ECON3102-005 Chapter 6: Economic Growth: the Solow Growth Model (Part 2)

Neha Bairoliya

Spring 2014

$$k' = \frac{szf(k)}{1+n} + \frac{(1-d)k}{1+n} \qquad (*)$$

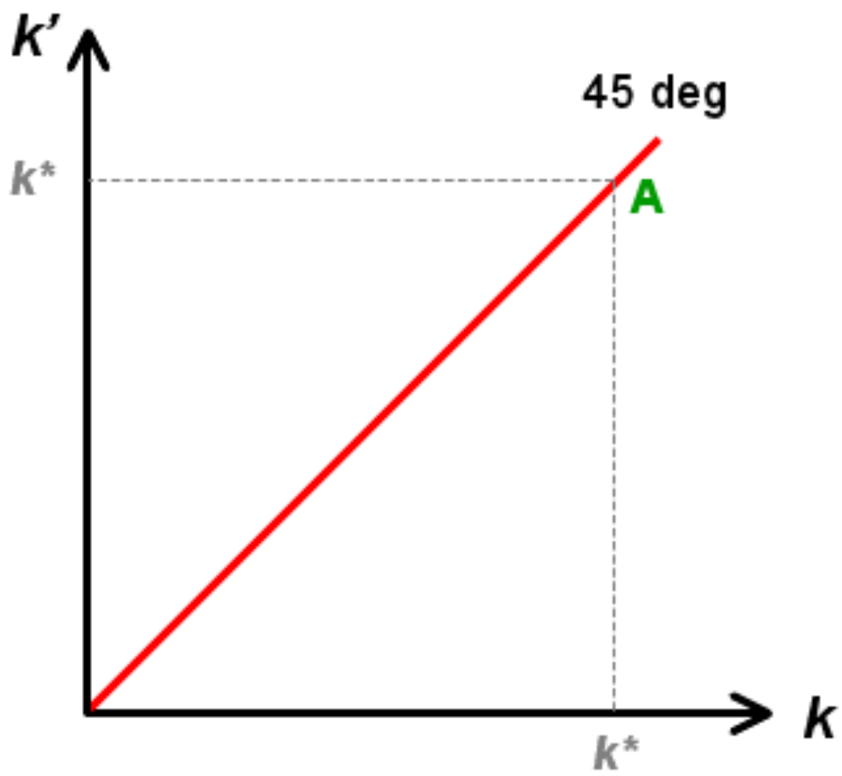
Steady States

We want to find the steady state of the model. This is, the point at which $k' = k = k^*$.

Steady States

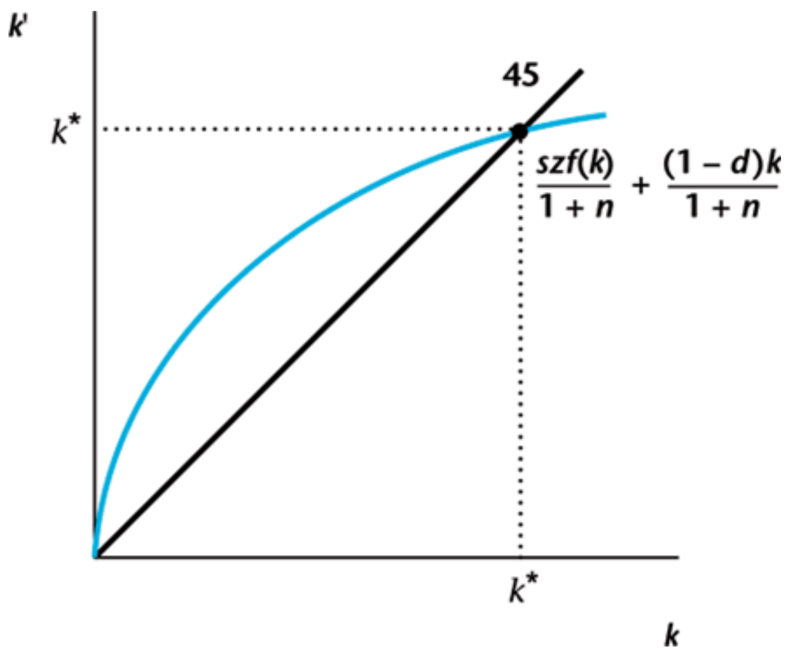
We want to find the steady state of the model. This is, the point at which $k' = k = k^*$.

- Note that when we graph in $k'k$ space, any point that crosses the 45 degree line satisfies $k' = k$.



Recall equation (*):

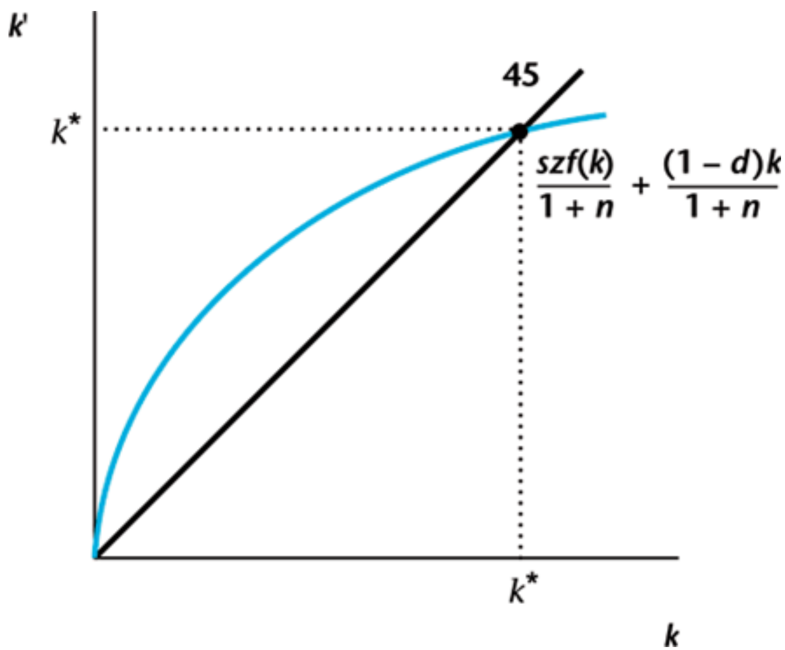
$$k' = \frac{szf(k)}{1+n} + \frac{(1-d)k}{1+n} \qquad (*)$$



