CS 2209A
Final exam
2:00 AM - 5:00 PM, December 18th, 2020

Instructions

1. No calculators or electronic devices allowed.
2. There is a cheat sheet for laws of propositional logic and rules of inference at the end of the question paper. With the appropriate translation you can use the same cheat sheet for set identities and laws of Boolean algebra.
3. Every proof must be justified carefully. Clarity in arguments is weighed heavily. Clearly state what your assumptions are and what you are trying to prove.

1. (14 points, 7 each) Prove the validity or invalidity of the following arguments.

   (i.)

$$\begin{array}{r} p \vee q \\ p \to r \\ \neg q \\ \hline \therefore r \end{array}$$

   (ii.)

$$\begin{array}{r} \forall x\ (P(x) \to (Q(x) \wedge R(x))) \\ \forall x\ (S(x) \vee P(x)) \\ \neg \forall x\ Q(x) \\ \hline \therefore \exists x\ S(x) \end{array}$$

2. (6 points, 3 each) Write the negation of the following statements without starting with the word 'not'.

   (i.) Every student has an official email address.

   (ii.) For every real number x there exists a real number y such that $x = y^2$.

3. (10 points) Let $p_1, \ldots, p_n$ be a collection of propositions. Prove that

$$\neg(p_1 \vee \ldots \vee p_n) = (\neg p_1 \wedge \ldots \wedge \neg p_n)$$

   for all $n \geq 2$ using induction.

4. (15 points) Let P be a predicate which depends on the set of variables $x_1, \ldots, x_n$. Let Q_n denote the statement $\forall x_n \forall x_{n-1} \ldots \forall x_1\ P$ for all $n \geq 1$. Prove that

$$\neg Q = \exists x_n \exists x_{n-1} \ldots \exists x_1 \neg P$$

   for all $n \geq 1$ using induction.

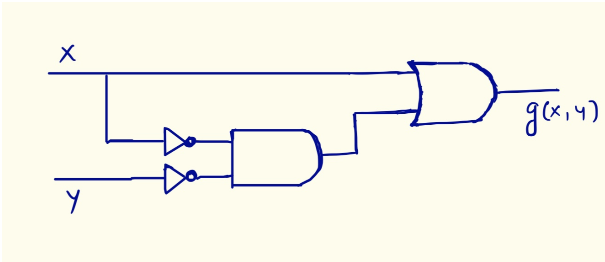
5. (20 points)

   (i.) (5 points) Write a function $f(x, y)$ which represents the following truth table in terms of (any) Boolean operations

x	y	$f(x,y)$
1	1	1
1	0	1
0	1	0
0	0	1

(ii.) (10 points) Construct a Boolean circuit that represents the Boolean function $f(x, y)$ as output ONLY using the NAND gate. The NAND operation is defined as $x \uparrow y = \overline{x.y}$.

(iii.) (5 points) The Boolean circuit below represents the Boolean function $g(x, y)$. Is $g(x, y)$ equivalent to the function $f(x, y)$ in 5(i)? Why?



6. (15 points)

   (i.) (5 points) Define functional completeness of a set of Boolean operations.

   (ii.) (10 points) The NAND operation is defined as $x \uparrow y = \overline{x.y}$ and the XOR (Exclusive OR) operation is defined as $xy = (x+y).\overline{x.y}$. Express the XOR operation in terms of the NAND operation only (i.e., with out using any other Boolean operations).

7. (10 points) Let $f(x, y)$ and $g(x, y)$ be arbitrary Boolean expressions in the variables x and y. Prove that if $f(x, y).g(x, y)$ is satisfiable then both $f(x, y)$ and $g(x, y)$ are satisfiable.
   Hint: Proof by contrapositive.

8. (10 points) Prove that for all natural numbers $n \geq 1$,

$$4 \text{ divides } 5^n - 1.$$

   i.e, for every $n \geq 1$, there exists an integer k such that $5^n - 1 = 4k$.

Rules of inference:

p $p \to q$ $------$ q	Modus ponens
$\neg q$ $p \to q$ $------$ $\neg p$	Modus tollens
p $------$ $p \lor q$	Addition
$p \land q$ $------$ p	Simplification
p q $--------$ $p \land q$	Conjunction

$p \to q$ $q \to r$ $-----------$ $p \to r$	Hypothetical Syllogism
$p \vee q$ $\neg p$ $---------$ q	Disjunctive Syllogism
$p \vee q$ $\neg p \vee r$ $-----------$ $q \vee r$	Resolution

Laws of propositional logic

Idempotent laws:	$p \vee p \equiv p$	$p \wedge p \equiv p$
Associative laws:	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws:	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
Distributive laws:	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws:	$p \vee F \equiv p$	$p \wedge T \equiv p$
Domination laws:	$p \wedge F \equiv F$	$p \vee T \equiv T$
Double negation law:	$\neg\neg p \equiv p$	
Complement laws:	$p \wedge \neg p \equiv F, \neg T \equiv F$	p $\vee \neg p \equiv T, \neg F \equiv T$
De Morgan's laws:	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
Absorption laws:	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Conditional identities:	$p \to q \equiv \neg p \vee q$	$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$