CHAPTER 5
Basic Quantificational Logic

Quantificational logic studies arguments whose validity depends on "all," "no," "some," and similar notions.[1] This chapter covers the basics, and the next adds relations and identity.

5.1 Easier translations

To help us evaluate quantificational arguments, we'll construct a little quantificational language. Our language builds on propositional logic and includes all the vocabulary, wffs, inference rules, and proofs of the latter. We add two new vocabulary items: small letters and "∃." Here are sample formulas:

Ir	=	Romeo is Italian.
Ix	=	x is Italian.
(x)Ix	=	For all x, x is Italian (all are Italian).
(∃x)Ix	=	For some x, x is Italian (some are Italian).

"Romeo is Italian" is "Ir"; we write the capital letter first. Here "I" is for the general category "Italian" and "r" is for the specific individual "Romeo":

Use capital letters for **general terms** (terms that *describe* or put in a *category*): I=an Italian C=charming R=drives a Rolls Use capitals for "a so and so," adjectives, and verbs.	Use small letters for **singular terms** (terms that pick out a *specific* person or thing): i=the richest Italian c=this child r=Romeo Use small letters for "the so and so," "this so and so," and proper names.

Capital and small letters have various uses in our quantificational language. Capitals can represent statements, general terms, or relations (but we won't study relations until the next chapter):

A capital letter alone (not followed by small letters) represents a *statement*.	S	*It is snowing.*
A capital letter followed by a single small letter represents a *general term*.	Ir	Romeo is *Italian.*
A capital letter followed by two or more small letters represents a *relation.*	Lrj	Romeo *loves* Juliet.

Similarly, small letters can be constants or variables:

A small letter from "a" to "w" is a **constant**—and stands for a specific person or thing.	Ir	*Romeo* is Italian.
A small letter from "x" to "z" is a **variable**—and stands for an unspecified member of a class of things.	Ix	*x* is Italian.

A variable stands for an unspecified person or thing. "Ix" ("x is Italian") is incomplete, and thus not true or false, since we haven't said whom we are talking about; but we can add a quantifier to complete the claim.

A **quantifier** is a sequence of the form "(x)" or "(∃x)"—where any variable may replace "x":

"(x)" is a **universal quantifier**. It claims that the formula that follows is true for *all* values of x.	"(∃x)" is an **existential quantifier**. It claims that the formula that follows is true for *at least one* value of x.
(x)Ix = For all x, x is Italian. = All are Italian.	(∃x)Ix = For some x, x is Italian. = Some are Italian.

Quantifiers express "all" and "some" by saying in how many cases the following formula is true.

As before, a grammatically correct formula is called a *wff,* or *well-formed formula.* For now, wffs are strings that we can construct using the propositional rules plus these two rules:

1. The result of writing a capital letter and then a small letter is a wff.
2. The result of writing a quantifier and then a wff is a wff.

These rules let us build wffs that we've already mentioned: "Ir," "Ix," "(x)Ix," and "(∃x)Ix." Don't use additional parentheses with these forms:

Right:	Ir	Ix	(x)Ix	(∃x)Ix
Wrong:	(Ir)	(Ix)	(x)(Ix) ((x)Ix)	((∃x)Ix)

Use a pair of parentheses for each quantifier and for each instance of "·," "∨," "⊃," and "≡"; use no other parentheses. Here are some further wffs:

~(x)Ix	=	Not all are Italian.
	=	It is not the case that, for all x, x is Italian.
~(∃x)Ix	=	No one is Italian.
	=	It is not the case that, for some x, x is Italian.

$(Ix \supset Lx)$	=	If x is Italian then x is a lover.
$(Ix \cdot Lx)$	=	x is Italian and x is a lover.

Translating from English sentences to wffs can be difficult. We'll begin with sentences that translate into wffs starting with a quantifier, or with "~" and then a quantifier. This rule tells where to put what quantifier:

If the English begins with	then begin the wff with
all (every)	(x)
not all (not every)	$\sim(x)$
some	$(\exists x)$
no	$\sim(\exists x)$

Here are basic examples:

All are Italian	=	$(x)Ix$	Some are Italian	=	
Not all are Italian	=	$\sim(x)Ix$	No one is Italian	=	$\sim(\exists x)Ix$

Here are harder examples:

All are rich or Italian	=	$(x)(Rx \vee Ix)$
Not everyone is non-Italian	=	$\sim(x)\sim Ix$
Some aren't rich	=	$(\exists x)\sim Rx$
No one is rich and non-Italian	=	$\sim(\exists x)(Rx \cdot \sim Ix)$

When the English begins with "all," "some," "not all," or "no," the quantifier must go *outside* all parentheses:

Right:	All are rich or Italian	=	$(x)(Rx \vee Ix)$
Wrong:	All are rich or Italian	=	$((x)Rx \vee Ix)$

The *wrong* formula means "Either everyone is rich, or x is Italian"—which isn't what we want to say.

If the English sentence specifies a logical connective (like "or," "and," or "if-then"), then use the corresponding logical symbol. When the English doesn't specify the connective, use these rules:

With "all…is…," use "⊃" for the *middle* connective.	*Otherwise* use "·" for the connective.

"All (every) A is B" uses "⊃," while "Some A is B" and "No A is B" use "·"; here are examples:

All Italians are lovers	=	(x)(Ix⊃Lx)
	=	For all x, *if* x is Italian *then* x is a lover.
Some Italians are lovers	=	(∃x)(Ix · Lx)
	=	For some x, x is Italian *and* x is a lover.
No Italians are lovers	=	~(∃x)(Ix · Lx)
	=	It is not the case that, for some x, x is Italian *and* x is a lover.

This next example illustrates both boxed rules:

All rich Italians are lovers	=	(x)((Rx·Ix)⊃Lx)
	=	For all x, *if* x is rich *and* Italian, *then* x is a lover.

We use "⊃" as the *middle* connective ("*If* rich Italian, *then* lover") and "·" in the *other* place ("If rich *and* Italian, then lover"). Note carefully the connectives in the next two examples:

Not all Italians are lovers	=	~(x)(Ix⊃Lx)
	=	It is not the case that, for all x, *if* x is Italian *then* x is a lover.
All are rich Italians	=	(x)(Rx·Ix)
	=	For all x, x is rich *and* Italian.

