MATH 2155
Sect 001, Code 001

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THE UNIVERSITY OF WESTERN ONTARIO
MATHEMATICS 2155A MIDTERM EXAMINATION
11 November 2021 7:00pm - 9:00pm

Please **PRINT VERY CLEARLY**:

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INSTRUCTIONS:

- This exam is 8 pages long. It is printed double-sided. **Check that your exam is complete.**
- There are **5 questions** worth **28 marks**. All questions will be graded.
- All questions must be answered in the space provided. If you need extra space, a blank white page is provided at the end of the booklet. Make sure to indicate in the original answer space that your solution is continued on the final page.
- Do not unstaple the exam booklet.
- Coloured paper is for rough work only.

MIDTERM RULES:

- You are **not allowed** to use: cell phones or other devices, calculators, ear buds, the textbook, or any notes.
- Remember that this exam must be done **on your own**, without asking other students for help or searching for solutions online. Giving information to other students or receiving it from other students is an **academic offence** that is taken very seriously.

WRITING SOLUTIONS:

- Answers are graded on **correctness**, **style** and **presentation**.
- Do not submit any **rough work**. Do your rough work on scratch paper, and only submit a **clearly** and **neatly** written answer.
- Your proofs should be written in **complete sentences**, starting with capitals and ending with periods. They should be in **paragraph form**, similar to the proofs in the textbook. For example, you shouldn't put each sentence on a new line.
- Don't use informal abbreviations such as three dots for "therefore." Logical symbols can be used when they clarify things to the reader, but English words can often be more effective.
- Include **all of the steps** that are needed to logically justify every claim you make. Do not include unnecessary steps. Try to be **concise** and **complete**.

For question 1 only, you do not need to show your work.

1. Let $A = \{1,2,3,4,5\}$, $B = \{0,2,4,6\}$, and $C = \{7,8,9\}$. Let R be the relation from A to B defined by aRb iff $|b-a| = 2$. Let S be the relation from C to B defined by $S = \{(7,0),(7,4),(8,4)\}$.

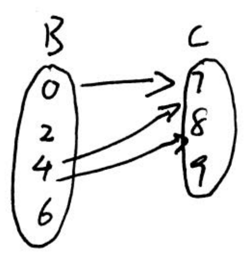
[2] (a) Write down R as a set of ordered pairs.

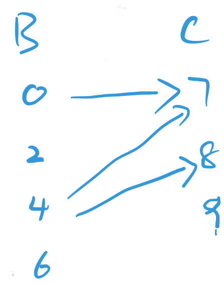
~~$R = \{(2,0),(2,4),(4,6)\}$.~~

$R = \{(2,0),(2,4),(4,6),(4,2)\}$.

[3] (b) Write down S^{-1} as a set of ordered pairs and draw a picture of S^{-1} using arrows.

$S^{-1} = \{(0,7),(4,7),(4,8)\}$





[2] (c) Write down $S^{-1} \circ R$ as a set of ordered pairs.

$S^{-1} \circ R = \{ \cancel{(2,0)}\ (2,7), (2,8)\}$

$R = \{(2,0),(2,4),(4,6),(4,2)\}$

$S^{-1} = \{(0,7),(4,7),(4,8)\}$.

$S^{-1} \circ R = \{(2,7),(4,7),(2,8)\}$.

[2] (d) What are the domain of R and the range of S?

the domain of R is $\{2,4\}$

the range of S is $\{0,4\}$.

$\{2,4\}$ $\{0,4\}$.

2. Consider the relation T on $\mathbb{N}$ defined by nTm iff $n - m$ is an even natural number. $>0.$

[2] (a) Is T reflexive? If it is, give a proof. If it is not, give a counterexample.

(a) Yes. assume an arbitrary natural number a.
$a-a=0$ for any value of a.
$aTa = \{(a,a) \mid a-a$ is an even natural number $\}$.
Since 0 is an even natural number, aTa holds.
Thus, T is reflexive.

reflexive:

[2] (b) Is T symmetric? If it is, give a proof. If it is not, give a counterexample.

(b) Yes. assume random number n, m such that there is a interger k $n-m=2k$. therefore, $m-n=-2k$.
$mTn = \{(m,n) \mid m-n$ is an even natural number $\}$.
Since $-2k$ is an even natural number, mTn holds.
Thus, T is symmetric.

No. Assume that $n=4, m=2$. $n-m=4-2=2$,
but $m-n=-2$, which is not a nature number.

