FUNCTIONS

OUTLINE:

1) Introduction to functions
2) Properties of functions
3) Operations on functions
4) Sequences

1. INTRODUCTION TO FUNCTIONS

Definition

- Let A and B be nonempty sets. A function f from A to B (denoted $f : A \rightarrow B$) is an assignment of exactly one element of B to each element of A.
- If b is the unique element of B assigned by the function f to the element $a \in A$, we write $f(a) = b$.
- Functions are also called maps, mapping, transformations

Functions as relations

- A function $f : A \to B$ can be seen as a particular relation $f \subseteq AxB$, satisfying 2 conditions:
  - $\forall a\ (a \in A \to \exists b\ (b \in B \wedge (a,b) \in f\ ))$ (every $a \in A$ appears as the first entry of a couple in the relation f)
  - $\forall a\ \forall b\ \forall c\ (((a,b) \in f \wedge (a,c) \in f) \to b=c)$ (no 2 distinct elements of the relation have the same first entry)
  - Equivalently in "condensed form", $\forall a \in A\ \exists! b \in B\ (\ (a,b) \in f\ )$ (for all a in A, there is a unique b in B such that a is in relation with b)

Terminology

- Given a function $f : A \rightarrow B$, we say f maps A to B
- A is the domain of f ($Dom(f)$). B is the codomain of f ($Codom(f)$)
- If, for $a \in A$ and $b \in B$, $f(a) = b$, then b is called the image of a and a is called a preimage of b.
- The range of f is the set $Range(f) = f(A) = \{f(a) \mid a \in A\}$ of the elements of B which are the image of some elements of A. Note that f(A) is a subset of B.
- Two functions are equal when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.

Representation

A function $f : A \to B$ can be represented

1) In set-theoretic notation (being a relation)

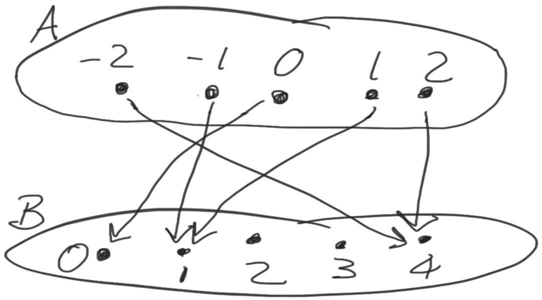
2) With a graph (being a relation)

3) With a formula

EX: Let A = {-2,-1,0,1,2} and B = {0,1,2,3,4}.

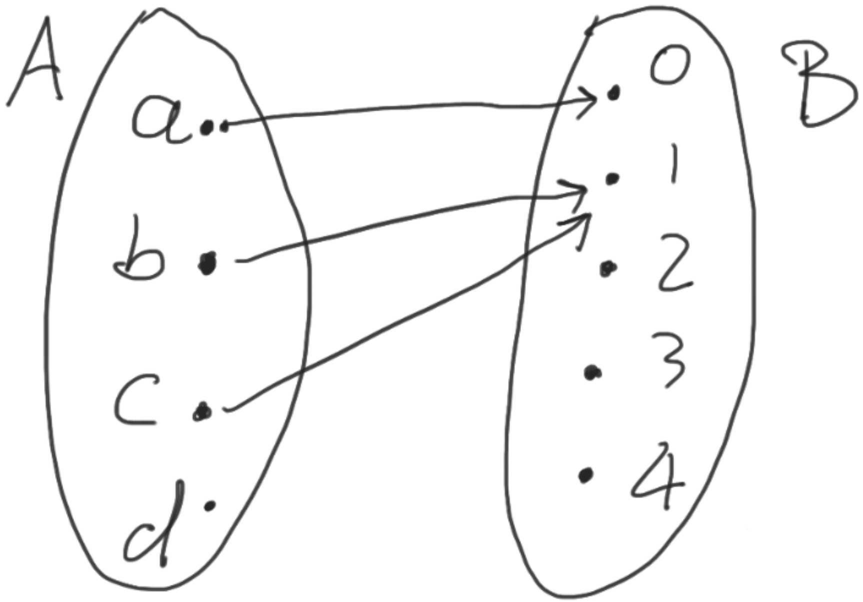
1) f = {(-2,4),(-1,1),(0,0),(1,1),(2,4)}

2)

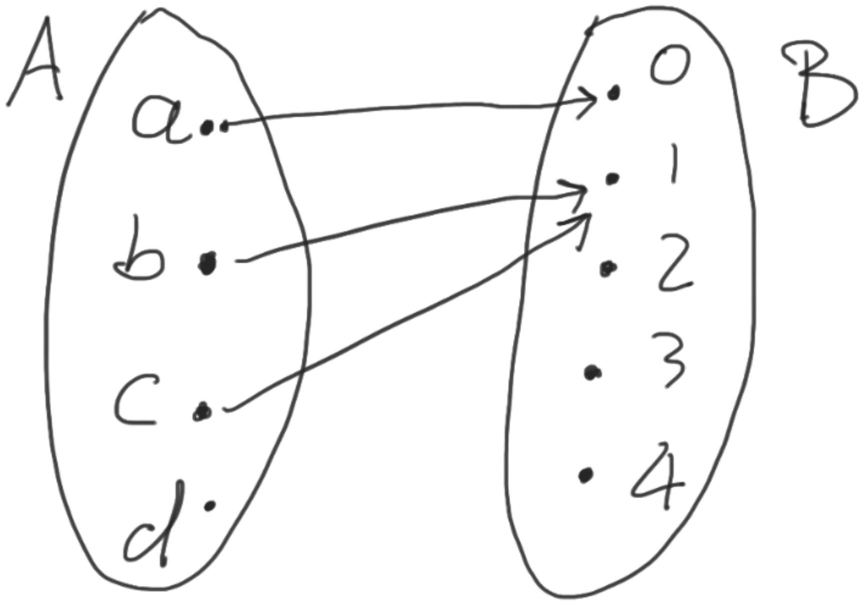


3) $f : A \to B$, $f(a) = a^2$

Is this a function?

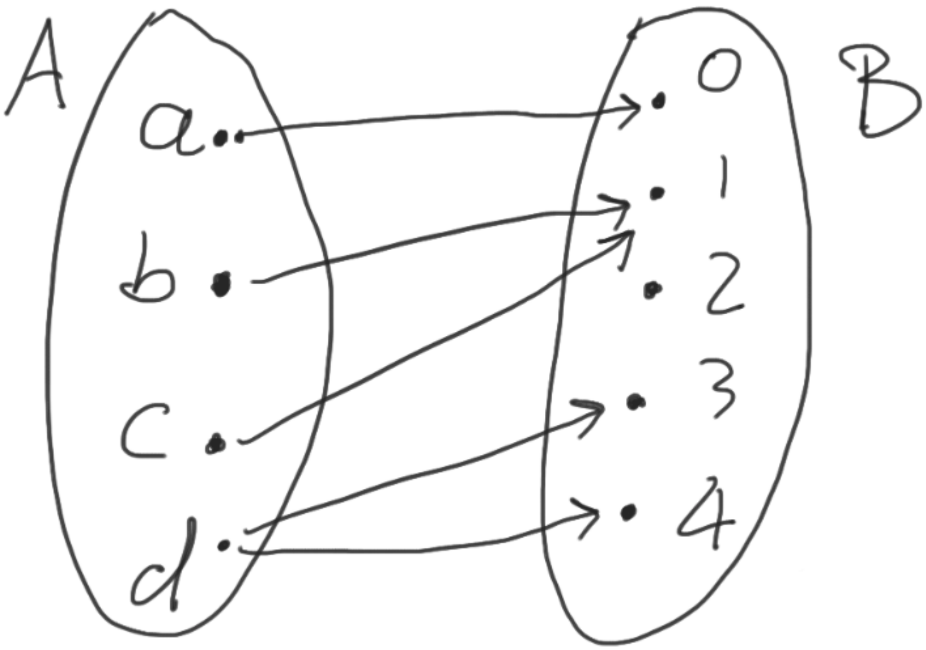


Is this a function?

