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CS3331 - Final Notes Dec 14/19

Chapter 17 - Turing Machine and undecidability

A TM M's behaviour will only be defined on input strings that are finite and only contain characters in M's input alphabet (Σ)

How does a TM work? In each step:

1. Choose its next state
2. Write on current square
3. Move read/write head left or right one square

Since δ (transition) is a function, NOT a relation. Thus a TM is deterministic

transition on a TM (graphically) r/w/m = read/write/moveTo

A TM is NOT guaranteed to halt.

What does a Macro TM mean? You can combine smaller TMs to form a more complex one.

Example of shifting:

input: $\square u \underline{\square} w \square$

output: $\square u w \underline{\square}$

(Ans) M: $> R \xrightarrow{x \leftarrow \neg\square} \square L x R$

$\downarrow \square$

L

A language is DECIDABLE iff a TM either accept OR reject every string.

A language is SEMI-DECIDABLE iff a TM either accept OR does not accept (either reject or loop).

A non-deterministic TM, M is DECIDABLE iff:

- ↳ M accepts $w \in \Sigma^*$ iff at least one computation accepts
- ↳ M rejects $w \in \Sigma^*$ iff all of its computations reject

A non-deterministic TM, M decides a language $L \subseteq \Sigma^*$ iff $\forall w \in \Sigma^*$
↳ There's a finite number of paths that M can follow on input w.
↳ All of those paths halt
↳ $w \in L$ iff M accepts w

A non-deterministic TM, M, SEMI-DECIDES a language $L \subseteq \Sigma^*$ iff $\forall w \in \Sigma^*$
↳ $w \in L$ iff $(s, \sqcup w)$ *starting state* yields at least one accepting configuration.

A non-deterministic TM, M, computes a function, f, iff $\forall w \in \Sigma^*$
↳ All of M's computations halt
↳ All of M's computations result in $f(w)$

Any computation by a TM with a two-way infinite tape can be simulated by a TM with a one-way tape

Universal Turing Machine:
1. A programmable TM that accepts (program, string) as input
2. It executes the program and produce a string as output

How to define an Universal Turing Machine, U?
1. Define an encoding operation for TMs
2. Describe the operation of U given input $\langle M, w \rangle$.

How to encode the states?
↳ Let t' be the binary string assigned to state t
↳ If t is the halting state y, assign it the string yt'.
↳ If t is the halting state n, assign it the string nt'.
↳ If t is any other state, assign it the string qt'.

Ex. Let say we have 9 states, and then this is the encoding of the states:
q0000 (s), q0001, q0010, y0011, n0100, q0101, q0110, q0111, q1000
* State 3 is y & State 4 is n *

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How to encode the tape alphabet (Γ)?

Example: Say we have a tape alphabet of $\Gamma = \{\sqcup, a, b, c\}$. Then the encoding is:

$\sqcup$ = a00

a = a01

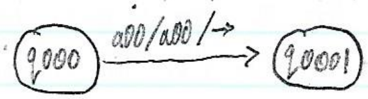
b = a10

c = a11

How to encode the transition?

Transition = (starting state, input string, destination state, ouput string, direction)

Ex. (q000, a00, q0001, a00, $\rightarrow$)



Theorem: There exists an infinite lexicographic enumeration of

1. All syntactically valid TMs
2. All syntactically valid TMs with specific input alphabet Σ.
3. All syntactically valid TMs with specific input AND tape alphabet, Σ & Γ.

Specification of the Universal TM:

On input $\langle M, w\rangle$, U must:

↳ Halt iff M halts on w

↳ If M is a deciding or semi-deciding machine, then:

  ↳ If M accepts, accept

  ↳ If M rejects, reject

↳ If M computes a function, then $U(\langle M, w\rangle)$ must equal $M(w)$.

How does an Universal TM work? It uses 3 tapes as follow $U(\langle M, w\rangle)$

Tape 1: M's tape

Tape 2: $\langle M\rangle$, the "program" that U's running

Tape 3: M's state

Step 1: Copy the encoding of $\langle M, w\rangle$ onto tape 1.

Step 2: Transfer $\langle M\rangle$ from tape 1 to tape 2 (erasing it from tape 1)

Step 3: Find out the # of states in M, then let i be the binary digit of the # of states in M. Write $q0^i$ (corresponding to the start state of M) on tape 3.

