University of Western Ontario

Departments of Applied Mathematics

Calculus 2402A
Fall 2020
Midterm Examination

(Online)

Code 111

October 24, 2020 | 3 hours

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Instructions

1. **Print your Name, Student Number in the box above**.
2. The Exam Booklet should have 14 pages (including the front page).
3. In Part A (Multiple Choice questions), **circle the correct answer for each multiple choice question**.
4. Part B must be answered in the space provided in the Exam Booklet. Unjustified answers will receive little or no credit.
5. Pages 13 and 14 of the Exam Booklet are blank and are to be used for Part B if you need extra space for presenting your answers for Part B. Indicate clearly which questions from Part B you are answering there.
6. Only scientific non-programmable calculators are permitted.
7. **Code of Conduct:** Students are not allowed to assist or communicate to each other during the exam time. This constitutes a scholastic offence subject to severe academic penalties.
8. I pledge on my honour that I have neither given nor received aid on this examination ☑.
   (You must check the box above before writing the exam. Otherwise, the exam is NOT graded.)
9. **Total Marks = Part A (40) + Part B (42) = 82 marks.**

Part A: 20 multiple choice questions (2 marks each) = 40 marks
Do your work in the Scratch Papers. Circle the correct answer for each multiple choice question.

A

A1: Find the domain of the function $f(x,y) = \ln(9 - x^2 - 9y^2)$.

A) $\{(x,y) : x^2 + 9y^2 < 9\}$ B) $\{(x,y) : \frac{1}{3}x^2 + y^2 < 3\}$

C) $\{(x,y) : x^2 + y^2 < 9\}$ D) $\{(x,y) : \frac{1}{3}x^2 + y^2 < 1\}$

E) $\{(x,y) : x^2 + 9y^2 < 1\}$

D

A2: Evaluate the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1 - e^{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}$. $\frac{1-e^u}{u}$ $u \to 0.$ $\frac{-e^u}{1}.$

A) 0 B) -2 C) ∞ D) -1 E) The limit does not exist.

B.

A3: Let $x - k.$ $x=0.$ $-k = k^2 - 2.$

$$f(x,y) = \begin{cases} \frac{\cancel{x^3} - kx^2 + \cancel{2xy}^2 - 2ky^2}{x^2+2y^2} & \text{if } (x,y) \neq (0,0) \\ k^2 - 2 & \text{if } (x,y) = (0,0) \end{cases}$$

$k^2 + k - 2 = 0.$
$(k+2)(k-1) = 0$
$k=-2 / k=1.$

where k is a constant real number. Find all values of k so that the function f is continuous at $(0,0)$.

A) $-1, 2$ B) $1, -2$ C) $\sqrt{3}, -\sqrt{3}$ D) $\sqrt{2}, -\sqrt{2}$ E) none of the above

B.

$e^x \cdot e^y.$

A4: Given $z = f(x,y) = x^3y - e^{xy}$, find $\frac{\partial z}{\partial x}$. $= -\frac{F_x}{F_z} = \frac{3x^2y - e^{xy}}{1}.$

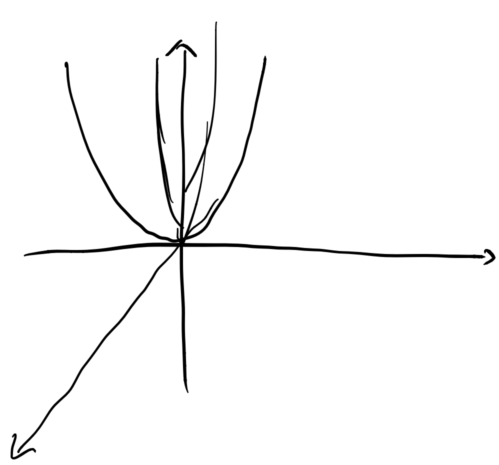
$F_x = x^3y - e^{xy} - z = 0.$

A) $3x^2y - ye^{xy}$ B) $3x^2y - e^{xy}$ C) $x^3 - xe^{xy}$ D) $x^3y - 2e^{xy}$ E) $x^2y - xe^{xy}$

B.

A5: Write the equation for the surface obtained by rotating the curve $z = y^2$ (in the $yz-$plane) about the $z-$axis.