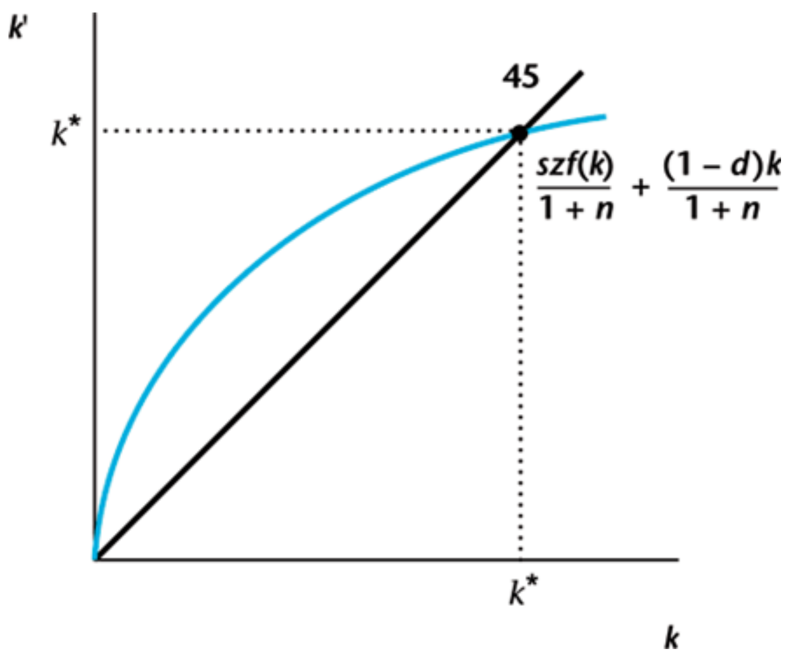
Steady State in the Solow Growth Model

Recall equation (*):

$$k' = \frac{szf(k)}{1+n} + \frac{(1-d)k}{1+n} \qquad (*)$$

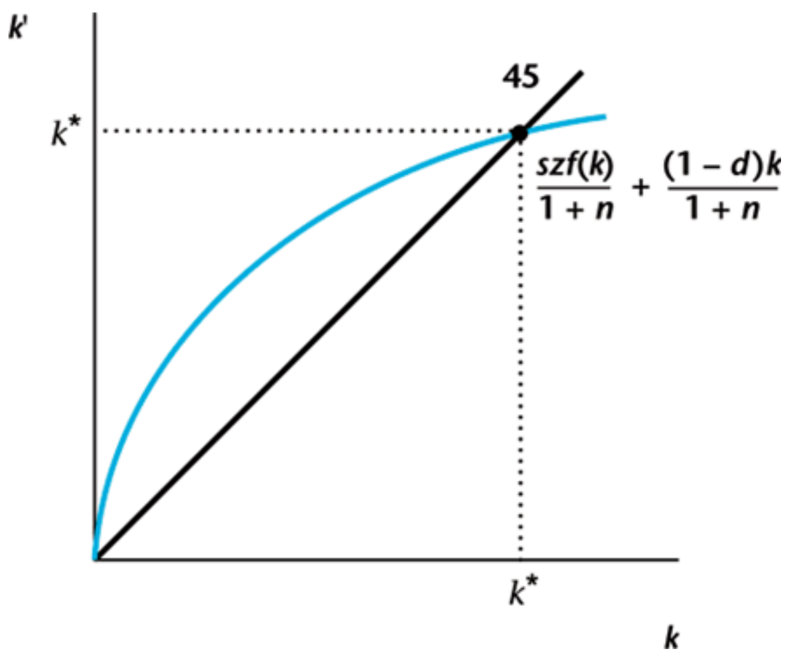


- At the steady state, $k = k^*$ and $k' = k^*$; k^* is the equilibrium level of capital in the economy.



- Suppose $k < k^*$. Then $k' > k$, and the capital stock increases from the current to the future period, until $k = k^*$.

Steady State in the Solow Growth Model



- Suppose $k < k^*$. Then $k' > k$, and the capital stock increases from the current to the future period, until $k = k^*$.
- Here, current investment is relatively large with respect to depreciation and labor force growth.

Steady State Growth Rates

- What is the growth rate of k^*?

Steady State Growth Rates

- What is the growth rate of k^*?
- The answer: zero.

Steady State Growth Rates

- What is the growth rate of k^*?
- The answer: zero.
- Why? Since its a steady state, it wont move from there.

Steady State Growth Rates

- What is the growth rate of k^*?
- The answer: zero.
- Why? Since its a steady state, it wont move from there.
- Another question: What is the growth rate of y^*?

Steady State Growth Rates

- What is the growth rate of k^*?
- The answer: zero.
- Why? Since its a steady state, it wont move from there.
- Another question: What is the growth rate of y^*?
- The answer: zero.

Steady State Growth Rates

- What is the growth rate of k^*?
- The answer: zero.
- Why? Since its a steady state, it wont move from there.
- Another question: What is the growth rate of y^*?
- The answer: zero.
- Why? Since $k = k^*$ in the long run, output per worker is constant at $y^* = zf(k^*)$.

Steady State Growth Rates

- What is the growth rate of k^*?
- The answer: zero.
- Why? Since its a steady state, it wont move from there.
- Another question: What is the growth rate of y^*?
- The answer: zero.
- Why? Since $k = k^*$ in the long run, output per worker is constant at $y^* = zf(k^*)$.
- So, theres no growth in here? Are we forgetting something?

Steady State Growth Rates

There is growth in this economy! In the long run, when $k = k^*$, all real aggregate quantities grow at a rate n. Why?

- The aggregate quantity of capital is $K = k^* N$. Since k^* is constant and N grows at a rate n, K should grow at a rate n.

Steady State Growth Rates

There is growth in this economy! In the long run, when $k = k^*$, all real aggregate quantities grow at a rate n. Why?

- The aggregate quantity of capital is $K = k^* N$. Since k^* is constant and N grows at a rate n, K should grow at a rate n.

- Aggregate real output is $Y = y * N = zf(k^*)N$, hence Y also grows at a rate n.

Steady State Growth Rates

There is growth in this economy! In the long run, when $k = k^*$, all real aggregate quantities grow at a rate n. Why?

- The aggregate quantity of capital is $K = k^* N$. Since k^* is constant and N grows at a rate n, K should grow at a rate n.
- Aggregate real output is $Y = y * N = zf(k^*)N$, hence Y also grows at a rate n.
- Consumption and investment follow the same logic:

$$I = sY = szf(k^*)N,$$

$$C = (1-s)Y = (1-s)zf(k^*)N.$$

Steady State Growth Rates

There is growth in this economy! In the long run, when $k = k^*$, all real aggregate quantities grow at a rate n. Why?

- The aggregate quantity of capital is $K = k^* N$. Since k^* is constant and N grows at a rate n, K should grow at a rate n.

- Aggregate real output is $Y = y * N = zf(k^*)N$, hence Y also grows at a rate n.

