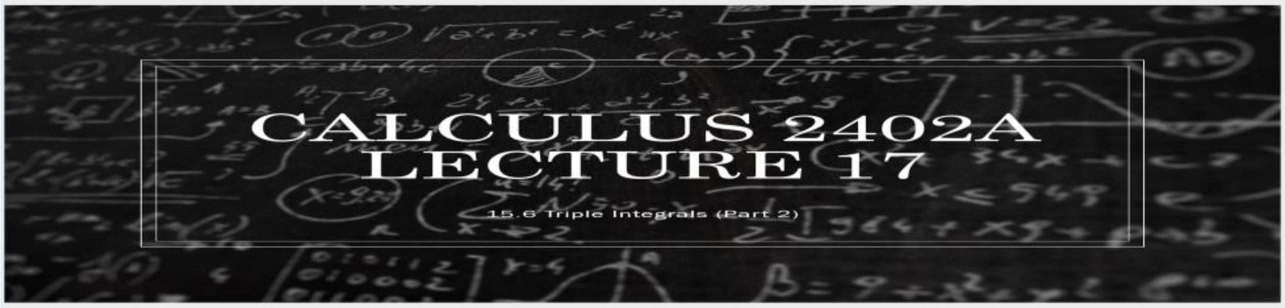
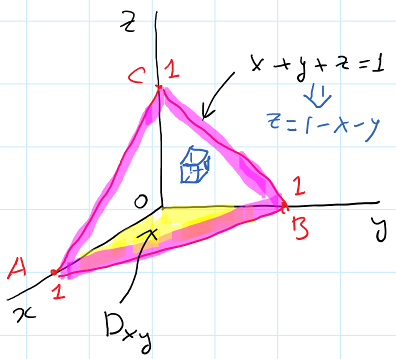
Lecture 17

October 19, 2020 11:35 AM



Ex 2: Find the mass of the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$ and the mass density is $\rho(x,y,z) = y$.

Solution

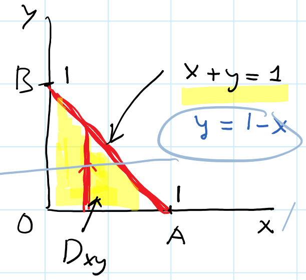


In this case, D is the tetrahedron OABC. We can consider D as a type I region so we integrate wrt z first.

Consider a typical volume element dV. It mass dm is

$$dm = (\text{mass density})\,\text{volume}$$
$$= \rho(x,y,z)\,dV$$
$$= y\,dV$$

$$\therefore \quad m = \iiint_D dm = \iiint_D y\,dV$$



$$= \iint_{D_{xy}} \left(\int_{z=0\ (\text{the } xy\text{-plane})}^{z=1-x-y\ (\text{the plane } ABC)} y\,dz \right) dA$$

$$= \int_0^1 \int_{y=0}^{1-x} \int_0^{1-x-y} y\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} y(1-x-y)\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left[(1-x)y - y^2\right] dy\,dx$$

$$= \int_0^1 \left((1-x)\frac{y^2}{2} - \frac{y^3}{3} \right) \Bigg|_{y=0}^{1-x} dx$$

$$= \int_0^1 \left[(1-x)\frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} \right] dx$$

$$= \int_0^1 \left[(1-x)\frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} \right] dx$$

$$= \int_0^1 \left[\frac{(1-x)^3}{2} - \frac{(1-x)^3}{3} \right] dx$$

$$= \int_0^1 \frac{(1-x)^3}{6}\, dx \qquad \text{let } \boxed{\begin{array}{l} \boxed{u = 1-x} \\ du = -dx \end{array}}$$

$$= \int_1^0 \frac{u^3}{6} (-du)$$

$$\underset{\substack{\uparrow \\ \text{Switching the limits}}}{=} \frac{1}{6}\int_0^1 u^3\, du = \frac{1}{6}\left(\frac{u^4}{4}\right)\Big|_0^1 = \frac{1}{6}\left(\frac{1}{4}\right) = \frac{1}{24} \quad // \text{Ans.}$$

In double integrals, there are two ways of integration. They are $dy\,dx$ and $dx\,dy$. However, for triple integrals, there are $3! = 6$ ways of integration. They are

$dx\,dy\,dz$	$dy\,dz\,dx$	$dz\,dx\,dy$
$dx\,dz\,dy$	$dy\,dx\,dz$	$dz\,dy\,dx$

Ex 3 : Consider the iterated integral

$$I = \int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\, dz\, dy\, dx \quad (\star)$$

Change the order of integration to $dz\,dx\,dy$, $dx\,dy\,dz$, $dx\,dz\,dy$, $dy\,dz\,dx$ and $dy\,dx\,dz$.

