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Tutorial 05: Floating-point Numbers

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Music: *"Corporate Success"* by *Scott Holmes*, used under *Attribution-NonCommercial License*

Example of Decimal to IEEE-754 Floating-point Conversion

- ***Example 1***: *Convert* $5.877472_{10} \times 10^{-39}$ into a *32-bit single-precision IEEE-754 FP* value.

$\text{Log}_2(10) = 1 / \log_{10}(2)$

$10^{-39} = 2^z \rightarrow \log_2(10^{-39}) = z \rightarrow -39 \times \log_2(10) = z \rightarrow z = -129.5551957$

$10^{-39} = 2^{-129.5551957} = 2^{-129} \times 2^{-0.5551957} = 2^{-129} \times 0.680564734_{10}$

$5.877472_{10} \times 10^{-39} = 5.877472_{10} \times 0.680564734_{10} \times 2^{-129}$

$= 4_{10} \times 2^{-129} = 2^2 \times 2^{-129} = 2^{-127} = 1_2 \times 2^{-127}$

  - Convert 1_2 into a fixed-point binary
    - $1_2 = 1.0_2$ (*already normalized*)
  - *True exponent is less than -126 ➔ underflow case*
    - *The exponent needs to be -126:* $-127_{10} = -126 - 1$
    - *Hence, the significant needs to be adjusted to compensate the -1*
    - After moving the radix point backward by 1 position ➔ 0.1_2
      i.e., $1.0_2 \times 2^{-127} = 0.1_2 \times 2^{-126}$
    - After Taking 23 bits ➔ $0.\ 100\ 0000\ 0000\ 0000\ 0000\ 0000_2$
  - The sign bit, S, is 0 because the number is positive
  - The final number is 0000 0000 0100 0000 0000 0000 0000 0000 or 00400000_{16}

Example of Decimal to IEEE-754 Floating-point Conversion

- ***Example 2***: *Convert* $9.0_{10} \times 10^{-44}$ into a *32-bit single-precision IEEE-754 FP* value.

$\text{Log}_2(10) = 1 / \log_{10}(2)$

$10^{-44}=2^{z} \rightarrow \log_2(10^{-44}) = z \rightarrow -44\times\log_2(10) = z \rightarrow z = -146.164836175$

$10^{-44} = 2^{-146.164836175} = 2^{-146} \times 2^{-0.164836175} = = 2^{-146} \times 0.892029808_{10}$

$9.0_{10} \times 10^{-44} = 9.0_{10} \times 0.892029808_{10} \times 2^{-146} = 8.028268272_{10} \times 2^{-146}$

  - Convert 8.028268272_{10} into a fixed-point binary
    - $8_{10} = 1000_2$ and
    - $0.028268272_{10} = 0.00000111001111001001\ldots_2$.
    - Therefore, $8.028268272_{10} = 1000.00000111001111001001\ldots_2$.
  - *Normalization:* $9.0_{10} \times 10^{-44} = 8.028268272_{10} \times 2^{-146} =$ $1000.00000111001111001001\ldots_2 \times 2^{-146} = 1.00000000111001111001001\ldots_2 \times 2^{-143}$
  - *True exponent is less than -126 ➔ underflow case*
    - *The exponent needs to be -126:* $-143_{10} = -126 - 17$
    - *Hence, the significant needs to be adjusted to compensate the -17*
    - After moving the radix point backward by 17 position ➔ 0. **0000 0000 0000 0000 1**000 0000 0111001111001001…$_2$

      Rounded to the nearest FP
    - After Taking only 23 bits ➔ 0. **000 0000 0000 0000 01**00 0000 0011…$_2$
  - The sign bit, S, is 0 because the number is positive
  - The final number is 0000 0000 0000 0000 0000 0000 0100 0000 or 00000040_{16}

Example of Decimal to IEEE-754 Floating-point Conversion

- ***Example 3***: *Convert* 3.6_{10} into a *32-bit single-precision IEEE-754 FP* value.
  - Convert 3.6_{10} into a fixed-point binary
    - $3_{10} = 11_2$ and
    - $0.6_{10} = 0.1001\ 1001\ \ldots\ _2$.
    - Therefore, $3.6_{10} = 11.1001\ 1001\ \ldots\ _2$
  - Normalize $11.1001\ 1001\ \ldots\ _2$ to
    $1.11001\ 1001\ \ldots\ _2 \times 2^1$.
  - The sign bit, S, is 0 because the number is positive
  - The *biased exponent* is the *true exponent* plus 127; that is,
    $1 + 127 = 128_{10} = 1000\ 0000_2$
  - The fractional significand is 110 0110 0110 0110 0110 0110 0110 0110 …
    - *the leading 1 was stripped* and
    - *to be rounded to 23 bits (rounded to nearest FP number)*.
  - The final number is 0100 0000 0110 0110 0110 0110 0110 0110,
    or 40666666_{16}. ➔ 3.599999904632568_{10}