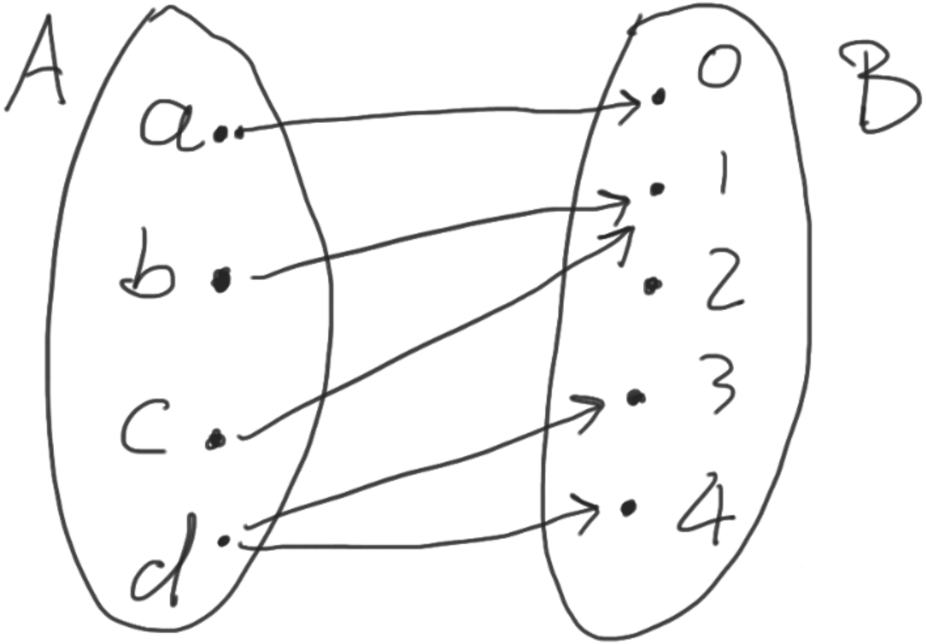


- No: d∈A has no image in B
- (it is a relation)
- In some texts, relations of this type are called partial functions, because restricting the domain to only the elements with image produces a function.

Is this a function?

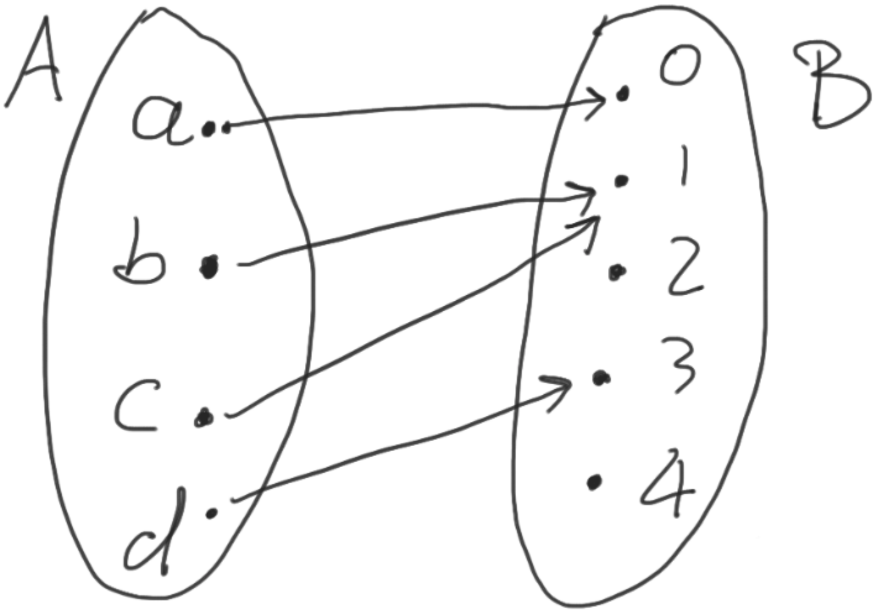


Is this a function?

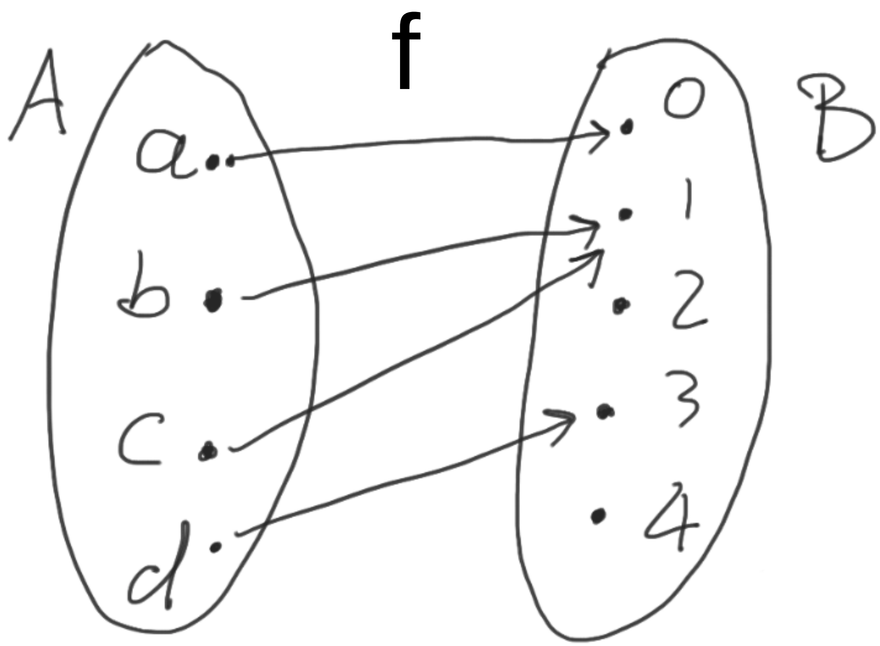


- No: $d \in A$ has more than one “image” in B
- (it is a relation)
- This is not a partial function

Is this a function?

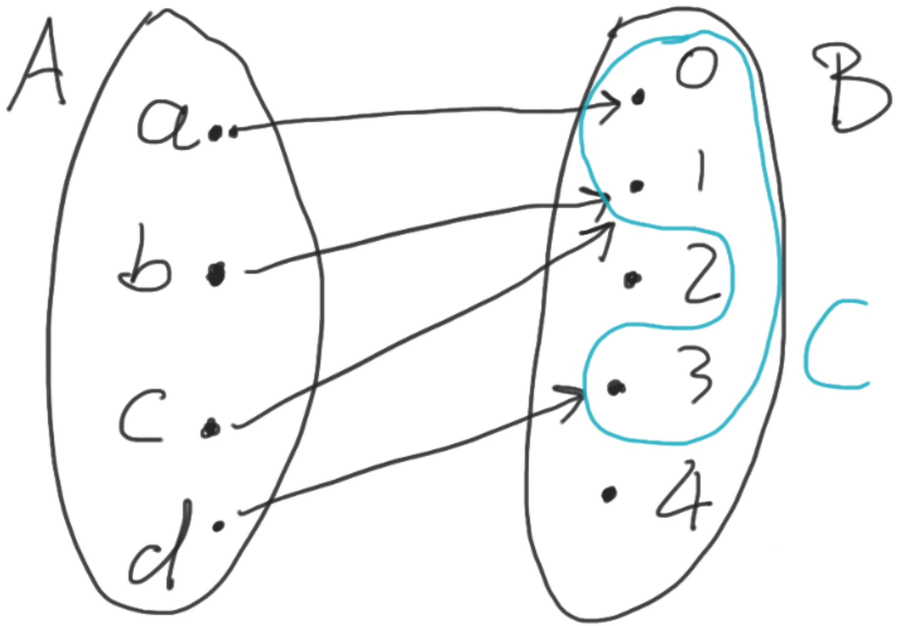


Is this a function?



