SETS

OUTLINE:

1) Introduction to sets
2) Defining and representing sets
3) Set operations
4) Sets and logic
5) Cardinality
6) Cartesian products and relations

1. INTRODUCTION TO SETS

Naive Set Theory

- In this course, we adopt the following informal definition of set:
- A set is a (unordered) collection of objects, called elements or members of the set. The set is said to contain its elements. The notation $a \in S$ means that the object a is an element of the set S. The notation $a \notin S$ means that a is not an element of the set S.
- Note that we do not specify what an "object" is. The theory that results from this intuitive definition of a set as a collection of objects is called naive set theory. This theory leads to logical inconsistencies (paradoxes).

Russell’s Paradox

- Many paradoxes of naive set theory are variations on a common theme.
- The problem is the (implicit) assumption that any property whatsoever may be used to form a set (that is, any collection of elements whatsoever can be grouped into a set, without restrictions).
- Here we illustrate the problem in that assumption via Russell’s Paradox.
- Preliminary remarks:

1) A set may contain other sets as elements.
2) A set may contain *itself* as an element.

Russell's Paradox

- Let then S be the set whose elements are exactly the sets which are not elements of themselves.
- Pop quiz: does S contain itself?

S

$\forall a \in S,\ a \in \neg S.$

Russell’s Paradox

- Let’s then define S to be the set whose elements are exactly the sets which are not elements of themselves.
- Pop quiz: does S contain itself?
  - Assume $S \in S$. Then, by definition of S, $S \notin S$, contradiction.
  - Assume $S \notin S$. Then, again by definition of S, $S \in S$, contradiction.

Axiomatic vs. Naive

- There is a more proper axiomatic set theory (actually, there are many versions of it) which gets rid of the known paradoxes of naive set theory.
- However, since
  - Naive set theory suffices for the everyday use of set theory concepts in contemporary mathematics (in particular, it does not give rise to inconsistencies <u>within the restricted scope of this course</u>);
  - Naive set theory is much more user-friendly than axiomatic set theory;
  - Naive set theory is a useful stepping stone towards more formal set theories;

  we are going to adopt naive set theory here (being aware of its limitations).

2. DEFINING AND REPRESENTING SETS

Set equality and subsets

- Two sets A and B are equal if and only if they have exactly the same elements:

$$A = B \ \mathit{iff} \ \forall x \left(x \in A \leftrightarrow x \in B \right)$$

- A set A is a subset of a set B (notation: A $\subseteq$ B) if and only if every element of A is also an element of B

$$A \subseteq B \ \mathit{iff} \ \forall x \left(x \in A \rightarrow x \in B \right)$$

Representing sets: the roster method

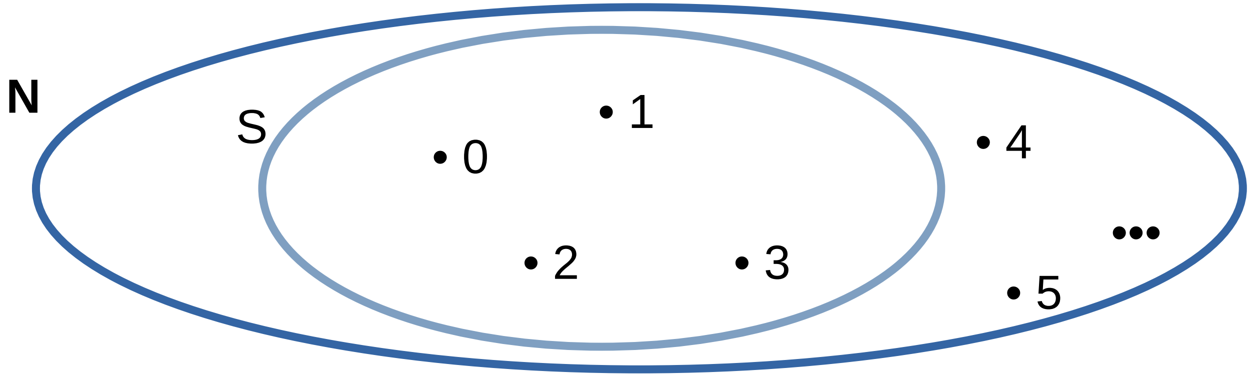
- The elements of a set S are listed explicitly within curly brackets. EX: $S = \{0,1,2,3\}$
- The order of elements does not matter: $S = \{0,1,2,3\} = \{2,0,1,3\}$
- Listing a member more than once does not change the set, and it is strongly discouraged: $S = \{0,1,2,3\} = \{0,0,0,1,1,2,3,3,3,3\}$
- For large sets (finite or infinite), we list enough elements to highlight the pattern and then we use ellipses (...) to leave the others implied:
  - $A = \{a,b,c,...,x,y,z\}$ (the set of english letters, i.e. the english alphabet)
  - $\mathbf{Z} = \{...,-3,-2,-1,0,1,2,3,...\}$ (the set of integer numbers)