In case of doubt, phrase out the symbolic formula to yourself and see if it means the same as the English sentence.

Sentences with a main verb other than "is" should be rephrased to make "is" the main verb—and then translated. Here's an example:

All dogs hate cats	=	All dogs are cat-haters.
	=	For all x, if x is a dog then x is a cat-hater.
	=	(x)(Dx⊃Cx)

The **universe of discourse** is the set of entities that words like "all," "some," and "no" range over in a given context. In translating arguments about some one kind of entity (such as persons or statements), we can simplify our formulas by restricting the universe of discourse to that one kind of entity. We did this implicitly when we translated "All are Italian" as "(x)Ix"—instead of as "(x)(Px⊃Ix)" ("All persons are Italians"); here our "(x)" really means "For all persons x." We'll often restrict the universe of discourse to persons.

English has many idiomatic expressions; so our translation rules are rough and don't always work. After you symbolize an English sentence, it's wise to read your formula carefully, to make sure it reflects what the English means.

5.1a Exercise—LogiCola H (EM & ET)

Using these equivalences, translate these English sentences into wffs.

Ex	=	x is evil	Cx	=	x is crazy
Lx	=	x is a logician	Rx	=	x runs

Not all logicians run.	~(x)(Lx⊃Rx)

1. x isn't evil.
2. x is either crazy or evil.
3. Someone is evil.
4. Someone isn't evil.
5. All are evil.
6. If x is a logician, then x is evil.
7. All logicians are evil.
8. No one is evil.
9. Some logicians are evil.
10. No logician is evil.
11. Some logicians are evil and crazy.
12. No logician who runs is crazy.
13. Not all run.
14. Every logician is crazy or evil.
15. Some who are crazy aren't evil logicians.
16. All crazy logicians are evil.
17. Not all non-logicians are evil.
18. Some logicians who aren't crazy run.
19. No one who is crazy or evil runs.
20. All who aren't logicians are evil.
21. Not everyone is crazy or evil.
22. Not all who are evil or crazy are logicians.
23. All evil people and crazy people are logicians.
24. All are evil logicians.
25. No one who isn't an evil logician is crazy.

5.2 Easier proofs

Quantificational proofs work much like propositional ones, but use four new inference rules for quantifiers.

These two reverse-squiggle (RS) rules hold regardless of what variable replaces "x" and what pair of contradictory wffs replaces "Fx"/"~Fx" (here "→" means we can infer whole lines from left to right):

Reverse squiggle	
~(x)Fx → (∃x)~Fx	
~(∃x)Fx → (x)~Fx	

So "Not everyone is funny" entails "Someone isn't funny." Similarly, "It is not the case that someone is funny" ("No one is funny") entails "Everyone is non-funny." We can reverse squiggles on more complicated formulas, so long as the whole formula begins with "~" and then a quantifier:

Right:	*Right:*	*Wrong:*
$\sim(\exists x)\sim Gx$	$\sim(x)(Lx \cdot \sim Mx)$	$(Ir \supset \sim(x)Gx)$
$\therefore (x)\sim\sim Gx$	$\therefore (\exists x)\sim(Lx \cdot \sim Mx)$	$\therefore (Ir \supset (\exists x)\sim Gx)$

It the first example, it would be simpler to conclude "(x)Gx" (eliminating the double negation). Reverse squiggles whenever you have a wff that begins with "~" and then a quantifier; reversing a squiggle moves the quantifier to the beginning of the formula, so we can later drop it.

We drop quantifiers using the next two rules (which hold regardless of what variable replaces "x" and what wffs replace "Fx"/"Fa"—provided that the two wffs are identical except that wherever the variable occurs freely[1] in the former the same constant occurs in the latter). Here's the drop-existential (DE) rule:

Drop existential: $\boxed{(\exists x)Fx \rightarrow Fa,}$ use a *new* constant

Suppose that *someone* robbed the bank; we can give this person a name—like "Al"—but it must be an *arbitrary name* that we make up. Likewise, when we drop an existential, we'll label this "someone" with a *new constant*—one that hasn't yet occurred in earlier lines of the proof. In proofs, we'll use the next unused letter in alphabetical order—starting with "a," and then "b," and so on:[1]

$(\exists x)Mx$		Someone is male, someone is female;
$(\exists x)Fx$		let's call the male "a" and the female "b."
$\therefore Ma$	←	"a" is OK since it occurs in no earlier line.
$\therefore Fb$	←	Since "a" has now occurred, we use "b."

We can drop existentials from complicated formulas if the quantifier begins the wff and we replace the variable with the same new constant throughout:

[1] *Technical footnote:* An instance of a variable occurs "freely" if it doesn't occur as part of a wff that begins with a quantifier using that variable; just the first instance of "x" in "(Fx·(x)Gx)" occurs freely. So we'd go from "(∃x)(Fx · (x)Gx)" to "(Fa·(x)Gx)"

[1] *Technical footnote:* This paragraph needs three qualifications. (1) If *someone* robbed the bank, then maybe *more than one* person did; then our name (or constant) will refer to a random *one* of the robbers. (2) Using a new name is consistent with the robber being someone mentioned in the argument so far; different names (like "Jim" and "Smith") might refer to the same individual. (3) Rule DE should be used only when there is at least one not-blocked-off assumption; otherwise, the symbolic version of "Someone is a thief, so Gensler is a thief" would be a two-line proof.

Right:	*Wrong:*	*Wrong:*
$(\exists x)(Fx \cdot Gx)$	$((\exists x)Fx \supset P)$	$(\exists x)(Fx \cdot Gx)$
$\therefore (Fa \cdot Ga)$	$\therefore (Fa \supset P)$	$\therefore (Fa \cdot Gb)$

The middle formula doesn't *begin* with a quantifier; instead, it begins with a left-hand parenthesis. Drop only *initial* quantifiers.

