1 (A ⊃ (B • C))
* 2 (B ⊃ (A • C))
[∴ ((A ∨ B) ⊃ C)
* 3 asm: ~((A ∨ B) ⊃ C)
* 4 ∴ (A ∨ B) {from 3}
5 ∴ ~C {from 3}
6 asm: A {break up 1}
7 ∴ (B • C) {from 1 and 6}
8 ∴ B {from 7}
9 ∴ C {from 7}
10 ∴ ~A {from 6; 5 contradicts 9} ⇦
11 ∴ B {from 4 and 10}
12 ∴ (A • C) {from 2 and 11}
13 ∴ A {from 12} ⇦
14 ∴ ((A ∨ B) ⊃ C) {from 3; 10 contradicts 13}

4.5a Exercise—LogiCola GHV

Prove each of these arguments to be valid (all are valid).

(B ∨ A)
(B ⊃ A)
∴ ~(A ⊃ ~A)

* 1 (B ∨ A) **Valid**
2 (B ⊃ A)
[∴ ~(A ⊃ ~A)
* 3 asm: (A ⊃ ~A)
4 asm: B {break up 1}
5 ∴ A {from 2 and 4}
6 ∴ ~A {from 3 and 5}
7 ∴ ~B {from 4; 5 contradicts 6}
8 ∴ A {from 1 and 7}
9 ∴ ~A {from 3 and 8}
10 ∴ ~(A ⊃ ~A) {from 3; 8 contradicts 9}

1. (A ⊃ B)
(A ∨ (A • C))
∴ (A • B)

2. (((A • B) ⊃ C) ⊃ (D ⊃ E))
D
∴ (C ⊃ E)

3. (B ⊃ A)
~(A • C)
(B ∨ C)
∴ (A ≡ B)

4. (A ∨ (D • E))
(A ⊃ (B • C))
∴ (D ∨ C)

5. ((A ⊃ B) ⊃ C)
(C ⊃ (D • E))
∴ (B ⊃ D)

6. (~(A ∨ B) ⊃ (C ⊃ D))
(~A • ~D)
∴ (~B ⊃ ~C)

7. (~A ≡ B)
∴ ~(A ≡ B)

8. (A ⊃ (B • ~C))
C
((D • ~E) ∨ A)
∴ D

4.5b Exercise—LogiCola GHV

First appraise intuitively. Then translate into logic (using the letters given) and prove to be valid (all are valid).

1. If the butler put the tablet into the drink and the tablet was poison, then the butler killed the deceased and the butler is guilty.
   The butler put the tablet into the drink.
   The tablet was poison.
   ∴ The butler is guilty.   [Use P, T, K, and G.]
2. If I'm coming down with a cold and I exercise, then I'll get worse and feel awful.
   If I don't exercise, then I'll suffer exercise deprivation and I'll feel awful.
   ∴ If I'm coming down with a cold, then I'll feel awful.   [Use C, E, W, A, and D.]
3. You'll get an A if and only if you either get a hundred on the final exam or else bribe the teacher.
   You won't get a hundred on the final exam.
   ∴ You'll get an A if and only if you bribe the teacher.   [Use A, H, and B.]
4. If President Nixon knew about the massive Watergate cover-up, then he lied to the American people on national television and he should resign.
   If President Nixon didn't know about the massive Watergate cover-up, then he was incompetently ignorant and he should resign.
   ∴ Nixon should resign.   [Use K, L, R, and I.]
5. Common sense assumes we have moral knowledge.
   There's no disproof of moral knowledge.
   If common sense assumes we have moral knowledge, then if there's no disproof of moral knowledge we should believe that we have moral knowledge.
   Any proof of a moral truth presupposes a more basic moral truth.
   We can't prove moral truths by more basic ones endlessly.
   If any proof of a moral truth presupposes a more basic moral truth and we can't prove moral truths by more basic ones endlessly, then if we should believe that we have moral knowledge we should accept self-evident moral truths.
   ∴ We should accept self-evident moral truths.   [Use C, D, B, P, E, and S; this argument defends ethical intuitionism.]
6. Moral judgments express either truth claims or feelings.
   If moral judgments express truth claims, then "ought" expresses either a concept from sense experience or an objective concept that isn't from sense experience.
   "Ought" doesn't express a concept from sense experience.
   "Ought" doesn't express an objective concept that isn't from sense experience.
   ∴ Moral judgments express feelings and not truth claims.   [Use T, F, S, and O.]
7. If Michigan either won or tied, then Michigan is going to the Rose Bowl and Gensler is happy.
   ∴ If Gensler isn't happy, then Michigan didn't tie.   [Use W, T, R, and H.]

