8. (x)Gaxb
∴ (∃x)(∃y)Gxcy

9. (x)(y)Lxy
∴ (∃x)Lax

10. Lab
Lbc
∴ (∃x)(Lax · Lxc)

11. (x)Lxx
∴ (x)(y)(Lxy ⊃ x=y)

12. (∃x)Lxa
~Laa
∴ (∃x)(~a=x · Lxa)

13. (x)(y)(z)((Lxy · Lyz) ⊃ Lxz)
(x)(y)(Kxy ⊃ Lyx)
∴ (x)Lxx

14. (x)Lxa
(x)(Lax ⊃ x=b)
∴ (x)Lxb

15. (x)(y)(Lxy ⊃ (Fx · ~Fy))
∴ (x)(y)(Lxy ⊃ ~Lyx)

6.4b Exercise—LogiCola I (RC & BC)

First appraise intuitively. Then translate into logic and say whether valid (and give a proof) or invalid (and give a refutation).

1. Juliet loves everyone.
∴ Someone loves you. [Use Lxy, j, and u.]
2. Nothing caused itself.
∴ There's nothing that caused everything. [Use Cxy.]
3. Alice is older than Betty.
∴ Betty isn't older than Alice. [Use Oxy, a, and b. What premise implicit would make this valid?]
4. There's something that everything depends on.
∴ Everything depends on something. [Use Dxy.]
5. Everything depends on something.
∴ There's something that everything depends on. [Use Dxy.]
6. Romeo loves all females. x Lrx ∃x(Lrx · ~Lxr).
No females love Romeo. ~∃xLxr
Juliet is female. Fj
∴ Romeo loves someone who doesn't love him. [Use Lxy, r, Fx, and j.]
7. In all cases, if a first thing caused a second then the first exists before the second.
Nothing exists before it exists.
∴ Nothing caused itself. [Use Cxy and Bxy (for "x exists before y exists").]
8. Everyone hates my enemy.
My enemy doesn't hate anyone besides me.
∴ My enemy is me. [Use Hxy, e, and m.]
9. Not everyone loves everyone.
∴ Not everyone loves you. [Use Lxy and u.]

10. There's someone that everyone loves.
∴ Some love themselves. [Use Lxy.]
11. Andy shaves all and only those who don't shave themselves.
∴ It is raining. [Use Sxy, a, and R.]
12. No one hates themselves.
I hate all logicians.
∴ I am not a logician. [Use Hxy, i, and Lx.]
13. Juliet loves everyone besides herself.
Juliet is Italian.
Romeo is my logic teacher.
My logic teacher isn't Italian.
∴ Juliet loves Romeo. [Use j, Lxy, Ix, r, and m.]
14. Romeo loves either Lisa or Colleen.
Romeo doesn't love anyone who isn't Italian.
Colleen isn't Italian.
∴ Romeo loves Lisa. [Use Lxy, r, 1, and c.]
15. Everyone loves all lovers.
Romeo loves Juliet.
∴ I love you. [Use Lxy, r, j, i, and u. This one is difficult.]
16. Everyone loves someone.
∴ Some love themselves. [Use Lxy.]
17. Nothing caused itself.
This chemical brain process caused this pain.
∴ This chemical brain process isn't identical to this pain. [Use Cxy, b, and p.]
18. For every positive contingent truth, something explains why it's true.
The existence of the world is a positive contingent truth.
If something explains the existence of the world, then some necessary being explains the existence of the world.
∴ Some necessary being explains the existence of the world. [Cx, Exy, e, and NX. This argument for the existence of God is from Richard Taylor.]
19. That girl is Miss Novak.
∴ If you don't like Miss Novak, then you don't like that girl. [Use t, m, u, and Lxy. This is from the movie, *The Little Shop around the Corner:* "If you don't like Miss Novak, I can tell you right now that you won't like that girl. Why? Because it is Miss Novak."]
20. Everyone who is wholly good prevents every evil that he can prevent.
Everyone who is omnipotent can prevent every evil.
If someone prevents every evil, then there's no evil.
There's evil.
∴ Either God isn't omnipotent, or God isn't wholly good. [Use Gx, Ex, Cxy (for "x can prevent y"), Pxy (for "x prevents y"), Ox, and g. This argument is from J.L.Mackie.]

21. Your friend is wholly good.
Your knee pain is evil.
Your friend can prevent your knee pain.
Your friend doesn't prevent your knee pain (since he could prevent it only by amputating your leg—which would bring about a worse situation).
∴ "Everyone who is wholly good prevents every evil that he can prevent" is false. [Use f, Gx, k, Ex, Cxy, and Pxy. Alvin Plantinga thus attacked premise 1 of the previous argument; he proposed instead roughly this: "Everyone who is wholly good prevents every evil that he knows about if he can do so without thereby eliminating a greater good or bringing about a greater evil."]

