The Foundations: Logic and Proofs

Chapter 1, Part II: Predicate Logic

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Motivating problems

Handling variables in propositional logic

1. Can statements with variables, eg $0 \leq x < 4$ implies $\sqrt{x} < 2$ be turned into formal *propositions*?
2. To do so, we need to specify the values of variables, like x above (using predicates).
3. We also need to learn how to work with propositions expressed using quantifiers, like all, every, some.

Automating reasoning

1. If "All men are mortal." and "Socrates is a man" does this imply that "Socrates is mortal"?
2. This seems a natural deduction when we speak in English, but how can we automate such a deduction in a software?
3. To do so, we will discuss some rules of inference with quantifiers.

Plan for Part II

1. Predicates and Quantifiers

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Plan for Part II

Extending Propositional Logic

1. Predicate logic uses the following features:
   a. variables: e.g. x, y, z
   b. which take values in a domain: e.g. the integer numbers
   c. predicates (action): e.g. *is mortal, greater than 4, is green.*
   d. propositional functions: $P(x)$, $Q(x,y)$, $M(z)$
   e. quantifiers: (see later).

Predicate is only a statement.

2. *Propositional functions* combine variables and predicates e.g.

$$P(x) = x \text{ is mortal}$$

$$Q(y) = y \text{ is greater than } 4 \text{ (that is, } y > 4\text{)}$$

3. A *propositional function* becomes a *proposition* when variables are replaced by elements from their domain (which is often denoted by U), e.g.:

$$P(Socrates) = \text{Socrates is mortal}$$

$$Q(5) = \text{5 is greater than 4}$$

$$Q(2) = \text{2 is greater than 4}$$

propositions.

statements that can be determined TRUE or FALSE.

Plan for Part II

Propositional functions

1. Propositional functions become propositions (and have truth values) when each of their variables is replaced by a value from the *domain* (or when *bound* by a quantifier, see later).
2. The statement *P(x)* is said to be the value of the propositional function P at x.
3. For instance, let *P(x)* denote "$x > 0$" and the domain be the integers. Then:

$$\begin{aligned} P(-3) &\text{ is false.} \\ P(0) &\text{ is false.} \\ P(3) &\text{ is true.} \end{aligned}$$

Examples of propositional functions

1. Let "$x + y = z$" be denoted by $R(x, y, z)$ and U (for all three variables) be the integers. Find these truth values:
   a. $R(2, -1, 5)$

      Solution: F
   b. $R(3, 4, 7)$

      Solution: T
   c. $R(x, 3, z)$

      Solution: Not a Proposition
2. Now let "x - $y = z$" be denoted by $Q(x, y, z)$, with U as the integers. Find these truth values:
   a. $Q(2, -1, 3)$

      Solution: T
   b. $Q(3, 4, 7)$

      Solution: F
   c. $Q(x, 3, z)$

      Solution: Not a Proposition

Compound expressions

1. Connectives from propositional logic carry over to predicate logic.
2. If $P(x)$ denotes "$x > 0$," find these truth values:
   a. $P(3) \vee P(-1)$

      Solution: T
   b. $P(3) \wedge P(-1)$

      Solution: F
   c. $P(3) \rightarrow P(-1)$

      Solution: F
   d. $P(-3) \rightarrow P(-1)$

      Solution: T
3. Expressions with variables are not propositions and therefore do not have truth values. For example:
   a. $P(3) \wedge P(y)$
   b. $P(x) \rightarrow P(y)$
4. Instead of specifying values of variables, one can convert a propositional function into a proposition using quantifiers (see next slide)

Plan for Part II

Quantifiers: introduction

Idea: converting propositional functions into propositions by specifying the values of variables.

Charles Peirce (1839-1914)

Example

propositional function	*variable*	*predicate P*
$P(x)$:	x	is mortal

P(Socrates)

proposition Socrates is mortal (specifying value of variable x)

For all x, P(x) domain is "men"

All men are mortal (quantifier "for all")

For some x, P(x)

Some men are mortal (quantifier "for some")

Quantifiers: introduction

Example

propositional function	*variable*	*predicate P*
$Q(y) =$	y	> 4

quantifier has no relation with the domain of the variable.

