v_3)$ be any vector in $\Re^3$. Then:

1. If ℓ is the line which passes through P parallel to $\vec{v}$ (so that $\vec{v}$ is a direction vector for ℓ), then
$$\vec{x}(t) = (p_1, p_2, p_3) + t(v_1, v_2, v_3)$$
is an equation for line ℓ in **point-parallel form**.

2. If line ℓ passes through point P and $\vec{v}$ is a direction vector for ℓ, then **parametric equations** of line ℓ are:
$$\begin{aligned} x &= p_1 + tv_1 \\ y &= p_2 + tv_2 \\ z &= p_3 + tv_3 \end{aligned}$$

3. If points P and Q are both on line ℓ then a **two-point** form equation for ℓ is
$$\vec{x}(t) = (1-t)(p_1, p_2, p_3) + t(q_1, q_2, q_3)$$

Example 3.10. Let ℓ be the line which passes through the points $P(1, 2, 3)$ and $Q(1, -1, 1)$. Write equations of line ℓ in two-point form and in point-parallel form.

Solution:
In two-point form, we get the equation for ℓ:
$$\vec{x}(t) = (1-t)(1, 2, 3) + t(1, -1, 1)$$

For a point-parallel form equation of line ℓ we first need to find a direction vector for ℓ. The directed line segment $\overrightarrow{PQ}$ is equivalent to
$$\vec{v} = \vec{q} - \vec{p} = (1, -1, 1) - (1, 2, 3) = (0, -3, -2)$$
which is parallel to (and hence is a direction vector for) ℓ. Using this direction vector and the point P which we know is on the line, we get
$$\vec{x}(t) = (1, 2, 3) + t(0, -3, -2)$$
(Of course, we could have used point Q instead of point P to write the point-parallel form equation. Likewise, we could have used $\vec{p} - \vec{q} = (0, 3, 2)$ as the direction vector. And in the two-point form equation, we could have switched the roles of P and Q.)

Example 3.11. Write parametric equations for the line through the point $(0, 1, -1)$ which is parallel to $\vec{v} = (2, 1, 0)$.

Solution:
An equation of the line in point-parallel form is $\vec{x}(t) = (0, 1, -1) + t(2, 1, 0)$. This tells us that a point (x, y, z) is on this line if there is some value of t for which $(x, y, z) = (0, 1, -1) + t(2, 1, 0)$. So it must be true that, for the same value of t, we have
$$\begin{aligned} x &= 0 + 2t \\ y &= 1 + 1t \\ z &= -1 + 0t \end{aligned}$$

That is, we can write parametric equations of the line as

$$\begin{aligned} x &= 2t \\ y &= 1+t \\ z &= -1 \end{aligned}$$

Example 3.12. ℓ_1 is the line $\vec{x}(t) = (1-t)(2,1,-1) + t(0,1,2)$. ℓ_2 is the line with parametric equations $x = 2t-2$, $y = 1$, $z = 5-3t$. Are ℓ_1 and ℓ_2 the same line?

Solution:
Hmm. That's different. Let's see. For ℓ_1 we recognize that what we've been given is a two-point form equation. (We can tell because of the $(1-t)$ multiplier.) From it we can see that $P(2,1,-1)$ and $Q(0,1,2)$ are two points on line ℓ_1. This also tells us that the vector

$$\vec{v} = \overrightarrow{PQ} = \vec{q} - \vec{p} = (0,1,2) - (2,1,-1) = (-2,0,3)$$

is parallel to line ℓ_1.

For ℓ_2 we're given parametric equations. It may be helpful to write these equations all the same way, with "constant + multiple of t" on the right hand side. We have:

$$\begin{aligned} x &= 2t-2 \\ y &= 1 \\ z &= 5-3t \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} x &= -2+2t \\ y &= 1+0t \\ y &= 5+(-3)t \end{aligned}$$

From the rearranged set of equations, using our knowledge of the form of parametric equations, we see that the point on ℓ_2 used to write these parametric equations is $R(-2,1,5)$. Also, the direction vector used for these equations is $\vec{u} = (2,0,-3)$.

Since $\vec{u} = (2,0,-3) = -(-2,0,3) = -\vec{v}$, we see that these vectors are scalar multiples of one another, so they are collinear. That is, the direction vector $\vec{u}$ used to write the equation of ℓ_2 is parallel to the vector which we know is parallel to ℓ_1. Therefore $\vec{u}$ is also parallel to ℓ_1, and thus lines ℓ_1 and ℓ_2 are parallel to one another. It's possible that they could be the same line. How can we tell whether they are?

Since ℓ_1 and ℓ_2 are parallel, then either they have no points in common or else they are the same line and have all points in common. So all we need to do is determine whether any point which is known to be on one line is also on the other. If it is, then they are actually the same line. But if it isn't, then they must be different, but parallel, lines.

