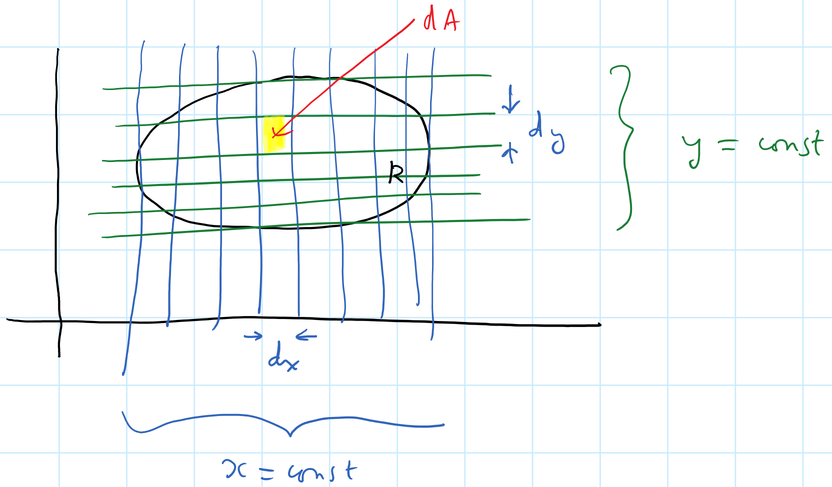
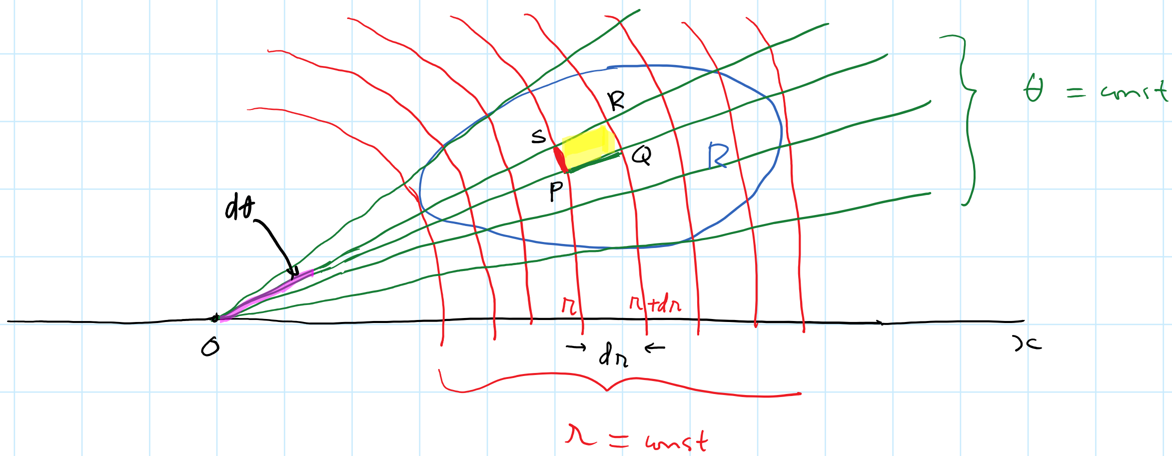
The region R over which a double integral is evaluated may be defined in terms of polar coordinates r, θ. Recall that the area element dA in Cartesian coordinates is $dx\,dy$, obtained by the lines $x = \text{const}$, $x + dx = \text{const}$, $y = \text{const}$, $y + dy = \text{constant}$.



In polar coordinates, we construct the curves $r = \text{const}$, $r + dr = \text{const}$, $\theta = \text{const}$, $\theta + d\theta = \text{const}$ to obtain the area element dA.



