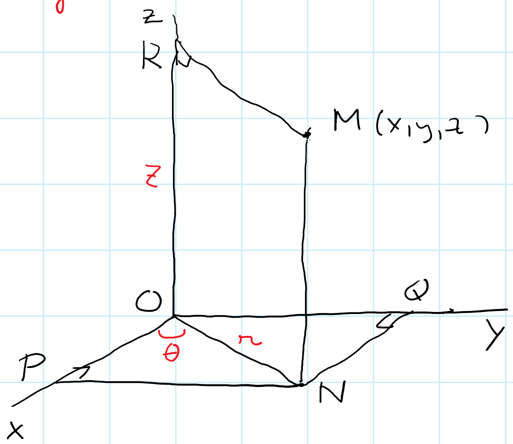
# Lecture 20

October 26, 2020 11:10 AM


Like double integrals, in some situations, a change of variables from Cartesian coordinates $(x, y, z)$ to cylindrical coordinates $(r, \theta, z)$ or spherical coordinates $(\rho, \phi, \theta)$ are essential.

## Cylindrical coordinates $(r, \theta, z)$




Consider a point $M(x, y, z)$ in $\mathbb{R}^3$. Let $N$ be the projection of $M$ onto the xy-plane and $R$ is the projection of $M$ onto the z-axis. Let $P$ and $Q$ be the projections of $N$ onto the x- and y- axes, respectively. Then

$$r = ON$$

$$\theta = \widehat{NOP}$$

$$z = OR = MN$$

These are cylindrical coordinates of $M$. We have,

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

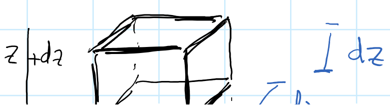
## Volume Element in Cylindrical coordinates

In Cartesian coordinates, the volume element is obtained by three families of plane

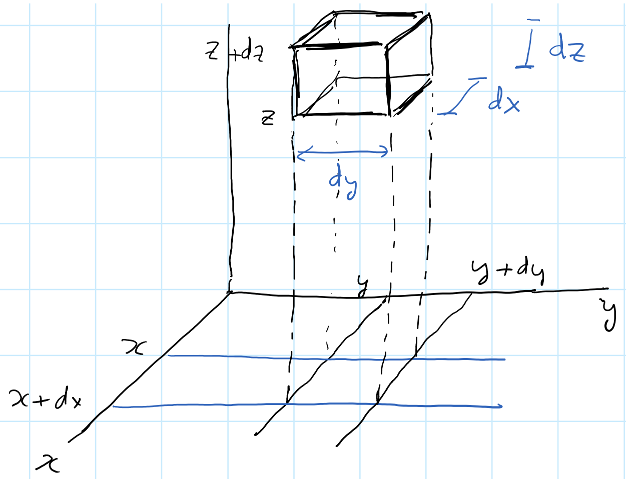
(i) $x = \text{const}$, $x + dx = \text{const}$

(ii) $y = \text{const}$, $y + dy = \text{const}$

(iii) $z = \text{const}$, $z + dz = \text{const}$




$dV = dx\,dy\,dz$



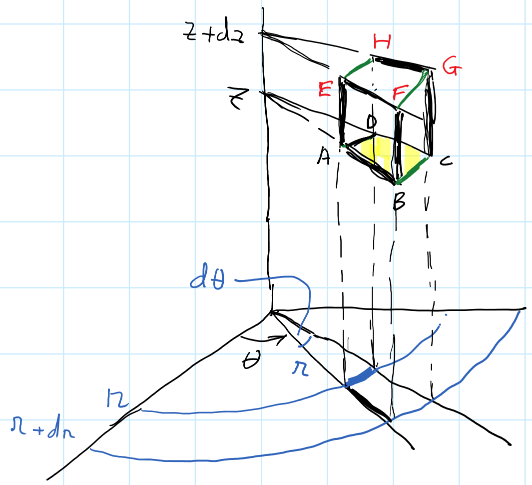

$dV = dx\,dy\,dz$

Similarly, the volume element in cylindrical coordinates is obtained by three families of surfaces

(i) $r = \text{const}$, $r + dr = \text{const}$ these are cylinders

(ii) $\theta = \text{const}$, $\theta + d\theta = \text{const}$ these are half planes passing thru the $z$-axis

(iii) $z = \text{const}$, $z + dz = \text{const}$ these are horizontal planes (|| to the xy-plane)




$$
\begin{aligned}
dV &= \text{volume}\,(ABCDEFGH) \\
&= \text{area}\,(ABCD)\,(AE) \\
&= (AB)\,(AD)\,(AE) \\
&= (dr)\,(r\,d\theta)\,(dz)
\end{aligned}
$$