[3] (c) Is T transitive? If it is, give a proof. If it is not, give a counterexample.

No.

Yes.

Assume x, y, z that xTy, yTz. Since xTy, we have $x-y=2k_1$ for some nature num k. Similarly, we have $y-z=2k_2$ for —

Since k_1, k_2 are nature num, $x-z=2(k_1+k_2)$ is an even nature num —.

[4] 3. Suppose x and y are real numbers. Prove that $x+y=xy+1$ iff either $x=1$ or $y=1$.

~~existence~~: if $x=1$, left hand side $= y+1$. right hand side $= y+1$.
if $y=1$, left hand side $= x+1$ right hand side $= x+1$.
so $x+y=xy+1$ holds if either $x=1$ or $y=1$.

~~uniqueness~~: $x+y=xy+1$

$x-xy+y-1=0$

$x(1-y)-(1-y)=0$.

$(x-1)(1-y)=0$.

In this case, the function holds only if $x=1$ or $y=1$.

Thus, $x+y=xy+1$ iff either $x=1$ or $y=1$ $\square$

$1 + xy^2 = xy.$

[4] 4. Prove that for all $y \in \mathbb{R}$, if $y \neq 0$ and $y \neq 1$ then there exists a unique $x \in \mathbb{R}$ such that $1 + xy^2 = xy$.

existence:

$1 + xy^2 = xy$

$x(y - y^2) = 1$

Since $y \neq 0$ and $y \neq 1$,

$(y - y^2) \neq 0$

$x = \frac{1}{y(1-y)}.$

$\forall y_0 \in \mathbb{R}, \exists x_0 = \frac{1}{y_0(1-y_0)}$ such that $1 + xy^2 = xy$

uniqueness: $y^2x - yx + 1 = 0.$

$x(y - \frac{1}{2})^2 = \frac{x}{4} - 1$

$(y - \frac{1}{2})^2 = \frac{x-4}{4x}$

$y = \pm\sqrt{\frac{x-4}{4x}} + \frac{1}{2}$

$1 + xy^2 = xy.$

$x(y - y^2) = 1$

$x = \frac{1}{y - y^2}.$

existence: $x = \frac{1}{y - y^2}.$

uniqueness: Assume $x = \frac{1}{y - y^2}.$

Since $y \neq 0$ && $y \neq 1$, this is valid.

left-hand:

right-hand –.

[4] 5. Let $\mathcal{F}$ and $\mathcal{G}$ be families of sets. Prove that $\bigcup\mathcal{F}$ and $\bigcup\mathcal{G}$ are disjoint iff for all $A \in \mathcal{F}$ and all $B \in \mathcal{G}$, A and B are disjoint.

assume an arbitrary $a \in A$. Because $a \in A$, $a \in \mathcal{F}$. Since $\bigcup\mathcal{F}$ and $\bigcup\mathcal{G}$ are disjoint, $a \notin B$, and a is an arbitrary element, it can be written as $\forall a \in A \rightarrow a \notin B$, thus, A and B are disjoint.

assume an arbitrary $a \in A$. Since A and B are disjoint, $\forall a \in A \rightarrow a \notin B$. Since $\forall f \in \mathcal{F} \rightarrow \exists A\ f \in A$, $\forall g \in \mathcal{G} \rightarrow \exists B\ g \in B$, $\forall f \in \mathcal{F} \rightarrow \neg \exists B\ f \in B$, so $\forall f \in \mathcal{F} \rightarrow f \notin \mathcal{G}$. Thus, $\bigcup\mathcal{F}$ and $\bigcup\mathcal{G}$ are disjoint.

$\rightarrow$: Assume $\bigcup\mathcal{F}$ and $\bigcup\mathcal{G}$ are disjoint, that is $\bigcup\mathcal{F} \cap \bigcup\mathcal{G} = \emptyset$. and $\mathcal{F} \cap \mathcal{G} = \emptyset$. Since $A \in \mathcal{F}$, $A \cap \mathcal{G} = \emptyset$, Since $B \in \mathcal{G}$, $A \cap G = \emptyset$.

$\leftarrow$ Assume for all $a \in \mathcal{F}$ $B \in \mathcal{G}$, $A \cap B = \emptyset$. Since A is random, $A \notin \mathcal{G}$

Use this page if you need extra space for your work.

Did you write your name, user name and student ID on the first page?
Did you give full explanations of all steps?