The theory of SLE

(2)

Rank of a matrix

Recall that every matrix can be reduced to a unique row reduced echelon form (RREF).

Rank of a matrix

Recall that every matrix can be reduced to a unique row reduced echelon form (RREF).

The *rank* of a matrix A is the number of non-zero rows in the row-reduced echelon form of the matrix A. We can use $r(A)$ to denote the rank of matrix A.

Also, we say that the $m \times n$ matrix A has *full rank* if $r(A) = n$.

Find the rank of each matrix. Which has a full rank?

(a) $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -2 & -3 & 4 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 3 & 3 & 3 \end{bmatrix}$

$4 - 21 + 6$

(c) $\left[A \mid \vec{b}\right] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 3 & 4 \end{array}\right]$ (d) $\left[A \mid \vec{b}\right] = \left[\begin{array}{cc|c} 3 & 4 & 1 \\ 2 & 1 & 2 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 2 & 1 & 3 \\ 3 & 3 & 4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 7 \\ 3 & 3 & 4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 7 \\ 0 & 0 & -11 \end{array}\right]$

Solution. Need to perform elementary row operations to tranform each matrix to a row-reduced echelon form (RREF).

(a).

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -2 & -3 & 4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The rank of A is $r(A) = 3$ and it is full rank.

(b).

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The rank of A is $r(A) = 1$ and since the number of columns of A is $n = 3 \neq r(A)$, it is not full rank.

(c).

$$\left[A \mid \vec{b}\right] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 3 & 4 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

?

The rank of $\left[A \mid \vec{b}\right]$ is 3 and it is full rank.

How about the matrix A? (the rank of A?)

(c).

$$\left[A \mid \vec{b}\right] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 3 & 4 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

The rank of $\left[A \mid \vec{b}\right]$ is 3 and it is full rank.

How about the matrix A? (the rank of A?)

The rank of A is $r(A) = 2$ and it is full rank.

(d).

$$\left[A \mid \vec{b}\right] = \left[\begin{array}{cc|c} 3 & 4 & 1 \\ 2 & 1 & 2 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & \frac{7}{5} \\ 0 & 1 & -\frac{4}{5} \end{array}\right]$$

The rank of $\left[A \mid \vec{b}\right]$ is 2 and it is not full rank.

The rank of A is $r(A) = 2$ and it is full rank.

We see that the rank $r(A)$ of an $m \times n$ matrix A satisfies

$$r(A) \leq m \text{ and } r(A) \leq n.$$

We see that the rank $r(A)$ of an $m \times n$ matrix A satisfies

$$r(A) \leq m \text{ and } r(A) \leq n.$$

This is because the rank of a matrix A is the number of leading ones in the row reduced echelon form of A. Each leading one sits at a certain row and a certain column.

We see that the rank $r(A)$ of an $m \times n$ matrix A satisfies

$$r(A) \leq m \text{ and } r(A) \leq n.$$

This is because the rank of a matrix A is the number of leading ones in the row reduced echelon form of A. Each leading one sits at a certain row and a certain column. So in the RREF of A, the number of leading ones will not be over the (total) number of rows and will not be over the (total) number of columns .

solutions of an SLE

The matrix equation $A\vec{x} = \vec{b}$ represents an SLE consisting of m linear equations with n variables x_1, x_2 …, x_n

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}$$

(Recall)

The coefficient matrix of an SLE is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

(Recall)

Its augmented matrix is

$$\left[A \mid \vec{b}\right] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

Theorem If an SLE has two different solutions, then it has infinitely many solutions.

solutions of an SLE

Theorem If an SLE has two different solutions, then it has infinitely many solutions.

Corollary For any system of linear equations there are exactly 3 possibilities:

- the SLE may have no solution
- the SLE may have a unique solution, or
- the SLE may have infinitely many solutions (r-parameter family of solutions).

Theorem If an SLE has two different solutions, then it has infinitely many solutions.

Corollary For any system of linear equations there are exactly 3 possibilities:

- the SLE may have no solution
- the SLE may have a unique solution, or
- the SLE may have infinitely many solutions (r-parameter family of solutions).

GOAL: use the rank of coefficient matrix A and the rank of augmented matrix $\left[A \mid \vec{b}\right]$ to check these possibilities of an SLE.