Here's the drop-universal (DU) rule:

Drop universal $\boxed{\begin{array}{c} (x)Fx \;\rightarrow\; F\overset{\downarrow}{a}, \\ \text{use any constant} \end{array}}$

If *everyone* is funny, then Al is funny, Bob is funny, and so on. From "(x)Fx" we can derive "Fa," "Fb," and so on—using any constant. As before, the quantifier must begin the wff and we must replace the variable with the same constant throughout:

Right:	*Wrong:*	*Wrong:*
$(x)(Fx \supset Gx)$	$((x)Fx \supset (x)Gx)$	$(x)(Fx \supset Gx)$
$\therefore (Fa \supset Ga)$	$\therefore (Fa \supset Ga)$	$\therefore (Fa \supset Gb)$

The middle inference is wrong because the quantifier doesn't *begin* the formula (a left-hand parenthesis begins it). "((x)Fx⊃(x)Gx)" is an if-then and follows the if-then rules: if we have the first part "(x)Fx" true, we can get the second true; if we have the second part "(x)Gx" false, we can get the first false; and if we get stuck, we need to make another assumption.

Here's an English version of a quantificational proof:

1 All logicians are funny.
2 Someone is a logician.
[∴ Someone is funny.
3 ┌ Assume: It is not the case that someone is funny.
4 │ ∴ Everyone is non-funny. {from 3, reverse squiggles}
5 │ ∴ a is a logician. {from 2, drop existential, call the logician "a"}
6 │ ∴ If a is a logician then a is funny. {from 1, drop universal}
7 │ ∴ a is funny. {from 5 and 6}
8 └ ∴ a is non-funny. {from 4, drop universal}
9 ∴ Someone is funny. {from 3; 7 contradicts 8}

The steps here should make sense. Our proof strategy is: first reverse squiggles, then drop existentials using new letters, and finally drop universals using the same old letters. Here's the proof in symbols:

```
    1   (x)(Lx ⊃ Fx)   Valid
*   2   (∃x)Lx
      [∴ (∃x)Fx
*   3  ┌ asm: ~(∃x)Fx
    4  │ ∴ (x)~Fx   {from 3}
    5  │ ∴ La   {from 2}
*   6  │ ∴ (La ⊃ Fa)   {from 1}
    7  │ ∴ Fa   {from 5 and 6}
    8  └ ∴ ~Fa   {from 4}
    9  ∴ (∃x)Fx   {from 3; 7 contradicts 8}
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After making the assumption (line 3), we reverse the squiggle to move the quantifier to the outside (line 4). We drop the existential quantifier using a new constant (line 5). We drop the universal quantifier in line 1 using this same constant—and then use an I-rule (lines 6 and 7). Then we drop the universal in line 4 using this same constant (line 8), which gives us a contradiction. RAA gives us the original conclusion (line 9).

We starred lines 2, 3, and 6; as before, starred lines largely can be ignored in deriving further steps. Here are the new starring rules—with examples:

Star any wff on which you reverse squiggles.	Star any wff from which you drop an existential.
* ~(x)Fx ――――― ∴ (∃x)~Fx	* (∃x)Fx ――――― ∴ Fa

When we reverse squiggles or drop existentials, the new line has the same information. Don't star when dropping a universal; we can never exhaust an "all" statement by deriving instances—and we may have to derive further things from it later in the proof.

Here's another quantificational proof.

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    1   (x)(Fx · Gx)
      [∴ (x)Fx
*   2  ┌ asm: ~(x)Fx
*   3  │ ∴ (∃x)~Fx        First reverse squiggles
    4  │ ∴ ~Fa            and drop existentials.
    5  │ ∴ (Fa · Ga)      Lastly drop universals.
    6  └ ∴ Fa
    7  ∴ (x)Fx   {from 2; 4 contradicts 6}
```

It would be wrong to switch lines 4 and 5. If we drop the universal first using "a," then we can't drop the existential next using "a" (since "a" would be old).

Our proof strategy works much like before. We first assume the opposite of the conclusion; then we use our four new rules plus the S- and I-rules to derive whatever we can. If we find a contradiction, we apply RAA. If we get stuck and need to break up a wff of the form "~(A·B)" or "(A∨B)" or "(A⊃B)," then we make another assumption. If we get no contradiction and yet can't do anything further, then we try to refute the argument.

Reverse squiggles and drop existentials first, drop universals last:

1. FIRST REVERSE SQUIGGLES: For each unstarred, not-blocked-off step that begins with "~" and then a quantifier, derive a step using the reverse-squiggle rules. Star the original step.

2. AND DROP EXISTENTIALS: For each unstarred, not-blocked-off step that begins with an existential quantifier, derive an instance using the next available *new* constant (unless some such instance already occurs in previous not-blocked-off steps). Star the original step.

   Note: Don't drop an existential if you already have a not-blocked-off instance in previous steps—there's no point in deriving a second instance. So don't drop "(∃x)Fx" if you already have "Fc."

3. LASTLY DROP UNIVERSALS: For each not-blocked-off step that begins with a universal quantifier, derive instances using each *old* constant. Don't star the original step; you might have to use it again.

   Note: Drop a universal using a *new* letter only if you've done everything else possible (making further assumptions if needed) and still have no old letters. This is unusual, but happens if we try to prove "(x)~Fx ∴ ~(x)Fx."

Be sure to drop existentials before universals. Introduce a new letter each time you drop an existential, and use the same old letters when you drop a universal. And drop only *initial* quantifiers.

We won't see invalid quantificational arguments until later.

5.2a Exercise—LogiCola IEV

Prove each of these arguments to be valid (all are valid).

