1 Few operations on Sets

1. $U = \{1,2,3\ldots 9\}$, $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7\}$

   a. $A \cup B = \{1,2,3,4,5,6,7\}$

   b. $A \cap B = \{4,5\}$

   c. $A - B = \{1,2,3\}$

   d. $B - A = \{6,7\}$

   e. $A^c = U - A = \{6,7,8,9\}$

   f. $B^c = U - B = \{1,2,3,8,9\}$

   g. $P(B) = \{\emptyset, \{4\}, \{5\}, \{6\}, \{7\}, \{4,5\}, \{4,6\}, \{4,7\}, \{5,6\}, \{5,7\}, \{6,7\}$

      $\{4,5,6\}, \{4,5,7\}, \{4,6,7\}, \{5,6,7\}, \{4,5,6,7\}\}$

2. ~~a~~ Is it possible to have $A \cap B = A \cap C$ even if $B \neq C$?

   Let $A = \{1,2\}$, $B = \{2,3\}$, $C = \{2,4\}$

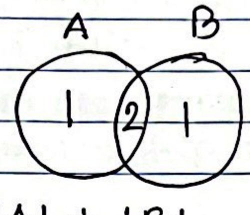
   Then $A \cap B = \{2\}$ and $A \cap C = \{2\}$

3. Is it possible to have $A \cup B = A \cup C$ ~~where~~ even if $B \neq C$?

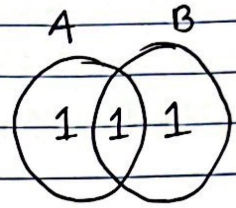
   Let $A = \{1,2\}$, $B = \{1,3\}$, $C = \{2,3\}$

   Then $A \cup B = \{1,2,3\}$, $A \cup C = \{1,2,3\}$

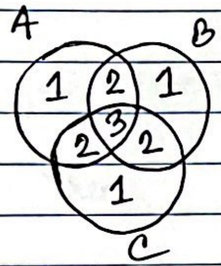
Cardinality of Union (Inclusion - Exclusion Principle)



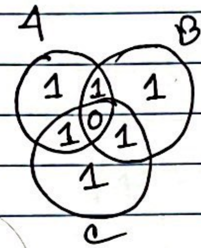
$|A| + |B|$



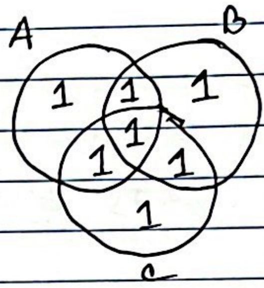
$|A|+|B|-|A\cap B|$



$|A|+|B|+|C|$



$|A|+|B|+|C|-|A\cap B|$
$-|A\cap C|-|B\cap C|$



$|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|$
$+|A\cap B\cap C|$

General Equation:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n}|A_i| - \sum_{1\le i<j\le n}|A_i\cap A_j| + \sum_{1\le i<j<k\le n}|A_i\cap A_j\cap A_k|$$

$$\dots + (-1)^{n-1}|A_1\cap A_2 \dots \cap A_n|$$

① How many integers from 1 to 50 are multiples of 2 or 3.

Lets have A = Set of integers from 1 to 50 that are multiples of 2
B = Set of integers from 1 to 50 that are multiples of 3

$|A| = 50/2 = 25$, $|B| = 50/3 = 16$

Multiples of both 2 and 3, $|A\cup B| = |A|+|B|-|A\cap B|$ ~~$= 25+16-8$~~
~~$=33$~~

[Note: There are numbers who are multiples of both 2 and 3 like 6, 12, 18 etc. If we simply sum up then those numbers will be counted twice. To avoid this, we should make sure those common multiples are counted once. So, we should substract the number of common multiples ~~by~~ once.]

$|A \cap B|$ = multiples of 2 and 3, that means 6.
$= 50/6 = 8$

So, $|A \cap B| = 8$

Then, $|A \cup B| = |A| + |B| - |A \cap B| = 25 + 16 - 8 = 33$

② In a group of 50 students, 24 like cold drinks and 36 like hot drinks. Each student likes one of the drinks. How many likes both?