- Consumption and investment follow the same logic:

$$I = sY = szf(k^*)N,$$

$$C = (1-s)Y = (1-s)zf(k^*)N.$$

- In this way, the Solow growth model is an exogenous growth model.

Steady State Analysis (1)

- Take the equilibrium time path of capital

$$k' = \frac{szf(k)}{1+n} + \frac{(1-d)k}{1+n}$$

Steady State Analysis (1)

- Take the equilibrium time path of capital

$$k' = \frac{szf(k)}{1+n} + \frac{(1-d)k}{1+n}$$

- and set $k' = k = k^*$:

$$\begin{aligned} k^* &= \frac{szf(k)}{1+n} + \frac{(1-d)k^*}{1+n} \\ (1+n)k^* &= szf(k) + (1-d)k^* \\ (1+n)k^* - (1-d)k^* &= szf(k) \\ (n+d)k^* &= szf(k^*) \qquad (1) \end{aligned}$$

Steady State Analysis (1)

- Take the equilibrium time path of capital

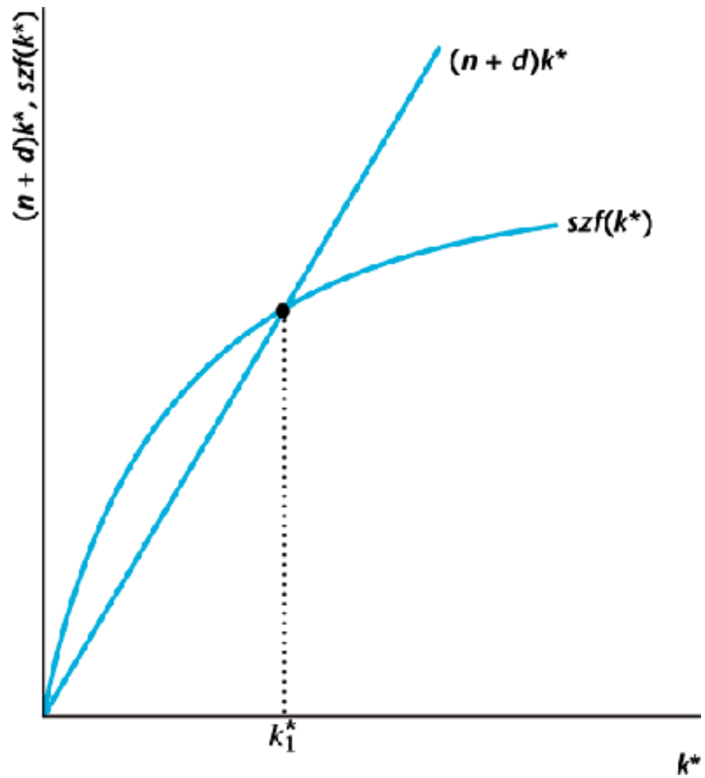
$$k' = \frac{szf(k)}{1+n} + \frac{(1-d)k}{1+n}$$

- and set $k' = k = k^*$:

$$\begin{aligned} k^* &= \frac{szf(k)}{1+n} + \frac{(1-d)k^*}{1+n} \\ (1+n)k^* &= szf(k) + (1-d)k^* \\ (1+n)k^* - (1-d)k^* &= szf(k) \\ (n+d)k^* &= szf(k^*) \qquad (1) \end{aligned}$$

- In equation [1], the right hand side is the per worker production function multiplied by the savings rate; the left hand side is an equation for a line with slope (n+d).

- Graphically, to determine $k*$ we match the left and right hand sides of equation [1]:



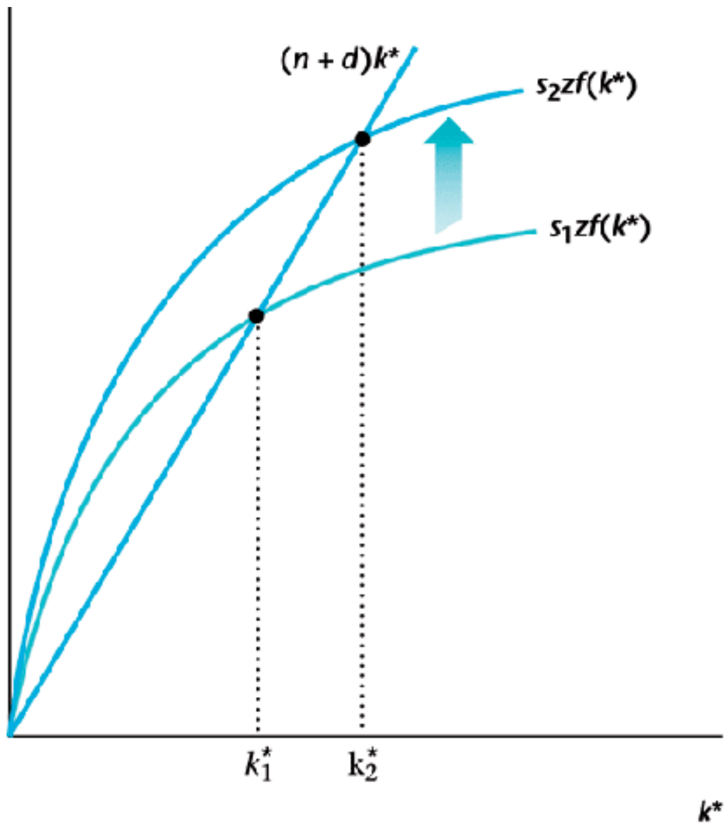
An Increase in the Savings Rate, s (1)

- We know what happens in the steady state. But now, let's see what happens when we change the savings rate, s.

An Increase in the Savings Rate, s (1)

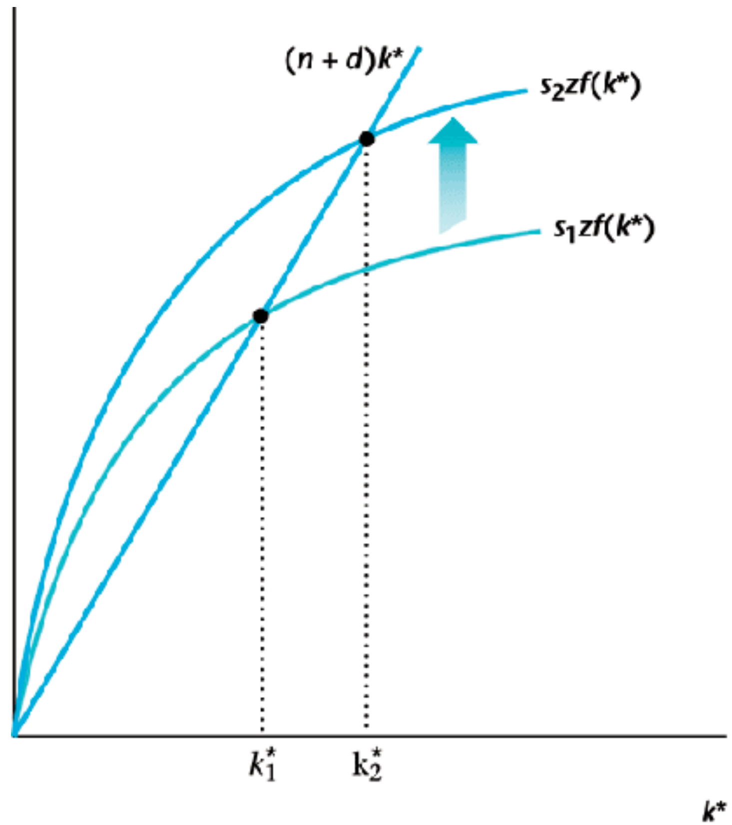
- We know what happens in the steady state. But now, let's see what happens when we change the savings rate, s.

