Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Chapter 2: Part II

UWO – February 9, 2021

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Plan for Part II

1. Functions

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1.2 Injections
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1.4 Bijection
1.5 Inverse Function
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3.3 Transpose of a Matrix

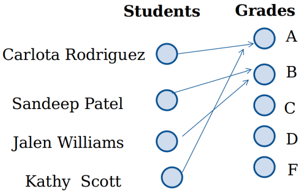
Plan for Part II

Functions

Definition: Let A and B be two nonempty sets.

1. A *function f from A to B*, denoted $f : A \to B$ is an assignment of each element of A to exactly one element of B.
2. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

- Functions are sometimes called *mappings* or *transformations*.



Functions

1. A function $f : A \to B$ can also be defined as a subset of $A \times B$, that is, a relation of $A \times B$.
2. This subset is restricted to be a relation, where no two elements of the relation have the same first element.
3. To be precise, a function f from A to Bcontains one, and only one ordered pair (a, b) for every element $a \in A$.

$$\forall x \quad (x \in A \;\; \to \;\; \exists y \; (y \in B \wedge (x, y) \in f))$$

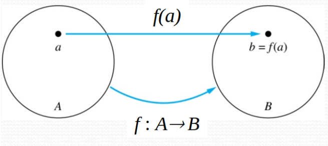
and

$$\forall x, y_1, y_2 \quad (((x, y_1) \in f \wedge (x, y_2) \in f) \;\; \to \;\; y_1 = y_2)$$

Functions: terminology

Given a function $f : A \to B$:

1. We say f *maps* A to B *or* f is a *mapping* from A to B.
2. A is called the *domain* of f.
3. B is called the *codomain* of f.
4. If $f(a)=b$, then b is called the *image* of a under f and a is called the *preimage* of b.
5. The range of f, denoted by $f(A)$, is the set of all images of points in A under f. The range is a subset of the codomain B.
6. Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.



Representing functions

Functions may be specified in different ways:

1. An explicit statement of the assignment, as in the students and grades example.
2. A formula, like in:
$$f(x) = x + 1.$$
3. A computer program.

```
int add(int a,int b)
{
int c;
c=a+b;
return c;
}
```

Questions

1. $f(a) = ?$

   Solution: z
2. The image of d is ?

   Solution: z
3. The domain of f is ?

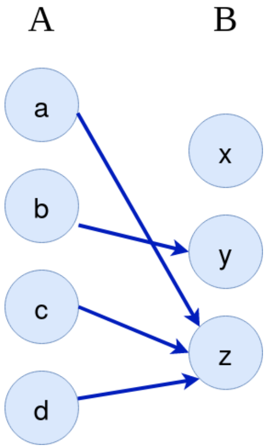
   Solution: A
4. The codomain of f is ?

   Solution: B
5. The preimage of y is ?

   Solution: b
6. $f(A) = ?$

   Solution: $\{y, z\}$
7. The preimage(s) of z is/are ?

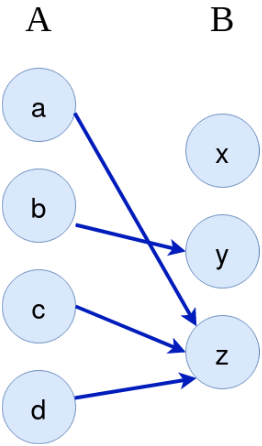
   Solution: $\{a, c, d\}$



Question on functions and sets

1. If $f : A \to B$ and S is a subset of A, then:

$$f(S) = \{f(s) \mid s \in S\}$$

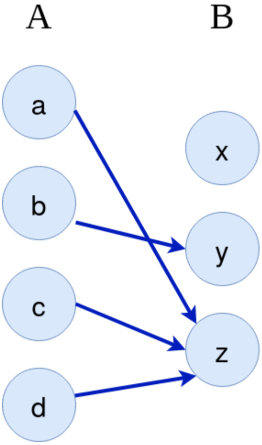


1. $f\{a, b, c\}$ is ?
   Solution: $\{y, z\}$
2. $f\{c, d\}$ is ?
   Solution: $\{z\}$

“many-to-one”

1. A function can map many elements in the domain on the same element in the codomain.
2. Such a function is called a *many-to-one mapping*.

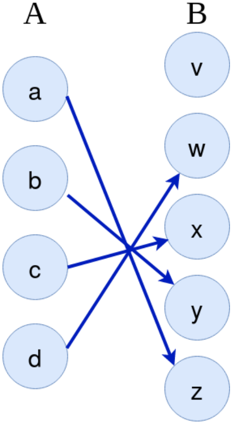
In this example, each of the elements a, c, d is mapped to z.