$Q(5)$

$5 > 4$ (specifying value of variable y)

$\forall\ y\ Q(y)$

$\forall y\ \ y > 4$ (quantifier $\forall$ – "for all")

$\exists\ y\ Q(y)$

$\exists y\ \ y > 4$ (quantifier $\exists$ – "for some")

Plan for Part II

Quantifiers

1. We need *quantifiers* to express the meaning of English words including *all* and *some*:
   a. "All men are Mortal." $\forall x, P(x).$
   b. "Some cats do not have fur." $\exists y, P(y).$
2. The two most important quantifiers are:
   a. *Universal Quantifier*, "For all," symbol: $\forall$
   b. *Existential Quantifier*, "There exists," symbol: $\exists$
3. We write as in $\forall x\, P(x)$ and $\exists x\, P(x)$.
4. $\forall x\, P(x)$ asserts $P(x)$ for every x in the *domain*.
5. $\exists x\, P(x)$ asserts $P(x)$ for some x in the *domain*.
6. The quantifiers are said to *bind* the variable x in these expressions.

The universal quantifier $\forall$

$\forall x\, P(x)$ is read as "for all x, P(x)" or "for every x, P(x)"

Examples:

domain

a. If $P(x)$ denotes "$x > 0$" and U is the integers, then proposition $\forall x\, P(x)$ is false.
b. If $P(x)$ denotes "$x > 0$" and U is the positive integers, then proposition $\forall x\, P(x)$ is true.
c. If $P(x)$ denotes "x is even" and U is the integers, then proposition $\forall x\, P(x)$ is false.

The existential quantifier $\exists$

$\exists x\, P(x)$ is read as "for some x, P(x)", or "there is an x such that P(x)," or "there exists x such that P(x)," or "for at least one x, P(x)."

Examples:

- (a) If $P(x)$ denotes "$x > 0$" and U is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers.
- (b) If $P(x)$ denotes "$x < 0$" and U is the positive integers, then proposition $\exists x\, P(x)$ is false.
- (c) If $P(x)$ denotes "x is even" and U is the integers, then proposition $\exists x\, P(x)$ is true.

Thinking about quantifiers

1. When the domain (of variable) is finite, we can think of quantification as looping through the elements of the domain.

2. To evaluate $\forall x\, P(x)$, we loop through all x in the domain.
   a. If at every step P(x) is true, then $\forall x\, P(x)$ is true.
   b. If at a step P(x) is false, then $\forall x\, P(x)$ is false and the loop terminates.

3. To evaluate $\exists x\, P(x)$, we loop through all x in the domain.
   a. If at some step, P(x) is true, then $\exists x\, P(x)$ is true and the loop terminates.
   b. If the loop ends without finding an x where P(x) is true, then $\exists x\, P(x)$ is false.

4. Even if the domains are infinite, we can still think of the quantifiers this fashion, but the loops will not terminate in some cases.

Properties of quantifiers

1. The truth value of $\exists x\, P(x)$ and $\forall\ x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain U.
2. **Examples**:
   a. If U is all integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall\ x\, P(x)$ is false.
   b. If U is the negative integers and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall\ x\, P(x)$ are true.
   c. If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x > 2$", then both $\exists x\, P(x)$ and $\forall\ x\, P(x)$ are true. But if $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall\ x\, P(x)$ are false.

Precedence of quantifiers

1. The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.
2. For example, $\boxed{\forall x\ P(x)} \lor Q(x)$ means $(\forall x P(x)) \lor Q(x)$
3. $\forall x\ (P(x) \lor Q(x))$ means something different.
4. Unfortunately, often people write $\forall x\ P(x) \lor Q(x)$ when they mean $\forall x(P(x) \lor Q(x))$.

Plan for Part II

Translating from English to Logic

Example 1: Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java."

Solution:

First decide on the domain U.

Solution 1: If U is all students in this class, define a propositional function J(x) denoting "x has taken a course in Java" and translate as $\forall x\ J(x)$.

Solution 2: But if U is all people, also define a propositional function S(x) denoting "x is a student in this class" and translate as $\forall x(S(x) \rightarrow J(x))$.

if that person is a student in this class.
then he has taken JAVA.

