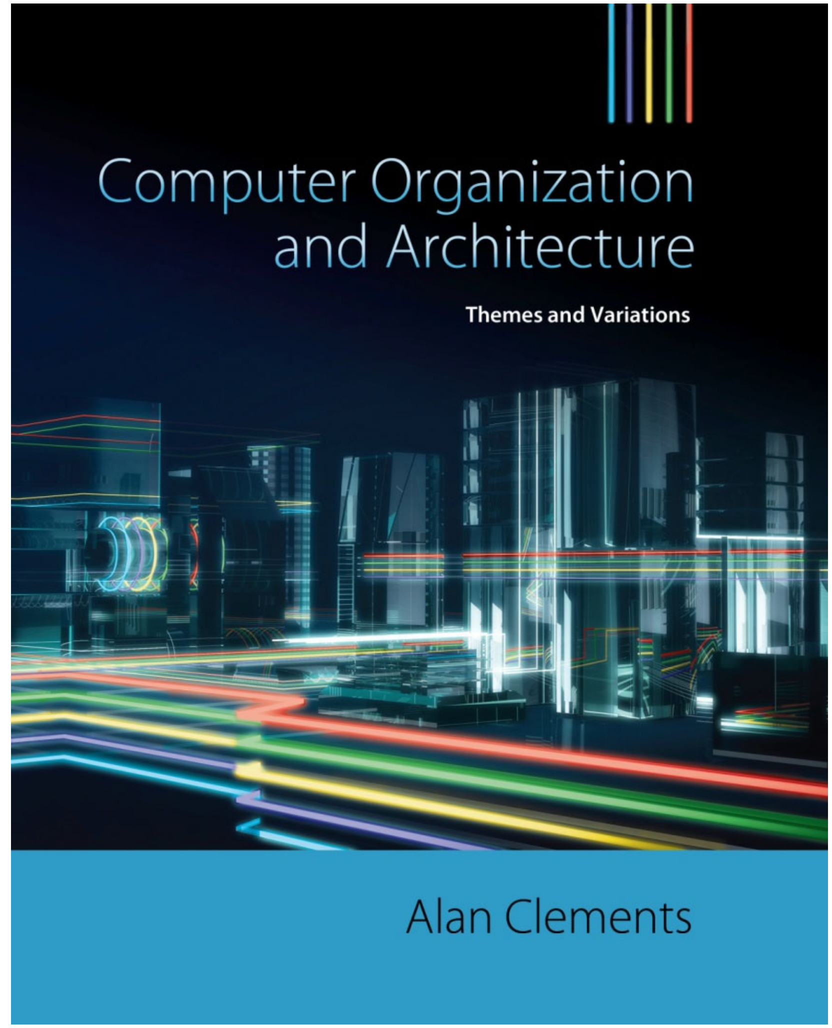
Part 3

CHAPTER 2

Computer Arithmetic and Digital Logic



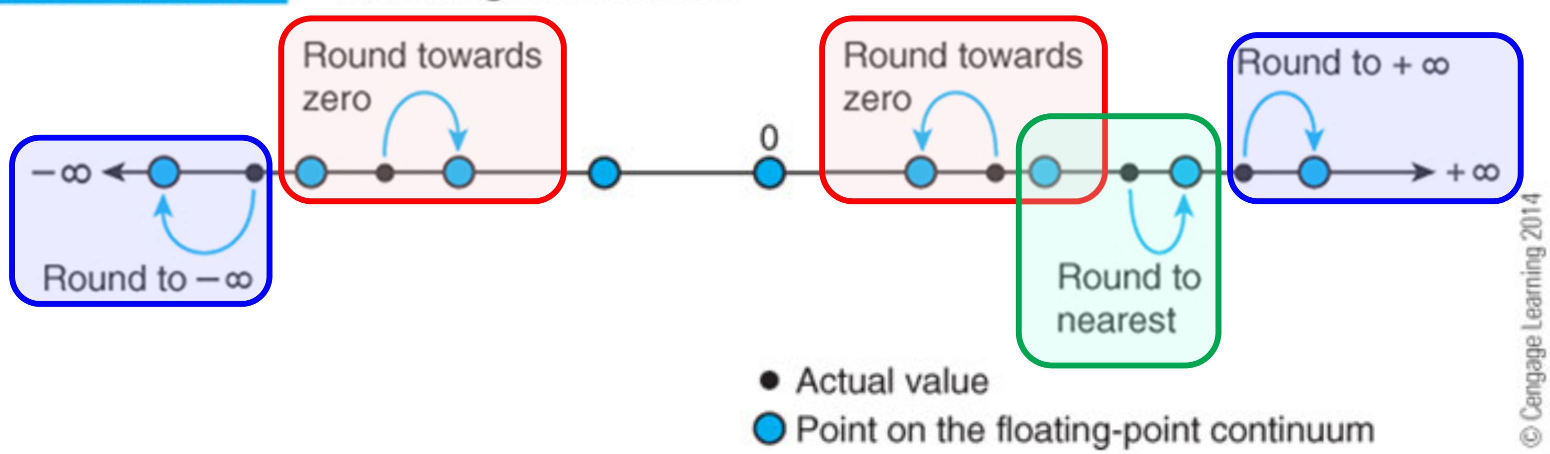
1

CENGAGE Learning

Rounding and Errors

- Floating-point arithmetic can lead to an increase in the number of bits in the fractional part
- To keep the number of fractional bits constant, rounding is needed
  - Error will be induced
- The rounding mechanisms include
  - *Truncation* (i.e., *dropping unwanted bits*) by *rounding towards zero*; a.k.a., *rounding down*
  - *Rounding towards positive or negative infinity*: the *nearest valid floating-point number* in the direction of positive infinity (for positive values) or negative infinity (for negative values) is chosen to decide the rounding; a.k.a., *rounding up*.
  - *Rounding to nearest*: the *closest valid floating-point number* to the actual value is used.

FIGURE 2.11 Rounding mechanisms



Rounding and Errors

- Fraction rounding examples to *2 digits* after the decimal point:

Rounding towards zero (i.e., *rounding down*)

- +4.7744 *truncation*, i.e., *rounded towards zero* ➔ **+4.77**
- –4.7744 *truncation*, i.e., *rounded towards zero* ➔ **–4.77**
- +4.7777 *truncation*, i.e., *rounded towards zero* ➔ **+4.77**
- –4.7777 *truncation*, i.e., *rounded towards zero* ➔ **–4.77**

In *truncation*, we just get rid of the extra digits (regardless the number is positive or negative). The result is *rounding towards zero*.

