Announcements :

1) Tentatively, you can collect your midterms on Thursday 3:30 - 5:00pm, from our TAs (James & Eli).

2) HW05 is due this Friday.

3) There will be Quiz 03 on the Monday after the Reading week.

Recall : 4.1

For absolute min / max in an interval :

① Find critical points

② Compare the y-values ~~wit~~ of critical points and end points.

For local min / max :

① Find critical points.

② At each critical point $x = a$, find the second derivative $f''(a)$.

③ If $f''(a) < 0 \Rightarrow a$ is local max

$f''(a) > 0 \Rightarrow a$ is local min.

A15) $f(x) = \sin(2x) + 3$, g is a function.

$g''(0) = -4$

$(fg)''(0) = 4$ ⟵ Product.

Q. Find $g'(0)$.

<u>Ans</u>. we need to use product rule

$(fg)' = f'g + g'f$ ⟵ Product rule

$(fg)'' = (f'g + g'f)'$

$= (f'g)' + (g'f)'$

$= f''\cdot g + f'g' + g''f + g'f'$ ⟵ Product Rule on each term

at $x = 0$

$$(fg)''(0) = f''(0)\cdot g(0) + f'(0)\cdot g'(0) + \underbrace{g''(0)}_{=-4}\cdot f(0) + g'(0)\cdot f'(0) \quad -(*)$$

$f(x) = \sin(2x) + 3$ at $x = 0$ $f(0) = 3$

$f'(x) = 2\cos(2x)$ ⟶ $f'(0) = 2$

$f''(x) = -2.2.\sin(2x)$ $f''(0) = 0$

Plugging everything back in (*)

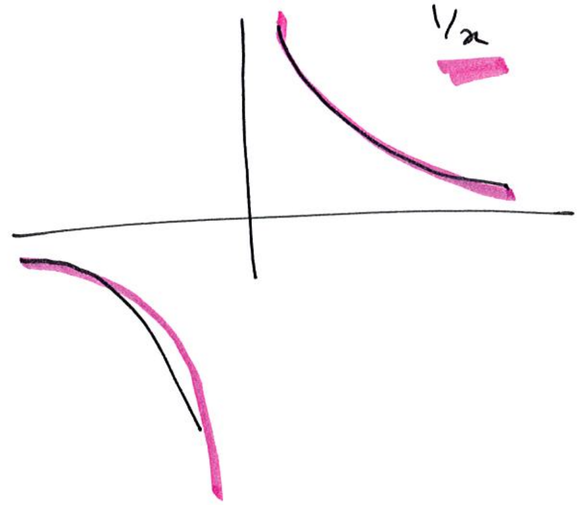
$$4 = 0 \cdot g(0) + \underbrace{2 \cdot g'(0)} + (-4)\cdot 3 + \underbrace{g'(0)\cdot 2}$$

$$\Rightarrow \quad 4 = 4 \cdot g'(0) - 12$$

$$\Rightarrow \quad 16 = 4 \cdot g'(0)$$

$$\Rightarrow \quad \boxed{4 = g'(0)}$$

- $\lim\limits_{x \to 0^-} e^{1/x} = 0$



Reason: as $x \to 0^-$,

$\frac{1}{x} \to -\infty$

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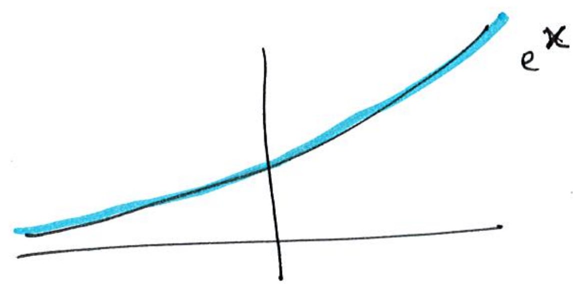
as $\frac{1}{x} \to -\infty$,

$e^{1/x} \to 0$.



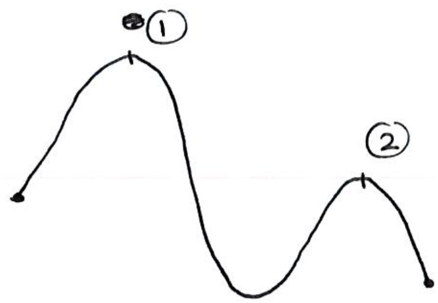
Definitions:

- **Local max** : $x=a$ is local max for $f(x)$ iff for all points near a, $f(x) \leq f(a)$.

