CS3388B: Lecture 1

January 10, 2023

1 Review of Linear Algebra

We're going to start with a review of linear algebra, vectors, and matrices. This mathematical basis for computer graphics, its implementation, and its manipulation is very important. As we explore topics within computer graphics, we will see new, varying, and specific applications of vectors and matrices. In this first section we'll just review the basics.

Linear algebra is so important in computer graphics mainly because of their ability to describe transformations. We'll revisit transformations in the 2D and 3D cases independently in further sections.

With this section we should re-familiarize ourselves with:

- Coordinate Systems
- Points, vectors, directions
- Dot product, cross product.
- Matrices and their arithmetic

1-dim: just a number line.

2-dim: cartesian plane:

origin & ortho vectors!

1.1 Coordinate Systems

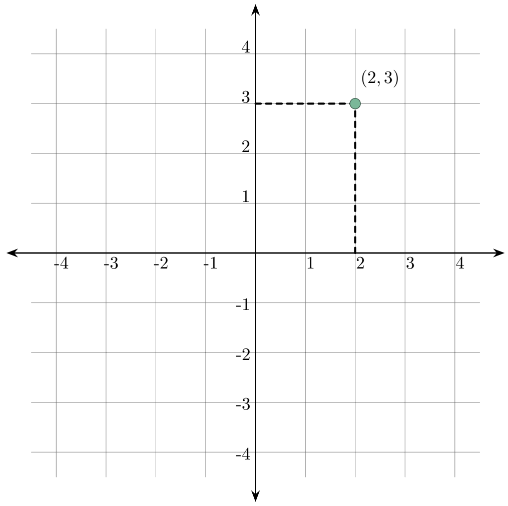
A coordinate system, in general, is a geometrical system where tuples of numbers are used to uniquely describe the positions of points, lines, surfaces, and other geometrical structures in some space. Each number in the tuples is *a* coordinate.

The *real number line* is really just a one-dimensional coordinate system. Each point in that one-dimensional space is determined by a single real number. For example, (3.14).

We should be very familiar with the *Cartesian plane* or two-dimensional *Cartesian coordinate system*. In this system, each point on the plane is given by a pair of real numbers (x, y).

Formally, this coordinate system is defined by an **origin** and two **orthonormal basis vectors**. In the two-dimensional case, those orthonormal basis vectors can be thought of just two perpendicular lines. The coordinates (x, y) represents the signed distance x to the first line, and the signed distance y to the second line. But, much like the real number line, you still need a starting place, a place for 0, an origin. In most coordinate systems the origin is given by the point where all coordinates are 0s.

Another way of thinking about coordinate systems is based much more explicitly on vectors. But we'll return to that point of view. First, let's just think about the Cartesian system as we

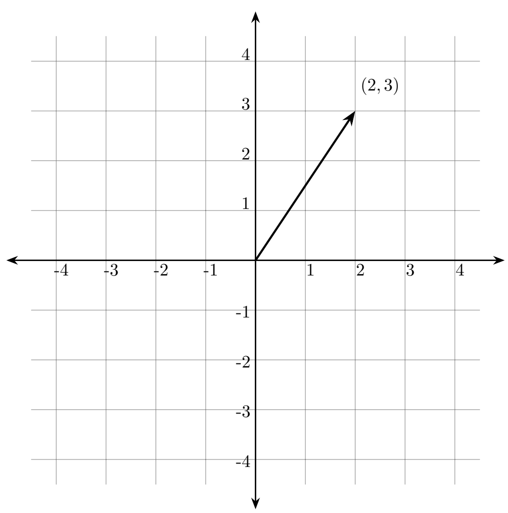


always have since grade school.

1.2 Vectors

From a geometric point of view, vectors are objects defined by their *length* and their *direction*. One could define vectors algebraically using *vector spaces*, but let's not think too hard about that and stay with the geometric ideas. This is a *graphics* course, after all.

In two dimensions, and on the Cartesian coordinate system, a vector points in a direction somewhere in the plane and has some length. The below vector points from (0, 0) to (2, 3).



