LOGIC

OUTLINE:

(1) Introduction to logic
(2) Formalization of logic
(3) Propositional logic
(4) Predicate logic
(5) Proof techniques

1. INTRODUCTION TO LOGIC

Arguments

- The goal of science is to extend human knowledge. But we want to be sure that the process of extending knowledge is reliable (i.e., we want to be sure that no false statements can sneak into our body of knowledge).
- Logic formalizes valid *deductive methods of reasoning* (called *deductive arguments*).
- A deductive argument is made of 2 ingredients:
  1) A sequence of statements from our previous body of knowledge, called the premises;
  2) A new statement, typically not included in our previous body of knowledge, called the conclusion.
- A deductive argument is valid if and only if it makes it impossible for the premises (the “old knowledge”) to be true and the conclusion (the “new knowledge”) to be false.
- ATTENTION: A valid deductive argument is not required to have premises that are actually true, but to have premises that, *if they were true*, would guarantee the truth of the argument's conclusion.

Example 1

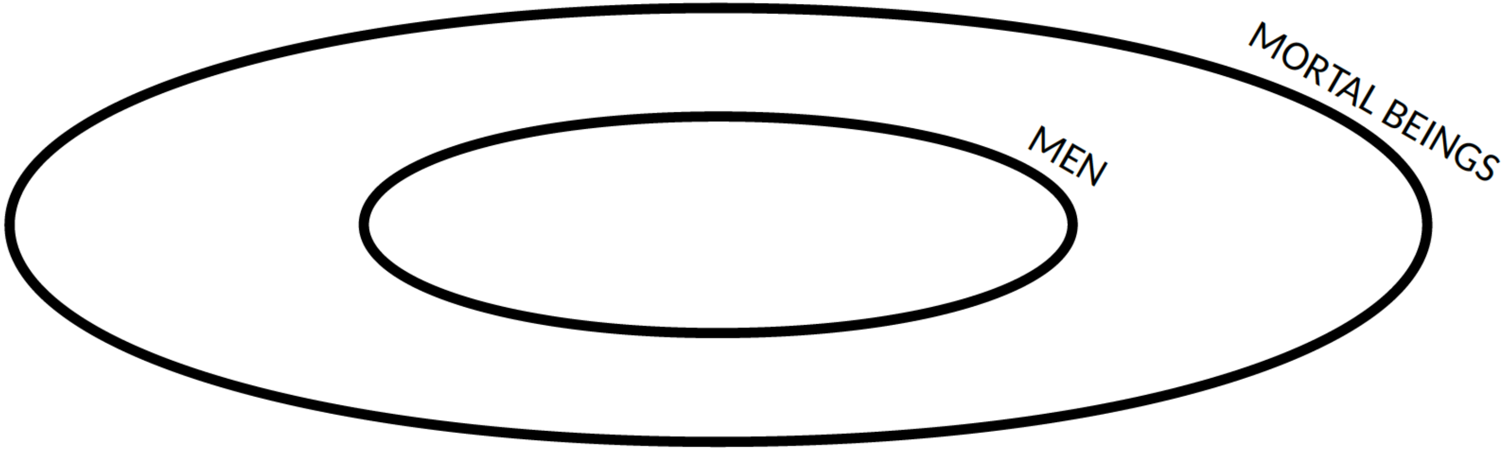
- All men are mortal. Socrates is a man. THEREFORE Socrates is mortal.
- This is an argument with 2 premises
  1) All men are mortal
  2) Socrates is a man

  and one conclusion

  3) Socrates is mortal
- We can check intuitively the validity of this argument with Euler-Venn set diagrams (more on sets in Chapter 3)

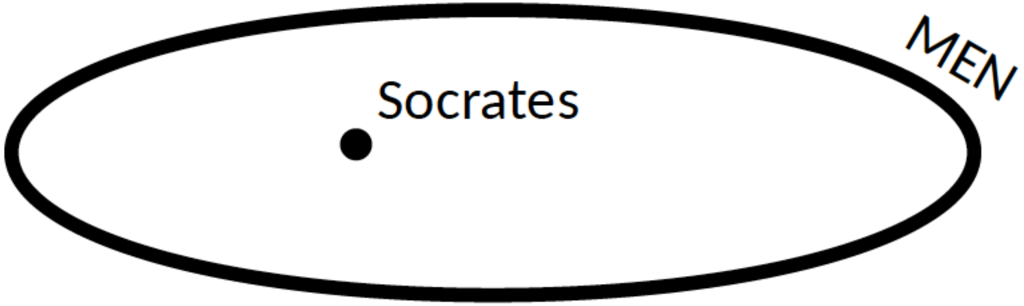
Example 1

- Premise (1) means: all elements of the universe that happen to be men also happen to be mortal, so the set of men is a subset of the set of mortal beings:



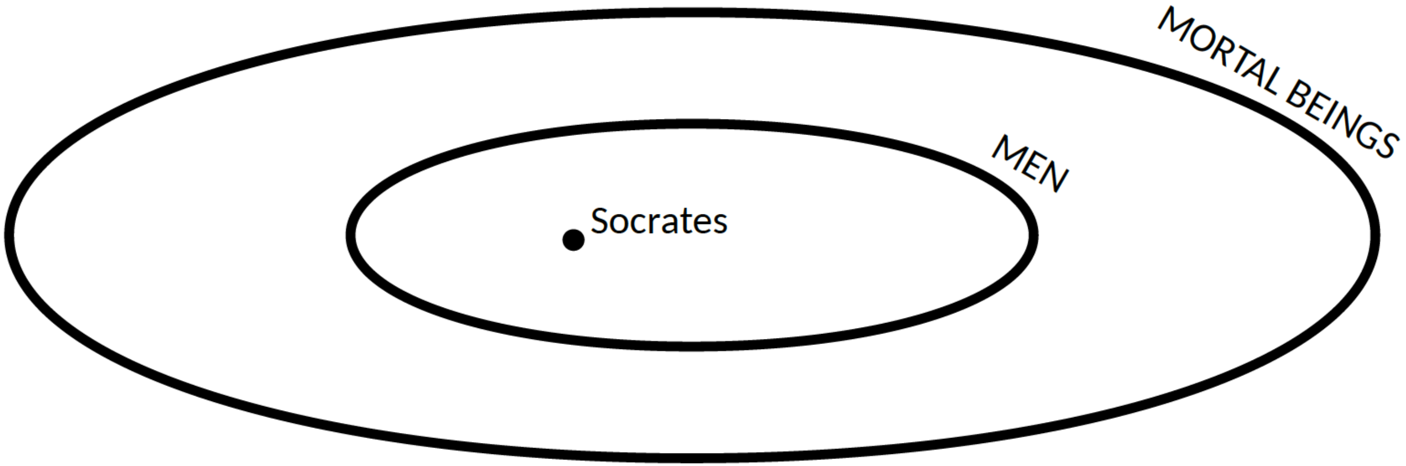
Example 1

- Premise (2) means: the element Socrates is contained in the set of Men:



Example 1

- Altogether, premises (1) and (2) force Socrates to be also an element of the set of mortal beings, as the conclusion claims:



- It is important to note that in Example 1 the conclusion is a formal consequence of the premises, that is, of their “abstract shape”, the way they are constructed.
- The conclusion does not depend on the meaning of the premises, i.e., their content. Let us clarify this concept with 3 additional examples.