8. There are moral obligations.
   If there are moral obligations and moral obligations are explainable, then either there's an explanation besides God's existence or else God's existence would explain moral obligations.
   God's existence wouldn't explain moral obligation.
   ∴ Either moral obligations aren't explainable, or else there's an explanation besides God's existence. [Use M, E, B, and G.]
9. If determinism is true and Dr Freudlov correctly predicts (using deterministic laws) what I'll do, then if she tells me her prediction I'll do something else.
   If Dr Freudlov tells me her prediction and yet I'll do something else, then Dr Freudlov doesn't correctly predict (using deterministic laws) what I'll do.
   ∴ If determinism is true, then either Dr Freudlov doesn't correctly predict (using deterministic laws) what I'll do or else she won't tell me her prediction. [Use D, P, T, and E.]
10. [The parents told their son that their precondition for financing his graduate education was that he leave his girlfriend Suzy. A friend of mine talked the parents out of their demand by using this argument.]
   If you make this demand on your son and he leaves Suzy, then he'll regret being forced to leave her and he'll always resent you.
   If you make this demand on your son and he doesn't leave Suzy, then he'll regret not going to graduate school and he'll always resent you.
   ∴ If you make this demand on your son, then he'll always resent you. [Use D, L, F, A, and G; this one is difficult.]

4.6 Harder refutations

Multiple-assumption invalid arguments work much like other invalid arguments—except that we need to make further assumptions before we reach our refutation. Here's an example:

If the butler was at the party, he fixed the drinks and poisoned the deceased.
If the butler wasn't at the party, he was at a neighbor's house.
∴ The butler poisoned the deceased.

	1	$(A^0 \supset (F \cdot P^0)) = 1$	**Invalid**
**	2	$(\sim A^0 \supset N^1) = 1$	$\boxed{N, \sim A, \sim P}$
		[$\therefore P^0 = 0$	
	3	asm: $\sim P$	
	4	asm: $\sim A$ {break up 1}	
	5	$\therefore N$ {from 2 and 4}	

We derive all we can and make additional assumptions when needed. But now we reach no contradiction; instead, we reach a refutation in which the butler was at a neighbor's house, wasn't at the party, and didn't poison the deceased. This refutation makes the premises all true and conclusion false.

As we work out our attempted proof, we can follow the five-step proof strategy of the previous section until one of two things happens:

- Every assumption leads to a contradiction. Then we have a proof of validity.
- We can derive nothing further and all complex wffs are either starred or already broken up (we already have one side or its negation). Then the remaining simple wffs will give a refutation that proves invalidity.

In the invalid case, additional assumptions can help to bring us our refutation.

Invalid arguments often need three or more assumptions, as in this example:

1 $(A^0 \supset B) = 1$ **Invalid**
2 $(C^0 \supset D) = 1$ $\boxed{E, \sim A, \sim C, \sim F}$
3 $(F^0 \supset (C^0 \cdot D)) = 1$
[$\therefore (E^1 \supset C^0) = 0$
* 4 asm: $\sim(E \supset C)$
5 $\therefore E$ {from 4}
6 $\therefore \sim C$ {from 4}
7 asm: $\sim A$ {break up 1}
8 asm: $\sim F$ {break up 3}

We keep going until we can derive nothing further and all complex wffs are either starred (like line 4) or already broken up (like lines 1–3).[1] While our refutation doesn't give us a value for "B" or "D," this is all right, since the refutation still makes the premises all true and conclusion false.