$0.6 \times 2 = 1.2$
$0.2 \times 2 = 0.4$
$0.4 \times 2 = 0.8$
$0.8 \times 2 = 1.6$
$0.6 \times 2 = 1.2$
…

Example of Decimal to IEEE-754 Floating-point Conversion

- *Example 4*:
  Convert 16777216.75_{10} into a *32-bit single-precision IEEE-754 FP* value.
  - Convert 16777216.75_{10} into a fixed-point binary
    - $16777216_{10} = 1\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000_2$ and
    - $0.75_{10} = 0.11_2$.
    - Therefore, $16777216.75_{10} = 1\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000.11_2$.
  - Normalize $1\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000.11_2$ to
    $1.0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 11_2 \times 2^{24}$.
  - The sign bit, S, is 0 because the number is positive
  - The *biased exponent* is the *true exponent* plus 127; that is,
    $24 + 127 = 151_{10} = 1001\ 0111_2$
  - The fractional significand is 000 0000 0000 0000 0000 0000 011
    - *the leading 1 was stripped* and
    - *to be rounded to 23 bits (rounded to nearest FP number)*.
  - The final number is 0100 1011 1000 0000 0000 0000 0000 0000,
    or $4B800000_{16}$ ➔ 16777216_{10} *(i.e., there is 0.75 rounding error)*

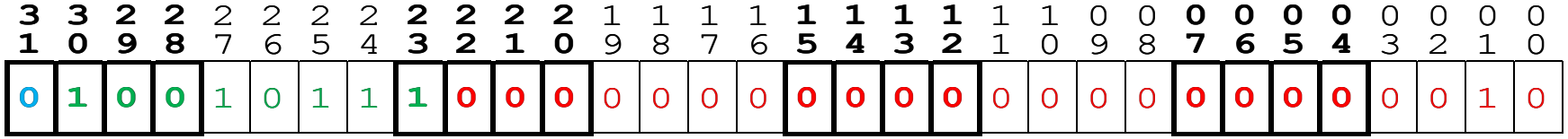
Example of Decimal to IEEE-754 Floating-point Conversion

- ***Example 5***:
  Convert 16777219_{10} into a *32-bit single-precision IEEE-754 FP* value.
  - Convert 16777219_{10} into a fixed-point binary
    - $16777219_{10} = 1\ 0000\ 0000\ 0000\ 0000\ 0000\ 0011_2$ and
  - Normalize $1\ 0000\ 0000\ 0000\ 0000\ 0000\ 0011_2$ to
    $1.0000\ 0000\ 0000\ 0000\ 0000\ 0011_2 \times 2^{24}$.
  - The sign bit, S, is 0 because the number is positive
  - The *biased exponent* is the *true exponent* plus 127; that is,
    $24 + 127 = 151_{10} = 1001\ 0111_2$
  - The fractional significand is 000 0000 0000 0000 0000 0001 1
    - *the leading 1 was stripped* and
    - *to be rounded to 23 bits (rounded to nearest FP number)*.
  - The final number is 0100 1011 1000 0000 0000 0000 0000 0010,
    or $4B800002_{16}$ ➔ 16777220_{10} *(i.e., there is 1.0 rounding error)*

Example of IEEE-754 FP to Decimal to IEEE-754 FP Conversion

- ***Example 6***:
  Convert $4B800002_{16}$ from the *32-bit single-precision IEEE-754 FP* representation into decimal representation. ***Then*** add 1.0_{10} to the result. And ***finally*** convert it back to the *32-bit single-precision IEEE-754 FP* representation.

  - Convert the hexadecimal number ($4B800002_{16}$) into binary form



  - Unpack the number into *sign bit*, *biased exponent*, and *fractional significand*.
    - S = 0
    - E = 1001 0111
    - F =000 0000 0000 0000 0000 0010
  - As the sign bit is 0, the number is positive.
  - We subtract 127 from the *biased exponent* $1001\ 0111_2$ to get the *true exponent* ➔ $1001\ 0111_2 - 0111\ 1111_2 = 0001\ 1000_2 = 24_{10}$.
  - The fractional significand is $.000\ 0000\ 0000\ 0000\ 0000\ 0010_2$.
  - Reinserting the leading one gives $\mathbf{1}.000\ 0000\ 0000\ 0000\ 0000\ 0010_2$.
  - The number is $+(\mathbf{1} + 2^{-22}) \times 2^{24} = 2^{24} + 2^2 = 1024_{10} \times 1024_{10} \times 16_{10} + 4_{10}$
    $= 16777216_{10} + 4_{10} = 16777220_{10}$