In polar coordinates $r = \text{const}$ is a family of concentric circles centered at the pole and $\theta = \text{const}$ is a family of rays (half-lines) originated from the pole. Then the area element dA is just the area of the parallelogram $PQRS$ in the above figure.

$$dA = \text{area}(PQRS)$$
$$= (PQ)(PS)$$
$$= (dr)(r\,d\theta)$$
$$\boxed{dA = r\,dr\,d\theta}$$

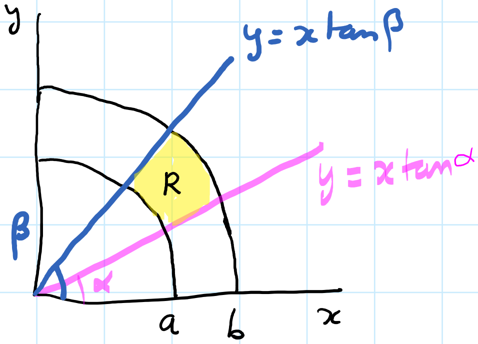
Then $z = f(x,y)$ can be written as $f(r\cos\theta, r\sin\theta)$ and

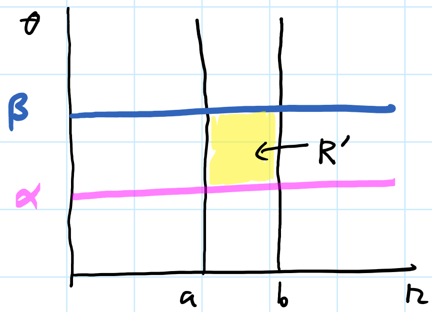
$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

where R' is the region in $r\theta$-plane corresponding to the region R in the xy-plane.

In particular, if R is the region bounded by two circles $x^2+y^2=a^2$ $(r=a)$ and $x^2+y^2=b^2$ $(r=b,\ a<b)$ and the lines $y = x\tan\alpha$ $(\theta=\alpha)$ and $y = x\tan\beta$ $(\theta=\beta,\ \alpha<\beta)$, then

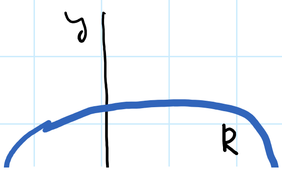
$$\iint_R f(x,y)\,dA = \iint_{R'} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta = \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

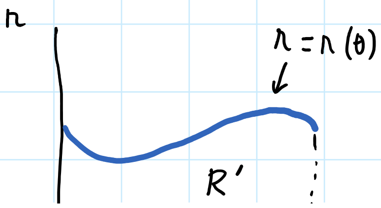


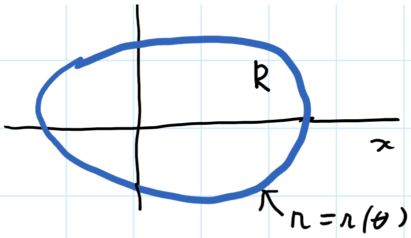


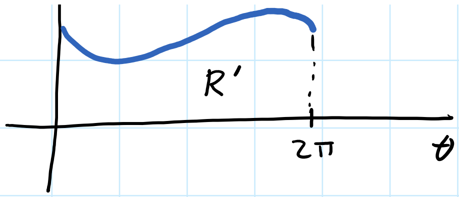
We note that R' is a rectangle defined by $a \le r \le b$, $\alpha \le \theta \le \beta$. This explains why a change of variables from Cartesian coordinates to polar coordinates is a necessity in some situations.

More generally, the region R may be bounded by a closed curve enclosing the pole given by $r = r(\theta)$ as shown in the figure below





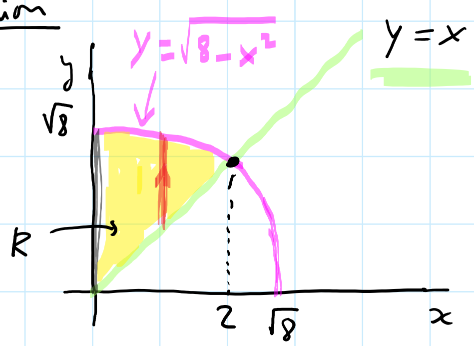




$$\iint_R f(x,y)\,dx\,dy = \int_0^{2\pi}\int_0^{r(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Example: Evaluate $I = \iint_R \frac{1}{5+x^2+y^2}\,dA$ where R is the region in the first quadrant bounded by $x^2+y^2=8$, $y=x$, $x=0$.