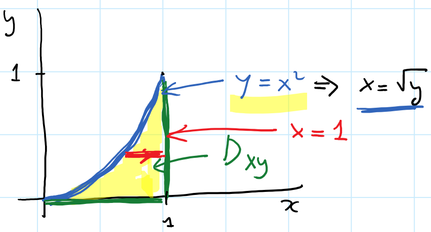
Solution

$(\star)$ can be written as

$$I = \iint_{D_{xy}} \left(\int_0^y f(x,y,z)\, dz \right) dA$$

where D_{xy} is the projection of D onto the xy-plane.

$$D_{xy} = \{ (x,y) \mid 0 \le x \le 1,\ 0 \le y \le x^2 \}$$



$$I = \int_0^1 \int_{x=\sqrt{y}}^{x=1} \int_0^y f(x,y,z)\, dz\, dx\, dy$$

$$\int_0 \int_{x=\sqrt{y}} \int_0 f(x,y,z)\, dz\, dx\, dy$$

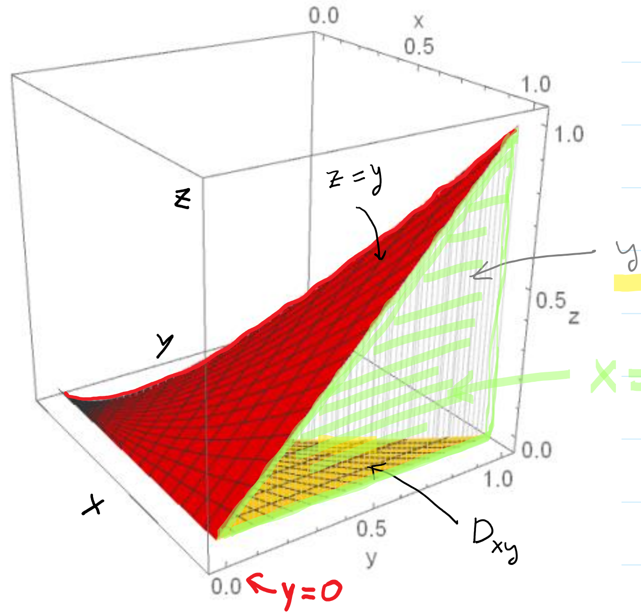
$$= \int_0^1 \int_{\sqrt{y}}^1 \int_0^y f(x,y,z)\, dz\, dx\, dy \qquad // \text{Ans.}$$

If we integrate wrt x first, then

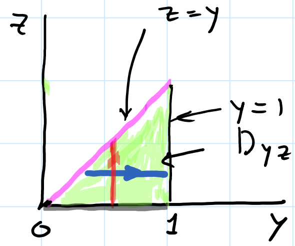
$$I = \iint_{D_{yz}\,?} \left(\int_{\sqrt{y}}^1 f(x,y,z)\, dx \right) dA = \int_0^1 \int_{y=z}^{y=1} \int_{\sqrt{y}}^1 f(x,y,z)\, dx\, dy\, dz$$

$$= \int_0^1 \int_z^1 \int_{\sqrt{y}}^1 f(x,y,z)\, dx\, dy\, dz \qquad // \text{Ans.}$$

$$I = \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\, dx\, dz\, dy \qquad // \text{Ans.}$$