- Similarly, local min

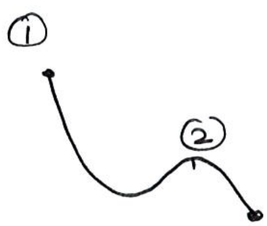
- **Absolute max/min** : $x=a$ is absolute max in an interval $[c,d)$ if $f(a) \geq f(x)$ for all $x \in [c,d]$.

eg:



1, 2 are both local max but only 1 is the absolute max.

- absolute max/min can also occur at the endpoints



1 is the absolute max.

∴ Similarly, there is absolute min.

① $x = a$ is critical point of $f(x)$ if $f'(a) = 0$

② $f'(a) > 0 \Rightarrow f(x)$ increasing near a.

$f'(a) < 0 \Rightarrow f(x)$ decreasing near a.

③ $x = a$ is inflection point of $f(x)$ if $f''(a) = 0$.

④ $f''(a) > 0 \Rightarrow f(x)$ is concave ~~downward~~ upward near a.

$f''(a) < 0 \Rightarrow f(x)$ is concave downward near a.

· at a local min/max $f(x)$ cannot be increasing nor can it be decreasing.

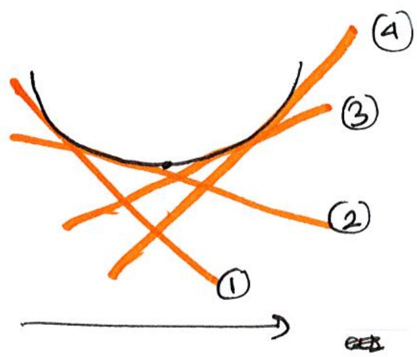
$\Rightarrow$ local min/max are critical points.

· How to tell if a $\underbrace{\text{critical point}}_{f'(a) = 0}$ is a min/max?

Ans: · $f'(x)$ is the rate of change of $f(x)$.

· $f''(x)$ is the rate of change of $f'(x)$.

At local min:



as we increase x the slope of the tangent increases

$\Rightarrow$ as we increase x, $f'(x)$ increases (at a local min)

$\Rightarrow$ $(f'(x))' = f''(x)$ is positive at a

$\Rightarrow$ i.e. $\boxed{\text{at local min } x=a, \quad f''(a) > 0}$

← This shape is called concave upward.

Basic example: $f(x) = x^2$

concave upward for all x

$f'(x) = 2x$

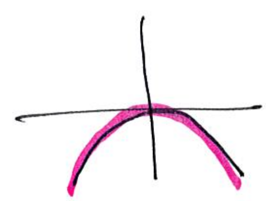
$f''(x) = 2 > 0$ for all x

$\boxed{\text{at local max, } x=a, \quad f''(a) < 0}$

← This shape is called concave downward.

Basic example: $f(x) = -x^2$

$f'(x) = -2x$



$f''(x) = -2 < 0$ $\Rightarrow$ for all x concave downward

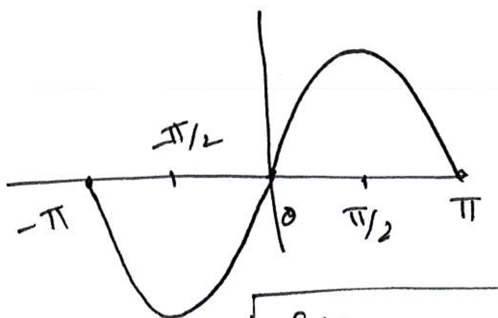
$f''(a) = 0 \Rightarrow$ neither concave upward nor concave downward.

$\downarrow$

inflection point $\equiv$ the curve is changing its concavity at the inflection point.

eg: $f(x) = \sin(x) \qquad x \in (-\pi, \pi)$

Find ~~ex~~ local min / max, concavity, inflection point etc.