However, the same vector can be constructed pointing from $(-2, -1)$ to $(0, 2)$. That is to say, they are the same vector. Vectors are only defined by direction and length. They don't actually have any real notion of "starting point" or "end point".

In that sense, vectors are usually constructed as *displacement vectors*: vectors that connect one

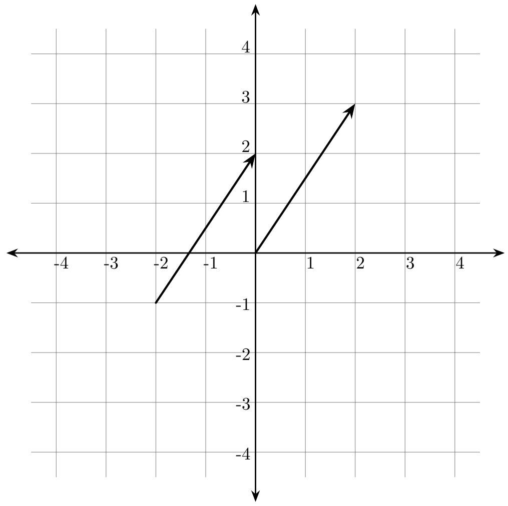


Figure 1: Two vectors in the plane.

point to another; or vectors that *displace* one point to another. In Figure 1, the left vector displaces the point $(-2, -1)$ to $(0, 2)$. The right vector displaces the point $(0, 0)$ to $(2, 3)$. But, they are the same vector!

The **difference** between two points is a vector. $\vec{v} = (2, 3) - (0, 0) = (0, 2) - (-2, -1)$.

The **sum** of a point and a vector is a another point. In this way, a vector is said to *displace* a point. The vector $\vec{v}$, as defined above and in Figure 1, displaces the origin $(0, 0)$ to $(2, 3)$.

$$(0, 0) + \vec{v} = (2, 3)$$

Even though the same vector can geometrically be represented in many ways, using any different choice of starting point, we can view vectors canonically as displacing the origin. Thus, we create a one-to-one correspondence between points and vectors in the Cartesian plane. Figure 1.2 shows the vector associated with the point $(2, 3)$.

We can represent vectors are *row matrices* or *column matrices*. That is, informally, a tuple of numbers viewed as either a row of numbers or a column of numbers.

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 2 & 3 \end{pmatrix}$$

This is a two-dimensional vector since there are two coordinates. Generally, vectors easily extend to any dimension by adding more coordinates.

An n-dimensional vector is an n-tuple: $\vec{v} = (v_1, v_2, \ldots, v_n)$.

So we get a vector's direction by talking about how a vector displaces a particular point. How do we get a vector's length?

Well it comes down to the particular coordinate system we are using, and the particular *norm* we are considering. But that's too much generic linear algebra for what we need. We'll assume length to be the standard *Euclidean norm*. For $\vec{v} = (v_1, v_2, \ldots, v_n)$ its length is:

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

1.2.1 Vector Operations

We have three basic operations with vectors:

1. Scalar multiplication
2. Vector addition
3. Vector subtraction

This operations are all performed *component-wise*.

Scalar multiplication can be viewed as *stretching* or *squishing* a vector. It does not change the direction* of the vector, it only changes the length of the vector.

The vector $\vec{v} = (1\ 1)$ has length 1. meanwhile the vector $2\vec{v} = (2, 2)$ has length 2. In scalar multiplication we simply multiply each coordinate of the vector by the same *scalar*.

