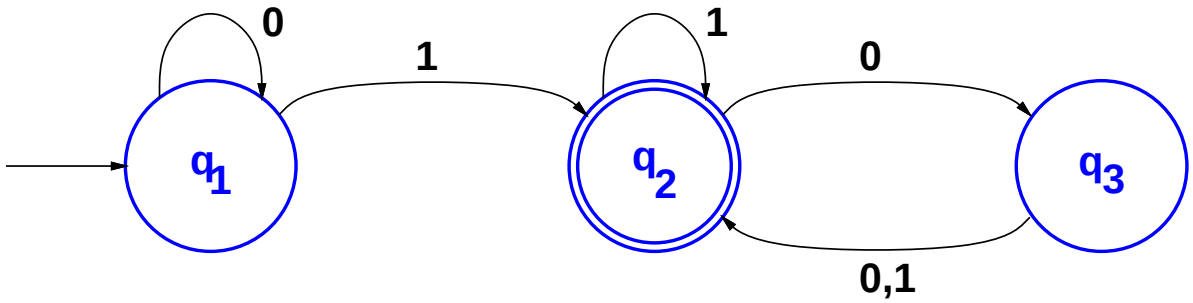
DFA Formal Definition (reminder)

A deterministic finite automaton (DFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

- Q is a finite set called the states,
- Σ is a finite set called the alphabet,
- $\delta : Q \times \Sigma \to Q$ is the transition function,
- $q_0 \in Q$ is the start state, and
- $F \subseteq Q$ is the set of accept states.

Back to M_1



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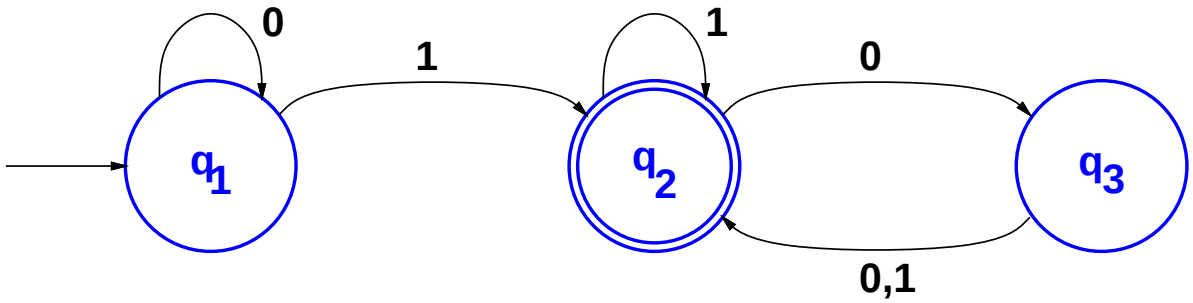

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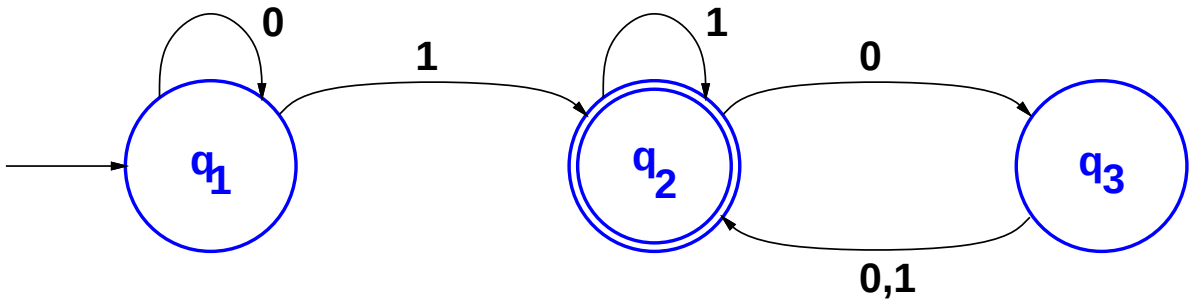

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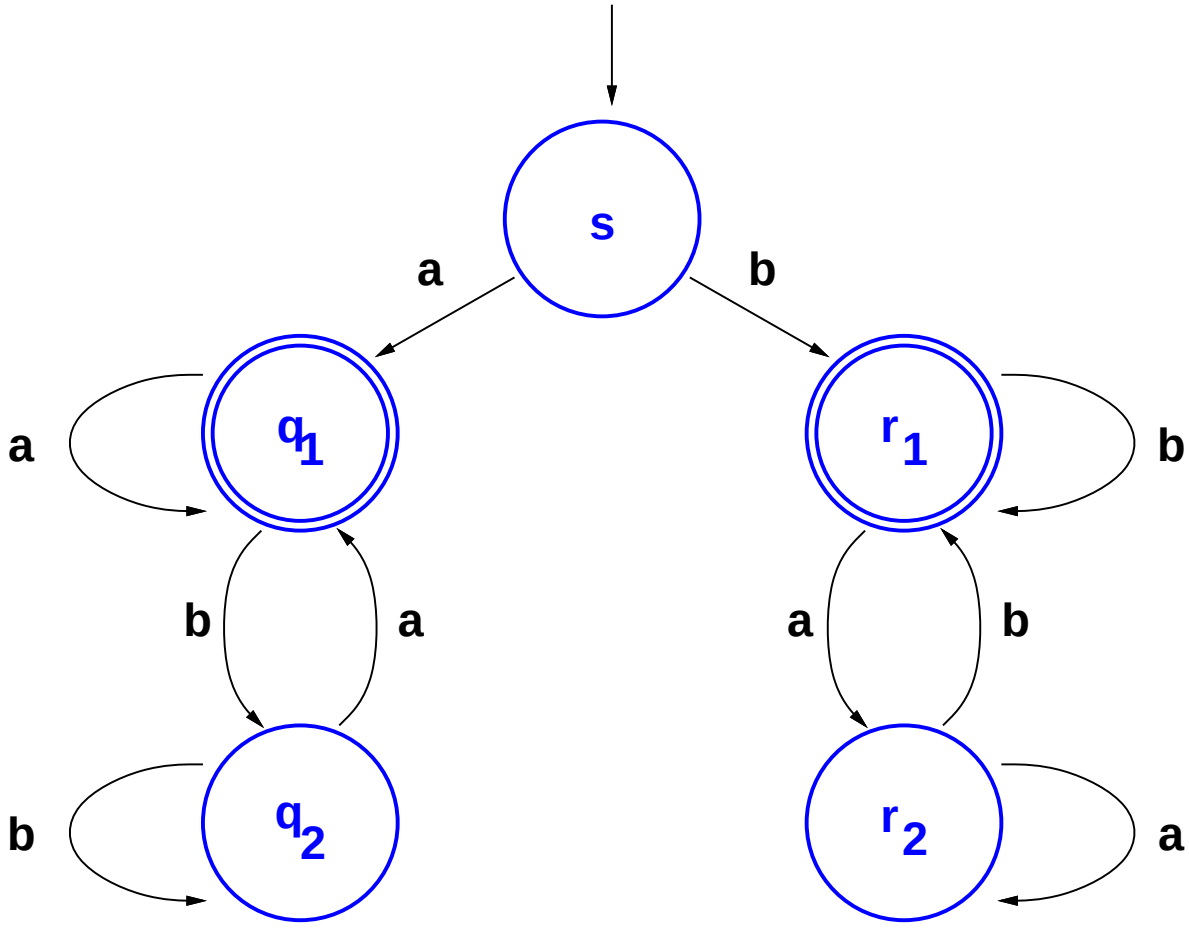