We know that the point $P(0,1,2)$ is on line ℓ_1. Is it also on line ℓ_2? If it is, then $(x,y,z) = (0,1,2)$ must satisfy the parametric equations for ℓ_2, using the same value of t for each component (equation). Since the second coordinate of P is 1, the equation $y = 1$ is satisfied. For the first coordinate, we see that we need to have $x = 2t-2$ satisfied for $x = 0$. This gives

$$0 = 2t-2 \qquad \Rightarrow \qquad 0+2 = 2t \qquad \Rightarrow \qquad 2t = 2 \qquad \Rightarrow \qquad t = 1$$

Now, if we substitute $t = 1$ into the third of the parametric equations, we get

$$z = 5 - 3(1) = 5 - 3 = 2$$

Since $z = 2$ is the third coordinate of point P, we see that the point $(x,y,z) = (0,1,2)$ *does* satisfy the parametric equations of ℓ_2. That is, we have

$$(0,1,2) = (-2,1,5) + 1(2,0,-3)$$

so $(x, y, z) = (0, 1, 2)$ satisfies $x = -2 + 2t$, $y = 1 + 0t$ and $z = 5 - 3t$ with $t = 1$. Because the point $(0, 1, 2)$ does satisfy the equations for ℓ_2, it is a point on line ℓ_2.

So now we know that ℓ_1 and ℓ_2 are parallel lines, with a point in common, which means that they must have all points in common and be the same line. That is, since ℓ_1 and ℓ_2 are parallel and intersect at point $P(0, 1, 2)$, they must intersect at all other points as well, and actually be the same line.

Planes in $\Re^3$

We know that in $\Re^2$, something of the form $ax + by = c$ is the standard form of an equation of a line. What about the 3-dimensional equivalent, $ax + by + cz = d$. Is that an equation of a line? Well, let's see.

Let's think about a specific, uncomplicated, example. Consider the equation $x + y + z = 0$. This equation is satisfied by the point $P(2, -2, 0)$ (because $2 + (-2) + 0 = 0$) and also by the point $Q(3, -3, 0)$. And we know that there's a unique line that passes through those 2 points. Let's call that line ℓ_1. Using $\vec{v} = \vec{q} - \vec{p} = (3, -3, 0) - (2, -2, 0) = (1, -1, 0)$ as a vector which is parallel to ℓ_1, we can write an equation of ℓ_1 as

$$\vec{x}(t) = (2, -2, 0) + t(1, -1, 0)$$

Notice that for any point (x, y, z) on line ℓ_1 we have

$$\begin{aligned} x &= 2 + t \\ y &= -2 - t \\ z &= 0 \end{aligned}$$

and so $x + y + z = (2 + t) + (-2 - t) + 0 = 2 - 2 + t - t = 0$. That is, every point on ℓ_1 satisfies the equation $x + y + z = 0$.

But those aren't the only points which satisfy that equation. For instance, the point $R(1, 0, -1)$ also satisfies this equation. And this point is not on ℓ_1. The easiest way to tell is because from the parametric equations of ℓ_1 we can see that every point on ℓ_1 has $z = 0$, but the third coordinate of point R isn't 0, so it is not a point on ℓ_1.

Hmm. Every point on line ℓ_1 satisfies $x + y + z = 0$, but it's not true that every point that satisfies $x + y + z = 0$ is on line ℓ_1. So $x + y + z = 0$ cannot be an equation of line ℓ_1. Then what is it? Well, actually, it's the equation of a plane. $\Re^2$ (i.e. 2-space) is just a single plane. But $\Re^3$, which is to say 3-space, contains infinitely many planes. (For instance, think of the walls, ceilings and floors of the building you're in. And also every other building you ever have been or ever could be in. And all the ramps you've ever seen. And what those ramps would look like if they were knocked off kilter. And ... Each of those things lies in some particular plane in $\Re^3$, and there are many other planes besides those.)

So $x + y + z = 0$ is an equation of a plane. Let's call it the plane Π. (Planes are often named Π, which is just the Greek letter P, just like lines are named ℓ. ℓ for line, Π for plane. Same idea.) Any plane contains infinitely many lines. One of the lines that lies in the particular plane Π we've been talking about is the line ℓ_1. But there are many others. For instance, we saw that $P(2, -2, 0)$ and $R(1, 0, -1)$ both lie on this plane, so the line on which those points lie is another line in plane Π. We can call that one ℓ_2. And the vector $\vec{r} - \vec{p} = (1, 0, -1) - (2, -2, 0) = (-1, 2, -1)$ is parallel to ℓ_2 so we can express ℓ_2 as $\vec{x}(t) = (1, 0, -1) + t(-1, 2, -1)$.

