1. Matrix $A = [a_{ij}]$ $m \times n$ $i = 1, 2, \cdots, m$
$j = 1, 2, \cdots, n.$

a_{ij} is the (i, j)-entry of A.

Ex: $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

$\alpha_1 = (a_{11}, a_{12}, a_{13})$ is the first row

$\alpha_2 = (a_{21}, a_{22}, a_{23})$ is the second row.

$\beta_1 = (a_{11}, a_{21})$ is the first column

$\beta_2 = (a_{12}, a_{22})$ is the second column

$\beta_3 = (a_{13}, a_{23})$ is the third column.

2. Row-reduced echelon form (RREF)

① "leading 1": For a row that does not contain entirely zeros, the first non zero number is 1.

looks like $0 \ 0 \cdots 0 \ 1 \ * \ * \ *$

② "zero off": For a column that contains a leading one, all other entries are zero.

③ "staircase": All leading ones form a shape of stairs down to the right.

$$\begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ 0 & 0 & 1 & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

④ Any rows that are entirely zero are placed at the bottom.

3. RREF and solutions of an SLE

Let $[A|\vec{b}]$ be an augmented matrix for an SLE.

$$[A|\vec{b}] \xrightarrow{\text{RREF}} [R|\vec{\beta}]$$

R is an RREF of A.

1) First observation in $[R|\vec{\beta}]$: $0\ 0\ 0 \cdots 0 \mid \underset{\text{nonzero}}{*}$

If there exists a row $0\ 0 \cdots 0 \mid \underset{\text{nonzero}}{*}$, then no solution.

2) If NOT:

- Count the number of leading ones in R.

If the number $r(A)$ of leading ones in R equals to the number n of variables, then it has unique solution.

If the number of leading ones in R is less than n,

Conclusion: The number of parameters of solutions is $n - r(A)$.

ex:

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$

$$\left[\begin{array}{ccccc|c} 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$x_1 = s$ $\qquad$ $x_3 + x_4 = 1$

$x_2 = t$ $\qquad$ $x_3 = -w + 1$

$x_4 = w$ $\qquad$ $x_5 = 2$

$(s, t, -w+1, w, 2)$

III Matrix operations.

addition $\quad A + B \quad A, B : m \times n$

subtraction $\quad A - B \quad A, B \quad m \times n$

scalar multiple $\quad cA = [ca_{ij}]$

matrix multiplication $\quad AB \quad A : m \times n$

$B : n \times s$

and $AB : m \times s$ matrix

Ex: $A : 3 \times 2$ $\qquad A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 3 & 4 \end{bmatrix}$ $\qquad B = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 2 & 1 & 1 \end{bmatrix}$

$B : 2 \times 4$

AB

$\alpha_1 = (1,2)$

$\alpha_2 = (1,0)$ are rows of A

$\alpha_3 = (3,4)$

$\beta_1 = (1,0)$

$\beta_2 = (-1,2)$ are columns of B

$\beta_3 = (0,1)$

$\beta_4 = (1,1)$

$C = AB = [c_{ij}]$ $\quad c_{ij} = \alpha_i \cdot \beta_j$

$c_{11} = \alpha_1 \cdot \beta_1 = (1,2) \cdot (1,0) = 1$

$c_{12} = \alpha_1 \cdot \beta_2 = (1,2) \cdot (-1,2) = 1\times(-1) + 2\times 2 = 3$

$c_{13} = \alpha_1 \cdot \beta_3 = (1,2) \cdot (0,1) = 2$

$\vdots$

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \end{bmatrix}$$