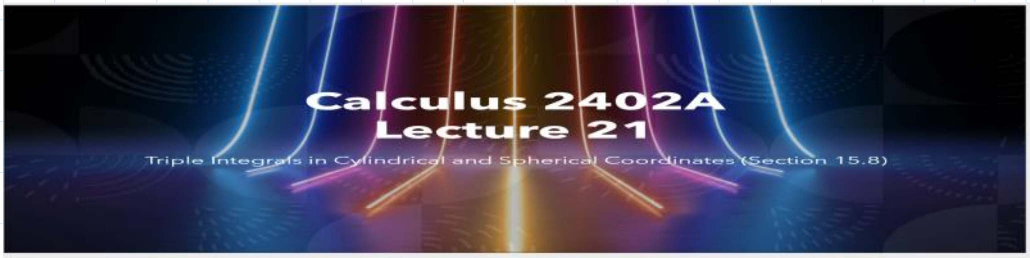
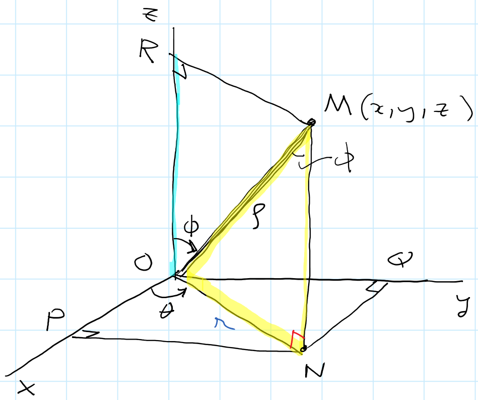


Spherical coordinates (ρ, ϕ, θ)



Consider a point $M(x,y,z)$ in $\mathbb{R}^3$.
Let N be the projection of M onto the xy-plane and R be the projection of M onto the z-axis.
Let P and Q be projections of N onto the x- and y-axes, respectively.
Then

$$\rho = OM, \quad \rho \geq 0$$
$$\phi = x\hat{O}M, \quad 0 \leq \phi \leq \pi$$
$$\theta = x\hat{O}N, \quad 0 \leq \theta \leq 2\pi$$

ρ, ϕ, θ are spherical coordinates of M.

Relationship between Cartesian coordinates (x,y,z) and spherical coordinates (ρ, ϕ, θ)

Consider the right $\triangle OMN$.

$$O\hat{M}N = \phi \quad (\text{alternating angles})$$
$$MN = z = \underbrace{OM}_{\rho} \cos\phi$$
$$\therefore \quad z = \rho\cos\phi$$

and we also note that

$$ON = r = \underbrace{OM}_{\rho} \sin\phi = \rho\sin\phi$$

$$\therefore \quad x = OP = ON\cos\theta = \rho\sin\phi\cos\theta$$
$$y = OQ = ON\sin\theta = \rho\sin\phi\sin\theta$$

Grouping these formulas together,

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases} \iff \begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ \phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \end{cases}$$

Ex1: Write an equation of the sphere $x^2+y^2+z^2 = a^2$ in spherical coordinates.

in spherical coordinates.

Solution

Replacing $x^2+y^2+z^2$ by ρ^2, we obtain

$$\rho^2 = a^2$$

$$\therefore \boxed{\rho = a}$$ which is the equation of the sphere $x^2+y^2+z^2=a^2$ in spherical coordinates. //Ans.

Ex2: Write an equation of the cone $z = \sqrt{x^2+y^2}$ in spherical coordinates.

Solution

Replacing z by $\rho\cos\phi$ and $\sqrt{x^2+y^2}$ by r $(r = \rho\sin\phi)$ into $z = \sqrt{x^2+y^2}$, we obtain

$$\cancel{\rho}\cos\phi = \cancel{\rho}\sin\phi$$

$$\frac{\sin\phi}{\cos\phi} = 1$$

$$\tan\phi = 1$$

$$\therefore \boxed{\phi = \frac{\pi}{4}}$$ which is an equation of the cone in spherical coordinates. //Ans.

Ex3: Describe in words the surface whose equation is

$$\rho^2 - 3\rho + 2 = 0$$

Solution

$$(\rho-1)(\rho-2) = 0$$

$\Rightarrow$ $\rho = 1$ a sphere of radius 1

or $\rho = 2$ a sphere of radius 2

This is a union of two spheres, $\rho = 1$ and $\rho = 2$. //Ans.