Example 2

- (1) All cats are green. (2) Prof. Pasini is a cat. THEREFORE (3) Prof. Pasini is green.

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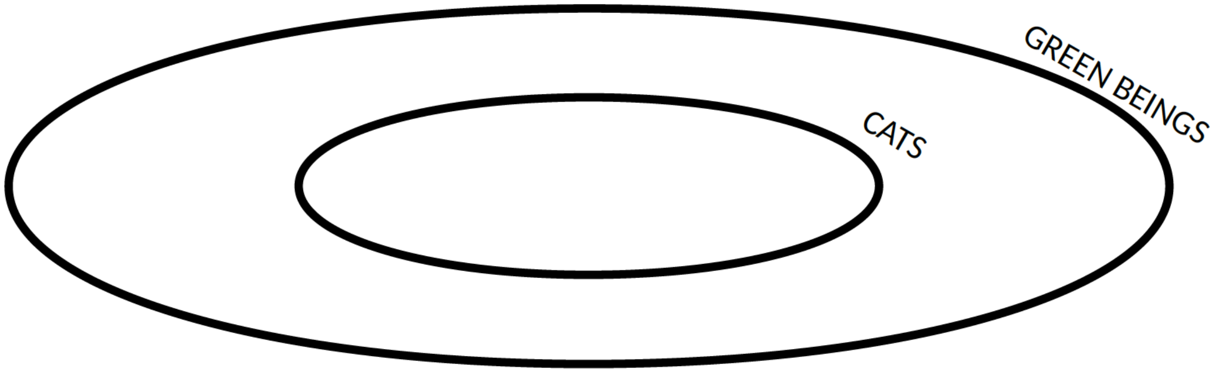
- (1) All cats are green. (2) Prof. Pasini is a cat. THEREFORE (3) Prof. Pasini is green.

- Although the *content* of this argument makes no sense, there is something convincing in the *formal shape* of the argument. Let's represent it via set diagrams.

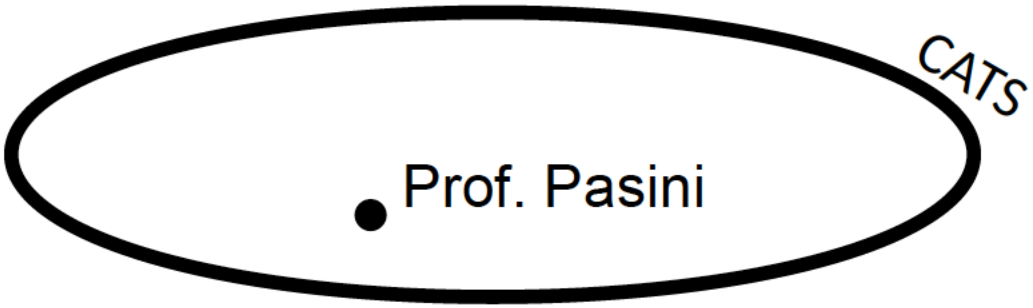
Example 2

- (1) All cats are green. (2) Prof. Pasini is a cat. THEREFORE (3) Prof. Pasini is green.
- Premise (1) means: all elements of the universe that happen to be cats also happen to be green, so the set of cats is a subset of the set of green beings.



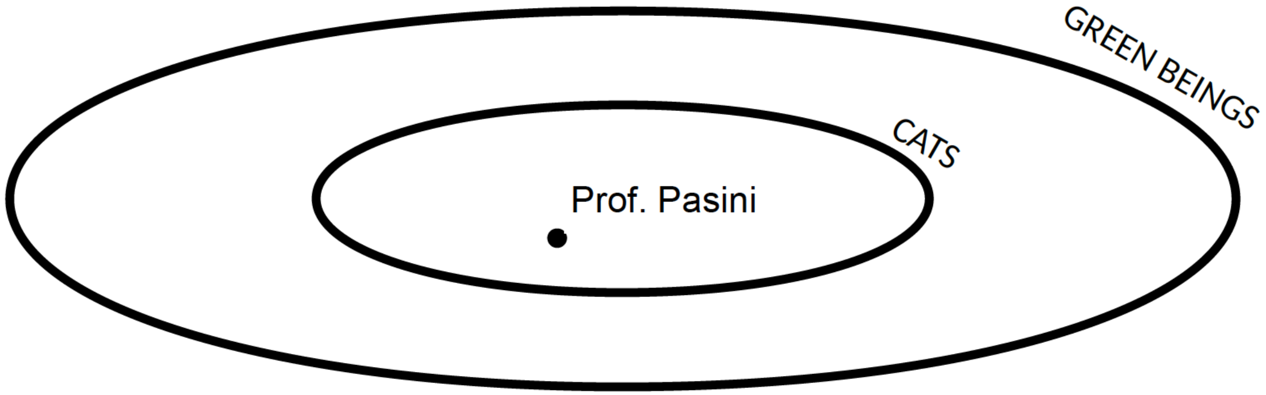
Example 2

- (1) All cats are green. (2) Prof. Pasini is a cat. THEREFORE (3) Prof. Pasini is green.
- Premise (2) means: the particular element Prof. Pasini is contained in the set of cats.



