# University of Western Ontario

**Departments of Applied Mathematics**

# Calculus 1301B Final Examination

**(Take-Home)**

## Code 111

April 13, 2020 24 hours

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## Instructions

1. **Print your Name, Student Number in the box above**.
2. **Circle your section number below.**

| | | | |
|---|---|---|---|
| 001 | Z. Krougly | 005 | K. Nguyen |
| 004 | U. Hussain | 006 | Z. Krougly |

3. The Exam Booklet should have 16 pages (including the front page).
4. In Part A (Multiple Choice questions), **circle the correct answer for each multiple choice question**.
5. Part B must be answered in the space provided in the Exam Booklet. Unjustified answers will receive little or no credit.
6. Pages 14, 15 and 16 of the Exam Booklet are blank and are to be used for Part B if you need extra space for presenting your answers for Part B. Indicate clearly which questions from Part B you are answering there.
7. **Total Marks = Part A (40) + Part B (50) = 90 marks.**

**Part A: 20 multiple choice questions (2 marks each) = 40 marks**
**Do your working in the Scratch Papers. Circle the correct answer for each multiple choice question.**

A1: Evaluate $\displaystyle\int_3^4 \frac{dx}{x-2}$

D

| | | |
|---|---|---|
| A: 0 | B: $-\ln 2$ | C: 2 |
| D: $\ln 2$ | E: diverges | |

A2: Which of the following is the most appropriate substitution to evaluate $\displaystyle\int \frac{7x^3}{4x^2+9}\,dx$.

C

| | | |
|---|---|---|
| A: $x = \frac{3}{2}\sin u$ | B: $u = \frac{2}{3}\sin x$ | C: $x = \frac{3}{2}\tan u$ |
| D: $x = \frac{2}{3}\tan u$ | E: $x = \frac{3}{2}\sec u$ | |

A3: Determine whether the sequence $\{a_n\}_{n=0}^{\infty} = \{\ln(n+1) - \ln(n+2)\}_{n=1}^{\infty}$ converges or diverges.

A

| | | |
|---|---|---|
| A: converges to 0 | B: converges to 1/4 | C: converges to 1/2 |
| D: converges to 1 | E: diverges | |

A4: The series $\displaystyle\sum_{n=1}^{\infty} 3^{n+1}4^{-n}$

D

| | | |
|---|---|---|
| A: converges to 12 | B: converges to 27/4 | C: converges to 3 |
| D: converges to 9 | E: diverges | |

A5: Use the Integral Test to determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ converges or diverges.

E

| | | |
|---|---|---|
| A: converges to 1 | B: converges to 1/4 | C: converges to 1/2 |
| D: converges to 2 | E: diverges | |

A6: The series $\sum_{n=1}^{\infty} n^{-\sin(1)}$

E

| | | |
|---|---|---|
| A: converges to 0 | B: converges to 1/2 | C: converges to 1 |
| D: converge to $1/\sqrt{2}$ | E: diverges | |

A7: Find the radius of convergence $R$ of the power series $\sum_{n=2}^{\infty} \frac{x^n}{(n+2)^{2n}}$.

E

| | | | | |
|---|---|---|---|---|
| A: 0 | B: 1 | C: 1/3 | D: 3 | E: $\infty$ |

A8: The coefficient of $x^3$ in the Maclaurin series of $f(x) = (1+x)^{3/4}$ is

N

| | | | | |
|---|---|---|---|---|
| A: 5/128 | B: $-5/128$ | C: 7/125 | D: $-8/125$ | E: 5/96 |

$y = (x-1)^2 - 2(x-1)$.

A9: The parametric curve $x = t+1$, $y = t^2 - 2t$ describes the graph of

C

| | | |
|---|---|---|
| A: $x = y^2 - 4y + 3$ | B: $y = x^2 - 1$ | C: $y = x^2 - 4x + 3$ |
| D: $x = y^2 - 1$ | E: $x = y^2 - 4y - 3$ | |

$x^2 - \cancel{2x} + 1 - \cancel{2x} + 2$.

$x^2 + 3$.

A10: Find the equation for the tangent line to the parametric curve $x = 2 + \ln t$, $y = t^2 + 2$ at the point $(2, 3)$.