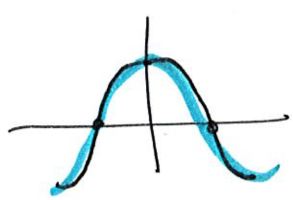
• Ans: ~~local~~

$$\boxed{\begin{aligned} f(x) &= \sin x \\ f'(x) &= +\cos x \\ f''(x) &= -\sin x \end{aligned}}$$

① Critical points:

$f'(x) = 0$

$\Rightarrow \cos x = 0$ in $(-\pi, \pi)$



$\Rightarrow \boxed{x = -\frac{\pi}{2}, \frac{\pi}{2}}$

② At the critical points

• $\boxed{x = -\frac{\pi}{2}}$

$f''(-\frac{\pi}{2}) = -\sin(-\frac{\pi}{2})$

$= -(-1)$

$= 1 > 0$

$\Rightarrow$ concave up

$\Rightarrow$ local min.

• $\boxed{x = \pi/2}$

$f''(\frac{\pi}{2}) = -\sin(\frac{\pi}{2}) = -1 < 0$

$\Rightarrow$ concave down

$\Rightarrow$ local max.

(3) Point of inflection:

$$f''(x) = 0$$

$\Rightarrow -\sin x = 0$ in $(-\pi, \pi)$

$\Rightarrow \boxed{x = 0}$ ← at $x = 0$ $\sin x$ goes from being concave upward to concave downward.

Ch 4. 4.1, 4.3, 4.5, 4.7 — critical points, local min/max, concavity

✓ 4.1 absolute min/max

✓ 4.3 local min/max

4.5 sketching graphs

4.7 Word problems

Q. Sketch $\boxed{f(x) = x + 2\sin x}$ in $[0, 2\pi]$.

- (Find critical points, check concavity / inflection.)

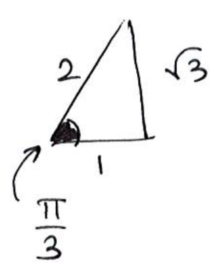
(1) Critical points:

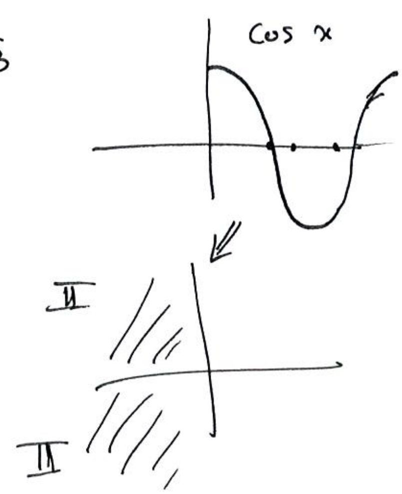
$$f'(x) = 0$$

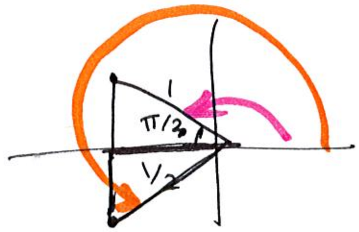
$\Rightarrow 1 + 2\cos x = 0$

$\Rightarrow 2\cos x = -1$

$\Rightarrow \boxed{\cos x = \frac{-1}{2}}$







$$\boxed{\pi - \frac{\pi}{3}} = \frac{2\pi}{3}$$

$$\boxed{\pi + \frac{\pi}{3}} = \frac{4\pi}{3}$$

The critical points are

$$x = \frac{2\pi}{3}, \ \frac{4\pi}{3}$$

(2) ~~$f''(x)$~~ $f(x) = x + 2\sin x$

$$f'(x) = 1 + 2\cos x$$

$$f''(x) = -2\sin x$$

i) at $x = 2\pi/3$

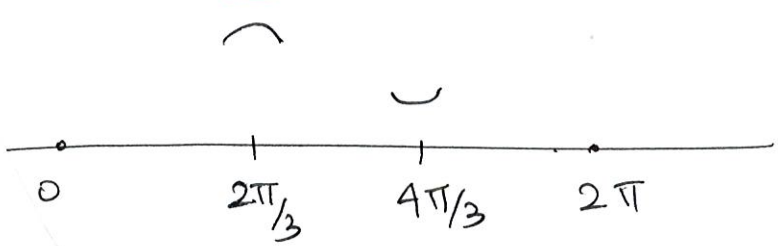
$$f''\left(\frac{2\pi}{3}\right) = -2\sin\left(\frac{2\pi}{3}\right) < 0$$ as sin is positive in second quadrant

