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Tutorial 05:

Floating-point Numbers

Computer Science Department

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Example of Decimal to IEEE-754 Floating-point Conversion

- **Example 1:** Convert $5.877472_{10} \times 10^{-39}$ into a 32-bit single-precision IEEE-754 FP value.

$$\log_2(10) = 1 / \log_{10}(2)$$

$$\begin{aligned} 10^{-39} &= 2^z \rightarrow \log_2(10^{-39}) = z \rightarrow -39 \times \log_2(10) = z \rightarrow z = -129.5551957 \\ 10^{-39} &= 2^{-129.5551957} = 2^{-129} \times 2^{-0.5551957} = 2^{-129} \times 0.680564734_{10} \\ 5.877472_{10} \times 10^{-39} &= 5.877472_{10} \times 0.680564734_{10} \times 2^{-129} \\ &= 4_{10} \times 2^{-129} = 2^2 \times 2^{-129} = 2^{-127} = 1_2 \times 2^{-127} \end{aligned}$$

- Convert 1_2 into a fixed-point binary
 - $1_2 = 1.0_2$ (*already normalized*)
- True exponent is less than -126 → underflow case
 - The exponent needs to be -126: $-127_{10} = -126 - 1$
 - Hence, the significant needs to be adjusted to compensate the -1
 - After moving the radix point backward by 1 position → 0.1_2
i.e., $1.0_2 \times 2^{-127} = 0.1_2 \times 2^{-126}$
 - After Taking 23 bits → $0.100\ 0000\ 0000\ 0000\ 0000\ 0000_2$
- The sign bit, S, is 0 because the number is positive
- The final number is **0000 0000 0100 0000 0000 0000 0000 0000** or 00400000_{16}

Example of Decimal to IEEE-754 Floating-point Conversion

- **Example 2:** Convert $9.0_{10} \times 10^{-44}$ into a 32-bit single-precision IEEE-754 FP value.

$$\log_2(10) = 1 / \log_{10}(2)$$

$$10^{-44} = 2^z \rightarrow \log_2(10^{-44}) = z \rightarrow -44 \times \log_2(10) = z \rightarrow z = -146.164836175$$

$$10^{-44} = 2^{-146.164836175} = 2^{-146} \times 2^{-0.164836175} = 2^{-146} \times 0.892029808_{10}$$

$$9.0_{10} \times 10^{-44} = 9.0_{10} \times 0.892029808_{10} \times 2^{-146} = 8.028268272_{10} \times 2^{-146}$$

- Convert 8.028268272_{10} into a fixed-point binary

- $8_{10} = 1000_2$ and
- $0.028268272_{10} = 0.00000111001111001001..._2$
- Therefore, $8.028268272_{10} = 1000.00000111001111001001..._2$

- **Normalization:** $9.0_{10} \times 10^{-44} = 8.028268272_{10} \times 2^{-146} = 1000.00000111001111001001..._2 \times 2^{-146} = 1.00000000111001111001001..._2 \times 2^{-143}$

- **True exponent is less than -126 → underflow case**

- **The exponent needs to be -126:** $-143_{10} = -126 - 17$
- **Hence, the significant needs to be adjusted to compensate the -17**
- After moving the radix point backward by 17 position **Rounded to the nearest FP**
 $\rightarrow 0.00000000000000010000000111001111001001..._2$
- After Taking only 23 bits $\rightarrow 0.0000000000000000100000000111..._2$

- The sign bit, **S**, is 0 because the number is positive

- The final number is **0000 0000 0000 0000 0000 0000 0100 0000** or 00000040_{16}

Example of Decimal to IEEE-754 Floating-point Conversion

□ **Example 3:** Convert 3.6_{10} into a 32-bit single-precision IEEE-754 FP value.