Plan for Part II

Injections (i.e. *one-to-one*)

Definition: A function f is said to be *one-to-one*, or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if it is one-to-one.

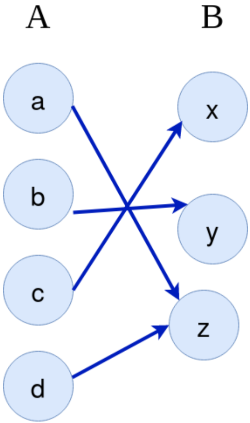


Plan for Part II

Surjections (i.e. *onto*)

Definition: A function f from A to B is called *onto* or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called a *surjection* if it is onto.

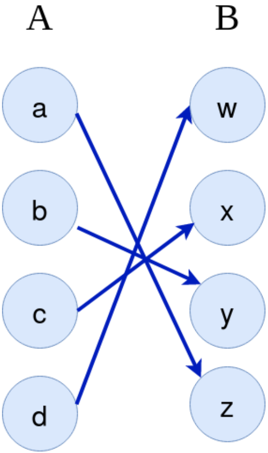
1. As illustrated by the example on the right, a function can be surjective (onto) but not injective (one-to-one).
2. Vice versa, the example on the previous slide shows that a function can be injective (one-to-one) but not surjective (onto).



Plan for Part II

Bijections

Definition: A function f is a *one-to-one correspondence* , or a *bijection*, if it is both *one-to-one* and *onto* (injective and surjective).



Showing that f is one-to-one or onto

Let A, B be two sets and $f : A \to B$ be a function from A to B

1. Showing that f is injective means proving that for all arbitrary $x, y \in A$ we have:
$$f(x) = f(y) \quad \to \quad x = y.$$
2. Showing that f is not injective means proving that there exist $x, y \in A$ so that:
$$f(x) = f(y) \ \text{ and } \ x \neq y.$$
3. Showing that f is surjective means proving that:
$$\forall y \in B \ \ \exists x \in A \ \ f(x) = y.$$
4. Showing that f is not surjective means proving that:
$$\exists y \in B \ \ \forall x \in A \ \ f(x) \neq y.$$

Showing that f is one-to-one or onto

1. **Example 1** : Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3, f(b) = 2, f(c) = 1$, and $f(d) = 3$. Is f an onto function?

   Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.

2. **Example 2** : Consider function $f : \mathbb{Z} \to \mathbb{Z}$ defined for any $x \in \mathbb{Z}$ by equation $f(x) = x^2$. Is this function *onto* $\mathbb{Z}$ (surjective)?

   Solution: No, f is not onto because there is no integer x with $x^2 = -1$, for example.

Showing that f is one-to-one or onto

1. **Example 3** : Consider the function $f : \mathbb{Z} \to \mathbb{Z}^+$ defined by equation $f(x) = x^2$. Is this function *onto*?

   Solution: No. There is no integer such that $x^2 = 2$, for example

2. **Example 4** : Consider function/mapping $f : \mathbb{R} \to \mathbb{R}^+$ defined by equation $f(x) = x^2$. Is this function a *onto*?

   Solution: Yes.

3. Is that same function f a bijection?

   Solution: No. It is *onto* but not *one-to-one*.

Showing that f is one-to-one or onto

1. **Example 5** : Consider the function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by equation $f(x) = x^2$. Is this function a *bijection*?

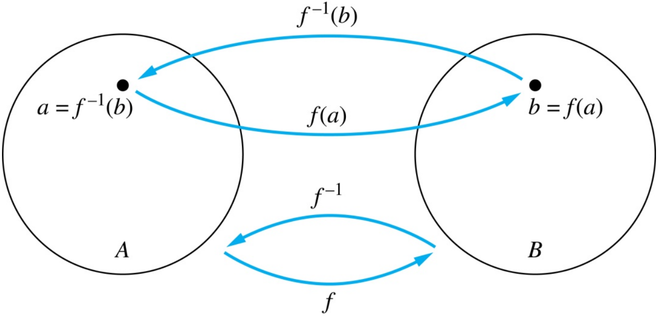
   Solution: Yes, Why?