- Suppose that at some time t_0 the savings rate increases from s_1 to s_2. (This could be due to a change in preferences.)



An Increase in the Savings Rate, s (1)

- We know what happens in the steady state. But now, let's see what happens when we change the savings rate, s.
- Suppose that at some time t_0 the savings rate increases from s_1 to s_2. (This could be due to a change in preferences.)



- The steady state capital level increases.

An Increase in the Savings Rate, s (2)

As $k_2^* > k_1^*$, it follows that

- Output per worker is higher: $y_2^* = xf(k_2^*) > y_1^*$

An Increase in the Savings Rate, s (2)

As $k_2^* > k_1^*$, it follows that

- Output per worker is higher: $y_2^* = xf(k_2^*) > y_1^*$
- Aggregate capital is higher: $K_2^* = k_2^* N > K_1^*$

An Increase in the Savings Rate, s (2)

As $k_2^* > k_1^*$, it follows that

- Output per worker is higher: $y_2^* = xf(k_2^*) > y_1^*$
- Aggregate capital is higher: $K_2^* = k_2^* N > K_1^*$
- Aggregate output is higher: $Y_2^* = y_2^* N > Y_1^*$

An Increase in the Savings Rate, s (2)

As $k_2^* > k_1^*$, it follows that

- Output per worker is higher: $y_2^* = xf(k_2^*) > y_1^*$
- Aggregate capital is higher: $K_2^* = k_2^* N > K_1^*$
- Aggregate output is higher: $Y_2^* = y_2^* N > Y_1^*$
- Note that, since in equilibrium $S = I$, this implies a positive relationship between I and Y (the first empirical evidence from data).

An Increase in the Savings Rate, s (3)

After this shock, what is the growth rate of K, Y?

An Increase in the Savings Rate, s (3)

After this shock, what is the growth rate of K, Y?

- K and Y still grow at a rate n.

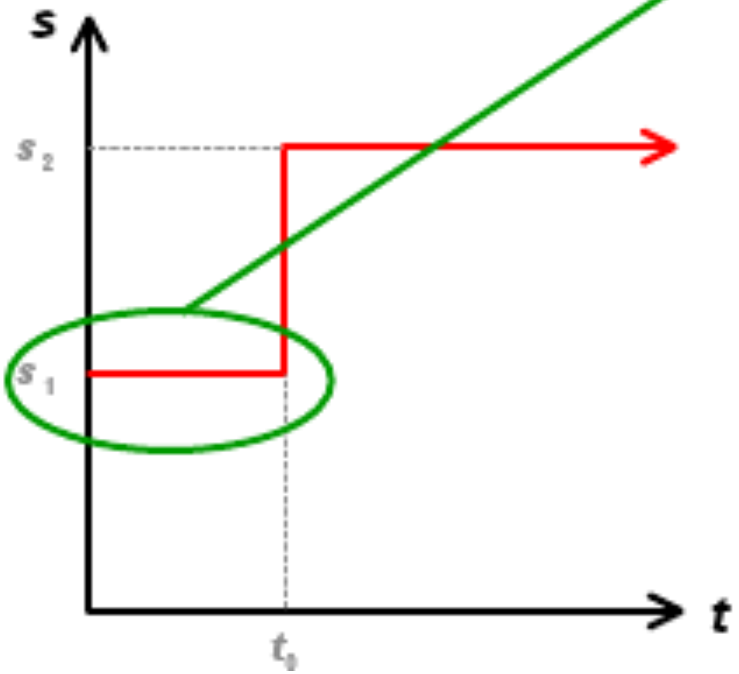
An Increase in the Savings Rate, s (3)

After this shock, what is the growth rate of K, Y?

- K and Y still grow at a rate n.
- However, it may take some time for the variables to adjust to the new steady state.

Transition between the Steady States: s

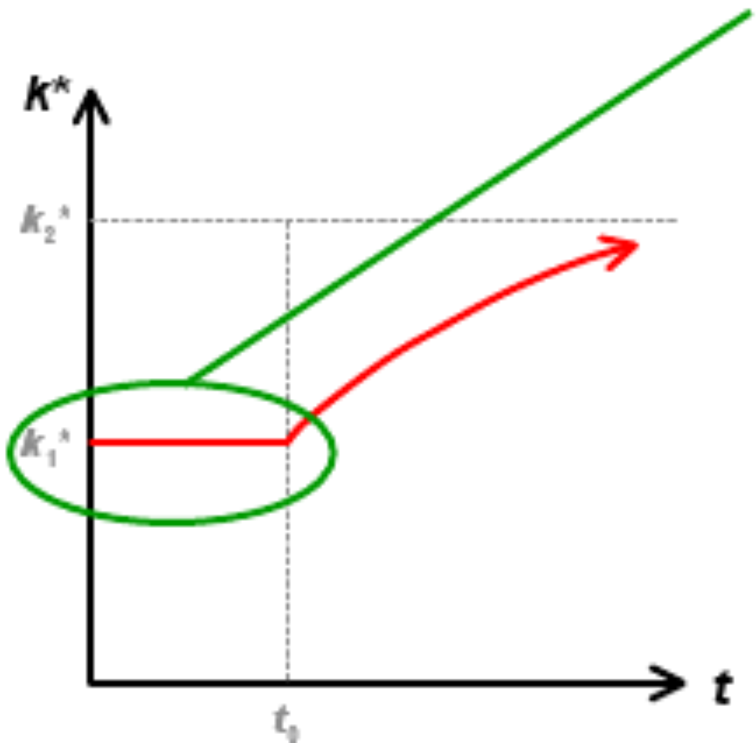
In time: The savings rate



The savings rate starts at a level s_1 ...