22. For everything contingent, there's some time at which it fails to exist.
∴ If everything is contingent, then there's some time at which everything fails to exist. [Use Cx for "x is contingent"; Ext for "x exists at time t"; t for a time variable; and t′, t″, t‴,…for time constants. This is a critical step in St Thomas Aquinas's third argument for the existence of God.]

23. If everything is contingent, then there's some time at which everything fails to exist.
If there's some time at which everything fails to exist, then there's nothing in existence now.
There's something in existence now.
Everything that isn't contingent is necessary.
∴ There's a necessary being. [Besides the letters for the previous argument, use Nx for "x is necessary" and n for "now." This continues St Thomas's argument; here premise 1 is from the previous argument.]

24. [The great logician Gottlob Frege tried to systematize mathematics. One of his axioms said that *every sentence with a free variable*[1] *determines a set*. So then "x is blue" determines a set: there's a set y containing all and only blue things. While this seems sensible, Bertrand Russell showed that Frege's axiom entails that "x doesn't contain x" determines a set—so there's a set y containing all and only those things that don't contain themselves—and this leads to the self-contradiction "y contains y if and only if y doesn't contain y." The foundations of mathematics haven't been the same since "Russell's paradox."]
If every sentence with a free variable determines a set, then there's a set y such that, for all x, y contains x if and only if x doesn't contain x.
∴ Not every sentence with a free variable determines a set. [Use D for "Every sentence with a free variable determines a set," Sx for "x is a set," and Cyx for "y contains x."]

25. All dogs are animals.
∴ All heads of dogs are heads of animals. [Use Dx, Ax, and Hxy (for "x is a head of y"). Translate "x is a head of a dog" as "for some y, y is a dog and x is a head of y." Augustus DeMorgan in the nineteenth century claimed that this was a valid argument that traditional logic couldn't validate.]

[1] An instance of a variable is "free" in a wff if it doesn't occur as part of a wff that begins with a quantifier using that variable; each instance of "x" is free in "Fx" but not in "(x)Fx."

6.5 Definite descriptions

Phrases of the form "the so and so" are called **definite descriptions,** since they're meant to pick out a definite (single) person or thing. This final section sketches Bertrand Russell's influential ideas on definite descriptions. While philosophical discussions about these and about proper names can get complex and controversial, I'll try to keep things fairly simple.

Consider these two sentences and how we've been symbolizing them:

Socrates is bald.	The king of France is bald.
Bs	Bk

The first sentence has a proper name ("Socrates") while the second has a definite description ("the king of France"); both seem to ascribe a *property* (baldness) to a particular *object* or entity. Russell argued that this object-property analysis is misleading in the second case;[1] sentences with definite descriptions (like "the king of France") were in reality more complicated and should be analyzed in terms of a complex of predicates and quantifiers:

The king of France is bald.
=There's exactly one king of France, and he is bald.
=For some x:x is king of France, there's no y such that y≠x and y is king of France, and x is bald.
= (∃x)((Kx • ~(∃y)(~y=x • Ky)) • Bx)

Russell saw his analysis as having several advantages; I'll mention two.

First, "The king of France is bald" might be false for any of three reasons:

1. There's no king of France;
2. there's more than one king of France; or
3. there's exactly one king of France, and he has hair on his head.

In fact, "The king of France is bald" is false for reason 1: France is a republic and has no king. This accords well with Russell's analysis. By contrast, the object-property analysis suggests that if "The king of France is bald" is false, then "The king of France isn't bald" would have to be true[2]—and so the king of France would have to have hair! So Russell's analysis seems to express better the logical complexity of definite descriptions.

[1] He also thought the analysis misleading in the first case; but I don't want to discuss this now.

[2] On Russell's analysis, "The king of France isn't bald" is false too—since it means "There's exactly one king of France, and he isn't bald."

Second, the object-property analysis of definite descriptions can lead us into metaphysical errors, like positing existing things that aren't real. The philosopher Alexius Meinong argued roughly as follows:

"The round square does not exist" is a true statement about the round square.
If there's a true statement about something, then that something has to exist.
∴ The round square exists.
But the round square isn't a real thing.
∴ Some things that exist aren't real things.

For a time, Russell accepted this argument. Later he came to see the belief in non-real existing things as foolish; he rejected Meinong's first premise and appealed to the theory of descriptions to clear up the confusion.

According to Russell, Meinong's error comes from his naïve object-property understanding of the following statement:

The round square does not exist.

This, Russell contended, isn't a true statement ascribing non-existence to some object called "the round square." If it were a true statement about the round square, then the round square would have to exist—which the statement denies. Instead, the statement just denies that there's exactly one round square. So Russell's analysis keeps us from having to accept that there are existing things that aren't real.