A) $z = -x^2 + y^2$ B) $z = x^2 + y^2$ C) $z = x + y^2$ D) $-x^2 - y^2$ E) $z = x^2 - y^2$



D

A6: Find an equation of the plane tangent to the surface $z = x^2 - y^2$ at the point $(2, 1, 3)$.

$F = x^2 - y^2 - z = 0$ $\quad F_x = 2x \quad F_y = -2y \quad F_z = -1.$

A) $4x + 2y + z = 13$ B) $4x + 2y - z = 7$ C) $4x - 2y + z = -9$

D) $4x - 2y + z = 9$ E) $4x - 2y - z = 3$

C

A7: The pressure p, the volume V, and the temperature T (in Kelvin) of a confined gas are related by the idea gas equation $pV = kT$, where k is a constant. If $p = 0.5$ Pascal when $V = 50$ cm^3 and $T = 360$ Kelvin, determine by approximately what percentage p changes if V and T change to 52 cm^3 and 351 Kelvin, respectively.

$0.5 \cdot 50 = k \cdot 360.$

$25 = k \cdot 360$

$k = \frac{25}{360}.$

A) 6.5% B) 1.5% C) −6.5% D) −4.5% E) −1.5%

$52\,p' = 351 \cdot \frac{25}{360}.$ $\quad 0.468 \quad \frac{.032}{0.5}.$

C

A8: Find the linear approximation $L(x, y)$ of $f(x,y) = \dfrac{1 + y}{1 + x}$ at $(1, 3)$.

$F_x = -\frac{1+y}{(1+x)^2} = -\frac{4}{4} = -1$

$F_y = \frac{1}{1+x} = \frac{3}{2}.$

A) $\cancel{-x + \frac{1}{2}y + \frac{3}{2}}$ B) $x + \frac{1}{2}y + \frac{3}{2}$ C) $-x - \frac{1}{2}y + \frac{3}{2}$

$(1 + \frac{3}{2} + \frac{3}{2}.$ $\quad -1 - \frac{3}{2} + \frac{3}{2}.$

D) $-x + \frac{1}{2}y - \frac{3}{2}$ E) $\cancel{-x + \frac{3}{2}y + \frac{1}{2}}$

$-1 + \frac{1}{2}.$ $\quad -1$

D

A9: Calculate and simplify $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2}$ if $z = \dfrac{x}{x^2 + y^2}$.

$F_{xx} = -\frac{2x(3y^2 - x^2)}{(x^2+y^2)^3}.$

A) $\dfrac{4x(x^2 - 3y^2)}{(x^2 + y^2)^3}$ B) $-\dfrac{x^2 - y^2 + 2xy}{(x^2 + y^2)^2}$ C) $\dfrac{x^2 - y^2 - 2xy}{(x^2 + y^2)^2}$

D) 0 E) $\dfrac{4x(3y^2 - x^2)}{(x^2 + y^2)^3}$

E.

A10: Evaluate $f_{yz}(3, 2, 1)$ if $f(x, y, z) = x \tan^{-1}(yz)$.

$\frac{x(-2^2y^2+1)}{(2^2y^2+1)^2}.$

A) $-\dfrac{6}{25}$ B) $\dfrac{7}{25}$ C) $\dfrac{8}{25}$ D) 0 E) $-\dfrac{9}{25}$

$\frac{3(-1 \cdot 4 + 1)}{(4+1)^2}.$

$= -\frac{9}{25}.$

D

A11: Find the directional derivative of $f(x, y, z) = \ln(xy + yz + zx)$ at the point $(1, 1, 1)$ in the direction from $(1, 1, 1)$ to $(-1, -2, 3)$.

$\vec{p_0p_1} = (2, 3, -2)$. $f_x = \dfrac{y+z}{xy+yz+xz}$.

A) $\dfrac{2}{\sqrt{17}}$ B) $-\dfrac{1}{\sqrt{17}}$ C) $-\dfrac{3}{\sqrt{17}}$ D) $\dfrac{1}{\sqrt{17}}$ E) $-\dfrac{2}{\sqrt{17}}$

D

A12: What function is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$?

A) $u(x, y) = x^2 + y^2$ B) $u(x, y) = x^3 + 3xy^2$
C) $u(x, y) = \sin(kx)\sin(aky)$ D) $u(x, y) = e^{-x}\cos y - e^{-y}\cos x$
E) $u(x, y) = y/(x^2 - a^2y^2)$

C

A13: If $f(x, y) = x^2y - x^2 - y^2 + y + 25$ then the Hessian of f at $(1, 1)$ is.