Representing sets: set-builder method

- The elements of a set S are implicitly defined as all and only the objects satisfying a property:
  - $S = \{x \mid x$ is an integer between 0 and 3 (included)$\} = \{x \mid x \in \mathbf{Z}$ and $0 \leq x \leq 3\} = \{0,1,2,3\}$
    - We are saying: "S is the set of the objects x such that (|) x is an integer and $0 \leq x \leq 3$"
  - $T = \{2x \mid x \in S\} = \{0,2,4,6\}$
    - We are saying: "T is the set of the objects of the form 2x for (|) x in S", that is, "T is the set of the doubles of the elements of S"
  - $\mathbf{Q} = \{a/b \mid a \in \mathbf{Z}, b \in \mathbf{Z}$ and $b \neq 0\}$ is the set of rational numbers
- The variable x is just a label (placeholder), which can be changed to any other symbol, especially to avoid clashes of notation:
  - A = {x | x is a letter of the English alphabet} not ideal, since the label x clashes with the English letter 'x'
  - Better: A = {@ | @ is a letter of the English alphabet}

Representing sets: Venn diagrams

- This is a graphical representation, useful to illustrate simple set relationships at a glance.
- Objects are drawns as dots and sets as closed curves encircling their elements



Empty set and universal set

- The empty set is the set with no elements. It is usually denoted $\varnothing$, or $\{\,\}$
  - Not to be confused with $\{\varnothing\}$, which is the set containing the empty set (and hence it has one element)
- The universal set is the set of all the objects under consideration.
  - Often denoted U. In Venn diagrams, represented with a rectangle
  - May be explicitly stated or left implicit
  - Depends on the context

Some important sets of numbers

- $\mathbb{N}$ or **N** = {0,1,2,3,...} = the set of natural numbers
- $\mathbb{Z}$ or **Z** = {...,-2,-1,0,1,2,...} = the set of integer numbers
- $\mathbf{Z}^{+}$ = {1,2,3,...} = the set of positive integers
- $\mathbf{Z}^{-}$ = {...,-3,-2,-1} = the set of negative integers
- $\mathbb{Q}$ or **Q** = $\{a/b \mid a \in \mathbf{Z}, b \in \mathbf{Z} \text{ and } b \neq 0\}$ = the set of rational numbers
- $\mathbf{Q}^{+}$ = the set of positive rational numbers, $\mathbf{Q}^{-}$ = the set of negative rational numbers
- $\mathbb{R}$ or **R** = the set of real numbers (very complicated definition)
- $\mathbf{R}^{+}$ = the set of positive real numbers, $\mathbf{R}^{-}$ = the set of negative real numbers
- $\mathbb{C}$ or **C** = $\{a + ib \mid a \in \mathbf{R}, b \in \mathbf{R}\}$ = the set of complex numbers; $i = \sqrt{-1}$