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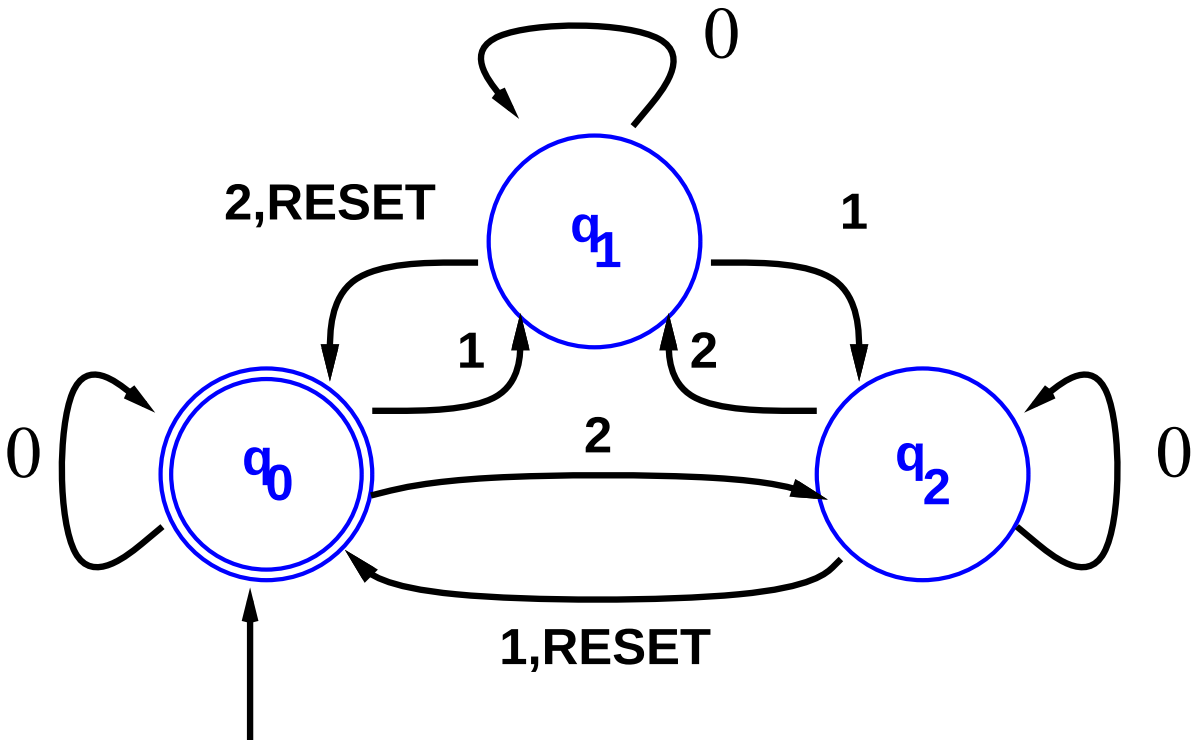

A Formal Model of Computation

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA, and
- let $w = w_1 w_2 \cdots w_n$ be a string over Σ.

We say that M accepts w if there is a sequence of states $r_0, \ldots, r_n$ ($r_i \in Q$) such that

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The Regular Operations

Let A and B be languages.

The union operation:

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

The concatenation operation:

$$A \circ B = \{xy | x \in A \text{ and } y \in B\}$$

The star operation:

$$A^* = \{x_1 x_2 \ldots x_k | k \geq 0 \text{ and each } x_i \in A\}$$

The Regular Operations – Examples

Let $A = \{\text{good}, \text{bad}\}$ and $B = \{\text{boy}, \text{girl}\}$.

Union

$$A \cup B = \{\text{good}, \text{bad}, \text{boy}, \text{girl}\}$$

Concatenation

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Star

$$A^* = \{\varepsilon, \text{good}, \text{bad}, \text{goodgood}, \text{goodbad}, \text{badbad}, \text{badgood}, \ldots\}$$

Claim: Closure Under Union

If A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

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Fix: Simulate both machines simultaneously.

Closure Under Union: Correct Proof

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(hey, why not choose $F = F_1 \times F_2$?)

What About Concatenation?

Thm: If L_1, L_2 are regular languages, so is $L_1 \circ L_2$.

Example: $L_1 = \{\text{good}, \text{bad}\}$ and $L_2 = \{\text{boy}, \text{girl}\}$.

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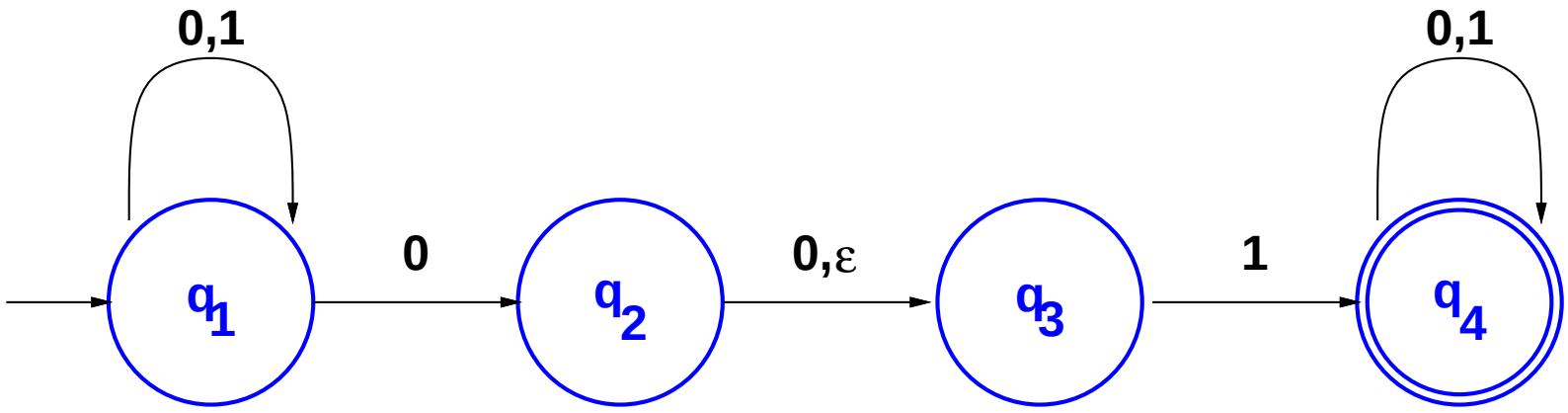
Problem: But when do you switch?

This leads us into non-determinism.

Non-Deterministic Finite Automata



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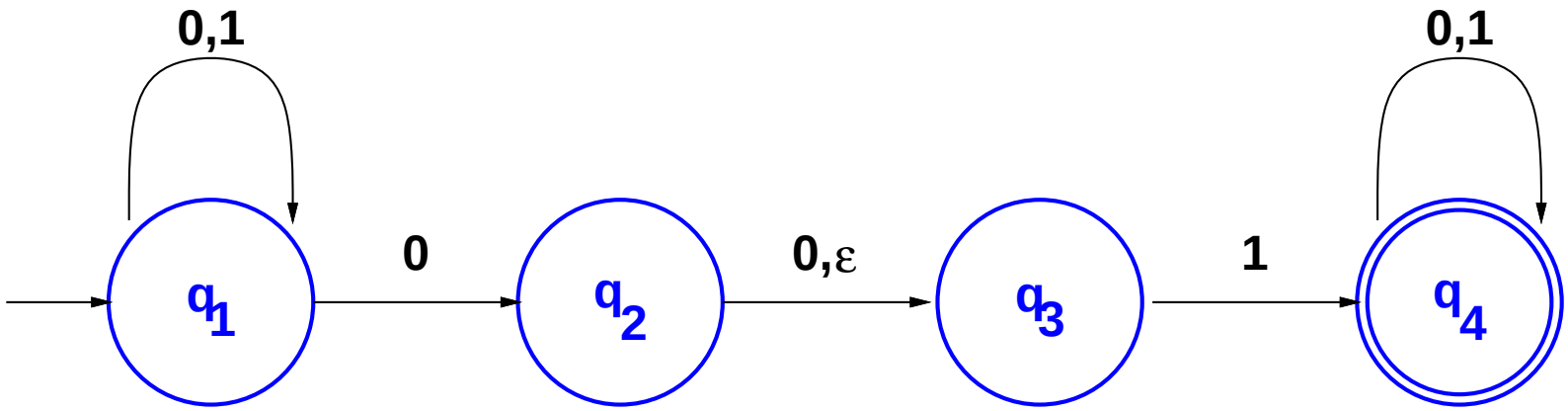