Solution



We note that R is a type I region. Next, we find the points of intersection of the circle $x^2+y^2=8$ and the line $y=x$,

$$x^2+x^2=8$$
$$2x^2=8$$
$$x^2=4 \Rightarrow x=\pm 2$$

we choose $x=2$.

$$I = \int_0^2 \boxed{\int_x^{\sqrt{8-x^2}} \frac{1}{5+x^2+y^2}\,dy}\,dx \qquad \int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)$$

$$= \int_0^2 \frac{1}{\sqrt{x^2+5}} \arctan\left(\frac{y}{\sqrt{x^2+5}}\right)\Bigg|_{y=x}^{\sqrt{8-x^2}} dx$$

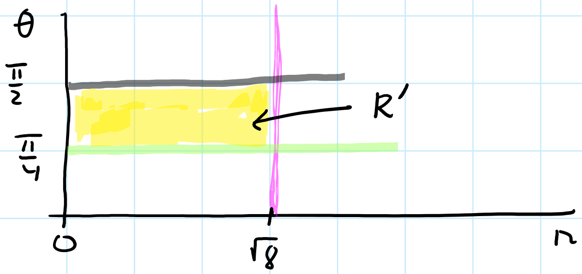
$$= \int_0^2 \frac{1}{\sqrt{x^2+5}}\left[\arctan\left(\frac{\sqrt{8-x^2}}{\sqrt{x^2+5}}\right) - \arctan\left(\frac{x}{\sqrt{x^2+5}}\right)\right] dx$$

This integral is too difficult to proceed further (both Mathematica and Maple fail badly!) So we need to change the double integral from Cartesian coordinates to polar coordinates

The circle $x^2+y^2=8 \Rightarrow r=\sqrt{8}$

The line $y = x \implies \theta = \frac{\pi}{4}$

the y-axis $\implies \theta = \frac{\pi}{2}$



$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\sqrt{8}} \frac{1}{5 + r^2}\, r \, dr\, d\theta$$

$$= \left(\int_{\frac{\pi}{4}}^{\pi/2} d\theta\right)\left(\frac{1}{2}\int_0^{\sqrt{8}} \frac{2r\,dr}{\underbrace{5 + r^2}_{u}}\right)$$

$$= \left(\frac{\pi}{4}\right) \frac{1}{2} \ln(5 + r^2)\Big|_{r=0}^{\sqrt{8}}$$

$$= \frac{\pi}{8}\left(\ln(5+8) - \ln(5)\right) = \frac{\pi}{8}\left(\ln(13) - \ln(5)\right)$$

$$= \frac{\pi}{8} \ln\left(\frac{13}{5}\right) \qquad // \text{Ans.}$$

Ex2: Evaluate $I = \int_0^\infty e^{-x^2} dx$.

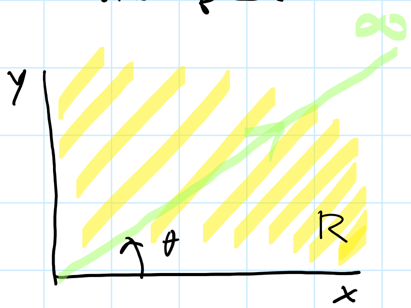
Soln We already know e^{-x^2} DOES NOT have an antiderivative in a "closed form". However, we can use polar coordinates to evaluate this improper integral.

To use double integral, we need compute I^2 instead of I.

$$I^2 = \left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-y^2} dy\right)$$

$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx\, dy$$

$$= \iint_R e^{-(x^2+y^2)} \, dA$$



where R is the 1st quadrant of the xy-plane

$$\overset{=}{\uparrow} \int_0^{\frac{\pi}{2}} \int_0^\infty \boxed{e^{-r^2} r}\, dr\, d\theta \quad = \text{Separable}$$

$$= \int_0^{2} \int_0 \boxed{e^{-r^2} r} \, dr \, d\theta$$

Change to polar coordinates

$$= \left(\int_0^{\pi/2} d\theta \right) \left(\int_0^{\infty} e^{-r^2} r \, dr \right)$$

$u = r^2$
$du = 2r\,dr$

$$= \left(\frac{\pi}{2} \right) \left(\int_0^{\infty} e^{-u} \frac{du}{2} \right)$$

$$= \frac{\pi}{4} \left(-e^{-u} \right) \Big|_0^{\infty} = \frac{\pi}{4} \left(-\underbrace{e^{-\infty}}_{0} + \underbrace{e^{0}}_{=1} \right) = \frac{\pi}{4}$$

$$\therefore \quad I = \int_0^{\infty} e^{-x^2} dx = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2} \quad // \text{Ans.}$$

See you on Friday!