Notice that $x+y+z = (1,1,1) \bullet (x,y,z)$. Let's think about the vector $(1,1,1)$ whose components are the coefficients in the equation of the plane Π. We know that $(1,-1,0)$ is parallel to line ℓ_1, which lies in plane Π. Notice that $(1,1,1) \bullet (1,-1,0) = 1(1) + 1(-1) + 1(0) = 1 - 1 + 0 = 0$, so the vector $(1,1,1)$ is a normal for (i.e. is perpendicular to) line ℓ_1. Likewise, we know that $(-1,2,-1)$ is parallel to line ℓ_2, which also lies in plane Π. Notice that $(1,1,1) \bullet (-1,2,-1) = 1(-1)+1(2)+1(-1) = -1+2-1 = 0$, so the vector $(1,1,1)$ is also a normal for (perpendicular to) line ℓ_2. However $(1,-1,0)$ is not a scalar multiple of $(-1,2,-1)$, so those vectors aren't orthogonal (i.e. parallel) and therefore lines ℓ_1 and ℓ_2 aren't parallel to one another. How can the same vector be perpendicular to both? Well, by being perpendicular to *the whole plane in which both lines lie.* This vector $(1,1,1)$ is actually perpendicular to, i.e. a normal for, the plane Π.

> **Definition:** A vector which is perpendicular to a particular plane in $\Re^3$ is said to be **normal** to the plane, and is called a **normal** for that plane, or a **normal vector** for the plane.

Point-Normal Form of an Equation of a Plane

At this point, it may not be too surprising to you to learn that if we write an equation using a normal vector and a point in $\Re^3$, what we get is an equation of a plane. That is, if we extend to $\Re^3$ the idea of the point-normal form of an equation of a line in $\Re^2$, the result is **not** an equation of a line in $\Re^3$, but rather an equation of a plane in $\Re^3$. This point-normal form equation of a plane in 3-space looks just like the point-normal form equation of a line in 2-space, except that the vectors have 3 components instead of 2. So it's the presence of that third component that distinguishes a point-normal form equation of a plane (with 3 components in the vectors) from a point-normal form equation of a line (with only 2 components in the vectors).

> **Definition:** The **point-normal form** of an equation of a plane Π in $\Re^3$, where $\vec{n} = (n_1, n_2, n_3)$ is any normal vector to the plane and $P(p_1, p_2, p_3)$ is any point on the plane, is given by
>
> $$\vec{n} \bullet (\vec{x} - \vec{p}) = 0 \qquad \textit{i.e.} \qquad (n_1, n_2, n_3) \bullet (\vec{x} - (p_1, p_2, p_3)) = 0$$

Example 3.13. Write an equation of the plane containing $P(1,2,3)$ with normal vector $\vec{n} = (1,0,-1)$ in point-normal form.

Solution:
The form of the equation is $\vec{n} \bullet (\vec{x} - \vec{p}) = 0$, so we get

$$(1,0,-1) \bullet (\vec{x} - (1,2,3)) = 0$$

Example 3.14. Write a point-normal form equation of the plane Π which contains the lines ℓ_1 represented by $\vec{x}(t) = (2,-2,0) + t(1,-1,0)$ and ℓ_2 represented by $\vec{x}(s) = (1,0,-1) + s(-1,2,-1)$.

Solution:
In order to write a point-normal form equation of the plane Π, we need a normal vector for the plane and a point that lies in the plane. Of course, the point-parallel form equations of ℓ_1 and ℓ_2 each give us a point that lies in the plane. That is, we know that $(2,-2,0)$ is a point in this plane, because this is a point on ℓ_1 which lies in plane Π, and likewise that $(1,0,-1)$ is another point on plane Π, because it is a point on line ℓ_2, which also lies in this plane. So we can use either one of these points in our equation of Π.

How do we find a normal for the plane? Well, we know from the equation of ℓ_1 that the vector $\vec{u} = (1, -1, 0)$ is parallel to ℓ_1, and likewise from the equation of ℓ_2 that the vector $\vec{v} = (-1, 2, -1)$ is parallel to ℓ_2. Of course any vector $\vec{n}$ which is a normal for Π (i.e. is perpendicular to this plane) must be perpendicular to any line that lies within Π. So if $\vec{n}$ is a normal for Π, then $\vec{n}$ is perpendicular to both ℓ_1 and ℓ_2 and therefore must be orthogonal to both $\vec{u}$ and $\vec{v}$. (That is, any vector which is perpendicular to ℓ_1 is also perpendicular to (orthogonal to) every vector that is parallel to ℓ_1. And similarly for ℓ_2.)

So how do we find a vector which is perpendicular to both $\vec{u}$ and $\vec{v}$? Well that's easy. We know that the vector $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$. So we can use

$$\begin{aligned} \vec{n} &= \vec{u} \times \vec{v} = (1, -1, 0) \times (-1, 2, -1) \\ &= ((-1)(-1), (0)(-1), (1)(2)) - ((2)(0), (-1)(1), (-1)(-1)) \\ &= (1, 0, 2) - (0, -1, 1) = (1, 1, 1) \end{aligned}$$

(Recall that we discussed previously that the vector $(1, 1, 1)$ was a normal for the plane containing these lines ℓ_1 and ℓ_2.)