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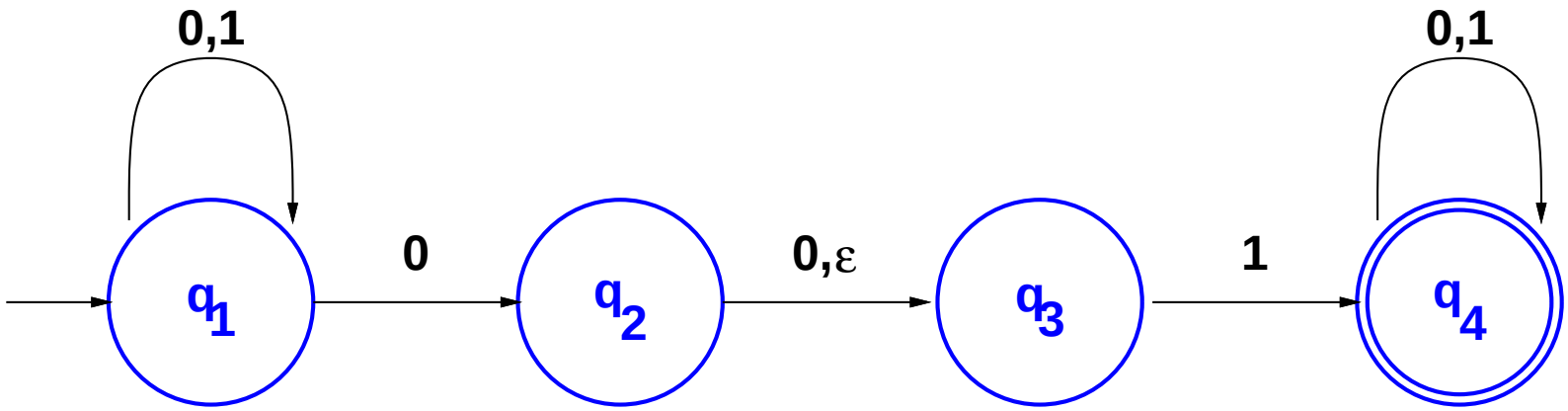

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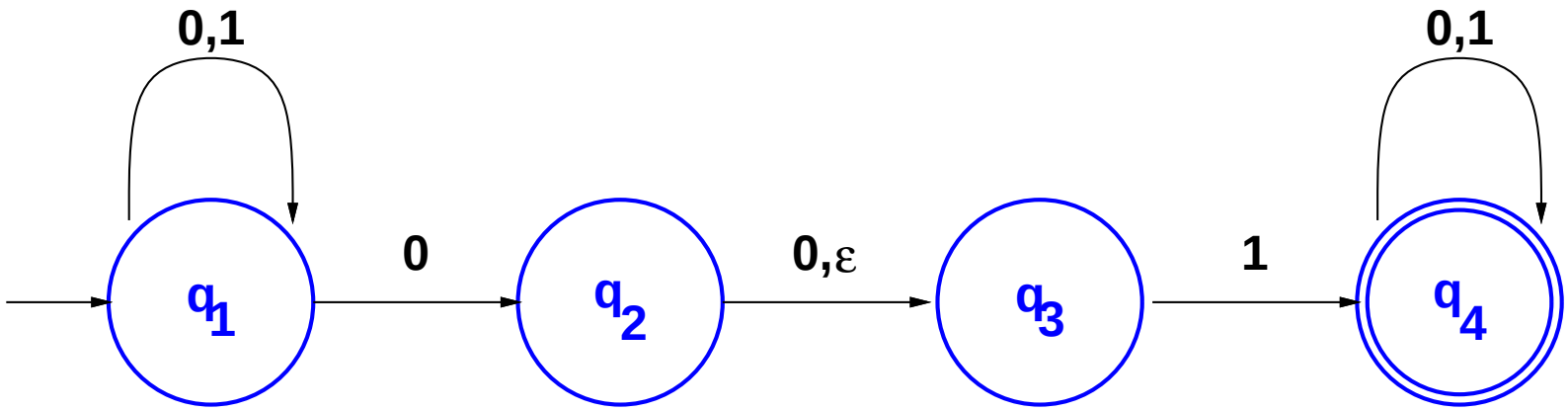

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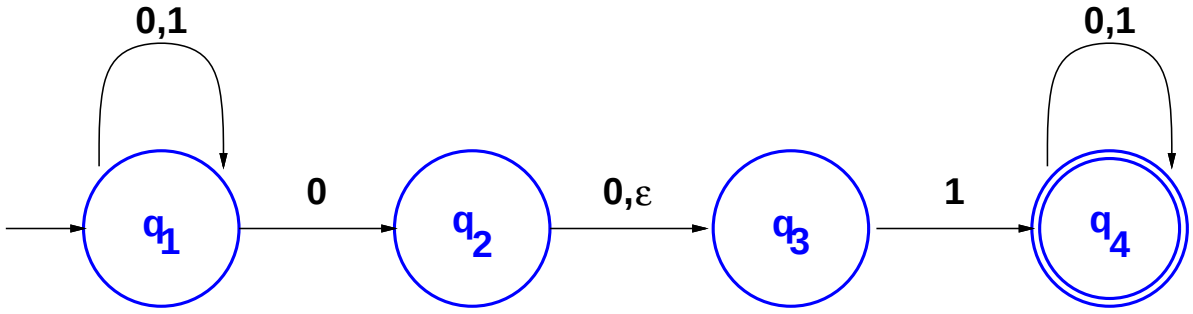
Comment: Every DFA is also a non-deterministic finite automata (NFA).

Non-Deterministic Computation



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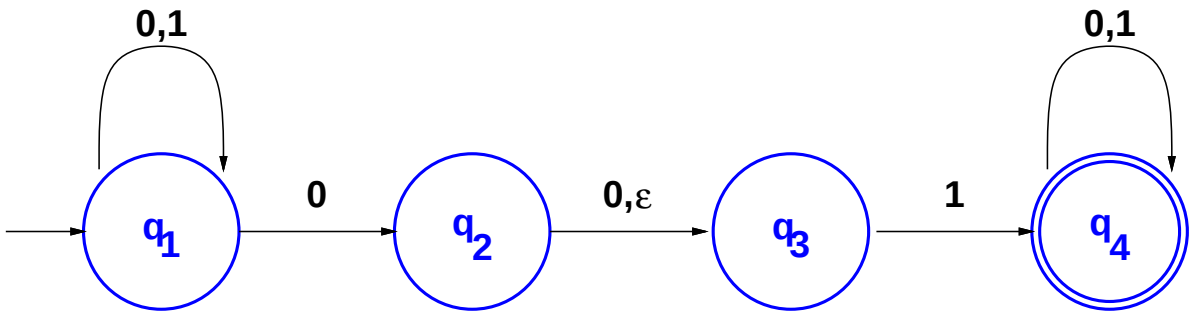

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- the machine “splits” into multiple copies
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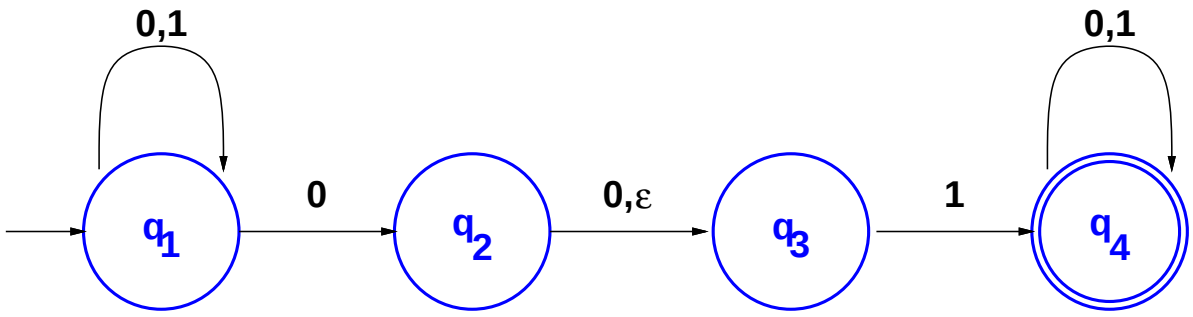

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What does an ε transition do?