2. The properties like being an injection, a surjection and a bijection depend on the function's domain and codomain.

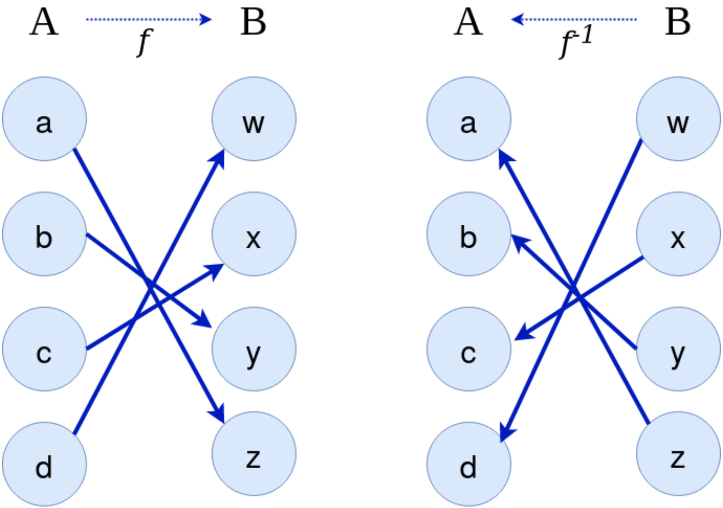
Plan for Part II

Inverse functions

1. **Definition**: Let f be a bijection from A to B. Then the *inverse* of f, denoted f^{-1}, is the function from B to A defined as $f^{-1}(y) = x$ iff $f(x) = y$.
2. if f was not surjective, then the relation
$$\{(y, x) \in B \times A \mid f(x) = y\}.$$
would miss to map some element from B to an element of A.
3. Moreover, if f was not injective, then the same relation would map some element from B to more than one element of A.



Inverse functions



Questions

Example 1 : Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a)$=2, $f(b)$ =3, and $f(c)$=1. Is f invertible and if so what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} is $f^{-1}(1) = c$, $f^{-1}(2) = a$, and$f^{-1}(3) = b$.

Questions

Example 2 : Let $f : \mathbb{Z} \to \mathbb{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

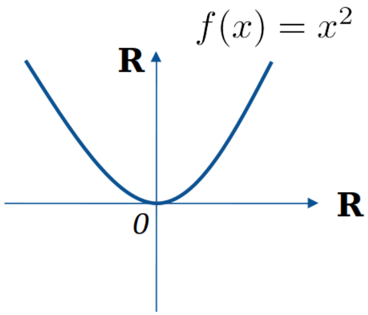
Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

Questions

Example 3 : Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x) = x^2$. Is f invertible, and if so, what is its inverse?

Solution: The function f is not invertible.

1. It is not injective since $f(2) = 4 = f(-2)$.
2. It is also not surjective since no $x \in \mathbb{R}$ has -1 as an image.

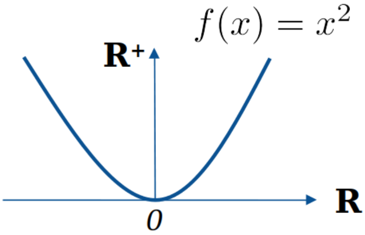


Questions

Example 4 : Let $f : \mathbb{R} \to \mathbb{R}^+$ be such that $f(x) = x^2$. Is f invertible, and if so, what is its inverse?

Solution: The function f is not invertible.

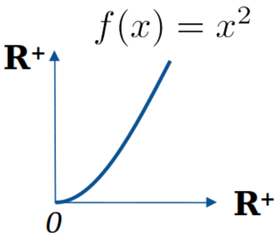
1. It is surjective since for every $y \in \mathbb{R}^+$ there exists $x \in \mathbb{R}$ so that $f(x) = y$, namely $\sqrt{y}$ and $-\sqrt{y}$.
2. It is not injective since $f(2) = 4 = f(-2)$.

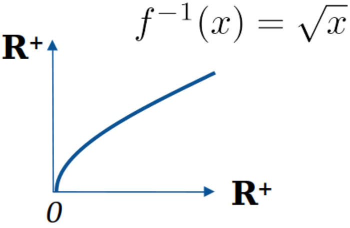


Questions

Example 5 : Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be such that $f(x) = x^2$. Is f invertible, and if so, what is its inverse?

Solution: Yes and the inverse is $f^{-1}(y) = \sqrt{y}$.



