e.g. $f(x)=\frac{3}{x^2-x-2}$

express $f(x)$ in sums of two partial fractions.

and combine these fractions.

$$f(x)=\frac{1}{x-2}-\frac{1}{x+1}.$$

$$-\frac{1}{x+1}=-\frac{1}{1-(-x)}=-\sum_{n=0}^{\infty}(-x)^n=-\sum_{n=0}^{\infty}(-1)^n x^n=\sum_{n=0}^{\infty}(-1)^{n+1}x^n. \quad |x|<1$$

$$\frac{1}{x-2}=\frac{1}{-2(1-\frac{x}{2})}=-\frac{1}{2}\left(\frac{1}{1-\frac{x}{2}}\right)=-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n \quad \left|\frac{x}{2}\right|<1.$$

$$f(x)=\sum_{n=0}^{\infty}(-1)^{n+1}x^n-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n.$$

$$=\sum_{n=0}^{\infty}\left[(-1)^{n+1}-\frac{1}{2^{n+1}}\right]x^n. \quad |x|<1.$$

e.g. $f(x)=\arctan x.$

Find the power series of $f(x)$.

$$\int\frac{1}{1+x^2}dx=\arctan x+C.$$

$$\frac{1}{1+x^2}=\frac{1}{1-(-x^2)}=\sum_{n=0}^{\infty}(-x^2)^n=\sum_{n=0}^{\infty}(-1)^n x^{2n}. \quad |x|<1.$$

$$\int\frac{1}{1+x^2}dx=\int\sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$\arctan x+C=\sum_{n=0}^{\infty}(-1)^n\cdot\int x^{2n}dx.$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

let $x=0$, then

$\arctan x+C=0$ when $C=0$

$\therefore \arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}\cdots\cdots$

let $x=1$. $\arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots\cdots$
$= \frac{\pi}{4}$.

e.g. J. $\lim\limits_{x\to 0} \frac{x - \arctan(x)}{x^3}$.

$\lim\limits_{x\to 0} x^3 = 0 \quad \lim\limits_{x\to 0} (x - \arctan x) = 0$.

(1). L'Hospital rule: $\lim\limits_{x\to 0} \frac{1 - \frac{1}{1+x^2}}{3x^2} = \frac{1}{3}$.

(2). Using power series:

$\lim\limits_{x\to 0} \frac{x - (x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \cdots)}{x^3}$

$= \lim\limits_{x\to 0} \frac{1}{x^3} (\frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} \cdots\cdots)$

$= \lim\limits_{x\to 0} (\frac{1}{3} - \frac{x^2}{5} + \frac{x^4}{7} \cdots\cdots)$.

$= \frac{1}{3}$. ⇐ 人才解法.

taylor & maclaurin series

Suppose that f is any differentiable function that be reperesent by a power series.

$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$.
$= \sum\limits_{n=0}^{\infty} c_n(x-a)^n$.

$f'(x) = [\sum\limits_{n=0}^{\infty} c_n(x-a)^n]' = \sum n c_n (x-a)^{n-1}$.

$f''(x) = \sum_{n=0}^{\infty} n(n-1) c_n (x-a)^{n-2}.$

$f'''(x) = \sum_{n=0}^{\infty} n(n-1)(n-2) c_n (x-a)^{n-3}.$

let $x=a$. Then

$f(a) = c_0$

$f'(a) = 1 \cdot c_1$

$f''(a) = 2\times1\cdot c_2.$

$f'''(a) = 3\times2\times1\cdot c_3.$

$\vdots$

$f^{(n)}(a) = n!\, c_n.$

$\sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$

开始蒙人了…

the power series $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is called the taylor series of function at the point $x=a$.

If $a=0$ $f(x)$ becomes

$f(x) = \sum_{x=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

this series is called Maclaurin series.

e.g. Find the Maclaurin series of e^x

let $f(x) = e^x$ $\quad f(0) = 1$

$f'(x) = e^x$ $\quad f'(0) = 1 \Rightarrow f(x) = \sum_{x=0}^{\infty} \frac{x^n}{n!} = a_n.$

$\vdots$ $\quad \vdots$

$f^{(n)}(x) = e^x.$ $\quad f^{(n)}(0) = 1$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x}{n+1}\right|$

$\because$ x is independent to n

$\therefore = |x| \lim\limits_{n\to\infty} \left|\frac{1}{n+1}\right| = 0 < 1$

$\therefore$ the power series converges everywhere (写过程!)

i.e. the interval of converges is $(-\infty, \infty)$.

$e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ let $x=1$ $e^1 = \sum\limits_{n=0}^{\infty} \frac{1}{n!}$

e.g. 2. $\sin x$. $f(0) = 0$

$f'(x) = \cos x$ $f'(0) = 1$

$f''(x) = -\sin x$ $f''(0) = 0$

$f'''(x) = -\cos x$ $f'''(0) = -1$

$f''''(x) = \sin x$. $f''''(0) = 0$.

$$\therefore \sin x = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{n}(0)}{n!}x^n.$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \cdots\cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{x^2}{2n+3}\right|$$

$$= x^2 \lim_{n\to\infty}\frac{1}{2n+3} = 0 < 1.$$

$\therefore$ converges on the interval $(-\infty, \infty)$.

e.g. 3. $\lim\limits_{x\to 0}\frac{\sin x}{x}$.

L'Hospital rule $= \lim\limits_{x\to 0} = \lim\limits_{x\to 0}\frac{\cos x}{1} = 1$.

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots}{x}$$

$$= \lim_{x \to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)$$

$$= 1.$$

e.g. 4. Find the Maclaurin series of $\cos x$. and its interval of convergence.

First we know $(\sin x)' = \cos x$.

the power series of $\sin x$: $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$$\therefore \cos x = \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\right]'$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{(2n)!} x^{2n}$$

$$\therefore \cos x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$$