Rounding towards ± infinity (i.e., *rounding up*)

It is the opposite of *rounding towards zero*

- +4.7744 *rounded towards + infinity* ➔ **+4.78**
- –4.7744 *rounded towards – infinity* ➔ **–4.78**
- +4.7777 *rounded towards + infinity* ➔ **+4.78**
- –4.7777 *rounded towards – infinity* ➔ **–4.78**

I ACTUALLY LIKE THIS CLASS

Rounding to nearest

- +4.7744 *rounded to nearest* ➔ **+4.77**
- –4.7744 *rounded to nearest* ➔ **–4.77**
- +4.7777 *rounded to nearest* ➔ **+4.78**
- –4.7777 *rounded to nearest* ➔ **–4.78**

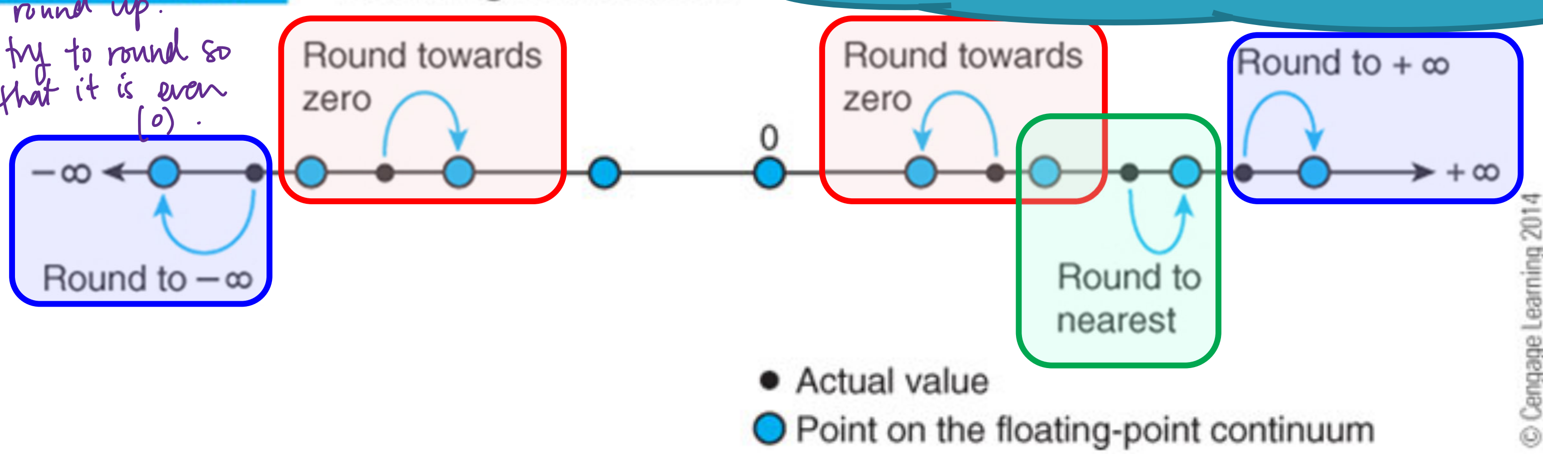
Compare 44 with 50

Compare 77 with 50

How about +4.7750 or -4.7750?

FIGURE 2.11 Rounding mechanisms

50 in decimal = round up.
50 in binary = try to round so that it is even (0).



Normalization

- A number is called normalized when it is written in *scientific notation* with a ***single non-zero digit before the radix point*** (i.e., ***the integer part consists of a single non-zero digit***).

Example 1:

- The number 123.456_{10} is not normalized, as the integer part is not a single non-zero digit.
- To normalize it, you need to move the decimal point two position to the left and to compensate this move by multiplying the number by 100, i.e.,
  - ✓ $1.23456_{10} \times 10^2$

Example 2:

- The number 0.00123_{10} is not normalized, as the integer part is not a single non-zero digit.
- To normalize it, you need to move the decimal point three position to the right and to compensate this move by dividing the number by 1000, i.e.,
  - ✓ $1.23_{10} \times 10^{-3}$

in binary can only be 1.

- In **base b**, a **normalized number** will have the form $\pm$ $\mathbf{b_0 . b_1 b_2 b_3 ... \times b^n}$ where $\mathbf{b_0 \neq 0}$, and $\mathbf{b_0, b_1, b_2, b_3}$ ... are integers between **0** and **b -1**

Floating-point Numbers

- Floating-point arithmetic lets you handle the very large and very small values found in scientific applications.
- Floating-point is also called *scientific notation*, because scientists use it to represent large numbers (e.g., 1.2345×10^{20}) and small numbers that are very close to zero, but not zero (e.g., $0.45679999 \times 10^{-50}$).
- A floating-point value is encoded as ***two*** components: *a number* and *an adjustment to the location of the radix point* within the number.
- A binary floating-point number is represented by

$$\textbf{mantissa} \times 2^{\textbf{exponent}}$$

  - for example, 101010.111110_2 can be represented by
    mantissa $\boxed{1.01010111110_2} \times 2^5$, where
    - the ***significan***t ***d***igits (or simply *significand*) is $\underbrace{1.01010111110}$ and
    - the *exponent* is 5 ($\underbrace{00000101_2}$ in 8-bit binary arithmetic).
- The term *mantissa* has been replaced by *significand* (= mantissa.) to indicate the number of *significant bits* in a floating-point number.
- Because a floating-point number is defined as the *product* of *two values*, a floating-point value is not unique; for example, $10.110_2 \times 2^4 = 1.011_2 \times 2^5$.

Standard for C & Java.

Normalization of Floating-point Numbers

exception: underflow # cannot be normalized.

- In the ***IEEE-754*** *Standard for Floating-Point Arithmetic*, the ***significand*** term is *always normalized* (***unless*** *it represents a zero* or *underflow*)
- A *normalized* binary ***significand*** always has a leading ***1*** *(i.e., **1** in the MSB)*
- The *normalized* absolute ***non-zero*** values of the ***IEEE-754*** FP numbers are always in *the range*

The book is missing the –ve sign here

The minimum absolute value

$$1.000\ldots0_2 \times 2^{-e} \text{ to } 1.111\ldots1_2 \times 2^{e}$$

The maximum absolute value

- The *floating-point* normalization leads to the highest available ***precision***, as all significant bits are utilized.
  - the un-normalized 8-bit significand 0.0000101 has only three significant bits, whereas
  - the normalized 8-bit significand 1.0100011 has eight significant bits.

three not four

  - If a floating-point calculation is to yield the value $0.110\ldots_2 \times 2^{e}$, the result would be normalized to give $1.10\ldots_2 \times 2^{e-1}$.
  - Similarly, the result $10.1\ldots_2 \times 2^{e}$ would be normalized to $1.01\ldots_2 \times 2^{e+1}$.

Signed magnitude and biased representation in floating point

Significand and Exponent Encoding

- The *significand* of an ***IEEE-754*** floating-point number is *represented in sign and magnitude* form.

- The *exponent* is *represented in a biased* form, by *adding a constant* to the *true exponent*.

- Suppose an 8-bit exponent is used and all exponents are biased by 127.
  - If the *true exponent* is 0, it will be encoded as $0 + 127 = 127$.
  - If the *true exponent* is –2, it will be encoded as $-2 + 127 = 125$.
  - If the *true exponent* is +2, it will be encoded as $+2 + 127 = 129$.

- A real number such as 1010.1111 is normalized to get $+1.0101111 \times 2^3$.

  - The *true exponent* is +3, which is encoded as a *biased exponent* of $3 + 127$; that is 130_{10} or 10000010 in binary form.

