Lecture 6

September 21, 2020 10:25 AM

Definitions: A function $f(x,y)$ has a local maximum at (a,b) if

$$f(x,y) \leq f(a,b)$$

when (x,y) is near (a,b) and $f(a,b)$ is a local maximum value.

Similarly, if $f(a,b) \leq f(x,y)$ when (x,y) is near (a,b) then f has a local minimum at (a,b) and $f(a,b)$ is a local minimum value.

Theorem: If f has a local maximum or minimum at (a,b) then $\nabla f(a,b) = \vec{0}$.

Proof

Let $g(t) = f(a+ht, b+kt)$ and we note that

$$g(0) = f(a,b)$$

Then $g(t)$ has a local max or a local min at $t=0$ if

$$\boxed{g'(0) = 0}$$

$$g'(t) = h f_x(a+ht, b+kt) + k f_y(a+ht, b+kt)$$

Let $t=0$, we have

$$h f_x(a,b) + k f_y(a,b) = 0$$

$$\underbrace{[f_x(a,b)\hat{i} + f_y(a,b)\hat{j}]}\cdot[\underbrace{h\hat{i}+k\hat{j}}] = 0$$

$$\nabla f(a,b)\cdot(h,k) = 0$$

Since (h,k) is an arbitrary vector, we must have

$$\nabla f(a,b) = \vec{0} \text{ or } f_x(a,b) = 0 \text{ and } f_y(a,b) = 0. \quad /QED/$$

More definitions

(i) A critical point (a,b) of f is a point satisfying

$$\nabla f(a,b) = \vec{0}, \text{ ie, } f_x(a,b) = 0 \text{ and } f_y(a,b) = 0$$

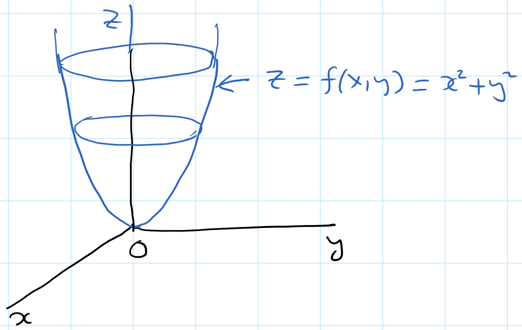
(ii) A singular point (a,b) of f is a point where $\nabla f(a,b)$ is undefined (ie, Does Not Exist).

Ex1: Let $f(x,y) = x^2 + y^2$. Then

$$f_x = 2x \quad \text{and} \quad f_y = 2y$$

To obtain the CPs, set $f_x = 0$ and $f_y = 0$

$$\begin{cases} 2x = 0 \\ 2y = 0 \end{cases} \Rightarrow x = 0,\ y = 0 \Rightarrow (0,0) \text{ is a CP.}$$



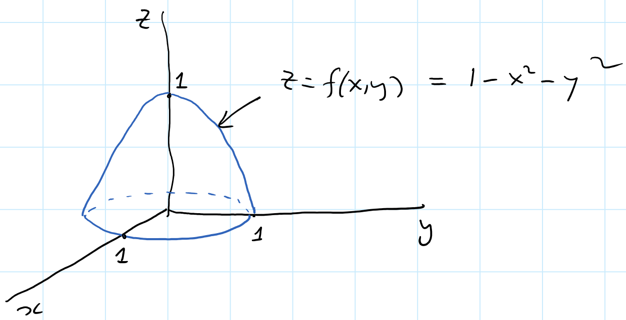
We note that f has a local minimum value which is 0 at $(0,0)$.

Ex2: Let $f(x,y) = 1 - x^2 - y^2$. Then

$$f_x = -2x \quad \text{and} \quad f_y = -2y$$

To obtain the CPs, set $f_x = 0$ and $f_y = 0$

$$\begin{cases} -2x = 0 \\ -2y = 0 \end{cases} \Rightarrow \begin{matrix} x = 0 \\ y = 0 \end{matrix} \Rightarrow (0,0) \text{ is a CP}$$



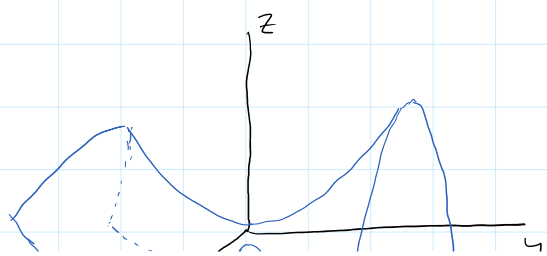
We note that f has a local maximum value at $(0,0)$.