Interval notation

- Used to concisely represent ranges of real numbers
- $[a,b] = \{x \in \mathbf{R} \mid a \leq x \leq b\}$ "closed interval": the extrema are included
- $(a,b] = \{x \in \mathbf{R} \mid a < x \leq b\}$
- $[a,b) = \{x \in \mathbf{R} \mid a \leq x < b\}$
- $(a,b) = \{x \in \mathbf{R} \mid a < x < b\}$ "open interval": the extrema are excluded
- Note that degenerate cases are possible:
  - if $a = b$, then $[a,b] = \{a\}$ (set with single element) and $(a,b] = [a,b) = (a,b) = \varnothing$;
  - if $b < a$, then $[a,b] = (a,b] = [a,b) = (a,b) = \varnothing$
- a or b can be infinite (as an excluded extremum): e.g., $(-\infty,4) = \{x \in \mathbf{R} \mid x < 4\}$; $(-\infty,+\infty) = \mathbf{R}$

Subsets and power set

- Recall the notion of subset: $A \subseteq B \; iff \; \forall x (x \in A \rightarrow x \in B)$
- The empty set is a subset of any set: $\forall A, \varnothing \subseteq A$
- A set is always a subset of itself: $A \subseteq A$
- A subset of A different from A itself is a proper subset of A (specific notation: $\subset$); e.g., $\varnothing \subset A$
- The power set of a set A is the set of all subsets of A, including $\varnothing$ and A (various notations: P(A), $\wp$(A), 2^A)

$$P(A) = \{ S \mid S \subseteq A \}$$

Examples

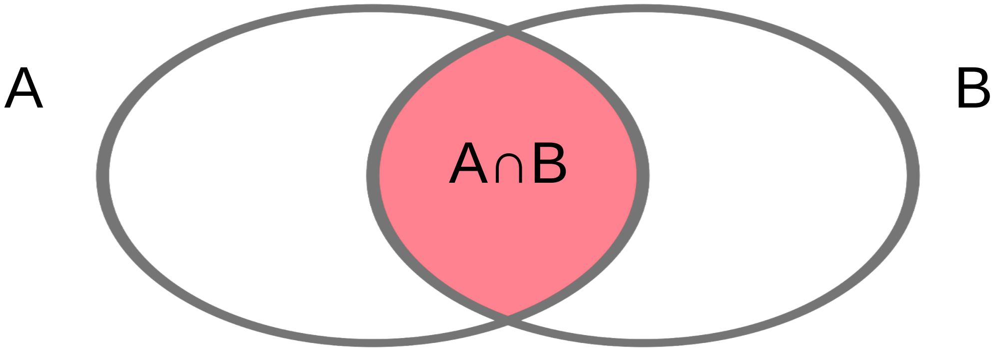
- $P(\varnothing) = \{S \mid S \subseteq \varnothing\} = \{\varnothing\}$
- $P(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$
- $P(\{0,1\}) = \{\varnothing, \{0\}, \{1\}, \{0,1\}\}$
- $P(\{0,1,2\}) = \{\varnothing, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$
- In general, if A has n elements, P(A) has 2^n elements (hence the name power set) [EX: prove this statement by induction on n]

3. SET OPERATIONS

Intersection

- The intersection of 2 sets A and B is the set of the objects which are elements of both A and B (notation: $A \cap B$).