- Likewise, if a *biased exponent* is 130_{10}, the *true exponent* is $130 - 127 = 3$

Significand and Exponent Encoding

- A 32-bit single-precision ***IEEE-754*** floating-point number is represented by the bit sequence

  S EEEEEEEE 1.FFFFFFFFFFFFFFFFFFFFFFF

  - **S** is the *sign bit*,
    - 0 means positive significand,
    - 1 means negative significand
  - **E** is an eight-bit *biased exponent* that tells you how to shift the binary point, and
  - **F** is a 23-bit ***fractional*** *significand*.
  - ***The leading 1 and the binary point in front of the significand are omitted when the number is encoded.***

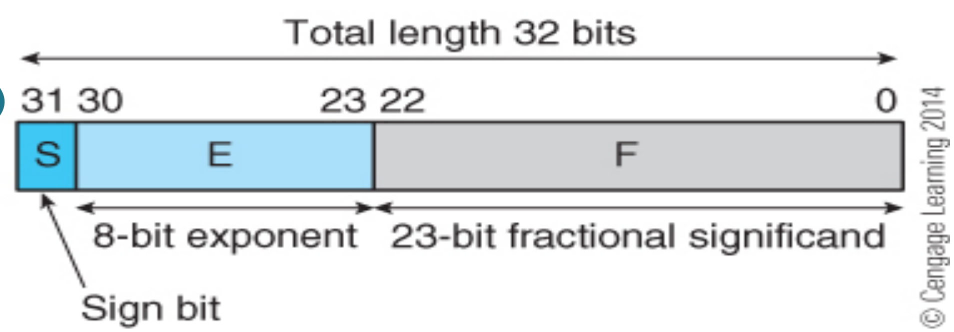
- A floating-point number X is defined as:

In this case, B is 127, i.e., excess-127 code

$$1 \le E \le 254 \iff X = (-1)^{S} \times 2^{(E-B)} \times 1.F$$

When $1 \le E \le 254$,
the *significand* =
1 + the fractional significand **F**

FIGURE 2.7 Structure of a 32-bit IEEE floating-point number



Significand and Exponent Encoding

- If the exponent **EEEEEEEE > 0**, the *significand* of an ***IEEE-754*** floating-point number is *normalized* in the range *1.0000…00* to *1.1111…11*,
- If the exponent **EEEEEEEE = 0**, the *significand* is represented ***without** normalization*.

Used when it is ***impossible*** to normalize the number.

  - In such cases, the floating-point number *X* is defined as:

underflow: too small that cannot fit in bits I have.

$$\textbf{S 00000000 0.FFFFFFFFFFFFFFFFFFFFFFF}$$

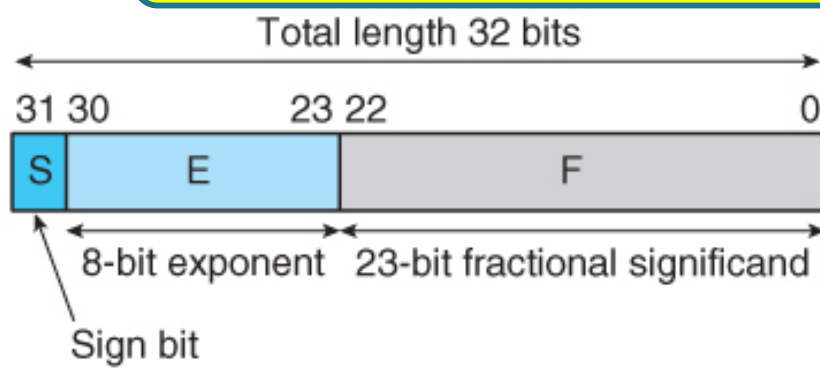
$$E = 0 \Leftrightarrow X = (-1)^S \times 2^{(0-(B-1))} \times 0.F$$

use excess 126, not excess 127.

When $E = 0$, the significand = 0 + the fractional significand F

In this case, B – 1 is 126, i.e., excess-126 code

where,
  - S is the sign bit,
    - 0 means positive significand,
    - 1 means negative significand
  - $E = 0$
    - the exponent was biased by $B - 1$
  - F is the fractional significand
    - As $E = 0$, the *significand* was encoded ***without** normalization*, i.e., 0.F *without* an *implicit leading one*



- When $E = 0$, $F \neq 0$ → ± ***Denormalized underflow number***

Significand and Exponent Encoding

- The floating-point value of ***zero*** is represented by

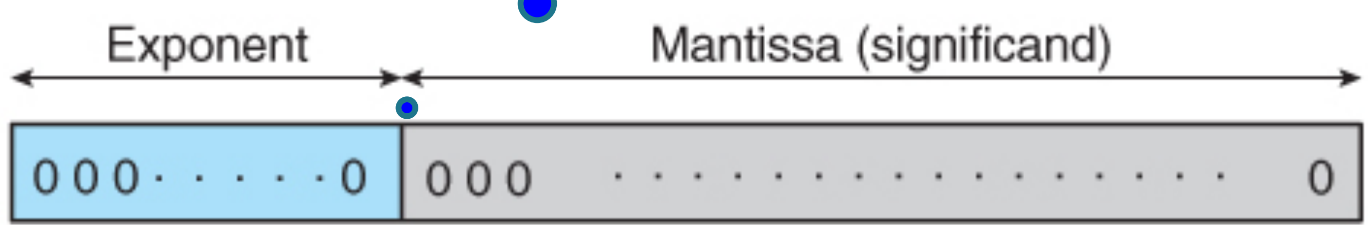
$$\mathbf{0.00...00 \times 2^{\text{most negative exponent}}}$$

i.e., the ***zero*** is represented by
- a ***zero*** *significand* and
- a ***zero*** *biased exponent*

as Figure 2.6 demonstrates.

In this floating-point representation,
how many zeros do we have?

FIGURE 2.6 Representing the value zero in floating-point arithmetic



Significand and Exponent Encoding

TABLE 2.7 IEEE Floating-Point Formats

float type in Java and C

double type in Java and C

	Single Precision	Double Precision (Single Extended)
Field width in bits		
S = sign	1	1
E = exponent	8	11
L = leading bit	1 (not stored)	1 (not stored)
F = fraction	23	52
Total width	32	64
Exponent		
Maximum E	255	2047
Minimum E	0	0
Bias	127	1023
E_{max}	127	1023
E_{min}	−126	−1022

Biased values

Unbiased values

The L value =1, if and only if E ≠ 0
The L value =0, if and only if E = 0

If E ≠ 0, True exponent = ***biased exponent – bias***

If E = 0, True exponent = ***0 – (bias – 1)***

The book flipped the meaning of S. It is S=0 for +ve and =1 for –ve.

S = sign bit (0 for a ~~negative~~ positive number, 1 for a ~~positive~~ negative number)

L = leading bit (always 1 in a normalized, non-zero significand)

F = fractional part of the significand

1 ➔ 254 for **NORMALIZED** numbers

The range of exponents is from the minimum $E + 1$ to the maximum $E - 1$

The number is represented by $-1^S \times 2^{E\ -\ \text{exponent}} \times L.F$

Zero is represented by the minimum exponent, $L = 0$, and $F = 0$

The maximum exponent, $E_{max} + 1$ represents signed infinity

When E = 255

In the IEEE single precision representation,

the ***largest*** ***normalized*** absolute number is

$2^{+127} \times 1.111\ldots1_2 \approx 2^{+128} = 10^{+38.5318394} \approx 3.4\times10^{+38}$.