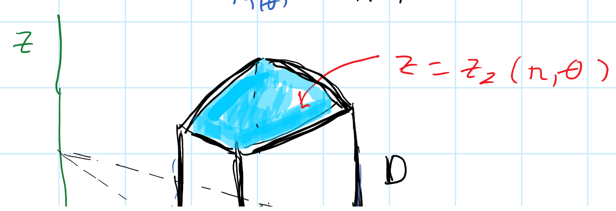
$$\therefore \boxed{dV = r\,dz\,dr\,d\theta}$$

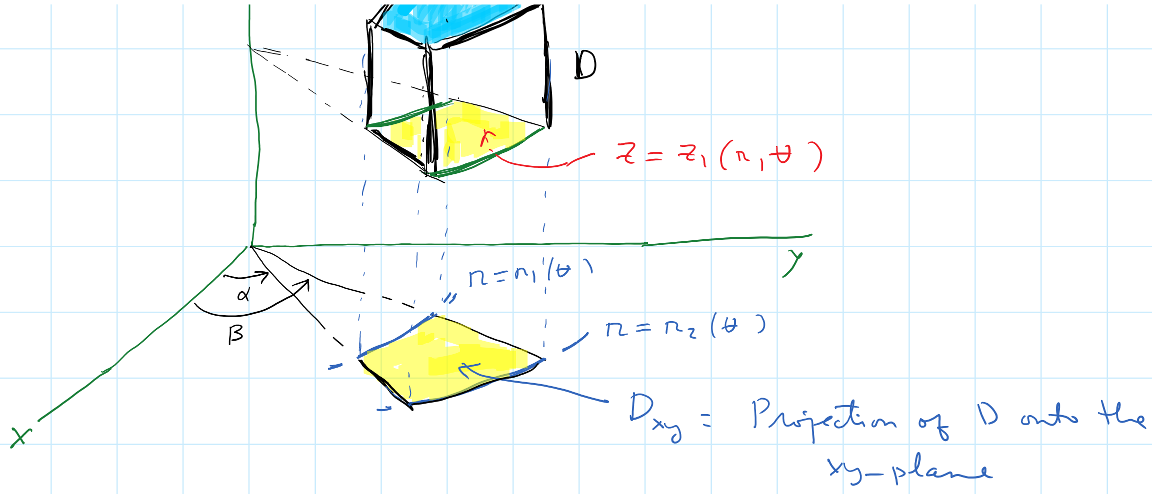
In general, if $D$ is defined by

$$D = \{ (r, \theta, z) \mid \alpha \le \theta \le \beta,\ r_1(\theta) \le r \le r_2(\theta),\ z_1(r,\theta) \le z \le z_2(r,\theta) \}$$

then

$$\iiint_D F(x,y,z)\,dV = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} \int_{z_1(r,\theta)}^{z_2(r,\theta)} F(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$$






**Ex1**: Sketch the solid described by the given inequalities

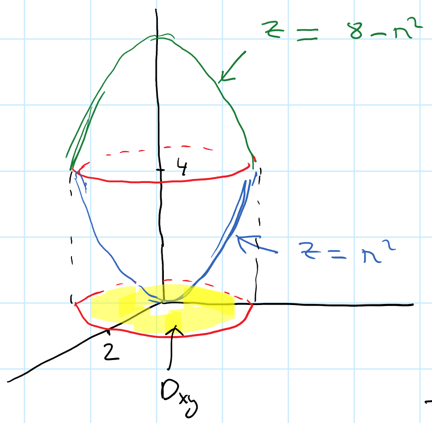
$$r^2 \leq z \leq 8 - r^2$$

and find the volume of the solid.

Solution

the lower surface $z = z_1(r,\theta) = r^2 \Rightarrow z = x^2 + y^2$ which is a paraboloid

the upper surface $z = z_2(r,\theta) = 8 - r^2 \Rightarrow z = 8 - x^2 - y^2$ an upside down paraboloid




The curve of intersection of these surfaces

$$r^2 = 8 - r^2$$

$$2r^2 = 8$$

$$r^2 = 4 \Rightarrow \boxed{r = 2}$$

The projection of D onto the xy-plane is $D_{xy}$, a circular disk with radius 2.