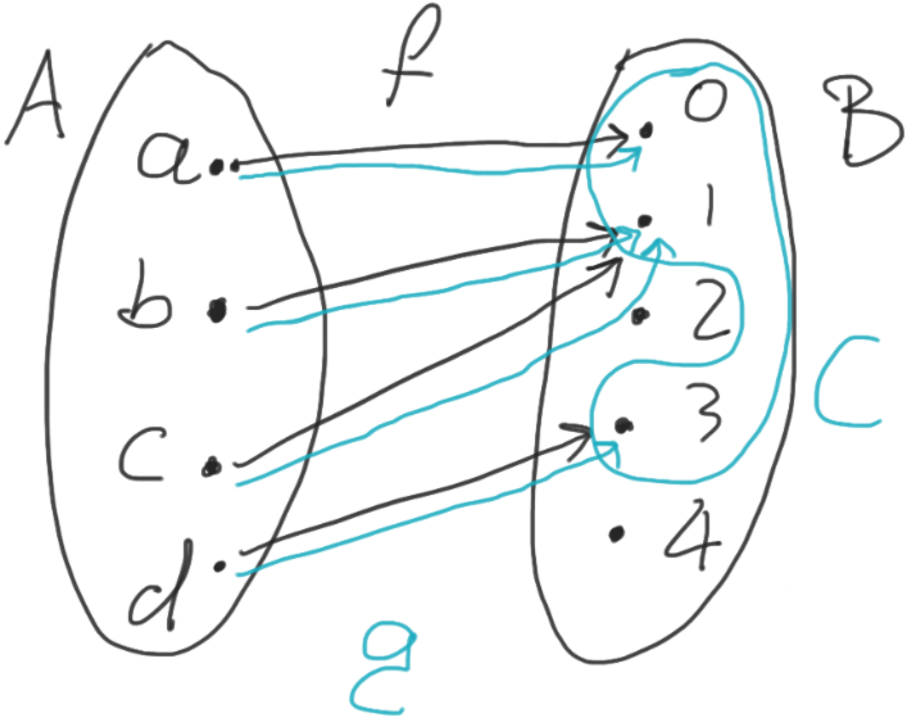
- Yes: each element of A has exactly one image in B.
- So we have a function $f : A \to B$
- Note that what happens in B does not matter for f to be a function:
  - the preimage of 0 is a
  - b and c are both preimages of 1
  - the preimage of 3 is d
  - 2 and 4 have no preimage
- The domain of f is A, its codomain is B, its range is {0,1,3}

Is this a function?



- If we restrict the codomain of f to C = Range(f) = f(A) (leaving everything else unchanged), we still have a function.
- This new function, though, is not equal to f, because it has a different codomain (even if the 2 functions act in the same way on any element of A!)

Is this a function?



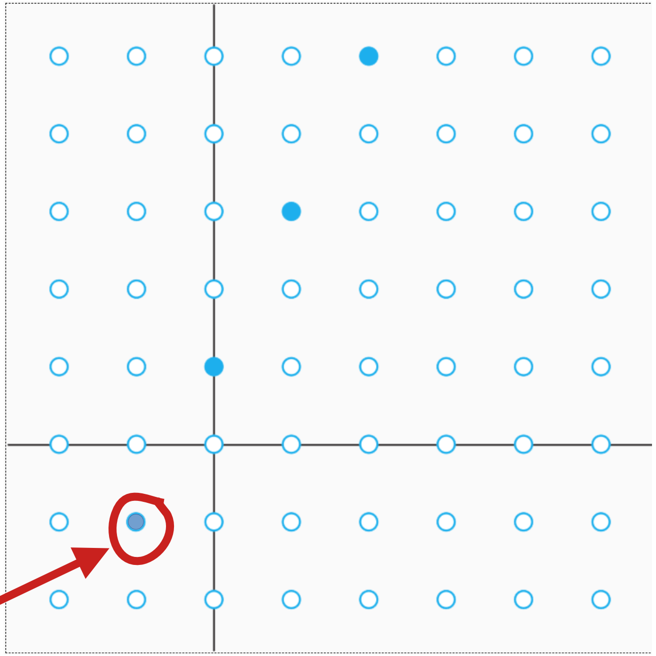
- So we have 2 different functions:
- $f : A \to B$
- $g : A \to C$
- $\forall x \in A\ (f(x) = g(x))$, but $f \neq g$

Graphs of functions

- Let $f : A \rightarrow B$ be a function. The graph of f is the set of pairs $\{(a,b) \mid a \in A \wedge f(a) = b\} \subseteq A x B$
- Note that the graph of f is the same set as the description of f as a relation
- We call it the graph of f because, if A and B are sufficiently well-behaved (e.g. subsets of **R**), it can be used to draw the function on a cartesian plane

Graphs of functions

- EX: graph of f : **Z** → **Z** , f(n) = 2n+1



This too!

(note that the figure on page 157 of the textbook is wrong)

2. PROPERTIES OF FUNCTIONS

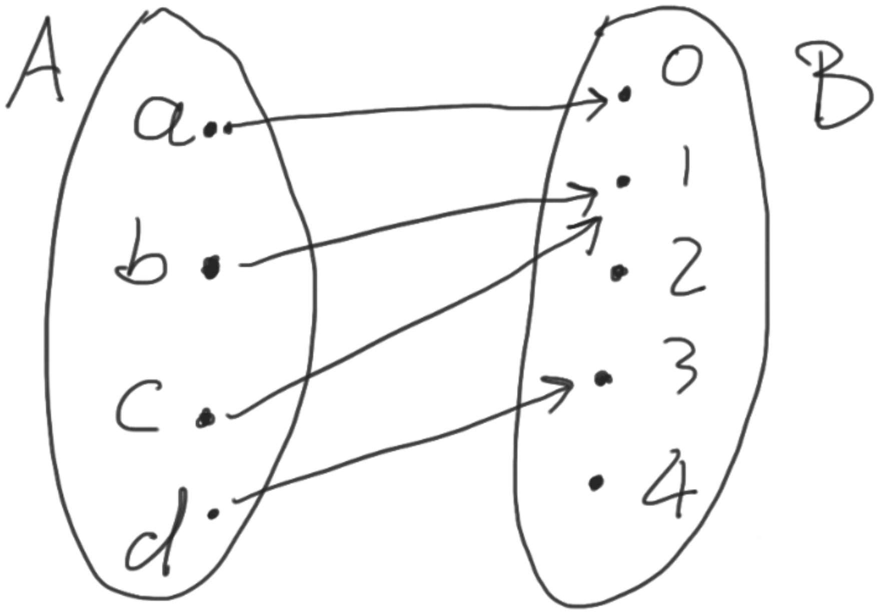
Injective functions

- A function f is said to be injective, or one-to-one, if and only if distinct elements of the domain of f have distinct images in the codomain of f. In other words, f is injective iff for all a and b in the domain of f,

$$f(a) = f(b) \text{ implies } a = b.$$

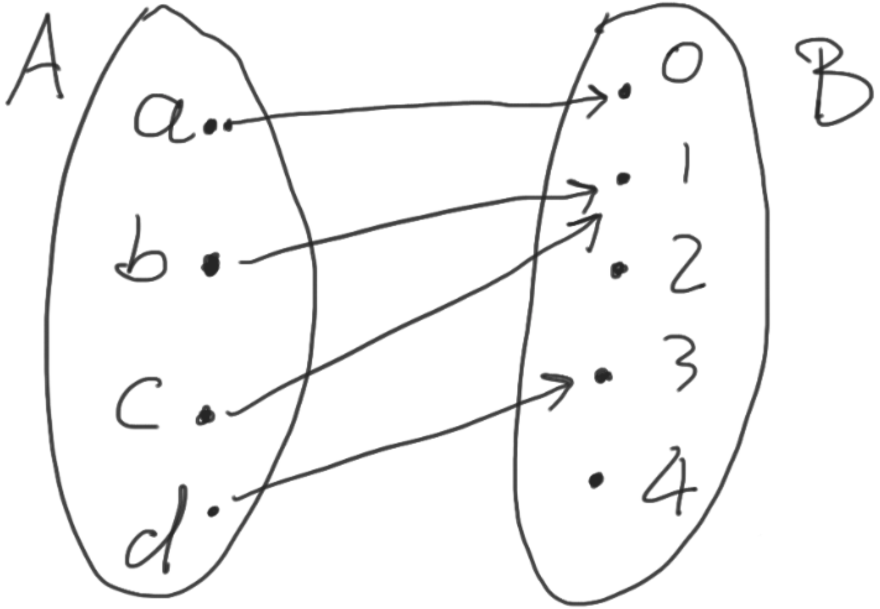
Injective functions

- Is this function injective?