- Convert 3.6_{10} into a fixed-point binary
 - $3_{10} = 11_2$ and
 - $0.6_{10} = 0.1001\ 1001\ \dots_2$
 - Therefore, $3.6_{10} = 11.1001\ 1001\ \dots_2$
- Normalize $11.1001\ 1001\ \dots_2$ to $1.11001\ 1001\ \dots_2 \times 2^1$.
- The sign bit, **S**, is 0 because the number is positive
- The **biased exponent** is the **true exponent** plus 127; that is, $1 + 127 = 128_{10} = 1000\ 0000_2$
- The fractional significand is **110 0110 0110 0110 0110 0110 0110 0110** ...
 - *the leading 1 was stripped* and
 - *to be rounded to 23 bits (rounded to nearest FP number).*
- The final number is **0100 0000 0110 0110 0110 0110 0110 0110**, or 40666666_{16} . $\rightarrow 3.5999999046325684_{10}$

$0.6 \times 2 = 1.2$
$0.2 \times 2 = 0.4$
$0.4 \times 2 = 0.8$
$0.8 \times 2 = 1.6$
$0.6 \times 2 = 1.2$
...

Example of Decimal to IEEE-754 Floating-point Conversion

□ Example 4:

Convert 16777216.75_{10} into a *32-bit single-precision IEEE-754 FP* value.

- Convert 16777216.75_{10} into a fixed-point binary
 - $16777216_{10} = 1\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000_2$ and
 - $0.75_{10} = 0.11_2$.
 - Therefore, $16777216.75_{10} = 1\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000.11_2$.
- Normalize $1\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000.11_2$ to $1.0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 11_2 \times 2^{24}$.
- The sign bit, **S**, is 0 because the number is positive
- The *biased exponent* is the *true exponent* plus 127; that is, $24 + 127 = 151_{10} = 1001\ 0111_2$
- The fractional significand is **000 0000 0000 0000 0000 0000 011**
 - *the leading 1 was stripped* and
 - *to be rounded to 23 bits (rounded to nearest FP number).*
- The final number is **0100 1011 1000 0000 0000 0000 0000 0000**, or $4B800000_{16} \rightarrow 16777216_{10}$ (*i.e., there is 0.75 rounding error*)

Example of Decimal to IEEE-754 Floating-point Conversion

□ Example 5:

Convert 16777219_{10} into a 32-bit single-precision IEEE-754 FP value.

- Convert 16777219_{10} into a fixed-point binary
 - $16777219_{10} = 1\ 0000\ 0000\ 0000\ 0000\ 0000\ 0011_2$ and
- Normalize $1\ 0000\ 0000\ 0000\ 0000\ 0000\ 0011_2$ to $1.0000\ 0000\ 0000\ 0000\ 0000\ 0011_2 \times 2^{24}$.
- The sign bit, S , is 0 because the number is positive
- The *biased exponent* is the *true exponent* plus 127; that is, $24 + 127 = 151_{10} = 1001\ 0111_2$
- The fractional significand is $000\ 0000\ 0000\ 0000\ 0000\ 0001\ 1000\ 0000\ 0000\ 0000\ 0000\ 0010$
 - *the leading 1 was stripped* and
 - *to be rounded to 23 bits (rounded to nearest FP number).*
- The final number is $0100\ 1011\ 1000\ 0000\ 0000\ 0000\ 0000\ 0010$, or $4B800002_{16} \rightarrow 16777220_{10}$ (*i.e., there is 1.0 rounding error*)

Mid-way →
round to even
significand

Example of IEEE-754 FP to Decimal to IEEE-754 FP Conversion

□ Example 6:

Convert $4B800002_{16}$ from the *32-bit single-precision IEEE-754 FP* representation into decimal representation. **Then** add 1.0_{10} to the result. And **finally** convert it back to the *32-bit single-precision IEEE-754 FP* representation.

- Convert the hexadecimal number ($4B800002_{16}$) into binary form

3	3	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0		
1	0	9	8	7	6	5	4	3	2	1	0	9	8	7	6	5	4	3	2	1	0	9	8	7	6	5	4	3	2	1	0
0	1	0	0	1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	

- Unpack the number into *sign bit*, *biased exponent*, and *fractional significand*.
 - $S = 0$
 - $E = 1001\ 0111$
 - $F = 000\ 0000\ 0000\ 0000\ 0000\ 0010$
- As the sign bit is 0, the number is positive.
- We subtract 127 from the *biased exponent* $1001\ 0111_2$ to get the *true exponent* $\rightarrow 1001\ 0111_2 - 0111\ 1111_2 = 0001\ 1000_2 = 24_{10}$.
- The fractional significand is $.000\ 0000\ 0000\ 0000\ 0000\ 0010_2$.
- Reinserting the leading one gives $1.000\ 0000\ 0000\ 0000\ 0000\ 0010_2$.
- The number is $+(1 + 2^{-22}) \times 2^{24} = 2^{24} + 2^2 = 1024_{10} \times 1024_{10} \times 16_{10} + 4_{10}$
 $= 16777216_{10} + 4_{10} = 16777220_{10}$