CHAPTER 7
Basic Modal Logic

Modal logic studies arguments whose validity depends on "necessary," "possible," and similar notions. This chapter covers the basics, and the next gets into further modal systems.

7.1 Translations

To help us evaluate modal arguments, we'll construct a little modal language. For now, our language will build on propositional logic, and thus include all the vocabulary, wffs, inference rules, and proofs of the latter. Our language adds two new vocabulary items: "◇" and "□" (diamond and box):

◇A	=	It's possible that A	=	A is true in some possible world.
A	=	It's true that A	=	A is true in the actual world.
□A	=	It's necessary that A	=	A is true in all possible worlds.

Calling something *possible* is a weak claim—weaker than calling it *true*. Calling something *necessary* is a strong claim; it says, not just that the thing is true, but that it *has* to be true—it *couldn't* be false.

"Possible" here means *logically possible (not self-contradictory)*. "I run a mile in two minutes" may be physically impossible; but there's no self-contradiction in the idea, so it's logically possible. Likewise, "necessary" means *logically necessary (self-contradictory to deny)*. "2+2=4" and "All bachelors are unmarried" are examples of **necessary truths;** such truths are based on logic, the meaning of concepts, or necessary connections between properties.

We can rephrase "possible" as *true in some possible world*—and "necessary" as *true in all possible worlds*. A **possible world** is a consistent and complete[1] description of how things might have been or might in fact be. Picture a possible world as a *consistent story* (or novel). The story is *consistent,* in that its statements don't entail self-contradictions; it describes a set of possible situations that are all possible together. The story may or may not be true. The **actual world** is the story that's true—the description of how things in fact are.

[1] Since we are finite beings, we will in practice only give partial (not "complete") descriptions.

As before, a grammatically correct formula is called a *wff,* or *well-formed formula*. For now, wffs are strings that we can construct using the propositional rules plus this additional rule:

1. The result of writing "◇" or "□," and then a wff, is a wff.

Don't use parentheses with "◇A" and "□A":

Right:	◇A	□A
Wrong:	◇(A) (◇A)	□(A) (□A)

Parentheses here would serve no purpose.

Now we'll focus on how to translate English sentences into modal logic. Here are some simpler examples:

A is possible (consistent, could be true) = ◇A
A is necessary (must be true, has to be true) = □A

A is impossible (self-contradictory) = ~◇A = A couldn't be true.
= □~A = A has to be false.

An impossible statement (like "2≠2") is one that's false in every possible world.

These examples are more complicated:

A is consistent (compatible) with B = It's possible that A and B are both true.
= ◇(A·B)

A entails B = It's necessary that if A then B.
= □(A⊃B)

"Entails" makes a stronger claim than plain "if-then." Compare these two:

"There's rain" entails "There's precipitation" = □(R⊃P)
If it's Saturday, then I don't teach class = (S⊃~T)

The first if-then is logically necessary; every conceivable situation with rain also has precipitation. The second if-then just happens to be true; we can consistently imagine me teaching on Saturday—even if in fact I never do.

These common forms negate the whole wff:

A is inconsistent with B = It's not possible that A and B are both true.
= ~◇(A·B)

A doesn't entail B = It's not necessary that if A then B.
= ~□(A⊃B)

Here is how we translate "contingent":

A is a contingent statement	=	A is possible and not-A is possible.
	=	$(\Diamond A \cdot \Diamond{\sim}A)$
A is a contingent truth	=	A is true but could have been false.
	=	$(A \cdot \Diamond{\sim}A)$

Statements are necessary, impossible, or contingent. But truths are only necessary or contingent (since impossible statements are false).

When translating, it's usually good to mimic the English word order:

necessary not	=	$\Box{\sim}$	necessary if	=	$\Box($
not necessary	=	${\sim}\Box$	if necessary	=	$(\Box$

Use a separate box or diamond for each "necessary" or "possible":

If A is necessary and B is possible, then C is possible=$((\Box A \cdot \Diamond B) \supset \Diamond C)$

The following tricky English forms are ambiguous; translate these into two modal wffs, and say that the English could mean one or the other:

"If A is true, then it's necessary (must be) that B" could mean "$(A \supset \Box B)$" or "$\Box(A \supset B)$."	*"If A is true, then it's impossible (couldn't be) that B"* could mean "$(A \supset \Box{\sim}B)$" or "$\Box(A \supset {\sim}B)$."

So this next sentence could have either of the following two meanings:

"If you're a bachelor, then you must be unmarried."

$(B \supset \Box U)$ = "If you're a bachelor, then you're *inherently unmarriable* (in no possible world would anyone ever marry you)."
= If B, then U (by itself) is necessary.

$\Box(B \supset U)$ = "It's necessary that *if* you're a bachelor *then* you're unmarried."
= It's necessary that if B then U.