... then jumps to the new level of savings s_2.

In time: Capital per worker

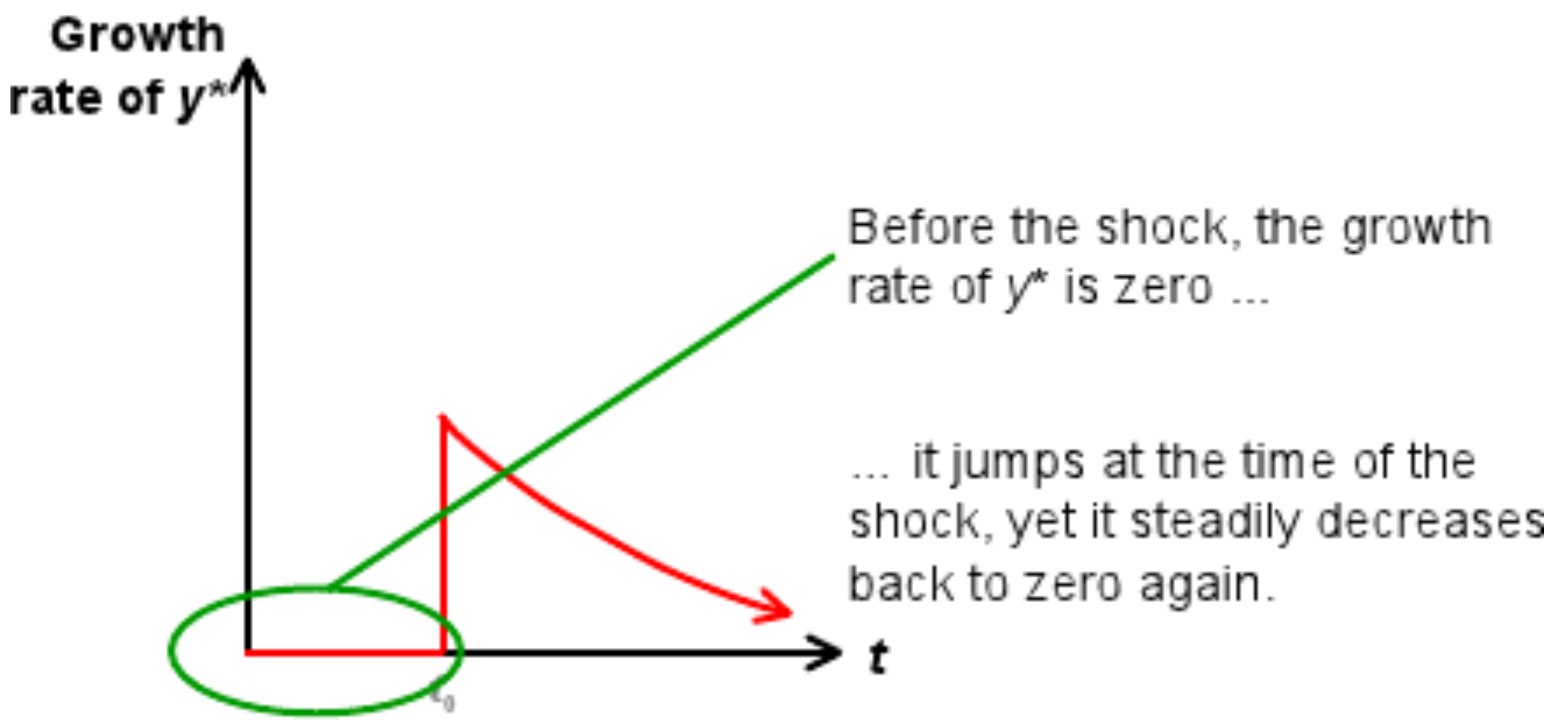


The capital per worker starts at a level k_1* ...

... then steadily increases towards its new level k_2*.

Transition between the Steady States: Growth Rate of y

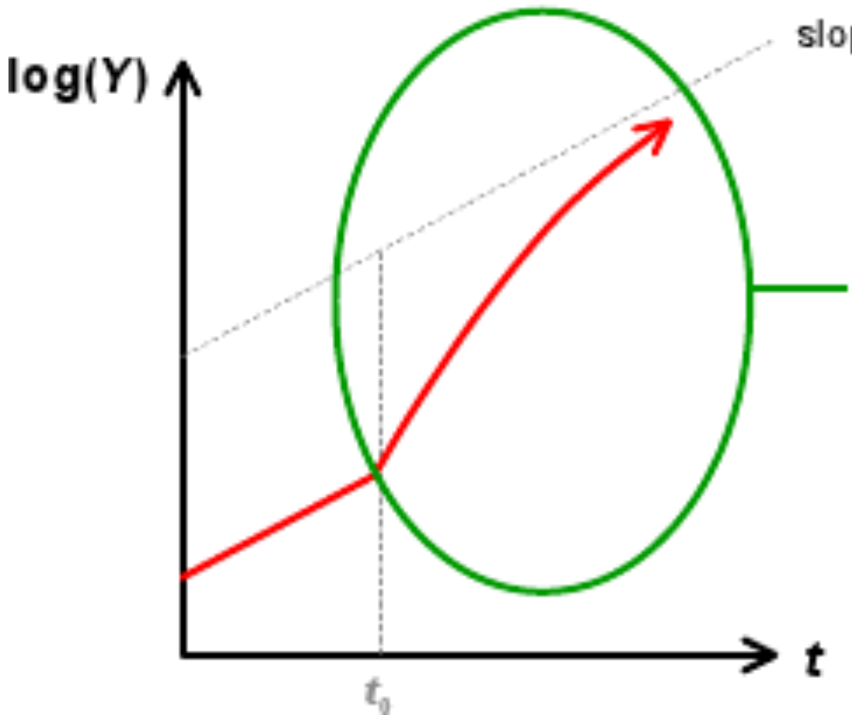
In time: Growth rate of y^*



Transition between the Steady States: $\log(Y)$

In time: $\log(Y)$

The growth rate of aggregate output is n ...



... after the shock, the growth rate of aggregate output increases from trend but then steadily reduces back to n.

Transition between the Steady States: c (1)

- What about consumption?