Injective functions

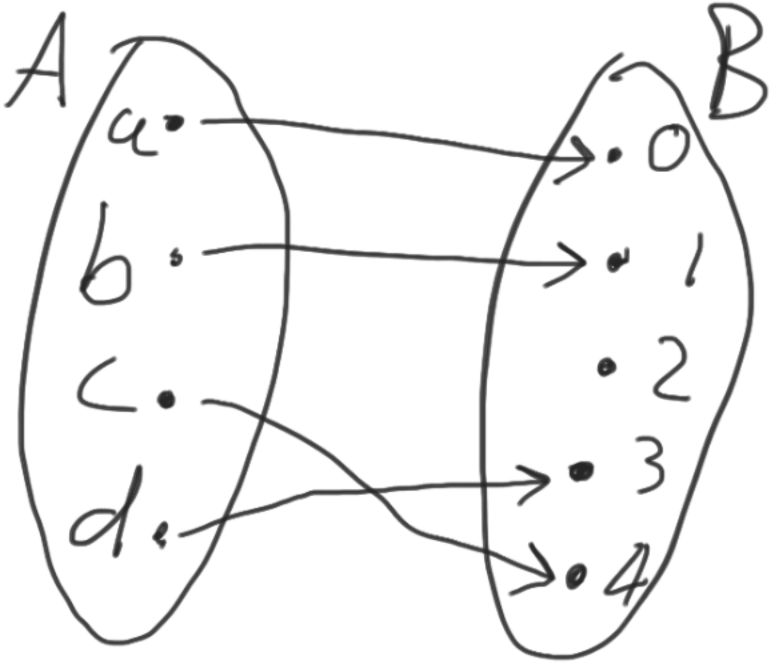
- Is this function injective?



NO, because b and c have the same image

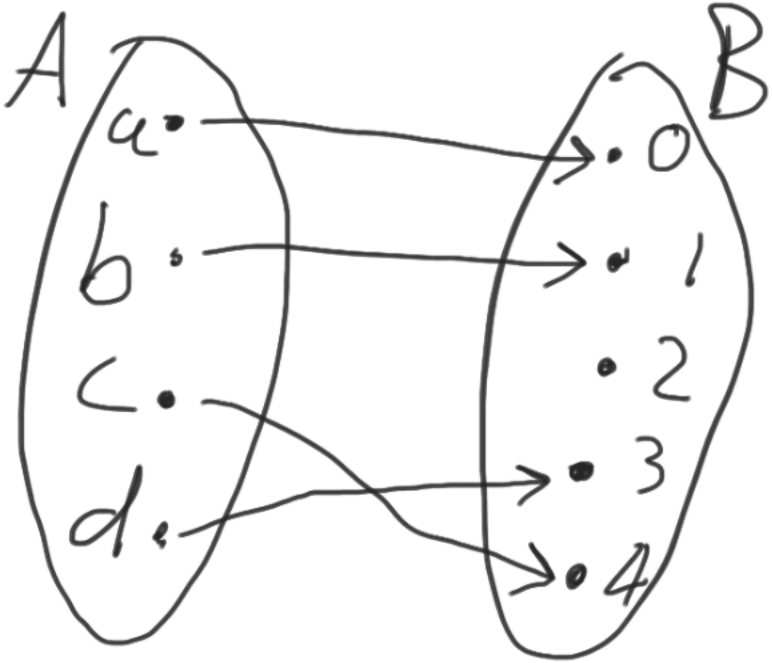
Injective functions

- Is this function injective?



Injective functions

- Is this function injective?



Yes: distinct elements of the domain have distinct images

Injectivity depends on the domain

- Is this function injective?
- $f : \mathbf{N} \to \mathbf{N}$, $f(x) = x^2$

Injectivity depends on the domain

- Is this function injective?
- $f : \mathbf{N} \rightarrow \mathbf{N}$, $f(x) = x^2$
- Yes: assume, for some $x,y \in \mathbf{N}$, $f(x) = f(y)$. This means $x^2 = y^2$, which happens only if $x = y$.

Injectivity depends on the domain

- Is this function injective?
- $f : \mathbf{Z} \rightarrow \mathbf{N}$, $f(x) = x^2$

Injectivity depends on the domain

- Is this function injective?
- $f : \mathbf{Z} \to \mathbf{N}$, $f(x) = x^2$
- No: assume, for some $x,y \in \mathbf{Z}$, $f(x) = f(y)$. This means $x^2 = y^2$, which happens if $x = y$, but also if $x = -y$.
- More explicitly, for example $f(-1) = f(1)$, so the 2 distinct integers -1 and 1 have the same image.

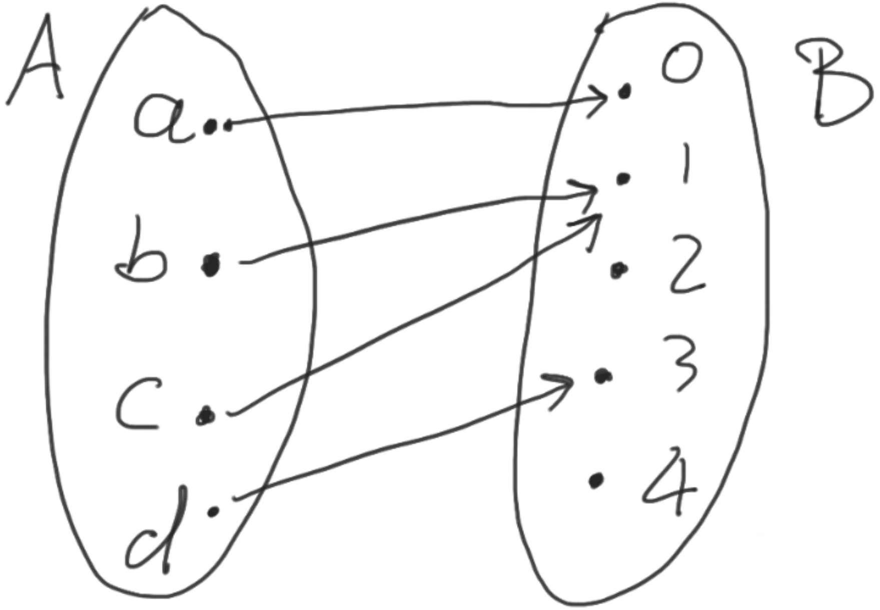
Surjective functions

- A function f is said to be surjective, or onto, if and only if every element of the codomain is in the range of f. In other words, f is surjective iff for any b in the codomain of f, there is an a in the domain of f such that

$$f(a) = b.$$

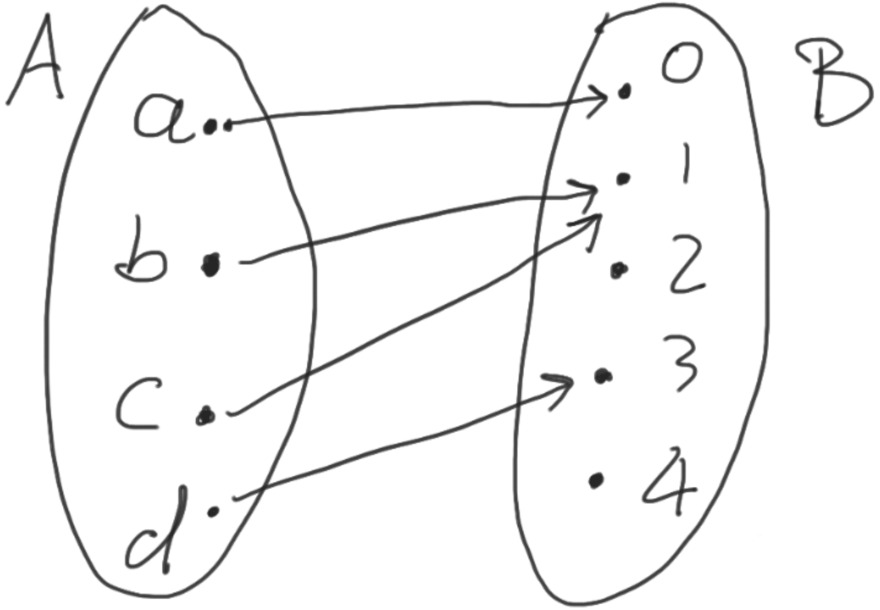
Surjective functions

- Is this function surjective?