the ***smallest*** ***normalized*** absolute number is

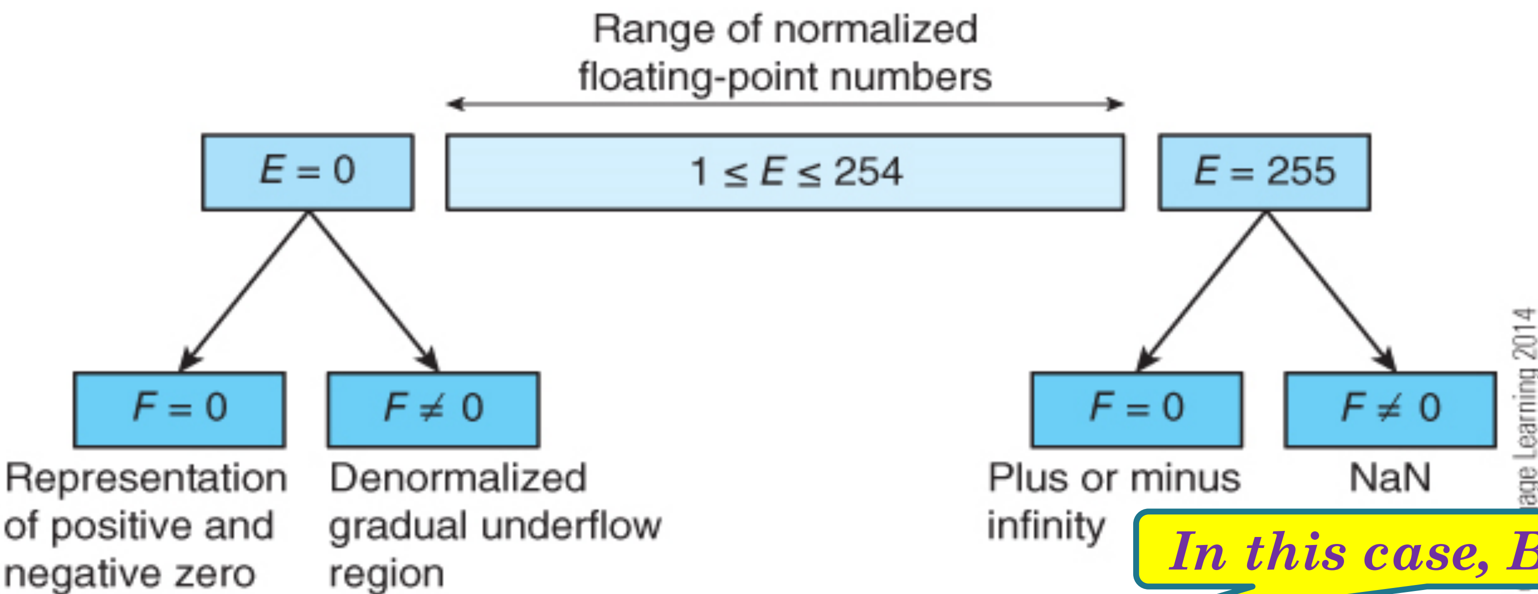
$2^{-126} \times 1.000\ldots0_2 = 2^{-126} = 10^{-37.9297794} \approx 1.17\times10^{-38}$.

This slide is a modified version of the orig...

underflow: $0 < \# < 1.17\times10^{-38}$

Significand and Exponent Encoding

FIGURE 2.8 IEEE floating-point number space for a single-precision number



In this case, B – 1 is 126

$$E = 0 \Leftrightarrow X = (-1)^{S} \times 2^{(0-(B-1))} \times 0.F$$

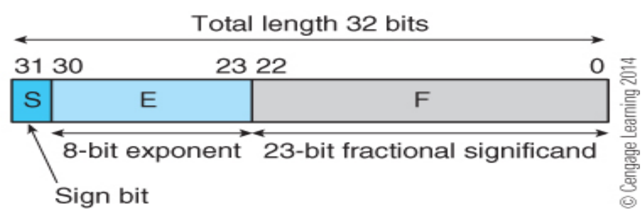
- *Underflow* occurs when the result of a calculation is a very small number; smaller in magnitude than the smallest value representable as a *normalized* floating-point number in the target data type.
- Replacing an *underflow* case by a **zero** might be *ok* from the *addition* point of view, but it is *not ok* from the *multiplication* point of view.
- **NaN** means *Not a Number*, e.g., $0 \div 0$, $\infty \div \infty$, $0 \times \infty$, or $\infty - \infty$
- In **NaN**, the value of F is ignored by applications.

From **Binary** to **32-bit IEEE-754 FP**

- ***Example 1(a):***
  Convert $-11110000111100.00111100001111_2$ into a *32-bit single-precision IEEE-754 FP* value.
  - The number is negative ➔ ***S = 1***
  - The *significand* is $11110000111100.00111100001111_2$
  - The normalized *significand* is
    $1.111000011110000111100001111_2 \times 2^{13}$
  - To encode the ***F*** value, we will *ignore* the leading 1 and we will only consider the first 23 bits after the binary point, i.e., $111000011110000111100001111_2$
  - The ignored part of the *significand* is *rounded to the nearest*, hence the value of ***F*** = $11100001111000011110001_2$
  - The *biased exponent* is the *true exponent* plus 127; that is, $13 + 127 = 140_{10} = 1000\ 1100_2$
    Hence, ***E*** = $1000\ 1100_2$
  - The final number is $1100\ 0110\ 0111\ 0000\ 1111\ 0000\ 1111\ 0001_2$, or $C670F0F1_{16}$.

0 = 0000	8 = 1000
1 = 0001	9 = 1001
2 = 0010	A = 1010
3 = 0011	B = 1011
4 = 0100	C = 1100
5 = 0101	D = 1101
6 = 0110	E = 1110
7 = 0111	F = 1111

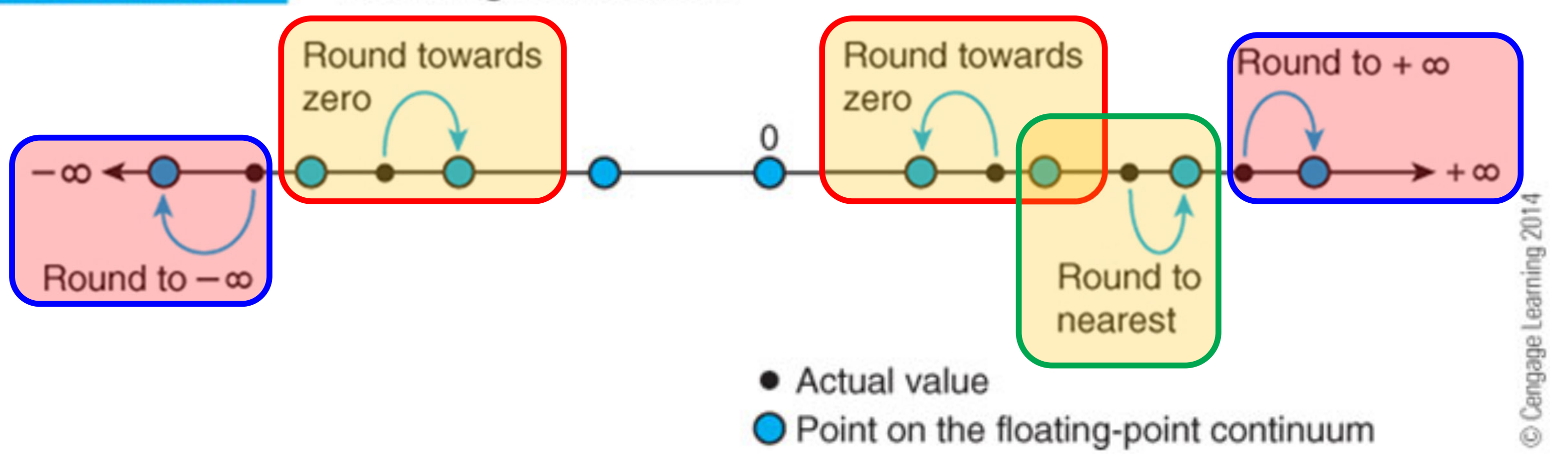
FIGURE 2.7 Structure of a 32-bit IEEE floating-point number



Rounding and Errors

- *When the number to be rounded is midway between two points on the floating-point line, IEEE rounding to the nearest selects the value whose least-significant digit is zero (i.e., rounding to an even binary significand).*
    - *For example:*
        - $0.1110000111100001111000\color{red}{0}\color{gray}{1000}_2$ will be rounded to $0.1110000111100001111000\color{red}{0}_2$
        - $0.1110000111100001111000\color{red}{1}\color{gray}{1000}_2$ will be rounded to $0.111000011110000111100\color{red}{10}_2$