Our proof strategy, if applied correctly, will always give a proof or refutation. How exactly these go may depend on which steps we do first and what we decide to assume; proofs and refutations may differ but still be correct.

4.6a Exercise—LogiCola GHI

Prove each of these arguments to be invalid (all are invalid).

$(A \vee \sim(B \supset C))$
$(D \supset (A \supset B))$
$\therefore (C \supset \sim(D \vee A))$

1 $(A^1 \vee \sim(B \supset C^1)) = 1$ **Invalid**
2 $(D^0 \supset (A^1 \supset B)) = 1$ $\boxed{A, C, \sim D}$
[$\therefore (C^1 \supset \sim(D^0 \vee A^1)) = 0$
* 3 asm: $\sim(C \supset \sim(D \vee A))$
4 $\therefore C$ {from 3}
5 $\therefore (D \vee A)$ {from 3}
6 asm: A {break up 1}
7 asm: $\sim D$ {break up 2}

[1] If you like, you can star a line when it becomes "broken up" (when you have one side or its negation, but not what is needed to infer something). Then you can continue until all assumptions die (then the argument is valid) or until you can derive nothing further and all complex wffs are starred (then the argument is invalid).

1. $\sim(A \cdot B)$
   $\therefore (\sim A \cdot \sim B)$

2. $(\sim A \supset B)$
   $\therefore \sim(A \supset B)$

3. $((A \cdot B) \supset \sim(C \cdot D))$
   C
   $(E \supset B)$
   $\therefore \sim E$

4. $\sim(A \cdot B)$
   $\therefore \sim(A \equiv B)$

5. $(A \supset B)$
   $(C \supset (\sim D \cdot E))$
   $\therefore (D \vee F)$

6. $(\sim A \vee \sim B)$
   $\therefore \sim(A \vee B)$

7. $(A \supset (B \cdot C))$
   $((D \supset E) \supset A)$
   $\therefore (E \vee C)$

8. $(A \supset (B \supset C))$
   $(B \vee \sim(C \supset D))$
   $\therefore (D \supset \sim(A \vee B))$

4.6b Exercise—LogiCola G (HC & MC)

First appraise intuitively. Then translate into logic (using the letters given) and say whether valid (and give a proof) or invalid (and give a refutation).

1. If the maid prepared the drink, then the butler didn't prepare it.
   The maid didn't prepare the drink.
   If the butler prepared the drink, then the butler poisoned the drink and the butler is guilty.
   ∴ The butler is guilty. [Use M, B, P, and G.]
2. If you tell your teacher that you like logic, then your teacher will think that you're insincere and you'll be in trouble.
   If you don't tell your teacher that you like logic, then your teacher will think that you dislike logic and you'll be in trouble.
   ∴ You'll be in trouble. [Use L, I, T, and D.]
3. If we don't get reinforcements, then the enemy will overwhelm us and we won't survive.
   ∴ If we do get reinforcements, then we'll conquer the enemy and we'll survive. [Use R, O, S, and C.]
4. If Socrates didn't approve of the laws of Athens, then he would have left Athens or would have tried to change the laws.
   If Socrates didn't leave Athens and didn't try to change the laws, then he agreed to obey the laws.
   Socrates didn't leave Athens.
   ∴ If Socrates didn't try to change the laws, then he approved of the laws and agreed to obey them. [Use A, L, C, and O. This argument is from Plato's *Crito,* which argued that Socrates shouldn't disobey the law by escaping from jail.]
5. If I hike the Appalachian Trail and go during late spring, then I'll get maximum daylight and maximum mosquitoes.
   If I'll get maximum mosquitoes, then I won't be comfortable.
   If I go right after school, then I'll go during late spring.
   ∴ If I hike the Appalachian Trail and don't go right after school, then I'll be comfortable. [Use A, L, D, M, C, and S.]