For $\vec{v} = (v_1, v_2, \ldots, v_n)$:

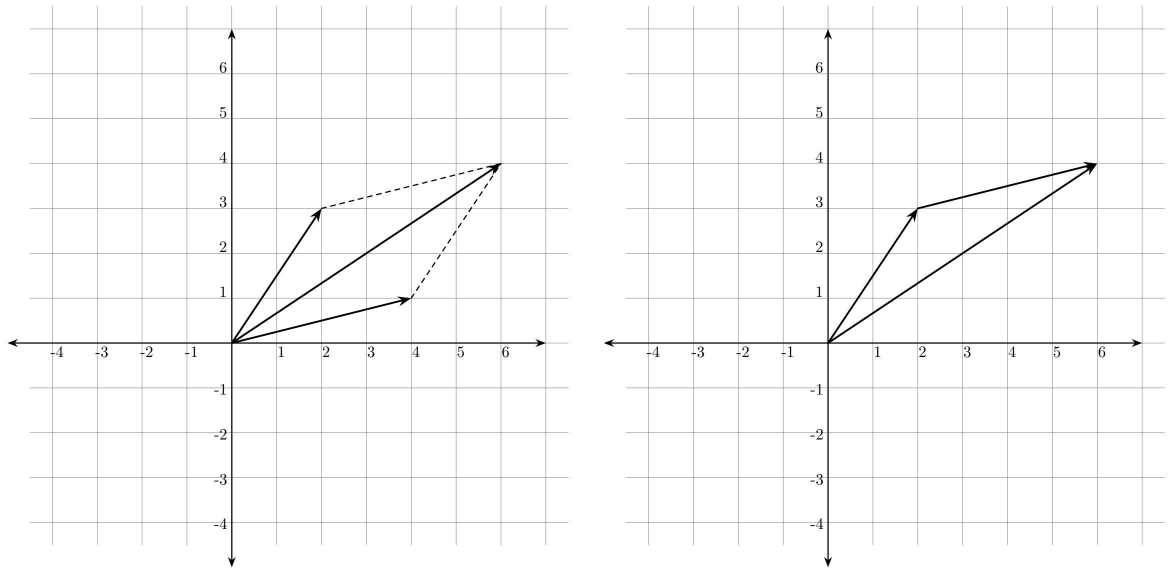
$$c\,\vec{v} = (cv_1, cv_2, \ldots, cv_n)$$

Because scalar multiplication only changes length, for any vector $\vec{v}$ and scalar c:

$$|c\,\vec{v}| = c\,|\vec{v}|$$

*Here's the caveat: if the scalar is negative, the vector flips around. It's still parallel to the original vector, but now points exactly 180° from its original direction. Well, 180° when the vector is in two dimensions. In more dimensions, you can thing about swapping around the tip and the tail of the vector.

Vector addition combines two vectors together to the have the "cumulative" effective of displacing a point by both vectors. Which vector is applied first, doesn't really matter.



For $\vec{u} = (2, 3)$, $\vec{v} = (4, 1)$:

$$\vec{u} + \vec{v} = (2 + 4, 3 + 1) = (6, 4) = (4 + 2, 1 + 3) = \vec{v} + \vec{u}$$

Vector subtraction behaves the same way you'd expect, subtracting elements component-wise. This time, order *does* matter.

For $\vec{u} = (2, 3)$, $\vec{v} = (4, 1)$:

$$\vec{u} - \vec{v} = (2 - 4, 3 - 1) = (-2, 2)$$

Now, the geometric interpretation of vector subtraction is a little different. We travel/displace *backwards* by the subtracting vector, and then travel/displace along the subtracted vector.

This makes more sense if we think about subtraction as first multiplying by -1 and then doing vector addition.

$$\vec{u} - \vec{v} = \vec{u} + (-1\vec{v})$$

So, we flip $\vec{v}$ 180° and then add the vectors together. After the flip it works just like addition.