Now we know both a normal vector for Π and a point in plane Π so we can write the point-normal form equation. We get

$$(1, 1, 1) \bullet (\vec{x} - (2, -2, 0)) = 0$$

<u>Standard Form Equation of a Plane</u>

Consider any point-normal form equation of a plane. For instance, let's work with the one we found in that last example. If we express the vector $\vec{x}$ as $\vec{x} = (x, y, z)$ and carry through the vector arithmetic, what do we get? Well let's see:

$$\begin{aligned} & (1, 1, 1) \bullet (\vec{x} - (2, -2, 0)) = 0 \\ \Rightarrow\; & (1, 1, 1) \bullet ((x, y, z) - (2, -2, 0)) = 0 \\ \Rightarrow\; & [(1, 1, 1) \bullet (x, y, z)] - [(1, 1, 1) \bullet (2, -2, 0)] = 0 \\ \Rightarrow\; & (1, 1, 1) \bullet (x, y, z) = (1, 1, 1) \bullet (2, -2, 0) \\ \Rightarrow\; & (1)(x) + (1)(y) + (1)(z) = (1)(2) + (1)(-2) + (1)(0) \\ \Rightarrow\; & x + y + z = 2 - 2 + 0 \\ \Rightarrow\; & x + y + z = 0 \end{aligned}$$

Ah, yes. We've seen that before. As an equation of the plane in which we found ℓ_1 and ℓ_2 to lie. That is, we started out our discussion of planes by wondering what $x + y + z = 0$ was the equation of, and we realized it was a plane. The equation looks just like the standard form equation of a line in $\Re^2$, except it has a z in it. So this form is called the standard form equation of a plane.

That is, as we said earlier, if we have an equation in the same form as a standard form equation of a line, but with z in it as well as x and y, then this equation isn't describing a single line. It is describing a whole plane in $\Re^3$. And now we see that the coefficients of x, y and z in this equation are the components of a normal vector for that plane.

Definition: The equation $ax + by + cz = d$ is a **standard form equation of a plane** and the vector (a, b, c) is a normal vector to this plane.

Example 3.15. Write an equation in standard form for the plane with normal vector $\vec{n} = (1, 2, 3)$ which contains the point $P(0, -1, 2)$.

Solution:
Since $\vec{n} = (1, 2, 3)$ is a normal vector for the plane, then the standard form equation must have the form $1x + 2y + 3z = d$ for some scalar d. But of course we would write that as $x + 2y + 3z = d$. How can we find the value of d? Well, we know that the point $P(0, -1, 2)$ lies on the plane, so $(x, y, z) = (0, -1, 2)$ must satisfy this equation. That is, we plug in $x = 0$, $y = -2$ and $z = 2$ to find the value of d. We get:

$$x + 2y + 3z = d \Rightarrow 0 + 2(-1) + 3(2) = d \Rightarrow -2 + 6 = d \Rightarrow d = 6 - 2 = 4$$

So a standard form equation of the plane is $x + 2y + 3z = 4$.

Notice: We could have used $\vec{n} \bullet \vec{p} = d$, from rearranging the point-normal equation for the plane. What we did here is just another explanation of the exact same arithmetic. (Look back at the examples in which we found point-normal equations of lines. We could have described the arithmetic we did there as "let $x = p_1$ and $y = p_2$" instead of "find $\vec{x} \bullet \vec{p}$".)

The Plane Determined by Three Points

In Example 3.14, we used the fact that if we know 2 vectors which are direction vectors for 2 lines contained in a plane, their cross product gives a normal for the plane. However that only works if the 2 vectors are non-collinear. That is, if the 2 vectors lie on the same line, then there are other vectors *in the same plane* that are orthogonal to both vectors. It's only if the 2 vectors are not parallel to one another that we can be sure that any (non-zero) vector which is perpendicular to both must be perpendicular to the whole plane.

Theorem 3.2. *If $\vec{u}$ is a direction vector for a line in some plane Π, and $\vec{v}$ is a direction vector for another line in plane Π, where $\vec{u}$ and $\vec{v}$ are not collinear, then $\vec{n} = \vec{u} \times \vec{v}$ is a normal vector for plane Π.*

We know that for any two points (whether in $\Re^2$ or in $\Re^3$), there is exactly one line which contains those two points. Consider 3 points in $\Re^3$. If the 3 points are all collinear, i.e. if they all lie on the same line, then there are many planes which contain those 3 points. (The infinitely many different planes that intersect along that line.) But if the 3 points are not all collinear, then there is only one plane that contains all three points.

Suppose we know three non-collinear points, P, Q and R. What do we need to do to find an equation of the plane containing those points? Well, the plane containing these points must contain the line on which P and Q both lie, and must also contain the line on which P and R both lie. (It must also contain the line on which Q and R both lie, but that's more lines than we need.) If the points are non-collinear, then these are different lines. So if we find direction vectors for those lines, then those vectors are not parallel, and we can use them to find a normal vector for the plane. Then we just use any one of the points, along with that normal vector, to write an equation of the plane. So finding an equation of the plane containing three specified points is a simple procedure. We refer to this as the plane **determined** by the three points.

Example 3.16. Find both a point-normal form equation and a standard form equation of the plane determined by the points $P(-1,0,1)$, $Q(1,2,3)$ and $R(2,-1,5)$.