$\Rightarrow$ $\boxed{x = \frac{2\pi}{3} \text{ is local max}}$ i.e. $f(x)$ is concave downward at $\frac{2\pi}{3}$

ii) at $x = 4\pi/3$

$$f''\left(\frac{4\pi}{3}\right) = -2\sin\left(\frac{4\pi}{3}\right) > 0$$ as sin is negative in third quadrant

$\Rightarrow$ $\boxed{x = \frac{4\pi}{3} \text{ is local min}}$ i.e. $f(x)$ is concave upward at $\frac{4\pi}{3}$



$f(x) = x + 2\sin x$

at $x = 0$, $f(0) = 0$

local max

$x = \frac{2\pi}{3}$, $f(x) = f\left(\frac{2\pi}{3}\right)$

$= \frac{2\pi}{3} + 2 \cdot \sin\left(\frac{2\pi}{3}\right)$

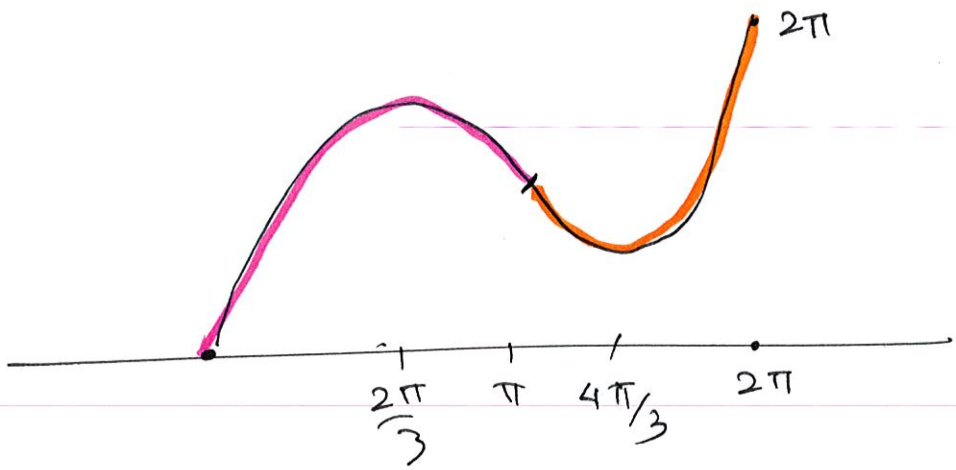
$= \frac{2\pi}{3} + \cancel{2} \cdot \frac{\sqrt{3}}{\cancel{2}}$

local min

$x = \frac{4\pi}{3}$, $f\left(\frac{4\pi}{3}\right)$ = something

$x = 2\pi$, $f(2\pi) = 2\pi + 2\sin(2\pi)$

$= 2\pi$



Point of inflection:

$f(x) = x + 2\sin x$

$f'(x) = 1 + 2\cos x$

$\boxed{f''(x) = -2\sin x}$

$f''(x) = 0$

$\Rightarrow \sin x = 0$

$\Rightarrow \boxed{x = \pi}$ ← curve goes from the concave down to concave up

Q. Sketch $f(x) = x^4 - 4x^3$

Ans :

$$f(x) = x^4 - 4x^3$$

$$f'(x) = 4x^3 - 12x^2$$

$$f''(x) = 12x^2 - 24x$$

Critical points :

$$f'(x) = 0$$

$$\Rightarrow 4x^3 - 12x^2 = 0$$

$$\Rightarrow 4x^2(x-3) = 0$$

$\Rightarrow x^2 = 0$, [redacted] $x - 3 = 0$

$\Rightarrow \boxed{x=0}$, [redacted] $\boxed{x=3}$

(1) $x = 0$, check concavity:

$$f''(0) = 12\cdot 0 - 24 \cdot 0$$

$$= 0$$

$\Rightarrow$ $x=0$ is a point of inflection

$f(x)$ is changing concavity at $x=0$.

