Partial Derivatives of the Loss Functions (OLS & LAD)

$$\hat{y}_i = b_0 + b_1 x_i \qquad \text{OLS} \Longrightarrow L = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} (y_i - b_0 - b_1 x_i)^2$$

Using the chain rule, $\frac{\partial}{\partial b_1}\left((y_i - b_0 - b_1 x_i)^2\right) = \frac{\partial u^2}{\partial u} \frac{\partial u}{\partial b_1}$, where $u = y_i - b_0 - b_1 x_i$

and $\frac{\partial}{\partial u}(u^2) = 2u$: we get $2(y_i - b_0 - b_1 x_i)\left(\frac{\partial}{\partial b_1}(y_i - b_0 - b_1 x_i)\right)$

Differentiate the sum term by term and factor out constants:

$2(y_i - b_0 - b_1 x_i)(-x_i) \Longrightarrow$ Therefore, $\frac{\partial L}{\partial b_1} = -2 \sum_{i=1}^{n} \underbrace{(y_i - b_0 - b_1 x_i)}_{r_i} x_i = -2 \sum_{i=1}^{n} r_i x_i$

$\frac{\partial}{\partial b_0}\left((y_i - b_0 - b_1 x_i)^2\right)$, using the chain rule $\frac{\partial}{\partial b_0}\left((y_i - b_0 - b_1 x_i)^2\right) = \frac{\partial u^2}{\partial u} \frac{\partial u}{\partial b_0}$, where

$u = y_i - b_0 - b_1 x_i$ and $\frac{\partial}{\partial u}(u^2) = 2u$: we get $2(y_i - b_0 - b_1 x_i)\left(\frac{\partial}{\partial b_0}(y_i - b_0 - b_1 x_i)\right)$

Differentiate the sum term by term and factor out constants: $2(y_i - b_0 - b_1 x_i)(-1)$

Therefore, $\frac{\partial L}{\partial b_0} = -2 \sum_{i=1}^{n} \underbrace{(y_i - b_0 - b_1 x_i)}_{r_i} = -2 \sum_{i=1}^{n} r_i$

$$\text{LAD} \Rightarrow L = \sum_{i=1}^{n} |y_i - \hat{y}_i| \quad \text{where } \hat{y}_i = b_0 + b_1 x_i \Rightarrow L = \sum_{i=1}^{n} |y_i - b_0 - b_1 x_i|$$

$$L = \begin{cases} \sum_{i=1}^{n} (y_i - b_0 - b_1 x_i) & \text{if } y_i > \hat{y}_i \\ \sum_{i=1}^{n} (-y_i + b_0 + b_1 x_i) & \text{if } y_i \leq \hat{y}_i \end{cases}$$

$$\frac{\partial L}{\partial b_0} = \begin{cases} \sum_{i=1}^{n} \frac{\partial (y_i - b_0 - b_1 x_i)}{\partial b_0} = -1 & \text{if } y_i > \hat{y}_i \\ \sum_{i=1}^{n} \frac{\partial (-y_i + b_0 + b_1 x_i)}{\partial b_0} = 1 & \text{if } y_i \leq \hat{y}_i \end{cases}$$

$$\frac{\partial L}{\partial b_1} = \begin{cases} \sum_{i=1}^{n} \frac{\partial (y_i - b_0 - b_1 x_i)}{\partial b_1} = -x_i & \text{if } y_i > \hat{y}_i \\ \sum_{i=1}^{n} \frac{\partial (-y_i + b_0 + b_1 x_i)}{\partial b_1} = x_i & \text{if } y_i \leq \hat{y}_i \end{cases}$$

Therefor $\frac{\partial L}{\partial b_0}$ can be simplified and written as $\frac{\partial L}{\partial b_0} = -\sum_{i=1}^{n} \text{sign}(r_i)$

similarly $\frac{\partial L}{\partial b_1}$ can be simplified and written as $\frac{\partial L}{\partial b_1} = -\sum_{i=1}^{n} \text{sign}(r_i) \cdot x_i$