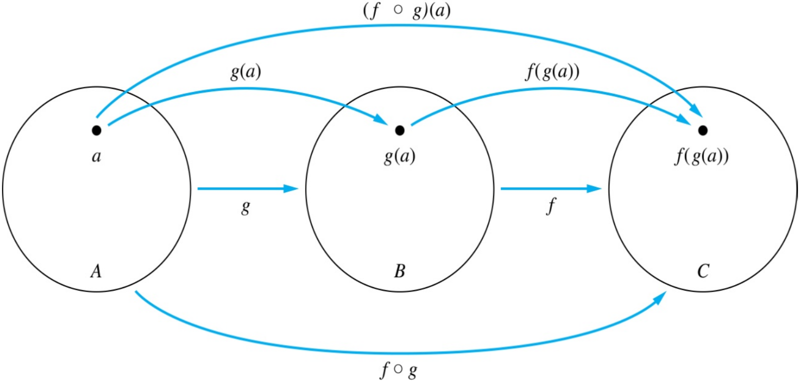


Plan for Part II

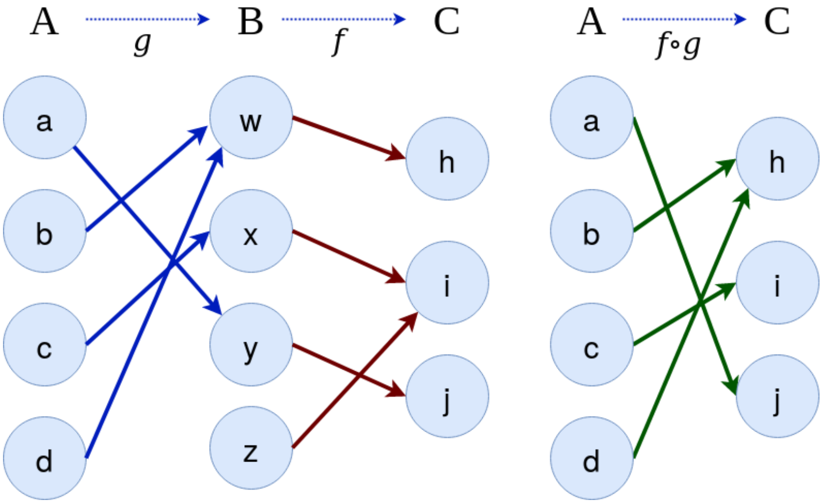
Composition

Definition: Let A, B, C be three sets.

1. Let $f : B \to C$ and $g : A \to B$ be two functions.
2. The *composition of f with g*, denoted $f \circ g$ is the function from A to C defined by
$$f \circ g(x) = f(g(x)).$$
3. One trick to remember the meaning of $f \circ g$ is to read the symbol $\circ$ as *origin*.



Composition



Composition

Example 1 : If $f(x) = x^2$ and $g(x) = 2x + 1$, then:

$$f(g(x)) = (2x+1)^2$$

and

$$g(f(x)) = 2x^2 + 1$$

Composition questions

1. Let g be the function from the set $\{a,b,c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$.
2. Let f be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a)$=3, $f(b)$=2, and $f(c)$= 1.
3. What is the composition of f with g ?
4. The composition $f \circ g$ is defined by
   a. $f \circ g(a) = f(g(a)) = f(b) = 2.$
   b. $f \circ g(b) = f(g(b)) = f(c) = 1.$
   c. $f \circ g(c) = f(g(c)) = f(a) = 3.$
5. Note that the composition $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Composition questions

1. Let f and g be functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.
2. What is the composition of f and g, and also the composition of g and f?
3. **Solution:**

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f(3x + 2) \\ &= 2(3x + 2) + 3 \\ &= 6x + 7 \end{aligned}$$

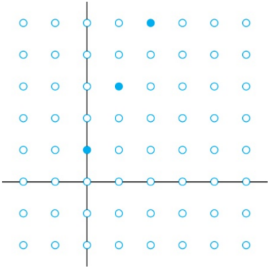
$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(2x + 3) \\ &= 3(2x + 3) + 2 \\ &= 6x + 11 \end{aligned}$$

Plan for Part II

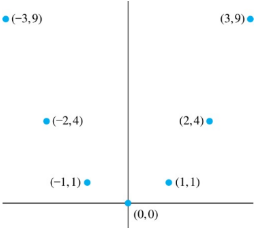
1. Functions

Graphs of functions

Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$ from $\mathbb{Z}$ to $\mathbb{Z}$



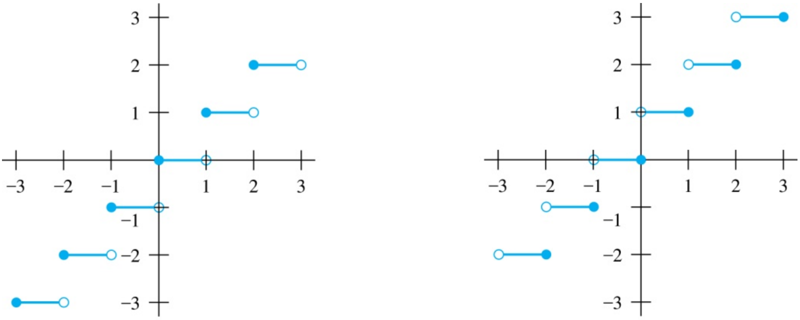
Graph of $f(x) = x^2$ from $\mathbb{Z}$ to $\mathbb{Z}$

Plan for Part II

The floor and ceiling functions

1. The *floor* function, denoted $\lfloor x \rfloor$, is the largest integer less than or equal to x.
2. The *ceiling* function, denoted $\lceil x \rceil$, is the smallest integer greater than or equal to x
3. Examples:
   a. $\lfloor 3.5 \rfloor = 3 \qquad \lceil 3.5 \rceil = 4$
   b. $\lfloor -1.5 \rfloor = -2 \qquad \lceil -1.5 \rceil = -1$
4. The floor and ceiling functions play a very important role in computer science, since they allow to approximate real numbers with integer numbers.
5. For instance, in computer graphics, calculations are performed with real numbers and plotting the results (on the screen pixels) requires to use floor or ceiling values.

