UWO CS2214

Tutorial #5

Problem 1 Consider the set of ordered pairs $(x,\, y)$ where x are y are real numbers. Such a pair can be seen as a point in the plane equipped with Cartesian coordinates $(x,\, y)$. For each of the following functions F_1, F_2, F_3, F_4 determine a (2×2)-matrix A so that the point of coordinates $(x,\, y)$ is sent to the point $(x',\, y')$ when we have

$$\left(\begin{array}{c} x' \\ y' \end{array} \right) = A \left(\begin{array}{c} x \\ y \end{array} \right)$$

where

$$A = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right)$$

1. $F_1(x, y) = (x, y)$,
2. $F_2(x, y) = (x, 0)$,
3. $F_3(x, y) = (0, y)$,
4. $F_4(x, y) = (y, x)$.

Solution 1 Note that we have:

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \left(\begin{array}{c} x \\ y \end{array} \right) = \left(\begin{array}{c} ax + by \\ cx + dy \end{array} \right)$$

Hence, for each question, we need to find a, b, c, d so that, we have:

$$\left(\begin{array}{c} ax + by \\ cx + dy \end{array} \right) = \left(\begin{array}{c} x' \\ y' \end{array} \right)$$

The solutions are:

1. Since $F_1(x, y) = (x, y)$, the matrix A defining F_1 satisfies

   $$A \left(\begin{array}{c} x \\ y \end{array} \right) = \left(\begin{array}{c} x \\ y \end{array} \right).$$

   According to what we saw in the lectures, A can be the identity matrix of order 2. that is:

   $$A = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right).$$

2. Since $F_2(x,y) = (x,0)$, the matrix A defining F_2 satisfies

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix},$$

that is:

$$\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

This suggests the following choice for A: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

3. Since $F_3(x,y) = (0,y)$, the matrix A defining F_3 satisfies

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix},$$

that is:

$$\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}.$$

This suggests the following choice for A: $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

4. Since $F_4(x,y) = (y,,x)$, the matrix A defining F_4 satisfies

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix},$$

that is:

$$\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}.$$

This suggests the following choice for A: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Problem 2 Following up on the previous problem, determine which of the above functions F is injective? surjective?

Solution 2

1. We study F_1:

   (a) F_1 is injective: Indeed, for all (x_1,y_1) and (x_2,y_2) if $F_1(x_1,y_1) = F_1(x_2,y_2)$ holds then we have $(x_1,y_1) = (x_2,y_2)$, which exactly means that F_1 is injective.

(b) F_1 is surjective: Indeed, every $(x',\, y')$ has a pre-image by F_1, namely itself, since $F_1(x',\, y') = (x',\, y')$ holds.

2. We study F_2:

   (a) F_2 is not injective: Indeed, we have $F_2(0, 1) = (0, 0) = F_2(0, 2)$, thus two different points, namely $(0, 1)$ and $(0, 2)$ have the same image by F_2, namely $(0, 0)$.

   (b) F_2 is not surjective: Indeed, $(1, 1)$ has no pre-image by F_2.

3. For similar reasons as those for F_2, F_3 is neither injective nor surjective.

4. We study F_4:

   (a) F_4 is injective: Indeed, for all (x_1, y_1) and (x_2, y_2) if $F_4(x_1, y_1) = F_4(x_2, y_2)$ holds then we have $(y_1, x_1) = (y_2, x_2)$ that is, $y_1 = y_2$ and $x_1 = x_2$, thus $(x_1, y_1) = (x_2, y_2)$, which exactly means that F_1 is injective.

   (b) F_4 is surjective: Indeed, every $(x',\, y')$ has a pre-image by F_4, namely $(y',\, x')$, since $F_1(y',\, x') = (x',\, y')$ holds.

Problem 3 Let x be a real number. Prove the following identities:

1. $\lceil -x \rceil \;=\; -\lfloor x \rfloor$
2. $\lfloor -x \rfloor \;=\; -\lceil x \rceil$

Solution 3

1. Remember that for every real number x there exists a unique integer n and a real number ε so that $0 \leq \varepsilon < 1$ and $x = n + \varepsilon$ hold; moreover, we have $\lfloor x \rfloor = n$. Hence, for manipulating $\lfloor x \rfloor$, it is useful to keep that property (namely the formula $x = n + \varepsilon$) in mind.
   Similarly, for every real number x there exists a unique integer n and a real number ε so that $0 < \varepsilon \leq 1$ and $x = n + \varepsilon$ hold; moreover, we have $\lceil x \rceil = n + 1$.
   Since the equality that we want to prove mixes the floor and ceiling functions, let us assume first that x is an integer n. Then, we have:

$$\lceil -x \rceil \;=\; \lceil -n \rceil \;= -n \;=\; -\lfloor n \rfloor \;=\; -\lfloor x \rfloor.$$