$\forall x(S(x) \land J(x))$ is not correct. What does it mean?

all student in class may take JAVA.

Translating from English to Logic

Example 2: Translate the following sentence into predicate logic: "Some student in this class has taken a course in Java."

Solution:

First decide on the domain U.

Solution 1: If U is all students in this class, translate as $\exists x J(x)$

Solution 2: But if U is all people, then translate as $\exists x (S(x) \wedge J(x))$

$\exists x (S(x) \rightarrow J(x))$ is not correct. What does it mean?

Returning to the Socrates Example

1. Introduce the propositional functions *Man(x)* denoting "x is a man" and *Mortal(x)* denoting "x is mortal." Specify the domain U as all people.
2. The two premises are:

   $\forall x\, Man(x) \rightarrow Mortal(x)$
   $Man(Socrates)$

3. The conclusion is:

   $Mortal(Socrates)$

NOTE: This is a valid argument form ("universal modus ponens") combining "universal instantiation" $\forall x\, P(x)\, then\, P(c)$ (for any c in U)and "modus ponens" (see Part I of Chapter 1)
$if\ p\, and\, p \rightarrow q\, then\, q$

Plan for Part II

Equivalences in predicate logic

1. Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value for...
   a. every predicate substituted into these statements

      and

   b. every domain used for the variables in the expressions.
2. The notation $S \equiv T$ indicates that S and T are logically equivalent.
3. **Example**: $\forall x \neg\neg S(x) \equiv \forall x S(x)$

Thinking about quantifiers as conjunctions and disjunctions

1. If the domain is finite, a universally quantified proposition is equivalent to a conjunction of propositions without quantifiers and an existentially quantified proposition is equivalent to a disjunction of propositions without quantifiers.
2. If U consists of the integers 1,2, and 3:

$$\forall x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$
$$\exists x P(x) \equiv P(1) \vee P(2) \vee P(3)$$

3. Even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long.

Plan for Part II

Negating Quantified Expressions

1. Consider: $\forall x\ J(x)$
   1. "Every student in your class has taken a course in Java."
   2. Here $J(x)$ is "x has taken a course in Java."
   3. The domain is "students in your class".
2. Negating the original statement gives "It is not the case that every student in your class has taken Java."
3. This implies that "There is a student in your class who has not taken Java."
4. Symbolically $\neg(\forall x\ J(x))$ and $\exists(x\ \neg J(x))$ are equivalent

Negating Quantified Expressions (*continued*)

1. Now Consider: $\exists\, x\, J(x)$
   a. "There is a student in this class who has taken a course in Java."
   b. Where $J(x)$ is "x has taken a course in Java."
2. Negating the original statement gives "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken Java"
3. Symbolically $\neg(\exists\, x\, J(x))$ and $\forall(x\, \neg J(x))$ are equivalent

De Morgan's Laws for Quantifiers

1. The rules for negating quantifiers are:

TABLE 2 **De Morgan's Laws for Quantifiers.**

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

2. The reasoning in the table shows that:

$$\boxed{\begin{array}{c} \neg\forall x P(x) \equiv \exists x \neg P(x) \\ \neg\exists x P(x) \equiv \forall x \neg P(x) \end{array}}$$

3. These are important. You will use these.

Plan for Part II

Translation from English to Logic

Examples:

1. "Some student in this class has visited Mexico."

   Solution: Let $M(x)$ denote "x has visited Mexico" and $S(x)$ denote "x is a student in this class," and U be all people.
   $\exists x \ (S(x) \wedge M(x))$

2. "Every student in this class has visited Canada or Mexico."

   Solution: Add $C(x)$ denoting "x has visited Canada."
   $\forall x \ (S(x) \rightarrow (M(x) \vee C(x)))$

   if a person is a student in this class,
   then ....

