ECON3102-005 Chapter 6:Economic Growth: the Solow Growth Model (Part 2)

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$$k' = \frac{szf(k)}{1+n} + \frac{(1-d)k}{1+n} \qquad (*)$$

Steady States

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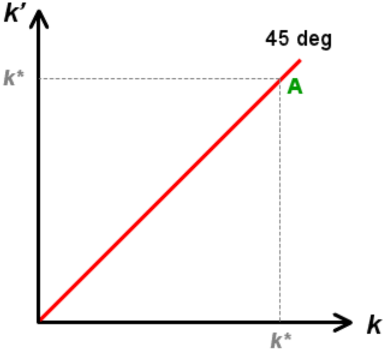
- Note that when we graph in $k'k$ space, any point that crosses the 45 degree line satisfies $k' = k$.



Steady State in the Solow Growth Model

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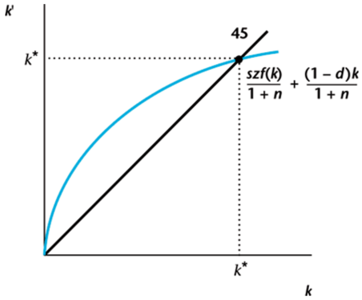

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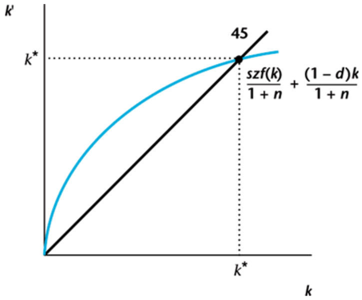

- At the steady state, $k = k^*$ and $k' = k^*$; k^* is the equilibrium level of capital in the economy.

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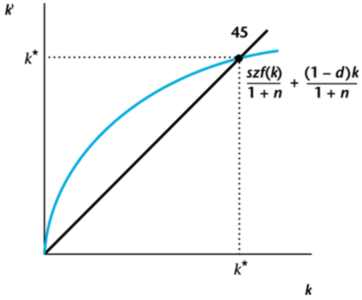

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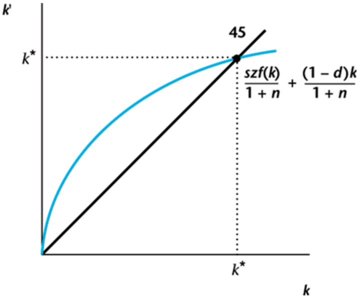

- Suppose $k < k^*$. Then $k' > k$, and the capital stock increases from the current to the future period, until $k = k^*$.
- Here, current investment is relatively large with respect to depreciation and labor force growth.

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- So, theres no growth in here? Are we forgetting something?

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There is growth in this economy! In the long run, when $k = k^*$, all real aggregate quantities grow at a rate n. Why?

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- In this way, the Solow growth model is an exogenous growth model.

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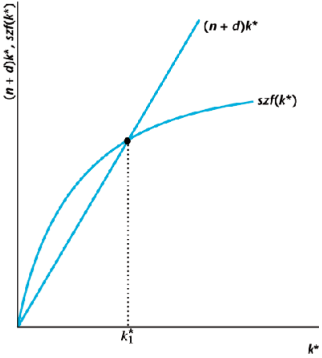
- In equation [1], the right hand side is the per worker production function multiplied by the savings rate; the left hand side is an equation for a line with slope (n+d).

STEADY STATE ANALYSIS (2)

- Graphically, to determine $k*$ we match the left and right hand sides of equation [1]:



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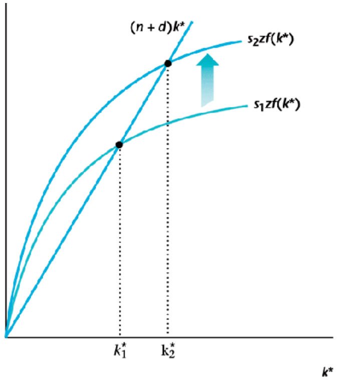
- Suppose that at some time t_0 the savings rate increases from s_1 to s_2. (This could be due to a change in preferences.)



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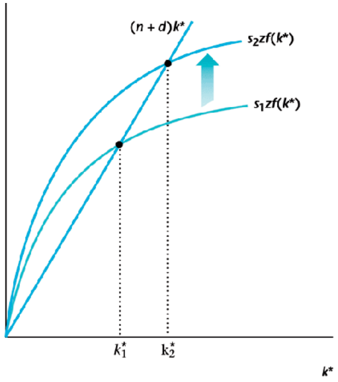

- The steady state capital level increases.

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- Aggregate output is higher: $Y_2^* = y_2^* N > Y_1^*$
- Note that, since in equilibrium $S = I$, this implies a positive relationship between I and Y (the first empirical evidence from data).

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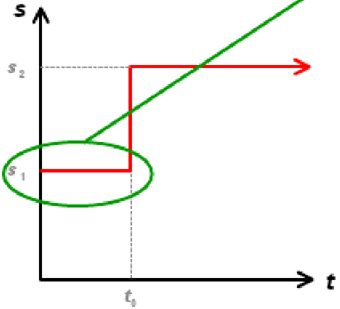
- K and Y still grow at a rate n.
- However, it may take some time for the variables to adjust to the new steady state.

Transition between the Steady States: s

In time: The savings rate

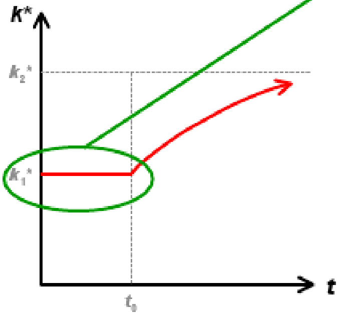


The savings rate starts at a level s_1 ...