It's important to remember that in all of this addition and subtraction, the actual start and end points of the vectors does not matter. What we are really interested in is the "cumulative effect"

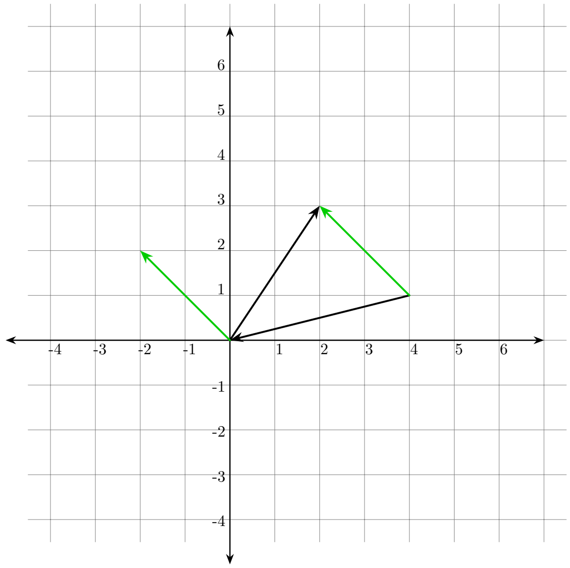


Figure 2:

of applying both vectors. So, in Figure 2, the green vector is the top vector minus the bottom vector. However, that green vector might displace the origin, or might displace the point (4, 1), or any other point. We only draw vector addition and subtraction as triangles to make things more visually friendly.

The final operation we consider in this section is a **linear combination**. It is scalar multiplication and vector addition used together. A linear combination of two vectors is the vector addition of some scalar multiples of the two vectors. For $\vec{u}$ and $\vec{v}$ and any two scalars a, b, a linear combination of $\vec{u}$ and $\vec{v}$ are:

$$a\vec{u} + b\vec{v}$$

1.3 Direction and Unit Vectors

We have seen already how to compute the *Euclidean norm* or *Euclidean length* of a vector. A very important class of vectors are the **unit vectors**. Vectors which have a length equal to 1.

To obtain a unit vector we can apply **normalization** to any vector with non-zero length. It's pretty simple. Just divide by its length (scalar multiply by the fraction 1 over its length).

$$norm(\vec{v}) = \frac{\vec{v}}{|\vec{v}|}$$

We often write normalized vectors using special notation: $\hat{v}$ to denote that it has length 1.

When a vector is a unit vector it can also be called *a* **direction**. Why? Because there is **exactly one** unit vector in any direction. Remember, vectors only care about direction and length. If we fix the length to be exactly 1, then there is exactly one vector in each possible direction.

Now, you might wonder that in all of this we haven't precisely defined direction. There's been no angles or north/south/east/west. This is intentional. Those kinds of directions only make sense when we are *relative* to some fixed position. To be north, it must be north *of something*. If you are in plane 10,000 feet above the north pole, which direction is north?

Instead, we let directions be the vectors themselves. remember that there is exactly one unit vector in each direction. Thus, two vectors point in the *same direction* if their normalized vectors are equal.

$$(2,4) \text{ and } (1,2) \text{ point in the same direction.}$$

In addition to two vectors pointing in the same direction, two other classifications are important: parallel and perpendicular vectors. We'll see those soon in the next two sections.

1.4 Dot product

The **dot product** or **inner product** of two vectors is a combination of those vectors that produces a single scalar value. It is the equal to the sum of component-wise products.

For $\vec{u} = (u_1, \ldots, u_n)$ and $\vec{v} = (v_1, \ldots, v_n)$ their dot product $\vec{u} \cdot \vec{v}$ is:

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i$$

The dot product has some nice properties:

- Commutativity: $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- Linearity (distributivity): $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
- Homogeneity: $c\vec{u} \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$

As a nice by-product of dot product, we get a different definition of length:

$$\vec{v} \cdot \vec{v} = |v|^2$$

As a second by-product of the dot product (details omitted), we finally get some angles. The angle θ *between* two vectors $\vec{u}$ and $\vec{v}$ can be found using the dot product and the following formula:

$\vec{u} \cdot \vec{v} = |u| \cdot |v| \cdot \cos$

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{|u|\,|v|}$$

For two unit vectors $\hat{u}$ and $\hat{v}$, their dot product exactly gives the cosine of the angle between them:

$$cos(\theta) = \hat{u} \cdot \hat{v}$$

If we recall the properties of the cosine function, we can easily gather some nice geometric information from the dot product of two vectors. Consider angles between 0° and 360°.

