# Extra slides for Chapter 3: Adequacy of connectives

Based on Prof. Lila Kari's slides

For CS2209A/B

By Dr. Charles Ling

# Adequate set of connectives

A remarkable property of the “standard set of connectives” $(\sim, \bullet, \vee, \supset, \equiv)$ is the fact that for every table

| P | Q | ... | |
|---|---|---|---|
| 1 | 1 | ... | * |
| | | | * |
| | | | * |
| 0 | 0 | ... | * |

there is a formula (depending on the variables P, Q,... and using only the standard connectives) that has exactly this truth table.
(There is Boolean function f(P, Q, …) with exactly this truth table).

Any set of connectives with the capability to express all truth tables is said to be **adequate**. As Post (1921) observed, the standard connectives are adequate.

We can show that another set S of connectives is adequate if we can express all the standard connectives $(\sim, \bullet, \vee, \supset, \equiv)$ in terms of S

# Adequate set of connectives

Formulas $(A \supset B)$ and $(\sim A \vee B)$ are tautologically equivalent.
Then $\supset$ is *definable* in terms of (*or is reducible to or can be expressed in terms of*) $\sim$ and $\vee$. (can verify by truth tables)
Similarly, $\vee$ is definable in terms of $\sim$ and $\supset$ because $(A \vee B)$ is tautologically equivalent to $(\sim A \supset B)$

**Theorem.** $\{\sim, \bullet, \vee\}$ is an adequate set of connectives (given the standard set of connectives is adequate)

***Proof.***

For any formulas A,B

$$(A \supset B) \quad = \quad (\sim A \vee B)$$

and

$$(A \equiv B) \quad = \quad ((A \supset B) \bullet (B \supset A))$$

**Corollary** $\{\sim, \bullet\}, \{\sim, \vee\} \text{and} \{\sim, \supset\}$ **are adequate, given….**
***Proof.*** **Exercise (need to know for exams)**

**So any adequate set of connectives is sufficient for wff!**

# Proving inadequacy

How do we show that a given set of connectives is not adequate?
Show that some standard connective cannot be expressed by S.

***Example.*** The set $S = \{\bullet\}$ is not adequate.
***Proof.*** To see this, note that a formula depending on only one variable and which uses only the connective $\bullet$ has the property that its truth value for a value assignment that makes P= 0 is always 0.

In order to define the negation $\sim P$ in terms of $\bullet$, there should exist a formula F depending on the variable P and using only the connective $\bullet$ such that

$$\sim P \quad = \quad F$$

However, for a value assignment $\upsilon$ such that $\upsilon(\sim P) = 1$, we have $\upsilon(P) = 0$ and therefore $\upsilon(F) = 0$, which shows that $\sim P$ and F cannot be tautologically equivalent.

# Adequate set of connectives

Schroder showed in 1880 that each of the standard connectives is definable in terms of a single binary connective $\downarrow$, where the truth table associated with $\downarrow$ is

| P | Q | $P \downarrow Q$ |
|---|---|---|
| 1 | 1 | 0 |
| 1 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 1 |

We can express $\downarrow$ in terms of the standard connectives by $(P \downarrow Q) = (\sim P \bullet \sim Q)$, and also the standard connectives in terms of $\downarrow$ by

$$\sim P = (P \downarrow P)$$
$$(P \bullet Q) = (P \downarrow P) \downarrow (Q \downarrow Q)$$
$$(P \vee Q) = (P \downarrow Q) \downarrow (P \downarrow Q)$$
$$(P \supset Q) = ((P \downarrow P) \downarrow Q) \downarrow ((P \downarrow P) \downarrow Q)$$
$$(P \equiv Q) = ((P \downarrow P) \downarrow Q) \downarrow ((Q \downarrow P) \downarrow P)$$

Thus it follows that a single connective $\downarrow$ is adequate.
Consequently, to test a given S for being adequate it suffices to test if $\downarrow$ can be expressed by S.
**So how to express (~ P ·Q) with ↓ ?**

# Adequate set of connectives

In 1913 Sheer showed that the Sheer stroke | with associated truth table

| P | Q | P \| Q |
|---|---|---|
| 1 | 1 | 0 |
| 1 | 0 | 1 |
| 0 | 1 | 1 |
| 0 | 0 | 1 |

is also a single binary connective in terms of which the standard connectives can be expressed. Prove it is adequate