6. [Logical positivism says *“Every genuine truth claim is either experimentally testable or true by definition.”* This view, while once popular, is self-refuting and hence not very popular today.]
   If LP (logical positivism) is true and is a genuine truth claim, then it’s either experimentally testable or true by definition.
   LP isn’t experimentally testable.
   LP isn’t true by definition.
   If LP isn’t a genuine truth claim, then it isn’t true.
   □ LP isn’t true. [Use T, G, E, and D.]
7. If you give a test, then students either do well or do poorly.
   If students do well, then you think you made the test too easy and you’re frustrated.
   If students do poorly, then you think they didn’t learn any logic and you’re frustrated.
   □ If you give a test, then you’re frustrated. [Use T, W, P, E, F, and L. This is from a class who tried to talk me out of giving a test.]
8. If the world contains moral goodness, then the world contains free creatures and the free creatures sometimes do wrong.
   If the free creatures sometimes do wrong, then the world is imperfect and the creator is imperfect.
   □ If the world doesn’t contain moral goodness, then the creator is imperfect. [Use M, F, S, W, and C.]
9. We’ll find a cause for your action, if and only if your action has a cause and we look hard enough.
   If all events have causes, then your action has a cause.
   All events have causes.
   □ We’ll find a cause for your action, if and only if we look hard enough. [Use F, H, L, and A.]
10. Herman sees that the piece of chalk is white.
   The piece of chalk is the smallest thing on the desk.
   Herman doesn’t see that the smallest thing on the desk is white. (He can’t see the whole desk and so can’t tell that the piece of chalk is the smallest thing on it.)
   If Herman sees a material thing, then if he sees that the piece of chalk is white and the piece of chalk is the smallest thing on the desk then he sees that the smallest thing on the desk is white.
   If Herman doesn’t see a material thing, then he sees a sense datum.
   □ Herman doesn’t see a material thing, but he does see a sense datum. [Use H, P, H′, M, and S. This argument attacks direct realism—the view that we directly perceive material things and not just sensations or sense data.]
11. If the final loading capacitor in the radio transmitter is arcing, then the SWR (standing wave ratio) is too high and the efficiency is lowered.
   If you hear a cracking sound, then the final loading capacitor in the radio transmitter is arcing.
   □ If you don’t hear a cracking sound, then the SWR isn’t too high. [Use A, H, L, and C.]

12. If we can know that God exists, then we can know God by experience or we can know God by logical inference from experience.
If we can't know God empirically, then we can't know God by experience and we can't know God by logical inference from experience.
If we can know God empirically, then "God exists" is a scientific hypothesis and is empirically falsifiable.
"God exists" isn't empirically falsifiable.
∴ We can't know that God exists. [Use K, E, L, M, S, and F.]
13. If I perceive, then my perception is either delusive or veridical.
If my perception is delusive, then I don't directly perceive a material object.
If my perception is veridical and I directly perceive a material object, then my experience in veridical perception would always differ qualitatively from my experience in delusive perception.
My experience in veridical perception doesn't always differ qualitatively from my experience in delusive perception.
If I perceive and I don't directly perceive a material object, then I directly perceive a sensation.
∴ If I perceive, then I directly perceive a sensation and I don't directly perceive a material object. [Use P, D, V, M, Q, and S. This form of the argument from illusion attacks direct realism—the view that we directly perceive material objects and not just sensations or sense data.]
14. If you're romantic and you're Italian, then Juliet will fall in love with you and will want to marry you.
If you're Italian, then you're romantic.
∴ If you're Italian, then Juliet will want to marry you. [Use R, I, F, and M.]
15. If emotions can rest on factual errors and factual errors can be criticized, then we can criticize emotions.
If we can criticize emotions and moral judgments are based on emotions, then beliefs about morality can be criticized and morality isn't entirely non-rational.
∴ If morality is entirely non-rational, then emotions can't rest on factual errors. [Use E, F, W, M, B, and N.]
16. If you backpack over spring break and don't study logic, then you won't know how to do proofs.
If you take the test and don't know how to do proofs, then you'll miss many problems and get a low grade.
∴ If you backpack over spring break, then you'll get a low grade. [Use B, S, K, T, M, and L.]

4.7 Other proof methods

The proof method in this book tries to combine the best features of two other methods: traditional proofs and truth trees. These three approaches, while differing in how they do proofs, can prove all the same arguments.

