Lecture 7

September 23, 2020 10:23 AM

The Second Derivative Test of functions of two variables

Suppose the 2nd partial derivatives of f are continuous on a disk with center (a,b) and suppose that $f_x(a,b) = 0$ and $f_y(a,b) = 0$, ie, (a,b) is a CP of f. Let

$$D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}(a,b)\, f_{yx}(a,b)$$

$$= f_{xx}(a,b)\, f_{yy}(a,b) - (f_{xy}(a,b))^2$$

a) If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum

b) If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum

c) If $D < 0$ then f has (a,b) as a saddle point

d) If $D = 0$ the test is inconclusive. We need further investigation

Ex6: Find the local min or max values and saddle points of

$$f(x,y) = x^2 + xy + y^2 + y$$

Solution

$$f_x = 2x + y = 0 \Rightarrow y = -2x \Rightarrow y = -\frac{2}{3}$$

$$f_y = x + 2y + 1 = 0 \Rightarrow x + 2(-2x) = -1$$

$$-3x = -1 \Rightarrow x = \frac{1}{3}$$

$\therefore$ The point $\left(\frac{1}{3}, -\frac{2}{3}\right)$ is a CP.

$$f_{xx} = 2, \quad f_{xy} = 1$$

$$f_{yy} = 2, \quad f_{yx} = 1$$

$$\therefore D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0$$

Because $f_{xx} = 2 > 0$, $\left(\frac{1}{3}, -\frac{2}{3}\right)$ is a local min

and $f\left(\frac{1}{3}, -\frac{2}{3}\right) = \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)\left(-\frac{2}{3}\right) + \left(-\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)$

$$= \frac{1}{9} - \frac{2}{9} + \frac{4}{9} - \frac{2}{3} = \frac{1}{3} - \frac{2}{3} = -\frac{1}{3} \quad // \text{Ans.}$$

We note that $f(x,y)$ can be written as

$$f(x,y) = \underbrace{\left(x + \frac{y}{2}\right)^2}_{\geq 0} + \frac{3}{4}\underbrace{\left(y + \frac{2}{3}\right)^2}_{\geq 0} - \frac{1}{3}$$

$f(x,y)$ has a local min value $-\frac{1}{3}$ if

$$x + \frac{y}{2} = 0 \Rightarrow x = -\frac{y}{2} = -\frac{1}{2}\left(-\frac{2}{3}\right) = \frac{1}{3} \checkmark$$

$$y + \frac{2}{3} = 0 \Rightarrow y = -\frac{2}{3} \checkmark$$

The Hessian

Consider the function $f(x,y)$. The matrix

$$H(x,y) = \begin{bmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{bmatrix}$$

is called the Hessian of f at (x,y). We define the trace of H, denoted as $Tr(H)$, as

$$Tr(H) = f_{xx}(x,y) + f_{yy}(x,y)$$

because $f_{xy}(x,y) = f_{yx}(x,y)$, $H(x,y)$ is a symmetric matrix.

$$\det(H(x,y)) = f_{xx}(x,y) f_{yy}(x,y) - (f_{xy}(x,y))^2 \quad (= D)$$

The Second Derivative Test in terms of the Hessian

Suppose that (a,b) is a CP of f and f has continuous 2nd order partial derivatives in some neighborhood of (a,b). Then

a) f has a local minimum value at (a,b) if $Tr(H(a,b)) > 0$ and $\det(H(a,b)) > 0$ (N.B: If $Tr(H(a,b)) > 0$ then $f_{xx}(a,b) > 0$)

b) f has a local maximum value at (a,b) if $Tr(H(a,b)) < 0$ and $\det(H(a,b)) > 0$

c) f has (a,b) as a saddle point if $\det(H(a,b)) < 0$

d) If $\det(H(a,b)) = 0$, the test is inconclusive.

Recall Eigenvalues and Eigenvectors

A non-zero vector $\underline{v}$ is called an eigenvector of a matrix A corresponding to an eigenvalue λ if

$$A\underline{v} = \lambda \underline{v} \qquad (1)$$

If all eigenvalues of A are positive, A is called +ve definite.

If all eigenvalues of A are negative, A is called −ve definite

If some are +ve and some are −ve, A is called indefinite

Rewriting (1) as

$$A\underline{v} = \lambda I \underline{v} \quad \text{where } I \text{ is the identity matrix}$$

$$A\underline{v} - \lambda I \underline{v} = \underline{0} \qquad (I\underline{v} = \underline{v})$$

$$(A - \lambda I)\,\underline{v} = \underline{0}$$

Since $\underline{v}$ is a non-zero vector, we must have

$$\boxed{\det(A - \lambda I) = 0} \qquad (2)$$

which is called the Characteristic Equation.

Consider A as a 2x2 matrix. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and $\underline{v} = \begin{bmatrix} h \\ k \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$Tr(A) = a + d$

$\det(A) = ad - bc$

Expanding

$$(a-\lambda)(d-\lambda) - bc = 0$$

$$ad - a\lambda - d\lambda + \lambda^2 - bc = 0$$

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

$$\boxed{\lambda^2 - \lambda\, Tr(A) + \det(A) = 0} \qquad (3)$$

$$\lambda_1 + \lambda_2 = Tr(A)$$

$$\lambda_1 \lambda_2 = \det(A)$$

A is +ve definite if $Tr(A) > 0$ and $\det(A) > 0$

A is −ve definite if $Tr(A) < 0$ and $\det(A) > 0$

A is indefinite if $\det(A) < 0$

Go back to the Hessian

$$H(x,y) = \begin{bmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{bmatrix}$$

At the CP (a,b)

$$H(a,b) = \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}$$

$$Tr(H(a,b)) = f_{xx}(a,b) + f_{yy}(a,b)$$

$$\det(H(a,b)) = f_{xx}(a,b)\, f_{yy}(a,b) - (f_{xy}(a,b))^2$$

(i) f has a local minimum value at (a,b) if $H(a,b)$ is +ve definite

(ii) f has a local maximum value at (a,b) if $H(a,b)$ is −ve definite

(iii) f has (a,b) as a saddle point if $H(a,b)$ is indefinite

We will generalize to the case of f is a function of 3 variables or more in the next lecture.

See you on Friday.