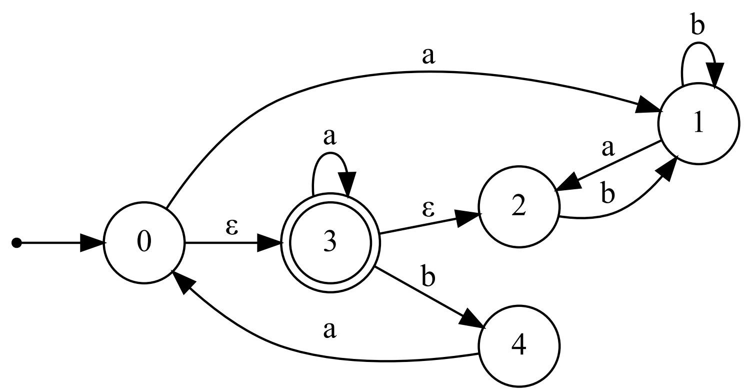
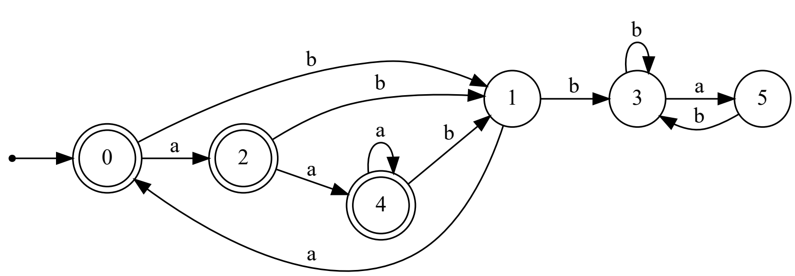
# Assignment 1 Solution

COMPSCI 3331

(5 marks) **1.** Given the $\varepsilon$-NFA below, construct a DFA that accepts the same language. Ensure that all states in the final DFA are appropriately labelled (i.e., with sets of states). Ensure your DFA is complete.




By applying the $\varepsilon$-NFA removal algorithm and the subset construction, we arrive at the following DFA:



(6 marks) **2.** Let $\Sigma$ be an alphabet and $a \in \Sigma$ be a letter from the alphabet. Let $p_a : 2^{\Sigma^*} \to 2^{\Sigma^*}$ be the following language operation:

$$p_a(L) = \{w \in \Sigma^* \ : \ \exists n \geq 0 \text{ such that } wa^n \in L\}$$

Show that the regular languages are closed under $p_a$. You do not need to give a formal proof, but you should give a explanation for how your construction works.

Let $M = (Q, \Sigma, \delta, q_0, F)$ be an arbitrary DFA. Then define

$$cl_a(q) = \{q' \in Q \ : \ \exists n \geq 0 \text{ such that } \delta(q', a^n) = q\}.$$

That is, $cl_a(q)$ is the set of states for which you can reach $q$ by reading only $a$s. This set is well-defined: it doesn't matter what value of $n$ gives $\delta(q', a^n) = q$, or if several values of $n$ give $\delta(q', a^n) = q$. As long as any value of $n$ gives this, then $q' \in cl_a(q)$.

Now define $M'$ to be the following DFA: $M' = (Q, \Sigma, \delta, q_0, F')$ where $F' = F \cup \{q \in Q \ : \ cl_a(q) \cap F \neq \emptyset\}$. That is, we add to $F'$ all the states $q$ from which you can reach a final state by reading only $a$s. Put another way, if there is a word $a^n$ such that $\delta(q', a^n) \in F$, then we add $q'$ as a final state. Again, we don't care what value of $n$ (or if multiple values of $n$ exist) satsifies $\delta(q', a^n) \in F$, as long as one exists.

Then $L(M') = p_a(L(M))$. This shows that the regular languages are closed under $p_a$. Informally, our construction works by ensuring that we add all the final states that could allow us to travel to an old final state of $M$ by $a$s only. In particular, suppose that $w \in p_a(L)$. Then $wa^n \in L$ for some $n$. But if $M$ goes from $q_0$ to $q$ on $w$ then $\delta(q, a^n) = q_f \in F$, since $wa^n$ is accepted by $M$. Therefore $cl_a(q) \cap F \supseteq \{q_f\}$ and so $q \in F'$. Thus, $w$ is accepted by $M'$.