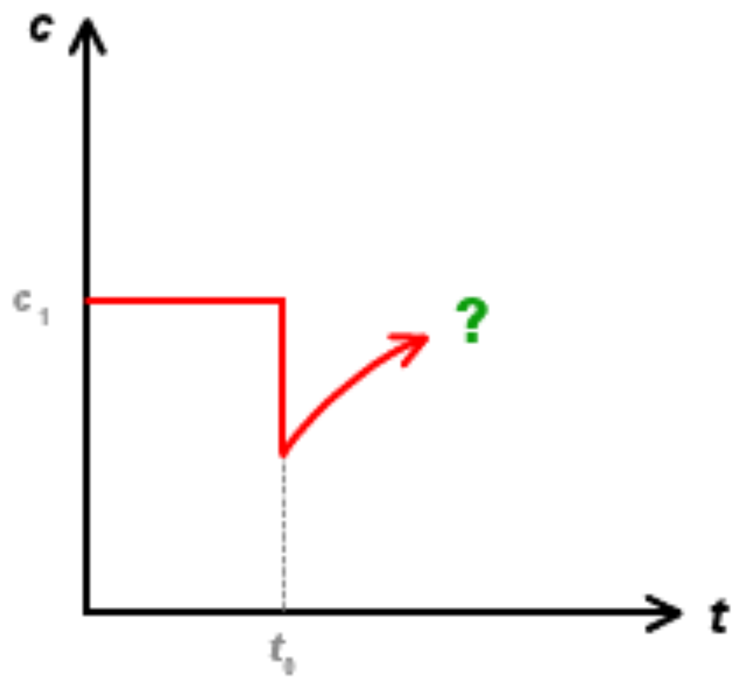
Transition between the Steady States: c (1)

- What about consumption?

- We know consumption per capita equals $(1-s)zf(k^*)$. Since s changes discontinuously at t_0 and k^* adjusts gradually, c falls initially but increases over time.

Transition between the Steady States: c (1)

- What about consumption?
- We know consumption per capita equals $(1-s)zf(k^*)$. Since s changes discontinuously at t_0 and k^* adjusts gradually, c falls initially but increases over time.
- Whether per capita consumption is higher or lower at the end is not immediately clear.



Transition between the Steady States: c (2)

- Let c^* denote consumption per capita on the steady state. Our main objective in this section is to see what happens to c^* as the savings rate changes.

Transition between the Steady States: c (2)

- Let c^* denote consumption per capita on the steady state. Our main objective in this section is to see what happens to c^* as the savings rate changes.

- In particular, we know that c^* equals

$$c^* = (1-s)zf(k^*) = zf(k^*) - szf(k^*)$$

Transition between the Steady States: c (2)

- Let c^* denote consumption per capita on the steady state. Our main objective in this section is to see what happens to c^* as the savings rate changes.

- In particular, we know that c^* equals

$$c^* = (1-s)zf(k^*) = zf(k^*) - szf(k^*)$$

- Since on the steady state, by equation (1), $(n+d)k^* = szf(k^*)$, we have that

$$c^* = zf(k^*) - (n+d)k^*$$

Transition between the Steady States: c (2)

- Let c^* denote consumption per capita on the steady state. Our main objective in this section is to see what happens to c^* as the savings rate changes.

- In particular, we know that c^* equals

$$c^* = (1-s)zf(k^*) = zf(k^*) - szf(k^*)$$

- Since on the steady state, by equation (1), $(n+d)k^* = szf(k^*)$, we have that

$$c^* = zf(k^*) - (n+d)k^*$$

- Hence, to achieve our goal, we take the partial derivative of c^* with respect to s!

Transition between the Steady States: c (3)

$$c^* = zf(k^*) - (n+d)k^*$$

Transition between the Steady States: c (3)

$$c^* = zf(k^*) - (n+d)k^*$$

- Take the partial derivative; this yields

$$\frac{\partial c^*}{\partial s} = zf'(k^*)\frac{dk^*}{ds} - (n+d)\frac{dk^*}{ds}$$

(note that k^* is itself a function of s, n, and d!)

Transition between the Steady States: c (3)

$$c^* = zf(k^*) - (n+d)k^*$$

- Take the partial derivative; this yields

$$\frac{\partial c^*}{\partial s} = zf'(k^*)\frac{dk^*}{ds} - (n+d)\frac{dk^*}{ds}$$

(note that k^* is itself a function of s, n, and d!)

- Rearranging a bit:

$$\frac{\partial c^*}{\partial s} = \left[zf'(k^*) - (n+d)\right]\frac{dk^*}{ds}$$

Transition between the Steady States: c (3)

$$c^* = zf(k^*) - (n+d)k^*$$

- Take the partial derivative; this yields

$$\frac{\partial c^*}{\partial s} = zf'(k^*)\frac{dk^*}{ds} - (n+d)\frac{dk^*}{ds}$$

(note that k^* is itself a function of s, n, and d!)

- Rearranging a bit:

$$\frac{\partial c^*}{\partial s} = \left[zf'(k^*) - (n+d)\right]\frac{dk^*}{ds}$$

- We now want to know the sign of the derivative above.

Transition between the Steady States: c (3)

$$c^* = zf(k^*) - (n+d)k^*$$

- Take the partial derivative; this yields

$$\frac{\partial c^*}{\partial s} = zf'(k^*)\frac{dk^*}{ds} - (n+d)\frac{dk^*}{ds}$$

(note that k^* is itself a function of s, n, and d!)

- Rearranging a bit:

$$\frac{\partial c^*}{\partial s} = \left[zf'(k^*) - (n+d)\right]\frac{dk^*}{ds}$$

- We now want to know the sign of the derivative above.
- From previous results, we know that $\frac{dk^*}{ds} > 0$. Hence, we just need to check whether $zf'(k^*)$ is bigger or smaller than $(n+d)$!

Transition between the Steady States: c (4)

So, for the equation:

$$\frac{\partial c^*}{\partial s} = \left[zf'(k^*) - (n+d)\right] \frac{dk^*}{ds}$$

we have 3 cases:

- Case 1: $zf'(k^*) < (n+d) \Rightarrow \frac{dc^*}{ds} < 0$

Transition between the Steady States: c (4)

So, for the equation:

$$\frac{\partial c^*}{\partial s} = \left[zf'(k^*) - (n+d)\right] \frac{dk^*}{ds}$$

we have 3 cases:

- Case 1: $zf'(k^*) < (n+d) \Rightarrow \frac{dc^*}{ds} < 0$
- Case 2: $zf'(k^*) = (n+d) \Rightarrow \frac{dc^*}{ds} = 0$

Transition between the Steady States: c (4)

So, for the equation:

$$\frac{\partial c^*}{\partial s} = \left[zf'(k^*) - (n+d)\right] \frac{dk^*}{ds}$$

we have 3 cases:

- Case 1: $zf'(k^*) < (n+d) \Rightarrow \frac{dc^*}{ds} < 0$
- Case 2: $zf'(k^*) = (n+d) \Rightarrow \frac{dc^*}{ds} = 0$
- Case 3: $zf'(k^*) > (n+d) \Rightarrow \frac{dc^*}{ds} > 0$

Transition between the Steady States: c (4)

So, for the equation:

$$\frac{\partial c^*}{\partial s} = \left[zf'(k^*) - (n+d)\right] \frac{dk^*}{ds}$$

we have 3 cases:

- Case 1: $zf'(k^*) < (n+d) \Rightarrow \frac{dc^*}{ds} < 0$
- Case 2: $zf'(k^*) = (n+d) \Rightarrow \frac{dc^*}{ds} = 0$
- Case 3: $zf'(k^*) > (n+d) \Rightarrow \frac{dc^*}{ds} > 0$
- How do we know which case occurs in equilibrium?

