UWO CS2214

Tutorial #12

Problem 1 Let $V = \{1, 2, \ldots, n\}$. How many different (simple, undirected) graphs with vertex set V are there?

Solution 1 The edges of G are subsets of V with two elements. Hence, the answer is $2^{\binom{n}{2}}$.

See details at: `http://www.maths.lse.ac.uk/Personal/jozef/MA210/06sol.pdf`

Problem 2 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs given by:

$$V_1 = \{1, 2, 3, 4, 5, 6\} \text{ and } E_1 = \{\{1, 2\}, \{2, 3\}, \{3, 1\}, \{4, 5\}, \{5, 6\}, \{6, 4\}\},$$

and

$$V_2 = \{a, b, c, d, e, f\} \text{ and } E_2 = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}, \{e, f\}, \{f, a\}\},$$

1. Let $G = (V, E)$ and $H = (W, F)$ be two isomorphic simple graphs. Let f be a one-to-one correspondence from V to W realizing that isomorphism. Let X be a non-empty subset of V. Let G_X be the subgraph of G induced by X. Let $Y = f(X)$ be the image of X in W. Let H_Y be the subgraph of H induced by Y. Prove that the the graphs G_X and H_Y are isomorphic.

2. Are G_1 and G_2 isomorphic? Justify your answer.

Solution 2

1. We need to prove the following statement: For every $v, u \in X$, with $u \neq v$, we have:

   $$\{v, u\} \text{ is an edge of } G_X \text{ if and only if } \{f(v), f(u)\} \text{ is an edge of } H_Y.$$

   So, let $v, u \in X$, with $u \neq v$. Since G and H are isomorphic The following statement is already true:

   $$\{v, u\} \text{ is an edge of } G \text{ if and only if } \{f(v), f(u)\} \text{ is an edge of } H.$$

By definition of G_X, we have:

$$\{v, u\} \text{ is an edge of } G \text{ if and only if } \{v, u\} \text{ is an edge of } G_X$$

Since $f(v)$ and $f(u)$ belong to Y, by definition of H_Y, we also have:

$$\{f(v), f(u)\} \text{ is an edge of } H \text{ if and only if } \{f(v), f(u)\} \text{ is an edge of } H_Y.$$

Combining the last 3 equivalence yields the desired one.

2. We apply the result of the previous question. We note that $\{1, 2, 3\}$ induces on G_1 a subgraph which is isomorphic to the complete graph K_3. But no subgraph of G_2 is isomorphic to K_3. Indeed, G_2 is isomorphic to the cycle C_6.

Problem 3 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two isomorphic simple graphs.

1. Show that if one is a **complete** bipartite graph, then the other is also a **complete** bipartite graph.
2. Show that if one is bipartite, then the other is also bipartite.
3. Show that if one is connected, then the other is also connected.

Solution 3

1. Assume that G_1 is a complete bipartite graph, with "colors" B (for blue) and R (for red). That is, $\{B, R\}$ is a partition of V_1 so that for every u, v in V_1 we have:

   $$\{v, u\} \text{ is an edge of } G \text{ if and only if } (u, v) \in R \times B \cup B \times R.$$

   In other words, the edges of G_1 connect the red and blue vertices.

   Let f be a one-to-one correspondence from V_1 to V_2 realizing that isomorphism between G_1 and G_2. Hence, for every u, v in V_1 we have:

   $$\{v, u\} \text{ is an edge of } G_1 \text{ if and only if } \{f(v), f(u)\} \text{ is an edge of } G_2.$$

   Clearly, $\{f(B), f(R)\}$ is a partition of V_2. Moreover, f realizes a one-to-one correspondence between B and $f(B)$ as well as a one-to-one correspondence between R and $f(R)$. Therefore, for every u, v in V_1 we have:

$$(u,v) \in R \times B \cup B \times R \text{ if and only if}$$
$$(f(u), f(v)) \in f(R) \times f(B) \cup f(B) \times f(R)$$

Putting everything together, we have for every u, v in V_1:

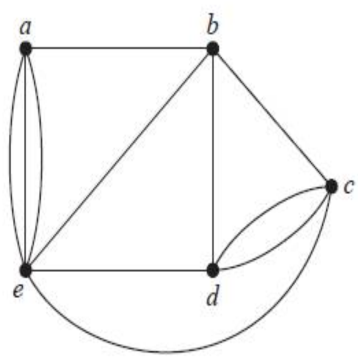
$$\{f(v), f(u)\} \text{ is an edge of } G_2 \text{ if and only if}$$
$$(f(u), f(v)) \in f(R) \times f(B) \cup f(B) \times f(R)$$