Some Fun with Translating from English into Logical Expressions

1. U = fleegles, snurds, thingamabobs
   a. $F(x)$: x is a fleegle
   b. $S(x)$: x is a snurd
   c. $T(x)$: x is a thingamabob
2. Translate "Everything is a fleegle"
3. **Solution**: $\forall x\ F(x)$

Translation (cont)

1. U $=$ fleegles, snurds, thingamabobs
   1. $F(x)$: x is a fleegle
   2. $S(x)$: x is a snurd
   3. $T(x)$: x is a thingamabob
2. Translate "Nothing is a snurd."
3. **Solution**: $\neg\exists x\ S(x)$
4. What is this equivalent to?
5. **Solution**: $\forall x\ \neg\ S(x)$

Translation (cont)

1. U = fleegles, snurds, thingamabobs
    a. *F(x)*: x is a fleegle
    b. *S(x)*: x is a snurd
    c. *T(x)*: x is a thingamabob
2. Translate "All fleegles are snurds."
3. **Solution**: $\forall x\ (F(x) \rightarrow S(x))$

   if ..... then.

Translation (cont)

1. U = fleegles, snurds, thingamabobs
   a. $F(x)$: x is a fleegle
   b. $S(x)$: x is a snurd
   c. $T(x)$: x is a thingamabob
2. Translate "Some fleegles are thingamabobs."
3. **Solution**: $\exists x\, (F(x) \wedge T(x))$

Translation (cont)

1. U = fleegles, snurds, thingamabobs
   1. $F(x)$: x is a fleegle
   2. $S(x)$: x is a snurd
   3. $T(x)$: x is a thingamabob
2. Translate "No snurd is a thingamabob."
3. **Solution**: $\neg \exists x\ (S(x) \wedge T(x))$ What is this equivalent to?
4. **Solution**: $\forall x\ (\neg S(x) \vee \neg T(x))$
5. *Another equivalent solution* $\forall x\ (S(x) \longrightarrow \neg T(x))$

Translation (cont)

1. U = fleegles, snurds, thingamabobs
   a. $F(x)$: x is a fleegle
   b. $S(x)$: x is a snurd
   c. $T(x)$: x is a thingamabob
2. Translate "If any fleegle is a snurd then it is also a thingamabob."
3. **Solution**: $\forall x\ ((F(x) \wedge S(x)) \rightarrow T(x))$

Plan for Part II

System Specification Example

1. Predicate logic is used for specifying properties that systems must satisfy.
2. For example, translate into predicate logic:
   a. "Every mail message larger than one megabyte will be compressed."
   b. "If a user is active, at least one network link will be available."
3. Decide on predicates and domains (left implicit here) for the variables:
   a. Let $L(m, y)$ be "Mail message m is larger than y megabytes."
   b. Let $C(m)$ denote "Mail message m will be compressed."
   c. Let $A(u)$ represent "User u is active."
   d. Let $S(n, x)$ represent "Network link n is state x.
4. Now we have:

$$\begin{array}{cccc} \forall m & L(m,1) & \rightarrow & C(m) \\ \exists u & A(u) & \rightarrow & \exists n \;\; S(n, available) \end{array}$$

exist an active user

Lewis Carroll Example

1. "All lions are fierce."
2. "Some lions do not drink coffee."
3. "Some fierce creatures do not drink coffee."

1. Here is one way to translate these statements to predicate logic. Let P(x), Q(x), and R(x) be the propositional functions "x is a lion," "x is fierce," and "x drinks coffee," respectively.
   a. $\forall x\ (P(x) \rightarrow Q(x))$
   b. $\exists x\ (P(x) \land \neg R(x))$
   c. $\exists x\ (Q(x) \land \neg R(x))$
2. Later we will see how to prove that 3 (the conclusion) follows from 1 and 2 (the premises).

Charles Lutwidge Dodgson (AKA Lewis Caroll, 1832-1898)

Plan for Part II

Plan for Part II

Nested Quantifiers

1. Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.
2. **Example**: "Every real number has an opposite" is

$$\forall x \; \exists y \quad (x + y = 0),$$

   where the domains of x and y are the real numbers.
3. We can also think of nested propositional functions:

$$\forall x \; \exists y \quad (x + y = 0),$$

   can be viewed as

$$\forall x \quad Q(x),$$

   where $Q(x)$ is $\exists y \quad P(x, y)$ and $P(x, y)$ is $(x + y = 0)$.

