Math 2155, Fall 2022: Homework 7

Instructions for GradeScope:

You may handwrite your answers or type them. Only type solutions if you are able to type the appropriate symbols. If **typed**, use at least a 12pt font, with **one question per page**. You can then directly submit the pdf file to GradeScope.

If **handwritten**:

- Don't erase or cross out more than a word or two.
- Only write on one side of the page. Use dark, large, neat writing.
- If you have a real scanner, great! Most people will scan their solutions using their phone. Use one of the recommended apps: For iOS, use Scannable by Evernote or Genius Scan. For Android, use Genius Scan. Do *not* just take regular photos.
- When scanning, have good lighting and try to avoid shadows.
- It is best to have **one question per scan**. If the solution is short, **fold the page in half** and scan just the half it is on, so there isn't so much blank space. Or, you can scan and then crop the scan. You don't need to scan parts (a), (b), etc. separately or put them on separate pages.
- It works well to scan each question separately and produce one pdf file with one question per page. Most scanning apps will automatically combine your images into one pdf file.
- You must **check the quality** of your scans afterwards, and rescan if needed.

Make sure you are going to `http://gradescope.ca` not `http://gradescope.com`. You can access it through the course OWL site.

You can **resubmit** your work any number of times until the deadline.

Don't forget to accurately **match questions to pages**. If you do this incorrectly, the grader will not see your solution and will give you zero.

See the GradeScope help website for lots of information: `https://help.gradescope.com/` Select "Student Center" and then either "Scanning Work on a Mobile Device" or "Submitting an Assignment".

Instructions for writing solutions:

- Homework is graded both on **correctness** and on **presentation/style**.
- Your proofs should be written in **complete sentences**, starting with capitals and ending with periods. They should be in **paragraph form**, similar to the proofs in the textbook. For example, you shouldn't put each sentence on a new line. Paragraphs should be used to break longer proofs into logical chunks.
- Don't use informal abbreviations such as three dots for "therefore." Logical symbols can be used when they clarify things to the reader, but English words can often be more effective.
- Include all of the steps that are needed to logically justify every claim you make. Do not include unnecessary steps. Try to be **concise and complete**.
- Do not submit any **rough work**. Do your rough work on scratch paper, and only submit a clearly and neatly written answer. Do not show a table of givens and goals. Do not use Venn diagrams.
- Do not cross out or erase more than a word or two. If you write each final solution on a new page, it's easy to start over on a fresh page when you've made a large error.
- You should do the work **on your own**. Read the course syllabus for the rules about scholastic offences, which include sharing solutions with others, uploading material to a website, viewing material of others or on a website (even if you don't use it), etc. The penalty for cheating on homework will be a grade of **0** on the homework set as well as a penalty of **negative 5%** on the overall course grade.
- We may not grade every question.

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Due **SATURDAY, October 29 at 11:59pm**. You can **resubmit** your work any number of times until the deadline.

Note: For HW7 only, I will accept submissions up to 24 hours past the deadline (so until SUNDAY, Oct 30 at 11:59pm), with a 10% penalty. To avoid the penalty, submit by Oct 29.

H7Q1 (6 marks): Let A, B and C be sets. Prove that the following are equivalent:

(a) $C \subseteq A \cup B$

(b) $(C \setminus A) \subseteq B$

(c) $(C \setminus B) \subseteq A$

(Hint: you can do it showing just three implications.)

Solution:

Proof. $(a) \implies (b)$: Assume $C \subseteq A \cup B$. Let $x \in C \setminus A$. Then $x \in C$ and $x \notin A$. Since $x \in C$, we have $x \in A \cup B$. Since $x \notin A$, we must have $x \in B$. Therefore, $(C \setminus A) \subseteq B$.

$(b) \implies (c)$: Assume $(C \setminus A) \subseteq B$. Let $x \in C \setminus B$. Then $x \in C$ and $x \notin B$. Since $x \notin B$, $x \notin C \setminus A$. But $x \in C$, so we must have $x \in A$. Therefore, $(C \setminus B) \subseteq A$.

$(c) \implies (a)$: Assume $(C \setminus B) \subseteq A$. Let $x \in C$. We must show that $x \in A \cup B$.

Case 1: $x \in B$. Then $x \in A \cup B$, so we are done.