Surjective functions

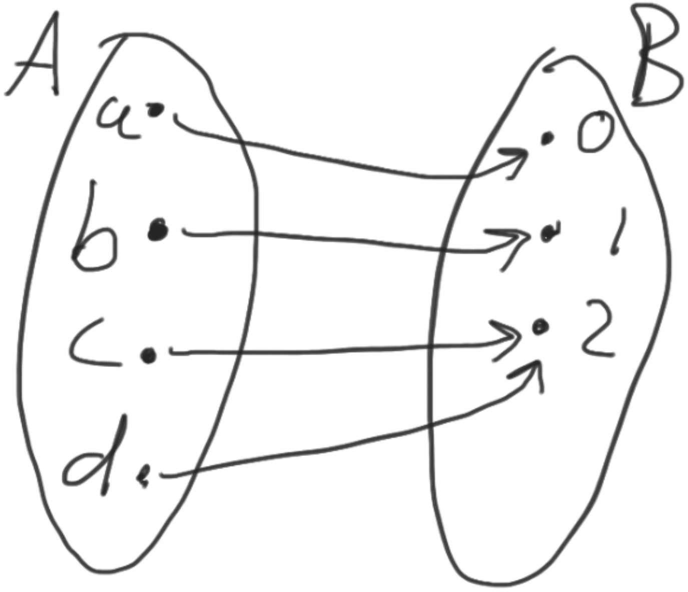
- Is this function surjective?



NO, because 2 and 4 have no preimage (i.e., 2 and 4 are in the codomain, but not in the range of the function)

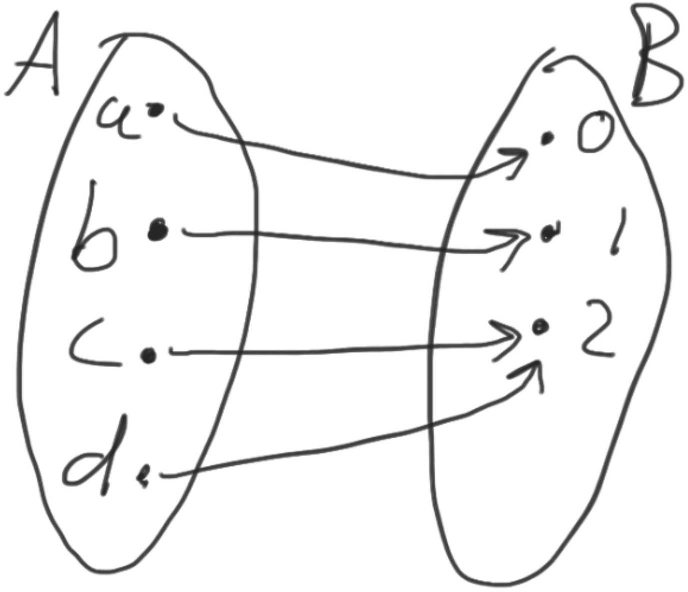
Surjective functions

- Is this function surjective?



Surjective functions

- Is this function surjective?



YES, because any element in the codomain (B) is the image of (at least) one element in the domain (A)

Surjectivity depends on the codomain

- Is this function surjective?
- f : $\mathbf{N} \rightarrow \mathbf{N}$, $f(x) = |x|$

Surjectivity depends on domain and codomain

- Is this function surjective?
- f : **N** → **N**, f(x) = |x|
- Yes: take a generic y in the codomain **N**. Then y = |y| = f(y), so y is the image of itself. In particular, any y∈**N** is in the range of f.

Surjectivity depends on domain and codomain

- Is this function surjective?
- f : **N** → **Z**, f(x) = |x|

Surjectivity depends on domain and codomain

- Is this function surjective?
- $f : \mathbf{N} \rightarrow \mathbf{Z}$, $f(x) = |x|$
- No, because by definition $|x| \geq 0$, so the negative integers are not in the range of f.

Surjectivity depends on domain and codomain

- Is this function surjective?
- f : $\mathbf{Z} \rightarrow \mathbf{Z}$, $f(x) = |x|$

Surjectivity depends on domain and codomain

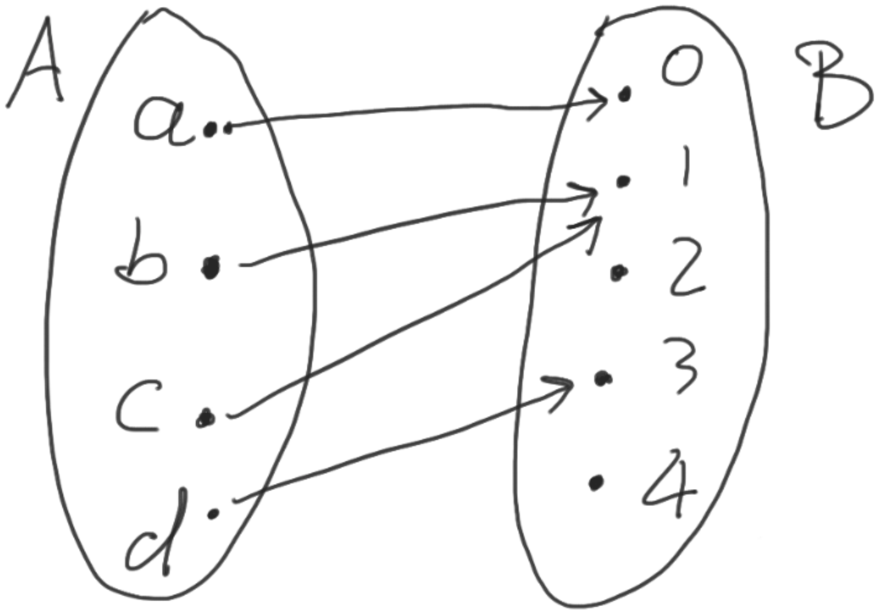
- Is this function surjective?
- $f : \mathbf{Z} \to \mathbf{Z}$, $f(x) = |x|$
- No, because by definition $|x| \geq 0$, so the negative integers are not in the range of f.

Bijective functions

- A function f is said to be bijective if and only if it is both injective and surjective.
- ATTENTION: some authors call bijective functions “one-to-one correspondences”. In order to not cause confusion with one-to-one functions (i.e., injective functions), we will use the terms “injective”, “surjective” and “bijective”.

