Tutorial

Def The floor function, denoted $\lfloor x \rfloor$, is the largest integer less than or equal to x.

Def the ceiling function, denoted $\lceil x \rceil$, is the smallest integer greater than or equal to x.

Properties :

(1) $\lfloor x \rfloor = n$ if and only if $n \leq x < n+1$

(2) $\lceil x \rceil = n$ if and only if $n-1 < x \leq n$

(3) $x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$

(4) $\lfloor -x \rfloor = -\lceil x \rceil$

$\lceil -x \rceil = -\lfloor x \rfloor$

(5) $\lfloor x+n \rfloor = \lfloor x \rfloor + n$

$\lceil x+n \rceil = \lceil x \rceil + n$

Exercise 1 prove property 4.

Sol Let x be a real number.

P1 Then, there exists a unique real number ε, with $0 < \varepsilon < 1$, and a unique integer n such that $x = n + \varepsilon$; moreover we have $\lfloor x \rfloor = n$

P2 Similarly, for every real number x, there exists a unique integer n and a real number ε such that $0 < \varepsilon < 1$ and $x = n + \varepsilon$; moreover $\lceil x \rceil = n+1$.

case $\lceil -x \rceil = -\lfloor x \rfloor$

First assume that x is an integer n. Then,

$$\lceil -x \rceil = \lceil -n \rceil = -n = -\lfloor n \rfloor = -\lfloor x \rfloor.$$

If n is not an integer, then there exists an integer n and a real number ε, so that $0 < \varepsilon < 1$ and $x = n + \varepsilon$ both hold. Then,

$$\lceil -x \rceil = \lceil -(n+\varepsilon) \rceil = \lceil (-n-1) + \underbrace{(1-\varepsilon)}_{0 < 1-\varepsilon < 1} \rceil$$

by P2 $= -n - 1 + 1$

$= -n$

$= -\lfloor n + \varepsilon \rfloor$ by P1

$= -\lfloor x \rfloor$

case $\lfloor -x \rfloor = -\lceil x \rceil$

Assume first x is an integer n. Then

$$\lfloor -x \rfloor = \lfloor -n \rfloor = -n = -\lceil n \rceil = -\lceil x \rceil.$$

If x is not an integer then exist an integer n and $0 < \varepsilon < 1$ so that $x = n + \varepsilon$ holds. Then

$$\lfloor -x \rfloor = \lfloor -(n+\varepsilon) \rfloor = \lfloor -n-1 + (1-\varepsilon) \rfloor \qquad 0 < 1-\varepsilon < 0$$

by P1 $= -n - 1$

$= -(n+1)$

by P2 $= -\lceil n + \varepsilon \rceil$

$= -\lceil x \rceil$.

Exercise 2 Let $f:\mathbb{Z} \to \mathbb{Z}$; $f(x)=x+1$. Guess a formula for f^n and prove it by induction.

$f^1(x) = x+1$
$f^2(x) = f\circ f(x) = f(x+1) = (x+1)+1 = x+2$
$f^3(x) = f(f^2(x)) = f(x+2) = (x+2)+1 = x+3$
$\vdots$

guess $\Rightarrow$ $f^n(x) = f(f^{n-1}(x)) = x+n$.

We are going to prove that $f^n(x) = x+n$ by induction, for $n \in \mathbb{Z}^+$.

Base Case $(n=1)$. $f^1(x) = f(x) = x+1$, then the base case holds.

Ind Hip Suppose that $f^k(x) = x+k$ for some $k \geq 1$. Then

$$f^{k+1}(x) = f(f^k(x)) = f(x+k) = (x+k)+1$$
$$= x+(k+1),$$

i.e. the case $n=k+1$ holds.

Conclusion. We have proved that $f^1(x) = x+1$, and $f^k(x) = x+k$ implies that $f^{k+1}(x) = x+k+1$, thus by induction we have that $f^n(x) = x+n$ for all $n \in \mathbb{Z}^+$.

Exercise 3 Which of the functions f below is injective? surjective? When f is invertive, determine its inverse.

1) $f_1 : \mathbb{Z} \to \mathbb{Z}$
$x \mapsto x+2$

2) $f_2 : \mathbb{Z} \to \mathbb{Z}$
$x \to x^3 - 1$

3) $f_3 : \mathbb{R} \to \mathbb{R}$
$x \mapsto \frac{x+2}{3}$

4) $f_4 : \mathbb{R} \to \mathbb{R}$
$x \mapsto \lceil x \rceil$

Sol

1) (a) Let $z_1, z_2 \in \mathbb{Z}$ such that $f_1(z_1) = f_1(z_2)$ then

$$f_1(z_1) = f_1(z_2) \Rightarrow z_1 + 2 = z_2 + 2$$
$$\Rightarrow z_1 = z_2 .$$

Hence f_1 is injective.

(b) Let $y \in \mathbb{Z}$. Set $x = y - 2$, then

$$f_1(x) = f_1(y-2) = y - 2 + 2 = y$$

Hence, f_1 is surjective.

(c) f_1 is bijective, then it is invertible with inverse given by

$$f_1^{-1} : \mathbb{Z} \longrightarrow \mathbb{Z}$$
$$y \longmapsto y-2$$

2) (a) f_2 is not injective since $f(1)=0=f(-1)$

(b) f_2 is not surjective since -2 has no pre-image by f_2. Indeed, $-2 = x^2-1$ has no solution in $\mathbb{Z}$.

3) (a) Let $x_1, x_2 \in \mathbb{R}$ such that $f_3(x_1)=f_3(x_2)$ then

$$f_3(x_1) = f_3(x_2) \Rightarrow \frac{x_1+2}{\cancel{3}} = \frac{x_2+2}{\cancel{3}}$$
$$\Rightarrow x_1+2 = x_2+2$$
$$\Rightarrow \boxed{x_1 = x_2}$$

Hence, f_3 is injective.

(b) Let $y \in \mathbb{R}$, and set $x = 3y-2$. Then

$$f_3(x) = f_3(3y-2) = \frac{(3y-2)+2}{3}$$
$$= \frac{3y}{3} = y.$$

Hence f_3 is surjective.

(c) f_3 is bijective, then it is invertible with inverse

$$f_3^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$$
$$y \longmapsto 3y-2.$$

4) (a) Not injective, since $f_4(\sqrt{2}) = 2 = f_4(2)$.

(b) Not surjective, since $\sqrt{2}$ has no pre-image by f_4.