- The cosine of 0° is 1.
- The cosine of 90° and 270° is 0.
- For angles less than 90° or greater than 270°, the cosine of that angle is positive.
- For angles greater than 90° the cosine of that angle is negative.

In the case of vectors, the *angle between them* is at most 180°. Why? Because we can always measure the angle between "the other way around". In Figure 3, we measure the angle between them as the smaller angle, not the larger angle shown in red.

Now we can relate dot products with the angle between two vectors $\vec{u}$ and $\vec{v}$.

- If $\vec{u} \cdot \vec{v} > 0$, the angle between them is less than 90°
- If $\vec{u} \cdot \vec{v} < 0$, the angle between them is greater than 90° (and less than 180°)
- If $\vec{u} \cdot \vec{v} = 0$, the angle between them is exactly 90°.

For two normalized vectors, there are two special cases as well:

- If $\hat{u} \cdot \hat{v} = 1$, the angle between them is exactly 0°.
- If $\hat{u} \cdot \hat{v} = -1$, the angle between them is exactly 180°.

These special cases have special names, of course.

Two vectors are **perpendicular** if the angle between them is 90°. Thus, if their dot product is 0.

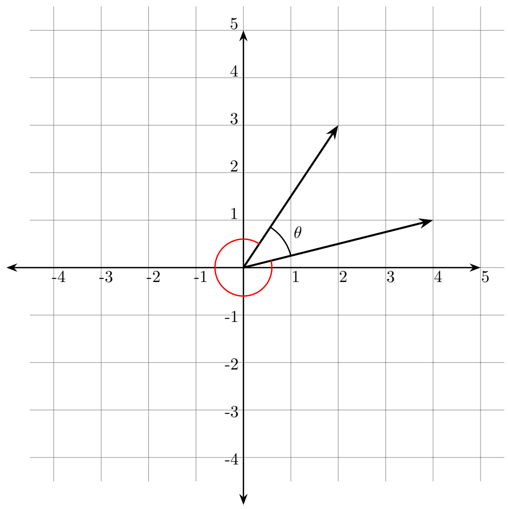


Figure 3:

Two vectors are **parallel** if the angle between them is 0° or 180°. Thus, if the absolute value of their dot product is equal to the product of their lengths.

Perpendicular vectors are particularly important in computer graphics. Two perpendicular vectors are also called **orthogonal** or **normal**. Don't get confused between a *normal vector* and a *normalized vector*. The first is perpendicular to some other vector, the second simply has its length equal to 1.

1.5 Orthonormal Bases

So now we know what orthogonal means. Looking back to Section 1.1, we can now make a little sense of how coordinate systems are defined. They defined using an origin and an orthonormal basis.

Orthonormal vectors are unit vectors which are orthogonal.

The Cartesian plane is defined with the origin at (0,0) and two orthonormal vectors denoted $\hat{\imath}$ and $\hat{\jmath}$:

$$\hat{\imath} = (1, 0) \qquad \hat{\jmath} = (0, 1)$$

These are exactly the unique normalized vectors pointing in the directions of the x-axis, and y-axis, respectively. It is not hard to show that these two vectors are perpendicular using their dot product.

In three dimensions, the Cartesian coordinate system has three basis vectors. We re-use $\hat{\imath}$, $\hat{\jmath}$, and add $\hat{k}$.

$$\hat{i} = (1, 0, 0) \qquad \hat{j} = (0, 1, 0) \qquad \hat{k} = (0, 0, 1)$$

Again, one can use dot products to show that these vectors are perpendicular to the other two.

When vectors are mutually orthogonal, they form an **orthogonal basis**. When normalized vectors are mutually orthogonal, they form an **orthonormal basis**.

In n dimensions we need n perpendicular vectors to form a orthogonal basis. You might recall this from linear algebra. But, an n-dimensional space can be described by many different bases. This will become important later when we start dealing with 3D transformations and multiple coordinate systems at the same time.