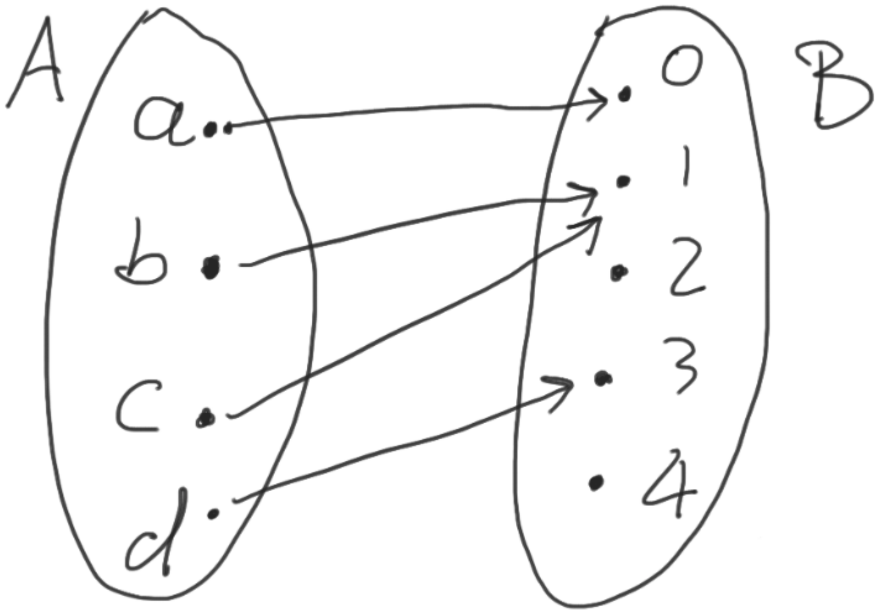
Bijective functions

- Is this function bijective?



Bijective functions

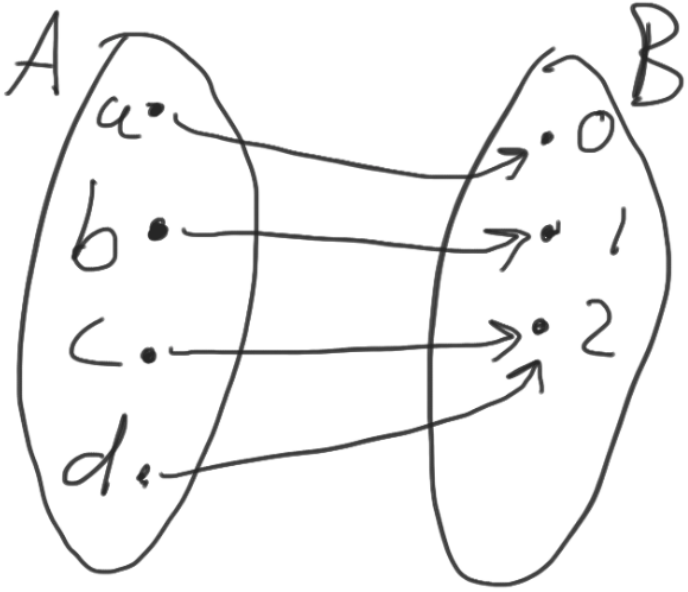
- Is this function bijective?



No, because it is neither injective nor surjective

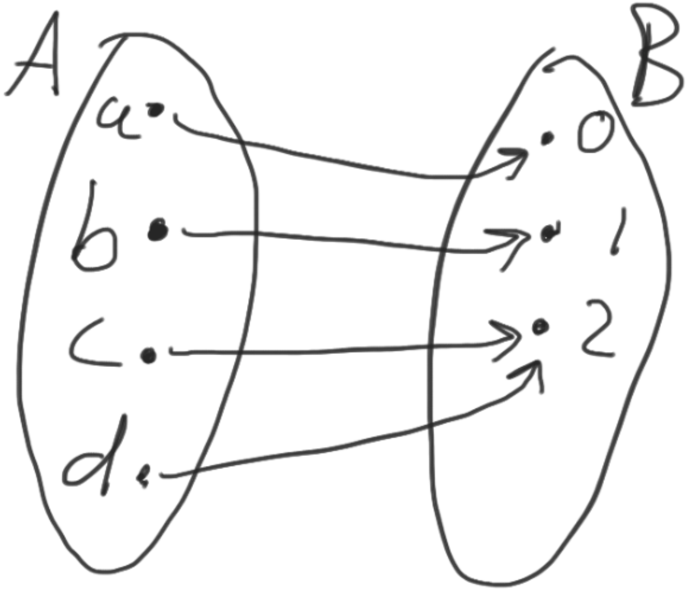
Bijective functions

- Is this function bijective?



Bijective functions

- Is this function bijective?



No, because it is not injective
(c and d have the same image)

Bijectivity depends on domain and codomain

- Is this function bijective?
- $f : \mathbf{N} \to \mathbf{N}$, $f(x) = x^2$

Bijectivity depends on domain and codomain

- Is this function bijective?
- $f : \mathbf{N} \to \mathbf{N}$, $f(x) = x^2$
- No, because it is not surjective. In fact, not every natural number is the square of a natural number: e.g., there is no natural number n such that $f(n) = n^2 = 2$.

Bijectivity depends on domain and codomain

- Is this function bijective?
- $f : \mathbf{R}^+ \rightarrow \mathbf{R}^+$, $f(x) = x^2$

Bijectivity depends on domain and codomain

- Is this function bijective?
- $f : \mathbf{R}^+ \to \mathbf{R}^+$, $f(x) = x^2$
- Yes!
- Injectivity: assume $x, y \in \mathbf{R}^+$ and $f(x) = f(y)$. This means $x^2 = y^2$, which happens only if $x = y$. In words, two positive real numbers have the same square iff they coincide.
- Surjectivity: for any positive real number r, the number $\sqrt{r}$ is well-defined and it is a positive real number; moreover, by definition of $\sqrt{\ }$, $f(\sqrt{r}) = (\sqrt{r})^2 = r$.

Bijectivity depends on domain and codomain

- Is this a bijective function?
- $f : \mathbf{R} \rightarrow \mathbf{R}^{+}$, $f(x) = x^2$

Bijectivity depends on domain and codomain

- Is this a bijective function?
- $f : \mathbf{R} \rightarrow \mathbf{R}^{+}$, $f(x) = x^2$
- No: it is not even a function! In fact, the element 0 of the domain has no image in the codomain (since $f(0)$ would be 0, but $0 \notin \mathbf{R}^{+}$, the image of 0 is not defined)

Bijectivity depends on domain and codomain

- Is this a bijective function?
- $f : \mathbf{R} \rightarrow \mathbf{R}^+ \cup \{0\}$, $f(x) = x^2$

Bijectivity depends on domain and codomain

- Is this a bijective function?
- $f : \mathbf{R} \rightarrow \mathbf{R}^+ \cup \{0\}$, $f(x) = x^2$
- NO. It is a function, since ant real number has exactly one square, which is a non-negative real number.
- However it is not injective, because two real numbers x, y have the same square if $x = y$, but also if $x = -y$. Concretely, $f(-1) = f(1) = 1$.
- (It is surjective)

3. OPERATIONS ON FUNCTIONS

Inverse function

- If $f : A \to B$ is a bijective function, then its inverse relation $f^{-1} : B \to A$ is also a function.
- In this case we also say that f is invertible.
- The surjectivity of f guarantees that f^{-1} is defined on all elements of B; the injectivity of f guarantees that f^{-1} maps each element of B to a unique element of A.