Example of IEEE-754 FP to Decimal to IEEE-754 FP Conversion

- *Example 6 (continution)*:
  Adding 1.0_{10} to the result ➔ $16777220_{10} + 1.0_{10} = 16777221_{10}$
  Converting the result back to the *32-bit single-precision IEEE-754 FP* format
  - Convert 16777221_{10} into a fixed-point binary
    - $16777221_{10} = 1\ 0000\ 0000\ 0000\ 0000\ 0000\ 0101_2$ and
  - Normalize $1\ 0000\ 0000\ 0000\ 0000\ 0000\ 0101_2$ to
    $1.0000\ 0000\ 0000\ 0000\ 0000\ 0101_2 \times 2^{24}$.
  - The sign bit, S, is 0 because the number is positive
  - The *biased exponent* is the *true exponent* plus 127; that is,
    $24 + 127 = 151_{10} = 1001\ 0111_2$
  - The fractional significand is 000 0000 0000 0000 0000 0010 1
    - *the leading 1 was stripped* and
    - *to be rounded to 23 bits (rounded to nearest FP number)*.
  - The final number is 0100 1011 1000 0000 0000 0000 0000 0010,
    or $4B800002_{16}$ ➔ 16777220_{10}

Mid-way ➔ round to even significand

This is the same FP number that we started with!!

$16777220_{10} + 1.0_{10} = 16777220_{10}$!!!
(This is due to the rounding error)

Example of IEEE-754 FP to Decimal to IEEE-754 FP Conversion

❑ ***Example 6 (continution)***:

- Run the following C program to verify ***Example 6***:

```
#include <stdio.h>
int main()
{
  float f = 16777220, ff;
  ff = f + 1;
  printf("%f %f \n", f, ff);
}
```

The output will be:
16777220.000000 16777220.000000

Change the "`float`" to "`int`" and the "`%f`" to "`%d`" and repeat executing the program again.

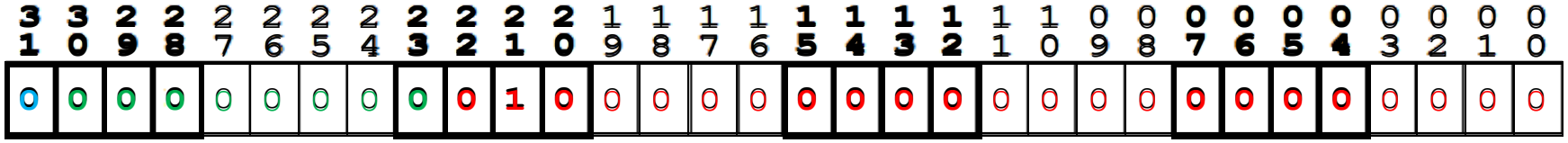
The output after the "`float`" to "`int`" change will be:
16777220 16777221

Change the "`float`" to "`double`" and the "`%f`" to "`%lf`" and repeat executing the program again.

The output after the "`float`" to "`double`" change will be:
16777220.000000 16777221.000000

Example of IEEE-754 Floating-point to Decimal Conversion

- ***Example 7***: *Convert* 00200000_{16} from *32-bit single-precision IEEE-754 FP* value into a decimal value.
  - Convert the hexadecimal number (00200000_{16}) into binary form



  - Unpack the number into *sign bit*, *biased exponent*, and *fractional significand*.
    - S = 0
    - E = 0000 0000
    - F =010 0000 0000 0000 0000 0000
  - As the sign bit is 0, the number is positive.
  - We subtract 126 from the *biased exponent* 0_2 to get the *true exponent* ➔ $0_2 - 0111\ 1110_2 = -126_{10}$.
    As the true exponent is -126, then the F is not normalized
  - The fractional significand is $.010\ 0000\ 0000\ 0000\ 0000\ 0000_2$.
  - The number is $.01_2 \times 2^{-126} = 2^{-2} \times 2^{-126} = 2^{-128}$

We are subtracting 126, not 127, from the biased exponent, because the biased exponent = 0.

$2^{-128} = 10^z \rightarrow \log_{10}(2^{-128}) = z \rightarrow z = -38.53183944$

$2^{-128} = 10^{-38.53183944} = 10^{-38} \times 10^{-0.53183944} = 10^{-38} \times 0.293873587$

$2^{-128} = 0.293873587 \times 10^{-38} = 2.9387358 \times 10^{-39}$

Final Word!!

- **How can you verify your FP conversion results?**
- There are many online converters between IEEE FP format to float and vice versa.
  - For example, https://www.h-schmidt.net/FloatConverter/IEEE754.html