# If-then-else

Let us use the symbol  for the ternary connective whose truth table is given by

| P | Q | R | S |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 |

It is easy to see that for any value assignment $\upsilon$ we have

$$\upsilon(S(P,Q,R)) = \upsilon(Q) \quad \text{if} \quad \upsilon(P) = 1 \quad \text{and} \quad \upsilon(R) \quad \text{if} \quad \upsilon(P) = 0$$

This is the familiar if-then-else connective from computer science, namely

*If P then Q else R*

# Propositional Calculus & Normal Forms

Based on Prof. Lila Kari’s slides

# Propositional calculus/logic

In standard algebra, expressions in which the variables and constants represent numbers are manipulated.
Consider for instance the expression

$$(A + B) - B$$

One sees at a glance that this expressions yields a. In fact, we are so accustomed to performing such algebraic manipulations that we are no longer aware of what is behind each step. Here we used the identities

$$(X + Y) - Z = X + (Y - Z)$$

$$Y - Y = 0$$

$$X + 0 = X$$

# Propositional calculus

In logic we manipulate formulas in which the constants and the variables represent truth values. Consider the following formula

$$((P \bullet Q) \bullet \sim Q)$$

This formula can be simplified in a similar way, except that (tauto)logical equivalences take place of algebraic identities. Specifically, the following equivalences are used: (can verify by the truth tables)

$$((A \bullet B) \bullet C) = (A \bullet (B \bullet C))$$

$$(A \bullet \sim A) = 0$$

$$A \bullet 0 = 0$$

We can now apply these equivalences to conclude

$$((P \bullet Q) \bullet \sim Q) = (P \bullet (Q \bullet \sim Q)) = P \bullet 0 = 0$$

# Removing conditionals and biconditionals

Since the symbolic treatment of conditionals and biconditionals is relatively cumbersome, one usually removes them before performing further formula manipulations. To remove the conditional, one uses the following logical equivalence:

$$(P \supset Q) = (\sim P \vee Q)$$

There are two ways to express the biconditional

$$(P \equiv Q) = ((P \bullet Q) \vee (\sim P \bullet \sim Q))$$

$$(P \equiv Q) = ((P \supset Q) \bullet (Q \supset P))$$

The first version expresses the fact that two formulas are equivalent if they have the same truth values. The second version stresses the fact that two formulas are equivalent if the first implies the second and the second implies the first.

Since we want to remove all $\supset$, we rewrite the last equivalence as

$$(P \equiv Q) = ((\sim P \vee Q) \bullet (P \vee \sim Q))$$

# Removing conditionals and biconditionals

***Example.*** Remove $\supset$ and $\equiv$ from the following formula:

$$(P \supset (Q \bullet R)) \vee ((R \equiv S) \bullet (Q \vee S))$$

***Solution***

$$(\sim P \vee (Q \bullet R)) \vee (((\sim R \vee S) \bullet (R \vee \sim S)) \bullet (Q \vee S))$$

# Essential laws for propositional calculus

| | | | |
|---|---|---|---|
| Excluded middle law | $(P \vee \sim P) = 1$ | Commutative laws | $(P \vee Q) = (Q \vee P)$ |
| Contradiction law | $(P \bullet \sim P) = 0$ | | $(P \bullet Q) = (Q \bullet P)$ |
| Identity laws | $(P \vee 0) = P$ | Associative laws | $((P \vee Q) \vee R) = (P \vee (Q \vee R))$ |
| | $(P \bullet 1) = P$ | | $((P \bullet Q) \bullet R) = (P \bullet (Q \bullet R))$ |
| Domination laws | $(P \vee 1) = 1$ | Distributive laws | $((P \vee Q) \bullet (P \vee R)) = (P \vee (Q \bullet R))$ |
| | $(P \bullet 0) = 0$ | | $((P \bullet Q) \vee (P \bullet R)) = (P \bullet (Q \vee R))$ |
| Idempotent laws | $(P \vee P) = P$ | De Morgan's laws | $\sim (P \bullet Q) = (\sim P \vee \sim Q)$ |
| | $(P \bullet P) = P$ | | $\sim (P \vee Q) = (\sim P \bullet \sim Q)$ |
| Double-negation law | $\sim (\sim P) = P$ | | |

***Treating ~ as –, v as +, • as * in arithmetic for many cases***

***Distributive laws and De Morgan's laws are important for the next part***

# Essential laws for propositional calculus

All laws can be proved by the truth table method.