$\forall x \mid Q(x)$
$\exists y \mid P(x, y)$
$x + y = 0.$

Thinking of Nested Quantification

1. To see if $\forall x\ \forall y\ \ P(x, y)$ is true, we loop through the values of x:
   a. At each step, we loop through the values of y.
   b. If for some pair of (x, y) the proposition $P(x, y)$ is false, then $\forall x\ \forall y\ \ P(x, y)$ is false as well, and both the outer and inner loops terminate.
   c. $\forall x\ \forall y\ \ P(x, y)$ is true if the outer loop ends after stepping through each value of x.
2. To see if $\forall x\ \exists y\ \ P(x, y)$ is true, we loop through the values of x:
   a. At each step, we loop through the values of y.
   b. If for some pair of (x, y) the proposition $P(x, y)$ is true, then the inner loop ends.
   c. If no y is found such that $P(x, y)$ is true, then he outer loop terminates as $\forall x\ \exists y\ \ P(x, y)$ has been shown to be false.
   d. $\forall x\ \exists y\ \ P(x, y)$ is true if the outer loop ends after stepping through each value of x.
3. If the domains of the variables are infinite, then this process can not actually be carried out.

Plan for Part II

Order of Quantifiers

Example

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers.
2. Then $\forall x\ \forall y P(x,y)$ and $\forall y\ \forall x P(x,y)$ have the same truth value.

Example

1. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers.
2. Then $\forall x\ \exists y P(x,y)$ is true, but $\exists y\ \forall x P(x,y)$ is false.

Questions on Order of Quantifiers

Example

1. Let U be the real numbers and define $P(x,y) : x \cdot y = 0$.
2. What is the truth value of the following:
   a. $\forall x \forall y P(x,y)$
      Solution: False
   b. $\forall x \exists y P(x,y)$
      Solution: True
   c. $\exists x \forall y \ P(x,y)$
      Solution: True
   d. $\exists x \exists \ y \ P(x,y)$
      Solution: True

Questions on Order of Quantifiers

Example

1. Let U be the real numbers, and define $P(x,y) : x \,/\, y = 1$.
2. What is the truth value of the following:
   a. $\forall x \forall y P(x,y)$
      Solution: False
   b. $\forall x \exists y P(x,y)$
      Solution: False
   c. $\exists x \forall y \; P(x,y)$
      Solution: False
   d. $\exists x \; \exists \; y \; P(x,y)$
      Solution: True

Plan for Part II

Quantifications of Two Variables

Statement A	When is A true?	When is A false? ($\neg$ A is true)
$\forall x \forall y\, P(x,y)$ $\forall y \forall x\, P(x,y)$	$P(x,y)$ is true for every pair (x,y).	$\exists x \exists y\, \neg P$ There is a pair (x,y) for which $P(x,y)$ is false.
$\forall x \exists y\, P(x,y)$	For every x there is a y for which $P(x,y)$ is true.	$\exists x \forall y\, \neg P$ There is an x such that $P(x,y)$ is false for every y.
$\exists x \forall y\, P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	$\forall x \exists y\, \neg P$ For every x there is a y for which $P(x,y)$ is false.
$\exists x \exists y\, P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair (x,y) for which $P(x,y)$ is true.	$\forall x \forall y\, \neg P$ $P(x,y)$ is false for every pair (x,y)

De Morgan's Law.

Plan for Part II

Translating Nested Quantifiers into English

Example 1: Translate the statement:

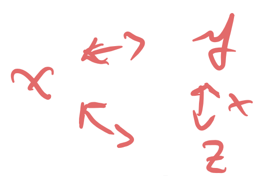
$$\forall x \quad (C(x) \lor \exists y \ (C(y) \land F(x,y)))$$

where C(x) is "x has a computer," and $F(x,y)$ is "x and y are friends," and the domain for both x and y consists of all students in your school.

Solution: Every student in your school has a computer or has a friend who has a computer.

Example 2: Translate the statement:

$$\exists x \ \ \forall y \ \ \forall z \ \ ((F(x,y) \land F(x,z) \land (y \neq z)) \ \rightarrow \ \neg F(y,z))$$



Solution: There is a student none of whose friends are also friends with each other.

