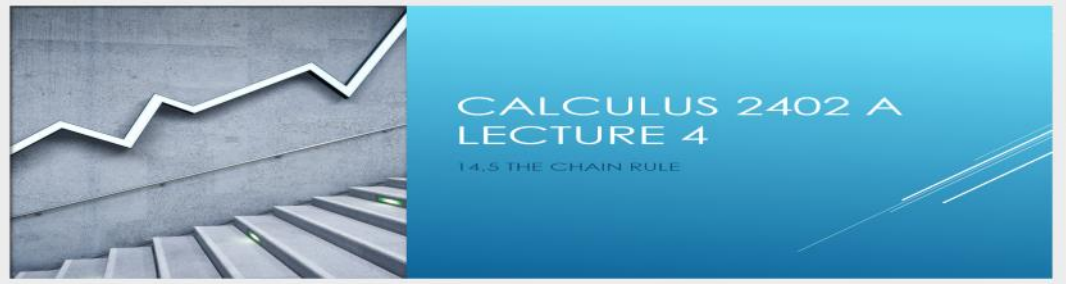


The Chain Rule (sec 14.5)

There are several versions of the CR.

(i) If $z = f(x,y)$ is a differentiable function of x and y and x and y are functions of t, then z is also a function of t.

We want to compute $\frac{dz}{dt}$.

From the definition of differentiable functions of two variables,

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where both $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

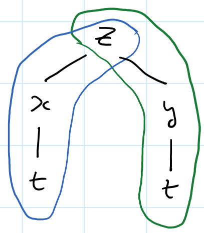
$$\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}$$

As $\Delta t \to 0$ then Δx and $\Delta y \to 0$ as well which gives

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t}$$

$$= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + (0)\frac{dx}{dt} + (0)\frac{dy}{dt}$$

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}} \qquad (\mathrm{I})$$



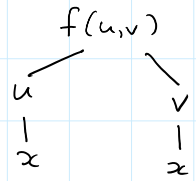
$$\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

To memorize (I), we use a tree diagram as shown in the adjacent figure. In this figure, we chose paths ending with t and there are two of them. The 1st path gives $\frac{\partial z}{\partial x}\frac{dx}{dt}$ and the 2nd path gives $\frac{\partial z}{\partial y}\frac{dy}{dt}$. Adding these terms, we obtain (I).

Ex1: Obtain the quotient rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$$

by using (I).



Solution

Let $f(u,v) = \frac{u}{v}$

Using (I), we have

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{d}{dx} f(u,v)$$

$$= \frac{\partial f}{\partial u}\frac{du}{dx} + \frac{\partial f}{\partial v}\frac{dv}{dx}$$

$$= \left(\frac{1}{v}\right) u' + \left(-\frac{u}{v^2}\right) v'$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2} \qquad // \text{Ans.}$$

(ii) If $z = f(x,y)$ is a differentiable function of x and y and x and y are functions of s and t then z is also a function of s and t. We want to compute $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Because z is a differentiable function of x and y

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \qquad (A)$$

Also x and y are functions of s and t,

$$dx = \frac{\partial x}{\partial s} ds + \frac{\partial x}{\partial t} dt \qquad (B1)$$

and

$$dy = \frac{\partial y}{\partial s} ds + \frac{\partial y}{\partial t} dt \qquad (B2)$$

Subst (B1), (B2) into (A),

$$dz = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s} ds + \frac{\partial x}{\partial t} dt\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial s} ds + \frac{\partial y}{\partial t} dt\right)$$

$$= \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\right) ds + \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right) dt \qquad (C)$$

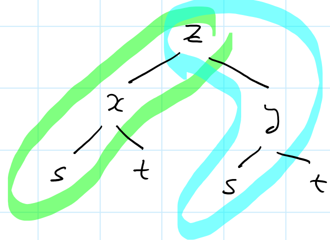
Since z is also a function of s and t,

$$dz = \frac{\partial z}{\partial s} ds + \frac{\partial z}{\partial t} dt \qquad (D)$$

Equating the coefficients of ds and dt in (C) & (D),

$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}} \qquad (\text{II a})$$

$$\boxed{\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad (\text{II}\,a) \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \qquad (\text{II}\,b) \end{aligned}}$$



To obtain $(\text{II}\,a)$, we use a tree diagram as shown in the above figure. In this diagram, we choose paths ending with s. The 1st path gives $\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}$ and the 2nd path gives us $\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$. Adding these terms, we obtain $(\text{II}\,a)$.