FIGURE 2.11 Rounding mechanisms



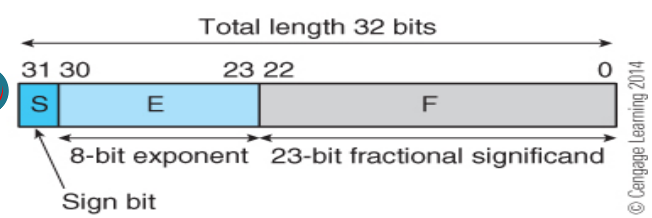
From 32-bit IEEE-754 FP to Binary

- ***Example 1(b):*** *Convert* $C670F0F1_{16}$ from a *32-bit single-precision IEEE-754 FP* value into a binary value
  ---This is the same value as in example 1(a)
  - Unpack the number into *sign bit*, *biased exponent*, and *fractional significand*:

    $C670F0F1_{16} \rightarrow 1100\ 0110\ 0111\ 0000\ 1111\ 0000\ 1111\ 0001_2$
    - S = 1
    - E = 100 0110 0
    - F =111 0000 1111 0000 1111 0001
  - As the sign bit is 1, the number is negative.
  - Subtract 127 from the *biased exponent* $100\ 0110\ 0_2$ to get the *true exponent* $\rightarrow 1000\ 1100_2 - 0111\ 1111_2 = 0000\ 1101_2 = 13_{10}$.
  - The fractional significand is $.111\ 0000\ 1111\ 0000\ 1111\ 0001_2$.
  - Reinserting the *leading one* gives $1.111\ 0000\ 1111\ 0000\ 1111\ 0001_2$.
  - The number is $-1.111\ 0000\ 1111\ 0000\ 1111\ 0001_2 \times 2^{13}$
    $= -1111\ 0000\ 1111\ 00.00\ 1111\ 0001_2$

Note that the correct answer is:
$-1111\ 0000\ 1111\ 00.00\ 1111\ 0001_2$ ***not***
$-1111\ 0000\ 1111\ 00.00\ 1111\ 00001111_2$
This is due to the rounding error.

FIGURE 2.7 Structure of a 32-bit IEEE floating-point number



From 32-bit IEEE-754 FP to Decimal

- ***Example 2:*** *Convert* $1111\ 1110\ 0110\ 0000\ 0000\ 0000\ 0000\ 0000_2$ from a *32-bit single-precision IEEE-754 FP* value into a decimal value.
  - Unpack the number into *sign bit*, *biased exponent*, and *fractional significand*.
    - S = 1
    - E = 1111 1100
    - F =110 0000 0000 0000 0000 0000
  - As the sign bit is 1, the number is negative.
  - Subtract 127 from the *biased exponent* $1111\ 1100_2$ to get the *true exponent* ➔ $1111\ 1100_2 - 0111\ 1111_2 = 0111\ 1101_2 = 125_{10}$.
  - The fractional significand is $.110\ 0000\ 0000\ 0000\ 0000\ 0000_2$.
  - Reinserting the *leading one* gives $1.110\ 0000\ 0000\ 0000\ 0000\ 0000_2$.
  - The number is $-1.11_2 \times 2^{125} = -1.75_{10} \times 2^{125}$

$2^{125} = 10^z$ ➔ $\log_{10}(2^{125}) = z$ ➔ $z = 125 \times 0.30103 = 37.62875$

$2^{125} = 10^{37.62875} = 10^{37} \times 10^{0.62875} = 10^{37} \times 4.25353$

$-1.75 \times 2^{125} = -1.75 \times 10^{37} \times 4.25353 = -7.4436775 \times 10^{37}$

From 32-bit IEEE-754 FP to Decimal

- ***Example 3:*** *Convert* 0000 0000 0110 0000 0000 0000 0000 0000_2 from a *32-bit single-precision IEEE-754 FP* value into a decimal value.
  - Unpack the number into *sign bit*, *biased exponent*, and *fractional significand*.
    - S = 0
    - E = 0000 0000 = 0 → significand is NOT normalized and biase is 126 NOT 127
    - F =110 0000 0000 0000 0000 0000
  - As the sign bit is 0, the number is positive.
  - As E = 0 ➔ *true exponent = 0 – (127 – 1) = –126*
  - The fractional significand is don't put 1 ↘ $.110\ 0000\ 0000\ 0000\ 0000\ 0000_2$.
  - As E = 0, the fractional significand is *not normalized*.

The L value =0, as E = 0

  - As E = 0 and F ≠ 0, it means that this is an *underflow* case.
  - The number is $+0.11_2 \times 2^{-126} = +0.75 \times 2^{-126}$

$2^{-126} = 10^z$ ➔ $\log_{10}(2^{-126}) = z$ ➔ $z = -126\times0.30103 = -37.92978$

$2^{-126} = 10^{-37.92978} = 10^{-37} \times 10^{-0.92978} = 10^{-37} \times 0.11755$

$+0.75 \times 2^{-126} = +0.75 \times 10^{-37} \times 0.11755 = +0.088162 \times 10^{-37}$

$= +0.88162 \times 10^{-38}$ < the ***smallest normalized*** value (1.17×10^{-38})

From 32-bit IEEE-754 FP to Decimal

- ***Example 4:*** *Convert* 0111 1111 1000 0000 0000 0000 0000 0000_2 from a *32-bit single-precision IEEE-754 FP* value into a decimal value.
  - Unpack the number into *sign bit*, *biased exponent*, and *fractional significand*.
    - S = 0
    - E = 1111 1111
    - F =000 0000 0000 0000 0000 0000
  - As the sign bit is 0, the number is positive.
  - As E = 255 ➔ either an infinity case or a NaN case
  - The fractional significand is .000 0000 0000 0000 0000 0000_2.
  - As the *bias exponent* is 255 and the F value is ***zero***, it means that this is an **+infinity** case, e.g., a number that is larger than ***3.4028235 ×10^{+38}***

↳ encode overflow as infinity for floating point

From 32-bit IEEE-754 FP to Decimal

makes it -∞

- ***Example 5:*** *Convert* 1111 1111 1110 0000 0000 0000 0000 0000_2 from a *32-bit single-precision IEEE-754 FP* value into a decimal value.
  - Unpack the number into *sign bit*, *biased exponent*, and *fractional significand*.
    - S = 1
    - E = 1111 1111
    - F =110 0000 0000 0000 0000 0000
  - As the sign bit is 1, the number is negative.
  - As E = 255 ➔ either an infinity case or a NaN case
  - The fractional significand is .110 0000 0000 0000 0000 0000_2.
  - As the *bias exponent* is 255 and the F value is ***NOT zero***, it means that this is a **NaN** case (*Not a Number*), e.g., the result of a ***0*** ÷ ***0***, ∞ ÷ ∞, ***0*** **×** ∞, or ∞ **—** ∞ operation.
  - In **NaN** cases, the value of F is ignored.
  - The value **–NaN**

From 32-bit IEEE-754 FP to Decimal

- ***Example 6:*** *Convert* $C46C0000_{16}$ from *32-bit single-precision IEEE-754 FP* value into a decimal value.
  - Convert the hexadecimal number into binary form $C46C0000_{16} = 1100\ 0100\ 0110\ 1100\ 0000\ 0000\ 0000\ 0000_2$.
  - Unpack the number into *sign bit*, *biased exponent*, and *fractional significand*.
    - S = 1
    - E = 1000 1000
    - F =110 1100 0000 0000 0000 0000
  - As the sign bit is 1, the number is negative.
  - We subtract 127 from the *biased exponent* $1000\ 1000_2$ to get the *true exponent* ➔ $1000\ 1000_2 - 0111\ 1111_2 = 0000\ 1001_2 = 9_{10}$.
  - The fractional significand is $.110\ 1100\ 0000\ 0000\ 0000\ 0000_2$.
  - Reinserting the leading one gives $1.110\ 1100\ 0000\ 0000\ 0000\ 0000_2$.
  - The number is $-1.110\ 1100\ 0000\ 0000\ 0000\ 0000_2 \times 2^9$,
    or $-1110\ 1100\ 00.00\ 0000\ 0000\ 0000_2$ (i.e., -944.0_{10}).

