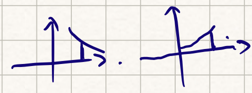
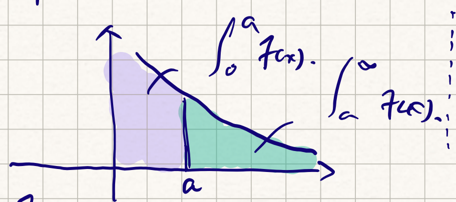
The p intragral theorem.



If $a \in (0, +\infty)$, then:



1. $\int_a^\infty \frac{1}{x^p}\,dx$ converses: if $p > 1$
   diverses: if $p \leq 1$
2. $\int_0^a \frac{1}{x^p}\,dx$ converses: if $p < 1$
   diverses: if $p \geq 1$.

proof:

$$\int_a^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_a^b x^{-p}\,dx.$$

$$= \lim \frac{x^{1-p}}{x^{1-p}} \Big|_{x=a}^{b}$$ constant.

$$= \frac{1}{1-p}\left(b^{1-p} - \underline{a^{1-p}}\right).$$

converses if $p < 1$.

$$b^{1-p} = \begin{cases} \infty & \text{if } p < 1 \\ 1 & \text{if } p = 1. \\ 0 & \text{if } p > 1. \end{cases}$$

if $p<1$: $= \frac{1}{1-p}(0 - a^{1-p}) = -\frac{a^{1-p}}{1-p}$ ← a finite number

if $p=1$: $= \int_a^\infty \frac{1}{x}\,dx. = \lim_{b\to\infty} \int_a^b \frac{1}{x}\,dx = \lim_{b\to\infty} [\ln|b| - \ln a]. = \infty.$

$$\int_0^a \frac{1}{x^p}\,dx = \lim_{\alpha\to 0}\left[\frac{1}{1-p}\left(a^{1-p} - \alpha^{1-p}\right)\right].$$

$$\alpha^{1-p} = \begin{cases} \infty & \text{if } p < 1 \\ 1 & \text{if } p = 1 \\ 0 & \text{if } p > 1. \end{cases}$$

diversis. if $p > 1$.

if $p<1$: $= \int_0^a \frac{1}{x^p}\,dx$ convert to $\frac{a^{1-p}}{1-p}$

if $p=1$ $= \int_0^a \frac{1}{x^p}\,dx = \lim_{\alpha\to 0} \ln|x| \Big|_\alpha^1 = \lim_{\alpha\to 0} (\ln|1| - \underset{-\infty.}{\ln\alpha}) = \infty.$

Some times, we're interested in the convertion of a given imporper intergral.

The comparison Theriom

Suppose that we have $f(x)$, $g(x)$ that are continnous $f(x) \leq g(x)$ for any $x \geq a$. Then:

1. if $\int_a^\infty g(x)dx$ converses then so does $\int_a^\infty f(x)dx$.
2. if $\int_a^\infty f(x)dx$ diverses then so does $\int_a^\infty g(x)dx$

e.g. $\int_0^\infty e^{-x^2}dx$ is convergent.

let $I = \int_0^\infty e^{-x^2}dx$. Then:

$$I = \underbrace{\int_0^1 e^{-x^2}dx}_{I_1} + \underbrace{\int_1^\infty e^{-x^2}dx}_{I_2}.$$

I_1: e^{-x^2} is continnous on $[0,1]$. and this intergral is finite.

$I_1 = \int_0^1 e^{-x^2}$ is finite

I_2: I_2 is inproper intergral of type I.

when $x \geq 1$ $\quad x^2 \geq x$

$$e^{x^2} \geq e^x.$$

$$\frac{1}{e^{x^2}} \leq \frac{1}{e^x}$$

e.g. $\int_0^\infty \frac{dx}{\sqrt{x+x^2}}$ is convergent. or divergent

$$\int_0^\infty \frac{dx}{\sqrt{x+x^2}} = \underbrace{\int_0^1 \frac{1}{\sqrt{x+x^2}}dx}_{I_1} + \underbrace{\int_1^\infty \frac{1}{\sqrt{x+x^2}}}_{I_2}.$$

$I_1 < \int_0^1 \frac{1}{\sqrt{x}} = \left| \frac{1}{x^{\frac{1}{2}}} \right| \Rightarrow$ convergent.

$\int_0^a \frac{1}{x^p}$ $p<1 \Rightarrow$ convergent.

$I_2 = \int_1^\infty \frac{1}{\sqrt{x+x^2}}\,dx < \int_1^\infty \frac{1}{x^{\frac{3}{2}}}\,dx \Rightarrow$ convergent.