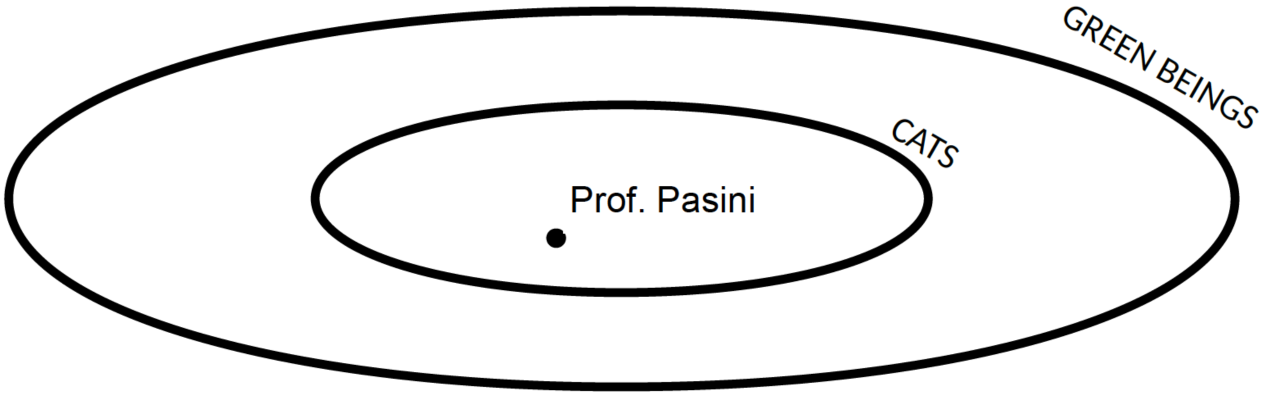
Example 2

- (1) All cats are green. (2) Prof. Pasini is a cat. THEREFORE (3) Prof. Pasini is green.
- The combined effect of the 2 premises forces Prof. Pasini to be also an element of the set of green things, which is exactly what the conclusion (3) means.



Example 2

- (1) All cats are green. (2) Prof. Pasini is a cat. THEREFORE (3) Prof. Pasini is green.
- In this sense the argument is valid: the premises, however crazy they are, force the conclusion, however crazy it is. If the premises were true, then the conclusion would be true as well.



- The previous two examples are actually two instances of the same argument. There is a common “formal skeleton”, i.e. a common abstract shape, which we can highlight by substituting all bits of content with meaningless (in the sense of “deprived of any meaning”) symbols:

Premise (1): all P are Q. (or also: all ♣ are ╩)

Premise (2): R is a P. (or also: ∫ is a ♣)

THEREFORE

Conclusion (3): R is a Q. (or also: ∫ is a ╩)

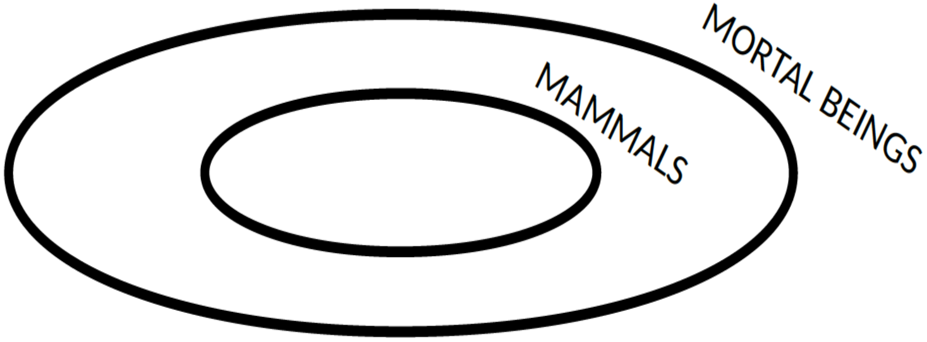
- In this abstract setting, the fact that the premises force the conclusion is much clearer, and it is a matter of how the 3 statements are interconnected, not of what the statements say.
- The MOTTO is: “The validity of an argument does not rely on the truth/falseness of its statements, but in the relationship between these statements.”

Example 3

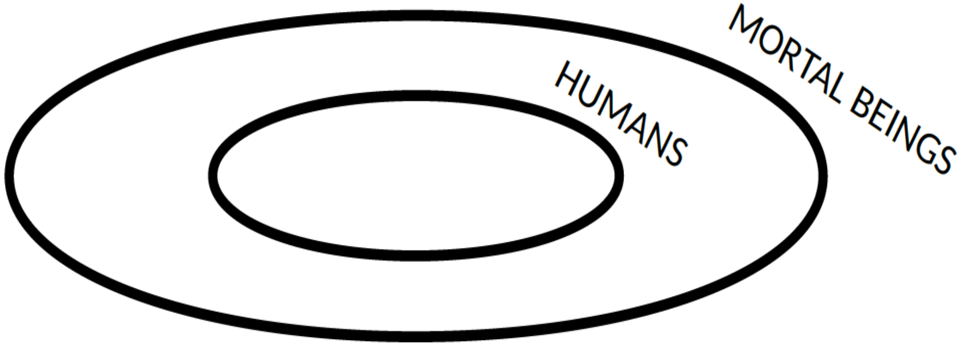
- (1) All mammals are mortal. (2) All humans are mortal. THEREFORE (3) all mammals are humans.

Example 3

- (1) All mammals are mortal. (2) All humans are mortal. THEREFORE (3) all mammals are humans.
- Premise 1 (biologically true)



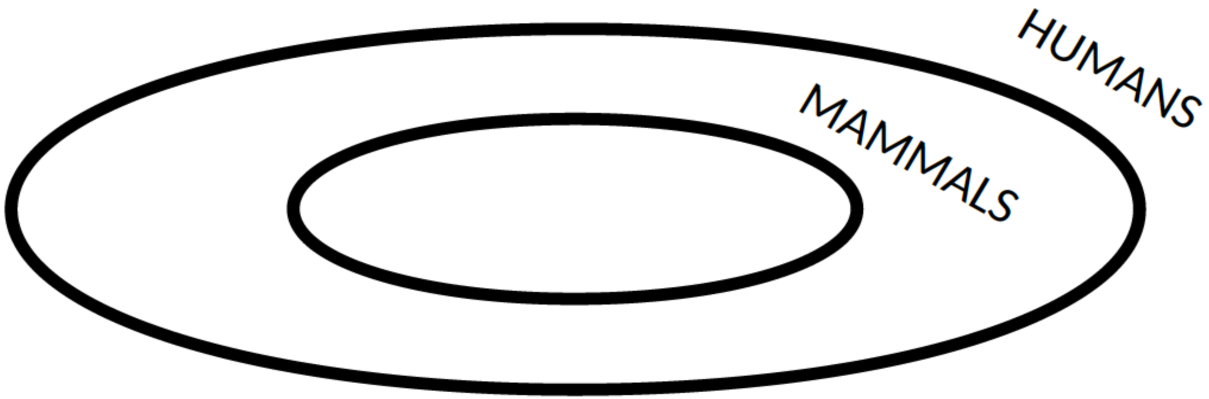
Premise 2 (biologically true)