Let, Set of students who like cold drinks • X •
Set of students who like hot drinks Y

Then $|X| = 24$, $|Y| = 36$, $|X \cup Y| = 50$

So,
$$|X \cup Y| = |X| + |Y| - |X \cap Y|$$
$$\Rightarrow |X \cap Y| = |X| + |Y| - |X \cup Y|$$
$$= 24 + 36 - 50$$
$$= 60 - 50$$
$$= 10$$

⊞ Cardinality of Power set with proof by induction.

If $|A| = n$ then, $|P(A)| = 2^n$

Base case:

Suppose $A = \{\}$

so, $|A| = 0$

here, $P(A) = \{\emptyset\}$

Because an empty set is a definite element of power set.

Then $|P(A)| = 1 = 2^0$

This satisfies the theorem

Induction Step:

If for any $|A| = n$, $|P(A)| = 2^n$ [Induction Hypothesis]

Then for any $|B| = n+1$, $|P(B)| = 2^{n+1}$ [We need to prove]

Let $A = \{a_1, a_2, a_3, \ldots\ a_n\}$

and $B = \{a_1, a_2, a_3, \ldots\ a_n, a_{n+1}\}$

Then we can say $B = A \cup \{a_{n+1}\}$

For B there two kinds of Subsets:

① Subsets that donot include a_{n+1}. These Subsets are exactly same as subsets of A. So, there should be 2^n Subsets as A has 2^n subsets.

② For the latter, we can say $X \cup \{a_{n+1}\}$ where $X \in P(A)$

As there 2^n possible choice for X, There must 2^n Subsets That includes a_{n+1}.

So, $|P(B)| = 2^n + 2^n = 2^n(1+1) = 2^n \cdot 2 = 2^{n+1}$

⊞ Double inclusion [if $A \subseteq B$ and $B \subseteq A$ then $A = B$.]

1. Prove $A \cup (A \cap B) = A$ with double inclusion.

Let, $x \in A \cup (A \cap B)$

$x \in A$ or $x \in (A \cap B)$

$x \in A$ or $(x \in A$ and $x \in B)$

So, $x \in A$ is true. or, $(x \in A$ and $x \in B)$ is true, or both true

In all cases, $x \in A$ is true

so, we can say $x \in A$ [$\vee$-intro, a rule of inference in Logic → Logic3.PDF, 18 page]

Then $A \cup (A \cap B) \subseteq A$

Again, Let $x \in A$

$x \in A$ or $x \in A \cap B$

$x \in A \cup (A \cap B)$

So, $A \subseteq A \cup (A \cap B)$

Then, $A = A \cup (A \cap B)$ [proved]

2 Prove that $(A \cup B) \cap \overline{A} = B - A$ with double inclusion

Let $x \in (A \cup B) \cap \overline{A}$

$\Rightarrow x \in (A \cup B)$ and $x \in \overline{A}$

$\Rightarrow (x \in A$ or $x \in B)$ and $x \notin A$

$\Rightarrow (x \in A$ and $x \notin A)$ or $(x \in B$ and $x \notin A)$

$\Rightarrow x \in B$ and $x \notin A$

$\Rightarrow x \in B - A$

So, $(A \cup B) \cap \overline{A} \subseteq B - A$

Again, Let $x \in B - A$

$\Rightarrow x \in B$ and $x \notin A$

$\Rightarrow (x \in A$ or $x \in B)$ and $x \in \overline{A}$ $\quad \frac{F}{F \vee G}$

$\Rightarrow x \in (A \cup B)$ and $x \in \overline{A}$

$\Rightarrow x \in (A \cup B) \cap \overline{A}$

Then, $B - A \subseteq (A \cup B) \cap \overline{A}$

So, $(A \cup B) \cap \overline{A} = B - A$ [proved.]

3 Prove that $(A \cup B) \cap \overline{A} = B - A$ with set builder

$$(A \cup B) \cap \overline{A} = \{x \mid x \in (A \cup B) \wedge x \in \overline{A}\}$$

$$= \{x \mid (x \in A \vee x \in B) \wedge x \in \overline{A}\} \quad \text{[Definition of Union]}$$

$$= \{x \mid (x \in A \wedge x \in \overline{A}) (x \in B \wedge x \in \overline{A})\} \quad \text{[Distributive]}$$

$$= \{x \mid x \in B \wedge x \in \overline{A}\}$$

$$= \{x \mid x \in (B - A)\}$$

$$= B - A$$