it is 9
not 7

0 = 0000
1 = 0001
2 = 0010
3 = 0011
4 = 0100
5 = 0101
6 = 0110
7 = 0111
8 = 1000
9 = 1001
A = 1010
B = 1011
C = 1100
D = 1101
E = 1110
F = 1111

From Binary to 32-bit IEEE-754 FP

- ***Example 7:*** *Convert* $0.0000\,1000\,0000\,0000\,0000\,0000\,0001\,11_2 \times 2^{-124}$ into a *32-bit single-precision IEEE-754 FP* value.
  - The number is positive ➔ ***S = 0***
  - The *fractional* part is $0.0000\,1000\,0000\,0000\,0000\,0000\,0001\,11_2$
    The normalized *fractional* part is
    $1.000\,0000\,0000\,0000\,0000\,0001\,11_2 \times 2^{-5}$
  - Hence the number will be $1.000\,0000\,0000\,0000\,0000\,0001\,11_2 \times 2^{-129}$
  - As the *exponent* is less than –126, the *fractional* part ***can NOT be*** represented as a ***normalized*** number (the number is ***too small***)
  - Instead, we will attempt to represent it as an ***un-normalized*** *underflow number* with *exponent* = –126
  - The number = $0.001\,0000\,0000\,0000\,0000\,0000\,0011\,1_2 \times 2^{-126}$
  - The encoded ***F*** value (23 bits) will be ***001 0000 0000 0000 0000 0000***

    rounded
  - As ***F*** is *un-normalized* the *biased exponent* will be the *true exponent* plus 127 – 1;
    that is, –126 + 127 – 1 = 0; Hence, ***E*** = $0000\,0000_2$
  - The final number is $0000\,0000\,0001\,0000\,0000\,0000\,0000\,0000_2$, or 00100000_{16}.

From Binary to 32-bit IEEE-754 FP

- ***Example 8:*** *Convert* 0.0000 0000 0000 0000 0000 0000 0001 $11_2 \times 2^{-124}$ into a *32-bit single-precision IEEE-754 FP* value.
  - The number is positive ➔ ***S = 0***
  - The *fractional* part is 0.0000 0000 0000 0000 0000 0000 0001 11_2
    The normalized *fractional* part is $1.11_2 \times 2^{-28}$
  - Hence the number will be $1.11_2 \times 2^{-152}$
  - As the *exponent* is less than –126, the *fractional* part ***can NOT be*** represented as a ***normalized*** number (the number is ***too small***)
  - Instead, we will attempt to represent it as an ***un-normalized*** *underflow number* with *exponent* = –126
  - The number = 0.000 0000 0000 0000 0000 0000 0011 $1_2 \times 2^{-126}$ rounded
  - The encoded ***F*** value (23 bits) will be ***000 0000 0000 0000 0000 0000***
  - As ***F*** is *un-normalized* the *biased exponent* will be the *true exponent* plus 127 – 1;
    that is, –126 + 127 – 1 = 0; Hence, ***E*** = $0000\ 0000_2$
  - The final number is 0000 0000 0000 0000 0000 0000 0000 0000_2,
    or 00000000_{16}.
    I.e., the number is encoded as ***+ZERO*** so small: smaller than smallest underflow # that can be represented

From Binary to 32-bit IEEE-754 FP

- ***Example 9:*** *Convert* 0.0000 0000 0000 0000 0000 0000 0111 11$_2$ × 2^{-124} into a *32-bit single-precision IEEE-754 FP* value.
  - The number is positive ➔ ***S = 0***
  - The *fractional* part is 0.0000 0000 0000 0000 0000 0000 0111 11$_2$
    The normalized *fractional* part is 1.1111$_2$ × 2^{-26}
  - Hence the number will be 1.1111$_2$ × 2^{-150}
  - As the *exponent* is less than –126, the *fractional* part ***can NOT be*** represented as a ***normalized*** number (the number is ***too small***)
  - Instead, we will attempt to represent it as an ***un-normalized*** *underflow number* with *exponent* = –126
  - The number = 0.000 0000 0000 0000 0000 0000 1111 1$_2$ × 2^{-126}

    rounded
  - The encoded ***F*** value (23 bits) will be ***000 0000 0000 0000 0000 0001***
  - As ***F*** is *un-normalized* the *biased exponent* will be the *true exponent* plus 127 – 1;
    that is, –126 + 127 – 1 = 0; Hence, ***E*** = 0000 0000$_2$
  - The final number is 0000 0000 0000 0000 0000 0000 0000 0001$_2$, or 00000001$_{16}$ ---the ***smallest non-zero absolute*** *un-normalized underflow number* ***(1.4012985×10^{-45})***

From Binary to 32-bit IEEE-754 FP

- ***Example 10:*** *Convert* $1111.1111\,1111\,1111\,1111\,1111\,011_2 \times 2^{124}$ into a *32-bit single-precision IEEE-754 FP* value.
  - The number is positive ➔ ***S = 0***
  - The *fractional* part is $1111.1111\,1111\,1111\,1111\,1111\,011_2$
    The normalized *fractional* part is
    $1.111\,1111\,1111\,1111\,1111\,1111\,011_2 \times 2^3$
  - Hence the number will be $1.111\,1111\,1111\,1111\,1111\,1111\,011_2 \times 2^{127}$
  - To encode the ***F*** value, we will ignore the leading 1 and we will only consider the first 23 bits after the binary point, i.e.,
    111 1111 1111 1111 1111 1111
    rounded
  - The *biased exponent* is the *true exponent* plus 127;
    that is, 127 + 127 = 254; Hence, ***E*** = $1111\,1110_2$
  - The final number is $0111\,1111\,0111\,1111\,1111\,1111\,1111\,1111_2$,
    or $7F7FFFFF_{16}$.
  - This number is the ***largest absolute*** *normalized* *number*
    ($3.4028235\times10^{+38}$)