N

| | | |
|---|---|---|
| A: $y = 2x - 1$ | B: $y = -2x + 7$ | C: $y = 2x + 1$ |
| D: $y = 8x - 13$ | E: $y = 3x - 3$ | |

$t = e^{(x-2)}$. $e^{2\cdot(x-2)}$.

$y = e^{2(x-2)} + 2$

$k = 2$

$y' = 2e^{(x-2)} \cdot e^{(x-2)}$

$2 \cdot 7$. $\boxed{2}$.

$y - 3 = 2(x-2)$

$y = 2x - 1$.

A11: Find a Cartesian equation for the curve $r^2 \cos 2\theta = a$, where $a$ is a real number.

A

$\cos 2\theta = \cos^2 - \sin^2$ $r^2\cos^2 - r^2\sin^2 = a$

| | | |
|---|---|---|
| A: $x^2 - y^2 = a$ | B: $2xy = a$ | C: $(x-1)^2 - y^2 = a$ |
| D: $(x-1)^2 - (y-1)^2 = a$ | E: $2x + 3y = a$ | |

A $x^2 - y^2 = a.$

A12: The arc length of the polar curve $r = 3\cos\theta$, $0 \le \theta \le \pi$ is

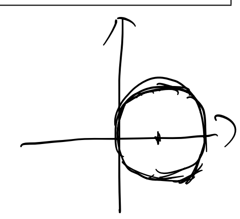
$2\pi \cdot 3/2.$

B

| | | | | |
|---|---|---|---|---|
| A: $3\pi/2$ | B: $3\pi$ | C: $2\pi$ | D: $\pi$ | E: $4\pi$ |

$x^2 + y^2 = 3x.$

$(x - 3/2)^2 + y^2 = \frac{9}{4}$



A13: Which of the following equations is separable and linear?

C

| | | | |
|---|---|---|---|
| A: $y' = x + y$ | B: $y' = e^{x-y}$ | C: $y' = \ln(x+y)$ | D: $y' = \sin(xy)$ |
| E: $y' = xy - 2x$ | | | |

A14: An integrating factor of the differential equation $xy' + xy\cos x = e^{-x}$ is

D

$x \cdot [y \cdot \sin x]' = e^{-x}.$

| | | | | |
|---|---|---|---|---|
| A: $e^{\sin x}$ | B: $e^{\cos x}$ | C: $e^{x \sin x + \cos x}$ | D: $e^{-\sin x}$ | E: $e^{-\cos x}$ |

A15: The arc length of the curve $y = \ln(\cos x)$ from $x = 0$ to $x = \pi/4$ is

A

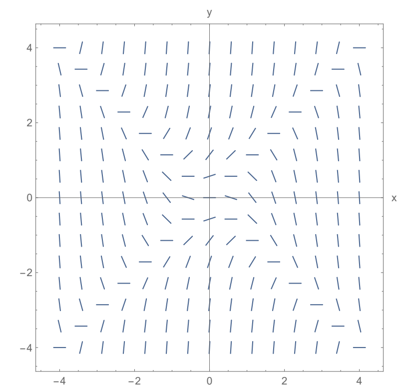
| | | | |
|---|---|---|---|
| A: $\ln(2+\sqrt{3})$ | B: $\ln(2+\sqrt{2})$ | C: $\ln(1+\sqrt{2})$ | D: $\ln(2+\sqrt{3}/3)$ |
| E: $\ln(\sqrt{3})$ | | | |

$\int_0^{\pi/4} \sqrt{1 + [\ln(\cos x)]'^2}\, dx.$

$\int_0^{\pi/4} 1 + 2\ln(\cos x) \cdot \frac{1}{\cos x} \cdot (-\sin x).\ dx.$

A16: Choose the differential equation which has the following direction field.

D




| | | | | |
|---|---|---|---|---|
| A: $y' = x + xy$ | B: $y' = 2 - y$ | C: $y' = x + y - 1$ | D: $y' = y^2 - x^2$ | E: $y' = x(2 - y)$ |

A17: Find the length of the curve $y = \frac{x^3}{6} + \frac{1}{2x}$, $1 \le x \le 2$.