Inverse function

- Let f : **R** → **R** , f(x) = x^3-2.
- f is bijective:
  - Injectivity: if f(x) = f(y) then x^3-2 = y^3-2, so x^3 = y^3. Since x,y are in **R**, this implies x = y.
  - Surjectivity: for any y∈R, we can find an x∈R such that f(x) = y "solving for x":

$$f(x)=y \quad iff \quad x^3-2=y \quad iff \quad x^3=y+2 \quad iff \quad x=\sqrt[3]{y+2}$$

- Therefore, f is invertible, with inverse given by "solving for x":

$$f^{-1}:\boldsymbol{R}\rightarrow\boldsymbol{R} \;\;,\;\; f^{-1}(y)=\sqrt[3]{y+2}$$

Inverse function

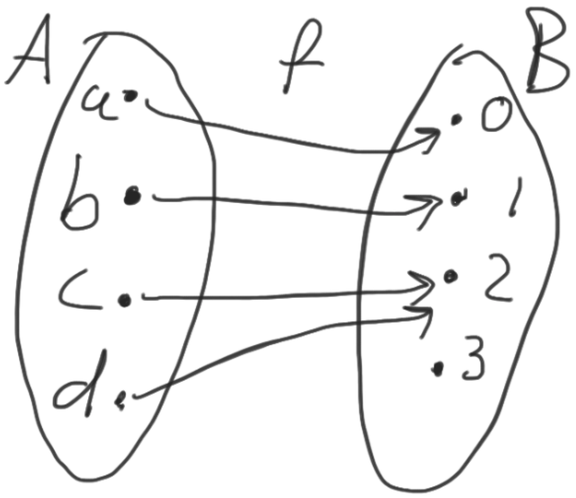
- If a function is not bijective, we would get into trouble when "solving for x"
- EX: f : **R** → **R** , f(x) = x^2
- This is neither injective nor surjective
- If we try to solve for x the equation f(x) = y for an arbitrary y, then we face 2 problems:

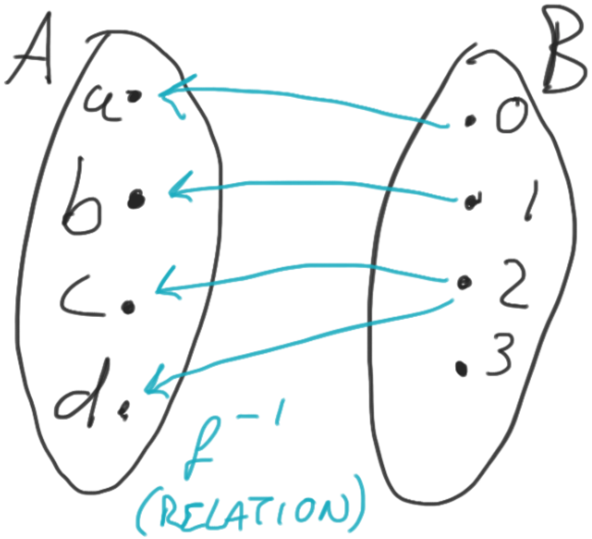
$$f(x)=y \quad iff \quad x^2=y \quad iff \quad ??x=\sqrt{y}??$$

  - If y < 0, then √y is not defined in R (this is due to f not being surjective)
  - Even if y > 0, so that √y is defined in R, this is not the only value for x: also x = -√y satisfies x^2 = y (this is due to f not being injective)

Inverse function

- CLARIFICATION: if a function is not bijective, it still has an inverse relation:





Inverse function

- CLARIFICATION: if a function is not bijective, it still has an inverse relation.
- However, it is often understood that a non bijective function does not have an inverse, because we tacitly imply “an inverse function”.
- It should be clear from the context whether we are allowing inverse relations which are not functions.

Inverse function

- Is f : $\mathbf{Z} \rightarrow \mathbf{Z}$, f(x) = x+1 invertible? If so, what is its inverse?
- f is injective: if f(x) = f(z), that is, x+1 = z+1, then x = z.
- f is surjective: for any y∈**Z**, we can solve for x the equation y = f(x): if y = x+1, then x = y-1. In other words, for any y∈**Z**, y = f(y-1)
- Therefore, f is invertible, with inverse

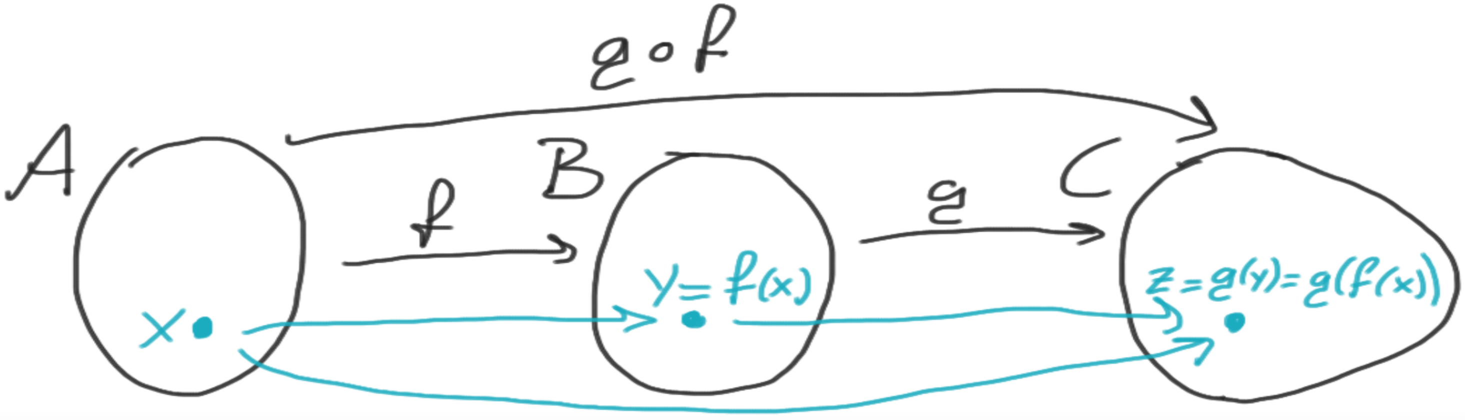
$$f^{-1}: Z \rightarrow Z \quad , \quad f^{-1}(y) = y - 1$$

Inverse function

- Is $f : \mathbf{N} \rightarrow \mathbf{N}$, $f(x) = x+1$ invertible? If so, what is its inverse?
- f is injective: if $f(x) = f(z)$, that is, $x+1 = z+1$, then $x = z$.
- f is not surjective: $0 \in \mathbf{N}$ has no preimage in $\mathbf{N}$
- Therefore, f is not invertible and has no inverse (function)

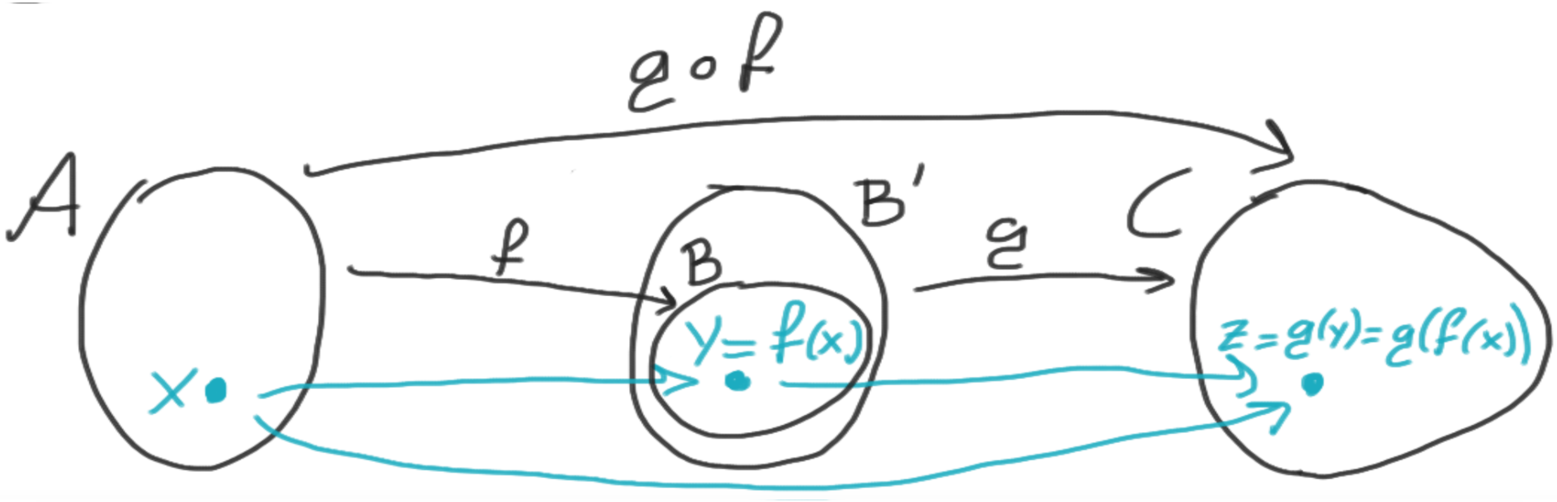
Composition of functions

- If f : A → B and g : B → C are functions, then the composition g∘f : A → C (which is always defined as a relation) is a function as well. It is defined as g∘f : A → C, g∘f (x) = g(f(x))