From Binary to 32-bit IEEE-754 FP

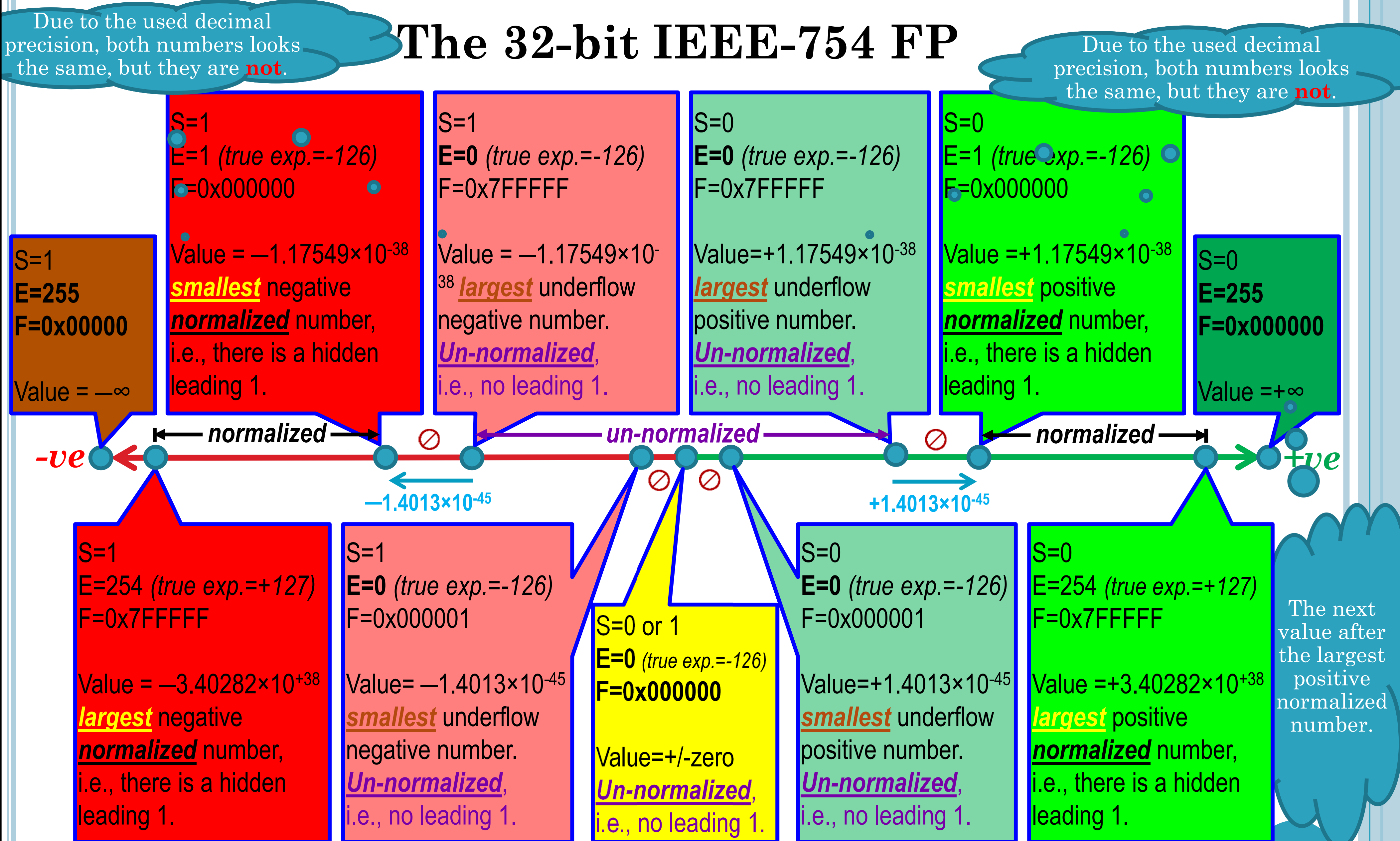
- ***Example 11:*** *Convert* 1111.1111 1111 1111 1111 1111 111$_2$ × 2^{124} into a *32-bit single-precision IEEE-754 FP* value.
  - The number is positive ➔ ***S = 0***
  - The *fractional* part is 1111.1111 1111 1111 1111 1111 111$_2$
    The normalized *fractional* part is
    1.111 1111 1111 1111 1111 1111 111$_2$ × 2^3
  - Hence the number will be 1.111 1111 1111 1111 1111 1111 111$_2$× 2^{127}
  - To encode the ***F*** value, we will only consider the first 23 bits after the binary point
  - Note that, the rounding here will add 1 to the fraction to make it 10.000 0000 0000 0000 0000 0000$_2$× 2^{127}
  - As a result of this, the number needs to renormalized again 1.0000 0000 0000 0000 0000 0000$_2$× 2^{128}
  - The true exponent of the normalized number is > 127, hence the number will be encoded as **+infinity**, i.e.,
    - the ***F*** value will be ***000 0000 0000 0000 0000 0000***
    - the ***E*** value will be 1111 1111$_2$
  - The final number is 0111 1111 1000 0000 0000 0000 0000 0000$_2$, i.e., **+infinity** (7F800000$_{16}$)

From Decimal to 32-bit IEEE-754 FP

- ***Example 12:*** *Convert* 4100.125_{10} into a *32-bit single-precision IEEE-754 FP* value.
  - Convert 4100.125_{10} into a fixed-point binary
    - $4100_{10} = 1\ 0000\ 0000\ 0100_2$ and
    - $0.125_{10} = 0.001_2$.
    - Therefore, $4100.125_{10} = 1000\ 0000\ 0010\ 0.001_2$.
  - Normalize $1000\ 0000\ 0010\ 0.001_2$ to $1.000\ 0000\ 0010\ 0001_2 \times 2^{12}$.
  - The sign bit, S, is 0 because the number is positive
  - The *biased exponent* is the *true exponent* plus 127; that is, $12_{10} + 127_{10} = 139_{10} = 1000\ 1011_2$
  - The fractional significand is 000 0000 0010 0001 **0000 0000**
    - *the leading 1 is stripped* and
    - *the significand is **expanded** to 23 bits*.
  - The final number is $0100\ 0101\ 1000\ 0000\ 0010\ 0001\ 0000\ 0000_2$, or 45802100_{16}.

0 = 0000
1 = 0001
2 = 0010
3 = 0011
4 = 0100
5 = 0101
6 = 0110
7 = 0111
8 = 1000
9 = 1001
A = 1010
B = 1011
C = 1100
D = 1101
E = 1110
F = 1111

The 32-bit IEEE-754 FP



The step-size between consecutive floating-point numbers is NOT always constant as in integer numbers.

To compare two floating-point values without fully decode them, you need to compare S, E, and then F values in order.

Floating-Point Arithmetic

- Consider an example using an *unsigned normalized 8-bit (**1 + 7 bits**) significand* and an ***unbiased** exponent* with $A = 1.010\ 1001_2 \times 2^4$ and $B = 1.100\ 1100_2 \times 2^3$
- To multiply these numbers,
  - you *multiply* the *significands* and
  - *add* the *exponents*
- $A \times B = 1.010\ 1001_2 \times 2^4 \times 1.100\ 1100_2 \times 2^3$

$$= 1.010\ 1001_2 \times 1.100\ 1100_2 \times 2^{3+4}$$

$$= 10.00\ 0110\ 1010\ 1100_2 \times 2^7$$

After normalization:

$$= 1.000\ 0110\ 1010\ 1100_2 \times 2^8.$$

After rounding using:

truncation, i.e., *rounding towards zero*:

$\rightarrow 1.000\ 0110_2 \times 2^8 = (268_{10})$

rounding up, i.e., *rounding toward infinity*:

$\rightarrow 1.000\ 0111_2 \times 2^8 = (270_{10})$

How about rounding to the nearest?

Why is it not rounded to 269_{10}?