(9 marks) **3.** (a) Let $\Sigma = \{a\}$ and $L = \{a^p \ : \ p \text{ is a prime number}\}$. Prove that $L$ is not regular.

Let $n$ be the constant defined by the pumping lemma. As there are infinitely many primes (by Euclid's theorem, let $p$ be a prime number greater than $n$. Then $x = a^p \in L$.

By the pumping lemma, we can write $x = uvw$ where $|uv| \leq n$ and $|v| > 0$. As all letters in $x$ are $a$, let $i, j$ be the lengths of $u$ and $v$, so that $u = a^i, v = a^j$ and $w = a^{p-i-j}$. Note that $i + j \leq n$ and $j > 0$.

Consider now $uv^{p+1}w = a^i a^{j(p+1)} a^{p-i-j} = a^{i+j(p+1)+p-j-i} = a^{p+jp}$. But as $p + jp = p(j+1)$, and $j + 1 \geq 2$, the length of $uv^{p+1}w$ is not prime. Thus, $uv^{p+1}w$ is not in $L$ and $L$ is not regular.

(b) Let $\Sigma = \{a\}$ and $L = \{a^n \ : \ n \text{ is a composite integer }\}$. Recall that an integer $n$ is composite if there exists another integer $m$ with $1 < m < n$ such that $m$ divides $n$. Prove that $L$ is not regular.

Assume that $L$ is regular. Then by the closure properties of the regular languages $\overline{L}$ is also regular. But $\overline{L} = \{\varepsilon, a\} \cup \{a^p \ : \ p \text{ is a prime number}\}$. Similarly, let $L_2 = L(aaa^*)$, i.e., the language accepted by the regular expression $aaa^*$. So $L_2$ is regular.

Then $\overline{L} \cap L_2$ is also regular, as the regular languages are closed under intersection. But then $\overline{L} \cap L_2 = \{a^p \ : \ p \text{ is a prime number}\}$. From part (a), this language is not regular, which is a contradiction. Therefore, $L$ is not a regular language.

(c) Let $\Sigma = \{a,b\}$ and $L \subseteq \Sigma^*$ be the language $L = \{a^i b^j \ : \ j \leq i \leq 2j\}$. Prove that $L$ is not regular.

Let $n$ be the constant defined by the pumpin lemma. Let $z = a^n b^n \in L$. Write $z$ as $z = uvw$ where $|uv| \leq n$ and $|v| \geq 0$. Thus $u = a^p$, $v = a^r$ and $w = a^n - p - rb^n$, for some $p, r \geq 0$ with $p + r \leq n$ and $r > 0$.

Consider now $z_0 = uw^0w = a^{n-r}b^n$. But now the number of $a$s in $z_0$ is less than the number of $b$s, so therefore, $z_0 \notin L$ and $L$ is not regular.

(4 marks) **4.** Let $L = \{wxw^R \ : \ w, x \in \{a,b\}^+\}$. Recall that $\{a,b\}^+$ is the positive Kleene closure, so that $w, x$ are not $\varepsilon$. Prove or disprove that $L$ is regular.

The language $L$ is regular. To prove this, we first establish an simplifed representation for $L$ that will make our proof easier. We claim that

$$L = \{uxu \ : \ u \in \{a,b\}, x \in \{a,b\}^+\}$$

Note that $u$ here is a single letter, not a whole word. Let $L_0 = \{uxu \ : \ u \in \{a,b\}, x \in \Sigma^*\} \cup \Sigma \cup \{\varepsilon\}$. We establish that $L = L_0$:

- $L_0 \subseteq L$ is clear. If we consider a word of the form $uxu$, then as $u$ is a single letter, then $u = u^R$ and thus any word of the form $uxu$ is also of the form $wxw^R$ with $w = u$.

- To show $L \subseteq L_0$, let $u = wxw^R \in L$. As $w, x$ are not empty, we must have that $w = aw'$ for some $a \in Sigma$ and so, $u = aw'x(w')^R a$ (by definition of reversal). Thus, $u$ has the form $aza$ for some $z = w'x(w')^R$. Thus, $u \in L_0$.

Now, we show that $L$ is regular. As $L = \{uxu \ : \ u \in \{a,b\}, x \in \{a,b\}^+\}$, we note that $L = a(a+b)^*(a+b)a + b(a+b)^*(a+b)b$ is a regular expression that matches $L$. Thus, $L$ is regular.