Transition between the Steady States: c (5)

- Well, if you were the consumer, you'd pretty much like to set up a savings rate s that would maximize your consumption in the steady state.

Transition between the Steady States: c (5)

- Well, if you were the consumer, you'd pretty much like to set up a savings rate s that would maximize your consumption in the steady state.

- We know that steady state consumption is a function of savings rate s.

Transition between the Steady States: c (5)

- Well, if you were the consumer, you'd pretty much like to set up a savings rate s that would maximize your consumption in the steady state.

- We know that steady state consumption is a function of savings rate s.

- If we want to find the x that maximizes some function $f(x)$, we set have the first order condition:
$$\frac{df(x)}{dx} = 0$$

- Well, if you were the consumer, you'd pretty much like to set up a savings rate s that would maximize your consumption in the steady state.
- We know that steady state consumption is a function of savings rate s.
- If we want to find the x that maximizes some function $f(x)$, we set have the first order condition:
$$\frac{df(x)}{dx} = 0$$
- Therefore, in steady state, we will have
$$\frac{\partial c^*}{\partial s} = 0$$

Transition between the Steady States: c (5)

- Well, if you were the consumer, you'd pretty much like to set up a savings rate s that would maximize your consumption in the steady state.
- We know that steady state consumption is a function of savings rate s.
- If we want to find the x that maximizes some function $f(x)$, we set have the first order condition:

$$\frac{df(x)}{dx} = 0$$

- Therefore, in steady state, we will have

$$\frac{\partial c^*}{\partial s} = 0$$

- WE ALREADY HAVE THIS:

TRANSITION BETWEEN THE STEADY STATES: c (6)

So, for the equation:

$$\frac{\partial c^*}{\partial s} = \left[zf'(k^*) - (n+d)\right] \frac{dk^*}{ds}$$

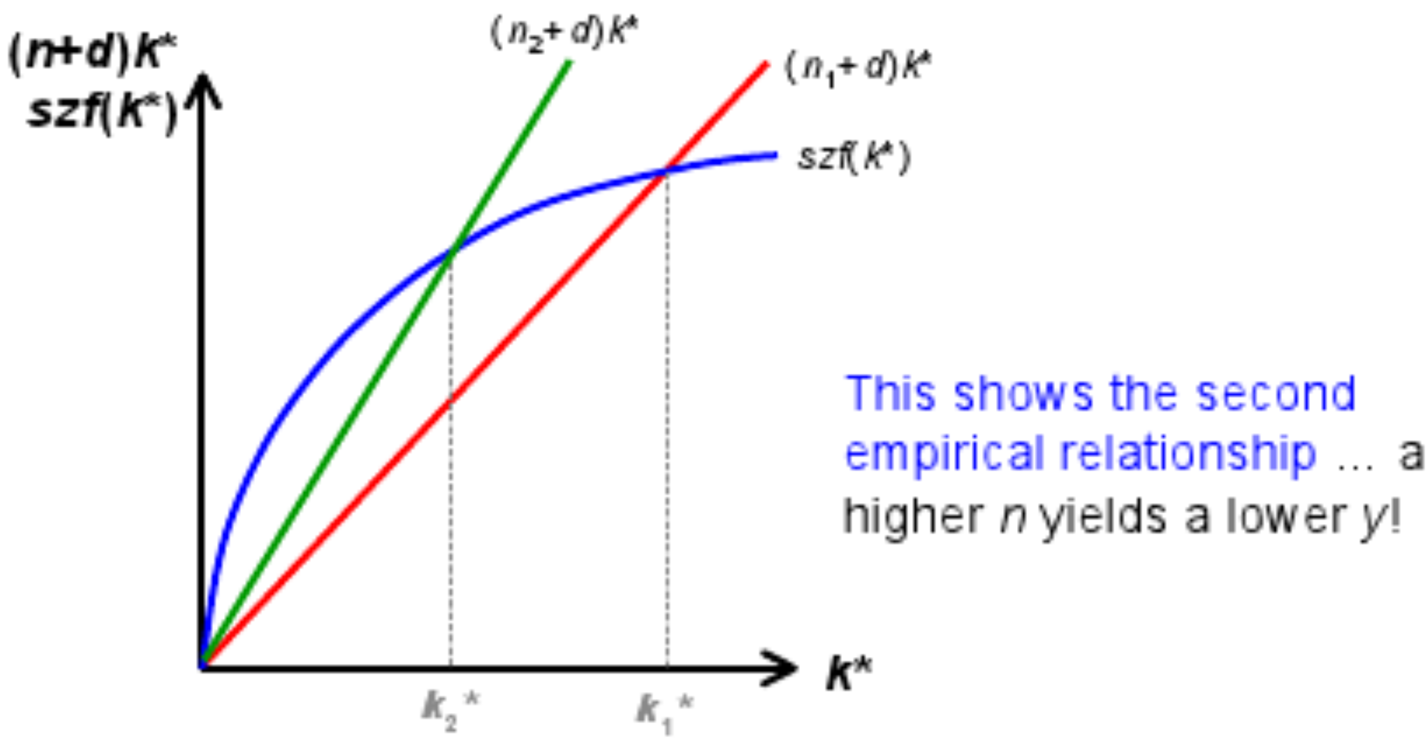
we have 3 cases:

- Case 1: $zf'(k^*) < (n+d) \Rightarrow \frac{dc^*}{ds} < 0$
- $\boxed{\text{Case 2: } \mathbf{zf'(k^*) = (n+d) \Rightarrow \frac{dc^*}{ds} = 0}}$
- Case 3:$zf'(k^*) > (n+d) \Rightarrow \frac{dc^*}{ds} > 0$

When the conditions of Case 2 are satisfied, consumption is at its maximum level for all possible steady states. Hence, consumption is maximized.