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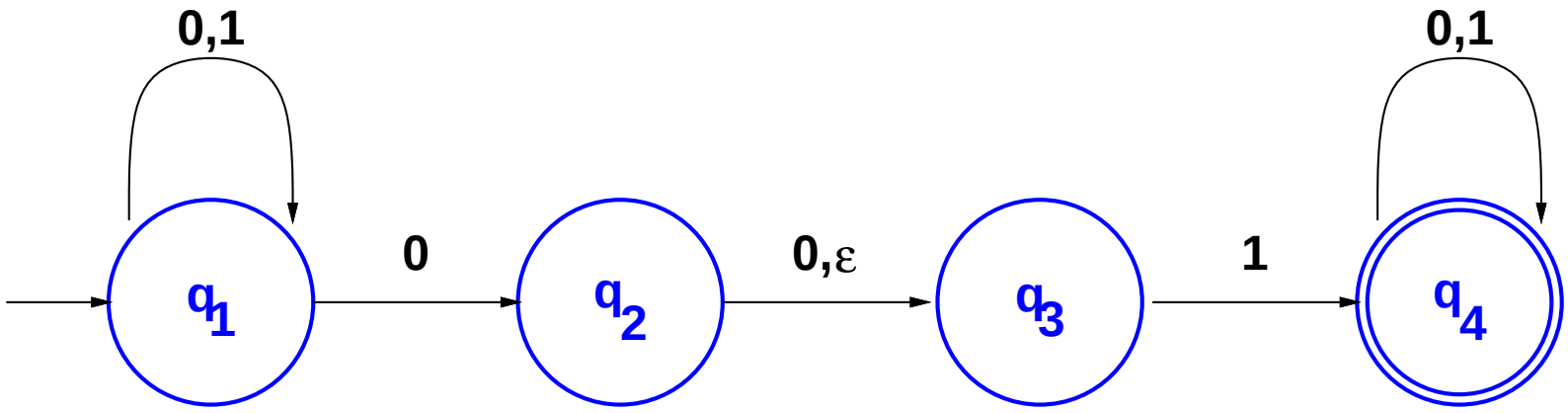
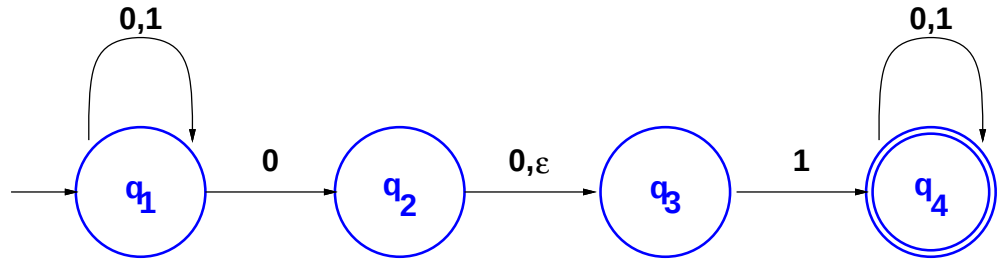


What happens on string 1001?

The String 1001



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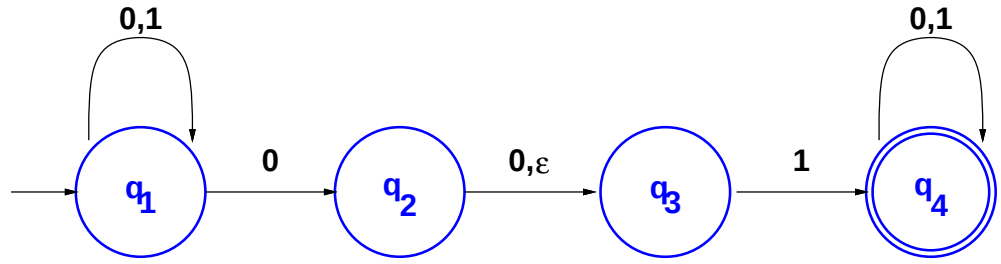

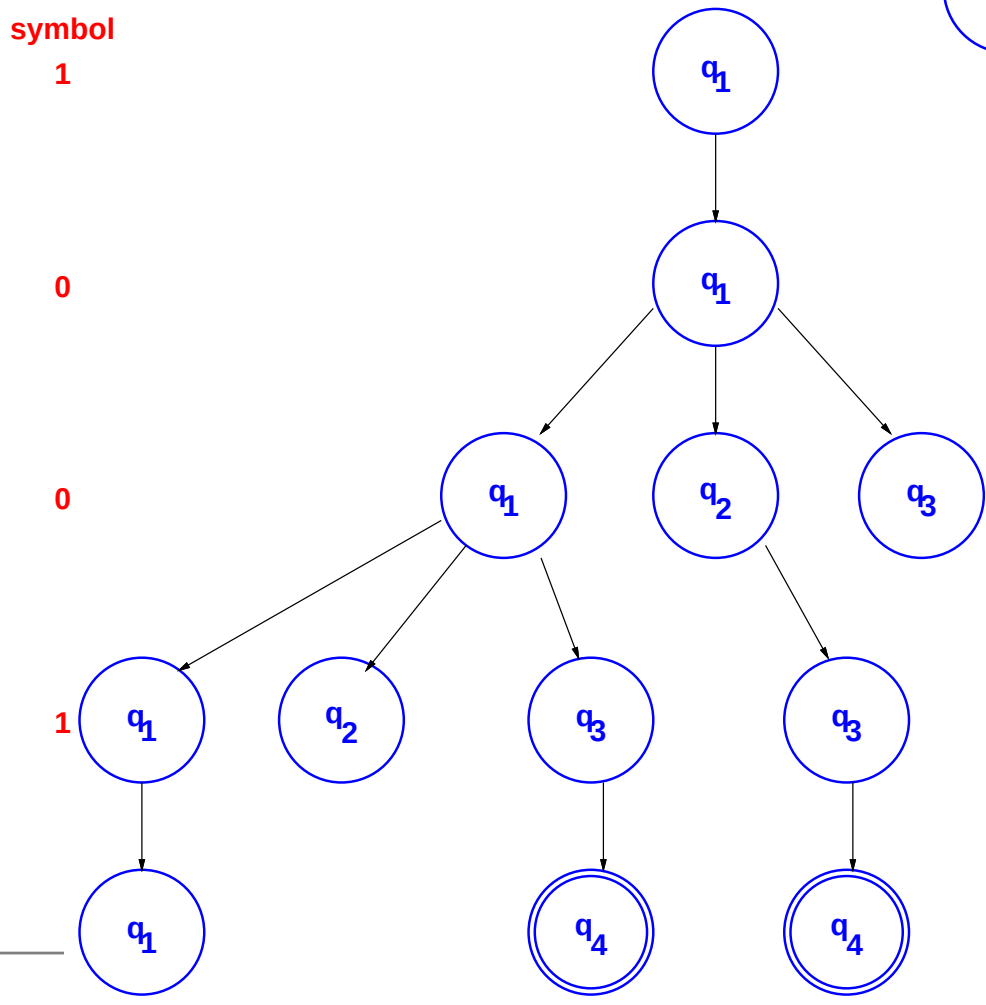

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Theorem: Deterministic and non-deterministic finite automata accept exactly the same set of languages.