Solution:
The line passing through points P and Q lies in this plane, and so any vector parallel to that line is also parallel to the plane. And if we let $\vec{u} = \vec{q} - \vec{p}$, then $\vec{u}$ is such a vector. Similarly, the vector $\vec{v} = \vec{r} - \vec{p}$ is parallel to the line which passes through both P and R, and since that line also lies in the plane, $\vec{v}$ is another vector which is parallel to the plane we need to describe. Also, we have

$$\begin{aligned} \vec{u} &= \vec{q} - \vec{p} = (1,2,3) - (-1,0,1) = (1-(-1), 2-0, 3-1) = (2,2,2) \\ \vec{v} &= \vec{r} - \vec{p} = (2,-1,5) - (-1,0,1) = (2-(-1), -1-0, 5-1) = (3,-1,4) \end{aligned}$$

and we can see that since $\vec{u}$ and $\vec{v}$ are not scalar multiples of one another then they are not collinear.

We use these two non-collinear vectors which are both parallel to the plane to find a normal for the plane:

$$\vec{n} = \vec{u} \times \vec{v} = (8-(-2), 6-8, -2-6) = (10,-2,-8)$$

Now we use this normal vector and any one of the three points to write a point-normal equation of the plane. For instance, using point P, the form $\vec{n} \bullet (\vec{x} - \vec{p}) = 0$ gives:

$$(10,-2,-8) \bullet (\vec{x} - (-1,0,1)) = 0$$

Finally, we can also rearrange this equation to standard form. Letting $\vec{x} = (x,y,z)$, we get:

$$\begin{aligned} (10,-2,-8) \bullet ((x,y,z) - (-1,0,1)) = 0 \quad &\Rightarrow \quad (10,-2,-8) \bullet (x,y,z) - (10,-2,-8) \bullet (-1,0,1) = 0 \\ &\Rightarrow \quad 10x - 2y - 8z = (10,-2,-8) \bullet (-1,0,1) \\ &\Rightarrow \quad 10x - 2y - 8z = -10 + 0 - 8 \\ &\Rightarrow \quad 10x - 2y - 8z = -18 \end{aligned}$$

(*Note:* We might prefer to divide through the equation by 2. That is, this plane would often be expressed as $5x - y - 4z = -9$.)

Determining the Distance between a Point and a Plane

Suppose we have some particular plane Π. Consider any point P which *does not* lie on this plane. How far is this point from the plane? That is, what is the shortest distance from the point to the plane?

The shortest way to get from the plane to point P is to start from the point on the plane which is nearest to point P. This will be the point P' on the plane such that the directed line segment from P' to P is normal (i.e. perpendicular) to Π. And the (shortest) distance from P to the plane will just be the length of that directed line segment, i.e. $\left|\left|\overrightarrow{P'P}\right|\right|$. However we don't want to have to actually find the point P', which is the point on the plane that is nearest to P, in order to find the distance from P to the plane. And in fact we don't need to.

Let plane Π have normal vector $\vec{n}$ and let Q be any known point on plane Π. Then it can be shown that the distance from P to P' is given by:

$$\left|\left|\overrightarrow{P'P}\right|\right| = \left|\left|\vec{p} - \vec{p'}\right|\right| = \frac{|\vec{n} \bullet (\vec{q} - \vec{p})|}{||\vec{n}||}$$

That is, we simply need to find the dot product of any normal vector to the plane with the vector equivalent to the directed line segment between the point P and *any* known point on the plane, discard the negative sign (if there is one), and divide by the magnitude of the normal vector used.

(*Notice:* We have not explained *why* this gives $\left|\left|\overrightarrow{P'P}\right|\right|$, so you should not be trying to understand that from the above. If you're interested, look at the explanation given in the text. All we've done here is to *assert* that *it can be shown that* this is true.)

Theorem 3.3. *Consider any plane Π. Let $\vec{n}$ be any normal vector for plane Π and let Q be any point on plane Π. Consider any other point P which is not on the plane Π. Then the distance between point P and plane Π is given by:*

$$distance = \frac{|\vec{n} \bullet (\vec{q} - \vec{p})|}{||\vec{n}||}$$

Example 3.17. Find the distance between the point $P(1, 2, 3)$ and the plane with point-normal form equation $(1, 2, 1) \bullet (\vec{x} - (3, -1, 0)) = 0$.

Solution:
We simply plug $\vec{n} = (1, 2, 1)$, $\vec{p} = (1, 2, 3)$ and $\vec{q} = (3, -1, 0)$ into the formula. We have:

$$\begin{aligned} \vec{q} - \vec{p} &= (3, -1, 0) - (1, 2, 3) = (2, -3, -3) \\ \text{so that } \vec{n} \bullet (\vec{q} - \vec{p}) &= (1, 2, 1) \bullet (2, -3, -3) = 1(2) + 2(-3) + 1(-3) = 2 - 6 - 3 = -7 \\ \text{and } ||\vec{n}|| &= \sqrt{1^2 + 2^2 + 1^2} = \sqrt{1 + 4 + 1} = \sqrt{6} \end{aligned}$$

and so the distance from P to the plane is

$$\text{distance} = \frac{|\vec{n} \bullet (\vec{q} - \vec{p})|}{||\vec{n}||} = \frac{|-7|}{\sqrt{6}} = \frac{7}{\sqrt{6}}$$

Example 3.18. Find the distance from the origin to the plane $x + y - z = 5$.