1.6 Cross Product

In three dimensions, the cross product is a kind of product between vectors that yields another vector. In particular, this new vector is perpendicular to both of the previous vectors. (You need three dimensions to be able to be perpendicular to two different vectors.)

For two vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$, their cross product is defined as:

$$\begin{aligned}\vec{u} \times \vec{v} &= (u_2 v_3 - u_3 v_2)\hat{i} + (u_3 v_1 - u_1 v_3)\hat{j} + (u_1 v_2 - u_2 v_1)\hat{k} \\ &= (u_2 v_3 - u_3 v_2\,, \quad u_3 v_1 - u_1 v_3\,, \quad u_1 v_2 - u_2 v_1)\end{aligned}$$

We'll revisit this definition later to show where it came from. What's most important in this section is the geometric interpretation of the cross product: it gives a third vector perpendicular to the first two.

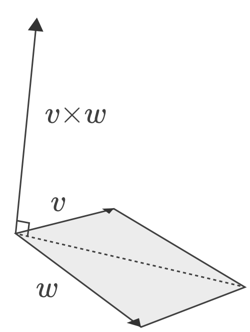


Figure 4:

For any two non-parallel vectors, say $\vec{v}$ and $\vec{w}$, they are contained in some *plane*. You can think of this as the two-dimensional space containing both $\vec{v}$ and $\vec{w}$. Geometrically, this represents all the possible points that can be obtained by a linear combination of $\vec{v}$ and $\vec{w}$.

Then, the cross product of $\vec{v}$ and $\vec{w}$ is the third vector perpendicular to $\vec{v}$ and $\vec{w}$. This it is a vector which is perpendicular to (or normal to) the plane spanned by $\vec{v}$ and $\vec{w}$.

But, you may now think: given a plane, there are two possible normal vectors, one pointing "up" and another pointing "down". Which to choose? Well, it is determined by the order of the cross product.

$$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$$

A quick way to determine the direction is shown in the next section. First, we conclude with one more formula about angles.

We can also use cross product to determine the angle θ between two vectors:

$$|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}|\, sin(\theta)$$

Again, θ here is always $\leq 180°$, by choosing to measure the angle from one vector to the other, or the other way around.

1.7 Handedness in Coorindate Systems

A coordinate system is also endowed with a "handedness" which gives us a cheap trick to determine the direction of cross products (and later, of rotations).

In a **right-hand coordinate system** we use our right hand. For $\vec{u} \times \vec{v}$, put your fingers in the direction of $\vec{u}$, with your palm facing toward $\vec{v}$ (in the direction where the angle between $\vec{u}$ and $\vec{v}$ is less than 180°). The thumb of your right hand points in the direction of their cross product.

If you repeat this procedure with $\vec{v} \times \vec{u}$, you'll see that you have to flip your palm in the opposite direction, and thus your thumb flips around 180°.

In a left-hand coordinate system, you do the exact same as above, but with the left hand.

This handedness also tells us about the orientation of that the coordinate system. In a right-hand coordinate system, we find that $\hat{k} = \hat{i} \times \hat{k}$.

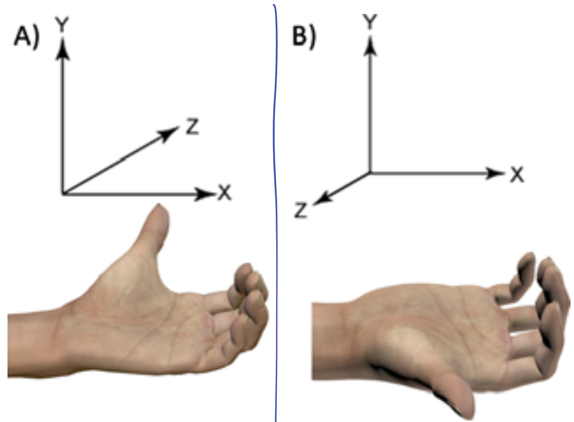


Figure 5: A left-hand coordinate system and a right-ahand coordinate system.

1.8 Matrices

We'll revisit matrices a lot in 3D graphics when we talk about *affine transforms*, and OpenGL. OpenGL *loves* matrices.