Ex4: Identify the surface whose equation is

$$\rho = \cos\phi$$

Solution

Using the equations of transformation,

$$\sqrt{x^2+y^2+z^2} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\therefore x^2+y^2+z^2 = z$$

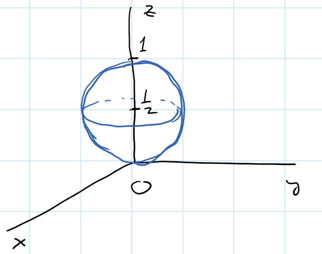
$$x^2+y^2+z^2-z = 0$$

$$x^2+y^2+z^2-2(z)\left(\tfrac{1}{2}\right) = 0$$

$$\therefore x^2+y^2+\underbrace{z^2-2(z)\left(\tfrac{1}{2}\right)+\left(\tfrac{1}{2}\right)^2}_{\left(z-\frac{1}{2}\right)^2} = \left(\tfrac{1}{2}\right)^2$$

$$\therefore x^2+y^2+\left(z-\tfrac{1}{2}\right)^2 = \left(\tfrac{1}{2}\right)^2$$

which is a sphere centered @ $(0,0,\frac{1}{2})$ and radius $\frac{1}{2}$.



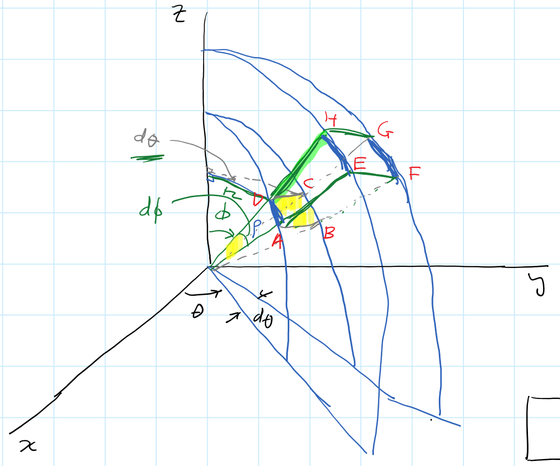
The volume element in Spherical coordinates

The volume element dV in spherical coordinates is obtained by three families of surface

(i) $\rho = \text{const}$, $\rho + d\rho = \text{const}$ these are concentric spheres

(ii) $\phi = \text{const}$, $\phi + d\phi = \text{const}$ these are cones

(iii) $\theta = \text{const}$, $\theta + d\theta = \text{const}$ these are half-planes passing thru the z-axis



$$
\begin{aligned}
dV &= \text{volume } (ABCDEFGH) \\
&= \text{area } (ABCD)\,(DH) \\
&= (AD)\,(CD)\,(DH) \\
&= (\rho\, d\phi)\,(r\, d\theta)\,(d\rho) \\
&= (\rho\, d\phi)\,(\rho \sin\phi\, d\theta)\,(d\rho)
\end{aligned}
$$

$$\boxed{dV = \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta}$$

In particular, if D is the region defined by

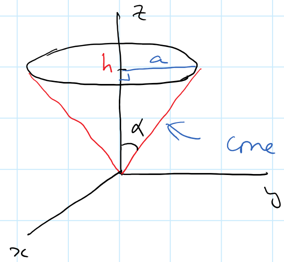
$$D = \{ (\rho, \phi, \theta) \mid \alpha \le \theta \le \beta,\ \phi_1(\theta) \le \phi \le \phi_2(\theta),\ \rho_1(\phi,\theta) \le \rho \le \rho_2(\phi,\theta) \}$$

then

$$\iiint_D F(x,y,z)\, dV = \int_\alpha^\beta \int_{\phi_1(\theta)}^{\phi_2(\theta)} \int_{\rho_1(\phi,\theta)}^{\rho_2(\phi,\theta)} F(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Ex 5: Find the volume of the cone with radius a and height h by using spherical coordinates.

Solution



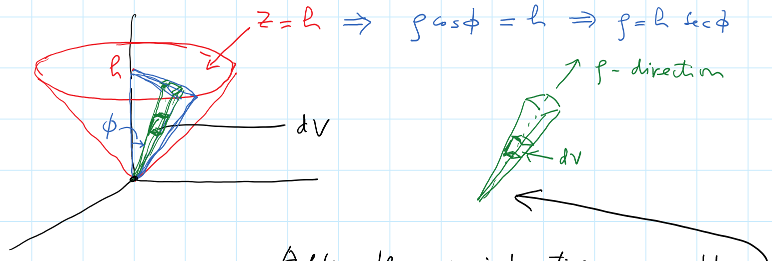
cone: $\phi = \alpha$

$\alpha = \tan^{-1}\left(\frac{a}{h}\right)$

or $\tan\alpha = \frac{a}{h}$

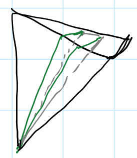
$$V = \iiint_{\text{Cone}} dV = \int_0^{2\pi} \int_0^{\alpha} \int_0^{h\sec\phi} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$z = h \Rightarrow \rho\cos\phi = h \Rightarrow \rho = h\sec\phi$