~(∃x)Fx
∴ (x)~(Fx · Gx)

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* 1   ~(∃x)Fx            Valid
    [ ∴ (x)~(Fx · Gx)
* 2   ┌ asm: ~(x)~(Fx · Gx)
* 3   │ ∴ (∃x)(Fx · Gx)   {from 2}
  4   │ ∴ (x)~Fx   {from 1}
  5   │ ∴ (Fa · Ga)   {from 3}
  6   │ ∴ ~Fa   {from 4}
  7   └ ∴ Fa   {from 5}
  8   ∴ (x)~(Fx · Gx)   {from 2; 6 contradicts 7}
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1. (x)Fx
   ∴ (x)(Gx ∨ Fx)

2. ~(∃x)(Fx · ~Gx)
   ∴ (x)(Fx ⊃ Gx)

3. ~(∃x)(Fx · Gx)
   (∃x)Fx
   ∴ (∃x)~Gx

4. (x)((Fx ∨ Gx) ⊃ Hx)
   ∴ (x)(~Hx ⊃ ~Fx)

5. (x)(Fx ⊃ Gx)
   (∃x)Fx
   ∴ (∃x)(Fx · Gx)

6. (x)(Fx ∨ Gx)
   ~(x)Fx
   ∴ (∃x)Gx

7. (x)~(Fx ∨ Gx)
   ∴ (x)~Fx

8. (x)(Fx ⊃ Gx)
   (x)(Fx ⊃ ~Gx)
   ∴ (x)~Fx

9. (x)(Fx ⊃ Gx)
   (x)(~Fx ⊃ Hx)
   ∴ (x)(Gx ∨ Hx)

10. (x)(Fx ≡ Gx)
    (∃x)~Gx
    ∴ (∃x)~Fx

5.2b Exercise—LogiCola IEV

First appraise intuitively. Then translate into logic (using the letters given) and prove to be valid (all are valid).

1. All who deliberate about alternatives believe in free will (at least implicitly).
   All deliberate about alternatives.
   ∴ All believe in free will. [Use Dx and Bx. This argument is from William James.]

2. Everyone makes mistakes.
   ∴ Every logic teacher makes mistakes. [Use MX and Lx.]

3. No feeling of pain is publicly observable.
   All chemical processes are publicly observable.
   ∴ No feeling of pain is a chemical process. [Use Fx, Ox, and Cx. This attacks a form of materialism that identifies mental events with material events. We also could test this argument using syllogistic logic (see Chapter 2).]

4. All (in the electoral college) who do their jobs are useless.
   All (in the electoral college) who don't do their jobs are dangerous.
   ∴ All (in the electoral college) are useless or dangerous. [Use Jx for "x does their job," Ux for "x is useless," and Dx for "x is dangerous." Use the universe of discourse of electoral college members: take "(x)" to mean "for every electoral college member x" and don't translate "in the electoral college."]

5. All that's known is experienced through the senses.
   Nothing that's experienced through the senses is known.
   ∴ Nothing is known. [Use Kx and Ex. Empiricism (premise 1) plus skepticism about the senses (premise 2) yields general skepticism.]

6. No pure water is burnable.
   Some Cuyahoga River water is burnable.
   ∴ Some Cuyahoga River water isn't pure water. [Use Px, Bx, and Cx. The Cuyahoga is a river in Cleveland that used to catch on fire.]

7. Everyone who isn't with me is against me.
   ∴ Everyone who isn't against me is with me. [Use Wx and Ax. These claims from the Gospels are sometimes thought to be incompatible.]

8. All basic laws depend on God's will.
   ∴ All basic laws about morality depend on God's will. [Use Bx, Dx, and MX.]

9. Some lies in unusual circumstances aren't wrong.
   ∴ Not all lies are wrong. [Use Lx, Ux, and Wx.]

10. Nothing based on sense experience is certain.
    Some logical inferences are certain.
    All certain things are truths of reason.

∴ Some truths of reason are certain and aren't based on sense experience. [Use Bx, Cx, Lx, and Rx.]

11. No truth by itself motivates us to action.
    Every categorical imperative would by itself motivate us to action.
    Every categorical imperative would be a truth.
    ∴ There are no categorical imperatives. [Use Tx, MX, and Cx. Immanuel Kant claimed that commonsense morality accepts categorical imperatives (objectively true moral judgment that command us to act and that we must follow if we are to be rational); but some thinkers argue against the idea.]

12. Every genuine truth claim is either experimentally testable or true by definition.
    No moral judgments are experimentally testable.
    No moral judgments are true by definition.
    ∴ No moral judgments are genuine truth claims. [Use Gx, Ex, Dx, and MX. This is logical positivism's argument against moral truths.]

13. Everyone who can think clearly would do well in logic.
    Everyone who would do well in logic ought to study logic.
    Everyone who can't think clearly ought to study logic.
    ∴ Everyone ought to study logic. [Use Tx, Wx, and Ox.]

5.3 Easier refutations

Applying our proof strategy to an invalid argument leads to a refutation:

All logicians are funny.
Someone is a logician.
∴ Everyone is funny.

	1	(x)(Lx ⊃ Fx)	**Invalid**
*	2	(∃x)Lx	a, b
		[∴ (x)Fx	
*	3	asm: ~(x)Fx	La, Fa
*	4	∴ (∃x)~Fx {from 3}	~Lb, ~Fb
	5	∴ La {from 2}	
	6	∴ ~Fb {from 4}	
*	7	∴ (La ⊃ Fa) {from 1}	
*	8	∴ (Lb ⊃ Fb) {from 1}	
	9	∴ Fa {from 5 and 7}	
	10	∴ ~Lb {from 6 and 8}	

After making the assumption (line 3), we reverse a squiggle to move a quantifier to the outside (line 4). Then we drop the two existential quantifiers, using a new and different constant each time (lines 5 and 6). We drop the universal quantifier twice, first using "a" and then using "b" (lines 7 and 8). Since we reach no contradiction, we gather the simple pieces to give a refutation. Our refutation is a little possible world with two people, a and b:

a is a logician. b isn't a logician.	a is funny. b isn't funny.

Here the premises are true, since all logicians are funny, and someone is a logician. The conclusion is false, since someone isn't funny. Since the premises are all true and conclusion false, our argument is invalid.

If we try to prove an invalid argument, we'll instead be led to a refutation—a little possible world with various individuals (like a and b) and simple truths about these individuals (like La and ~Lb) that would make the premises all true and conclusion false. In evaluating the premises and conclusion, evaluate each wff (or part of a wff) that starts with a quantifier according to these rules:

An *existential* wff is true if and only if *at least one case* is true.	A *universal* wff is true if and only if *all cases* are true.

In our world, universal premise "(x)(Lx⊃Fx)" is true, since all cases are true:

$$(La \supset Fa) \;=\; (1 \supset 1) \;=\; 1$$
$$(Lb \supset Fb) \;=\; (0 \supset 0) \;=\; 1$$

And existential premise "(∃x)Lx" is true, since at least one case is true (we have "La"—"a is a logician"). But universal conclusion "(x)Fx" is false, since at least one case is false (we have "~Fb"—"b isn't funny"). So our possible world makes the premises all true and conclusion false.