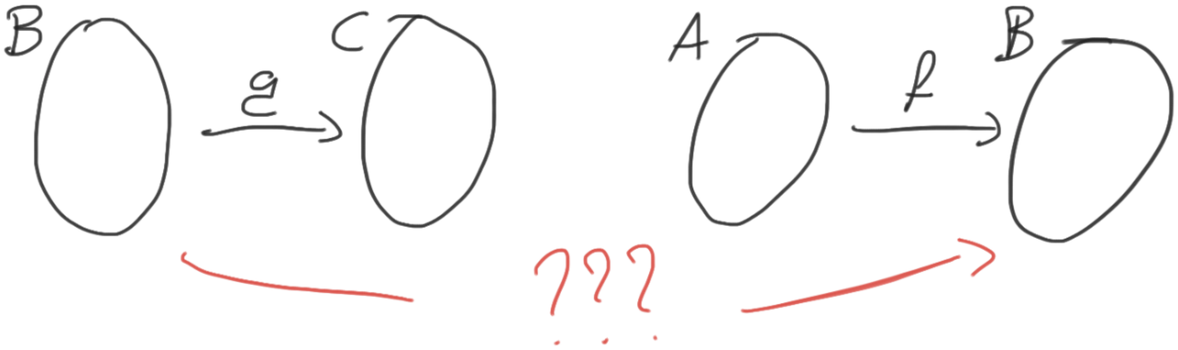
Composition of functions

- Slight generalization: if $f : A \to B$ and $g : B' \to C$ are functions, with $B \subseteq B'$, then the composition $g \circ f : A \to C$ is a well-defined function as well:



Composition of functions

- Composition of functions is not commutative
- EX: If $f : A \rightarrow B$ and $g : B \rightarrow C$ are functions, with $C \not\subseteq A$, then the composition $g \circ f : A \rightarrow C$ exists, but the composition $f \circ g$ does not exist



Composition of functions

- Even if both $g \circ f$ and $f \circ g$ exist, they need not coincide
- EX: If $f : A \to B$ and $g : B \to A$ are functions, with $A \neq B$, then the compositions $g \circ f : A \to A$ and $f \circ g : B \to B$ have different domain and codomain

Composition of functions

- Even if $f : A \rightarrow A$ and $g : A \rightarrow A$ are functions, then the compositions $g \circ f : A \rightarrow A$ and $f \circ g : A \rightarrow A$ need not coincide
- EX: If $f : \mathbf{N} \rightarrow \mathbf{N}$, $f(x) = x+1$ and $g : \mathbf{N} \rightarrow \mathbf{N}$, $g(x) = 2x$, then
- $g \circ f : \mathbf{N} \rightarrow \mathbf{N}$, $g \circ f\ (x) = g(f(x)) = g(x+1) = 2(x+1) = 2x+2$
- $f \circ g : \mathbf{N} \rightarrow \mathbf{N}$, $f \circ g\ (x) = f(g(x)) = f(2x) = 2x+1$

Powers of functions

- If $f : A \to A$ (or slightly more generally if $\mathrm{Codom}(f) \subseteq \mathrm{Dom}(f)$), then we can compose f with itself:
  $f^2 = f \circ f$, $f^3 = f \circ f \circ f = f^2 \circ f = f \circ f^2$, ...
- EX: if $f : \mathbf{Z} \to \mathbf{Z}$, $f(x) = 2x$, then
  $f^3 : \mathbf{Z} \to \mathbf{Z}$, $f^3(x) = f \circ f \circ f\ (x) = f(f(f(x))) = f(f(2x)) = f(4x) = 8x$
- If $f : A \to A$ is invertible, then also the negative powers of f are defined:
  f^{-1} is the inverse of f, $f^{-2} = f^{-1} \circ f^{-1}$, ...

4. SEQUENCES

Sequences

- A sequence is a function whose domain is a subset of **N**.
- Typically, sequences are functions f : **N** → **R**
- Typically, The notation a_n (or b_n, or similar) is used to denote the image of the natural number n, that is, if f is a sequence, then $a_n = f(n)$.
- a_n is the n^{th} term of the sequence.
- Besides the notation as a function, a sequence can be also denoted as the list of its terms: $a_0, a_1, a_2, a_3, \ldots$

Important sequence: geometric progression

- A geometric progression is a sequence of the form

  $t, tr, tr^2, tr^3, tr^4, \ldots$ ($a_n = t \cdot r^n$)

  where the initial term t and the common ratio r are real numbers.

- Geometric progressions can be defined recursively:

  $a_0 = t$, $a_{n+1} = r \cdot a_n$ for $n \in \mathbf{N}$

- Note that the ratio between a term and the preceding one is constant (it is the common ratio r)

Important sequence: geometric progression

- EX: the geometric progression with initial term t = 1 and common ratio r = -1 is

  1, -1, 1, -1, 1, -1, ... i.e.: $a_{2n} = 1$, $a_{2n+1} = -1$

- EX: the geometric progression with initial term t = 1 and common ratio r = 1/2 is

  1, 1/2, 1/4, 1/8, ... i.e.: $a_n = 2^{-n}$

Important sequence: arithmetic progression

- An arithmetic progression is a sequence of the form

  $t, t+d, t+2d, t+3d, t+4d, \ldots$ $(a_n = t+nd)$

  where the initial term t and the common difference d are real numbers.
- Arithmetic progressions can be defined recursively:

  $a_0 = t, \quad a_{n+1} = d+a_n$
- Note that the difference between a term and the preceding one is constant (it is the common difference d)

Important sequence: geometric progression

- EX: the arithmetic progression with initial term t = 1 and common difference d = -1 is

  1, 0, -1, -2, -3, ... i.e.: $a_n = 1-n$

- EX: the arithmetic progression with initial term t = 1 and common difference d = 1/2 is

  1, 3/2, 2, 5/2, ... i.e.: $a_n = 1+n/2$

Recursively defined sequences

- In applications, it often happens that a sequence we are interested in knowing is not given directly as an explicit formula

$$a_n = \textit{something}$$

  but rather in terms of a recurrence relation expressing a_n in terms of previous terms in the sequence
- Finding a formula for the n^{th} term of the sequence defined by a recurrence relation is called solving the recurrence relation. Such a formula is called a closed formula for the sequence.

Recursively defined sequences

- Finding closed formulas for recursively defined sequences is in general a hard problem, which can be attacked by the following strategy:

1) guess a closed formula looking at the first few terms output by the recurrence relation
2) prove the guess is correct using induction

Recursively defined sequences

- EX: find a closed formula for the sequence defined by the recurrence relation

$$a_0 = 0, \quad a_{n+1} = a_n + (2n+1)$$

Recursively defined sequences

- EX: find a closed formula for the sequence defined by the recurrence relation

$$a_0 = 0, \quad a_{n+1} = a_n+(2n+1)$$

1) Guess part:

$a_0 = 0,$

$a_1 = 0+(0+1) = 1,$

$a_2 = 1+(2+1) = 4,$

$a_3 = 4+(4+1) = 9,$

$a_4 = 9+(6+1) = 16,$

$a_5 = 16+(8+1) = 25$

$\boxed{\text{Maybe } a_n = n^2?}$

Recursively defined sequences

- EX: find a closed formula for the sequence defined by the recurrence relation

$$a_0 = 0, \quad a_{n+1} = a_n+(2n+1)$$

1) Proof part (by induction):

- Base case: $a_0 = 0 = 0^2$
- Induction step: assume, for $k \in N$, $a_k = k^2$ (IH). We want to show that $a_{k+1} = (k+1)^2$

  $a_{k+1} = a_k+(2k+1) = k^2+2k+1 = (k+1)^2$
- Therefore, indeed, a closed formula for the sequence is $a_n = n^2$