$$A \cap B = \{ x \mid x \in A \land x \in B \}$$



Examples

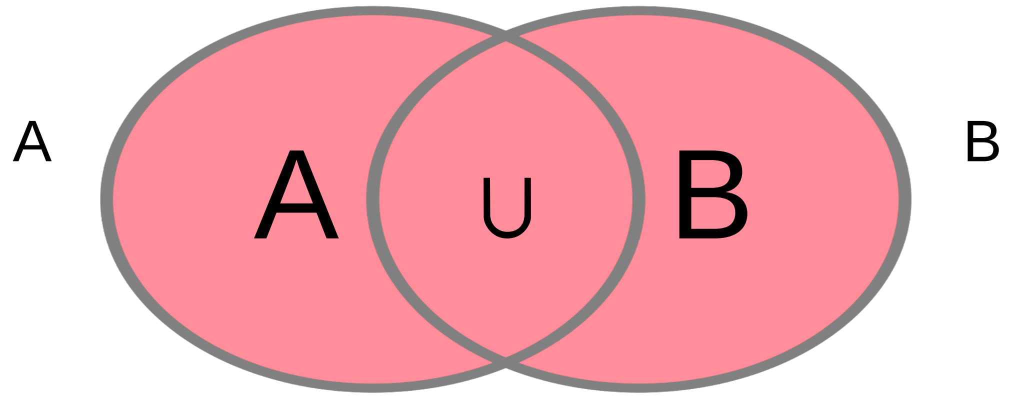
- $\{0,1,2\} \cap \{1,2,3\} = \{1,2\}$
- $\{0,1,2\} \cap \{4,5,6\} = \emptyset$. Sets with empty intersection are said to be disjoint
- $[1,5] \cap (2,6] = \{x \in \mathbf{R} \mid 1 \leq x \leq 5 \text{ and } 2 < x \leq 6\} = (2,5]$
- $\{x \in \mathbf{N} \mid x \text{ odd}\} \cap \{x^2 \mid x \in \{0,1,2,3\}\} = \{x \in \mathbf{N} \mid x \text{ odd}\} \cap \{0,1,4,9\} = \{1,9\}$
- If $A \subseteq B$, then $A \cap B = A$ [QUESTION: does the converse (if $A \cap B = A$, then $A \subseteq B$) hold?]
- Intersection is associative: $(A \cap B) \cap C = A \cap (B \cap C)$. Therefore, usually brackets are omitted: $(A \cap B) \cap C = A \cap (B \cap C) = A \cap B \cap C$
- This allows to intersect an arbitrary (even infinite) number of sets:

$$\bigcap_i A_i = \{x \mid \forall i (x \in A_i)\}$$

Union

- The union of 2 sets A and B is the set of the objects which are elements of A or of B (notation: A∪B).

$$A \cup B = \{ x \mid x \in A \vee x \in B \}$$



Examples

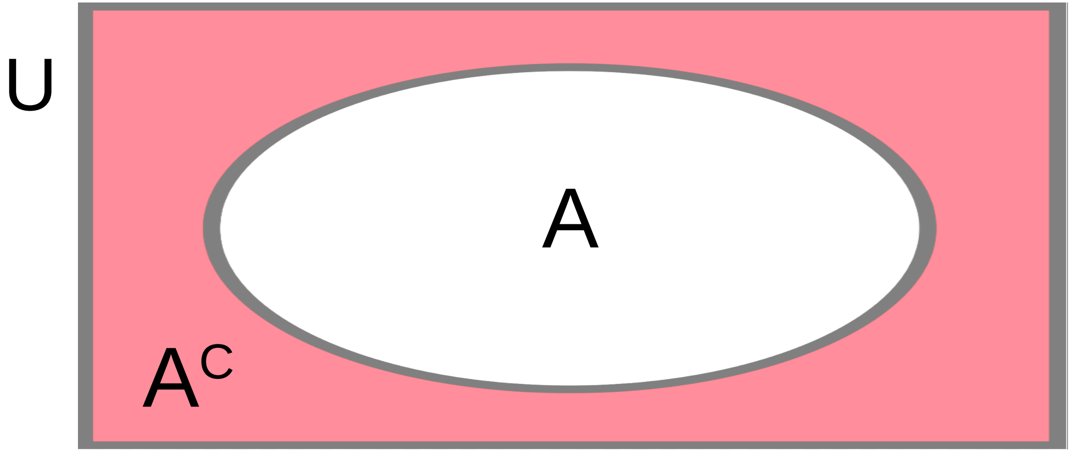
- $\{0,1,2\} \cup \{1,2,3\} = \{0,1,2,3\}$
- $\{0,1,2\} \cup \{4,5,6\} = \{0,1,2,4,5,6\}$
- $[1,5] \cup (2,6] = \{x \in \mathbf{R} \mid 1 \leq x \leq 5 \text{ or } 2 < x \leq 6\} = [1,6]$
- $\{x \in \mathbf{N} \mid x \text{ odd}\} \cup \{x^2 \mid x \in \{0,1,2,3\}\} = \{x \in \mathbf{N} \mid x \text{ odd}\} \cup \{0,1,4,9\} = \{x \in N \mid x \text{ odd}\} \cup \{0,4\}$
- If $A \subseteq B$, then $A \cup B = B$ [QUESTION: does the converse (if $A \cup B = B$, then $A \subseteq B$) hold?]
- Union is associative: $(A \cup B) \cup C = A \cup (B \cup C)$. Therefore, usually brackets are omitted: $(A \cup B) \cup C = A \cup (B \cup C) = A \cup B \cup C$
- This allows to unite an arbitrary (even infinite) number of sets:

$$\bigcup_i A_i = \{x \mid \exists i (x \in A_i)\}$$

Complement

- Given a universe U, the complement of a set A (with respect to U) is the set of the elements (of U) which are not in A (notation: A^c or $\overline{A}$)

$$A^C = \{ x \color{blue}{[\in U]} \mid x \notin A \}$$



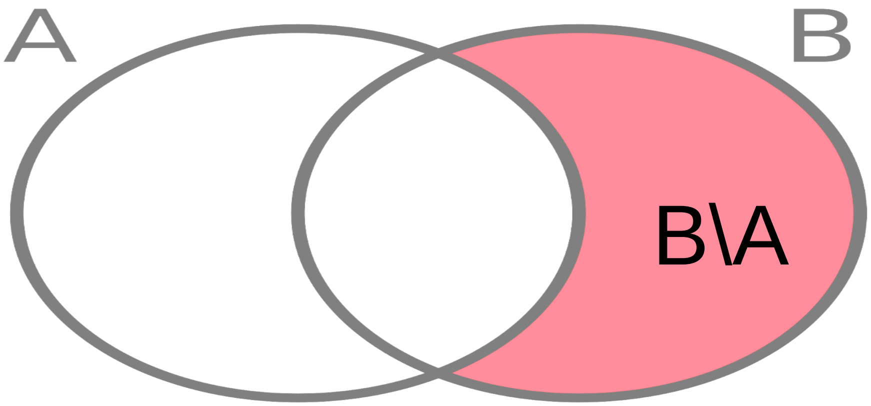
Examples

- For any universe U, $U^C = \varnothing$ and $\varnothing^C = U$
- For any universe U and any set $A \subseteq U$, $(A^C)^C = A$ [EX: prove this]
- In the universe **N** of natural numbers, $\{x \in \mathbf{N} \mid x \text{ odd}\}^C = \{x \in \mathbf{N} \mid x \text{ even}\}$ and $\{x \in \mathbf{N} \mid x \text{ even}\}^C = \{x \in \mathbf{N} \mid x \text{ odd}\}$
- In the universe **R** of real numbers, $[1,5)^C = \{x \in \mathbf{R} \mid \neg(1 \leq x < 5)\}$ $= \{x \in \mathbf{R} \mid x < 1 \lor x \geq 5\} = (-\infty, 1) \cup [5,+\infty)$

Difference

- Given 2 sets A and B, the difference of A in B is the set of the elements of B which are not in A (notation: B\A or B-A). Note that it is not required that A⊆B.

$$B \setminus A = \{x \mid x \in B \wedge x \notin A\} = \{x \in B \mid x \notin A\}$$



Examples

- The difference with respect to the universe is the complement
- $B \backslash A = \{x \mid x \in B \wedge x \notin A\} = \{x \mid x \in B \wedge (x \in U \wedge x \notin A)\} = B \cap A^C$
- $\{0,1,2,3,4\} \backslash \{2,3\} = \{0,1,4\}$
- $\{0,1,2,3,4\} \backslash \{5,6,8\} = \{0,1,2,3,4\}$
- $\{0,1,2,3,4\} \backslash \mathbf{N} = \varnothing$
- $\{0,1,2,3,4\} \backslash \varnothing = \{0,1,2,3,4\}$

Notable set identities (U = universe)

