Lecture 10

September 30, 2020 10:24 AM

Multiple Integrals (Chapter 15)

15.1 Double integrals

Consider a function $z = f(x,y)$ which represents a surface in 3-d space. Assume the surface is above the xy-plane, ie, $f(x,y) \geq 0$. let $f(x,y)$ be defined on a region R whose boundary is a closed curve (C).

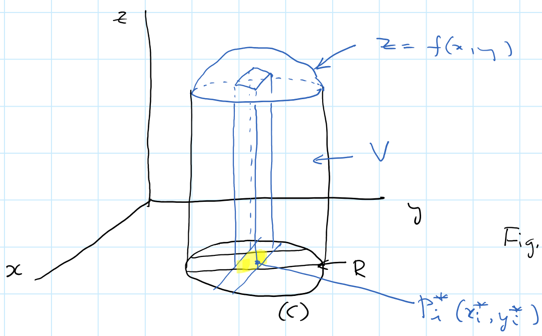


Fig. 1

We want to compute the volume V of a cylinder standing on (C) and under the surface $z = f(x,y)$.

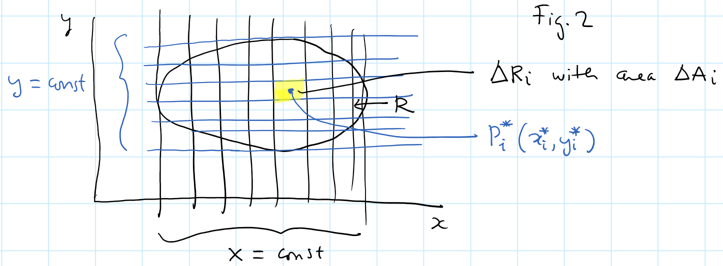


Fig. 2

Dividing R into n subregions by two families of lines : $x = \text{const}$ and $y = \text{const}$. Let $P_i^*(x_i^*, y_i^*)$ be an arbitrary point in the subregion ΔR_i with area ΔA_i. Then the volume of the column shown in Figure 1 is

$$\Delta V_i = f(P_i^*)\,\Delta A_i = f(x_i^*, y_i^*)\,\Delta A_i$$

Suming such contribution for all these columns, we obtain

$$\sum_{i=1}^{n} \Delta V_i = \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta A_i$$

which is an approximation to V. If we let the number of subregions go to infinity, ie, $n \to \infty$ in such a way that every subregion is shrinking to a point, then the above Riemann sum tends to a unique limit V, which is the required volume. In notation, we write

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta A_i$$

Estimate a double integral over a rectangular region — The Midpoint rule.

Ex1: Estimate $\iint\limits_R (16 - x^2 - 2y^2)\,dA$ where R is the rectangle defined by $0 \le x \le 2$, $0 \le y \le 2$ (or $R = [0,2] \times [0,2]$).

Solution

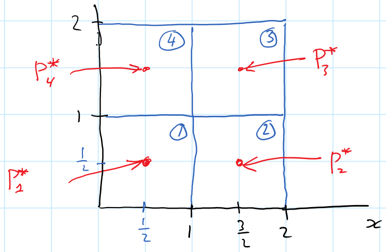


Fig. 3

Let's divide R into 4 subregions as shown in Fig. 3. We label these subregions as ①, ②, ③ and ④.

Consider the subregion ①. We can pick any point in ① as P_1^* to estimate ΔV_1 but the best way is to pick P_1^* as the midpoint of ① and similarly for ②, ③ and ④. Then

$$P_1^*(x_1^*, y_1^*) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right), \quad P_2^*(x_2^*, y_2^*) = \left(\tfrac{3}{2}, \tfrac{1}{2}\right)$$

$$P_3^*(x_3^*, y_3^*) = \left(\tfrac{3}{2}, \tfrac{3}{2}\right), \quad P_4^*(x_4^*, y_4^*) = \left(\tfrac{1}{2}, \tfrac{3}{2}\right)$$