With the exception of the double-negation law all laws come in pairs, called dual pairs. For each formula depending only on the connectives $\sim, \bullet, \vee$ , the dual is found by replacing all 1 by 0 and all 0 by 1 and by replacing all $\bullet$ by $\vee$ and all $\vee$ by $\bullet$ .

The laws allow one to simplify a formula and it is normally a good idea to apply them whenever it is possible. For instance, the formula $(\sim\sim P \bullet (Q \vee \sim Q))$ is logically equivalent to p.

The commutative, associative and distributive laws have their equivalents in standard algebra. In fact, the connective $\vee$ is often treated like +, and the connective $\bullet$ is often treated like $\times$ . (The analogy sometimes breaks down.)

# Essential laws for propositional calculus

From these laws one can derive further laws, for example, the absorption laws

$$(P \vee (P \bullet Q)) = P$$

$$(P \bullet (P \vee Q)) = P$$

(Hint: use identity law, distributive law, domination law and identity law again.)
Another important law:

$$((P \bullet Q) \vee (\sim P \bullet Q)) = Q$$

$$((P \vee Q) \bullet (\sim P \vee Q)) = Q$$

# Shortcuts for manipulating formulas

***Definition.*** A formula is called a **literal** if it is of the form p or ~p, where p is a propositional variable. The two formulas p and ~p are called complementary literals.

The following rules are available to simplify conjunctions containing only literals. Before applying these rules, it is best to sort the literals lexicographically according to variable names.

If a conjunction contains complementary literals or if it contains the propositional constant 0, it always yields 0; that is, it is a contradiction.

All instances of the propositional constant 1 and all duplicate copies of any literal, may be dropped.

If a disjunction contains complementary literals or if it contains the propositional constant 1, it always yields 1; that is, it is a tautology.

All instances of the propositional constant 0 and all duplicate copies of any literal may be dropped.

# Shortcuts for manipulating formulas

***Example.*** Simplify ***(OK to be less strict in wff notation)***

$$((P_3 \bullet \sim P_2 \bullet P_3 \bullet \sim P_1) \vee (P_1 \bullet P_3 \bullet \sim P_1))$$

***Solution***

$$(\sim P_1 \bullet \sim P_2 \bullet P_3)$$

# Normal forms

Formulas can be transformed into standard forms so that they become more convenient for symbolic manipulations and make identification and comparison of two formulas easier. There are two types of normal forms in propositional calculus: the disjunctive and the conjunctive normal form.

***Definition.*** A formula is said to be in **disjunctive normal form** (**DNF**) if it is written as a disjunction, in which all the terms are conjunctions of literals.

***Example:*** $(P \bullet Q) \vee (P \bullet \sim Q), P \vee (Q \bullet R), \sim P \vee T$ are in disjunctive normal forms. The disjunction : $(\sim (P \bullet Q) \vee R)$ is not in normal form.

In general a formula in disjunctive normal form is

$$((A_{11} \bullet \ldots \bullet A_{1n_1}) \vee \ldots \vee (A_{k_1} \bullet \ldots \bullet A_{kn_k}))$$

**Need to be able to convert any WFF to DNF and CNF in exams.**

# Normal forms

***Definition.*** Disjunctions (conjunctions) with literals as disjuncts (conjuncts) are called disjunctive (conjunctive) clauses. Disjunctive and conjunctive clauses are simply called clauses.

***Definition.*** A conjunction with disjunctive clauses as its conjuncts is said to be in **conjunctive normal form (CNF)**.

***Example:*** $(P \bullet (Q \vee R))$ and $P \bullet 0$ are in conjunctive normal form.
However $(P \bullet (R \vee (P \bullet Q)))$ is not in conjunctive normal form.

In general, a formula in conjunctive normal form is

$$((A_{11} \vee \ldots \vee A_{1n_1}) \bullet \ldots \bullet (A_{k_1} \vee \ldots \vee A_{kn_k}))$$

# Normal forms

***Examples:***
Observe the following formulas:

$$(1) P$$
$$(2) (\sim P \vee Q)$$
$$(3) (\sim P \bullet (Q \bullet \sim R))$$
$$(4) (\sim P \vee (Q \bullet \sim R))$$
$$(5) (\sim P \bullet (Q \vee \sim R) \bullet (\sim Q \vee R))$$

(1) is an atom, and therefore a literal. It is a disjunction with only one disjunct. It is also a conjunction with only one conjunct. Hence it is a disjunctive or conjunctive clause with one literal. It is a formula in disjunctive normal form
with one conjunctive clause P. It is also a formula in conjunctive normal form with one disjunctive clause P.