Traditional proofs use a standard set of inference rules and equivalence rules. The nine inference rules are like our S- and I-rules, in that they let us infer whole lines from previous whole lines:

$$
\begin{array}{cc}
(P \cdot Q) \rightarrow P & (P \supset Q), P \rightarrow Q \\
P, Q \rightarrow (P \cdot Q) & (P \supset Q), {\sim}Q \rightarrow {\sim}P \\
(P \vee Q), {\sim}P \rightarrow Q & (P \supset Q), (Q \supset R) \rightarrow (P \supset R) \\
P \rightarrow (P \vee Q) & (P \supset Q) \rightarrow (P \supset (P \cdot Q)) \\
\end{array}
$$
$$((P \supset Q) \cdot (R \supset S)), (P \vee R) \rightarrow (Q \vee S)$$

The sixteen equivalence rules let us replace parts of formulas with equivalent parts (I've dropped outer parentheses here to promote readability):

$$
\begin{array}{cc}
P \equiv {\sim}{\sim}P & (P \supset Q) \equiv ({\sim}P \vee Q) \\
P \equiv (P \cdot P) & (P \cdot (Q \cdot R)) \equiv ((P \cdot Q) \cdot R) \\
P \equiv (P \vee P) & (P \vee (Q \vee R)) \equiv ((P \vee Q) \vee R) \\
(P \cdot Q) \equiv (Q \cdot P) & (P \cdot (Q \vee R)) \equiv ((P \cdot Q) \vee (P \cdot R)) \\
(P \vee Q) \equiv (Q \vee P) & (P \vee (Q \cdot R)) \equiv ((P \vee Q) \cdot (P \vee R)) \\
{\sim}(P \cdot Q) \equiv ({\sim}P \vee {\sim}Q) & (P \equiv Q) \equiv ((P \supset Q) \cdot (Q \supset P)) \\
{\sim}(P \vee Q) \equiv ({\sim}P \cdot {\sim}Q) & (P \equiv Q) \equiv ((P \cdot Q) \vee ({\sim}P \cdot {\sim}Q)) \\
(P \supset Q) \equiv ({\sim}Q \supset {\sim}P) & ((P \cdot Q) \supset R) \equiv (P \supset (Q \supset R)) \\
\end{array}
$$

Our approach uses a simpler and more understandable set of rules.

Traditional proofs can use **indirect proofs** (where we assume the opposite of the conclusion and then derive a contradiction); but they more often use **direct proofs** (where we just derive things from the premises and eventually derive the desired conclusion) or **conditional proofs** (where we prove "(P□Q)" by assuming "P" and then deriving "Q"). Here's an argument proved two ways:

Traditional proof

1 $((A \supset B) \cdot (C \supset D))$
[∴ $((A \cdot C) \supset (B \vee D))$
2 ∴ $(A \supset B)$ {from 1}
3 ∴ $({\sim}A \vee B)$ {from 2}
4 ∴ $(({\sim}A \vee B) \vee D)$ {from 3}
5 ∴ $({\sim}A \vee (B \vee D))$ {from 4}
6 ∴ $(({\sim}A \vee (B \vee D)) \vee {\sim}C)$ {from 5}
7 ∴ $({\sim}C \vee ({\sim}A \vee (B \vee D)))$ {from 6}
8 ∴ $(({\sim}C \vee {\sim}A) \vee (B \vee D))$ {from 7}
9 ∴ $(({\sim}A \vee {\sim}C) \vee (B \vee D))$ {from 8}
10 ∴ $({\sim}(A \cdot C) \vee (B \vee D))$ {from 9}
11 ∴ $((A \cdot C) \supset (B \vee D))$ {from 10, giving the original conclusion}

Our proof

* 1 $((A \supset B) \cdot (C \supset D))$
[∴ $((A \cdot C) \supset (B \vee D))$
* 2 ┌ asm: ${\sim}((A \cdot C) \supset (B \vee D))$
* 3 │ ∴ $(A \supset B)$ {from 1}
4 │ ∴ $(C \supset D)$ {from 1}
* 5 │ ∴ $(A \cdot C)$ {from 2}
6 │ ∴ ${\sim}(B \vee D)$ {from 2}
7 │ ∴ A {from 5}
8 │ ∴ C {from 5}
9 │ ∴ B {from 3 and 7}
10 └ ∴ ~B {from 6}
11 ∴ $((A \cdot C) \supset (B \vee D))$ {from 2; 9 contradicts 10}