Be sure to check that your refutation works. If you don't get premises all 1 and conclusion 0, then you did something wrong—and the source of the problem is likely what you did with the formula that came out wrong.

Here's another example:

*	1	~(∃x)(Fx · Gx)	**Invalid**
*	2	(∃x)Fx	a, b
		[∴ ~(∃x)Gx	
*	3	asm: (∃x)Gx	Fa, ~Ga Gb, ~Fb
	4	∴ (x)~(Fx · Gx) {from 1}	
	5	∴ Fa {from 2}	
	6	∴ Gb {from 3}	
*	7	∴ ~(Fa · Ga) {from 4}	
*	8	∴ ~(Fb · Gb) {from 4}	
	9	∴ ~Ga {from 5 and 7}	
	10	∴ ~Fb {from 6 and 8}	

In this world, some things are F and some things are G, but nothing is both at once. In evaluating the "~(∃x)(Fx·Gx)" premise, first evaluate the part starting with the quantifier. "(∃x)(Fx·Gx)" is false since no case is true:

(Fa · Ga) = (1 · 0) = 0
(Fb · Gb) = (0 · 1) = 0

So the denial "~(∃ x)(Fx·Gx)" is true. The "(∃ x)Fx" premise is true since at least one case is true (namely, Fa). In evaluating the "~(∃ x)Gx" conclusion, first evaluate the part starting with the quantifier. "(∃ x)Gx" is true since at least one case is true (namely, Gb); so the denial "~(∃ x)Gx" is false. So we get the premises all true and conclusion false.

These two rules are crucial for working out proofs and refutations:

(1) For each *existential quantifier,* introduce a *new constant.*
(2) For each *universal quantifier,* derive an instance for each *old constant.*

In our last example, we'd violate (1) if we derived "Ga" in step 6—since "a" at this point is old; then we'd "prove" the argument to be valid. We'd violate (2) if we didn't derive "~(Fb·Gb)" in line 8. Then our refutation would have no truth value for "Fb"; so "Fb" and premise 1 would both be "?" (unknown truth value)—showing that we had to do something further with premise 1.

Possible worlds for refutations must contain at least one entity. Seldom do we need more than two entities.

5.3a Exercise—LogiCola IEI

Prove each of these arguments to be invalid (all are invalid).

~(x)(Fx ∨ Gx)
∴ ~(∃x)Gx

```
*  1   ~(x)(Fx ∨ Gx)                    Invalid
     [ ∴ ~(∃x)Gx
*  2   asm: (∃x)Gx                        a, b
*  3   ∴ (∃x)~(Fx ∨ Gx)  {from 1}
*  4   ∴ ~(Fa ∨ Ga)  {from 3}           ~Fa, ~Ga
   5   ∴ ~Fa  {from 4}                     Gb
   6   ∴ ~Ga  {from 4}
   7   ∴ Gb  {from 2}
```

1. (∃x)Fx
   ∴ (x)Fx

2. (∃x)Fx
   (∃x)Gx
   ∴ (∃x)(Fx · Gx)

3. (∃x)(Fx ∨ Gx)
   ~(x)Fx
   ∴ (∃x)Gx

4. (∃x)Fx
   ∴ (∃x)~Fx

5. ~(∃x)(Fx · Gx)
   (x)~Fx
   ∴ (x)Gx

6. (x)(Fx ⊃ Gx)
   ~(x)Gx
   ∴ (x)~(Fx · Gx)

7. (x)((Fx · Gx) ⊃ Hx)
   (∃x)Fx
   (∃x)Gx
   ∴ (∃x)Hx

8. (∃x)(Fx ∨ ~Gx)
   (x)(~Gx ⊃ Hx)
   (∃x)(Fx ⊃ Hx)
   ∴ (∃x)Hx

9. (∃x)~(Fx ∨ Gx)
   (∃x)Hx
   ~(∃x)Fx
   ∴ ~(x)(Hx ⊃ Gx)

10. (∃x)~Fx
    (∃x)~Gx
    ∴ (∃x)(Fx ≡ Gx)

5.3b Exercise—LogiCola IEC

First appraise intuitively. Then translate into logic (using the letters given) and say whether valid (and give a proof) or invalid (and give a refutation).

1. No pure water is burnable.
   Some Cuyahoga River water isn't burnable.
   ∴ Some Cuyahoga River water is pure water. [Use Px, Bx, and Cx.]

2. No material thing is infinite.
   Not everything is material.
   ∴ Something is infinite. [Use MX and Ix.]

3. Some smoke.
   Not all have clean lungs.
   ∴ Some who smoke don't have clean lungs. [Use Sx and Cx.]

4. Some Marxists plot violent revolution.
   Some faculty members are Marxists.
   ∴ Some faculty members plot violent revolution. [Use MX, Px, and Fx.]

5. All valid arguments that have "ought" in the conclusion also have "ought" in the premises.
   All arguments that seek to deduce an "ought" from an "is" have "ought" in the conclusion but don't have "ought" in the premises.
   ∴ No argument that seeks to deduce an "ought" from an "is" is valid. [Use Vx for "x is valid," Cx for "x has 'ought' in the conclusion," Px for "x has 'ought' in the premises," Dx for "x seeks to deduce an 'ought' from an 'is,'" and the universe of discourse of arguments. This one is difficult to translate.]

6. Every kick returner who is successful is fast.
   ∴ Every kick returner who is fast is successful. [Use Kx, Sx, and Fx.]

7. All exceptionless duties are based on the categorical imperative.
   All non-exceptionless duties are based on the categorical imperative.
   ∴ All duties are based on the categorical imperative. [Use Ex, Bx, and the universe of discourse of duties. This argument is from Kant, who based all duties on his supreme moral principle, called "the categorical imperative."]

8. All who aren't crazy agree with me.
   ∴ No one who is crazy agrees with me. [Use Cx and Ax.]

9. Everything can be conceived.
   Everything that can be conceived is mental.
   ∴ Everything is mental. [Use Cx and Mx. This is from George Berkeley, who attacked materialism by arguing that everything is mental and that matter doesn't exist apart from mental sensations; so a chair is just a collection of experiences. Bertrand Russell thought that premise 2 was confused.]