- *Identity laws:* $A \cap U = A$; $A \cup \varnothing = A$
- *Domination laws:* $A \cup U = U$; $A \cap \varnothing = \varnothing$
- *Law of disjointness (unofficial name):* $A \cap A^C = \varnothing$
- *Law of partition (unofficial name):* $A \cup A^C = U$
- *Idempotent laws:* $A \cap A = A$; $A \cup A = A$
- *Complementation law:* $(A^C)^C = A$
- *Associative laws:* $A \cap (B \cap C) = (A \cap B) \cap C$; $A \cup (B \cup C) = (A \cup B) \cup C$
- *Commutative laws:* $A \cap B = B \cap A$; $A \cup B = B \cup A$
- *Distributive laws:* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$; $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- *Absorption laws:* $A \cap (A \cup B) = A$; $A \cup (A \cap B) = A$
- *De Morgan's laws:* $(A \cap B)^C = A^C \cup B^C$; $(A \cup B)^C = A^C \cap B^C$

Proving set identities

- 3 ways:

1) Prove that the LHS set is included in the RHS set and vice-versa

2) Use set-builder notation and logic

3) Use membership tables (deprecated by Prof. Pasini)

Proving set identities

1) Prove that the LHS set is included in the RHS set and vice-versa

- EX: $(A^C)^C = A$.
- $(A^C)^C \subseteq A$: if $x \in (A^C)^C$, then $x \notin A^C$ (def. of complement), so $x \in A$ (def. of complement).
- $A \subseteq (A^C)^C$: if $x \in A$, then $x \notin A^C$ (def. of complement), so $x \in (A^C)^C$ (def. of complement).

Proving set identities

2) Use set-builder notation and logic

- EX: prove that $A \cap (B \cap C) = (A \cap B) \cap C$
- $A \cap (B \cap C) = \{x \mid x \in A \wedge x \in (B \cap C)\}$ (def. of $\cap$)
  $= \{x \mid x \in A \wedge (x \in B \wedge x \in C)\}$ (def. of $\cap$)
  $= \{x \mid (x \in A \wedge x \in B) \wedge x \in C\}$ (associativity of $\wedge$)
  $= \{x \mid x \in (A \cap B) \wedge x \in C\}$ (def. of $\cap$)
  $= (A \cap B) \cap C$ (def. of $\cap$)

Proving set identities

3) Use membership tables (deprecated by Prof. Pasini). Membership tables are “truth tables” for sets.

- EX: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

4. SETS AND LOGIC

Relationship between sets and logic

- Each set S is associated to a predicate $P_S(x)$ = "$x \in S$".
- Conversely, given a domain D and a predicate P (interpreted on D), we can form the truth set of P, which is the set of elements of D (thought of as the universe) for which P is true: $S_P = \{x \in D \mid P(x)\}$.
  - EX: If D = **N** (the natural numbers) and P(x) = "$x \leq 3$", then $S_P = \{x \in \mathbf{N} \mid P(x)\} = \{x \in \mathbf{N} \mid x \leq 3\} = \{0,1,2,3\}$.
- "Compound" predicates may be formed using the logical connectives and the quantifiers:
  - EX (cont.): If Q(x) = "$x > 2$", and we set $R(x) = P(x) \wedge Q(x)$, then $S_R = \{x \in \mathbf{N} \mid P(x) \wedge Q(x)\} = \{x \in \mathbf{N} \mid 2 < x \leq 3\} = \{3\}$.
  - The domain can also be encoded in a predicate instead of being explicitly stated: if I(x) = "x is a natural number", and we set $R'(x) = I(x) \wedge P(x) \wedge Q(x)$, then
  - $S_{R'} = \{x \mid I(x) \wedge P(x) \wedge Q(x)\} = \{x \mid x \in \mathbf{N} \wedge 2 < x \leq 3\} = \{x \in \mathbf{N} \mid 2 < x \leq 3\} = S_R$