For now, let's just recall the basics of matrices and dealing with them.

An m by n **matrix** is a rectangular array of entries organized into m rows and n columns. An m by n matrix can be denoted as $A_{m \times n}$

We give special names to matrices with certain dimensions.

1. When $m = 1$ we have a **row vector**. Below is a row vector with 7 entries.

$$\begin{bmatrix} 4 & 1 & 4 & 2 & 3 & 7 & 8 \end{bmatrix}$$

2. When $n = 1$ we have a **column vector**. Below is a column vector with 4 entries.

$$\begin{bmatrix} 5 \\ -3 \\ 4 \\ -11 \end{bmatrix}$$

3. When $m = n$, we have a **square matrix**. We say a square matrix is *of order n* for a matrix with dimension n by n. A square matrix of order 3 is shown below.

$$\begin{bmatrix} 9 & -1 & 16 \\ 3 & -5 & 12 \\ 11 & 4 & -8 \end{bmatrix}$$

We are almost always going to use square matrices in this course.

For a particular (i, j) index, we can say the (i, j)th entry of a matrix is the element of the matrix in the ith row and the jth column.

$$A = \begin{bmatrix} 3 & 5 & 12 \\ -1 & -7 & 4 \end{bmatrix}$$

In the above 2 by 3 matrix, 5 is the (1, 2) entry, and is often denoted $A_{1,2}$. -7 is the (2, 1) entry, denoted $A_{2,1}$.

1.9 Scalar Multiplication of Matrices

Scalar multiplication is a simple operation which multiplies a single number against each entry of a matrix to produce another matrix of the same dimensions.

Say $A = (a_{i,j})$ is an m by n integer matrix. Given some other integer c, cA is another m by n matrix where:

$$cA = (c \cdot a_{i,j}) = c \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

$$= \begin{bmatrix} c \cdot a_{1,1} & c \cdot a_{1,2} & \cdots & c \cdot a_{1,n} \\ c \cdot a_{2,1} & c \cdot a_{2,2} & \cdots & c \cdot a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c \cdot a_{m,1} & c \cdot a_{m,2} & \cdots & c \cdot a_{m,n} \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 5 & 12 \\ -1 & -7 & 4 \end{bmatrix} \qquad 4A = \begin{bmatrix} 12 & 20 & 48 \\ -4 & -28 & 16 \end{bmatrix}$$

1.10 Matrix Addition

Two matrices can be added or subtracted when they have the same dimensions. For two m by n matrices, $A = (a_{i,j})$ and $B = (b_{i,j})$, their sum $A + B$ is equal to $(a_{i,j} + b_{i,j})$. That is, each entry of A is added to the corresponding entry in B in the same position.

$$
\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix} =
$$

$$
\begin{bmatrix} a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} & \cdots & a_{1,n} + b_{1,n} \\ a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} & \cdots & a_{2,n} + b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} + b_{m,1} & a_{m,2} + b_{m,2} & \cdots & a_{m,n} + b_{m,n} \end{bmatrix}
$$

Thanks to scalar multiplication, we can define subtraction via addition.

$$
A - B = A + (-1)B
$$

To add or subtract matrices they must be of the same dimensions. Every entry in the first matrix must have a corresponding entry in the second matrix to be added to.

$$
\begin{bmatrix} 1 & -4 & 8 \\ 11 & 2 & 24 \\ 12 & 4 & 1 \end{bmatrix} + \begin{bmatrix} -9 & 8 & 6 \\ 0 & 15 & 2 \\ 3 & 14 & 0 \end{bmatrix} = \begin{bmatrix} -8 & 4 & 14 \\ 11 & 17 & 26 \\ 15 & 18 & 1 \end{bmatrix}
$$

1.11 Matrix Multiplication

Matrix multiplication is much more involved than addition. First, we must consider under which conditions two matrices can be multiples.