10. All sound arguments are valid.
∴ All invalid arguments are unsound. [Use Sx and Vx and the universe of discourse of arguments.]

11. All trespassers are eaten.
∴ Some trespassers are eaten. [Use Tx and Ex. The premise is from a sign on the Appalachian Trail in northern Virginia. Traditional logic (see Section 2.8) takes "all A is B" to entail "some A is B"; modern logic takes "all A is B" to mean "whatever is A also is B"—which can be true even if there are no A's.]

12. Some necessary being exists.
All necessary beings are perfect beings.
∴ Some perfect being exists. [Use Nx and Px. Kant claimed that the cosmological argument for God's existence at most proves premise 1; it doesn't prove the existence of God (a perfect being) unless we add premise 2. But premise 2, by the next argument, presupposes the central claim of the ontological argument—that a perfect being necessarily exists. So, Kant claimed, the cosmological argument presupposes the ontological argument.]

13. All necessary beings are perfect beings.
∴ Some perfect beings are necessary beings. [Use Nx and Px. Kant followed traditional logic (see problem 11) in taking "all A is B" to entail "some A is B."]

14. No one who isn't a logical positivist holds the verifiability criterion of meaning.
∴ All who hold the verifiability criterion of meaning are logical positivists. [Use Lx and Hx. The verifiability criterion of meaning says that every genuine truth claim is either experimentally testable or true by definition.]

5.4 Harder translations

We'll now start using statement letters (like "S" for "It is snowing") and individual constants (like "r" for "Romeo") in our translations and proofs:

If it's snowing, then Romeo is cold = $(S \supset Cr)$

Here "S," since it's a capital letter not followed by a small letter, represents a whole statement. And "r," since it's a small letter between "a" and "w," is a constant that stands for a specific person or thing.

We'll also start using multiple and non-initial quantifiers. From now on, use this expanded rule about what quantifier to use and where to put it:

Wherever the English has	put this in the wff
all (every)	(x)
not all (not every)	$\sim(x)$
some	$(\exists x)$
no	$\sim(\exists x)$

Here's an example:

If all are Italian, then Romeo is Italian = ((x)Ix⊃Ir)

Since "if" translates as "(," likewise "if all" translates as "((x)." As you translate, mimic the English word order:

all not	=	(x)~	all either	=	(x)(	if all either	=	((x)(
not all	=	~(x)	either all	=	((x)	if either all	=	(((x)

Use a separate quantifier for each "all," "some," and "no":

If all are Italian then all are lovers	=	((x)Ix ⊃ (x)Lx)
If not everyone is Italian, then some aren't lovers.	=	(~(x)Ix ⊃ (∃x)~Lx)
If no Italians are lovers, then some Italians are not lovers	=	(~(∃x)(Ix • Lx) ⊃ (∃x)(Ix • ~Lx))

"Any" differs in subtle ways from "all." "All" translates into a "(x)" that mirrors where "all" occurs in the English sentence. "Any" is governed by two different but equivalent rules; the easier rule goes as follows:

(1) To translate a sentence with "any," first rephrase it so it means the same thing but doesn't use "any"; then translate the second sentence.	"Not any…"="No…" "If any…"="If some…" "Any…"="All…"

Here are examples:

Not anyone is rich	=	No one is rich.
	=	~(∃x)Rx
Not any Italian is a lover	=	No Italian is a lover.
	=	~(∃x)(Ix • Lx)
It anyone is just, there will be peace	=	If someone is just, there will be peace.
	=	((∃x)Jx ⊃ P)

Our second rule usually gives a different formula, but an equivalent one:

> (2) To translate a sentence with "any," put a "(x)" at the *beginning* of the wff, regardless of where the "any" occurs in the sentence.

Here are the same examples worked out using the second rule:

Not anyone is rich	=	(x)~Rx
	=	For all x, x isn't rich.
Not any Italian is a lover	=	(x)~(Ix·Lx) ← Note the "·" here!
	=	For all x, x isn't both Italian and a lover.
If anyone is just, there will be peace	=	(x)(Jx⊃P)
	=	For all x, it x is just there will be peace.

"Any" at the beginning of a sentence usually just means "all." So "Any Italian is a lover" just means "All Italians are lovers."

5.4a Exercise—LogiCola H (HM & HT)

Using the equivalences below, translate these English sentences into wffs. Recall that our rules are rough guides and sometimes don't work; so read your formula carefully to make sure it reflects what the English means.

Cx	=	x is crazy		g	=	Gensler
Ex	=	x is evil		R	=	It will rain
Lx	=	x is a logician				

If everyone is evil, then Gensler is evil.	((x)Ex⊃Eg)

1. Gensler is either crazy or evil.
2. If Gensler is a logician, then some logicians are evil.
3. If everyone is a logician, then everyone is evil.
4. If all logicians are evil, then some logicians are evil.
5. If someone is evil, it will rain.
6. If everyone is evil, it will rain.
7. If anyone is evil, it will rain.
8. If Gensler is a logician, then someone is a logician.
9. If no one is evil, then no one is an evil logician.
10. If all are evil, then all logicians are evil.
11. If some are logicians, then some are evil.
12. All crazy logicians are evil.
13. Everyone who isn't a logician is evil.
14. Not everyone is evil.
15. Not anyone is evil.
16. If Gensler is a logician, then he's evil.
17. If anyone is a logician, then Gensler is a logician.
18. If someone is a logician, then he or she is evil.
19. Everyone is an evil logician.
20. Not any logician is evil.

5.5 Harder proofs

Now we come to proofs using formulas with multiple or non-initial quantifiers. Such proofs, while they require no new inference rules, often are tricky and require multiple assumptions. As before, drop only initial quantifiers:

$$\frac{((x)Fx \supset (x)Gx)}{\therefore (Fa \supset (x)Gx)} \qquad \textit{(both wrong)} \qquad \frac{((x)Fx \supset (x)Gx)}{\therefore (Fa \supset Ga)}$$

The formula "((x)Fx⊃(x)Gx)" is an if-then; to infer with it, we need the first part true or the second part false:

$$\frac{\begin{array}{c}((x)Fx \supset (x)Gx)\\ (x)Fx\end{array}}{\therefore (x)Gx} \qquad \textit{(both right)} \qquad \frac{\begin{array}{c}((x)Fx \supset (x)Gx)\\ \sim(x)Gx\end{array}}{\therefore \sim(x)Fx}$$

If we get stuck, we may need to assume one side or its negation.