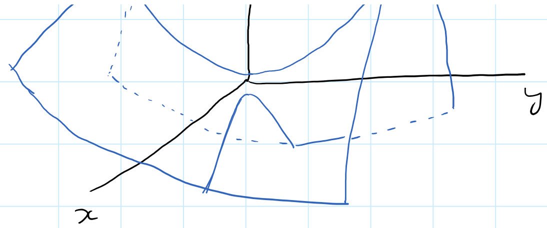
Ex3: Let $f(x,y) = y^2 - x^2$. Then

$$f_x = -2x \quad \text{and} \quad f_y = 2y$$

To obtain the CPs, set $f_x = 0$ and $f_y = 0$

$$\begin{cases} -2x = 0 \\ 2y = 0 \end{cases} \Rightarrow \begin{matrix} x = 0 \\ y = 0 \end{matrix} \Rightarrow (0,0) \text{ is a CP.}$$





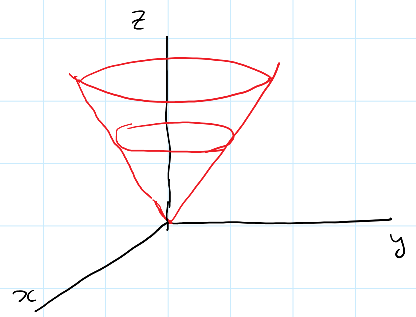
We note that f has neither a local max nor a local min at $(0,0)$. The point $(0,0)$ is called a saddle point.

Ex 4 : Consider $f(x,y) = \sqrt{x^2+y^2}$. Then

$$f_x = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}(\cancel{2}x) = \frac{x}{\sqrt{x^2+y^2}}$$

$$f_y = \frac{y}{\sqrt{x^2+y^2}}$$

Both f_x and f_y are undefined at $(0,0)$. Hence, $(0,0)$ is a singular point (SP).

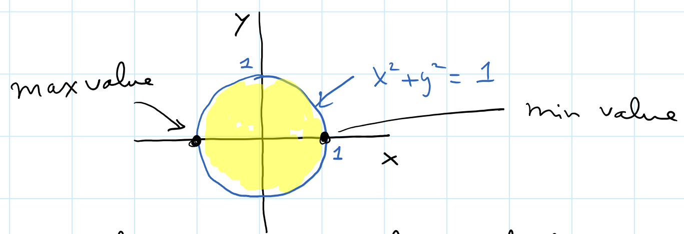


We note that f has a local minimum value at $(0,0)$.

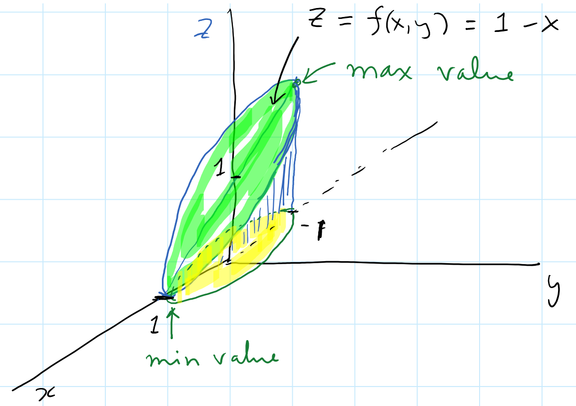
Ex 5 : Consider $f(x,y) = 1-x$. Then

$$f_x = -1 \neq 0 \quad , \quad f_y = 0$$

This function is defined everywhere and it has neither a CP nor a SP because $\nabla f = -\hat{\imath} \neq \vec{0}$. Hence, it has no extreme values in $\mathbb{R}^2$. However, if we restrict the domain of f to the set of points in the disk $x^2+y^2 \leq 1$



then f has a local min value which is 0 at $(1,0)$ and a local max value which is 2 at $(-1,0)$.



From the above examples, we have a rule to find the points where extreme values can occur.

A function $f(x,y)$ can have a local or absolute extreme value at a point (a,b) in its domain if (a,b) is one of the following

(i) a CP of f

(ii) a SP of f

(iii) a boundary point of the domain of f.

See you on Wednesday