(2) $x = 3$, check concavity :

$$f''(3) = 12\cdot(3)^2 - 24\cdot 3$$

$$= 12\cdot 3\cdot(3 - 2\cdot 1)$$

$= 12\cdot 3\cdot 1 > 0$ $\Rightarrow$ concave up

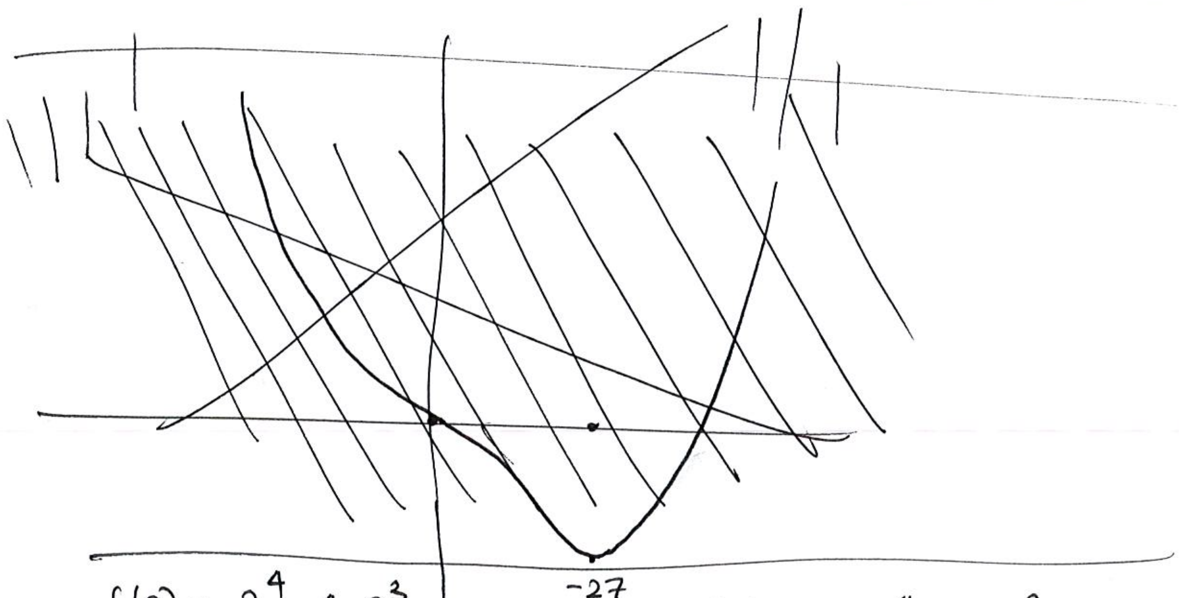
$\Rightarrow$ $x=3$ is local min

(4) What happens at $+\infty$, $-\infty$?

$$\lim_{x\to\infty} x^4 - 4x^3 = \lim_{x\to\infty} \underbrace{x^3}_{\infty}\underbrace{(x-4)}_{\infty} = \infty$$

$$\lim_{x \to -\infty} x^4 - 4x^3 = \lim_{x \to -\infty} \underbrace{x^3}_{-\infty} \underbrace{(x-4)}_{-\infty} = +\infty$$

- at both $\infty, -\infty$, $\lim f(x)$ is $+\infty$
- at $x = 3$ local min
- at $x = 0$, both critical point, inflection point.