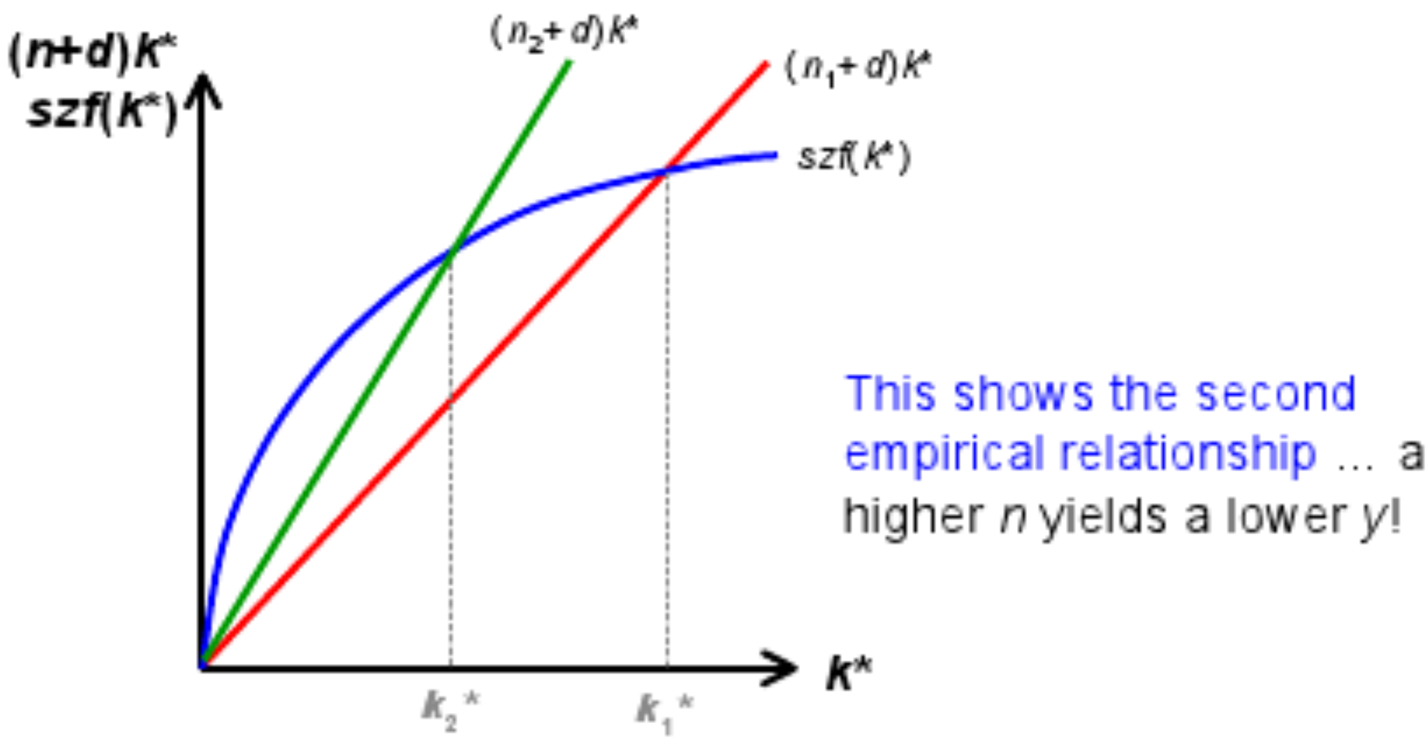
- This value of k^* is known as the golden rule capital per worker.

What happens in the steady state when we change the population growth rate, n?



An Increase in the Population Growth Rate, n

What happens in the steady state when we change the population growth rate, n?



- However, aggregate income Y is growing at a faster rate n_2. This shows that higher growth in aggregate income need not be associated with higher income per worker in the long run.

- So we've analyzed how things change when we move the savings rate and the population growth rate.

Long-term Growth

- So we've analyzed how things change when we move the savings rate and the population growth rate.

- If an economy needs to grow (income per capita), then we need to save more and control for population growth.

Long-term Growth

- So we've analyzed how things change when we move the savings rate and the population growth rate.
- If an economy needs to grow (income per capita), then we need to save more and control for population growth.
- However, since $s \leq 1$ and there is a "natural limit" on how much we can reduce n. How can we sustain a long-term growth?

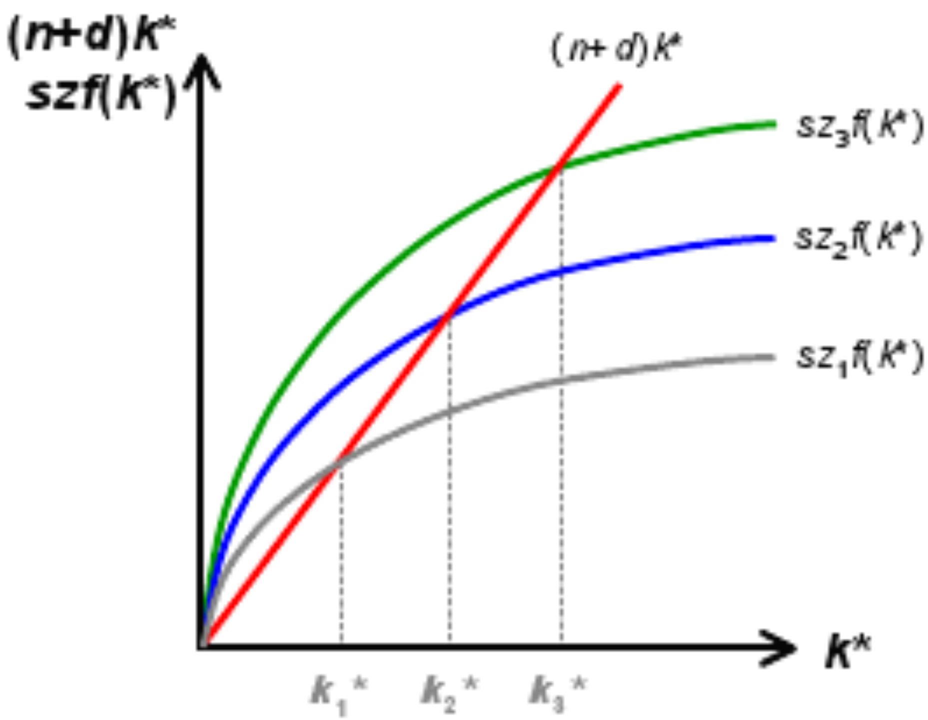
Long-term Growth and a Shock in TFP (1)

Let's see what happens when there is a positive technological shock; i.e., an increase in z.

Long-term Growth and a Shock in TFP (1)

Let's see what happens when there is a positive technological shock; i.e., an increase in z.

- For $z_1 < z_2 < z_3$:



Long-term Growth and a Shock in TFP (2)

Hence, increases in z translate into long-run growth (sustained growth in per capita income).

- Improvements in a country's standard of living are brought about in the long run by technological progress.