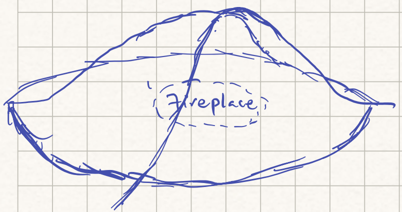
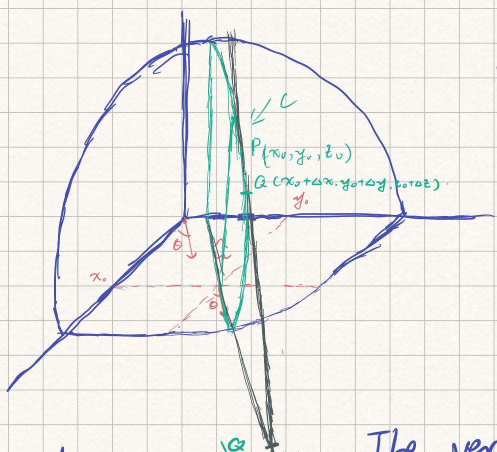
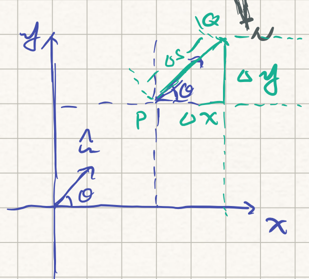
Directional derivatives and the gradient



Direction of the derivatives: the rate of change of temperature with distance depends on the direction moved



Supposed that we want to find the rate of change of $f(x, y)$ at (x_0, y_0) in the direction of a unit vector $\hat{u} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$



$\Delta x = \Delta s \cos\theta$

$\Delta y = \Delta s \sin\theta$

The vertical plane passing through $P(x_0, y_0, z_0)$ where $z_0 = f(x_0, y_0)$ in the $\hat{u}$ direction cuts the surface $z = f(x, y)$ along the curve c
the slope of tangent line to c at N is the rate of change of $z = f(x, y)$ in the $\hat{u}$ direction.

$\overrightarrow{PQ} \parallel \hat{u}$ and if we let $PQ = \Delta s$

$$\begin{aligned}\overrightarrow{PQ} &= \Delta s \cdot \hat{u} \\ &= \Delta s \cdot \cos\theta\,\hat{i} + \Delta s \sin\theta\,\hat{j} \\ &= \Delta x + \Delta y.\end{aligned}$$

Then the direction of the derivative of $z = f(x, y)$ in the direction of $\hat{u}$ is

$$D_{\hat{u}} f(x_0, y_0) = \lim_{\Delta s \to 0} \frac{\Delta f}{\Delta s}$$

$$= \lim_{\Delta s \to 0} \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)}{\Delta s}$$

$$= \lim_{\Delta s \to 0} \frac{f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 x + \varepsilon_2 y}{\Delta s}.$$

$$= \lim_{\Delta s \to 0} \frac{f_x(x_0, y_0)\Delta s \cos\theta + f_y(x_0, y_0)\Delta s \sin\theta + \underset{\downarrow 0}{\varepsilon_1} \Delta s \cos\theta + \underset{\downarrow 0}{\varepsilon_2} \Delta s \sin\theta}{\Delta s}$$

$$= f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta$$

The above expression can be written as a dot product:

$$D_{\hat{u}} f(x_0, y_0) = \Big[\underbrace{f_x(x_0, y_0)\hat{i} + f_y(x_0, y_0)\hat{j}}_{\nabla f(x_0, y_0)}\Big] \cdot \Big[\underbrace{\cos\hat{i} + \sin\hat{j}}_{\hat{u}}\Big].$$

$$D_{\hat{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \hat{u}$$

where $\nabla f(x_0, y_0) = f_x(x_0, y_0)\hat{i} + f_y(x_0, y_0)\hat{j}$

is called the gradient vector of f at the point (x, y).

i) if $\hat{u} = \hat{i}$, $\cos\theta = 1$, $\sin\theta = 0$

$D_{\hat{u}} f(x_0, y_0) = f_x(x_0, y_0)$.

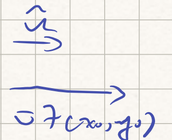
ii) if $\hat{u} = \hat{j}$, $\cos\theta = 0$, $\sin\theta = 1$

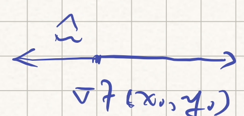
$D_{\hat{u}} f(x_0, y_0) = f_y(x_0, y_0)$.

$$D_{\hat{u}} f(x_0, y_0) = \|\nabla f(x_0, y_0)\| \cdot \overset{1}{\|\hat{u}\|} \cdot \cos\alpha$$

$$= \|\nabla f(x_0, y_0)\| \cos\alpha$$

$\Rightarrow$ max when $\alpha = 0$, min when $\alpha = \pi$





e.g.1 $f(x, y) = \frac{x}{y}$. Find i) gradient of f at $P(2, 1)$

2) the rate of change of f at P in the direction of $\vec{u} = 3\hat{i} + 4\hat{j}$

