Lines and planes in $\mathbb{R}^3$

Lines in $\mathbb{R}^2$ or $\mathbb{R}^3$

lines in $\mathbb{R}^2$

Point-parallel form	$\vec{x}(t) = (p_1, p_2) + t(v_1, v_2)$
Parametric form	$x = p_1 + tv_2$ and $y = p_2 + tv_2$
Two-point form	$\vec{x}(t) = (1 - t)(p_1, p_2) + t(q_1, q_2)$
Point-normal form	$(n_1, n_2) \cdot (\vec{x} - (p_1, p_2)) = 0$
Standard form	$ax + by = c$

- $P(p_1, p_2)$ and $Q(q_1, q_2)$ are two different points on the line.
- $\vec{v} = (v_1, v_2)$ is the *direction vector* of the line.
- $\vec{n} = (n_1, n_2)$ is the *normal vector* of the line.
- t is a parameter.

Lines in $\mathbb{R}^2$ or $\mathbb{R}^3$

	lines in $\mathbb{R}^2$	lines in $\mathbb{R}^3$
Point-parallel form	$\vec{x}(t) = (p_1, p_2) + t(v_1, v_2)$	$\vec{x}(t) = (p_1, p_2, p_3) + t(v_1, v_2, v_3)$
Parametric form	$x = p_1 + tv_2$ and $y = p_2 + tv_2$	$x = p_1 + tv_2, y = p_2 + tv_2$ and $z = p_3 + tv_3$
Two-point form	$\vec{x}(t) = (1 - t)(p_1, p_2) + t(q_1, q_2)$	$\vec{x}(t) = (1 - t)(p_1, p_2, p_3) + t(q_1, q_2, q_3)$
Point-normal form	$(n_1, n_2) \cdot (\vec{x} - (p_1, p_2)) = 0$	?
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- t is a parameter.

Definition If $\vec{v} \in \mathbb{R}^3$ is parallel to some line L in $\mathbb{R}^3$, we say that $\vec{v}$ is a *direction vector* for L.

Examples

1. Give a point-parallel form for the line through the point $P(2, 2, 0)$ and parallel to the vector $\vec{v} = (3, -1, 2)$.

$(2,2,0) + t(3,-1,2)$

2. What are the parametric form and two-point form of it?

$x = 2 + 3t$

3. L_1 is the line $\vec{x}(t) = (1 - t)(2, 1, 1) + t(0, 1, 2)$. L_2 is the line with parametric equations $x = 2t - 2, y = 1, z = 5 - 3t$. Are L_1 and L_2 the same line?

Planes in $\mathbb{R}^3$

Recall in $\mathbb{R}^2$, we have the point-normal form of a line L

$$\vec{n} \cdot (\vec{x} - (p_1, p_2)) = 0$$

where $\vec{n}$ is the normal vector to L and (p_1, p_2) is on the line.

What about $\vec{n} \cdot (\vec{x} - (p_1, p_2, p_3)) = 0$ in $\mathbb{R}^3$?

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Definition A vector which is perpendicular to a particular plane in $\mathbb{R}^3$ is said to be *normal* to the plane, and is called a normal for that plane, or a *normal vector* for the plane.

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Definition The *point-normal form* of an equation of a plane Π in $\mathbb{R}^3$, where $\vec{n} = (n_1, n_2, n_3)$ is any normal vector to the plane and $P(p_1, p_2, p_3)$ is any point on the plane, is given by

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0, \text{ i.e., } (n_1, n_2, n_3) \cdot (\vec{x} - (p_1, p_2, p_3)) = 0.$$

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Examples Write an equation in a point-normal form and a standard form for a plane with a normal vector $\vec{n} = (1, 0, 3)$ which contains the point $P(-1, 2, 3)$.

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Theorem If $\vec{u}$ is a direction vector for a line in some plane Π, and $\vec{v}$ is a direction vector for another line in plane Π, where $\vec{u}$ and $\vec{v}$ are not collinear, then $\vec{n} = \vec{u} \times \vec{v}$ is a normal vector Π.

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Examples 1. Find both a point-normal form equation and a standard form equation of the plane determined by the points $P(1,0,1), Q(1,2,3)$ and $R(2,1,5)$.

2. What is the point-normal form of the plane $2x - y + 3z = 1$?

Examples

1. Give a normal vector of the line $(x, y) = (3, 2) + t(7, 1)$.

2. Find a standard form equation of the plane containing the points $P(-2, 0, 1)$, $Q(0, 1, 3)$ and $R(-1, 1, -3)$.

3. What is a standard form equation for the plane through point $P(3, 4, -1)$ with normal vector $\vec{n} = (-2, 3, 4)$?

4. Convert the standard form equation $x - 2y + z = 6$ to point-normal form for the plane in $\mathbb{R}^3$.

5. Convert the line $\vec{x}(t) = (2, 3) + t(1, -3)$ into point-parallel form and the point-normal form.