The floor and ceiling functions



(a) $y = \lfloor x \rfloor$ (b) $y = \lceil x \rceil$

Graph of (a) Floor and (b) Ceiling Functions

The floor and ceiling functions

TABLE 1 **Useful Properties of the Floor and Ceiling Functions.** (n is an integer, x is a real number)
(1a) $\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$ (1b) $\lceil x \rceil = n$ if and only if $n - 1 < x \leq n$ (1c) $\lfloor x \rfloor = n$ if and only if $x - 1 < n \leq x$ (1d) $\lceil x \rceil = n$ if and only if $x \leq n < x + 1$
(2) $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
(3a) $\lfloor -x \rfloor = -\lceil x \rceil$ (3b) $\lceil -x \rceil = -\lfloor x \rfloor$
(4a) $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ (4b) $\lceil x + n \rceil = \lceil x \rceil + n$

Proving properties of functions

1. Prove that if x is a real number, then we have:

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$$

1. **Proof**: Let $x = n + \epsilon$, where n is an integer and $0 \leq \epsilon < 1$.
2. With $2x = 2n + 2\epsilon$, we need to discuss whether $2\epsilon < 1$ holds or not.
3. Case 1: $\epsilon < \frac{1}{2}$
   a. $2x = 2n + 2\epsilon$ and $\lfloor 2x \rfloor = 2n$, since $0 \leq 2\epsilon < 1$.
   b. $\lfloor x + \frac{1}{2} \rfloor = n$, since $x + \frac{1}{2} = n + (\frac{1}{2} + \epsilon)$ and $0 \leq \frac{1}{2} + \epsilon < 1$.
   c. Hence, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = n + n = 2n$.
4. Case 2: $\epsilon \geq \frac{1}{2}$
   a. $2x = 2n + 2\epsilon = (2n+1) + (2\epsilon - 1)$ and $\lfloor 2x \rfloor = 2n + 1$, since $0 \leq 2\epsilon - 1 < 1$.
   b. $\lfloor x + \frac{1}{2} \rfloor = \lfloor n + (\frac{1}{2} + \epsilon) \rfloor = \lfloor n + 1 + (\epsilon - \frac{1}{2}) \rfloor = n + 1$, since $0 \leq \epsilon - \frac{1}{2} < 1$.
   c. Hence, $\lfloor 2x \rfloor = 2n + 1$ and $\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = n + (n+1) = 2n + 1$. Q.E.D.

The factorial function

Definition: $f : \mathbb{N} \to \mathbb{Z}^+$, denoted by $f(n) = n!$ is the product of the first n positive integers when n is a non-negative integer.

$$f(n) = 1 \cdot 2 \cdots (n-1) \cdot n, \; f(0) = 0! = 1$$

Examples:

$f(1) = 1! = 1$

$f(2) = 2! = 1 \cdot 2 = 2$

$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$

$f(20) = 2,432,902,008,176,640,000$

Stirling's Formula:

$$g(n) = \sqrt{2\pi n}(\frac{n}{e})^n$$

$$f(n) = n! \sim g(n)$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$$

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3. Matrices

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Introduction

1. Sequences are ordered lists of elements.
   a. $1, 2, 3, 5, 8$
   b. $1, 3, 9, 27, 81, \ldots$
2. Sequences are not tuples; sequences generally have infinitely many terms.
3. Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from botany to music.
4. We will introduce the terminology to represent sequences and sums of the terms in the sequences.

Plan for Part II

Sequences

1. **Definition**: A *sequence* is a function from a subset of the integers (usually either the set $\{0, 1, 2, 3, 4, \ldots\}$ or $\{1, 2, 3, 4, \ldots\}$) to a set S, that is, $f : \mathbb{N} \to S$ or $f : \mathbb{Z}^{++} \to S$
2. The notation a_n is used to denote the image of the integer n.
3. We can think of a_n as the equivalent of $f(n)$ where f is a function $f : \mathbb{N} \to S$.
4. We call a_n a *term* of the sequence.

$$a_n = f(n)$$

Sequences

Example: Consider the sequence $\{a_n\}$ where:

$$a_n = \frac{1}{n} \qquad \{a_n\} = \{a_1, a_2, a_3, \ldots\}, \quad \text{for} \quad n \in \mathbb{Z}^+$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

Plan for Part II

Geometric progressions

Definition: A *geometric progression* is a sequence of the form:

$$a, ar, ar^2, \ldots, ar^n, \ldots \qquad a_n = ar^n$$

where the *initial term a* and the *common ratio r* are real numbers.