Example 3

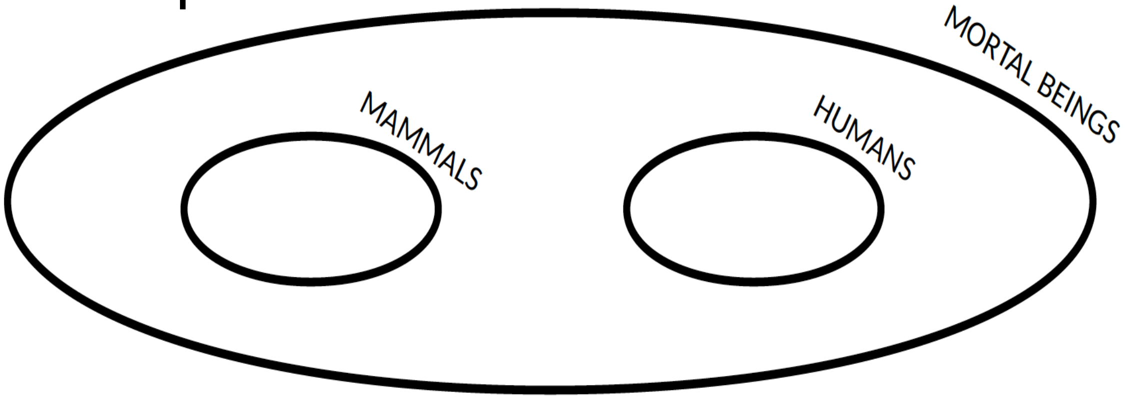
- (1) All mammals are mortal. (2) All humans are mortal. THEREFORE (3) all mammals are humans.

- Conclusion 3 (biologically false):



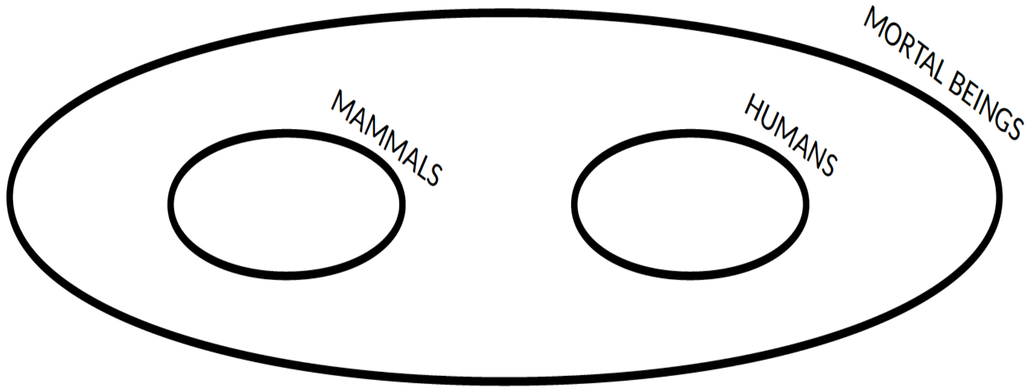
Example 3

- (1) All mammals are mortal. (2) All humans are mortal. THEREFORE (3) all mammals are humans.
- However, the 2 premises do not force the conclusion, because they can be combined a set diagram which is not compatible with the diagram of the conclusion, for example:



This diagram is compatible with both premises (i.e., it makes both premises true), but not with the conclusion (i.e., it makes the conclusion false).

Example 3



This diagram is compatible with both premises (i.e., it makes both premises true), but not with the conclusion (i.e., it makes the conclusion false).

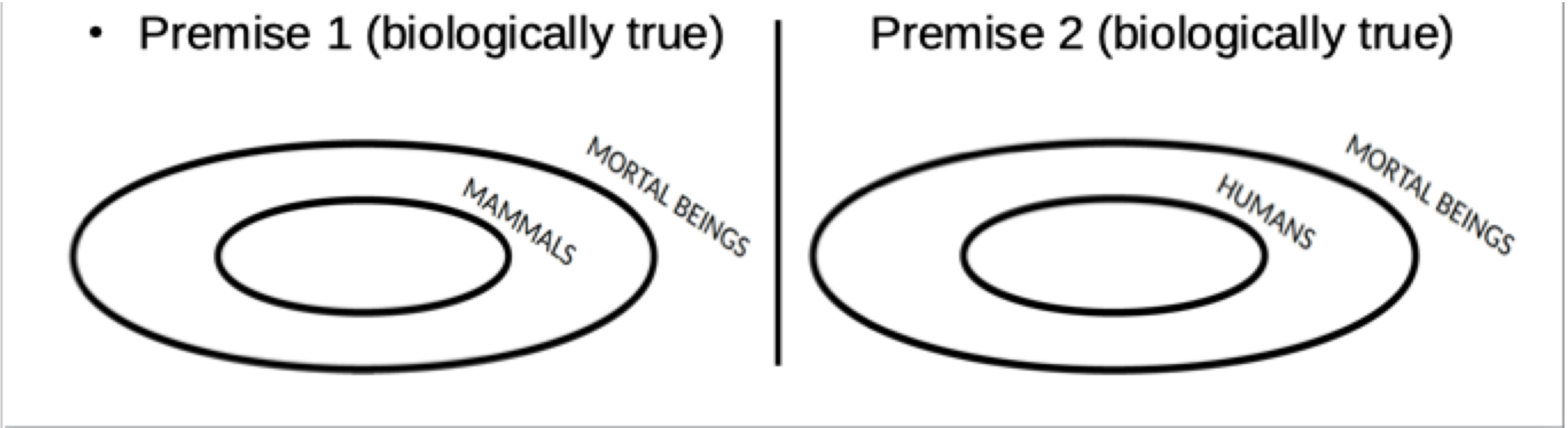
- This is not the only possible diagram with these properties, but that doesn’t matter. What matters is that there exists one set diagram (one is enough) which satisfies (i.e., is compatible with) both premises but not the conclusion. This means that the premises do not force the conclusion, and hence the proposed argument is not valid.
- Note that the 2 premises are true and the conclusion is false.

Example 4

- (1) All mammals are mortal. (2) All humans are mortal. THEREFORE (3) all humans are mammals.