Translating Mathematical Statements into Predicate Logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Quantifiers and domains should be made explicit as in:

   "For every two integers, if these integers are both positive, then the sum of these integers is positive."

2. Introduce variables x and y:

   "For every integers x and y, if $x>0$ and y>0, then $x+y>0$."

3. Using symbols for connectives and quantifiers we get

   $\forall x \; \forall \; y \; ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$

   where the domain for x and y are the integers

Translating English into Logical Expressions Example

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."
Solution:

1. Let $P(w, f)$ be "w has taken f" and $Q(f, a)$ be "f is a flight on airline a."
2. The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.
3. Then the statement can be expressed as:

   $\exists w \ \ \forall a \ \ \exists f \ \ (P(w, f) \wedge Q(f, a))$

Order of quantifies is very important! Try changing it and see what that means.

Plan for Part II

Negating Nested Quantifiers

Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

$$\neg \exists w \forall a \exists f (P(w, f) \land Q(f, a))$$

Use De Morgan's Laws to move the negation as far inwards as possible.

$\neg \exists w \forall a \exists f (P(w, f) \land Q(f, a))$	$\equiv \forall w \neg \forall a \exists f (P(w, f) \land Q(f, a))$	De Morgan's law for $\exists$
	$\equiv \forall w \exists a \neg \exists f (P(w, f) \land Q(f, a))$	De Morgan's law for $\forall$
	$\equiv \forall w \exists a \forall f \neg (P(w, f) \land Q(f, a))$	De Morgan's law for $\exists$
	$\equiv \forall w \exists a \forall f (\neg P(w, f) \lor \neg Q(f, a))$	De Morgan's law for $\land$

Can you translate the result back into English?

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"

Plan for Part II

Questions on Translation from English

Using the predicate B, S, L, express in predicate logic the sentences below:

$B(x,y)$: x and y are brothers, $S(x,y)$: x and y are siblings, $L(x,y)$: x loves y

$\exists x \exists y \, (S(x,y) \rightarrow (L(x,y) \wedge L(y,x))).$

$x =$ $y =$

Example 1: "Brothers are siblings."

If x and y are brothers, then ----- are siblings.

Solution: $\forall x \; \forall y \; (B(\text{x,y}) \rightarrow S(\text{x,y}))$

Example 2: "Siblinghood is symmetric."

Solution: $\forall x \; \forall y \; (S(x,y) \rightarrow S(y,x))$

Example 3: "Everybody loves somebody."

Solution: $\forall x \; \exists y \; L(x,y)$

Example 4: "There is someone who is loved by everyone."

Solution: $\exists y \; \forall x \; L(x,y)$

Example 5: "There is someone who loves someone."

Solution: $\exists x \; \exists y \; L(x,y)$

Example 6: "Everyone loves her/himself"

Solution: $\forall x \; L(x,x)$

Plan for Part II

Some Questions about Quantifiers

Can you switch the order of quantifiers?

IT DEPENDS . . .

1. Is this a valid equivalence? $\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$

   Solution: Yes! The left and the right side will always have the same truth value. The order in which x and y are picked does not matter.

2. Is this a valid equivalence? $\forall x \exists y P(x, y) \equiv \exists y \forall x P(x, y)$

   Solution: No! The left and the right side may have different truth values for some propositional functions for P. Try "x + y = 0" for *P(x,y)* with U being the integers. The order in which the values of x and y are picked does matter.

Some Questions about Quantifiers Cont...

Can you distribute quantifiers over logical connectives?

IT DEPENDS . . .

1. Is this a valid equivalence?
   $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$

   Solution: Yes! The left and the right side will always have the same truth value no matter what propositional functions are denoted by *P(x)* and *Q(x)*.

2. Is this a valid equivalence?
   $\forall x(P(x) \rightarrow Q(x)) \equiv \forall x P(x) \rightarrow \forall x Q(x)$

   Solution: No! The left and the right side may have different truth values. Pick “x is a fish” for P(x) and “x has scales” for Q(x) with the domain of discourse being <u>all animals</u>. Then the left side is false, because there are some fish that do not have scales. But the right side is true since not all animals are fish.