1. Determine $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$.

$a_n = \left(\frac{n^2+1}{2n^2+1}\right)^n$ ⇐ Root test

$= \left(1 - \frac{n^2}{2n^2+1}\right)^n$

$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{n^2+1}{2n^2+1}\right)$

$= \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{2+\frac{1}{n^2}} = \frac{1}{2} < 1.$

⇒ Absolute convergent

$1^{\infty} \Rightarrow \mathrm{Inf}$
$\Rightarrow$ finite number

2. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$.

$a_n = \left(1+\frac{1}{n}\right)^{n^2}$ ⇐ Root test.

$\lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^{n^2}\right]^{\frac{1}{n}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

$y = \left(1+\frac{1}{x}\right)^x.$

$\ln y = x \ln\left(1+\frac{1}{x}\right).$

$= \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}.$ ✓ 洛必达

$\lim_{x\to 0} \ln y = \frac{\left(\frac{1}{x}+1\right)\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = \frac{1}{x}+1 = 1$

$\Rightarrow \lim_{x\to 0} \ln y = 1$

$\lim_{x\to 0} y = e \approx 2.7 > 1.$

⇒ diverses. .

Review.

$$\int \frac{x^2+2x+2}{x^2+2x-1}\,dx.$$

$$=\int\left(1+\frac{3}{x^2+2x-1}\right)dx.$$

$$=x+3\int\frac{1}{x^2+2x-1}\,dx.$$

$$\int\frac{1}{x^2+2x-1}$$

$$=\int\frac{1}{(x+1)^2-2}$$

$u=x+1$.

$du=dx$.

$u=\sqrt{2}\sec\theta$

$du=\sqrt{2}\sec\theta\tan\theta\,d\theta$

$$\Rightarrow\ =\int\frac{\sqrt{2}\sec\theta\tan\theta\,d\theta}{2\sec^2\theta-2}$$

$$=\int\frac{\sqrt{2}\sec\theta\tan\theta}{2\tan^2\theta}\,d\theta$$

$$=\frac{\sqrt{2}}{2}\int\frac{\sec\theta}{\tan\theta}\,d\theta.$$

$$\Rightarrow x+\frac{3}{2}\sqrt{2}\int\frac{\sec\theta}{\tan\theta}\,d\theta.$$

$$=x+\frac{3}{2}\sqrt{2}\int\frac{1}{\sin\theta}\,d\theta$$

$$=x+\frac{3}{2}\sqrt{2}\ln|\csc\theta-\cot\theta|+C$$

$$=x+\frac{3}{2}\sqrt{2}\ln\left|\frac{u}{\sqrt{u^2-2}}-\frac{\sqrt{2}}{\sqrt{u^2-2}}\right|+C$$

$$=x+\frac{3}{2}\sqrt{2}\ln\left|\frac{x+1-\sqrt{2}}{\sqrt{x^2+2x-1}}\right|+C$$

$$=x+\frac{3}{2}\sqrt{2}\left[\ln|x+1-\sqrt{2}|-\frac{1}{2}\ln|x+2x-1|\right]+C.$$

$$\int\sec x\,dx=\ln|\sec x+\tan x|+C.$$

$$\int\csc x\,dx=\ln|\csc x-\cot x|+C.$$

2. $\dfrac{x^3-1}{x-1}=x^2+x+1.$

Synthetic Division:

$x^3+0x^2+0x-1.$

1 0 0 -

3. $\displaystyle\int_{-\infty}^{\infty}e^{4-|x|}=e^4\cdot\int_{-\infty}^{\infty}e^{-|x|}=e^4\left(\underbrace{\int_{-\infty}^{0}e^{-|x|}}_{\mathrm{I}}+\underbrace{\int_{0}^{\infty}e^{-|x|}}_{\mathrm{II}}\right)$

I: $\displaystyle=\int_{-\infty}^{0}e^{x}=\lim_{b\to-\infty}\left(\int_{b}^{0}e^{x}\right)=e^0=1$

II: $\displaystyle=\int^{\infty}e^{-x}\ \ \int^{b}e^{-x}$

$\int_0 e = \lim_{b\to\infty} \int_0 e = 1$

$\Rightarrow = e^4 \cdot 2 = 2e^4.$

3. $\int \frac{1}{\sqrt{e^x-1}}\, dx.$

try: $u^2 = e^x - 1.$ $\quad 2u\,du = e^x dx. \Rightarrow dx = \frac{2u\,du}{u^2+1}.$

$du = e^x$

$= \int \frac{2u\,du}{u\cdot(u^2+1)} = 2\int \frac{1}{u^2+1}\, du. = 2\tan^{-1}\sqrt{e^x-1} + C.$

4. Determine $\int_0^\infty \frac{\cos x}{x^4}$

$0 \le \left|\int_0^\infty \frac{\cos x}{x^4}\right| dx \le \int_0^\infty \left|\frac{\cos x}{x^4}\right| dx \le \int_0^\infty \left|\frac{1}{x^4}\right| dx.$

$\because 4 > 1$ $\therefore$ converges.

p intergral.