$f(3) = 3^4 - 4 \cdot 3^3$

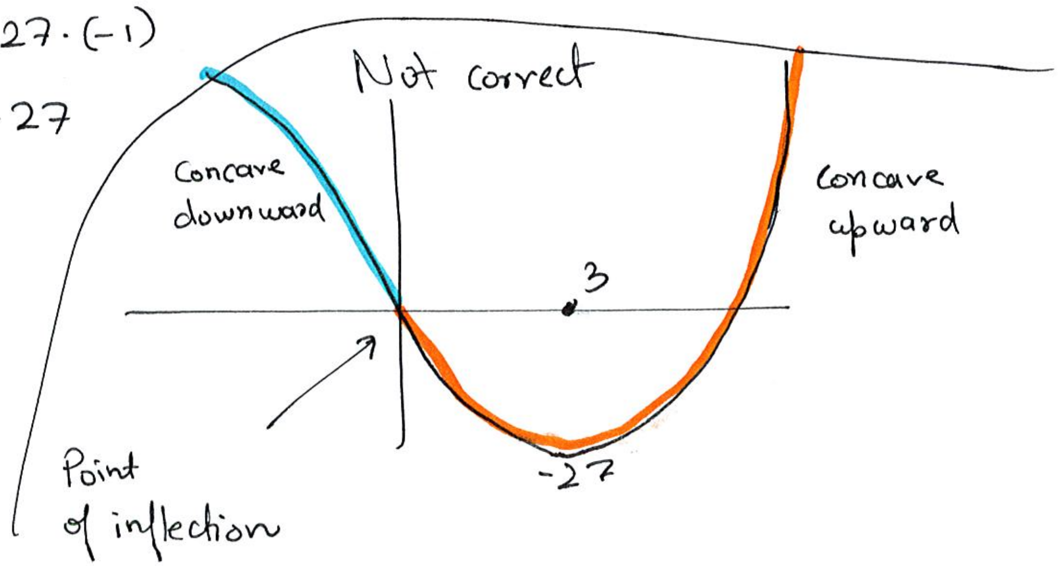
$= 3^3(3-4)$

$= 27 \cdot (-1)$

$= -27$

$f(0) = 0^4 - 4 \cdot 0^3$

$= 0$



Step 5 Points of infection:

$$f''(x) = 0$$

$$\Rightarrow \quad 12x^2 - 24x = 0$$

$$\Rightarrow \quad 12x(x-2) = 0$$

$$\Rightarrow \quad x = 0 \quad , \quad x - 2 = 0$$

$$\Rightarrow \quad x = 0, \quad x = 2$$

<u>Summary</u> :

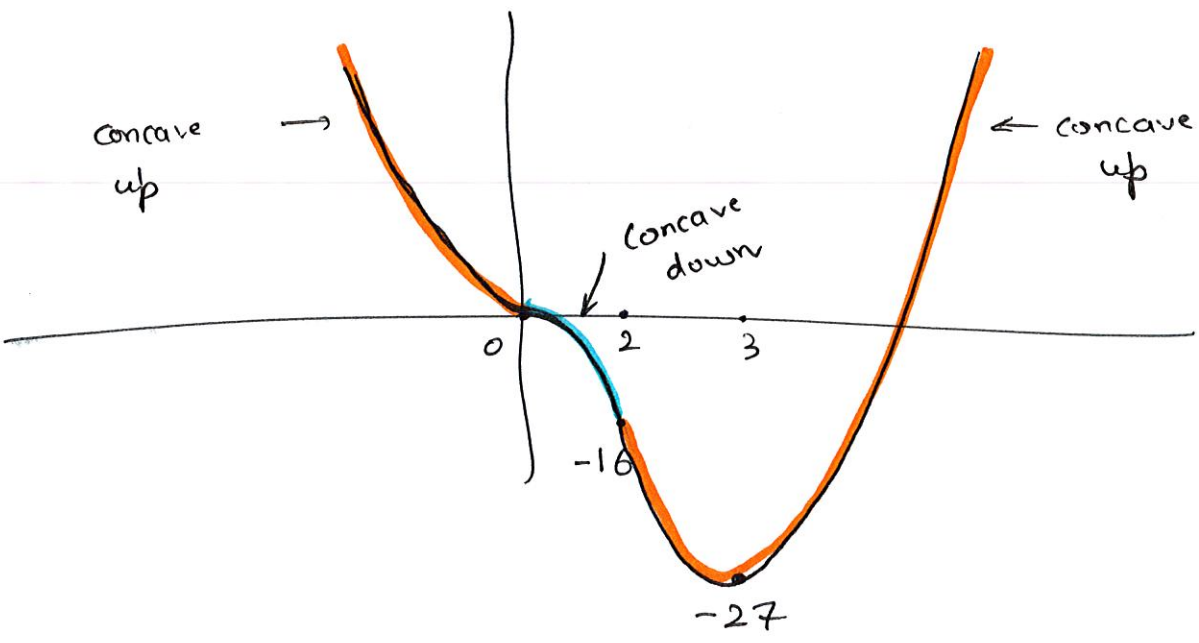
$x = 0$ critical point & point of inflection

$x = 2$ only point of inflection

$x = 3$ only critical point, local min

$\lim_{x \to \infty}$, $\lim_{x \to -\infty}$ are both $+\infty$

$$\begin{aligned} f(2) &= 2^4 - 4 \cdot 2^3 \\ &= 2^3(2-4) \\ &= 2^3 \cdot (-2) \\ &= -16 \end{aligned}$$



~~At a critical po~~

- To sketch functions
  - find critical points, find concavity at these critical points
  - find inflection points,
  - find "y-values" at all of the above points.
  - If we have endpoints find y-values at them.

    else find $\lim_{x \to \infty} f(x)$, $\lim_{x \to -\infty} f(x)$
  - find points of discontinuities, (vertical asymptotes).