$A = 1.010\ 1001_2 \times 2^4$
$= 1010\ 1.001_2$
$= 21.125_{10}$

$B = 1.100\ 1100_2 \times 2^3$
$= 1100.\ 1100_2$
$= 12.75_{10}$

$A \times B = \mathbf{269.34375_{10}}$

$269_{10} = 1\ 0000\ 1101_2$
$0.34375_{10} = 0.010\ 1100_2$

$A \times B =$
$\mathbf{1000\ 0110\ 1.010\ 1100_2}$

Floating-Point Arithmetic

- Now let's look at the addition.
- If these two floating-point numbers ($A = 1.010\ 1001_2 \times 2^4$ and $B = 1.100\ 1100_2 \times 2^3$) were to be added *by hand*, we would ***automatically align the binary points*** of A and B as follows.

$$\begin{array}{r} 10101.001_2 \\ +\ 1100.1100_2 \\ \hline 100001.1110_2 \end{array}$$

$A = 1.010\ 1001_2 \times 2^4$
$= 1010\ 1.001_2$
$= 21.125_{10}$

$B = 1.100\ 1100_2 \times 2^3$
$= 1100.\ 1100_2$
$= 12.75_{10}$

A + B = **33.875_{10}**

$33_{10} = 1\ 00001_2$
$0.875_{10} = 0.111_2$

A + B =
100001.111_2

Floating-Point Arithmetic

- However, as these numbers are held in a *normalized* floating-point format the computer has to carry out the following steps to ***equalize exponents***:

$$A = \quad 1.0101001_2 \times 2^4$$
$$B = \underline{+1.1001100_2} \times 2^3$$

1. *Identify* the number with *the smaller exponent*.
2. *Make the smaller exponent equal to the larger exponent* by dividing the significand of the smaller number by the same factor by which its exponent was increased, ***i.e., un-normalizing the small number to have the same exponent value as the large number.*** ($1.100\ 1100_2 \times 2^3 \rightarrow 0.110\ 0110\ 0_2 \times 2^4 \rightarrow 0.110\ 0110_2 \times 2^4$).
3. *Add (or subtract) the significands.*
4. If necessary, *normalize the result.*

- We can now add *A* to the denormalized *B*.

$$A = \quad 1.010\ 1001_2 \times 2^4$$
$$B = \underline{+\ 0.110\ 0110_2 \times 2^4}$$
$$10.000\ 1111_2 \times 2^4 \rightarrow 1.000\ 0111\ 1_2 \times 2^5 = \mathbf{33.875_{10}}$$

- After rounding using *truncation*, i.e., *rounding towards zero*:

$$\rightarrow 1.000\ 0111_2 \times 2^5 = (33.75_{10})$$

rounding up, i.e., *rounding toward infinity*:

$$\rightarrow 1.000\ 1000_2 \times 2^5 = (34_{10})$$

How about rounding to the nearest?

Floating-Point Arithmetic

- Consider another example using an *unsigned normalized 8-bit (**1+7 bits**) significand* and an ***unbiased** exponent*
  with A=$1.010\ 1001_2 \times 2^4$ & C=$1.100\ 1100_2 \times 2^{13}$

$$\begin{aligned} A &= 1.0101001_2 \times 2^4 \\ C &= \underline{+1.1001100_2} \times 2^{13} \end{aligned}$$

difference between 4 and 13 is 9. > 8 bits.

do not do addition.

1. *Identify* the number with *the smaller exponent*.
2. *Make the smaller exponent equal to the larger exponent* by dividing the significand of the smaller number by the same factor by which its exponent was increased, ***i.e., un-normalizing the small number to have the same exponent value as the large number.***
   ($1.010\ 1001_2 \times 2^4$ ➔ $0.000\ 0000\ 010101001_2 \times 2^{13}$ ➔ $0.000\ 0000_2 \times 2^{13}$)
3. *Add (or subtract) the significands.*

- We can now add C to the un-normalized A.

$$\begin{aligned} A &= 0.000\ 0000_2 \times 2^{13} \\ C &= \underline{+\ 1.100\ 1100_2 \times 2^{13}} \\ &\ 1.100\ 1100_2 \times 2^{13} \rightarrow C \end{aligned}$$

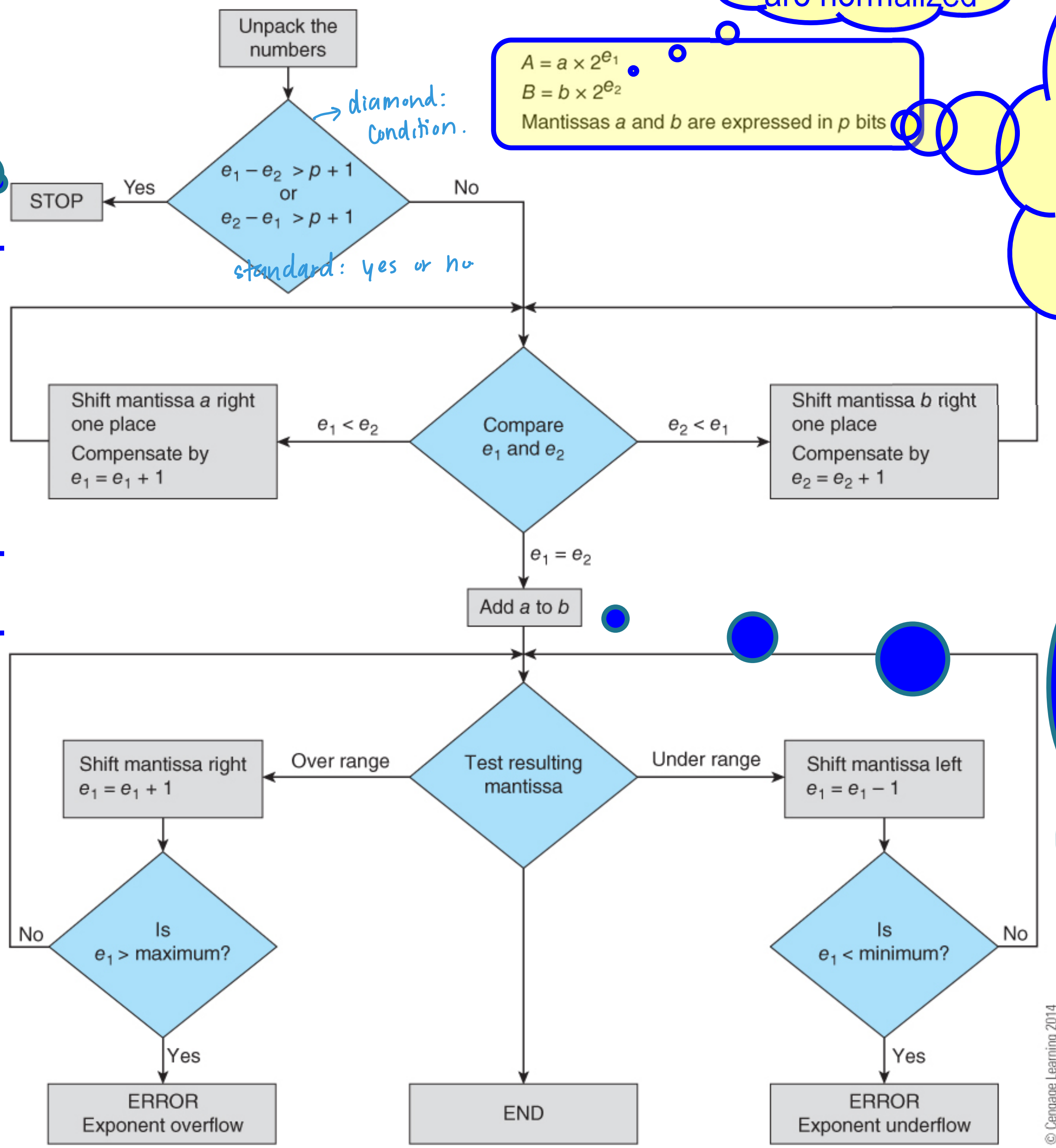
- If the difference between the two exponents of the ***normalized*** two numbers is *greater than* the number of significant bits (i.e., 7 + 1) ➔ the addition result of these two numbers will be the larger of them.

The result is the larger number of the two.

Equalize exponents

post normalization, if necessary

FIGURE 2.10 Flowchart for floating-point addition and subtraction



Both A and B are normalized

p here is the number of bits without the omitted leading 1, i.e., the number of significant fractional bits, i.e., in single-precision IEEE FP standard, **p** = 23

For subtraction, you need to perform the two's complement on the negative number.

Then, the result is converted back to sign-magnitude again to be ready for the normalization step, if needed.