After the ρ-integration, we obtain the volume of the wedge like this

After the ϕ-integration, we obtain a wedge like this



Then we add these wedges together (we integrate wrt θ from 0 to 2π) to get the volume of the cone.

$$V = \int_0^{2\pi}\int_0^{\alpha} \sin\phi \left(\frac{\rho^3}{3}\right)\Bigg|_0^{h\sec\phi} d\phi\, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\alpha} \sin\phi \;\; h^3 \;\; \sec^3\phi \;\; d\phi\, d\theta$$

$$= \frac{h^3}{3}\int_0^{2\pi}\int_0^{\alpha} \tan\phi\, \sec^2\phi\, d\phi\, d\theta$$

$$= \frac{h^3}{3}\int_0^{2\pi} \left(\frac{1}{2}\tan^2\phi\right)\Bigg|_0^{\alpha} d\theta \qquad \tan\alpha = \frac{a}{h}$$

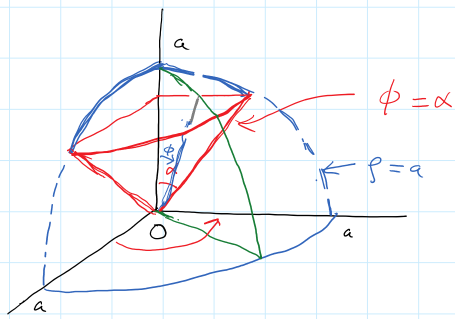
$$= \frac{h^3}{3}\int_0^{2\pi} \frac{1}{2}\left(\tan^2\alpha\right) d\theta$$

$$= \frac{h^3}{3}\int_0^{2\pi} \left(\frac{1}{2}\;\frac{a^2}{h^2}\right) d\theta$$

$$= \frac{h^3}{3}\;\frac{a^2}{2h^2}\underbrace{\left(\int_0^{2\pi} d\theta\right)}_{2\pi} = \frac{h^{\cancel{3}}}{3}\;\frac{a^2}{\cancel{2h^2}}\left(\cancel{2}\pi\right) = \frac{1}{3}\pi a^2 h \quad // \text{Ans.}$$

Ex 6 a) Find the volume of the solid bounded above by the sphere $x^2+y^2+z^2=a^2$ and below the cone $\phi = \alpha$.

b) From (a), find the volume of the sphere of radius a.



$$V = 4 \int^{\frac{\pi}{2}} \int^{\alpha} \int^{a} \rho^2 \sin\phi \; d\rho\; d\phi\; d\theta$$

$$V = 4 \int_0^{\frac{\pi}{2}} \int_0^{\alpha} \int_0^{a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Symmetry

$$= 4 \int_0^{\pi/2} \int_0^{\alpha} \sin\phi \left(\frac{\rho^3}{3} \right)\Big|_0^a \, d\phi \, d\theta$$

$$= \frac{4}{3} \int_0^{\pi/2} \int_0^{\alpha} \sin\phi \; a^3 \, d\phi \, d\theta$$

$$= \frac{4}{3} a^3 \left(\int_0^{\pi/2} d\theta \right) \left(\int_0^{\alpha} \sin\phi \, d\phi \right)$$

$$= \frac{4}{3} a^3 \left(\frac{\pi}{2} \right) \left(-\cos\phi \right)\Big|_0^{\alpha}$$

$$= \frac{2}{3} a^3 \pi \left(-\cos\alpha + 1 \right) \qquad // \text{Ans.}$$

b) If we let $\alpha = \frac{\pi}{2} \Rightarrow$ the cone $\phi = \frac{\pi}{2}$ is flat (which is the xy-plane), then V becomes the volume of the upper hemisphere

$$V_{\text{hemisphere}} = \frac{2 a^3 \pi}{3} \left(-\cancel{\cos\frac{\pi}{2}}^{\,0} + 1 \right) = \frac{2 \pi a^3}{3}$$

$$\therefore \quad V_{\text{sphere}} = 2 \, V_{\text{hemisphere}} = 2 \left(\frac{2\pi a^3}{3} \right) = \frac{4}{3} \pi a^3 \qquad // \text{Ans.}$$

See you on Friday!