Integration By Parts. Sec 7.1.

$\frac{d}{dx}(x) = x'$

$\int x = \int x\,dx$

$= \frac{1}{2}x^2 + C.$

$$\int u\,dv = uv - \int v\,du.$$

Prove: We start with the product rule.

$v'dx \Rightarrow dv.$

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + \frac{dv}{dx}v \qquad \frac{du}{dx} = u'$$

$$\frac{d}{dx}(uv) - \frac{du}{dx}v = \frac{dv}{dx}v$$

$$\int u \underbrace{\boxed{\frac{dv}{dx}dx}}_{dv} = \int \underbrace{\boxed{\frac{d}{dx}(uv)dx}}_{uv} - \int v \underbrace{\boxed{\frac{du}{dx}dx}}_{du.}$$

$$\int u\,dv = uv - \int v\,du.$$

$$uv = \int u\,dv + \int v\,du.$$

e.g. 1: Evaluate $\int xe^{-x}dx.$ 反导.

$u\int dv = uv - v\int du.$

$v'dx = dv.$

$= \int e^{-x}\,x\,dx.$

$v = -e^{-x}$

$du = x\,dx,$ $\quad = -x\cdot(e^{-x}) - (-\int e^{-x}dx)$

$= -xe^{-x} - e^{-x} + C.$

$\int xe^{-x}dx.$

$x = u \quad e^{-x}dx = v'dx. = dv$

$dv = -de^{-x}. \quad v = -e^{-x}.$

$u = x.$

$\int u\,dv = vu - \int v\,du.$

$= -xe^{-x} - \int -e^{-x}dx.$

$= -xe^{-x} + \int e^{-x}dx$

$= -xe^{-x} - e^{-x} + C.$

2. Evaluate $\int \underset{u}{\ln x}\,\underset{dv}{dx}.$

$u = \ln x \qquad = \underbrace{\ln x\cdot x}_{uv} - \int 1.\,dx.$

$du = \frac{1}{x}dx. \qquad = x\ln x. - x + C.$

$dv = dx$

$v = x.$

$\int \underline{u}\,\underline{v'dx}$

$= \int u\,dv$

应找出u和$v'dx$.

3. Evaluate. $\int \underset{u^2}{\cos^2 x}\,\underset{dv}{dx}.$

$u = \cos x$ | $uv - (-\int \sin x \cdot \sin x\,dx)$.

$du = -\sin x$ | $= \cos x \sin x + \int \sin^2 x\,dx$.

$v = \sin x$. | $= \cos x \sin x + \int (1-\cos^2 x)\,dx$.

$dv = \cos x\,dx$. | $= \cos x \sin x + \int dx - \int \cos^2 dx. = \int \cos^2 dx$

$$\int \cos^2 dx = \frac{\sin x \cos x + x}{2} + C.$$

4. Evaluate $\int \sin^2 x\,dx$. $\int \sin x\,dx$.

$I_n = \int f(x, n)\,dx$ where n is a positive integer.

a reduction formula

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a relation between I_n and I_{n-1}.

$\int \sin^n x\,dx$

$= \int \underbrace{\sin^{n-1} x}_{u}\ \underbrace{\sin x\,dx}_{dv}$. $v = -\cos x$.

$du = (n-1)\sin^{n-2} x \cdot (\cos x)\,dx$.

$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,(1-\sin^2 x)\,dx$

$\int \sin^n x\,dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x - (n-1)\int \sin^n x$.

$n\int \sin^n x\,dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x$.

$I_n = \sin^n x$

$$\int \sin^n x\,dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x$$

Evaluate $\int x^3 e^{-x}\,dx$.

adjacent table

$\frac{d}{dx}$ u	dv $\int x$.	
x^3	e^{-x}	
$3x^2$	$-e^{-x}$	$+$
$6x$	e^{-x}	$-$
6	$-e^{-x}$	$+$
0	e^{-x}	$-$

$$\int x^3 e^{-x}\,dx = -x^3 e^{-x} - 3x^2 e^{-x} - 6x e^{-x} - 6e^{-x} + C$$

$$= (-x^3 - 3x^2 - 6x - 6)e^{-x} + C.$$

Addition formulas.