Solution:
The origin is the point $P(0, 0, 0)$. Notice that we know that this point is not on the plane because $(x, y, z) = (0, 0, 0)$ does not satisfy the equation of the plane. Also, we recognize from the standard form equation that $\vec{n} = (1, 1, -1)$ is a normal for the plane. So now we just need to find any point that's on the plane. Simply pick any values of x, y and z that, when taken together, *do* satisfy the equation of the plane. For instance, for the point $(x, y, z) = (5, 0, 0)$ we get $x + y - z = 5 + 0 - 0 = 5$, so $Q(5, 0, 0)$ is a point on the plane.

Now we just use the formula. We see that the distance from the origin to the plane $x + y - z = 5$ is:

$$\frac{|\vec{n} \bullet (\vec{q} - \vec{p})|}{||\vec{n}||} = \frac{|(1, 1, -1) \bullet ((5, 0, 0) - (0, 0, 0))|}{||(1, 1, -1)||} = \frac{|(1, 1, -1) \bullet (5, 0, 0)|}{\sqrt{1^2 + 1^2 + (-1)^2}} = \frac{5}{\sqrt{3}}$$

Notice: If we accidentally try to find the distance from a point P to a plane Π for a point P which actually *lies on the plane* Π, we will simply get the answer 0. In that case the directed line segment from P to another point Q known to be on the plane lies on the plane, so any normal for the plane is orthogonal to $\vec{q} - \vec{p}$. And we know that if two vectors are orthogonal then their dot product is 0, so the numerator of the formula will be 0, giving the answer 0 as the distance.

Finding the Distance Between a Point and a Line

We have already seen some ways in which planes in $\Re^3$ behave like lines in $\Re^2$. That is, we know that in $\Re^2$, a point-normal form equation corresponds to a line, but the similar form in $\Re^3$ corresponds to a plane. And the same is true for a standard form equation. In $\Re^2$ it represents a line, but in $\Re^3$ it represents a plane. And both of these forms use a normal vector. In fact, in $\Re^2$ we talk about a normal vector for a line, but in $\Re^3$ we only talk of a normal vector for a plane.

So it may not surprise you to learn that the same distance formula given in Theorem 3.3, which refers to the distance between a point and a plane in $\Re^3$, can also be used in $\Re^2$, but there it gives the distance between a point and a *line*.

Theorem 3.4. *Consider any line ℓ. Let $\vec{n}$ be any normal vector for line ℓ and let Q be any point on line ℓ. Consider any other point P which is not on line ℓ. Then the distance between point P and line ℓ is given by:*

$$distance = \frac{|\vec{n} \bullet (\vec{q} - \vec{p})|}{||\vec{n}||}$$

Example 3.19. Find the distance between the point $P(1,2)$ and the line ℓ described by $2x + y = 1$.

Solution:
Line ℓ has normal $\vec{n} = (2,1)$. We need to find some point Q on line ℓ. Letting $x = 0$ we get $2(0) + y = 1$, so $y = 1$. That is, the point on line ℓ which has x-coordinate 0 has y-coordinate 1, so the point $Q(0,1)$ is a point on line ℓ. (Notice that for $(x,y) = (1,2)$ we have $2x+y = 2(1)+2 = 4 \neq 1$, so $P(1,2)$ is not on line ℓ.)

The distance between P and ℓ is

$$\frac{|\vec{n} \bullet (\vec{q} - \vec{p})|}{||\vec{n}||} = \frac{|(2,1) \bullet ((0,1) - (1,2))|}{||(2,1)||} = \frac{|(2,1) \bullet (-1,-1)|}{\sqrt{2^2 + 1^2}} = \frac{|-2-1|}{\sqrt{4+1}} = \frac{3}{\sqrt{5}}$$

Finding the Intersection of Two Lines

Any 2 lines which lie in the same plane and are not parallel intersect at exactly 1 point. To find this point of intersection, we simply need to find a point which is on both lines. We use parametric equations of at least one of the lines when we do this. We can either: (1) Use parametric equations of *both* lines to equate corresponding components of a point, and solve for the values of the parameters which satisfy all of those equations; or (2) Use parametric equations of one line and a standard form equation for the other line (if they are lines in $\Re^2$), and substitute for x and y in terms of the parameter, then solve for the value of the parameter.

Example 3.20. Find the point of intersection of the line ℓ_1: $\vec{x}(t) = (1,0) + t(2,1)$ with the line ℓ_2: $\vec{x}(s) = (1,1) + s(-1,0)$.

Solution:

For ℓ_1 we have parametric equations $\begin{array}{rcl} x & = & 1+2t \\ y & = & t \end{array}$ and for ℓ_2 we have $\begin{array}{rcl} x & = & 1-s \\ y & = & 1 \end{array}$.