Examples:

1. Let $a = 1$ and $r = -1$. Then:

$$\{b_n\} = \{b_0, b_1, b_2, b_3, b_4, \ldots\} = \{1, -1, 1, -1, 1, \ldots\}$$

2. Let $a = 2$ and $r = 5$. Then:

$$\{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{2, 10, 50, 250, 1250, \ldots\}$$

3. Let $a = 6$ and $r = \frac{1}{3}$. Then:

$$\{d_n\} = \{d_0, d_1, d_2, d_3, d_4, \ldots\} = \{6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots\}$$

Arithmetic progressions

Definition: A *arithmetic progression* is a sequence of the form:

$$a, a+d, a+2d, \ldots, a+nd, \ldots \qquad a_n = a + nd$$

where *initial term* a and *common difference* d are real numbers.

Examples:

1. Let $a = -1$ and $d = 4$. Then:

$$\{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \ldots\} = \{-1, 3, 7, 11, 15, \ldots\}$$

2. Let $a = 7$ and $d = -3$. Then:

$$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \ldots\} = \{7, 4, 1, -2, -5, \ldots\}$$

3. Let $a = 1$ and $d = 2$. Then:

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Strings

Definition: A *string* is a finite sequence of characters from a finite set (usually called an *alphabet*).

1. Sequences of characters or bits are important in computer science.
2. The *empty string* is represented by λ.
3. The string *abcde* has *length* 5.

Plan for Part II

Recurrence relations

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a non-negative integer.

1. A <u>sequence</u> is called a solution of a recurrence relation if its terms satisfy the recurrence relation.
2. The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Questions about recurrence relations

Example 1: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, 4, \dots$ and suppose that $a_0 = 2$. What are a_1, a_2 and a_3?

[Here $a_0 = 2$ is the initial condition.]

Solution: We see from the recurrence relation that:

$$a_1 = a_0 + 3 = 2 + 3 = 5$$

$$a_2 = 5 + 3 = 8$$

$$a_3 = 8 + 3 = 11$$

Questions about recurrence relations

Example 2: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

[Here the initial conditions are $a_0 = 3$ and $a_1 = 5$.] **Solution**: We see from the recurrence relation that:

$$a_2 = a_1 - a_0 = 5 - 3 = 2$$

$$a_3 = a_2 - a_1 = 2 - 5 = -3$$

The Fibonacci sequence

Definition: Define the *Fibonacci sequence*, $f_0, f_1, f_2, \ldots$ by*:*

1. Initial Conditions: $f_0 = 0, f_1 = 1$
2. Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

Example: Find f_2, f_3, f_4, f_5 and f_6 .

Solution:

$$f_0 = 0$$
$$f_1 = 1$$
$$f_2 = f_1 + f_0 = 1 + 0 = 1$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5$$
$$f_6 = f_5 + f_4 = 5 + 3 = 8$$

Solving recurrence relations

1. Finding a formula for the n^{th} term of the sequence generated by a recurrence relation is called *solving the recurrence relation*.
2. Such a formula is called a *closed formula*.
3. Various methods for solving recurrence relations will be covered in Chapter 5 where recurrence relations will be studied in greater depth.
4. Here we illustrate by example the method of iteration in which we need to guess the formula. The guess can be proved correct by the method of induction (Chapter 5).

Iterative solution example

Method 1 : Working upward (forward substitution)

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2, 3, 4, \ldots$ and suppose that $a_1 = 2$.

$$
\begin{aligned}
a_1 &= 2 \\
a_2 &= 2 + 3 \\
a_3 &= (2 + 3) + 3 &&= 2 + 3 \cdot 2 \\
a_4 &= (2 + 2 \cdot 3) + 3 &&= 2 + 3 \cdot 3 \\
&\ \ \vdots && \textit{observed pattern (guess):} \quad a_m = 2 + 3(m-1) \\
a_n &= a_{n-1} + 3 &&= (2 + 3 \cdot (n-2)) + 3 \qquad = 2 + 3(n-1)
\end{aligned}
$$

(confirmed)

(proof by induction covered in Chapter 5)

Iterative solution example

Method 2 : Working downward (backward substitution)