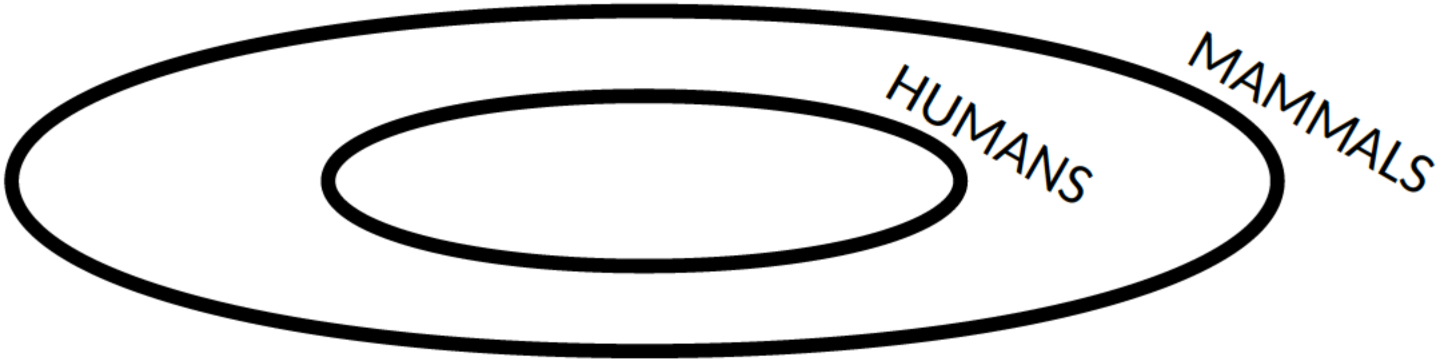
Example 4

- (1) All mammals are mortal. (2) All humans are mortal. THEREFORE (3) all humans are mammals.
- Premises: identical to the ones of Ex. 3:



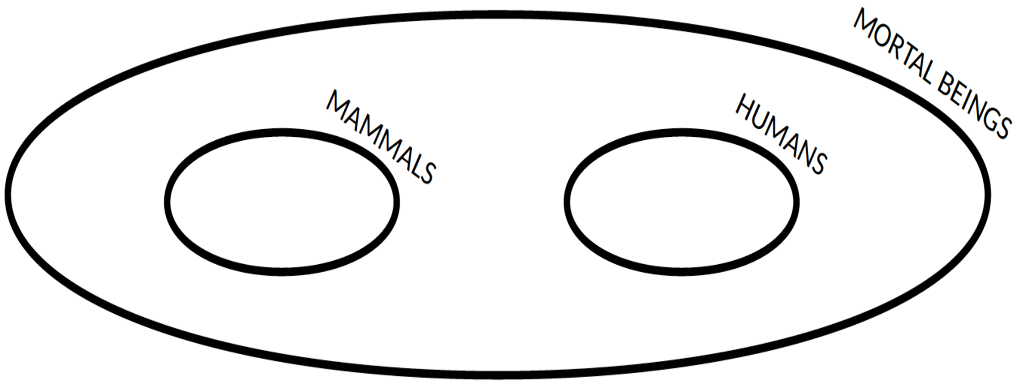
Example 4

- (1) All mammals are mortal. (2) All humans are mortal. THEREFORE (3) all humans are mammals.
- Conclusion: swaps the roles of “humans” and “mammals” in the conclusion of Ex. 3, and is biologically true.



Example 4

- (1) All mammals are mortal. (2) All humans are mortal. THEREFORE (3) all mammals are humans.
- Again, the following diagram (same as in Ex. 3) satisfies both premises but not the conclusion, therefore the argument is not valid, even though both premises and the conclusion are true.



The invalidity of the argument just tells that the truth of the conclusion is not due to the premises (or only to the premises), but relies on other facts.

Exercise

- Highlight the formal skeletons of the arguments of Examples 3 and 4, as we did for Examples 1 and 2.
- Are the two skeletons equal? Can you grasp their invalidity better in this abstract rewriting?

2. FORMALIZATION OF LOGIC

Formalization

- Why does logic need to be formalized?

  1) To make the abstract skeleton of arguments more evident;

  2) Because reasoning expressed in informal natural language can be flawed (natural language is ambiguous and tricky).

  EX: nothing is better than Chuck Norris. Something is better than nothing. THEREFORE something is better than Chuck Norris.

Formalization

Formalization is done on 3 levels:

1) DEFINITION OF THE LANGUAGE: formalize the concept of meaningful statement or well-formed formula (WFF) (grammatically sound sentence).
2) SEMANTICS: formalize the concept of truth of a statement through an interpretation of the language. This gives rise to the notion of true / false WFF (in the particular interpretation chosen), tautology (a WFF which is true in every possible interpretation) and contradiction (a WFF which is false in every possible interpretation).
3) PROOF THEORY: formalize the concept of validity of a reasoning (combining various WFFs). This requires defining axioms (statement declared to hold) and rules of inference (rules to pass from old WFFs to new WFFs) and gives rise to the notion of theorem (a WFF obtained applying a finite sequence of inference rules starting from a finite sequence of axioms) and proof (the sequence of WFFs and inference rules leading to a theorem).

Formalization

- There is not a unique choice of formalization. Different choices give rise to different logical systems, each of which can be the right choice to respond to a particular task with a specific situation.
- Logical systems were developed with the understanding that different sets of axioms would lead to different theorems.
- In this course, we introduce 2 logical systems: classical propositional logic, and classical first-order (a.k.a. predicate) logic.

3. PROPOSITIONAL LOGIC

3.1 Propositional logic: the language

(Textbook reference: Section 1.1)

- Each complex sentence in any natural language is made up of atomic propositions, a.k.a. atoms (the smallest syntactical units of a speech that convey an independent piece of information, and that can be assigned an unambiguous truth value of True or False), joined together by conjunctions (or substitutes thereof, e.g. punctuation or locutions).

3.1 Propositional logic: the language

- EX: “You are enrolled in CS2214B, and, if you have already attended CS2209A, then you know propositional logic.”
- This is a complex sentence which is made of 3 atomic sentences:
  1) “You are enrolled in CS2214B”
  2) “You have already attended CS2209A”
  3) “You know propositional logic”

  joined together by the conjunction “and” and the complex construct with conjunction function “if ... then”. Let’s emphasize this structure with the use of brackets:
- (You are enrolled in CS2214B) **AND** ((you have already attended CS2209A) **IF...THEN** (you know propositional logic)).
- REMARK: The weird way in which we wrote the IF...THEN construct, in between the two joined sentences, is a step forward in the direction of formalization. In formal languages we write symbols of conjunction *in between* the two propositions that they connect.