... then jumps to the new level of savings s_2.

TRANSITION BETWEEN THE STEADY STATES: k

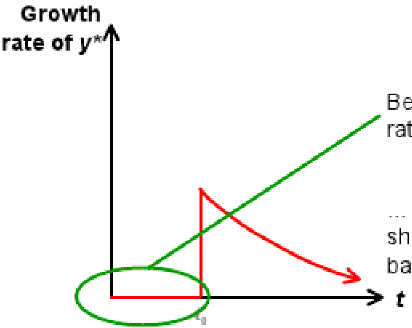
In time: Capital per worker



The capital per worker starts at a level k_1* ...

... then steadily increases towards its new level k_2*.

In time: Growth rate of y^*



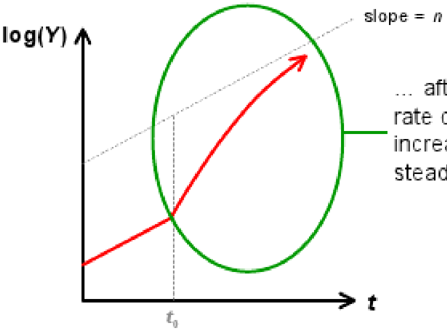
Before the shock, the growth rate of y^* is zero ...

... it jumps at the time of the shock, yet it steadily decreases back to zero again.

Transition between the Steady States: $\log(Y)$

In time: $\log(Y)$

The growth rate of aggregate output is n ...



... after the shock, the growth rate of aggregate output increases from trend but then steadily reduces back to n.

Transition between the Steady States: c (1)

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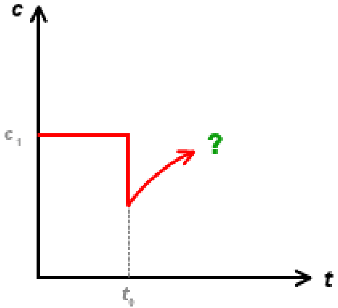
- Whether per capita consumption is higher or lower at the end is not immediately clear.



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- Hence, to achieve our goal, we take the partial derivative of c^* with respect to s!

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- From previous results, we know that $\frac{dk^*}{ds} > 0$. Hence, we just need to check whether $zf'(k^*)$ is bigger or smaller than $(n+d)$!

TRANSITION BETWEEN THE STEADY STATES: c (4)

So, for the equation:

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we have 3 cases:

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- How do we know which case occurs in equilibrium?

Transition between the Steady States: c (5)

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Transition between the Steady States: c (6)

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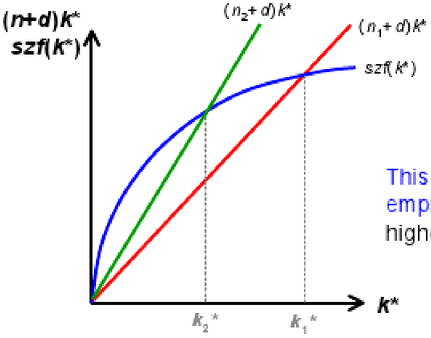
- Case 3:$zf'(k^*) > (n+d) \Rightarrow \frac{dc^*}{ds} > 0$

When the conditions of Case 2 are satisfied, consumption is at its maximum level for all possible steady states. Hence, consumption is maximized.

- This value of k^* is known as <u>the golden rule capital per worker</u>.

An Increase in the Population Growth Rate, n

What happens in the steady state when we change the population growth rate, n?



This shows the second empirical relationship ... a higher n yields a lower y!

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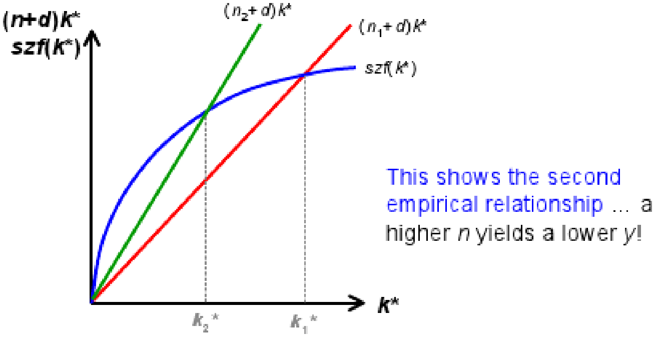

- However, aggregate income Y is growing at a faster rate n_2. This shows that higher growth in aggregate income need not be associated with higher income per worker in the long run.

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- If an economy needs to grow (income per capita), then we need to save more and control for population growth.
- However, since $s \leq 1$ and there is a "natural limit" on how much we can reduce n. How can we sustain a long-term growth?

Long-term Growth and a Shock in TFP (1)

Let's see what happens when there is a positive technological shock; i.e., an increase in z.

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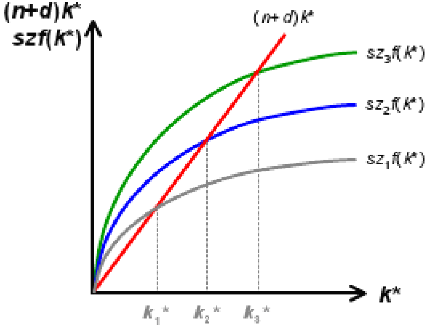
- For $z_1 < z_2 < z_3$:



Long-term Growth and a Shock in TFP (2)

Hence, increases in z translate into long-run growth (sustained growth in per capita income).

- Improvements in a country's standard of living are brought about in the long run by technological progress.