The chain rule

There are serval versions of chain rule:

(1) if $z = f(x, y)$ is a differentiable function of x and y, and x and y are functions of t, then z is also a function of t.

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y.$$

where both $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$

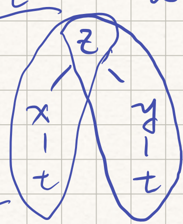
$$\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}.$$

as $\Delta t \to 0$, $\Delta x, \Delta y \to 0$

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t}$$

$$= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} + \underset{\varepsilon_1}{0}\cdot\frac{dx}{dt} + \underset{\varepsilon_2}{0}\cdot\frac{dy}{dt}.$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$



e.g.1. obtain the quotient rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}.$$

let $f(u, v) = \frac{u}{v}$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\partial z}{\partial u}\frac{du}{dx} + \frac{\partial z}{\partial v}\frac{dv}{dx}$$

$$= \frac{1}{v}\cdot u' + \left(-\frac{u}{v^2}\right)v'$$

$$= \frac{vu' - uv'}{v^2}.$$

e.g.2. if $z = f(x, y)$ is a differentiable of x, y, and x, y are functions of s and t., then z is also a function of s and t. compute $\frac{dz}{ds}$ and $\frac{dz}{dt}$.

$$dz = \frac{dz}{dx}dx + \frac{dz}{dy}dy.$$

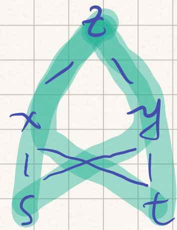
also, x and y are function of s and t,

$$dx = \frac{dx}{ds}ds + \frac{dx}{dt}dt, \quad dy = \frac{dy}{ds}ds + \frac{dy}{dt}dt.$$

z is also a function of s and t ?

$$dz = \frac{dz}{ds}ds + \frac{dz}{dt}dt + \frac{dz}{ds}ds + \frac{dz}{dt}dt.$$

$$= 2\left(\frac{dz}{ds}ds + \frac{dz}{dt}dt\right).$$



$$\frac{dz}{ds} = \frac{dz}{dx}\frac{dx}{ds} + \frac{dz}{dy}\frac{dy}{ds}. \qquad \frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt}.$$

e.g. 3. Use the chain rule to attain $\frac{dz}{ds}$ and $\frac{dz}{dt}$ where

$z = \arcsin(x-y)$, $x = s^2+t^2$, $y = 1-2st$

$$\frac{dz}{ds} = \frac{dz}{dx}\frac{dx}{ds} + \frac{dz}{dy}\frac{dy}{ds}.$$

$$= \frac{1}{\sqrt{1-(x-y)^2}}\cdot \underbrace{(1)}_{(x-y)'}(2s) + \frac{1}{\sqrt{1-(x-y)^2}}\cdot(-1)\cdot(-2t)$$

$$\frac{dz}{ds} = \frac{2s+2t}{\sqrt{1-(x-y)^2}}.$$

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt}$$

$$= \frac{1\cdot 2t}{\sqrt{1-(x-y)^2}} + \frac{(-1)\cdot -2s}{\sqrt{1-(x-y)^2}}$$

$$= \frac{2s+2t}{\sqrt{1-(x-y)^2}}$$

The general version: If u is a differentiable of n variables $x_1, x_2 \cdots x_n$, each of them are function of n variables $t_1, t_2 \cdots t_n$, and each x_i is a differentiables of n variables and u is differentiable for $t_1, t_2 \cdots t_n$.

then
$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}.$$
$$= \sum_{k=1}^{n} \frac{\partial u}{\partial x_k}\frac{\partial x_k}{\partial t_i}$$

e.g. 3. Given $R = f(x, y, z, t)$ where $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$, $t = t(u, v, w)$

Find $\frac{\partial R}{\partial u}$, $\frac{\partial R}{\partial v}$, $\frac{\partial R}{\partial w}$

R

x y z t

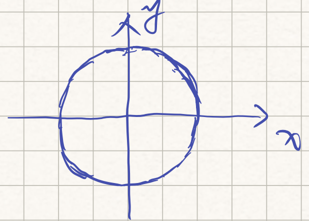
u v w

$$\frac{\partial R}{\partial u} = \frac{\partial R}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial R}{\partial t}\frac{\partial t}{\partial u}$$

$$\frac{\partial R}{\partial v} = \frac{\partial R}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial R}{\partial t}\frac{\partial t}{\partial v}$$

$$\frac{\partial R}{\partial w} = \frac{\partial R}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial w} + \frac{\partial R}{\partial t}\frac{\partial t}{\partial w}$$

Implicit Differentiation



Consider the circle $F(x, y) = x^2 + y^2 - 1 = 0$

By using the virtical test, we know that the above equation does not represent a function. However, we can express the circle as a set of two functions

$y = \sqrt{1-x^2}$ upper semi circle

$y = -\sqrt{1-x^2}$ lower semi circle.

We say $F(x,y)=0$ define y implicitly as a function of x $y=f(x)$ if $F(x,f(x))=0$ for any x in the domain of f

Differentiating $F(x,y)=0$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0.$$

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

The implict function therom states that if $F(x,y)=0$ at (x_0,y_0) and $\frac{\partial F}{\partial y} \neq 0$ at this point then y is a function of x in a neighborhood of (x_0,y_0).

e.g. 4 Find y' if $x^3+y^3=6xy$.

i) $x^3+y^3=6xy$

$$3x^2+3y^2y' = 6y+6xy'$$

$$y' = \frac{6y-3x^2}{3y^2-6x} = \frac{2y-x^2}{y^2-2x}.$$

ii)

$$y' = -\frac{F_x}{F_y} = \frac{6y-3x^2}{3y^2-6x} = \frac{2y-x^2}{y^2-2x}.$$

e.g. 5. Given $xyz = \cos(x+y+z)$, Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$F(x,y,z) = xyz - \cos(x+y+z) = 0$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$