If some point $P(x,y)$ is on both these lines, then it must be true that there are some values of t and s which give the same values of x and y. So we must have $1+2t = 1-s$ and $t = 1$. Since $t = 1$, then $1+2t = 3$, so $1-s = 3$ and we see that $s = 1-3 = -2$.

Notice that we've found values of the parameters, s and t, but we have not yet found the point on the line which corresponds to these values. That is, we know the value of t that gives the point on line ℓ_1 at which the two lines intersect, and likewise we know the value of s that gives that same point on line ℓ_2. But we were asked to find the actual point at which the two lines intersect. We're not finished until we've done that. And we have more information than we need to find the point, since we know two ways to get it. So we can use the value of t we found, in the equation for ℓ_1, to get the point P. And then we can use the value of s we found, in the equation of ℓ_2, to *check* our work. We get:

$$t = 1 \qquad \Rightarrow \qquad (x,y) = (1,0) + t(2,1) = (1,0) + 1(2,1) = (3,1)$$

as the point on ℓ_1 which we were looking for. We check that the point on ℓ_2 is the same point:

$$s = -2 \qquad \Rightarrow \qquad (x,y) = (1,1) + s(-1,0) = (1,1) + (-2)(-1,0) = (1,1) + (2,0) = (3,1)$$

Since we did find the same point on each line, this is the point we were looking for. We see that ℓ_1 and ℓ_2 intersect at the point $P(3,1)$.

Note: As we observed above, we found values of both parameters, but really we only need one. As we have seen, the other allows us to *check* our work. We're just checking that we didn't make an arithmetic error. If we got a different point on ℓ_2 than the one on ℓ_1 that would tell us that somewhere in our calculations we made an arithmetic mistake. Either in finding the points, or (more likely) in finding the values of the parameters. We would need to re-do our calculations until we find the mistake, and then finish the problem (including the check) again.

Example 3.21. Find the point of intersection of the line ℓ_1: $\vec{x}(t) = (1,1,2) + t(2,1,-1)$ with the line ℓ_2: $\vec{x}(s) = (0,1,2) + s(1,-1,1)$.

Solution:

For ℓ_1 we have $\begin{array}{rcl} x & = & 1+2t \\ y & = & 1+t \\ z & = & 2-t \end{array}$ and for ℓ_2 we have $\begin{array}{rcl} x & = & s \\ y & = & 1-s \\ z & = & 2+s \end{array}$.

The point of intersection of ℓ_1 and ℓ_2 is a point $P(x,y,z)$ which satisfies both sets of equations at the same time, so we must have:

$$\begin{aligned} 1+2t &= s && (1) \\ 1+t &= 1-s && (2) \\ 2-t &= 2+s && (3) \end{aligned}$$

Equation (1) says that $s = 1+2t$, so that $1-s = 1-(1+2t) = 0-2t = -2t$. Therefore equation (2) gives $1+t = -2t$, so $1 = -3t$ and thus $t = -\frac{1}{3}$. And then substituting $t = -\frac{1}{3}$ into $s = 1+2t$

gives $s = 1 + 2\left(-\frac{1}{3}\right) = 1 - \frac{2}{3} = \frac{1}{3}$. Checking these values in (3) we get

$$\begin{array}{rcl} 2 - t & = & 2 - \left(-\dfrac{1}{3}\right) = 2 + \dfrac{1}{3} = \dfrac{6}{3} + \dfrac{1}{3} = \dfrac{7}{3} \\ \text{and } 2 + s & = & 2 + \dfrac{1}{3} = \dfrac{7}{3} \\ \text{so it's true that } 2 - t & = & 2 + s \end{array}$$

Now, using $s = \frac{1}{3}$ in the equation for ℓ_2 we get the point

$$(x, y, z) = (0, 1, 2) + s(1, -1, 1) = (0, 1, 2) + \left(\frac{1}{3}\right)(1, -1, 1) = \left(\frac{0}{3}, \frac{3}{3}, \frac{6}{3}\right) + \left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right) = \left(\frac{1}{3}, \frac{2}{3}, \frac{7}{3}\right)$$

As before, we can use $t = -\frac{1}{3}$ to check that we haven't made any mistakes:

$$(x, y, z) = (1, 1, 2) + t(2, 1, -1) = (1, 1, 2) + \left(-\frac{1}{3}\right)(2, 1, -1) = \left(\frac{3}{3}, \frac{3}{3}, \frac{6}{3}\right) + \left(-\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}\right) = \left(\frac{1}{3}, \frac{2}{3}, \frac{7}{3}\right)$$

We see that ℓ_1 and ℓ_2 intersect at this common point, $P\left(\frac{1}{3}, \frac{2}{3}, \frac{7}{3}\right)$.

Note: Any 2 non-parallel lines in $\Re^2$ always intersect at a single point. For 2 non-parallel lines in $\Re^3$, there are 2 possibilities. If they lie in the same plane, then they intersect at a single point. But they may lie in parallel planes and not intersect at all. In that case, it will be impossible to find values of s and t that satisfy all 3 equations at the same time.