Both proofs have the same number of steps; but this can vary, depending on how we do each proof. Our steps generally use shorter formulas; our proofs tend to simplify larger formulas into smaller ones—while traditional proofs tend to manipulate longer formulas (often by substituting equivalents) to get the desired result. Our proofs are easier to do, since they use an automatic proof-strategy that students learn quickly; traditional proofs require guesswork and intuition. Also, our system refutes invalid arguments; it can separate valid from invalid arguments, prove valid ones to be valid, and refute invalid ones. In contrast, the traditional system is only a proof method; if we try to prove an invalid argument, we'll fail but won't necessarily learn that the argument is invalid.

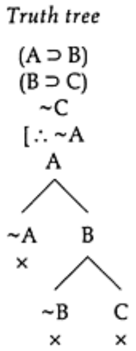
Another common approach is **truth trees,** which decompose formulas into the cases that make them true. Truth trees use simplifying rules and branching rules. The simplifying rules are like our S-rules, in that they let us simplify a formula into smaller parts and then ignore the original formula. These four simplifying rules (which apply to whole lines) are used:

$$\begin{array}{c} \sim\sim P \rightarrow P \\ (P \cdot Q) \rightarrow P, Q \\ \sim(P \vee Q) \rightarrow \sim P, \sim Q \\ \sim(P \supset Q) \rightarrow P, \sim Q \end{array}$$

Each form that can't be simplified is branched into the two sub-cases that would make it true; for example, since "~(P·Q)" is true just if "~P" is true or "~Q" is true, it branches into these two formulas. There are five branching rules:

$\sim(P \cdot Q)$	$(P \vee Q)$	$(P \supset Q)$	$(P \equiv Q)$	$\sim(P \equiv Q)$
$\sim P$ / $\sim Q$	P / Q	$\sim P$ / Q	P, Q / $\sim P, \sim Q$	$P, \sim Q$ / $Q, \sim P$

To test an argument, we write the premises, block off the original conclusion (showing that it is to be ignored in constructing the tree), and add the denial of the conclusion. Then we apply the simplifying and branching rules to each formula, and to each further formula that we get, until every branch either dies (contains a pair of contradictory wffs) or contains only simple wffs (letters or their negation). The argument is valid if and only if every branch dies. Here's an argument proved two ways:



Our proof

*	1	(A ⊃ B)
*	2	(B ⊃ C)
	3	~C
		[∴ ~A
	4	┌ asm: A
	5	│ ∴ B {from 1 and 4}
	6	└ ∴ ~B {from 2 and 3}
	7	∴ ~A {from 4; 5 contradicts 6}

In the truth tree, we write the premises, block off the original "□ A" conclusion (and henceforth ignore it), and add its contradictory "A." Then we branch "(A⊃B)" into its two sub-cases: "~A" and "B." The left branch dies, since it contains "A" and "~A"; we indicate this by putting "×" at its bottom. Then we branch "(B⊃C)" into its two sub-cases: "~B" and "C." Each branch dies; the left branch has "B" and "~B," while the right has "C" and "~C." Since every branch of the tree dies, no possible truth conditions would make the premises all true and conclusion false, and so the argument is valid.

An argument is invalid if some branch of the tree doesn't die. Then the simple wffs on each live branch give a refutation of the argument—truth conditions making the premises all true and conclusion false.

As compared with traditional proofs, truth trees give a simple and efficient way to decide whether an argument is valid or invalid—a way that uses an automatic strategy instead of guesswork and intuition. But truth trees don't mirror ordinary reasoning very well; they give a mechanical way to test validity instead of a way to help develop reasoning skills. And branching can get messy. Our method avoids these disadvantages but keeps the main advantages of truth trees over traditional proofs.