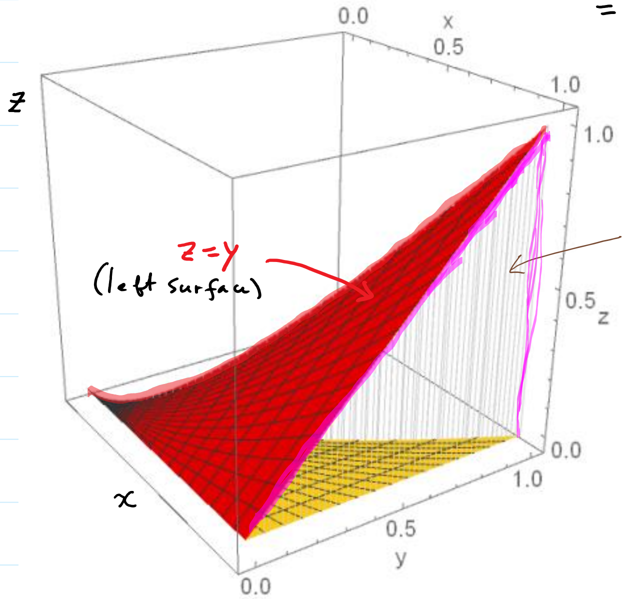


$y = x^2 \Rightarrow x = \sqrt{y}$

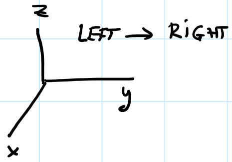


$$I = \iint_{D_{xz}\,?} \left(\int_z^{x^2} f(x,y,z)\, dy \right) dA = \int_0^1 \int_{z=0}^{z=x^2} \int_z^{x^2} f(x,y,z)\, dy\, dz\, dx$$

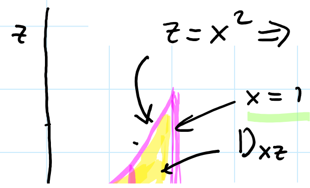
$$= \int_0^1 \int_0^{x^2} \int_z^{x^2} f(x,y,z)\, dy\, dz\, dx \qquad // \text{Ans.}$$



$y = x^2$ (a cylinder)
The right surface

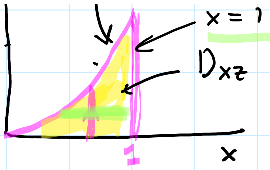


The curve of intersection of the surfaces $z = y$ and $y = x^2$ is $z = x^2$



$z = x^2 \Rightarrow x = \sqrt{z}$

$$I = \int^1 \int^{x=1} \int^{x^2} f(x,y,z)\, dy\, dx\, dz$$



$$I = \int_0^1 \int_{x=\sqrt{z}}^{x=1} \int_z^{x^2} f(x,y,z)\, dy\, dx\, dz$$

$$= \int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} f(x,y,z)\, dy\, dx\, dz \quad // \text{Ans.}$$

N.B: If $f(x,y,z) \equiv 1$ throughout the region D then the triple integral becomes $\iiint_D dV$ which is the volume V of D.

$$V = \int_0^1 \int_0^{x^2} \int_0^y dz\, dy\, dx = \int_0^1 \int_0^{x^2} y\, dy\, dx$$

$$= \int_0^1 \left(\frac{y^2}{2}\right)\Bigg|_{y=0}^{x^2} dx = \int_0^1 \frac{x^4}{2}\, dx = \frac{1}{2}\frac{x^5}{5}\Bigg|_0^1 = \frac{1}{10} \quad // \text{Ans.}$$

OR

$$V = \int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} dy\, dx\, dz = \int_0^1 \int_{\sqrt{z}}^1 (x^2 - z)\, dx\, dz$$

$\sqrt{z} = z^{\frac{1}{2}}$

$$= \int_0^1 \left(\frac{x^3}{3} - zx\right)\Bigg|_{x=\sqrt{z}}^1 dz = \int_0^1 \left[\frac{1}{3} - z - \underbrace{\left(\frac{z^{3/2}}{3} - z z^{\frac{1}{2}}\right)}_{-\frac{2}{3} z^{3/2}}\right] dz$$

$$= \int_0^1 \left(\frac{1}{3} - z + \frac{2}{3} z^{3/2}\right) dz = \left(\frac{1}{3} z - \frac{z^2}{2} + \frac{2}{3}\,\frac{2}{5}\, z^{5/2}\right)\Bigg|_0^1$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{4}{15} = \frac{10 - 15 + 8}{30} = \frac{3}{30} = \frac{1}{10} \quad // \text{Ans.}$$

the same!

See you on Wednesday!