The general takeaway

- A proposition is a declarative sentence (i.e., a sentence asserting a fact) that is either true or false.
  - Exercise: which of the following are propositions?
    1) “Prof. Pasini’s beard is awesome”
    2) “You should wear warmer clothes”
    3) “Wise men say only fools rush in, but I can’t help falling in love with you”
    4) “Everybody, let’s rock!”
    5) “Would I still see suspicion in your eyes?”
    6) “Mount Fairweather is a high peak”
    7) “Mount Fairweather is the highest peak in Nova Scotia”
    8) “x is a prime number”
    9) “3 is a prime number”

The general takeaway

- A proposition is a declarative sentence (i.e., a sentence asserting a fact) that is either true or false.
- All propositions are constructed assembling atomic propositions (atoms) via connectives (the formal logic official term for “conjunctions”).
- The truth value of a proposition is uniquely and algorithmically determined by its connectives and by the truth values of its atomic constituents.

The alphabet

The alphabet of propositional logic is made of

- Propositional symbols, representing atomic propositions, and denoted with lowercase letters, possibly with subscripts. Among them, 2 distinguished symbols:
  - The true, also denoted $\top$, and representing the atom that is always true by definition.
  - The false, also denoted $\bot$, and representing the atom that is always false by definition.
- Connectives, a.k.a. Boolean operators, representing conjunctions:

negation $\neg$	biconditional $\leftrightarrow$
conjunction $\wedge$	exclusive or $\oplus$
disjunction $\vee$	nor $\downarrow$
conditional $\rightarrow$	nand $\uparrow$

- The grouping symbols '(' and ')'

CAUTION: these symbols are not universally accepted; some authors, especially from of old, use different symbols for these connectives.

The well-formed formulas

The well-formed formulas (WFFs) are the sentences that make grammatical sense (as opposed to, e.g., “If I like ice cream and not”). They are defined as follows:

- Each atom is a WFF
- If A and B are WFFs, then so are $\neg A$, $A \wedge B$, $A \vee B$, $A \rightarrow B$ [and also $A \leftrightarrow B$, $A \oplus \mathrm{B}$, $A \downarrow B$, $A \uparrow B$]
- Nothing else is a WFF

The well-formed formulas

(BAD) EX: “If I like ice cream and not”.

- Let p denote the atom “I like ice cream”
- Then “If I like ice cream and not” is denoted by $p \rightarrow \wedge \neg$
- This is not a WFF

Precedence of connectives

- The usual order of precedence among connectives is

1	2	3	4	5
$\neg$	$\wedge$ [and $\uparrow$]	$\vee$ [and $\downarrow$]	$\rightarrow$	$\leftrightarrow$ [and $\oplus$]

EX: The formula $a\vee\neg b\wedge a\rightarrow\neg c\wedge d$ means $(a\vee((\neg b)\wedge a))\rightarrow((\neg c)\wedge d)$

- If a different meaning is intended, brackets can be used to modify the order: $(a\vee\neg b)\wedge a\rightarrow\neg(c\wedge d) \;=\; ((a\vee(\neg b))\wedge a)\rightarrow(\neg(c\wedge d))$
- Connectives with the same precedence associate to the right: $a\wedge b\wedge c \;=\; a\wedge(b\wedge c)$; $a\rightarrow b\rightarrow c \;=\; a\rightarrow(b\rightarrow c)$; $a\oplus b\leftrightarrow c \;=\; a\oplus(b\leftrightarrow c)$
- Brackets can be used to enhance clarity even where they are not strictly necessary: $(a\wedge b)\vee(\neg c\wedge d)$ is the same as $a\wedge b\vee\neg c\wedge d$, but more easily readable

3.2 Propositional logic: semantics

- In logic, the semantics (i.e. the meaning) of a proposition is limited to its truth value, that is, its being true (notation: 0, or T) or false (notation: 0, or F).
- Each atom (being an *abstract* proposition) can have either meaning, with the exception of $\top$ (not to be confused with the truth value T!!), which can only be true, and $\bot$, which can only be false.
- The semantics of connectives is defined by how they manipulate the truth values of the input propositions to produce the truth value of the output proposition. The semantics of the connectives is nicely represented by truth tables, i.e., tabulations of all possible combinations of input truth values and the relative output.

Negation

Negation (¬) is a unary connective (it requires 1 proposition as input). It models the locution "it is not the case that..."

Its truth table is

A	$\neg A$
0	1
1	0

which means that, if the input proposition A is false (respectively, true), then the output proposition $\neg A$ is true (respectively, false).

Negation

- EX: If *p* denotes *“all students of CS2214 shall pass the course”*, then what is $\neg p$?

Negation

- EX:
  - If p denotes *"all students of CS2214 shall pass the course"*,
  - then $\neg p$ denotes *"it is not the case that all students of CS2214 shall pass the course"* =

    = *"not all students of CS2214 shall pass the course"*

Negation

- EX:
  - If p denotes *“all students of CS2214 shall pass the course”*,
  - then $\neg p$ denotes *“it is not the case that all students of CS2214 shall pass the course”* =

    = *“not all students of CS2214 shall pass the course”* =

    = *“at least one student of CS2214 shall not pass the course”*

Conjunction

Conjunction ($\wedge$) is a binary connective (it requires 2 propositions as input). It models the conjunction “and”.

Its truth table is

A	B	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

which means that the output proposition $A \wedge B$ is true when both input propositions A and B are true, and false otherwise.

Conjunction

EX: If *p* denotes *"it is snowing" and q* denotes *"I am walking home", then* $p \wedge q$ denotes *"It is snowing and I am walking home".*

Conjunction

WARNING: mind that in logic we don’t want, and cannot, capture all the nuances of meaning of the conjunctions of a natural language; what will matter to us is just the truth or falseness of sentences.

EX: Let p denote *“Prof. Pasini is generous”* and q denote *“Prof. Pasini is clever”.* Then a logician who likes me might render the sentence *“Prof. Pasini is generous and clever”* as $p \wedge q$. However, a con artist choosing their next victim may think *“Prof. Pasini is generous [implied: this makes the job easier], but he is also clever [implied: he may discover my scam]“*, and this sentence may also be rendered as $p \wedge q$.

Even if in English language the conjunctions “and” and “but” convey very different nuances, from a formal language perspective they are perfectly equivalent: the proposition “p and q” holds true if and only if both p and q are true, and the same can be said of the proposition “p but q”. So there is no need to distinguish between “and” and “but”, they can be formalized with the same connective.