$\sin(a+b) = \sin a \cos b + \cos a \sin b.$

$\cos(a+b) = \cos a \cos b - \sin a \sin b.$

$\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1.$

$ax + by = e$

$cx + dy = f.$

$$\frac{\det\begin{bmatrix} e & b \\ f & d \end{bmatrix}}{\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}} = x. \qquad \frac{\det\begin{bmatrix} a & e \\ c & f \end{bmatrix}}{\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}} = y.$$

$I = \int e^{ax} \cos bx\, dx.\ J = \int e^{ax} \sin bx\, dx.$

$u = e^{ax}. \quad u'dx = a e^{ax} dx.$

$v'dx = \cos bx\, dx.$

$v = \frac{1}{b} \sin x.$

$I + \frac{a}{b} J = \frac{1}{b} e^{ax} \sin bx.$

$v'dx = dv.$ $\qquad \underline{\int x\,dx = \frac{1}{2}x^2 + C.}$

$uv = \int u\,dv + \int v\,du$

$= \int uv'dx + \int vu'dx.$

例1: $\int xe^{-x}dx.$

$$\underline{u} \quad \overline{v'dx}$$

$u = x \qquad du = u'dx = 1.$

$dv = v'dx = e^{-x}dx \qquad v = -e^{-x}$

$$\int \underset{u}{x}\,\underset{v'dx}{e^{-x}dx} = \underset{u}{x} \cdot \underset{v}{(-e^{-x})} - \int \underset{v}{-e^{-x}} \cdot \underset{u'dx}{dx}.$$

$$= -xe^{-x} - e^{-x} + C$$

当遇到 sin/cos 时，左右侧分别分布积分，再整合.

$1 + \tan^2\theta = \sec^2\theta.$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx$$

$x^2 = a^2\cos^2 x.$

Patial Franchims. & intergrals of rational form. (See 7.4)

In this section, we evaluate:

$$\int \frac{P_m(x)}{Q_n(x)}\,dx \text{ where } P_m(x), Q_n(x) \text{ are polynomials}$$

consider two fractions $\frac{21}{6}$ and $\frac{3}{5}$.

$\uparrow$ in proper => greater than 1. $\qquad \uparrow$ proper.

$\frac{21}{6} \Rightarrow 3\frac{1}{2}.$ (proper).

if $m \geq n$. we use long division to convert it to

$a \overline{)\, b}$

a proper form (the sum of a polynomial and a proper fraction).

e.g. $\dfrac{x^2+1}{x^2+3x+1} = 1 - \dfrac{3x}{x^2+3x+1}$. ← proper form.

proper fractions.

rules of decomposing a proper fraction into PFs.

if $Q_m(x) = (x-\alpha)(x-\beta)(x-\gamma)$ where α, β, γ are distinct.

$$\frac{P_m(x)}{Q_m(x)} = \frac{P_m(x)}{(x-\alpha)(x-\beta)(x-\gamma)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{x-\gamma}.$$

if $Q_m(x) = (x-\alpha)(x-\alpha)(x-\gamma)(x-f) = (x-\alpha)^2(x-\gamma)(x-f)$

$$\frac{P_m(x)}{Q_m(x)} = \frac{P_m(x)}{(x-\alpha)^2(x-\gamma)(x-f)} = \frac{A}{(x-\alpha)^2} + \frac{B}{(x-\alpha)} + \frac{C}{x-\gamma} + \frac{D}{x-f}$$

Similarly, if $Q_m(x) = (x-\alpha)^3(x-\beta)$.

