MATH 1600 Linear Algebra — Winter 2020

Tutorial 7 - Wednesday

The Inverse of a Matrix

1. In exercises (a)–(d), let

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 \\ -4 & -3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } E = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}.$$

   (a) Find the inverse of each of the matrices given above (if it exists).

   (b) Compute $(A+B)^{-1}$ and show that $(A+B)^{-1} \neq A^{-1} + B^{-1}$.

   (c) Compute $(AB)^{-1}$ and show that $(AB)^{-1} \neq A^{-1}B^{-1}$.

   (d) Show that $DC = EC$, yet $D \neq E$.

2. Let $A = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 2 & 1 & 0 & 2 \\ 1 & -1 & 3 & 0 \\ 0 & -1 & -1 & 1 \end{pmatrix}$.

   (a) Use the Gauss-Jordan method to find A^{-1}.

   (b) Check that $AA^{-1} = I_4 = A^{-1}A$ by direct multiplication.

3. Let $A = \begin{pmatrix} 0 & 3 & 2 \\ 1 & -4 & -2 \\ -3 & 4 & 1 \end{pmatrix}$, $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

   (a) Find A^{-1}.

   (b) Use your answer in (a) to solve the three systems $A\mathbf{x} = \mathbf{e}_1$, $A\mathbf{x} = \mathbf{e}_2$ and $A\mathbf{x} = \mathbf{e}_3$.

4. Let A be an $n \times n$ invertible matrix. Either prove the statement or give an example showing it is false:

   claim: If $\mathbf{v}$ is a vector in $\mathbb{R}^n$ such that $A\mathbf{v} = 0$, then $\mathbf{v} = 0$.

Subspaces

5. In each case, determine whether the given set is a subspace of $\mathbb{R}^2$. Explain.

   (a) $S_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0 \right\}$.

   (b) $S_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x \leq 0 \text{ and } y \leq 0 \right\}$.

   (c) $S_3 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : \begin{pmatrix} x \\ y \end{pmatrix} \text{ is in } S_1 \text{ and/or } S_2 \right\}$.

   (d) $S_4 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : y = 2x + 1 \right\}$.

6. In each case, determine whether the given set is a subspace of $\mathbb{R}^3$. Explain.
   (a) $S_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x = y = z \right\}$.
   (b) $S_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + y + z = 0 \right\}$.
7. In each case, determine whether the given set is a subspace of $\mathbb{R}^{n,n}$. Explain.
   (a) The set of all $n \times n$ invertible matrices.
   (b) The set of all $n \times n$ diagonal matrices.
   (c) The set of all $n \times n$ upper triangular matrices.
   (d) The set of all $n \times n$ matrices that are either zero or invertible.
8. Let A be an $m \times n$ matrix. In each case, determine whether the given set is a subspace of $\mathbb{R}^n$. Explain.
   (a) S is the set of solutions of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$.
   (b) S is the set of solutions of the non-homogeneous linear system $A\mathbf{x} = \mathbf{b}$.
   (c) S is the set spanned by the rows of A.