B

| | | | | |
|---|---|---|---|---|
| A: 87/8 | B: 17/12 | C: 56/5 | D: 14/3 | E: $2\sqrt{3}$ |

$\int_1^2 \left[1 + 2\left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{x^2}{2} - \frac{1}{2x^2}\right)\right]$.

A18: The solution of the differential equation $y' = \dfrac{\ln x}{xy}$ is

A

| | | |
|---|---|---|
| A: $y^2 = (\ln x)^2 + C$ | B: $y^2 = -(\ln x)^2 + C$ | C: $x^2 = (\ln y)^2 + C$ |
| D: $y = \ln x^2 + C$ | E: $y = -\ln x^2 + C$ | |

A19: The solution of $y' = xe^{-y}$ is

C

| | | |
|---|---|---|
| A: $y = \ln(2x^2 + C)$ | B: $y = \ln(x^2 + C)$ | C: $y = \ln(x^2/2 + C)$ |
| D: $y = -\ln(-x^2/2 + C)$ | E: $y = 2\ln(x^2 + C)$ | |

A20: The solution of the differential equation $x^2y' - y = 2x^3e^{-1/x}$ is

B

| | | |
|---|---|---|
| A: $y = Ce^{-2x}$ | B: $y = e^{-1/x}(x^2 + C)$ | C: $y = (xe^x - e^x + C)/x$ |
| D: $y = e^{1/x}(x^2 + C)$ | E: $y = Ce^{2x} + x^2$ | |

**Part B: Show all your work for each of the following questions. Total: 50 marks. Do all the 8 questions between B1 and B8.**

B1: (6 marks) Evaluate $\int \frac{\sqrt{1-x^2}}{x^2}\,dx$.

$x = \sin u$

$dx = x'\,du = \cos u\,du$

$$\int \frac{\sqrt{1-x^2}}{x^2}\,dx = \int \frac{\cos u}{\sin^2 u} \cdot \cos u\,du = \int \frac{1}{\tan^2 u}\,du.$$

$$= -\frac{u\sin u + \cos u}{\sin u}$$

$$= -\frac{x\arcsin x + \cos(\arcsin x)}{x}$$

B2: (6 marks) Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty}(-1)^n\frac{4^n x^n}{\sqrt{n}}$.

$a_n = (-1)^n \frac{4^n x^n}{\sqrt{n}}$

$a_{n+1} = (-1)^{n+1} \frac{4^{n+1} x^{n+1}}{\sqrt{n+1}}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{4\sqrt{n}}{\sqrt{n+1}}|x|\right)$

$= \lim_{n\to\infty}\left(4\frac{1}{\sqrt{1+\frac{1}{n}}}|x|\right) = 4|x|.$

$1 < 4|x|$ $x \in (-1/4, 1/4)$.

$\therefore$ $1/4$ is the radius of convergence.

At $x = 1/4$: the power series $\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}} \Rightarrow$ converges

$x = -1/4$: the power series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} \Rightarrow$ converges.

$\therefore$ the series converges on $x \in [-1/4, 1/4]$.

B3: (6 marks)

(a) Find the Maclaurin series of $\sin x$ and its radius of convergence.

(b) Evaluate $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n+1}}{4^{2n+1}(2n+1)!}$.

(a) $\sin x = \frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots\cdots$

$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$a_n = (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$a_{n+1} = (-1)^{n+1} \frac{x^{2n+3}}{(2n+3)!}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^2}{2n+3}\right|$

$= x^2 \lim_{n\to\infty} \frac{1}{2n+3} = 0 < 1$

$\therefore$ converges on $(-\infty, \infty)$

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(b). $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{4^{2n+1}(2n+1)!}$

$a_n = (-1)^n \frac{\pi^{2n+1}}{4^{2n+1}(2n+1)!}$

$a_{n+1} = (-1)^{n+1} \frac{\pi^{2n+3}}{4^{2n+3}(2n+3)!}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{\pi^2}{4^2(2n+3)} = 0 < 1$

$\therefore$ converges.

B4: (8 marks) Consider the parametric curve defined by $x = 3t - t^3$, $y = 3t^2$.

(a) Find $dy/dx$ in terms of $t$.