(2) is a disjunction with two disjuncts, and a disjunctive normal form with two clauses, each with one literal. It is also a conjunction with one conjunct, and a formula in conjunctive normal form which consists of two literals.

# Normal forms

(3) is a conjunction and a formula in conjunctive normal form. It is also a disjunction and a formula in disjunctive normal form.

(4) is a formula in disjunctive normal form, but not in conjunctive normal form.

(5) is a formula in conjunctive normal form but not in disjunctive normal form.

If $\vee$ is exchanged for $\bullet$ in (4) and (5), then (4) becomes a formula in conjunctive normal form and (5) a formula in disjunctive normal form.

# Existence of normal forms

***Theorem.*** Any formula $A \in \text{Form}(L^p)$ is tautologically equivalent to some formula in disjunctive normal form.

***Theorem.*** Any formula $A \in \text{Form}(L^p)$ is tautologically equivalent to some formula in conjunctive normal form.

# How to obtain normal forms?

Use the following tautological equivalences:

$$(1)(A \supset B) = (\sim A \vee B)$$

$$(2)(A \equiv B) = (\sim A \vee B) \bullet (A \vee \sim B)$$

$$(3)(A \equiv B) = (A \bullet B) \vee (\sim A \bullet \sim B)$$

$$(4) \sim (\sim A) = A$$

$$(5) \sim (A_1 \bullet \ldots \bullet A_n) = (\sim A_1 \vee \ldots \vee \sim A_n)$$

$$(6) \sim (A_1 \vee \ldots \vee A_n) = (\sim A_1 \bullet \ldots \bullet \sim A_n)$$

$$(7)(A \bullet (B_1 \vee \ldots \vee B_n)) = (A \bullet B_1) \vee \ldots \vee (A \bullet B_n)$$

$$((B_1 \vee \ldots \vee B_n) \bullet A) = (B_1 \bullet A) \vee \ldots \vee (B_n \bullet A)$$

$$(8)(A \vee (B_1 \bullet \ldots \bullet B_n)) = (A \vee B_1) \bullet \ldots \bullet (A \vee B_n)$$

$$((B_1 \bullet \ldots \bullet B_n) \vee A) = (B_1 \vee A) \bullet \ldots \bullet (B_n \vee A)$$

# How to obtain normal forms?

By the replaceability of tautological equivalences, we can replace the preceding formulas on the left with the corresponding ones on the right to yield a formula tautologically equivalent to the original one.

By (1)-(3) we eliminate $\supset$ and $\equiv$.

By (4){(6) we eliminate $\sim, \vee, \bullet$ from the scope of $\sim$ such that any $\sim$ has only an atom as its scope.

By (7) we eliminate $\vee$ from the scope of $\bullet$.

By (8) we eliminate $\bullet$ from the scope of $\vee$.

This method leads to obtaining the disjunctive or conjunctive normal forms.

# How to obtain normal forms?

***Example.*** Convert the following formula into a conjunctive normal form.

$$\sim((P \vee \sim Q) \bullet \sim R)$$

The conjunctive normal form can be found by the following derivations:

$$\sim((P \vee \sim Q) \bullet \sim R) =$$

$$= (\sim(P \vee \sim Q) \vee \sim\sim R) \quad \text{De} \quad \text{Morgan}$$

$$= (\sim(P \vee \sim Q) \vee R) \quad \text{Double} \quad \text{Negation}$$

$$= (\sim P \bullet \sim\sim Q) \vee R) \quad \text{De} \quad \text{Morgan}$$

$$= (\sim P \bullet Q) \vee R) \quad \text{Double} \quad \text{Negation}$$

$$= ((\sim P \vee R) \bullet (Q \vee R)) \quad \text{Distributivity}$$

# Example

Convert the following formula into conjunctive normal form.

$$((P_1 \bullet P_2) \vee (P_3 \bullet (P_4 \vee P_5)))$$

***Solution:*** $((P_1 \vee P_3) \bullet (P_1 \vee P_4 \vee P_5) \bullet (P_2 \vee P_3) \bullet (P_2 \vee P_4 \vee P_5))$

Once a conjunctive normal form is obtained, it pays to check if further simplifications are possible.