Relationship between sets and logic

- Intersection of sets corresponds to conjunction of predicates:
  - If $S = \{x \mid P(x)\}$ and $T = \{x \mid Q(x)\}$, then $S \cap T = \{x \mid P(x) \wedge Q(x)\}$
- Union of sets corresponds to disjunction of predicates:
  - If $S = \{x \mid P(x)\}$ and $T = \{x \mid Q(x)\}$, then $S \cup T = \{x \mid P(x) \vee Q(x)\}$
- Complementation of sets corresponds to negation:
  - If $S = \{x \mid P(x)\}$, then $S^C = \{x \mid \neg P(x)\}$
- We could define other set operations corresponding to the other connectives, but they are not customary.
- Multiple intersection of sets corresponds to universal quantifier
  - If $S_i = \{x \mid P_i(x)\}$ for $i = 1,\dots,n$, then $\cap S_i = \{x \mid \forall i\ P_i(x)\}$
- Multiple union of sets corresponds to existential quantifier
  - If $S_i = \{x \mid P_i(x)\}$ for $i = 1,\dots,n$, then $\cup S_i = \{x \mid \exists i\ P_i(x)\}$

5. CARDINALITY

- We say that a set is finite if it has n elements for some natural number n. Otherwise, we say that the set is infinite.
- The cardinality of a finite set S, denoted |S|, is the number of its elements.
  - EX: $|\varnothing| = 0$, $|\{\varnothing\}| = 1$, $|\{0,1,2,3\}| = 4$, $|P(S)| = 2^S$, $|S \times T| = |S| \cdot |T|$
- The cardinality of an infinite set is infinite, but there are various “grades” of infinity:
  - $|\mathbf{N}| = |\mathbf{Z}| = |\mathbf{Q}| < |(0,1)| = |\mathbf{R}| = |\mathbf{C}| = |\mathbf{R}^3| < |P(\mathbf{R})|$ ....

6. CARTESIAN PRODUCTS AND RELATIONS

Tuples

- An [ordered] n-tuple $(a_1,a_2,...,a_n)$ is a collection of objects in which one (a_1) can be identified as the 1^{st}, one (a_2) as the 2^{nd}, and so on up to the n^{th} (a_n). In other words, differently than for sets, for tuples the order of the elements matter:
  - $\{0,1,2\} = \{1,2,0\}$ (sets: only the elements matter, not the order in which they appear)
  - $(0,1,2) \neq (1,2,0)$ (3-tuples: although the appearing objects are the same, they appear in different orders)
- Two tuples are equal iff they [have the same number of elements and] have the same elements in the same positions: $(a_1,a_2,...,a_n) = (b_1,b_2,...,b_k)$ iff $n=k \wedge a_1=b_1 \wedge a_2=b_2 \wedge ... \wedge a_n=b_n$
- 2-tuples are also called couples or pairs, 3-tuples are also called triplets.

Cartesian product

- The cartesian product of 2 sets A and B is the set of ordered couples (a,b) for $a \in A$ and $b \in B$ (notation: AxB).

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

- More generally, the cartesian product of n sets $A_1,\ldots,A_n$ is the set of n-tuples $(a_1,\ldots a_n)$ with $a_i \in A_i$

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \ \mathit{for\ all}\ i = 1, \ldots, n\}$$

Examples

- $\{0\} \times \{1,2,3,4\} = \{(0,1),(0,2),(0,3),(0,4)\}$
- $\{a,b,c\} \times \{0,1\} = \{(a,0\},(b,0\},(c,0),(a,1),(b,1),(c,1)\}$
- $\emptyset \times A = \emptyset$
- $\mathbf{R} \times \mathbf{R} = \mathbf{R}^2$, which can be also interpreted geometrically: the cartesian product of 2 lines is a plane (the cartesian plane)
- $\{0,1\} \times \{0,1\} \times \{0,1\} = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$
- Cartesian products of infinitely many sets are also defined

Relations

- A subset R of the cartesian product A x B is called a binary relation on A and B (or from A to B).

- More generally, a subset R of the cartesian product A_1 x ... x A_n is called an n-ary relation on $A_1,\ldots,A_n$. The number n is the arity of the relation.