$\begin{matrix} 0 & 2 \\ 2 & -2 \end{matrix}$.

A) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ B) $\begin{bmatrix} 0 & -2 \\ -2 & -12 \end{bmatrix}$ C) $\begin{bmatrix} 0 & 2 \\ 2 & -2 \end{bmatrix}$

D) $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ E) none of the above

$F_x = 2xy - 2x$
$F_y = x^2 - 2y + 1$

$F_{xx} = 2y - 2 = 0$.
$F_{xy} = 2x = 2$.
$F_{yx} = 2x = 2$
$F_{yy} = -2$

C

A14: If $f(x, y) = -x^2 + xy - y^2 + 2x - y$ then the critical point of f is

$F_x = -2x + y + 2$. $F_y = x - 2y - 1$.

A) a saddle point B) a local minimum C) a local maximum D) none of the above

$f_{xx} = -2 < 0$ $F_{yy} = -2$ $f_{xy} = 1$.
$F_{xx}F_{yy} - F_{xy}^2 = 3 > 0$.

B.

A15: The shortest distance from the point $(2, 1, -3)$ to the plane $x + y + z = 1$ is

A) $\dfrac{\sqrt{2}}{3}$ B) $\dfrac{\sqrt{3}}{3}$ C) 2 D) $\dfrac{4}{3}$ E) $\dfrac{3}{5}$

$\dfrac{|2+1-3-1|}{\sqrt{3}}$.

A16: Find $\iint_R xy\,dA$, where R is the rectangle defined by $0 \le x \le 6$ and $0 \le y \le 4$.

C

$\int_0^6 \int_0^4 xy\,dy\,dx = \int_0^6 \cdot 8x\,dx$ 24.

A) 36 B) 168 C) 144 D) 72 E) 288

$\frac{1}{2}y^2 x.$ $4x^2 \cdot \boxed{4\times 36}$.

$\frac{1}{2}\cdot 16x.$

A17: Evaluate $\iint_R (1-x-y)\,dA$ where R is the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$.

B

$\int_0^1 \int_0^1 (1-x-y)\,dy\,dx = \int_0^1 (\frac{1}{2}-x)\,dx.$

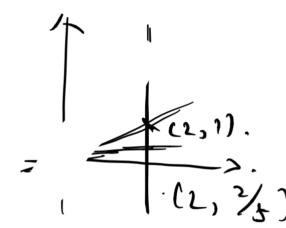
A) $\frac{1}{2}$ B) $\frac{1}{6}$ C) $\frac{1}{3}$ D) $\frac{2}{3}$ E) $\frac{1}{4}$

$1-x-y.$

A18: Evaluate $\iint_R y^2\,dA$ where R is the region bounded by $y = 2x$, $y = 5x$ and $x = 2$.

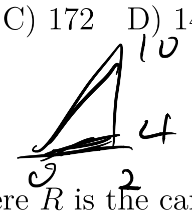
C

$\int_0^2 \int_{2/5}^{1} y^2\,dy\,dx$



$y^2 = 2x - y.$
$y^2 = 5x - y.$

A) 184 B) 160 C) 172 D) 144 E) 156



A19: Find $\iint_R r\,dA$, where R is the cardioid $r = 1 + \cos\theta$.

E

$r^2 = r + \cos\theta r.$

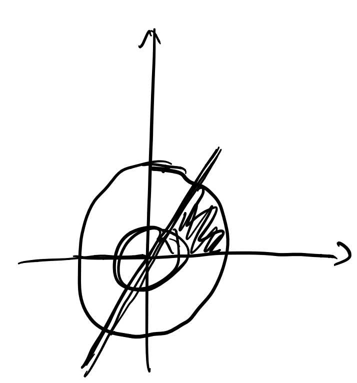
A) $\frac{\pi}{2}$ B) $\frac{4\pi}{3}$ C) π D) $\frac{5\pi}{3}$ E) $\frac{3\pi}{2}$

$\iint (1+\cos\theta)\,d\theta.$

A20: Find $\iint_R \frac{y}{x}\,dA$, where R is the region in the first quadrant lying between the circles $x^2+y^2 = 1$ and $x^2+y^2 = 4$ and between the lines $y = 0$ and $y = x$.