Here's a proof using a formula with multiple quantifiers:

If some are enslaved, then all have their freedom threatened.
∴ If this person is enslaved, then I have my freedom threatened.

* 1 ((∃x)Sx ⊃ (x)Tx) **Valid**
[∴ (St ⊃ Ti)
* 2 ┌ asm: ~(St ⊃ Ti)
3 │ ∴ St {from 2}
4 │ ∴ ~Ti {from 2}
5 │ ┌ asm: ~(∃x)Sx {break up 1}
6 │ │ ∴ (x)~Sx {from 5}
7 │ └ ∴ ~St {from 6}
8 │ ∴ (∃x)Sx {from 5; 3 contradicts 7}
9 │ ∴ (x)Tx {from 1 and 8}
10 └ ∴ Ti {from 9}
11 ∴ (St ⊃ Ti) {from 2; 4 contradicts 10}

After making the assumption, we apply an S-rule to get lines 3 and 4. Then we're stuck, since we can't drop the non-initial quantifiers in 1. So we make a second assumption in line 5, get a contradiction, and derive 8. We soon get a second contradiction to complete the proof.

Here's a similar invalid argument:

If *all* are enslaved, then all have their freedom threatened.
∴ If this person is enslaved, then I have my freedom threatened.

1 ((x)Sx ⊃ (x)Tx) **Invalid**
[∴ (St ⊃ Ti)
* 2 asm: ~(St ⊃ Ti)
3 ∴ St {from 2}
4 ∴ ~Ti {from 2}
** 5 asm: ~(x)Sx {break up 1}
** 6 ∴ (∃x)~Sx {from 5}
7 ∴ ~Sa {from 6}

t, i, a

St, ~Ti, ~Sa

In evaluating the premise here, we first evaluate parts starting with quantifiers:

((x)Sx⊃(x)Tx)	←	Our premise. We first evaluate "(x)Sx" and "(x)Tx": "(x)Sx" is false because "Sa" is false. "(x)Tx" is false because "Ti" is false.
(0⊃0)	←	So we substitute "0" for "(x)Sx" and "0" for "(x)Tx."
1	←	So "((x)Sx⊃(x)Tx)" is true.

So the premise is true. Since the conclusion is false, the argument is invalid.

5.5a Exercise—LogiCola I (HC & MC)

Say whether each is valid (and give a proof) or invalid (and give a refutation).

(x)(Mx ∨ Fx)
∴ ((x)Mx ∨ (x)Fx)

(This is like arguing that, since everyone is male or female, thus either everyone is male or everyone is female.)

	1	(x)(Mx ∨ Fx)	Invalid
		[∴ ((x)Mx ∨ (x)Fx)	a, b
*	2	asm: ~((x)Mx ∨ (x)Fx)	
*	3	∴ ~(x)Mx {from 2}	Fa, ~Ma Mb, ~Fb
*	4	∴ ~(x)Fx {from 2}	
*	5	∴ (∃x)~Mx {from 3}	
*	6	∴ (∃x)~Fx {from 4}	
	7	∴ ~Ma {from 5}	
	8	∴ ~Fb {from 6}	
*	9	∴ (Ma ∨ Fa) {from 1}	
*	10	∴ (Mb ∨ Fb) {from 1}	
	11	∴ Fa {from 7 and 9}	
	12	∴ Mb {from 8 and 10}	

1. (x)(Fx ∨ Gx)
   ~Fa
   ∴ (∃x)Gx

2. (x)(Ex ⊃ R)
   ∴ ((∃x)Ex ⊃ R)

3. ((x)Ex ⊃ R)
   ∴ (x)(Ex ⊃ R)

4. ((∃x)Fx ∨ (∃x)Gx)
   ∴ (∃x)(Fx ∨ Gx)

5. ((∃x)Fx ⊃ (∃x)Gx)
   ∴ (x)(Fx ⊃ Gx)

6. (x)((Fx ∨ Gx) ⊃ Hx)
   Fm
   ∴ Hm

7. Fj
   (∃x)Gx
   (x)((Fx · Gx) ⊃ Hx)
   ∴ (∃x)Hx

8. ((∃x)Fx ⊃ (x)Gx)
   ~Gp
   ∴ ~Fp

9. (∃x)(Fx ∨ Gx)
   ∴ ((x)~Gx ⊃ (∃x)Fx)

10. ~(∃x)(Fx · Gx)
    ~Fd
    ∴ Gd

11. (x)(Ex ⊃ R)
    ∴ ((x)Ex ⊃ R)

12. (x)(Fx · Gx)
    ∴ ((x)Fx · (x)Gx)

13. (R ⊃ (x)Ex)
    ∴ (x)(R ⊃ Ex)

14. ((x)Fx ∨ (x)Gx)
    ∴ (x)(Fx ∨ Gx)

15. ((∃x)Ex ⊃ R)
    ∴ (x)(Ex ⊃ R)

5.5b Exercise—LogiCola I (HC & MC)

First appraise intuitively. Then translate into logic (using the letters given) and say whether valid (and give a proof) or invalid (and give a refutation).

1. Everything has a cause.
   If the world has a cause, then there is a God.
   ∴ There is a God. [Use Cx for "x has a cause," w for "the world," and G for "There is a God" (which here we needn't break down into "(∃x)Gx"—"For some x, x is a God"). A student of mine suggested this argument; but the next example shows that premise 1 can as easily lead to the opposite conclusion.]

2. Everything has a cause.
   If there is a God, then something doesn't have a cause (namely, God).
   ∴ There is no God. [Use Cx and G. The next example qualifies "Everything has a cause" to avoid the problem; some prefer an argument based on "Every *contingent being or set of such beings* has a cause."]