As stated earlier, if we wish to find the point of intersection of 2 lines in $\Re^2$ and one of the lines is in point-normal form or standard form, we don't need to find a direction vector and write parametric equations of that line. Instead we can just use the standard form directly, as shown in the next example.

Example 3.22. Find the point of intersection of the lines ℓ_1: $\vec{x}(t) = (1, 0) + t(3, -1)$ and ℓ_2: $2x - y = 5$.

Solution:
For ℓ_1 we have $\begin{array}{rcl} x & = & 1 + 3t \\ y & = & -t \end{array}$. Substituting for x and y in the standard form equation for ℓ_2 we have:

$$2x - y = 5 \quad \Rightarrow \quad 2(1 + 3t) - (-t) = 5 \quad \Rightarrow \quad 2 + 6t + t = 5 \quad \Rightarrow \quad 7t = 3 \quad \Rightarrow \quad t = \frac{3}{7}$$

Now we find the point on ℓ_1 corresponding to $t = \frac{3}{7}$:

$$(x, y) = (1, 0) + t(3, -1) = (1, 0) + \left(\frac{3}{7}\right)(3, -1) = \left(\frac{7}{7}, \frac{0}{7}\right) + \left(\frac{9}{7}, -\frac{3}{7}\right) = \left(\frac{16}{7}, -\frac{3}{7}\right)$$

Of course, we should check that this really is a point on ℓ_2 (i.e. check for arithmetic mistakes). For $(x, y) = \left(\frac{16}{7}, -\frac{3}{7}\right)$ we get:

$$2x - y = 2\left(\frac{16}{7}\right) - \left(-\frac{3}{7}\right) = \frac{32}{7} + \frac{3}{7} = \frac{35}{7} = 5$$

Since $2x - y = 5$, we see that $(x, y) = \left(\frac{16}{7}, -\frac{3}{7}\right)$ *is* a point on ℓ_2. So ℓ_1 and ℓ_2 intersect at the point $\left(\frac{16}{7}, -\frac{3}{7}\right)$.

The Intersection of a Line with a Plane

If line ℓ lies in plane Π, then all points on line ℓ are on plane Π. If line ℓ lies on a plane parallel to plane Π, then no point on line ℓ lies on plane Π. But if line ℓ does not lie on plane Π or on any plane parallel to Π, then ℓ intersects Π at a single point. That is, there is only a single point on line ℓ which lies on plane Π.

To find the point of intersection of line ℓ with plane Π, we use parametric equations of ℓ to express the coordinates of the point in terms of the parameter, then use the standard form equation of Π to solve for the value of the parameter. (This is exactly what we did in Example 3.22. But now there's z as well as x and y.)

Example 3.23. Find the point at which the line ℓ described by $\vec{x}(t) = (1,0,1) + t(3,2,1)$ intersects the plane Π described by $x + y - 2z = 3$.

Solution:
Parametric equations of line ℓ give:

$$\begin{aligned} x &= 1 + 3t \\ y &= 2t \\ z &= 1 + t \end{aligned}$$

We substitute these expressions into the equation for Π:

$$\begin{aligned} x + y - 2z = 3 \quad &\Rightarrow \quad (1+3t) + 2t - 2(1+t) = 3 \quad &&\Rightarrow \quad 1 + 3t + 2t - 2 - 2t = 3 \\ &\Rightarrow \quad 3t = 3 - (-1) = 4 &&\Rightarrow \quad t = \tfrac{4}{3} \end{aligned}$$

For this value of t we find the point on line ℓ to be

$$(x,y,z) = (1,0,1) + t(3,2,1) = (1,0,1) + \left(\frac{4}{3}\right)(3,2,1) = (1,0,1) + \left(4, \frac{8}{3}, \frac{4}{3}\right) = \left(5, \frac{8}{3}, \frac{7}{3}\right)$$

Therefore ℓ and Π intersect at the point $\left(5, \frac{8}{3}, \frac{7}{3}\right)$.

Check: We check that this point really is on plane Π:

$$x + y - 2z = 5 + \frac{8}{3} - 2\left(\frac{7}{3}\right) = \frac{15}{3} + \frac{8}{3} - \frac{14}{3} = \frac{15 + 8 - 14}{3} = \frac{9}{3} = 3$$

Since $(x,y,z) = \left(5, \frac{8}{3}, \frac{7}{3}\right)$ does give $x + y - 2z = 3$, this point is on plane Π.

Notice: This procedure only works when the line ℓ does not lie in the plane Π or in any plane parallel to Π. If line ℓ lies in plane Π, then when we substitute into the equation for Π and try to solve for t, the t's will all disappear and we'll be left with something like $3 = 3$. This equation is satisfied for *all* values of t, telling us that all points on ℓ satisfy the equation for Π. But if line ℓ lies on a plane parallel to Π, then again all the $t's$ will disappear, but we'll be left with something like $2 = 3$. This equation isn't satisfied for any value of t, telling us that there is *no* value of t for which $\vec{x}(t)$ is a point on plane Π.