D.

A) $\frac{3}{2}\ln 2$ B) $\frac{3}{16}\ln 2$ C) $\frac{5}{8}\ln 2$ D) $\frac{3}{8}\ln 2$ E) $\frac{3}{4}\ln 2$



Part B: Show all your work for each of the following questions. Total: 42 marks. Do all the 7 questions between B1 and B7.

B1: (6 marks) Given $f(x, y, z) = x^2 + y^2 + z^2 - xy + x - 2z$, find the critical points and classify them.

$f(x,y,z) = x^2+y^2+z^2-xy+x-2z.$

$$\begin{cases} f_x = 2x - y + 1 = 0. \\ f_y = 2y - x = 0. \\ f_z = 2z - 2 = 0 \end{cases} \Rightarrow \begin{cases} x = -\frac{2}{3}. \\ y = -\frac{1}{3}. \\ z = 1. \end{cases}$$

$f_{xx} = 2 = A$ $\quad f_{xy} = -1 = D$ $\quad f_{xz} = 0 = F$

$f_{yy} = 2 = B$ $\quad f_{yz} = 0 = E$

$f_{zz} = 2 = C$

$$\begin{cases} A - D^2 - 1 = 3 > 0. \\ B - E^2 - 1 = 4 > 0. \\ C - F^2 - 1 = 4 > 0 \end{cases}$$

$f_{(-\frac{2}{3}, -\frac{1}{3}, 1)} = \frac{4}{9} + \frac{1}{9} + 1 - \frac{2}{9} - \frac{2}{3} - 2$

$= -\frac{4}{3}$

$f(x,y,z)$ has its minimum value $-\frac{4}{3}$ at the point $(-\frac{2}{3}, -\frac{1}{3}, 1)$.

B2: (6 marks) Find the highest and lowest points on the ellipse which is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$.

$$f_{(x,y)} = z = 1 - x - y.$$

$$F_{(x,y)} = 1 - x - y + \lambda(x^2 + y^2 - 1).$$

$$\begin{cases} F_x = -1 + 2\lambda x = 0. \\ F_y = -1 + 2\lambda x = 0. \\ F_\lambda = x^2 + y^2 - 1 = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{\sqrt{2}}{2} \\ y = \frac{\sqrt{2}}{2} \\ \lambda = \frac{\sqrt{2}}{2}. \end{cases} \text{ or } \begin{cases} x = -\frac{\sqrt{2}}{2} \\ y = -\frac{\sqrt{2}}{2} \\ \lambda = -\frac{\sqrt{2}}{2}. \end{cases}$$

$$f_{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})} = 1 - \sqrt{2}.$$

$$f_{(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})} = 1 + \sqrt{2}.$$

$\therefore$ the highest point $A: (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1+\sqrt{2})$

the lowest point $B: (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1-\sqrt{2})$.

B3: (6 marks) The temperature at a point (x, y, y) in a space is given by $T(x, y, z) = \dfrac{500}{x^2 + y^2 + z^2}$.

(a) Find the rate of change of T at $(2, 3, 3)$ in the direction of the vector $3\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$.

$4 + 9 + 9 = \frac{5000}{22} \cdot \cancel{4} \times 3 \quad 1000 \cdot 2$

(b) In which direction from $(2, 3, 3)$ does the temperature of T increase most rapidly?

(c) At $(2, 3, 3)$ what is the maximum rate of change?

(a). $\vec{v} = (3, 1, 1)$.

$\vec{a} = \left(\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right)$.

$$T_x = -\frac{500 \cdot 2x}{(x^2+y^2+z^2)^2} = \frac{500}{121}$$

$$T_y = -\frac{500 \cdot 2y}{(x^2+y^2+z^2)^2} = \frac{750}{121}$$

$$T_z = -\frac{500 \cdot 2z}{(x^2+y^2+z^2)^2} = \frac{750}{121}$$

$$T_u = \frac{3000}{121\sqrt{11}}$$

(b). When the direction is the same as vector $\vec{a} = \left(\frac{500}{121}, \frac{750}{121}, \frac{750}{121}\right)$.