***Example.*** Simplify the following conjunctive normal form:

$$((P \vee Q) \bullet P \bullet (Q \vee R) \bullet (P \vee \sim P \vee R) \bullet (\sim Q \vee R))$$

***Solution:*** $(P \bullet R)$

# Disjunctive normal forms from truth tables

So far we have shown how to find the truth table of a logical formula. The reverse is also possible. One can convert any given truth table into a formula. The formula obtained in this way is already in disjunctive normal form.

In fact, the conceptually easiest method to find the normal form of a formula is by using truth tables. Unfortunately, truth tables grow exponentially with the number of variables, which makes this method unattractive for formulas with many variables.

# Truth (Boolean) functions

Generally, a truth table gives truth values of some formula for all possible assignments. The table below gives an example of truth table for a certain formula F. The truth values of F depends on the three variables P, Q, R.

| P | Q | R | F |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 |

How to make a disjunctive normal form of F?

For each row with F = 1, write the conjunction of literals that makes it 1
Make a disjunction of all the terms above to form a disjunctive normal form of F.

How to make a conjunctive normal form of F?

Do the DNF for ~F as above
Apply ~ to the resulting DNF, to obtain CNF

**Exercise…**

# Boolean functions

To convert a Boolean function into a formula, one makes use of *minterms*.

***Definition.*** A *minterm* is a conjunction of literals in which each variable is represented exactly once.

***Example:*** If a Boolean function has the variables P, Q, R then $(P\bullet \sim Q \bullet R)$ is a minterm but $(P\bullet \sim Q)$ and $(P \bullet P\bullet \sim Q)$ are not.

Each minterm is true for exactly one assignment. For example, $(P\bullet \sim Q \bullet R)$ is true if p is 1, Q is 0 and r is 1. Any deviation from this assignment would make this particular minterm false.

A disjunction of minterms is true only if at least one of its constituents minterms is true. For example,

$$((P \bullet Q \bullet R) \vee (P\bullet \sim Q \bullet R) \vee (\sim P\bullet \sim Q \bullet R))$$

is only true if at least one of $(P \bullet Q \bullet R), (P\bullet \sim Q \bullet R)$ or $(\sim P\bullet \sim Q \bullet R)$ true

# Boolean functions

If a function, such as F, is given by truth table, we know exactly for which assignments it is true.
Consequently, we can select the minterms that make the function true and form the disjunction of these minterms.

The function f, for instance, is true for three assignments:
1. $P, Q, R$ are all true.
2. $P, \sim Q, R$ are all true.
3. $\sim P, \sim Q, R$ are all true.

The disjunction of the corresponding minterms is tautologically equivalent to F, which means that we have the following formula for F:

$$F = ((P \bullet Q \bullet R) \vee (P \bullet \sim Q \bullet R) \vee (\sim P \bullet \sim Q \bullet R))$$

Since the minterms are conjunctions, we have expressed the function in question in disjunctive normal form. Actually, we have a special type of normal form, the *full disjunctive* normal form.

***Definition.*** If a Boolean function is expressed as a disjunction of minterms, it is said to be in full disjunctive normal form.

# Conjunctive normal form and complementation

*Complementation* can be used to obtain conjunctive normal forms from truth tables.
If A is a formula containing only the connectives $\sim, \vee$ and $\bullet$ then its complement is formed by replacing all $\vee$ by $\bullet$, all $\bullet$ by $\vee$ and all atoms by their complements.

***Example:*** Find the complement of the formula

$$A = ((P \bullet Q) \vee \sim R)$$

Complementation can be used to find the conjunctive normal form from the truth table of some truth function (Boolean function) F.

One first determines the disjunctive normal form for $\sim F$. If the resulting disjunctive normal form is A, then $A = \sim F$, and the complement of A must be logically equivalent to F.

# Example

Find the full conjunctive normal form for f1 given by the table

| P | Q | R | $F_1$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 1 |

***Solution:*** $\sim F_1$ is true for the following assignments:

$$P = 1, Q = 0, R = 1$$

$$P = 1, Q = 0, R = 0$$

$$P = 0, Q = 0, R = 1$$

The disjunctive normal form of :f1 is therefore

$$((P \bullet \sim Q \bullet R) \vee (P \bullet \sim Q \bullet \sim R) \vee (\sim P \bullet \sim Q \bullet R))$$

This formula has the complement

$$F_1 = ((\sim P \vee Q \vee \sim R) \bullet (\sim P \vee Q \vee R) \bullet (P \vee Q \vee \sim R))$$

which is the desired conjunctive normal form.