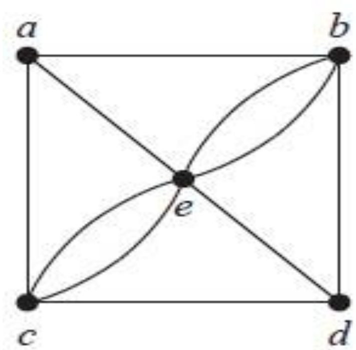
In other words, the edges of G_1 connect vertices from $f(B)$ to vertices from $f(R)$. Therefore, G_2 is a complete bipartite graph.

2. The proof is similar to the previous one. One can use the following characterization of a bipartite graph: there is a partition $\{B, R\}$ of the vertex set so that the induced graphs G_R and G_B have no edges.

3. The proof is similar to the previous one. One can use the following characterization of a connected graph. The simple graph $G = (V, E)$ is connected if for $v, u \in V$, with $u \neq v$, there is a path from u to v.

Problem 4 For each of the following two graphs, determine whether or not it has an Euler circuit. Justify your answers. If the graph has an Euler circuit, use the algorithm described in class to find it, including drawings of intermediate subgraphs.





Solution 4

1. First, consider the graph on the left. Every node has an even degree, hence there exists an Euler circuit, say from a to a. We use the algorithm seen in class. Observe that the following circuits partition of the set of the edges:

$$(a, b, d, e, a), (\mathbf{b}, \mathbf{c}, \mathbf{e}, \mathbf{b}), (c, d, c), (\mathbf{a}, \mathbf{e}, \mathbf{a}).$$

From there we deduce am Euler circuit from a to a, by first merging the first two elementary circuits:

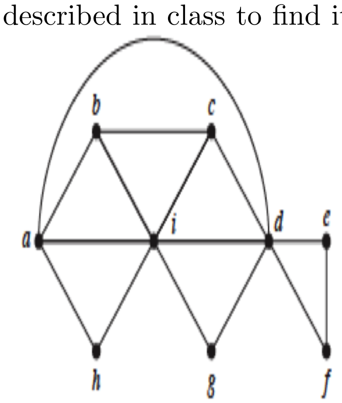
$$(a, \mathbf{b}, \mathbf{c}, \mathbf{e}, \mathbf{b}, d, e, a), (c, d, c), (\mathbf{a}, \mathbf{e}, \mathbf{a}).$$

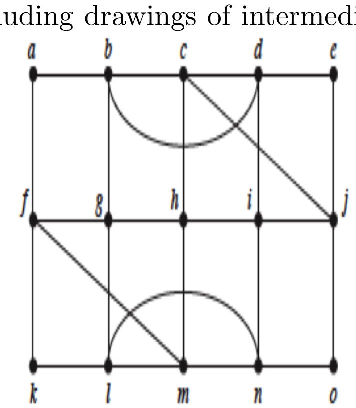
Then, adding the two circuits of length 2 yields the Euler circuit:

$$(a, b, c, d, c, e, b, d, e, a, e, a)$$

2. Second, consider the graph on the right. Every node has an even degree, except a and d. Hence there exists an Euler path from a to d. Removing the edge (a, f) we build an Euler circuit around a. Using the algorithm, we have the following circuits partition of the set of the edges:
$$(a, b, d, c, a), (b, e, c, e, b).$$
From there we deduce am Euler circuit from a to a:
$$(a, b, e, c, e, b, d, c, a)$$
and an Euler path from a and d:
$$(a, b, e, c, e, b, d, c, a, d)$$

Problem 5 For each of the following two graphs, determine whether or not it has an Euler circuit. Justify your answers. If the graph has an Euler circuit, use the algorithm described in class to find it, including drawings of intermediate subgraphs. If the graph has an Euler path, use the algorithm described in class to find it, including drawings of intermediate subgraphs.





Solution 5 1. *Removing* (b, c) *makes every degree even. Now we build an Euler circuit around* b. *We start with* b, i, h, a, b *We insert at* i *the Euler circuit around* i*:* i, c, d, e, j, d, g, i *We insert at* a *the Euler circuit around* a*:* a, d, i, a.

2. *Every degree even, hence there exists an Euler circuit. Now we build an Euler circuit around* a. *We start with the "frontier" :* $a, b, c, d, e, j, o, n, m, l, k, f$. *Next, we go around* b *using 4 vertical edges and the circular edges. We are left with two triangles*