Lecture 26.

Def 4.5. $R \subseteq A \times A$
R is equlation relation
if it is symmetric, reflex and transitive.
e.g. $E_A = \{(a,a) \mid a \in A\}$.

$W = \{$ all English words $\}$

$R_1 = \{(x,y) \mid$ start with same letter $\}$. reflexive (xRx) symmetric ($xRy \to yRx$) transitive ($xRy, yRz \to xRz$)
$R_2 = \{(x,y) \mid$ have a letter in common $\}$ reflexive symmetric transitive ×
$R_3 = \{(x,y) \mid$ have same number of elements $\}$ reflexive symmetric transitive

Def 4.5.9. Fix $m \in \mathbb{Z}^+$ $x, y \in \mathbb{Z}$ x is equlation if $m \mid (x-y)$.
i.e. $\exists k \in \mathbb{Z}$, $x - y = km \Rightarrow x \equiv_m y$ or $x \equiv y \pmod m$.

e.g. $4 \equiv_3 1$, $2 \equiv_{10} 12$ ...... $5 \not\equiv_2 2$

Def. 4.5.10 For each $m \in \mathbb{Z}^+$, $\equiv_m$ is an equivalence relation on $\mathbb{Z}$.
Proof: reflexive: Let $x \in \mathbb{Z}$, then $x - x = 0$, $0 = 0 \cdot m$,
so $m \mid (x-x)$. so $x \equiv_m x$
symmetric: Let $x, y \in \mathbb{Z}$. assume $x \equiv_m y$. so $m \mid (x-y)$.
so there exist $k \in \mathbb{Z}$ $x - y = km$. $y - x = -km$.
Since $-k$ is also an interger, thus $y \equiv_m x$.
transitive: ......

Def 4.5.3. For $x \in A$, the equivalence class of $x \in R$ is.
$[x]_R = \{y \in A \mid xRy\}$.
write $[x]_R$ if R is understand.
$A/R = \{[x]_R \mid x \in A\}$.

e.g. $\equiv_3$ on $\mathbb{Z}$
$[0] = \{y \in \mathbb{Z} \mid 0 \equiv_3 y\} = \{y \in \mathbb{Z} \mid 3 \mid 0 - y\} = \{y \in \mathbb{Z} \mid 3 \mid y\}$.
which is an infinite set $\{-3, 0, 3, 6 \cdots\}$.
$[1] = \{y \in \mathbb{Z} \mid 1 \equiv_3 y\} = \{y \in \mathbb{Z} \mid 3 \mid 1 - y\} = \{y \in \mathbb{Z} \mid 3 \mid y - 1\}$.
etc ....
$[3] = [0]$, $3 \equiv_3 0$.
$\mathbb{Z}/\equiv_3 = \{[0], [1], [2]\}$.

$= \{ \{-6, -3, 0, 3, 6 \cdots\},$
$\{-5, -2, 1, 4, 7 \cdots\},$
$\{-4, -1, 2, 5, 8 \cdots\} \}.$

Notice: $x \in [x]$. $\forall x \in \mathbb{Z}$ Since it is a reflexive relation

disjoint: $[0] \cap [1] = \phi$
$[1] \cap [2] = \phi$ $\Rightarrow [a], [b]$ are either equal or disjoint.
$[2] \cap [0] = \phi$.

$[0] \cup [1] \cup [2] = \mathbb{Z}$
For $x, y \in \mathbb{Z}$, $[x] = [y]$ iff $x \equiv_3 y$ iff $y \in [x]$.

e.g. $W = \{$ English words $\}$.
$R_1 = \{$ same first letter $\}$.

$W/R_1 = \{ [ant], [bat], [cat] \cdots\cdots [zoo] \}$. $\Leftarrow$ 26 elements.
$[bat] = [boy]$ $[bat] R_1 [boy]$.

Lemma 4.5.5. $R \subseteq A \times A$, equiv reln
1. $\forall x \in A$, $x \in [x]_R$
2. $\forall x, y \in A$ $y \in [x]_R$ iff $[y] = [x]$.

Proof: 1: $x \in A$. Since R is reflexive, xRx, so $x \in [x]_R$.
2: Let $x, y \in A$, $y \in [x]$ iff xRy.
So $xRy \to [x] = [y]$.
To show $[x] \subseteq [y]$, $z \in [x] \to xRz \to zRx$ (symmetric).
$\to zRy$ (transitive).
$\to yRz$ (symmetric).
iff $z \in [y]$.