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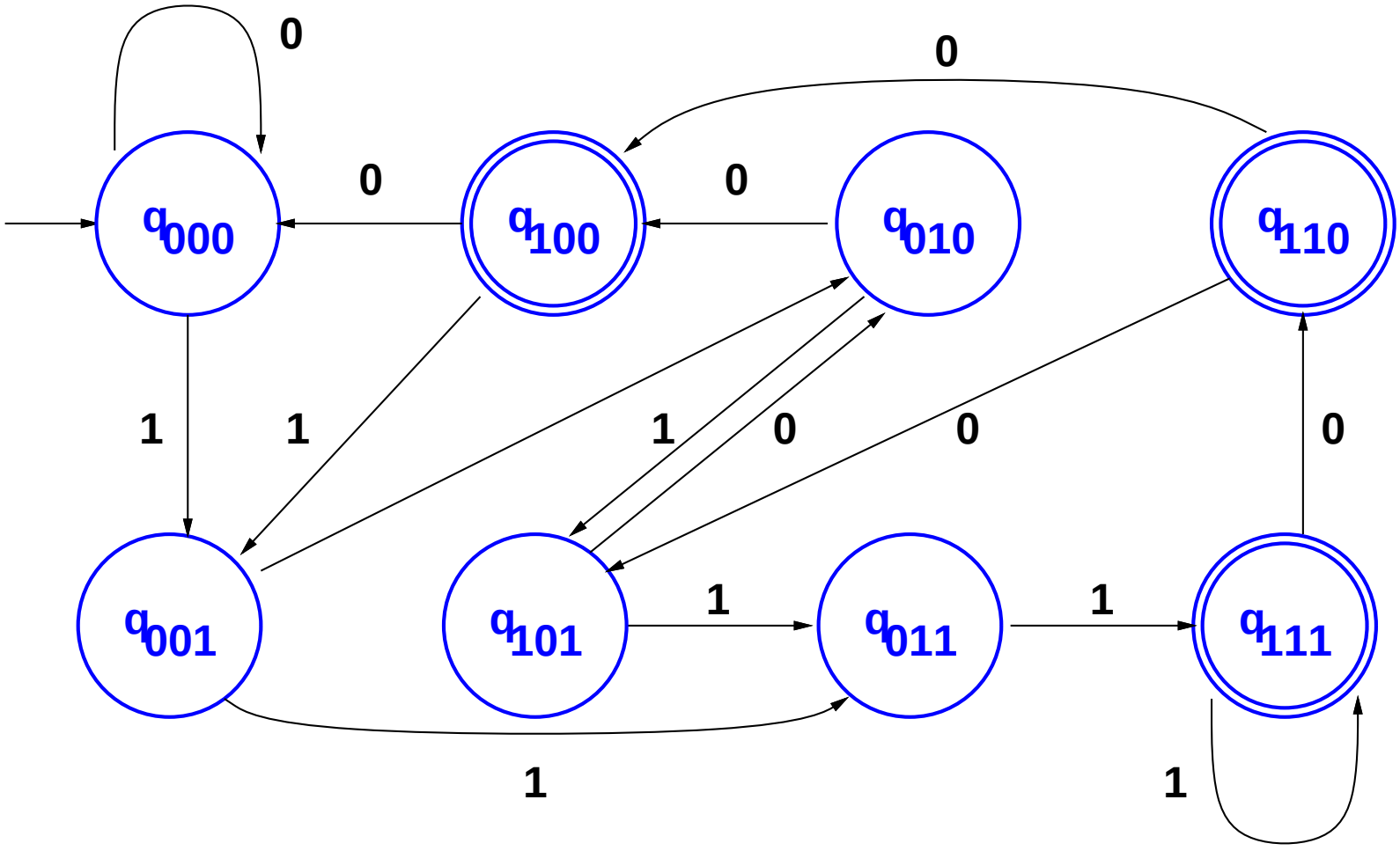
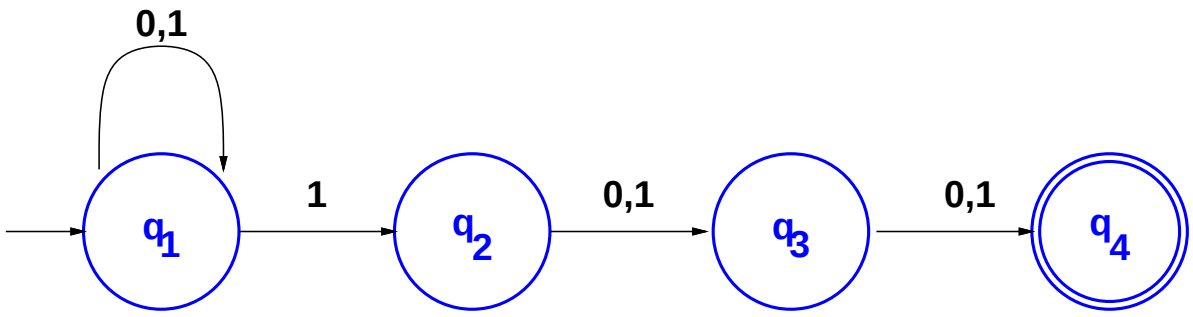
Example: Design a finite automaton that accepts all strings with a 1 in their third-to-the-last position?

A Deterministic Automaton



(there are a few errors, *e.g.* q_{101} should be an accept state, but overall it is OK.)

A Non-Deterministic Automaton



- "Guesses" which symbol is third from the last, and
- checks that it's a 1.

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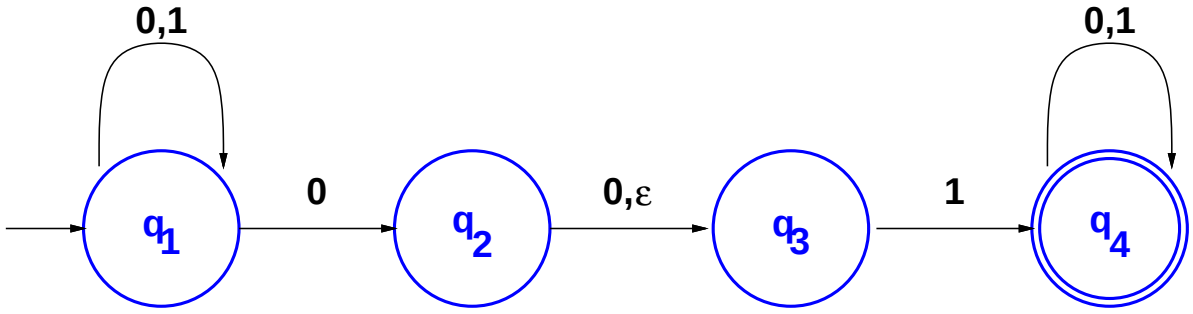

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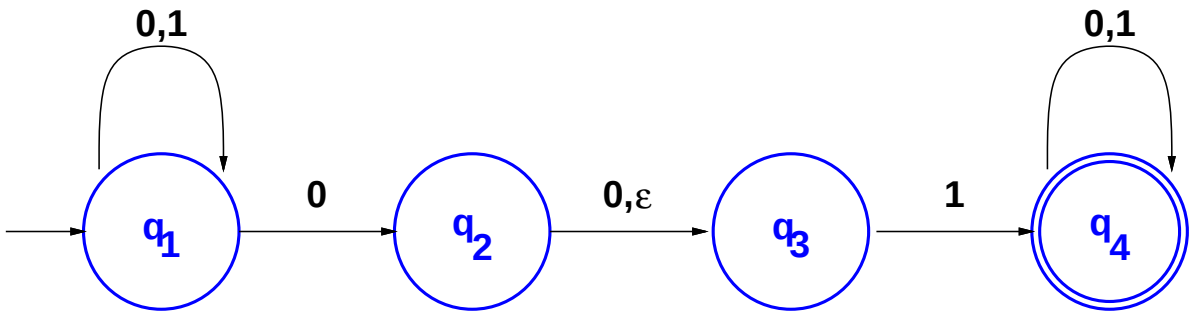

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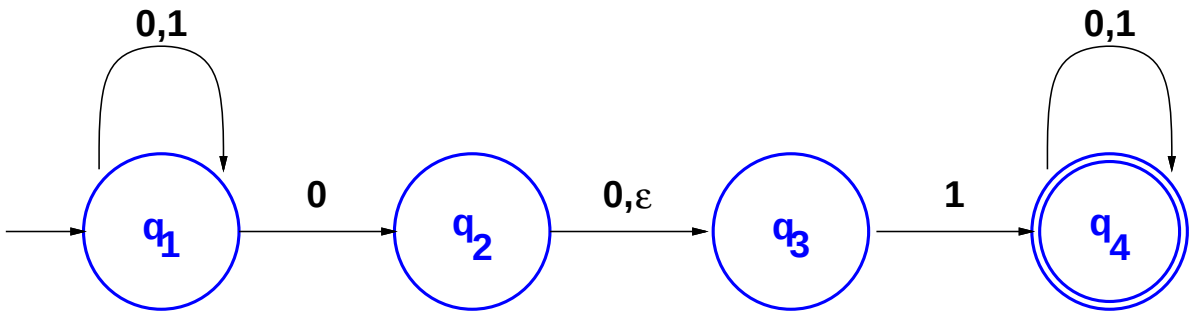

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- q_1 is the start state, and $F = \{q_4\}$.