Disjunction

Disjunction (∨) is a binary connective (it requires 2 propositions as input). It models the conjunction "or".

Its truth table is

A	B	$A \vee B$
0	0	0
0	1	1
1	0	1
1	1	1

which means that the output proposition $A \vee B$ is false when both input propositions A and B are false, and true otherwise.

Exclusive or

In English, there is a different use of the word “or”, namely in the locution “either...or...” (tacitly implying “but not both”). This different meaning is encoded in the binary connective exclusive or ($\oplus$).

Its truth table is

A	B	$A \oplus B$
0	0	0
0	1	1
1	0	1
1	1	0

which means that the output proposition $A \oplus B$ is true when exactly one of the input propositions A,B is true, and false otherwise.

Disjunction vs exclusive or

EX: Let p denote *“You can have a sandwich”* and q denote *“you can have a pasta”*.

Then, at a party with friends, the sentence *“you can have a sandwich or [you can have] a pasta”* is not forbidding you from eating both, and can be rendered as $p \vee q$.

However, on a plane, the same sentence told by a flight assistant usually means that you can choose between the two options, but you may not have both, so the correct rendering is $p \oplus q$.

Conditional

Conditional ($\rightarrow$) is a binary connective (it requires 2 propositions as input). It models the locution “if...then...”.

Its truth table is

A	B	$A \rightarrow B$
0	0	1
0	1	1
1	0	0
1	1	1

which means that the output proposition $A \rightarrow B$ is false when the 1st input proposition A is true and the 2nd input B is false, and true otherwise.

Conditional

Conditional ($\rightarrow$) is a binary connective (it requires 2 propositions as input). It models the locution "if...then...".

Its truth table is

A	B	$A \rightarrow B$
0	0	1
0	1	1
1	0	0
1	1	1

NOTE:
In a conditional proposition, the roles of the 2 inputs are not equal. The 1st input (A) is called the antecedent, or hypothesis, or premise, the 2nd input (B) is called the consequence, or thesis, or conclusion.

which means that the output proposition $A \rightarrow B$ is false when the 1st input proposition A is true and the 2nd input B is false, and true otherwise.

Understanding conditional

The semantics of the conditional may be surprising, so let's clarify a few points.

- Differently than in natural language, in logic a conditional statement does not imply a cause-and-effect relationship between the premise and the conclusion. The meaning of a conditional statement depends only on the truth values of its inputs. Sentences like "If I am a human, then the spin of a proton is 1/2" are perfectly fine, although I have no effect on the spin of subatomic particles.

➔ Motto: a conditional statement is a sentence, not a theorem.

Understanding conditional

The semantics of the conditional may be surprising, so let’s clarify a few points.

- The 2nd line in the truth table is counterintuitive: a false premise and a true conclusion make a true conditional. You can think of this choice as follows: if the premise is false, there are not enough data to consider the conditional statement false. The principle behind this choice may be viewed as the logical counterpart of a fundamental legal principle called *presumption of innocence*: “every person accused of any crime is considered innocent until proven guilty”.
- Treat the conditional with a presumption of truth principle: “every conditional is considered true until proven false”.
- The textbook has a different but equivalent view: consider a conditional as a contract with a premise and a conclusion. You are pledging to perform the conclusion whenever the premise is realized (i.e., if the premise is true). If the premise is not realized (i.e., it is false), nobody can charge you with breach of contract, whether you choose to perform the conclusion or not.

Conditional expressions in English

The conditional $A \to B$ is rendered in many ways:

- If *A*, [then] *B*
- *A* implies *B*
- *A*, therefore *B*
- *A* only if *B*
- *A* is a sufficient condition for *B*
- *B* if *A*
- *B*, provided *A*
- *B* when[ever] *A*
- *B* follows from *A*
- *B* is a necessary condition for *A*

Conditional expressions in English

EX: Let A denote "if life gives you lemons, make lemonade".

- Let p denote "life gives you lemons" and q denote "make lemonade"
- Then $A = p \rightarrow q$

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- Let p denote "life gives you lemons" and q denote "make lemonade"
- Then $A = p \to q$

WAIT A MINUTE! "make lemonade" is not a declarative sentence, it is an order (imperative sentence, undefined truth value)

Conditional expressions in English

EX: Let A denote "if life gives you lemons, you [can] make lemonade".

- Let p denote "life gives you lemons" and q denote "you [can] make lemonade" (now q is a proposition)
- Then $A = p \rightarrow q$

Biconditional

Biconditional ($\leftrightarrow$) is a binary connective (it requires 2 propositions as input). It models the locution “... if and only if...”.

Its truth table is

A	B	$A \leftrightarrow B$
0	0	1
0	1	0
1	0	0
1	1	1

which means that the output proposition $A \leftrightarrow B$ is true when the 2 input propositions A and B have the same truth value, and false otherwise.

Biconditional expressions in English

The biconditional $A \leftrightarrow B$ is rendered in many ways:

- *A* if and only if *B*
- *A* implies *B* and conversely [or viceversa]
- If *A*, then *B* and conversely [or viceversa]
- *A* is necessary and sufficient for *B*
- *A* iff *B*

Biconditional expressions in English

EX: Let *A* denote "You will pass this course if and only if you attend classes, study the notes and do the textbook exercises".

- Let *p* denote "you will pass this course"
- Let *q* denote "you attend classes"
- Let *r* denote "you study the notes"
- Let *s* denote "you do the textbook exercises"
- Then $A \;=\; p \leftrightarrow q \wedge r \wedge s \;=\; p \leftrightarrow (q \wedge r \wedge s)$

Truth tables

- We can construct a truth table for any compound proposition.
- We need a row for any possible combination of truth values of its atoms
  - 1 atom -> 2 possibilities, 0 and 1 -> 2 rows
  - 2 atoms -> 4 possibilities, (0,0),(0,1),(1,0),(1,1) -> 4 rows
  - ...
  - k atoms -> 2^k possibilities -> 2^k rows
- We need a column for each intermediate sub-formula. In that column, we write the truth values of the sub-formula applying the semantics of the appropriate connective

Example: truth table of $(a \wedge b) \vee (a \wedge c)$

- We first list all possible combinations of truth values of the atoms appearing in the formula. In this case, 3 distinct atoms give $2^3=8$ rows.

a	b	c
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Example: truth table of $(a \wedge b) \vee (a \wedge c)$