Theorem. Consider any SLE with m equations in n unknowns. Let A be the coefficient matrix of the system and $\left[A \mid \vec{b}\right]$ be the augmented matrix for the system. Then the SLE $A\vec{x} = \vec{b}$ has:

- no solution if $r(A) < r(\left[A \mid \vec{b}\right])$

Theorem. Consider any SLE with m equations in n unknowns. Let A be the coefficient matrix of the system and $\left[A \mid \vec{b}\right]$ be the augmented matrix for the system. Then the SLE $A\vec{x} = \vec{b}$ has:

- no solution if $r(A) < r(\left[A \mid \vec{b}\right])$

For example,

$$\left[A \mid \vec{b}\right] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 3 & 4 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

The last row corresponds to an equation $0 = 1$, which has no solution.

- a unique solution if $r(A) = r(\left[A \mid \vec{b}\right])$

- a unique solution if $r(A) = r(\begin{bmatrix} A \mid \vec{b} \end{bmatrix}) = n$

For example,

$$\begin{bmatrix} A \mid \vec{b} \end{bmatrix} = \left[\begin{array}{cc|c} 3 & 4 & 1 \\ 2 & 1 & 2 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & \frac{7}{5} \\ 0 & 1 & -\frac{4}{5} \end{array}\right]$$

Indeed, in this case, A is an invertible $n \times n$ matrix.

- infinitely many solutions if $r(A) = r(\left[A \mid \vec{b}\right])$ and $r(A) < n$.

For example,

$$\left[A \mid \vec{b}\right] = \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 4 & 2 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Then $r(A) = \left[A \mid \vec{b}\right] = 1 < 2$.

- infinitely many solutions if $r(A) = r(\left[A \mid \vec{b}\right])$ and $r(A) < n$.

For example,

$$\left[A \mid \vec{b}\right] = \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 4 & 2 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Then $r(A) = \left[A \mid \vec{b}\right] = 1 < 2.$

Indeed, in this case, the SLE $A\vec{x} = \vec{b}$ has an $(n - r(A))$-parameter of solutions.

Theorem. Consider any SLE with m equations in n unknowns. Let A be the coefficient matrix of the system and $\left[A \mid \vec{b}\right]$ be the augmented matrix for the system. Then the SLE $A\vec{x} = \vec{b}$ has:

- no solution if $r(A) < r(\left[A \mid \vec{b}\right])$
- a unique solution if $r(A) = r(\left[A \mid \vec{b}\right]) = n$
- infinitely many solutions if $r(A) = r(\left[A \mid \vec{b}\right])$ and $r(A) < n$.

(**cf.** lecture note Example 9.2.)

Determine how many solutions $A\vec{x} = \vec{b}$ has in each of the following.

(a) A is a 3×4 matrix with $r(A) = 3$ and $r(\left[A \mid \vec{b}\right]) = 3$. 1

(b) A is a 4×3 matrix with $r(A) = 3$ and $r(\left[A \mid \vec{b}\right]) = 4$. ~~inf~~ DNE

(c) A is a 4×3 matrix with $r(A) = 3$ and $r(\left[A \mid \vec{b}\right]) = 3$. inf

Examples

Let A be a 4×5 matrix. Consider the equation $A\vec{x} = \vec{0}$. Which of the following is true about the number of solutions?

- no solution;
- a unique solution;
- infinitely many solutions;
- depends on the rank of A

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

四式子五未知数 → 无穷解

No solution.

Find the value(s) of k such that the SLE $A\vec{x} = \vec{b}$ has no solution, unique solution or infinitely many solutions, where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & k \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 1 \\ 2 \\ k \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & k & k \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & k-2 & k-4 \end{array}\right]$$

$k-2=0$: $k=2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

$k \neq 2$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & k & k \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & k-2 & k-4 \end{array}\right]$$

$k-2=0 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -2 \end{array}\right]$

$\Rightarrow$ infinity solution.

$k-2 \neq 0 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1-\frac{2}{k-2} \end{array}\right]$

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2}{k-2} \\ 0 & 1 & 0 & 1+\frac{2}{k-2} \\ 0 & 0 & 1 & 1-\frac{2}{k-2} \end{array}\right].$