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Then we say that M accepts u.

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Equivalence of NFA's and DFA's

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- As consequence of the construction, if the NFA has k states, the DFA has 2^k states.

Equivalence of NFA's and DFA's

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Change start state to

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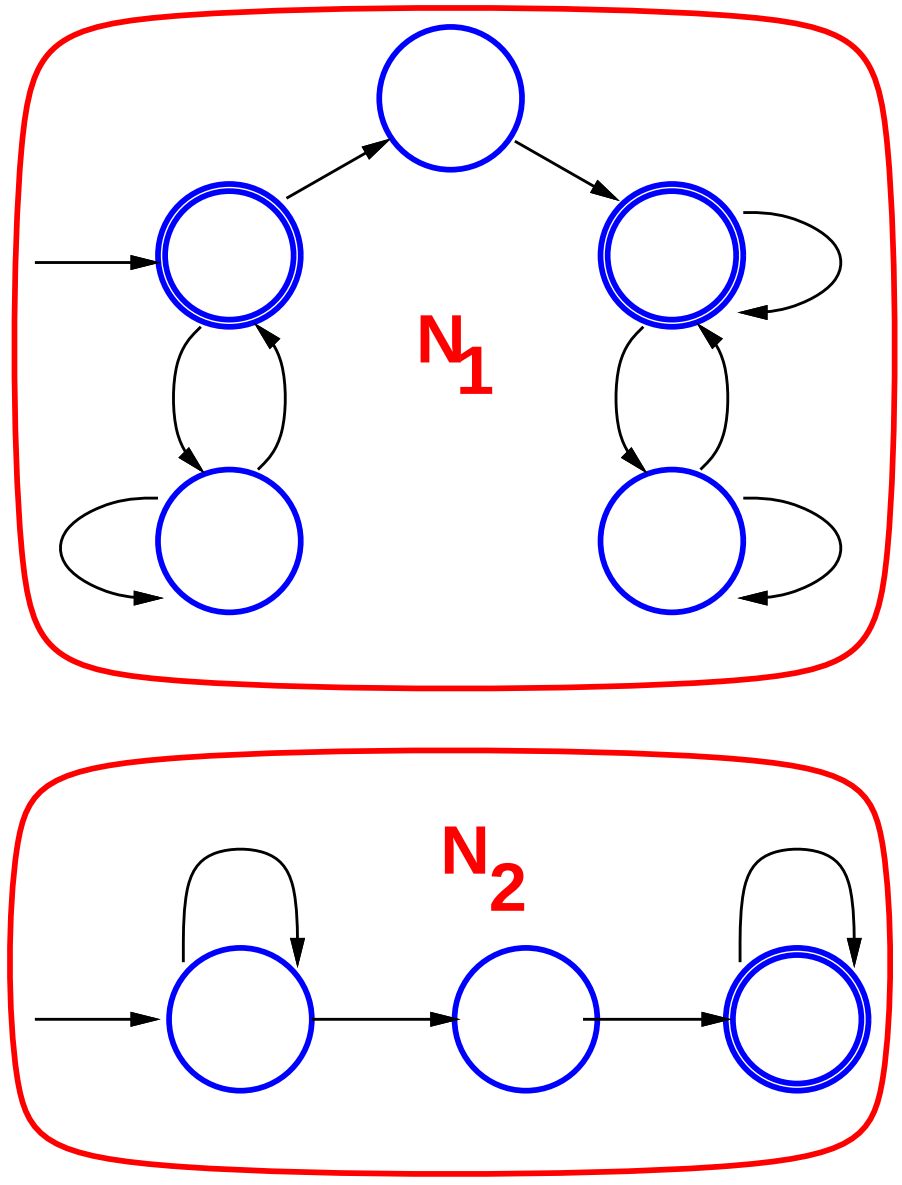
We will now use the equivalence to show that regular languages are closed under the regular operations (union, concatenation, star).

Regular Languages Closed Under Union



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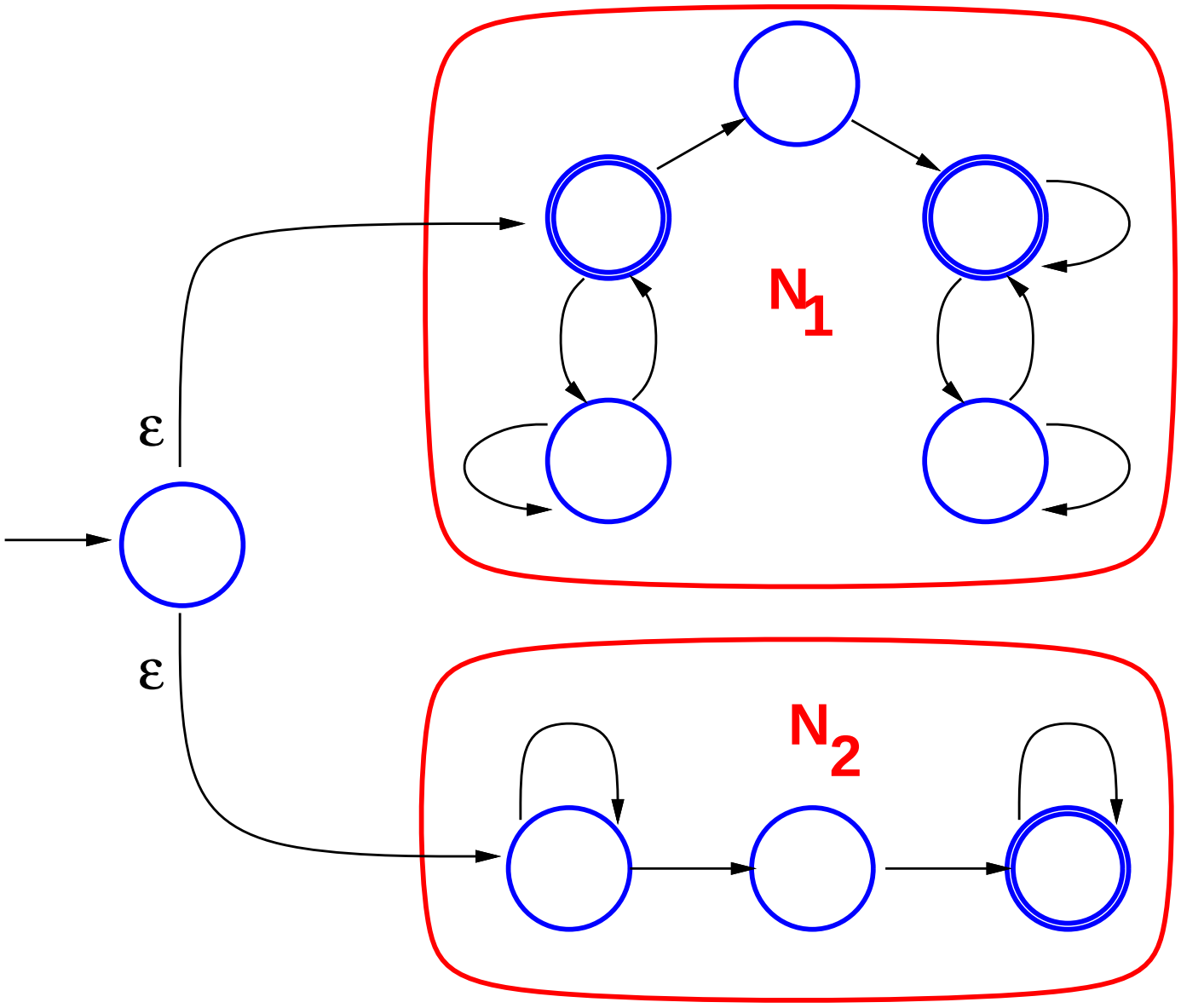