$$\therefore\ f(x_1^*, y_1^*) = f\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = 16 - \left(\tfrac{1}{2}\right)^2 - 2\left(\tfrac{1}{2}\right)^2 = \frac{61}{4}$$

$$f(x_2^*, y_2^*) = f\left(\tfrac{3}{2}, \tfrac{1}{2}\right) = 16 - \left(\tfrac{3}{2}\right)^2 - 2\left(\tfrac{1}{2}\right)^2 = \frac{53}{4}$$

$$f(x_3^*, y_3^*) = f\left(\tfrac{3}{2}, \tfrac{3}{2}\right) = 16 - \left(\tfrac{3}{2}\right)^2 - 2\left(\tfrac{3}{2}\right)^2 = \frac{37}{4}$$

$$f(x_4^*, y_4^*) = f\left(\tfrac{1}{2}, \tfrac{3}{2}\right) = 16 - \left(\tfrac{1}{2}\right)^2 - 2\left(\tfrac{3}{2}\right)^2 = \frac{45}{4}$$

$$\Delta A_1 = \Delta A_2 = \Delta A_3 = \Delta A_4 = \Delta A = (1)(1) = 1$$

$$\therefore\ V = \iint\limits_R (16 - x^2 - 2y^2)\,dA \approx \sum_{i=1}^{4} f(x_i^*, y_i^*)\,\Delta A_i$$

$$\approx \left(\frac{61}{4}\right)(1) + \left(\frac{53}{4}\right)(1) + \left(\frac{37}{4}\right)(1) + \left(\frac{45}{4}\right)(1)$$

$$\approx 49 \quad // \text{Ans.}$$

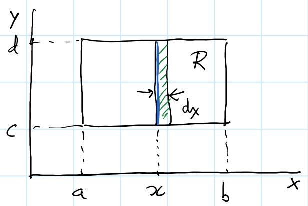
The exact value of the integral is 48.

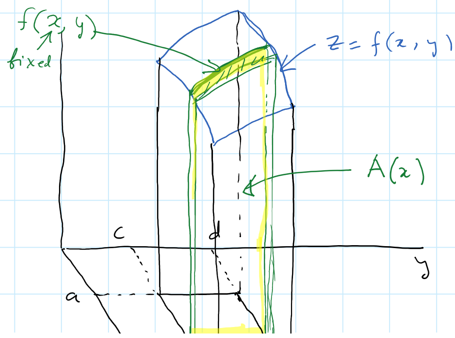
The percentage relative error $= \left|\dfrac{48 - 49}{48}\right| \times 100 \approx 2.1\%$

which is in very good agreement.

FUBINI'S THEOREM FOR A DOUBLE INTEGRAL (FOR A RECTANGULAR REGION)

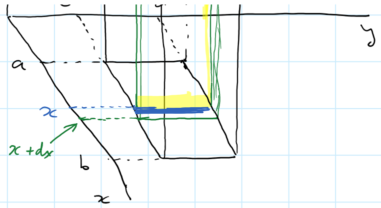
Consider the rectangular region R defined by $a \le x \le b$, $c \le y \le d$ (or $R = [a,b] \times [c,d]$).





Let A(x) be the area of

the section ...



Let $A(x)$ be the area of the section standing on the ordinate at x between $y = c$ and $y = d$. Then the volume dV of a typical slab with thickness dx is

$$dV = A(x)\, dx$$

$$\therefore \quad V = \int dV = \int_a^b A(x)\, dx \qquad (1)$$

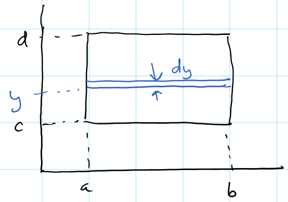
On the other hand, the cross-sectional area $A(x)$ is