Case 2: $x \notin B$. Then, since $x \in C$, we have $x \in C \setminus B$. So, by assumption, $x \in A$, from which it follows that $x \in A \cup B$.

These cases are exhaustive, and in either case $x \in A \cup B$. Therefore, $C \subseteq A \cup B$. $\square$

The third part could be briefer, using a strategy for proving goals of the form $P \vee Q$:

$(c) \implies (a)$: Assume $(C \setminus B) \subseteq A$. Let $x \in C$. We must show that $x \in A \cup B$. If $x \in B$, then this is clear, so assume that $x \notin B$. Then, since $x \in C$, we have $x \in C \setminus B$. So, by assumption, $x \in A$, from which it follows that $x \in A \cup B$. Therefore, $C \subseteq A \cup B$.

H7Q2 (5 marks): Consider the following statements.

Statement 1: There exists a unique $x \in \mathbb{R}$ such that $x^2 - 6x = -9$.

Statement 2: There exists a unique $x \in \mathbb{R}$ such that $x^2 - 6x + 1 < 0$.

One of these is correct, and one is wrong. Prove the correct one, and give a counterexample for the incorrect one.

Solution: Statement 1 is correct.

Proof. **Existence:** Let $x = 3$. Then $x^2 - 6x = (3)^2 - 6(3) = 9 - 18 = -9$, as required.

Uniqueness: Let $x \in \mathbb{R}$ be such that $x^2 - 6x = -9$. Then $x^2 - 6x + 9 = 0$. Factoring, we see that $(x - 3)^2 = 0$, which implies that $x - 3 = 0$, which shows that $x = 3$. $\square$

The uniqueness part of Statement 2 is false. When $x = 1$, we get $x^2 - 6x + 1 = 1 - 6 + 1 = -4 < 0$. When $x = 2$, we get $x^2 - 6x + 1 = 4 - 12 + 1 = -7 < 0$. So there are at least two values of x that satisfy $x^2 - 6x + 1 < 0$.

H7Q3 (6 marks): Prove the following:

Theorem 1: For every real number x, if $x \neq 0$ and $x \neq 1$, there is a unique real number y such that $\frac{y-1}{x} = y - x$.

Theorem 2: There exists a unique real number y such that for every real number x, $(y + x)(x - 3) = x^2 - 9$.

Solution:

Proof of Theorem 1. Let $x \in \mathbb{R}$ be arbitrary, and assume that $x \neq 0$ and $x \neq 1$.

Existence: Let $y = 1 + x$. Then $\frac{y-1}{x} = \frac{1+x-1}{x} = \frac{x}{x} = 1$ and $y - x = 1 + x - x = 1$, so $\frac{y-1}{x} = y - x$, as required.

Uniqueness: Suppose that $z \in \mathbb{R}$ satisfies $\frac{z-1}{x} = z - x$. Then $z - 1 = xz - x^2$. Therefore, $z(1 - x) = 1 - x^2$. Since $x \neq 1$, it follows that $z = \frac{1-x^2}{1-x} = 1 + x$. $\square$

[It's ok to give the solution in the form $y = \frac{1-x^2}{1-x}$, without simplifying, as long as you are self-consistent.]

Proof of Theorem 2. Existence: Let $y = 3$. Then, for any real number x, we have that

$$(y + x)(x - 3) = (3 + x)(x - 3) = x^2 - 9,$$

as required.

Uniqueness: Let z be a real number with the property that $(z + x)(x - 3) = x^2 - 9$ holds for every real number x. In particular, this must hold when $x = 0$, which gives that $z(-3) = -9$. Therefore, we must have $z = 3$, as required. $\square$

H7Q4 (4 marks): Prove that there exists a unique integer m such that $m > 1$, $m \mid 10$ and $m \mid 15$.

Solution:

Proof. **Existence:** Let $m = 5$. Then $m > 1$ and clearly $m \mid 10$ and $m \mid 15$, since $10 = 2(5)$ and $15 = 3(5)$.

Uniqueness: Suppose that k is an integer such that $k > 1$, $k \mid 10$ and $k \mid 15$. The positive divisors of 10 are $\{1, 2, 5, 10\}$ and the positive divisors of 15 are $\{1, 3, 5, 15\}$. Since $k > 1$ and k is in both sets, we must have that $k = 5$. $\square$