Ex2: Use the CR to obtain $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ where

$z = \arcsin(x-y)$, $x = s^2 + t^2$, $y = 1 - 2st$.

Solution Using $(\text{II}\,a)$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= \frac{1}{\sqrt{1-(x-y)^2}}(1)(2s) + \frac{1}{\sqrt{1-(x-y)^2}}(-1)(-2t)$$

$$\frac{\partial z}{\partial s} = \frac{2s + 2t}{\sqrt{1-(x-y)^2}} \qquad // \text{Ans.}$$

Using $(\text{II}\,b)$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= \frac{1}{\sqrt{1-(x-y)^2}}(2t) + \frac{-1}{\sqrt{1-(x-y)^2}}(-2s)$$

$$\frac{\partial z}{\partial t} = \frac{2t + 2s}{\sqrt{1-(x-y)^2}} \qquad // \text{Ans.}$$

(iii) The general version: If u is a differentiable function of n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of m variables $t_1, t_2, \ldots, t_m$ then u is also a differentiable function of $t_1, t_2, \ldots, t_m$ then

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

$$= \sum_{k=1}^{n} \frac{\partial u}{\partial x_k}\frac{\partial x_k}{\partial t_i}$$

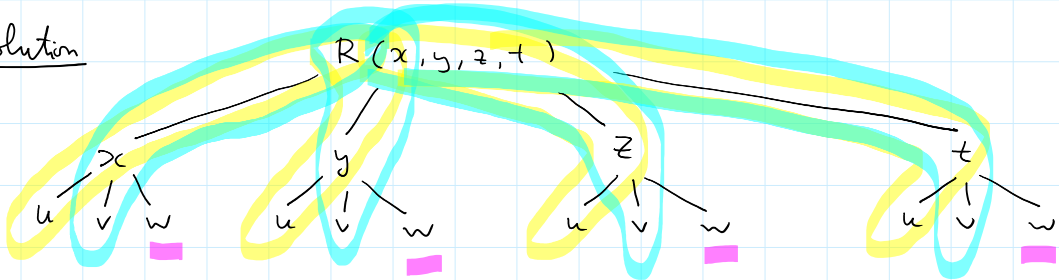
for $i = 1, 2, \ldots, m$

Ex 3: Given $R = f(x, y, z, t)$ where

$x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$, $t = t(u,v,w)$

find $\frac{\partial R}{\partial u}$, $\frac{\partial R}{\partial v}$ and $\frac{\partial R}{\partial w}$.

Solution

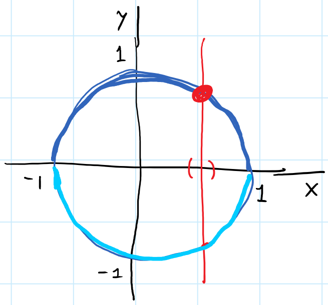


$$\frac{\partial R}{\partial u} = \frac{\partial R}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial R}{\partial t}\frac{\partial t}{\partial u} \quad // \text{Ans.}$$

$$\frac{\partial R}{\partial v} = \frac{\partial R}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial R}{\partial t}\frac{\partial t}{\partial v} \quad // \text{Ans.}$$

$$\frac{\partial R}{\partial w} = \frac{\partial R}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial w} + \frac{\partial R}{\partial t}\frac{\partial t}{\partial w} \quad // \text{Ans.}$$

Implicit Differentiation



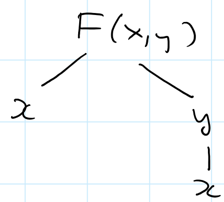
Consider the circle $F(x,y) = x^2 + y^2 - 1 = 0$. By using the vertical test, we know that the above equation does NOT represent a function. However, we can express the circle as a set of two functions

$y = \sqrt{1-x^2}$ which is the upper semi-circle

$y = -\sqrt{1-x^2}$ which is the lower semi-circle

We say $F(x,y) = 0$ defines y implicitly as a function of x, ie, $y = f(x)$ if $F(x, f(x)) = 0$ for any x in the domain of f.