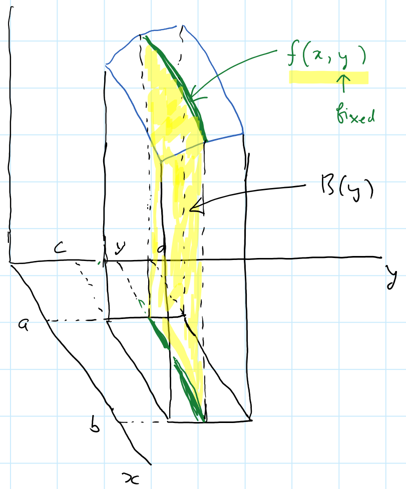
$$A(x) = \int_c^d f(\underset{\substack{\uparrow \\ \text{fixed}}}{x}, y)\, dy \qquad (2)$$

Subst this into (1),

$$V = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx \qquad (3)$$

If we use a horizontal strip





Let $B(y)$ be the area of the section standing on the absissa at y between $x = a$ and $x = b$. Then the volume dV of a typical slab with thickness dy is

$$dV = B(y)\, dy$$

$$\therefore \quad V = \int dV = \int_c^d B(y)\, dy \qquad (4)$$

where $$B(y) = \int_a^b f(x, \underset{\substack{\uparrow \\ \text{fixed}}}{y})\, dx \qquad (5)$$

Subst (5) into (4),

$$V = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy \qquad (6)$$

Combining (3) & (6),

$$\boxed{\iint_R f(x,y)\, dA = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy} \qquad (7)$$

which is Fubini's theorem of a double integral over a rectangular region.

<u>N.B</u>: When R is a rectangular region and $f(x,y)$ is separable, ie,

$$\underline{f(x,y) = g(x)\, h(y)}$$

Then (7) becomes

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Then (7) becomes

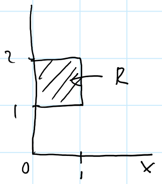
$$\iint_R f(x,y)\,dA = \int_c^d \left(\int_a^b g(x)\,h(y)\,dx \right) dy$$

$$= \int_c^d h(y) \left(\underbrace{\int_a^b g(x)\,dx}_{\text{it is a number}} \right) dy$$

$$= \left(\int_a^b g(x)\,dx \right) \left(\int_c^d h(y)\,dy \right)$$

Ex2: Evaluate $\iint_R (4 - x - y)\,dA$ where R is the square defined by $0 \le x \le 1$, $1 \le y \le 2$ (or $R = [0,1] \times [1,2]$).

Solution



$$\iint_R (4-x-y)\,dA = \int_0^1 \int_1^2 (4-x-y)\,dy\,dx$$

$$= \int_0^1 \left((4-x)y - \frac{y^2}{2} \right) \Big|_{y=1}^{2} dx$$

$$= \int_0^1 \left[(4-x)(2) - \frac{(2)^2}{2} - \left((4-x)(1) - \frac{(1)^2}{2} \right) \right] dx$$

$$= \int_0^1 \left(\frac{5}{2} - x \right) dx = \left(\frac{5}{2}x - \frac{x^2}{2} \right) \Big|_0^1 = \frac{5}{2} - \frac{1}{2} = 2 \text{ // Ans.}$$

OR we can change the order of integration

$$\iint_R (4-x-y)\,dA = \int_1^2 \left(\int_0^1 (4-x-y)\,dx \right) dy$$

$$= \int_1^2 \left((4-y)x - \frac{x^2}{2} \right) \Big|_{x=0}^{1} dy$$

$$= \int_1^2 \left[(4-y)(2) - \frac{(2)^2}{2} - 0 \right] dy$$

$$= \int_1^2 \left(\frac{7}{2} - y \right) dy = \left(\frac{7}{2}y - \frac{y^2}{2} \right) \Big|_1^2$$

$$= \left(\frac{7}{2}(2) - \frac{(2)^2}{2} \right) - \left(\frac{7}{2} - \frac{1}{2} \right)$$

$$= (7-2) - (3) = 2 \text{ // Ans.}$$

The same!

See you on Friday!