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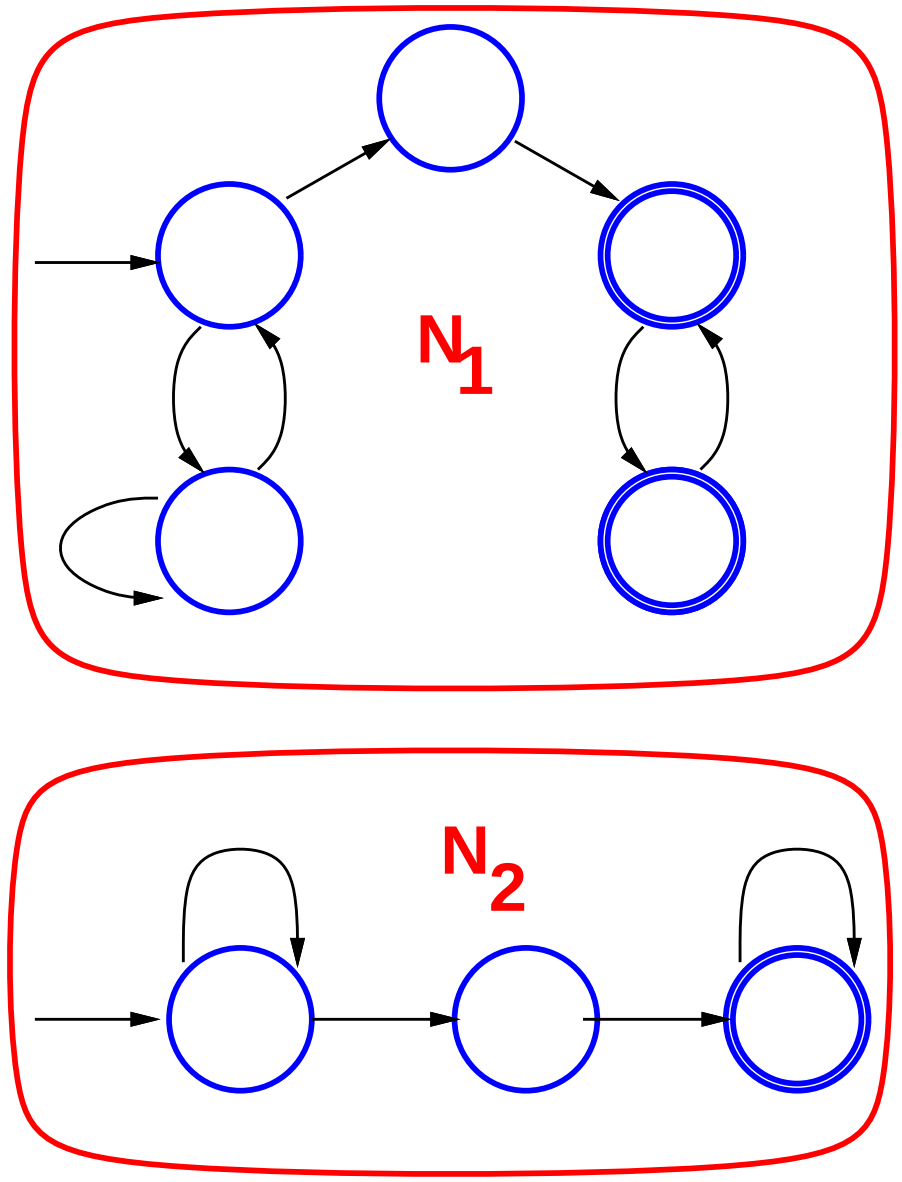
Suppose

- $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ accept L_1, and
- $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ accept L_2.

Define $N = (Q, \Sigma, \delta, q_0, F)$:

- $Q = q_0 \cup Q_1 \cup Q_2$
- Σ is the same, q_0 is the start state
- $F = F_1 \cup F_2$
- $$\delta'(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \\ \delta_2(q,a) & q \in Q_2 \\ \{q_1, q_2\} & q = q_0 \text{ and } a = \varepsilon \\ \emptyset & q = q_0 \text{ and } a \neq \varepsilon \end{cases}$$

Regular Languages Closed UnderConcatenation



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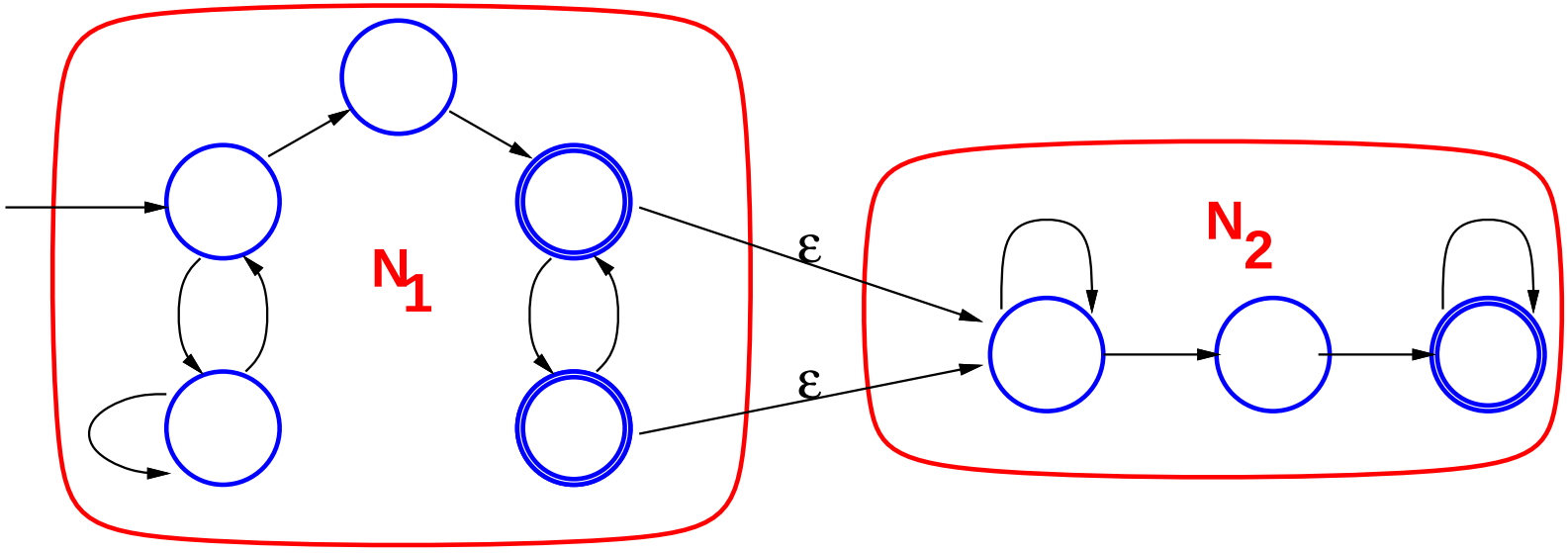

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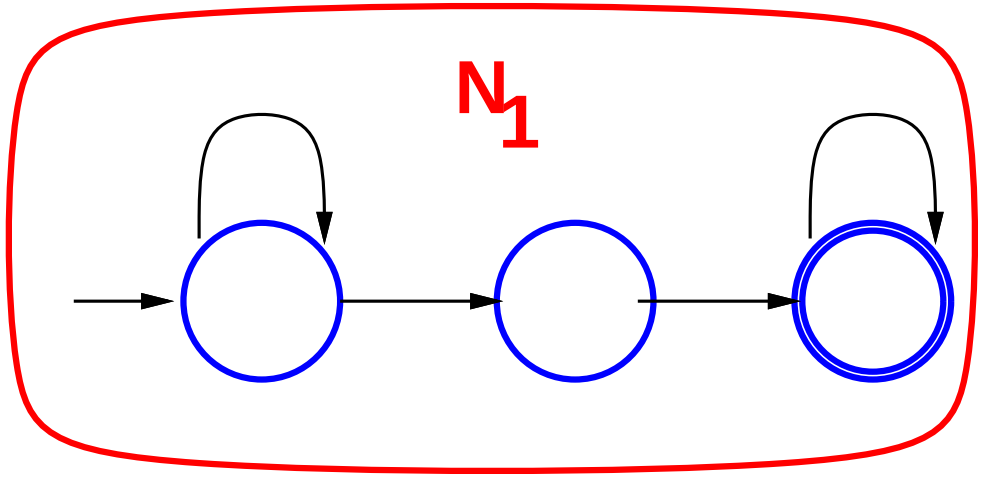