3. Everything that began to exist has a cause.
   The world began to exist.
   If the world has a cause, then there is a God.
   ∴ There is a God. [Use Bx, Cx, w, and G. This "kalam argument" is from William Craig and James Moreland; they defend premise 2 by appealing to the big-bang theory, the law of entropy, and the impossibility of an actual infinite.]

4. If everyone litters, then the world will be dirty.
   ∴ If you litter, then the world will be dirty. [Use Lx, D, and u.]

5. Anything enjoyable is either immoral or fattening.
   ∴ If nothing is immoral, then everything that isn't fattening isn't enjoyable. [Use Ex, Ix, and Fx.]

6. Anything that can be explained either can be explained as caused by scientific laws or can be explained as resulting from a free choice of a rational being.
   The totality of basic scientific laws can't be explained as caused by scientific laws (since this would be circular).
   ∴ Either the totality of basic scientific laws can't be explained or else it can be explained as resulting from a free choice of a rational being (God). [Use Ex for "x can be explained," Sx for "x can be explained as caused by scientific laws," Fx for "x can be explained as resulting from a free choice of a rational being," and t for "the totality of scientific laws." This one is from R.G.Swinburne.]

7. If someone knows the future, then no one has free will.
   ∴ No one who knows the future has free will. [Use Kx and Fx.]

8. If everyone teaches philosophy, then everyone will starve.
   ∴ Everyone who teaches philosophy will starve. [Use Tx and Sx.]

9. No proposition with empirical content is logically necessary.
   ∴ Either no mathematical proposition has empirical content, or no mathematical proposition is logically necessary. [Use Ex for "x has empirical content," Nx

for "x is logically necessary," Mx for "x is mathematical," and the universe of propositions. This argument is from the logical positivist A.J.Ayer.]

10. Any basic social rule that people would agree to if they were free and rational but ignorant of their place in society (whether rich or poor, white or black, male or female) is a principle of justice.
    The equal-liberty principle and the difference principle are basic social rules that people would agree to if they were free and rational but ignorant of their place in society.
    ∴ The equal-liberty principle and the difference principle are principles of justice. [Use Ax, Px, e, and d. This argument is from John Rawls. The equal-liberty principle says that each person is entitled to the greatest liberty compatible with an equal liberty for all others. The difference principle says that wealth is to be distributed equally, except for inequalities that serve as incentives that ultimately benefit everyone and are equally open to all.]

11. If there are no necessary beings, then there are no contingent beings.
    ∴ All contingent beings are necessary beings. [Use Nx and Cx. Aquinas accepted the premise but not the conclusion.]

12. Anything not disproved that is of practical value to one's life to believe ought to be believed.
    Free will isn't disproved.
    ∴ If free will is of practical value to one's life to believe, then it ought to be believed. [Use Dx, Vx, Ox, f (for "free will"), and the universe of discourse of beliefs. This argument is from William James.]

13. If the world had no temporal beginning, then some series of moments before the present moment is a completed infinite series.
    There's no completed infinite series.
    ∴ The world had a temporal beginning. [Use Tx for "x had a temporal beginning," w for "the world," MX for "x is a series of moments before the present moment," and Ix for "x is a completed infinite series." This one and the next are from Immanuel Kant, who thought our intuitive metaphysical principles lead to conflicting conclusions and thus can't be trusted.]

14. Everything that had a temporal beginning was caused to exist by something previously in existence.
    If the world was caused to exist by something previously in existence, then there was time before the world began.
    If the world had a temporal beginning, then there was no time before the world began.
    ∴ The world didn't have a temporal beginning. [Use Tx for "x had a temporal beginning," Cx for "x was caused to exist by something previously in existence," w for "the world," and B for "There was time before the world began."]

15. If emotivism is true, then "X is good" means "Hurrah for X!" and all moral judgments are exclamations.
    All exclamations are inherently emotional.
    "This dishonest income tax exemption is wrong" is a moral judgment.

"This dishonest income tax exemption is wrong" isn't inherently emotional.
∴ Emotivism isn't true. [Use T, H, MX, Ex, Ix, and t.]

16. If everything is material, then all prime numbers are composed of physical particles.
Seven is a prime number.
Seven isn't composed of physical particles.
∴ Not everything is material. [Use MX, Px, Cx, and s.]

17. If everyone lies, the results will be disastrous.
∴ If anyone lies, the results will be disastrous. [Use Lx and D.]

18. Everyone makes moral judgments.
Moral judgments logically presuppose beliefs about God.
If moral judgments logically presuppose beliefs about God, then everyone who makes moral judgments believes (at least implicitly) that there is a God.
∴ Everyone believes (at least implicitly) that there is a God. [Use Mx for "x makes moral judgments," L for "Moral judgments logically presuppose beliefs about God," and Bx for "x believes (at least implicitly) that there is a God." This argument is from the Jesuit theologian Karl Rahner.]

19. "x=x" is a basic law.
"x=x" is true in itself, and not true because someone made it true.
If "x=x" depends on God's will, then "x=x" is true because someone made it true.
∴ Some basic laws don't depend on God's will. [Use e (for "x=x"), Bx, Tx, Mx, and Dx.]

20. Nothing that isn't caused can be integrated into the unity of our experience.
Everything that we could experientially know can be integrated into the unity of our experience.
∴ Everything that we could experientially know is caused. [Use Cx, Ix, and Ex. This argument is from Immanuel Kant. The conclusion is limited to objects of possible experience—since it says "Everything *that we could experientially know* is caused"; Kant thought that the unqualified "Everything is caused" leads to contradictions (see problems 1 and 2).]

21. If everyone deliberates about alternatives, then everyone believes (at least implicitly) in free will.
∴ Everyone who deliberates about alternatives believes (at least implicitly) in free will. [Use Dx and Bx.]

22. All who are consistent and think that abortion is normally permissible will consent to the idea of their having been aborted in normal circumstances.
You don't consent to the idea of your having been aborted in normal circumstances.
∴ If you're consistent, then you won't think that abortion is normally permissible. [Use Cx, Px, Ix, and u. See my article in January 1986 *Philosophical Studies* or the last chapter of my *Ethics: A Contemporary Introduction* (London and New York: Routledge, 1998).]