(b) Write the equations of the horizontal tangent lines to the curve.

(c) Write the equations of the vertical tangent lines to the curve.

(d) Using the results in (a), (b) and (c), sketch the curve for $-2 \le t \le 2$.

(1) $\begin{cases} x = 3t - t^2 \\ y = 3t^2 \end{cases} \Rightarrow \begin{cases} dx = 3 - 2t \, dt \\ dy = 6t \, dt \end{cases}$ $\quad \frac{dy}{dx} = \frac{6t}{3-2t}$.

(2). $t = \sqrt{y/3}$ $\quad x = \sqrt{3y} - \frac{y}{3} \cdot \sqrt{\frac{y}{3}}$

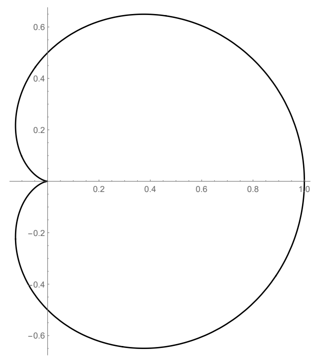
$9x = 9\sqrt{3y} - y\sqrt{3y}$.

(3)

B5: (6 marks) Consider the polar curve $r = \cos^2(\theta/2),\ 0 \le \theta \le 2\pi$ as shown in the figure below.

(a) Find the arc length of the curve.

(b) Find the area enclosed by the curve.




(a) $C=\int_0^{2\pi}\sqrt{r^2+(r')^2}\,d\theta$

$=\int_0^{2\pi}\sqrt{\cos^2(\frac{\theta}{2})+2\cos(\frac{\theta}{2})\cdot(-\sin(\frac{\theta}{2}))\cdot\frac{1}{2}\cdot\,d\theta}$

$=\int_0^{2\pi}\sqrt{\frac{1}{2}(\cos\theta-\sin\theta+1)}\,d\theta$

(b).

B6: (6 marks) Solve the initial value problem $\frac{dy}{dx} = y(2-y),\ y(0) = 1$. Express $y$ as a function of $x$ explicitly. Find $\lim_{x\to\infty} y$.

$$\frac{1}{y(2-y)}\, dy = 1 \cdot dx.$$

$$\int \left(\frac{1}{2} \cdot \frac{1}{y} + \frac{1}{2} \cdot \frac{1}{2-y}\right) dy = \int 1 \cdot dx$$

$$\frac{1}{2} \cdot \ln|y| - \frac{1}{2} \ln|2-y| = x.$$

$$\ln\left|\frac{y}{2-y}\right| = 2x.$$

$$y = 2 - \frac{2}{1-e^{2x}}$$

$$\lim_{x\to\infty} y = 2 - 0 = 2.$$

B7: (6 marks) Solve the initial value problem $\dfrac{dy}{dx} - \dfrac{2}{x}y = x^2 \sin 3x$ subject to the initial condition $y(\pi/3) = 0$.

$1 \cdot dy = x^2 \sin 3x \, dx + \frac{2}{x} y \cdot dx.$

B8: (6 marks) A tank is filled with 10 gallons of brine in which is dissolved 5 lb of salt. Brine containing 3 lb of salt per gallon enters the tank at a rate of 2 gal per minute, and the well-stirred mixture is pumped out at the same rate.

(a) Find the amount of salt in the tank at any time $t$.

(b) How much salt is in the tank after 10 minutes?

(c) How much salt is in the tank after a long time?

$\frac{5}{10}$ $\frac{3}{2}$ /min.

(a) assume there's $y$ lb of salt per gallon

$$y = \frac{5+3t-2yt}{10+2t-2t}$$

$$y = \frac{5+3t}{10+2t}$$

amount of salt $x = \frac{5+3t}{10+2t} \times 10$

(b) $x = \frac{5+30}{30} \times 10 = \frac{35}{3}$.

(c) $x = 10 \times \frac{5+3t}{10+2t}$.

$$\lim_{t \to \infty} \left( \frac{50+30t}{10+2t} \right) = 15.$$

This page is for answers for Part B questions which you could not fit in the space provided. Indicate these clearly. Rough work for Part B questions (not to be graded) should also be done in the Scratch Papers.

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