The box-inside "$(B \supset \Box U)$" posits an *inherent necessity*, given that the antecedent is true, "You're unmarried" is inherently necessary. This version is insulting and presumably false. The box-outside "$\Box(B \supset U)$" posits a *relative necessity*, what is necessary is, not "You're a bachelor" or "You're unmarried" by itself, but only the connection between the two. This version is trivially true because "bachelor" means *unmarried man.*

The medievals called the box-inside form the "necessity of the *consequent*" (the second part being necessary); they called the box-outside form the "necessity of the *consequence*" (the whole if-then being necessary). The ambiguity is important philosophically; several intriguing but fallacious philosophical arguments depend on the ambiguity for their plausibility.

It's not ambiguous if you say the second part "by itself" is necessary or impossible—or if you use "entails" or start with "necessary." These forms aren't ambiguous:

If A, then B (by itself) is necessary	=	$(A \supset \Box B)$
A entails B	=	$\Box(A \supset B)$
Necessarily, if A then B	=	$\Box(A \supset B)$
It's necessary that if A then B	=	$\Box(A \supset B)$
"If A then B" is a necessary truth	=	$\Box(A \supset B)$

The ambiguous forms have if-then with a strong modal term (like "necessary," "must," "impossible," or "can't") in the then-part, like these:[1]

If A, then it's necessary that B.
If A, then it must be that B.

If A, then it's impossible that B.
If A, then it can't be that B.

When you translate an ambiguous English sentence, say that it's ambiguous and give both translations. When you do an English argument with an ambiguous statement, give both translations and work out both arguments.

7.1a Exercise—LogiCola J (BM & BT)

Using these equivalences, translate these English sentences into wffs. Be sure to translate ambiguous forms both ways.

G	=	There s a God (God exists)
E	=	There's evil (Evil exists)
M	=	There's matter (Matter exists)
R	=	There s rain
P	=	There's precipitation

"God exists and evil doesn't exist" entails "There's no matter."	$\Box((G \cdot {\sim}E) \supset {\sim}M)$

1. It's necessary that God exists.
2. "There's a God" is self-contradictory.
3. It isn't necessary that there's matter.
4. It's necessary that there's no matter.
5. "There's rain" entails "There's precipitation."
6. "There's precipitation" doesn't entail "There's rain."
7. "There's no precipitation" entails "There's no rain."
8. If rain is possible, then precipitation is possible.

[1] There's an exception to this rule: if the if-part is a claim about necessity or possibility, then just use the box-inside form. So "If A is necessary then B is necessary" is just "$(\Box A \supset {\sim}\Box B)$"—and "If A is possible then B is impossible" is just "$(\Diamond A \supset {\sim}\Diamond B)$."

9. God exists.
10. If there's rain, then there must be rain.
11. It isn't possible that there's evil.
12. It's possible that there's no evil.
13. If there's rain, then it's possible that there's rain.
14. "There's matter" is compatible with "There's evil."
15. "There's a God" is inconsistent with "There's evil."
16. Necessarily, if there's a God then there's no evil.
17. If there's a God, then there can't be evil.
18. If there must be matter, then there's evil.
19. Necessarily, if there's a God then "There's evil" (by itself) is self-contradictory.
20. It's necessary that either there's a God or there's matter.
21. Either it's necessary that there's a God or it's necessary that there's matter.
22. "There's rain" is a contingent statement.
23. "There's rain" is a contingent truth.
24. "If there's rain, then there's evil" is a necessary truth.
25. If there's rain, then "There's evil" (by itself) is logically necessary.
26. If there's rain, then it's necessary that there's evil.
27. It's necessary that it's possible that there's matter.
28. "There's a God" isn't a contingent truth.
29. If there's a God, then it must be that there's a God.
30. It's necessary that if there's a God then "There's a God" (by itself) is necessary.

7.2 Proofs

Modal proofs work much like propositional proofs; but we need to add possible worlds and four new inference rules.

A **world prefix** is a string of zero or more instances of "W." So " " (zero instances), "W," "WW," and so on are world prefixes; these represent possible worlds, with the blank world prefix (" ") representing the actual world. A *derived step* is now a line consisting of a world prefix and then "∴" and then a wff. And an *assumption* is now a line consisting of a world prefix and then "asm:" and then a wff. Here are examples of derived steps and assumptions:

∴A	(So A is true in the actual world.)	asm: A	(Assume A is true in the actual world.)
W ∴ A	(So A is true in world W.)	W asm: A	(Assume A is true in world W.)
WW ∴ A	(So A is true in world WW.)	WW asm: A	(Assume A is true in world WW.)

Seldom do we need to assume something in another world.

We'll still use the S- and I-rules and RAA in modal proofs. Unless otherwise specified, we can use an inference rule only within a given world; so if we have