Differentiating $F(x,y) = 0$ wrt x and considering y as a function of x, we obtain

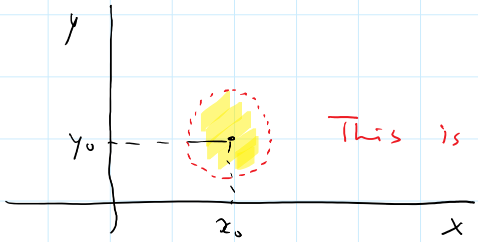


$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

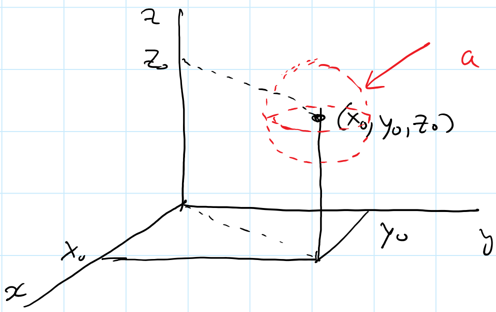
$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

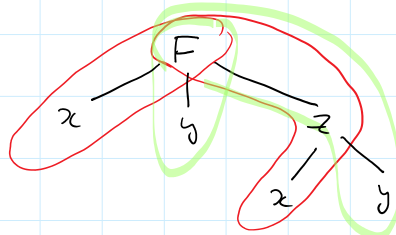
$$\therefore \quad \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y} \quad \text{provided } F_y \neq 0$$

The Implicit Function Theorem states that if $F(x,y) = 0$ at (x_0, y_0) and $\frac{\partial F}{\partial y} \neq 0$ at this point then y is a function of x in a neighborhood of (x_0, y_0).



For functions of three variables, if $F(x,y,z) = 0$ at the point (x_0, y_0, z_0) and $\frac{\partial F}{\partial z} \neq 0$ at (x_0, y_0, z_0) then z is a function of x and y in a neighborhood of (x_0, y_0, z_0).





Differentiating $F(x,y,z) = 0$ wrt x

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\therefore \quad \boxed{\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z}} \quad \text{provided } F_z \neq 0$$

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0$$

$$\therefore \quad \boxed{\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z}} \quad \text{provided } F_z \neq 0$$

Ex 4: Find y' if $x^3 + y^3 = 6xy$. $\leftarrow$

Soln

(i) $\quad 3x^2 + 3y^2 y' = 6y + 6xy'$

Soln

(i) $3x^2 + \underline{3y^2 y'} = 6y + \underline{6xy'}$

$$(3y^2 - 6x)\,y' = 6y - 3x^2$$

$$\therefore \quad y' = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{\cancel{3}(2y - x^2)}{\cancel{3}(y^2 - 2x)} = \frac{2y - x^2}{y^2 - 2x} \quad // \text{Ans.}$$

(ii) $y' = -\dfrac{F_x}{F_y}$ where $F(x,y) = x^3 + y^3 - 6xy = 0$

$$F_x = 3x^2 - 6y$$

$$F_y = 3y^2 - 6x$$

$$\therefore \quad y' = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{\cancel{3}(x^2 - 2y)}{\cancel{3}(y^2 - 2x)} = \frac{2y - x^2}{y^2 - 2x} \quad // \text{Ans.}$$

Ex 5: Given $xyz = \cos(x+y+z)$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Soln Rewriting the above eqn in the form of $F(x,y,z) = 0$.

$$F(x,y,z) = xyz - \cos(x+y+z) = 0$$

then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \qquad (F_z \neq 0)$$

$$F_x = yz + \sin(x+y+z)$$

$$F_y = xz + \sin(x+y+z)$$

$$F_z = xy + \sin(x+y+z)$$

$$\therefore \quad \frac{\partial z}{\partial x} = -\frac{yz + \sin(x+y+z)}{xy + \sin(x+y+z)} \quad // \text{Ans.}$$

$$\frac{\partial z}{\partial y} = -\frac{xz + \sin(x+y+z)}{xy + \sin(x+y+z)} \quad // \text{Ans.}$$

See you on Friday!