- Then, using the semantics of the connectives, for each row we compute the truth values of increasingly complex subformulas

a	b	c	$a \wedge b$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

1) Application of $\wedge$ to the truth values of a and b

Example: truth table of $(a \wedge b) \vee (a \wedge c)$

- Then, using the semantics of the connectives, for each row we compute the truth values of increasingly complex subformulas

a	b	c	$a \wedge b$	$a \wedge c$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	0	0
1	0	0	0	0
1	0	1	0	1
1	1	0	1	0
1	1	1	1	1

2) Application of $\wedge$ to the truth values of a and c

Example: truth table of $(a \wedge b) \vee (a \wedge c)$

- Then, using the semantics of the connectives, for each row we compute the truth values of increasingly complex subformulas

a	b	c	$a \wedge b$	$a \wedge c$	$(a \wedge b) \vee (a \wedge c)$
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	0	0
0	1	1	0	0	0
1	0	0	0	0	0
1	0	1	0	1	1
1	1	0	1	0	1
1	1	1	1	1	1

3) Application of $\vee$ to the truth values of $a \wedge b$ and $a \wedge c$

Equivalent formulas

- 2 formulas *A* and *B are* logically equivalent if they always have the same truth value (that is, for any possible combination of truth values of their atoms, the truth values of *A* and *B* coincide).
- Notation: $A \equiv B$
- *In propositional logic* (but not in other logical systems), equivalence can be checked via truth tables: $A \equiv B$ if and only if the columns of truth values of A and B in a compound truth table coincide.

Example 1

- Is it true that $(a \wedge b) \wedge c \equiv a \wedge (b \wedge c)$?

a	b	c	$a \wedge b$	$(a \wedge b) \wedge c$	$b \wedge c$	$a \wedge (b \wedge c)$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	0	0	1	0
1	0	0	0	0	0	0
1	0	1	0	0	0	0
1	1	0	1	0	0	0
1	1	1	1	1	1	1

Example 2

- Is it true that $(a \to b) \to c \equiv a \to (b \to c)$?

a	b	c	$a \to b$	$(a \to b) \to c$	$b \to c$	$a \to (b \to c)$
0	0	0	1	0	1	1
0	0	1	1	1	1	1
0	1	0	1	0	0	1
0	1	1	1	1	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	0	0	0
1	1	1	1	1	1	1

Notable logical equivalences

- *Identity laws:* $A \wedge \top \equiv A$; $A \vee \bot \equiv A$
- *Domination laws:* $A \vee \top \equiv \top$; $A \wedge \bot \equiv \bot$
- *Law of non-contradiction:* $A \wedge \neg A \equiv \bot$
- *Law of excluded middle:* $A \vee \neg A \equiv \top$
- *Idempotent laws:* $A \wedge A \equiv A$; $A \vee A \equiv A$
- *Double negation law:* $\neg(\neg A) \equiv A$
- *Associative laws:* $A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$; $A \vee (B \vee C) \equiv (A \vee B) \vee C$
- *Commutative laws:* $A \wedge B \equiv B \wedge A$; $A \vee B \equiv B \vee A$
- *Distributive laws:* $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$; $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$
- *Absorption laws:* $A \wedge (A \vee B) \equiv A$; $A \vee (A \wedge B) \equiv A$
- *De Morgan's laws:* $\neg(A \wedge B) \equiv \neg A \vee \neg B$; $\neg(A \vee B) \equiv \neg A \wedge \neg B$

Notable logical equivalences

- $A \to B \equiv \neg B \to \neg A$
- $A \to B \equiv \neg A \lor B$
- $(A \to B) \land (A \to C) \equiv A \to (B \land C)$; $(A \to B) \lor (A \to C) \equiv A \to (B \lor C)$
- $(A \to C) \land (B \to C) \equiv (A \lor B) \to C$; $(A \to C) \lor (B \to C) \equiv (A \land B) \to C$
- $A \leftrightarrow B \equiv (A \to B) \land (B \to A)$
- $A \leftrightarrow B \equiv \neg A \leftrightarrow \neg B$
- $A \leftrightarrow B \equiv (A \land B) \lor (\neg A \land \neg B)$
- $\neg(A \leftrightarrow B) \equiv A \leftrightarrow \neg B \equiv \neg A \leftrightarrow B$

Mega exercise

- Using truth tables, prove each of the previously stated notable logical equivalences.

Transitivity of equivalence

- Logical equivalence is transitive: if $A_1 \equiv A_2$, $A_2 \equiv A_3, \ldots, A_{n-1} \equiv A_n$, *then* $A_1 \equiv A_n$ *[more on transitivity property in the next episodes].*
- This allows to prove that 2 formulas are logically equivalent without truth tables, using the notable logical equivalences.
- EX: prove that $p \vee (\neg p \wedge q) \equiv p \vee q$

$$
\begin{aligned}
p \vee (\neg p \wedge q) &\equiv (p \vee \neg p) \wedge (p \vee q) && \text{[distributive law]} \\
&\equiv \top \wedge (p \vee q) && \text{[law of excluded middle]} \\
&\equiv (p \vee q) && \text{[domination law]}
\end{aligned}
$$

Tautologies, contingencies, contradictions

- A tautology is a formula which is always true
  - EX: $p \vee \neg p$
- A contradiction is a formula which is always false
  - EX: $p \wedge \neg p$
- A contingency is a formula which is neither a tautology nor a contradiction (i.e., it can be both true and false)
  - EX: p, $p \rightarrow \neg p$

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$	$p \rightarrow \neg p$
0	1	1	0	1
1	0	1	0	0

- Remark: $A \equiv B$ if and only if $A \leftrightarrow B$ is a tautology.

Satisfiability

- A formula is said to be satisfiable if there is an assignment of truth values to its atoms that makes the formula true. When no such assignments exist, the formula is said to be unsatisfiable.
- Remark: A formula is satisfiable iff it is either a tautology or a contingency, iff it is not a contradiction. A formula is unsatisfiable iff it is a contradiction, iff it is neither a tautology nor a contingency.
- The Boolean satisfiability problem (SAT), that is the problem of determining whether a propositional formula is satisfiable, is a major hit in computer science:
  - it is the 1st problem that was proven to be NP-complete;
  - there is no known algorithm that efficiently solves SAT in general
  - There are several interesting heuristic algorithms for SAT which work well for special classes of formulas
  - SAT has concrete applications in AI, circuit design, cryptography, and automatic theorem proving.
- More details in CS2209! [check also the Wikipedia page]