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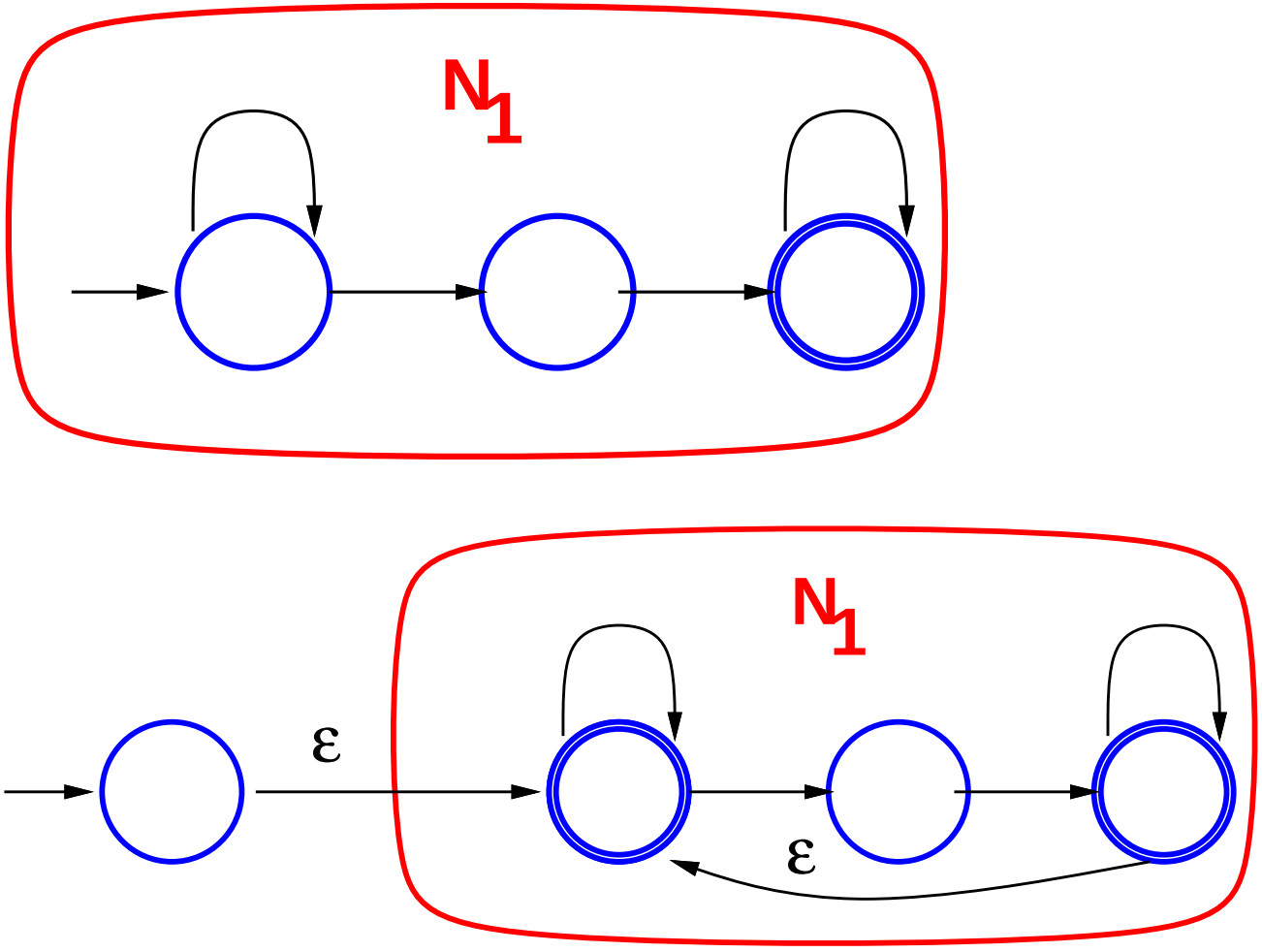

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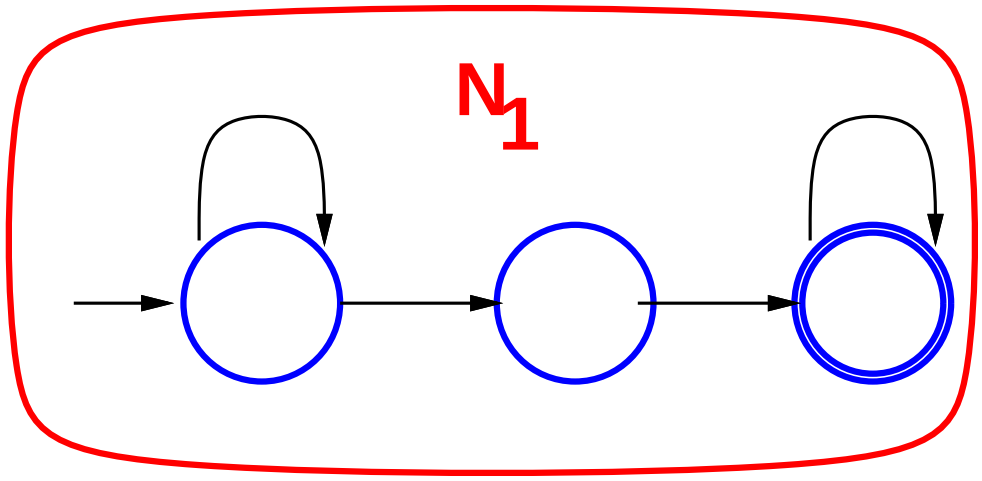

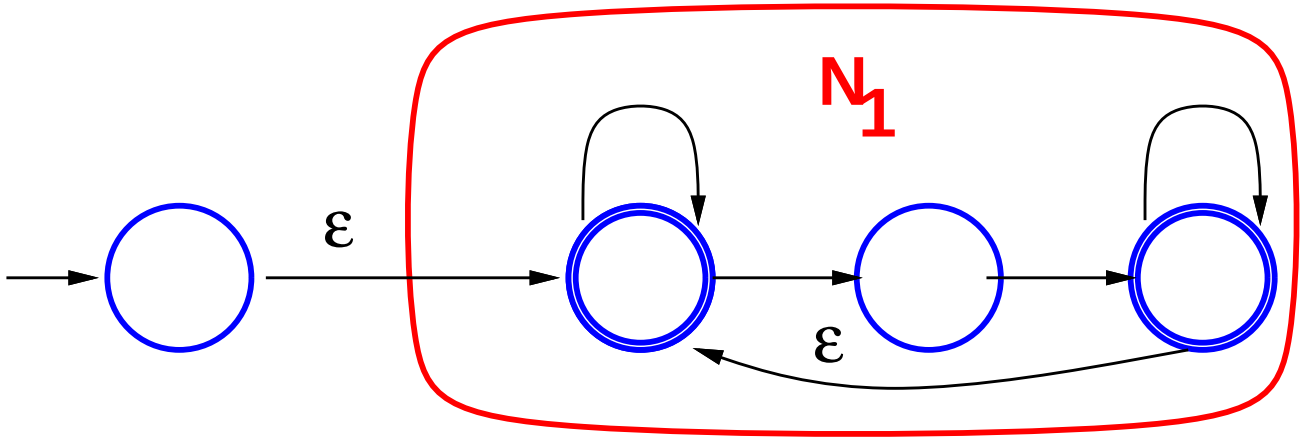

Oops - bad construction. How do we fix it?

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  - but much easier to use in some proofs and constructions.

Regular Expressions

A notation for building up languages by describing them as expressions, *e.g.* $(0 \cup 1)0^*$.

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Remark: Regular expressions are often used in text editors or shell scripts.

More Examples

Let Σ be an alphabet.

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Just like in arithmetic, operations have precedence:

- star first
- concatenation next
- union last
- parentheses used to change usual order

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$\Longleftarrow$ Given a regular language, L, construct an equivalent regular expression.