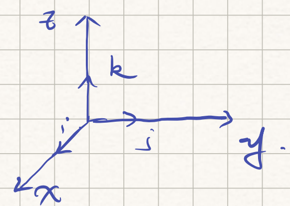
$$\nabla f(x_0, y_0) = f_x \hat{i} + f_y \hat{j}$$
$$= \frac{1}{y}\hat{i} + \left(-\frac{x}{y^2}\right)\hat{j}$$
$$= \hat{i} - 2\hat{j}$$

$$D_{\hat{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \frac{\vec{u}}{\|\vec{u}\|} \Leftarrow \hat{u}.$$

$$= (\hat{i} - 2\hat{j}) \cdot \left(\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}\right)$$
$$= (1, -2) \cdot \left(\frac{3}{5}, \frac{4}{5}\right)$$
$$= \frac{3}{5} - \frac{8}{5} = -1$$

For three vectors, $D_{\hat{u}} f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \hat{u}$.

where $\nabla f(x_0, y_0, z_0) = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$.



e.g. 2 Find the directional derivative of $f(x, y, z) = \sqrt{xyz}$ at $(3, 2, 6)$ in the direction of vector $\vec{v} = (-1, -2, 2)$

$$\nabla f = f_x i + f_y j + f_z k \qquad yz.$$
$$= \hat{i} + \frac{3}{2}\hat{j} + \frac{1}{2}\hat{k}$$

$$D_u f(x_0, y_0) = \left(1, \frac{3}{2}, \frac{1}{2}\right) \cdot (-1, -2, 2).$$
$$= -3$$

Tangent plane to the level surface

Given a function of three variables $w = F(x, y, z)$.

Then $F(x, y, z) = 0$ is a level surface. (S) of w

$P_0 = (x_0, y_0, z_0)$ be a point on the surface S.

C be any curve on s and it pass through P_0.

Then the paramatric equation of C:

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t). \end{cases}$$

$$F(x(t), y(t), z(t)) = 0$$

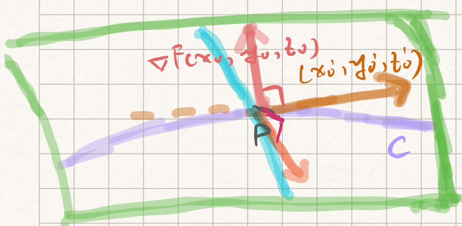
$$F_x (x)' + F_y (y)' + F_z (z)' = 0$$

$$\Downarrow$$

$$(F_x, F_y, F_z) \cdot (x', y', z') = 0.$$

$t = t_0$, $\quad (F_{x_0}, F_{y_0}, F_{z_0})(x_0', y_0', z_0') = 0$

$\nabla F_{(x_0, y_0, z_0)} \cdot$ tangent vector of C at $P_0 = 0$



For any C, $\nabla F(x_0, y_0, z_0) \perp$ tangent $\frac{d\vec{r}}{dt}\Big|_{t_0}$ at P_0

$\Downarrow$

the gradient vector is parpendicular to the tangent surface at P_0.

$\therefore$ the equation of the tangent plane to the level surface is $F(x, y, z) = C$ at $P(x_0, y_0, C_0)$ is:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

e.g. 3. Find equations of the tangent plane and the normal line to the surface (S) $y = x^2 - z^2$ at the point $(4, 7, 3)$.

$-x^2 + y - z^2 = 0.$

$-2x + 1 - 2z = 0.$

$-8\hat{i} + \hat{j} - 6\hat{k} = 0.$

$-8(x - 4) + (y - 7) - 6(z - 3) = 0$

$8x - y + 6z + 7 = 0$

normal line: $\dfrac{x-4}{8} = \dfrac{y-7}{-1} = \dfrac{z-3}{-6}$

paramatic form: $\begin{cases} x = 4 + 8t \\ y = 7 - t \end{cases}$

$z = 3-6t$

e.g. 5. The temperature at a point (x, y, z) is given by

$$T(x,y,z) = 200\, e^{-x^2-3y^2-9z^2}$$

a) Find the rate of change of temperature at the point $(2,-1,2)$ in the direction toward $(5,-3,3)$

b) In which direction does the temperature grows fastest?

c) Find the maximum rate of increase.

$$\nabla T(x,y,z) = T_x \hat{i} + T_y \hat{j} + T_z \hat{k}$$

$$= -800 e^{-43} \hat{i} + 1200 e^{-43} \hat{j} - 7200 e^{-43} \hat{k}.$$

$$D_{\hat{u}} T(x,y,z) = \langle -800 e^{-43}, 1200 e^{-43}, -7200 e^{-43} \rangle \cdot \frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|}.$$

$$= -\frac{5200\sqrt{6}}{3} e^{-43}.$$