Matrix multiplication between two matrices is only defined when the number of columns in the left-hand matrix equals the number of rows in the right-hand matrix. The result of the multiplication is another matrix whose number of rows equals the left-hand matrix's and whose number of columns equals the right-hand matrix's.

$$
A_{m\times n} \cdot B_{n\times p} = C_{m\times p}
$$

In this notation, the "inner" dimensions must be the same, and the "outer" dimensions give the dimensions of the product. In this case, $n = n$ are the inner dimensions, and m, p are the outer dimensions.

But how do we define the entries of the product $C = (c_{i,j})$? Each entry of the matrix product $c_{i,j}$ is an *inner product* of the ith row of A and the jth column of B.

$$c_{i,j} = \begin{bmatrix} a_{i,1} & a_{i,2} & a_{i,3} & \cdots & a_{i,n} \end{bmatrix} \cdot \begin{bmatrix} b_{1,j} \\ b_{2,j} \\ b_{3,j} \\ \vdots \\ b_{n,j} \end{bmatrix}$$

$$= \left(a_{i,1} \cdot b_{1,j} + a_{i,2} \cdot b_{2,j} + \cdots + a_{i,n} \cdot b_{n,j}\right)$$

$$= \sum_{k=1}^{n} a_{i,k} \cdot b_{k,j}$$

Therefore, the matrix multiplication $A_{m\times n} \cdot B_{n\times p} = C_{m\times p}$ is actually $m \times p$ individual inner products.

$$\begin{bmatrix} 0 & 4 & 1 & 0 \\ \boxed{6 \quad 5 \quad 1 \quad 8} \\ 5 & 2 & 7 & 9 \\ 0 & 2 & 4 & 7 \end{bmatrix} \times \begin{bmatrix} 3 & 6 & 0 & 3 \\ 7 & 5 & 7 & 1 \\ 3 & 9 & 2 & 9 \\ 6 & 7 & 8 & 4 \end{bmatrix} = \begin{bmatrix} c_{1,1} & c_{2,1} & c_{3,1} & c_{4,1} \\ c_{1,2} & \boxed{126} & c_{3,2} & c_{4,2} \\ c_{1,3} & c_{2,3} & c_{3,3} & c_{4,3} \\ c_{1,4} & c_{2,4} & c_{3,4} & c_{4,4} \end{bmatrix}$$

$$c_{2,2} = (6 \cdot 6) + (5 \cdot 5) + (1 \cdot 9) + (8 \cdot 7) = 126$$

$$\begin{bmatrix} -9 & 6 \\ 0 & 2 \\ 3 & 0 \end{bmatrix}_{3\times 2} \times \begin{bmatrix} 1 & -4 & 5 \\ 7 & 2 & 2 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} (-9 \cdot 1) + (6 \cdot 7) & (-9 \cdot -4) + (6 \cdot 2) & (-9 \cdot 5) + (6 \cdot 2) \\ (0 \cdot 1) + (2 \cdot 7) & (0 \cdot -4) + (2 \cdot 2) & (0 \cdot 5) + (2 \cdot 2) \\ (3 \cdot 1) + (0 \cdot 7) & (3 \cdot -4) + (0 \cdot 2) & (3 \cdot 5) + (0 \cdot 2) \end{bmatrix}_{3\times 3}$$

$$= \begin{bmatrix} 33 & 48 & -33 \\ 14 & 4 & 4 \\ 3 & -12 & 15 \end{bmatrix}_{3\times 3}$$

Square matrices are particularly important in matrix multiplication. Why? Because matrices must have compatible dimensions to be multiplied, and their product, in general, has different dimensions. However, for square matrices of order n, their product is also a square matrix of order n.

The consequence of this is that the multiplication of square matrices is *associative*. For square matrices A, B, C, we have:

$$ABC = A(BC) = (AB)C$$

However, note that matrix multiplication is still not commutative in general.

$$ABC \neq CBA$$

This will become very very important when we talk about 3D transformations.

With that, you know everything you need to know about linear algebra for graphics. The world is your oyster. Check your knowledge now by completing this week's problem set.