(c). $\nabla T = \frac{500}{121}\hat{i} + \frac{750}{121}\hat{j} + \frac{750}{121}\hat{k}$

B4: (6 marks) If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, find

(a) $\dfrac{\partial z}{\partial r}$,

(b) $\dfrac{\partial^2 z}{\partial r^2}$.

(a) $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial r}$

$= \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta.$

(b). $\dfrac{\partial^2 z}{\partial r^2} = \dfrac{\partial^2 z}{\partial x^2}\cos^2\theta + 2\dfrac{\partial^2 z}{\partial x\partial y}\cos\theta\sin\theta + \dfrac{\partial^2 z}{\partial y^2}\sin^2\theta.$

B5: (6 marks) Find the third degree Taylor polynomial of $f(x,y) = \dfrac{1}{1+x-y}$ near the point $(2,-1)$.

$$T_3(x,y) = f(2,-1) + \sum_{n=1}^{3} \frac{1}{n!}\left((x-2)\frac{\partial}{\partial x} + (y+1)\frac{\partial}{\partial y}\right)^n f(2,1)$$

$$f(2,-1) = \frac{1}{4}.$$

$$f_x = -\frac{1}{(1+x-y)^2} = -\frac{1}{16}. \quad f_y = \frac{1}{(1+x-y)^2} = \frac{1}{16}.$$

$$f_{xx} = 2(1+x-y)^{-3} = \frac{1}{32} \quad f_{xy} = -2(1+x-y)^{-3} = -\frac{1}{32} \quad f_{yy} = 2(1+x-y)^{-3} = \frac{1}{32}.$$

$$f_{xxx} = -6(1+x-y)^{-4} = -\frac{3}{128} \quad f_{xyx} = 6(1+x-y)^{-4} = \frac{3}{128}.$$

$$f_{xyy} = -6(1+x-y)^{-4} = -\frac{3}{128} \quad f_{yyy} = 6(1+x-y)^{-4} = \frac{3}{128}.$$

$$T_3(x,y) = \frac{1}{4} + \left((x-2)\cdot\left(-\frac{1}{16}\right) + (y+1)\left(\frac{1}{16}\right)\right) + \frac{1}{2!}\left((x-2)^2\left(\frac{1}{32}\right) + (y+1)^2\left(\frac{1}{32}\right) + 2(x-2)(y+1)\left(-\frac{1}{32}\right)\right)$$

$$+ \frac{1}{3!}\left((x-2)^3\left(\frac{1}{32}\right) + 3(x-2)^2(y+1)\left(\frac{3}{128}\right) + 3(x-2)(y+1)^2\left(-\frac{3}{128}\right) + (y+1)^3\left(\frac{3}{128}\right)\right)$$

B6: (6 marks) Evaluate the following integral by reversing the order of integration and sketch the region of integration.

$$\int_0^6 \int_{x/3}^2 e^{y^2}\, dy\, dx$$

$y = \frac{x}{3}$ $x = 3y$. $\because x \in [0,6]$ $\therefore y \in [0,2]$.

$f_{(x,y)} = \int_0^2 \int_0^{3y} e^{y^2}\, dx\, dy$

$= \int_0^2 3y e^{y^2}\, dy$

$= \frac{3}{2}(e^4 - 1)$.

the region is a triangle.

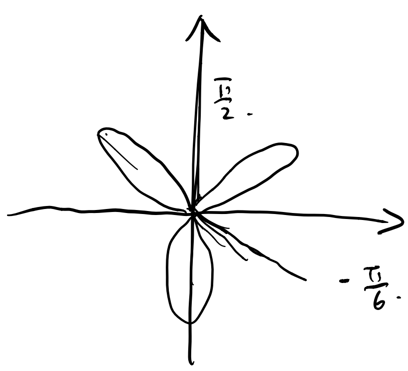
B7: (6 marks) Use double integral in polar coordinates to find the area of the region of one loop of the rose $r = \sin(3\theta)$ and sketch the region of integration.

$$A = \int_{-\pi/6}^{\pi/2} \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \int_{-\pi/6}^{\pi/2} \sin^2 3\theta \, d\theta$$

$$= \frac{1}{4} \int_{-\pi/6}^{\pi/2} (1 - \cos 6\theta) \, d\theta$$

$$= \frac{1}{4}\left(\frac{\pi}{3}\right).$$

$$= \frac{\pi}{12}.$$



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