$$V = \iiint_D dV = \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r(8 - r^2 - r^2)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \underbrace{\boxed{(8r - 2r^3)}}_{\text{separable}}\,dr\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^2 (8r - 2r^3)\,dr\right)$$

separable

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^2 (8r - 2r^3)\, dr\right)$$

$$= (2\pi)\left(4r^2 - \frac{r^4}{2}\right)\Big|_0^2 = (2\pi)\left(16 - \frac{16}{2}\right) = (2\pi)(8)$$

$$= 16\pi \quad // \text{Ans.}$$

Ex2: Sketch the solid whose volume is given by the following integral

$$\int_0^{2\pi} \int_0^2 \int_0^r r\, dz\, dr\, d\theta$$

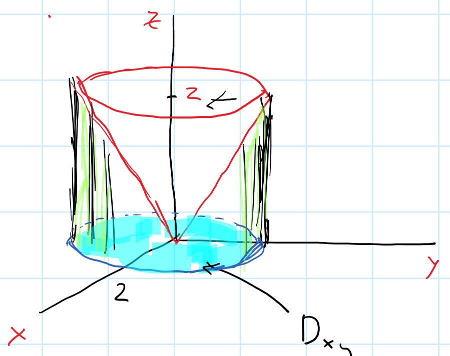
and evaluate the integral.

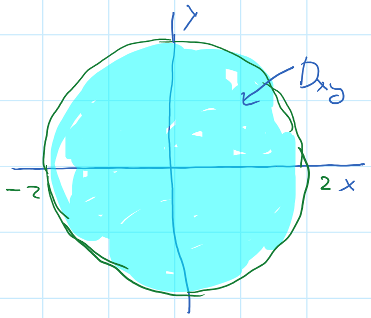
Solution

$$D = \{ (r, \theta, z) \mid \underbrace{0 \le \theta \le 2\pi, \ 0 \le r \le 2}_{D_{xy}}, \ 0 \le z \le r \}$$

the lower surface $z = z_1 = 0$ which is the xy-plane

the upper surface $z = z_2 = r = \sqrt{x^2+y^2}$ which is a cone.




$$V = \int_0^{2\pi} \int_0^2 \int_0^r r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^2 \boxed{r\,(r)}\, dr\, d\theta$$

separable

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^2 r^2\, dr\right) = (2\pi)\left(\frac{r^3}{3}\right)\Big|_0^2 = (2\pi)\left(\frac{8}{3}\right)$$

$$= \frac{16\pi}{3} \quad // \text{Ans.}$$

check: $V =$ Volume of the cylinder $-$ volume of the cone

$$= \pi r^2 h - \frac{1}{3}\pi r^2 h$$

$$= \pi (2)^2 (2) - \frac{1}{3}\pi (2)^2 (2)$$

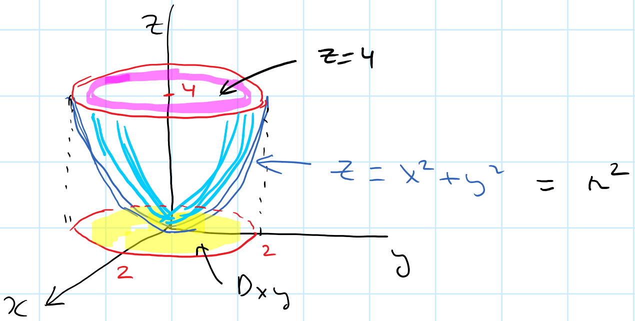
$$= 8\pi - \frac{8\pi}{3} = \frac{16\pi}{3} \quad //$$

the same!

Ex3: Evaluate $\iiint z\, dV$ where D is enclosed by the paraboloid

$z = x^2 + y^2$ and the plane $\underset{D}{}$ $z = 4$.

Solution




The curve of intersection between the paraboloid $z = x^2+y^2$ and the plane $z = 4$ which is

$$x^2+y^2 = 4$$

Hence, $D_{xy}$ is a circular disk of radius 2.

$$\iiint_D z \, dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 z \, r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^2 r \left(\frac{z^2}{2}\right)\Bigg|_{z=r^2}^{4} dr \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^2 \boxed{r\left(16 - r^4\right)} \, dr \, d\theta$$

separable

$$= \frac{1}{2}\left(\int_0^{2\pi} d\theta\right)\left(\int_0^2 (16r - r^5)\, dr\right)$$

$$= \frac{1}{\cancel{2}}\left(\cancel{2}\pi\right)\left(8r^2 - \frac{r^6}{6}\right)\Bigg|_0^2$$

$$= \pi\left(32 - \frac{64}{6}\right) = \pi\left(32 - \frac{32}{3}\right) = \pi(32)\left(\frac{2}{3}\right)$$

$$= \frac{64\pi}{3} \quad // \text{Ans.}$$

See you on Wednesday!