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2, 3, 4, \ldots$ and suppose that $a_1 = 2$.

$$
\begin{array}{llll}
a_n & = a_{n-1} + 3 & & \\
& = (a_{n-2} + 3) + 3 & = a_{n-2} + 3 \cdot 2 & \\
& = (a_{n-3} + 3) + 3 \cdot 2 & = a_{n-3} + 3 \cdot 3 & \\
& \vdots & \textit{pattern:} & a_n = a_{n-m} + 3 \cdot m \\
& a_2 + 3(n-2) & = (a_1 + 3) + 3(n-2) & = 2 + 3(n-1)
\end{array}
$$

Financial application

Example: Suppose that a person deposits \$10,000.00 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

1. Let P_n denote the amount in the account after n years. P_n satisfies the following recurrence relation:

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)P_{n-1}$$

2. We know our initial condition is $P_0 = 10,000$.

Continued on next slide ↪

Financial application

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)P_{n-1}, \quad \textit{with } P_0 = 10,000$$

Solution: Forward Substitution

$$\begin{aligned}
P_1 &= (1.11)P_0 \\
P_2 &= (1.11)P_1 &&= (1.11)^2 P_0 \\
P_3 &= (1.11)P_2 &&= (1.11)^3 P_0 \\
&\vdots && \textit{observed pattern (guess):} \quad P_m = (1.11)^m P_0 \\
P_n &= (1.11)P_{n-1} &&= (1.11)(1.11)^{n-1}P_0 = (1.11)^n P_0 \\
& && \textit{(confirmed)} \\
P_n &= (1.11)^n 10,000 \\
P_{30} &= (1.11)^{30} 10,000 &&= \$228,992.97
\end{aligned}$$

(proof by induction covered in Chapter 5)

Useful sequences

TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Plan for Part II

Summations

1. Given a sequence $\{a_n\}$, given two indices $m \leq n$, we are interested in the sum of the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_{n-1}, a_n$.
2. Three possible notations:
$$\sum_{j=m}^{n} a_j \qquad \textstyle\sum_{j=m}^{n} a_j \qquad \textstyle\sum_{m \leq j \leq n} a_j$$
3. Each of them represents
$$a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$
4. The variable *j* is called the *index of summation*. It runs through all the integers starting with its *lower limit m* and ending with its *upper limit n*.

Summations

More generally for a set S:

$$\sum_{j \in S} a_j$$

Examples:

1. $\sum_{0}^{n} r^j = r^0 + r^1 + r^2 + r^3 + \cdots + r^n$

2. $\sum_{1}^{\infty} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

3. if $S = \{2, 5, 7, 10\}$, then $\sum_{j \in S} a_j = a_2 + a_5 + a_7 + a_{10}$

Product notation

1. Product of the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_{n-1}, a_n$ from the sequence $\{a_n\}$
2. Three possible notation:

$$\prod_{j=m}^{n} a_j \qquad \textstyle\prod_{j=m}^{n} a_j \qquad \prod_{m \leq j \leq n} a_j$$

3. Each of them represents

$$a_m \times a_{m+1} \times a_{m+2} \times \cdots \times a_n$$

Geometric series

Sums of the terms of a geometric progression:

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & r \neq 1 \\ a(n+1) & r = 1 \end{cases}$$

Proof:

$$\text{Let } S_n = \sum_{j=0}^{n} ar^j$$

$$rS_n = r\sum_{j=0}^{n} ar^j \qquad \textit{Multiply by } r.$$

$$= \sum_{j=0}^{n} ar^{j+1} \qquad \textit{Move new } r \textit{ into exponent.}$$

Continued on next slide ↪

Geometric series

$$= \sum_{j=0}^{n} ar^{j+1} \qquad \textit{From previous slide.}$$

$$= \sum_{k=1}^{n+1} ar^{k} \qquad \textit{Shift index of summation with } k = j+1.$$

$$= (\sum_{k=0}^{n} ar^{k}) + (ar^{n+1} - a) \qquad \textit{Remove } k = n+1 \text{ term and add } k = 0 \text{ term.}$$

$$= S_n + (ar^{n+1} - a) \qquad \textit{Substitute } S \textit{ for the summation.}$$

$$\therefore\ rS_n = S_n + (ar^{n+1} - a)$$

$$S_n = \frac{ar^{n+1} - a}{r - 1} \qquad \textbf{if } r \neq 1.$$

$$S_n = \sum_{j=0}^{n} ar^{j} = \sum_{j=0}^{n} a = a(n+1) \qquad \textbf{if } r = 1.$$

Some useful summation formulae

TABLE 2 **Some Useful Summation Formulae.**

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1}-a}{r-1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$	$\frac{1}{(1-x)^2}$

1. The first is the Geometric Series we just proved.
2. We will prove some of these later by induction.
3. The last two have a proof in the textbook (required calculus knowledge).

