Recall: $f: A \to B$ is one-to-one (or injective) if
$\neg \exists a_1 \in A\ \exists a_2 \in A\ (f(a_1) = f(a_2) \wedge a_2 \neq a_1)$
Equality: $\forall a_1, a_2 \in A\ (f(a_1) = f(a_2) \to a_1 = a_2)$
f is onto if: $\forall b \in B\ \exists a \in A\ (f(a) = b)$ (every element in range is taken)

Ex $A = \{1,2,3\}$, $B = \{4,5,6\}$. $f = \{(1,5),(2,4),(3,5)\}$.
* f is a function
is not injective : $f(1) = f(3)$.
not onto : $! \exists a \in A\ f(a) = 6$.

Ex2 any A, $i_A: A \to A$. i.e. $i(a) = a$.
one-to-one + onto

Ex3 $\mathbb{R} \to \mathbb{R}$, $f(x) = x^2$
injective ✗ onto ✗
$f(1) = f(-1)$ $\quad ! \exists x \in \mathbb{R}\ f(x) = -1$.

Ex4 $h: \mathbb{R} \to \mathbb{R}$ $h(x) = 2x+3$.
injective ✓ onto ✓
$g: \mathbb{Z} \to \mathbb{R}$ $g(x) = 2x+3$
✓ ✗ $g(x) \in \mathbb{Z}$

Ex 5.1.4. $A = \mathbb{R} \setminus \{-1\}$. $f: A \to \mathbb{R}$, $f(a) = \frac{2a}{a+1}$
Proof: f is one-to-one but not onto.
Goal: $a_1, a_2 \in A$ $f(a_1) = f(a_2) \to a_1 = a_2$.
Proof: let $a_1, a_2 \in A$. Assume $f(a_1) = f(a_2)$. So $\frac{2a_1}{a_1+1} = \frac{2a_2}{a_2+1}$,
therefore $2a_1a_2 + 2a_1 = 2a_1a_2 + 2a_2$, $a_1 = a_2$. □
onto: roughwork. $b \in B$. $f(a) = b$. $\frac{2a}{a+1} = b$.
$2a = ba + b$
$a = \frac{b}{2-b}$ if $b \neq 2$.
f is not onto: Assume $a \in A$ gives us $f(a) = 2$.
Then $\frac{2a}{a+1} = 2$, $0 = 2$, which is a contradiction,
no such a exist. So it is not onto □.

Theorem 5.2.5 Let $f: A \to B$, $g: B \to C$.
If f, g are one-to-one, then $g \circ f$ is one to one.
onto onto.

Proof 1: Suppose f, g are one-to-one
Let $a_1, a_2 \in A$, Assume $g \circ f(a_1) = g \circ f(a_2)$,
so $g(f(a_1)) = g(f(a_2))$. Since g is one-to-one. $f(a_1) = f(a_2)$
Since f is one-to-one, $a_1 = a_2$. So $g \circ f$ is one-to-one $\square$

2: Suppose f, g are onto. Let $c \in C$, $a \in A$. Since g is onto, $\exists\, b \in B$ that $g(b) = c$. Since f is onto, there exist $a \in A$ that $f(a) = b$, that is, $g(f(a)) = c$, so $g \circ f(a) = c$.

§5.3. inverse of functions.

$$f:\quad 1 \to 5,\quad 2 \to 4,\quad 3 \to 6 \qquad\qquad f^{-1}:\quad 4 \to 2,\quad 5 \to 1,\quad 6 \to 3$$

f^{-1} is a function.

Thm 5.3.1 Suppose $f: A \to B$. If f is one-to-one and onto, then $f^{-1}: B \to A$ is also a function.

Proof: Assume f is one-to-one and onto, Let $b \in B$.
Existence: Since f is onto, there is $a \in A$, $f(a) = b$.
so $(a, b) \in f$, $(b, a) \in f^{-1}$
Uniqueness: Assume $(b, a_1), (b, a_2) \in f^{-1}$, so $(a_1, b), (a_2, b)$ are in f. Since f is one-to-one, $a_1 = a_2$.