	□	□	□	□	<w	---	--	-w>	
	0	0	0	0	1	0	0	0	
□	<M	--	--	-M>	□	□	□	□	□
	1	0	0	0	0	0	0	0	
	q	0	0	0	□	□	□	□	
	1	0	0	0	0	0	0	0	

How long does U take to simulate the computation of M? (If M halts in k steps)

↳ Ans: $O(|M| \cdot k)$ steps

↳ Explanation: Since U has to loop k times and each time U also has to loop through the entire <M> to find the corresponding computation. Hence $|M| \cdot k$

Chapter 19

$L_1 = \{\langle M, w\rangle : \text{TM } M \text{ halts on input string } w\}$

For a string x to be in L_1, it must:

↳ Be syntactically well-formed

↳ Encode a machine M and a string w such that M halts when started on w.

Theorem: $H = \{\langle M, w\rangle : \text{TM } M \text{ halts on input string } w\}$

↳ Is semi-decidable AND not decidable

Theorem: If H were in D, then every SD language would be in D.

↳ Proof: Suppose L is any SD language, then there's M_L that's semi-decidable.

Suppose H were in D, then there's O (oracle) that's decidable.

Pass $\langle M_L, w\rangle$ as input into O if M_L halts on w.

M'(w: string) =

1. Run O on $\langle M_L, w\rangle$
2. If O accepts (which it will iff M_L halts on w), then:
   - 2.1 Run M_L on w
   - 2.2 If M_L accepts, then accept. Else reject
3. Else Reject

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Chapter 20

Theorem: The set of context-free languages is a proper subset of D.

3 relations between SD and D:

1. D is a subset of SD. In other words, every decidable language is also semi-decidable.
2. There exists at least one language that is in SD/D (Eg. H)
3. There exists languages that are not in SD.

Theorem: The class D is closed under complement

↳ What it means: If a language is decidable, then so is its complement

Theorem: The class SD is NOT closed under complement

Theorem: A language is in D iff both it and its complement are in SD

Theorem: ¬H is not in SD

↳ Proof: We know that H is in SD, then if ¬H is also in SD, this would mean H is also in D but we know that H is NOT in D. Therefore ¬H cannot be in SD.

Theorem: A language is in SD iff it is Turing-Enumerable

↳ Proof (Right-to-Left): If a language, L, is Turing-Enumerable, then there's some TM that enumerates it. We convert M to M' that SD L,

M'(w: string) =

1. Save input w on a second tape
2. Run M. If a match is found from the result of "Running M" to w, then halt and accept.

* Running M means enumerate all strings in L.

∵ M' is SD.

∴ L is SD.

↳ Proof (Left-to-Right): If L is in SD, then there's a SD TM, M. We now construct M' using M to enumerate L in a lexicongraphical order.

M'() =

1. Enumerate all $w \in \Sigma^*$ lexicongraphically. For each string w_i:
   1.1 Start a copy of M with w_i as input
   1.2 Run M_i that are initiated, except those that are halted.
2. Whenever M_i accepts, output w_i

Theorem : A language is in D iff it is lexicongraphically Turing - enumerable

Chapter 21

How to use Reduction as a part of proof by contradiction that SD/D?

A Reduction, R, from L_1 to L_2 consists of one or more TMs with the following properties :

1. If there exists a TM, Oracle, that decides or semi-decides L_2,
2. Then the TMs in R can be composed with Oracle to build a decidable TM for L_1.

$(L_1 \leq L_2) \wedge (L_2 \text{ is in } D) \rightarrow (L_1 \text{ is in } D)$

* L_1 is reducable to L_2 *

Step 1 : Assume Oracle that decides L_2 exists

Step 2 : Choose a language L_1 :

↳ That is known to not be in D, and

↳ Can be reduced to L_2

Step 3 : Define Reduction R

Step 4 : Describe the composition C of R with Oracle :

$$C(x) = \text{Oracle}(R(x))$$

Step 5 : Show that C does correctly decide L_1 if Oracle exists. We do this by showing :

- R can be implemented by Turing Machines,
- C is correct :
  - ↳ If $x \in L_1$, then $C(x)$ accepts, and
  - ↳ If $x \notin L_1$, then $C(x)$ rejects

Rice Theorem

Any languages that are : $\{\langle M \rangle : P(L(M)) = \text{True}\}$

Non-trivial Property :

↳ true of at least one language

↳ false of at least one language