If x is not an integer, then there exist an integer n and a real number ε so that $0 < \varepsilon < 1$ and $x = n + \varepsilon$ both hold. In this case, we have:

$$\begin{array}{rl} \lceil -x \rceil & = \\ \lceil -(n+\varepsilon) \rceil & = \\ \lceil -n-1+(1-\varepsilon) \rceil & = \\ (-n-1)+1 & = \\ -n & = \\ -\lfloor n+\varepsilon \rfloor & \\ -\lfloor x \rfloor . & \end{array}$$

2. Assume first that x is an integer n. Then, we have:

$$\lfloor -x \rfloor \; = \; \lfloor -n \rfloor \; = -n \; = \; -\lceil n \rceil \; = \; -\lceil x \rceil .$$

If x is not an integer, then exist an integer n and $0 < \varepsilon < 1$ so that $x = n + \varepsilon$ holds. In this case, we have:

$$\begin{array}{rl} \lfloor -x \rfloor & = \\ \lfloor -(n+\varepsilon) \rfloor & = \\ \lfloor -n-1+(1-\varepsilon) \rfloor & = \\ -n-1 & = \\ -(n+1) & = \\ -\lceil n+\varepsilon \rceil & = \\ -\lceil x \rceil . & \end{array}$$

Problem 4 Let x be a real number and n be an integer. Prove the following identities:

1. $\lceil x+n \rceil \; = \; \lceil x \rceil + n$
2. $\lfloor x+n \rfloor \; = \; \lfloor x \rfloor + n$

Solution 4

1. There exist an unique integer m and $0 < \varepsilon \leq 1$ so that $x = m + \varepsilon$ holds; moreover, we have $\lceil x \rceil = m + 1$. Then, we have

   $$\lceil x+n \rceil \; = \; \lceil (m+n)+\varepsilon \rceil \; = \; m+n+1 \; = \; \lceil m+\varepsilon \rceil + n \; = \; \lceil x \rceil + n.$$

2. The proof is similar to that of the previous claim.

Problem 5 Which of the functions f below is injective? surjective? When f is bijective, determine its inverse

1. $f_1 : \begin{array}{ccc} \mathbb{Z} & \to & \mathbb{Z} \\ x & \longmapsto & x+2 \end{array}$

2. $f_2 : \begin{array}{ccc} \mathbb{Z} & \to & \mathbb{Z} \\ x & \longmapsto & x^2-1 \end{array}$

3. $f_3 : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \longmapsto & \frac{x+2}{3} \end{array}$

4. $f_4 : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \longmapsto & \lceil x \rceil \end{array}$

Solution 5

1. We study f_1 below:
   (a) f_1 is injective. Indeed, for all $x_1, x_2 \in \mathbb{Z}$, if we have $f_1(x_1) = f_1(x_2)$, we deduce $x_1 + 2 = x_2 + 2$, that is, $x_1 = x_2$.
   (b) f_1 is surjective. Indeed, given $y \in \mathbb{Z}$, there exists $x \in \mathbb{Z}$ so that $f_1(x) = y$, namely $x = y - 2$.
   (c) it follows that f_1 is bijective and that the inverse function of f_1 is: $f_1^{-1} : \begin{array}{ccc} \mathbb{Z} & \to & \mathbb{Z} \\ y & \longmapsto & y-2 \end{array}$
2. We study f_2 below:
   (a) f_2 is not injective since $f_2(1) = 0 = f_2(-1)$.
   (b) f_2 is not surjective since -2 has no pre-image by f_2. Indeed $-2 = x^2 - 1$ has no solution in $\mathbb{Z}$ (and even in $\mathbb{R}$).
3. We study f_3 below:
   (a) f_3 is injective. Indeed, for all $x_1, x_2 \in \mathbb{R}$, if we have $f_3(x_1) = f_3(x_2)$, then we have $\frac{x_1+2}{3} = \frac{x_2+2}{3}$, that is, $x_1 + 2 = x_2 + 2$, that is, $x_1 = x_2$.
   (b) Indeed, given $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ so that $f_3(x) = y$, namely $x = 3y - 2$.
   (c) it follows that f_3 is bijective and that the inverse function of f_3 is: $f_3^{-1} : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ y & \longmapsto & 3y-2 \end{array}$
4. f_4 is not injective since $f_4(\sqrt{2}) = 2 = f_4(2)$. f_4 is not surjective since $\sqrt{2}$ has no pre-image by f_4.