Example of IEEE-754 FP to Decimal to IEEE-754 FP Conversion

□ Example 6 (continuation):

Adding 1.0_{10} to the result $\rightarrow 16777220_{10} + 1.0_{10} = 16777221_{10}$

Converting the result back to the *32-bit single-precision IEEE-754 FP* format

- Convert 16777221_{10} into a fixed-point binary
 - $16777221_{10} = 1\ 0000\ 0000\ 0000\ 0000\ 0000\ 0101_2$ and
- Normalize $1\ 0000\ 0000\ 0000\ 0000\ 0000\ 0101_2$ to $1.0000\ 0000\ 0000\ 0000\ 0000\ 0101_2 \times 2^{24}$.
- The sign bit, *S*, is 0 because the number is positive
- The *biased exponent* is the *true exponent* plus 127; that is, $24 + 127 = 151_{10} = 1001\ 0111_2$
- The fractional significand is $000\ 0000\ 0000\ 0000\ 0000\ 0010\ 1$
 - *the leading 1 was stripped* and
 - *to be rounded to 23 bits (rounded to nearest FP number).*
- The final number is $0100\ 1011\ 1000\ 0000\ 0000\ 0000\ 0000\ 0010$, or $4B800002_{16} \rightarrow 16777220_{10}$

Mid-way \rightarrow
round to even
significand

$$16777220_{10} + 1.0_{10} = 16777220_{10}!!!$$

(This is due to the rounding error)

This is the same FP number
that we started with!!

Example of IEEE-754 FP to Decimal to IEEE-754 FP Conversion

□ Example 6 (continuation):

- Run the following C program to verify **Example 6**:

```
#include <stdio.h>
int main()
{
    float f = 16777220, ff;
    ff = f + 1;
    printf("%f %f \n", f, ff);
}
```

The output will be:

16777220.000000 16777220.000000



Change the “float” to “int” and the “%f” to “%d” and repeat executing the program again.

The output after the “float” to “int” change will be:

16777220 16777221



Change the “float” to “double” and the “%f” to “%lf” and repeat executing the program again.

The output after the “float” to “double” change will be:

16777220.000000 16777221.000000



Example of IEEE-754 Floating-point to Decimal Conversion

□ **Example 7:** Convert 00200000_{16} from 32-bit single-precision IEEE-754 FP value into a decimal value.

- Convert the hexadecimal number (00200000_{16}) into binary form

3	3	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
1	0	9	8	7	6	5	4	3	2	1	0	9	8	7	6	5	4	3	2	1	0	9	8	7	6	5	4	3	2	1	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	

- Unpack the number into *sign bit*, *biased exponent*, and *fractional significand*.
 - $S = 0$
 - $E = 0000\ 0000$
 - $F = 010\ 0000\ 0000\ 0000\ 0000\ 0000$
- As the sign bit is 0, the number is positive.
- We subtract 126 from the *biased exponent* 0_2 to get the *true exponent* $\rightarrow 0_2 - 0111\ 1110_2 = -126_{10}$.
As the true exponent is -126, then the F is not normalized
- The fractional significand is $.010\ 0000\ 0000\ 0000\ 0000\ 0000_2$.
- The number is $.01_2 \times 2^{-126} = 2^{-2} \times 2^{-126} = 2^{-128}$

We are subtracting 126, not 127, from the biased exponent, because the biased exponent = 0.

$$\begin{aligned}
 2^{-128} &= 10^z \rightarrow \log_{10}(2^{-128}) = z \rightarrow z = -38.53183944 \\
 2^{-128} &= 10^{-38.53183944} = 10^{-38} \times 10^{-0.53183944} = 10^{-38} \times 0.293873587 \\
 2^{-128} &= 0.293873587 \times 10^{-38} = 2.9387358 \times 10^{-39}
 \end{aligned}$$

Final Word!!

- ❑ **How can you verify your FP conversion results?**
- ❑ There are many online converters between IEEE FP format to float and vice versa.
 - For example, <https://www.h-schmidt.net/FloatConverter/IEEE754.html>