Plan for Part II

1. Functions

1.1 Definition and representation
1.2 Injections
1.3 Surjections
1.4 Bijection
1.5 Inverse Function
1.6 Composition
1.7 Graphs of Functions
1.8 Some Important Functions

2. Sequences and Summations

2.1 Sequences
2.2 Arithmetic and Geometric Progressions
2.3 Recurrence Relations
2.4 Summations

3. Matrices

3.1 Definition
3.2 Matrix Arithmetic
3.3 Transpose of a Matrix

Plan for Part II

Matrices

1. Matrices are useful discrete structures that can be used in many ways. For example, they are used to:
   a. describe certain types of functions known as linear transformations.
   b. express which vertices of a graph are connected by edges (see Chapter 10).
   c. represent systems of linear equations and their solutions
2. In later chapters, we will see matrices used to build models of transportation systems and communication networks.
3. Algorithms based on matrix models will be presented in later chapters.
4. Here we cover the aspect of matrix arithmetic that will be needed later.

Matrix

Definition: A *matrix* is a rectangular array of numbers.

1. A matrix with m rows and n columns is called an $m \times n$ matrix.
2. The plural of matrix is *matrices*.
3. A matrix with the same number of rows as columns is called *square*.
4. Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

$$3 \times 2 \text{ matrix} \quad \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$$

Notation

1. Let m and n be positive integers and let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

2. The i-th row of **A** is the $1 \times n$ matrix $[a_{i1}, a_{i2}, \ldots, a_{in}]$.
3. The j-th column of **A** is the $m \times 1$ matrix: $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$
4. The (i,j)-th *element* or *entry* of **A** is the element a_{ij}.
5. We can use $\mathbf{A} = [a_{ij}]$ to denote the matrix with its (i,j)th element equal to a_{ij}.

Plan for Part II

Matrix arithmetic: addition

Definition: Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be $m \times n$ matrices.

1. The sum of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} + \mathbf{B}$, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its (i, j)-th element.
2. In other words, if $\mathbf{A} + \mathbf{B} = [c_{ij}]$ then $c_{ij} = a_{ij} + b_{ij}$.

Example:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

Note that matrices of different sizes can not be added.

Matrix multiplication

Definition: Let $\mathbf{A}$ be an $m \times k$ matrix and $\mathbf{B}$ be a $k \times n$ matrix.

1. The *product* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{AB}$, is the $m \times n$ matrix that has its (i,j)-th element equal to the sum of the products of the corresponding elements from the i-th row of $\mathbf{A}$ and the j-th column of $\mathbf{B}$.
2. In other words, if $\mathbf{AB} = [c_{ij}]$ then:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

Example:

$$c_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}$$

$$\underset{4 \times 3}{\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}} \quad \underset{3 \times 2}{\begin{bmatrix} 2 & 4 \\ 1 & 2 \\ 3 & 0 \end{bmatrix}} = \underset{4 \times 2}{\begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}}$$

The product of two matrices is **undefined** when **the number of columns in the first matrix** is not the same as **the number of rows in the second .**

Illustration of matrix multiplication

The Product of $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ik} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mk} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{kj} & \cdots & b_{kn} \end{bmatrix}$$

$$AB = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & c_{ij} & \vdots \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix} \qquad c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

Matrix multiplication is not commutative

Example:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Does **AB** = **BA** ?

Solution:

$$AB = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \quad BA = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

AB $\neq$ **BA**

Identity matrix and powers of matrices

Definition: The *identity matrix of order n* is the $n \times n$ matrix $I_n = [\delta_{ij}]$**, where** $\delta_{ij} = 1$ **if** $i = j$ **and** $\delta_{ij} = 0$ **if** $i \neq j$**.**

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 1 \end{bmatrix}$$

$AI_n = I_m A = A$ when A is an $m \times n$ matrix

Powers of square matrices can be defined. When A is an $n \times n$ matrix, we have: $A^0 = I_n$ $\quad A^r = AAA \cdots A$ (r times)

Plan for Part II

Transpose of a matrix

Definition: Let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ matrix. The *transpose* of $\mathbf{A}$, denoted by A^t, is the $n \times m$ matrix obtained by interchanging the rows and columns of $\mathbf{A}$.

If $A^t = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

The transpose of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

Transpose of a matrix

Definition: A square matrix **A** is called symmetric if **A** = A^t.
Thus **A** = $[a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$.

The matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is square and symmetric.

(Square) symmetric matrices do not change when their rows and columns are interchanged.