Systems of Linear Equations

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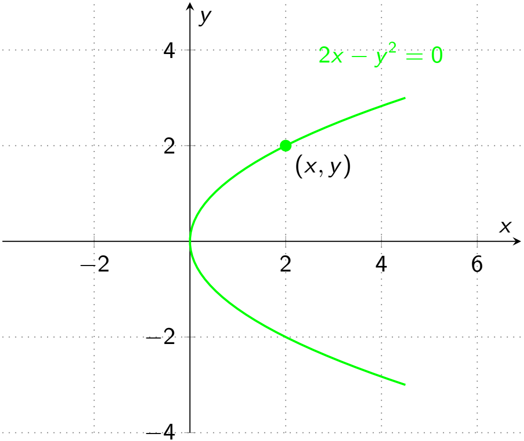

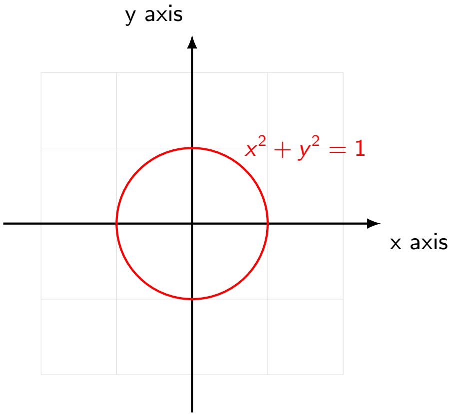

They are NOT straight lines in $\mathbb{R}^2$!

Definition A *linear equation* in the variables $x_1, x_2, \ldots x_n$ is an equation which can be put into the form

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• You could think that x_i are unknown.
• An n-vector $\vec{v} = (v_1, v_2, \ldots, v_n)$ is a solution to a linear equation if it satisfies the linear equation.
• If $n = 3$, all the solutions to a linear equation $ax + by + cz = d$ form a **PLANE** in $\mathbb{R}^3$.
• For any positive integer $n \geq 2$, all the solutions to a linear equation $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$ forms a **HYPERPLANE**.

Examples

Are these equations linear or not?

1. $x_1 x_2 + x_3 = \sqrt{7}$;
2. $x_1 + x_2 + \sqrt{x_4} = 3$;
3. sinx+y=0;
4. $2x_3 + \frac{2}{3}x_6 = cos\frac{\pi}{3}$;
5. $x_1 + \frac{2}{x_4} - x_6 = 0$;
6. $\sqrt[3]{12}x_1 + 2x_2 - x_3 = -3$;
7. $\sqrt[3]{12x_1} + 2x_2 - x_5 = -3$;

System of linear equations

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• A system of m linear equations in n variables is called an $m \times n$ SLE (pronounced "m by n").

• An n-vector $(v_1, v_2, \ldots, v_n)$ is called a solution to a particular SLE with n variables if it satisfies all of the equations in the system.

For example, we did the question to find the intersection of the line $2x + y = 1$ and the line $x - 2y = 4$.

Examples

Is each system of equations linear?

1. $$\begin{aligned} x_1 + x_4 &= 1 \\ x_2 - x_3 + 2x_4 &= 2 \\ x_1 - x_2x_3 + x_4 &= 0 \end{aligned}$$
2. $$\begin{aligned} (sin2)x_1 + x_2 + 9x_3 &= cos3 \\ 2x_1 + cosx_2 - x_3 &= \sqrt{2} \\ \frac{1}{2}x + (cos4)y - z &= 1 \end{aligned}$$
3. $$\begin{aligned} \frac{1}{2}x + cos4y - z &= 1 \\ x + y + z &= 1 \end{aligned}$$

Standard form

A SLE is in standard form if

• all of the equations have all of the variables appearing on the left hand side of the equation;

• all the variables appear in the same order for all equations, with spaces left for any variables missing in that equation;

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A standard-form SLE is

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & 5 \\ -x & & & + & z & = & 2 \end{array}$$

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It has "many" solutions. Any vectors in a form of $(-t-1, t, 2t-1)$ are solutions, where t can be any arbitrary real numbers. It is consistent.

Examples

Put the following SLEs into a standard form. Are they consistent or inconsistent?

$$1)\qquad \begin{aligned} 2 - x + 3y &= -1 \\ x &= z + 2 \end{aligned}$$

$$2)\qquad \begin{aligned} x_1 - 2x_2 &= 0 \\ x_2 + 3x_3 &= 2 \\ x_1 + x_3 &= -1 \end{aligned}$$

$$3)\qquad \begin{aligned} 1 + 4y - x &= 0 \\ 8y + 2 &= 2x \end{aligned}$$

Definition When a solution to a system of equations can be stated with one or more parameters, such that any value of the parameter(s) gives a solution to the system, the system is said to have an *r-parameter* family of solutions, where r is the number of parameters in the solution.

For example, the SLE

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Definition Two SLEs are *equivalent* if they have the same solutions.