$$\frac{P_m(x)}{Q_m(x)} = \frac{A}{(x-\alpha)^3} + \frac{B}{(x-\alpha)^2} + \frac{C}{(x-\alpha)} + \frac{D}{(x-\beta)}.$$

if $Q_m(x) = (\underbrace{ax^2+bx+c}_{\text{irreducible}})(x-\alpha)(x-\beta)$

irreducible. (i.e. cannot be factorized => $b^2-4ac<0$.

$$\frac{P_m(x)}{Q_m(x)} = \frac{P_m(x)}{(ax^2+bx+c)(x-\alpha)(x-\beta)} = \frac{Ax+B}{ax^2+bx+c} + \frac{C}{x-\alpha} + \frac{D}{x-\beta}.$$

if $Q_m(x) = (ax^2+bx+c)^2(x-\alpha)(x-\beta)$.

$$\frac{P_m(x)}{Q_m(x)} = \frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2} + \frac{E}{(x-\alpha)} + \frac{F}{(x-\beta)}.$$

e.g. Decompose $\dfrac{x+1}{x^2-4x+3}$ into partial fraction and evaluate this intergral.

$$\int \frac{x+1}{(x-1)(x-3)} \qquad \frac{x+1}{(x-1)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-3)} = \frac{-1}{(x-1)} + \frac{2}{x-3}.$$

$$\frac{A}{(x-1)} + \frac{B}{(x-3)} = \frac{Ax-3A+Bx-B}{(x-1)(x-3)}.$$

$x+1 = A(x-3) + B(x-1).$

$\begin{cases} A+B=1 \\ -3A-B=1. \end{cases} \Rightarrow \begin{matrix} A=-1 \\ B=2. \end{matrix}$

plug in $x=3 \Rightarrow B$

$x=1 \Rightarrow A.$

$$-\int \frac{1}{x-1}dx + 2\int \frac{1}{x-3}\,dx.$$

$$= -\ln|x-1| + 2\ln|x-3| + C.$$

e.g. 2 $\displaystyle\int \frac{4x^2+2x+3}{(x+3)(x-2)^2}$

$$\frac{4x^2+2x+3}{(x+3)(x-2)^2} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x+3)}.$$

$(x-2)\ (x+3)\ A + \quad (x+3)\ B + (x-2)^3\ C = 4x^2+2x+3.$

$x=2 \Rightarrow B = \frac{23}{5}.$ $\quad A = \frac{67}{25}.$

$x=-3 \Rightarrow C = \frac{33}{25}.$

rewrite it into decending of x.

$4x^2+2x+3 = A(x^2-4x+4) + B(x^2+x-6) + C(x+3).$

$$= \frac{33}{25}\int \frac{1}{x+3}\,dx + \frac{67}{25}\int \frac{1}{x-2}dx + \frac{23}{5}\int \frac{1}{(x-2)^2}dx.$$

$$= \frac{33}{25}\ln|x+3| + \frac{67}{25}\ln|x-2| - \frac{23}{5}\cdot\frac{1}{(x-2)} + C.$$

e.g. 3 $\displaystyle\int \frac{1}{x^3-1}\,dx.$ $\qquad \dfrac{1}{(x-1)(x^2+x+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+1}$

$$= \int \frac{1}{(x-1)(x^2+x+1)}.$$

$1 = A(x^2+x+1) + (Bx+C)(x-1).$

$x=1 \Rightarrow A = \frac{1}{3}.$

$x=0 \Rightarrow 1 = A - C. \Rightarrow C = -\frac{2}{3}.$

$x=-1 \Rightarrow B = -\frac{1}{3}$

$$\int \frac{1}{1-x^2}\,dx = \frac{\ln|x+1|}{2} - \frac{\ln|x-1|}{2}$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x.$$

$$\int \frac{1}{x^2+x+1}\,dx.$$

$$= \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx.$$

$$= \int \frac{1}{u^2+\frac{3}{4}}\,du = \int \frac{\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta}{\frac{3}{4}\tan^2\theta+\frac{3}{4}}.$$