Matrix equation

SLE

Inverse matrix (2)

Recap

Consider m linear equations with n variables x_1, x_2 ..., x_n

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

The coefficient matrix of an SLE is

$$
A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}
$$

We have a matrix equation

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix} = \vec{b}$$

where $\vec{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$ and $\vec{b} = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix}^T$.

The matrix equation $A\vec{x} = \vec{b}$ exactly represents the SLE.

In particular, if the coefficient matrix A is a square matrix and A is invertible, then we mutiply A^{-1} on the both sides of the equation

$$\begin{aligned} A\vec{x} &= \vec{b} \\ A^{-1}(A\vec{x}) &= A^{-1}\vec{b} \\ (A^{-1}A)\vec{x} &= A^{-1}\vec{b} \\ I\vec{x} &= A^{-1}\vec{b} \\ \vec{x} &= A^{-1}\vec{b} \end{aligned}$$

Hence, the SLE has a unique solution $\vec{x} = A^{-1}\vec{b}$.

Example

Now we consider the SLE

$$\begin{aligned} x - y + z &= 3 \\ 2y - z &= 0 \\ 2x + y + 2z &= 1 \end{aligned}$$

The matrix equation is

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

Use the method of inverse matrix to find the solution.

Examples

(Lecture note Example 8.6)

For what value(s) of c does the following SLE not have a unique solution?

$$\begin{aligned} x \quad + \ y + z &= 1 \\ x + 2y + 3z &= 1 \\ x + 2y + cz &= 1 \end{aligned}$$

The theory of SLE

- Tell some information for SLEs from matrix equations

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Now let $\vec{w}(t) = (1 - t)\vec{x}_1 + t\vec{x}_2$, where t is a parameter.

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Let us check

$$\begin{aligned} A\vec{w}(t) &= A((1-t)\vec{x}_1 + t\vec{x}_2) \ \textit{(definition of } \vec{w}(t)) \\ &= (1-t)(A\vec{x}_1) + t(A\vec{x}_2) \ \textit{(Distributivity and scalars factor out)} \\ &= (1-t)\vec{b} + t\vec{b} \ \textit{(because } A\vec{x}_1 = \vec{b} = A\vec{x}_2) \\ &= \vec{b} - t\vec{b} + t\vec{b} = \vec{b}. \end{aligned}$$

Corollary For any system of linear equations there are exactly 3 possibilities:

• the SLE may have no solution

• the SLE may have a unique solution, or

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Next, we see how to determine which situation an SLE has.

Rank of a matrix

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Also, we say that the $m \times n$ matrix A has *full rank* if $r(A) = n$, i.e. if every column of the RREF of A contains the leading one for some row.

Examples

Find the rank of each matrix. Which has a full rank?

$$\text{(a)}\quad A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -2 & -3 & 4 \end{bmatrix} \qquad \text{(b)}\quad A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 3 & 3 & 3 \end{bmatrix}$$

$$\text{(c)}\quad \left[A \mid \vec{b}\right] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 3 & 4 \end{array}\right] \qquad \text{(d)}\quad \left[A \mid \vec{b}\right] = \left[\begin{array}{cc|c} 3 & 4 & 1 \\ 2 & 1 & 2 \end{array}\right]$$

We wish to solve a system of linear equations of the form $A\vec{x} = \vec{b}$, where A is a 4×4 nonsingular coefficient matrix.
If we know that the inverse of A is

$$A^{-1} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

find a formula for the solution vector $\vec{x}$ in terms of components of the right hand side vector $\vec{b}$.
What is the coefficient matrix A?