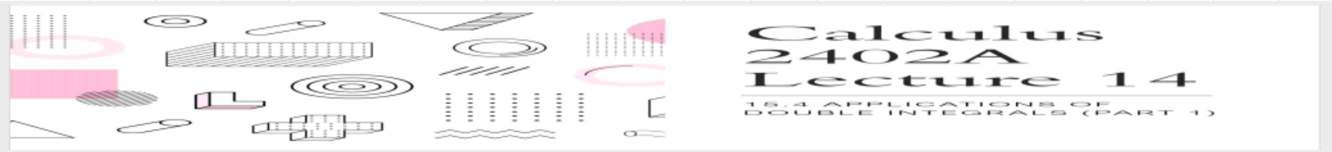
Lecture 14

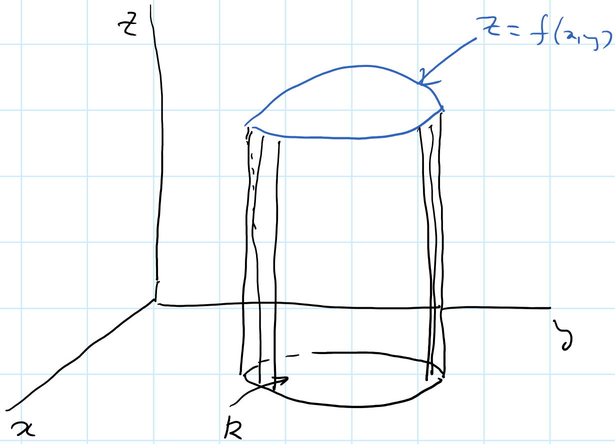
October 9, 2020 10:29 AM



Applications of double integrals (sec 15.4)

1. Volume of a solid

If $z = f(x,y) \geq 0$ defined over a region R then the volume V of the solid bounded above by $z = f(x,y)$ with base as the region R is



$$V = \iint_R f(x,y)\, dA$$

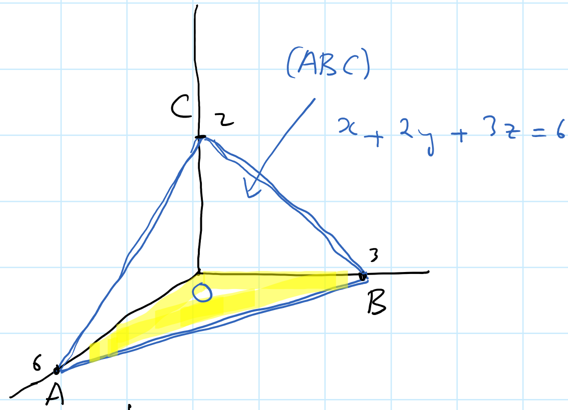
Ex1: Find the volume V of the tetrahydron with vertices $O(0,0,0)$, $A(6,0,0)$, $B(0,3,0)$ and $C(0,0,2)$.

Solution

The equation of the plane (ABC) is

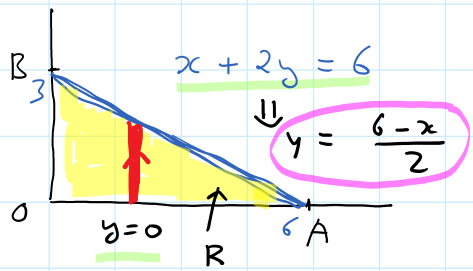
$$x + 2y + 3z = 6$$

$$\Rightarrow \quad z = \frac{6 - x - 2y}{3}$$



$$\therefore \quad V = \iint_R z\, dA$$

where R is the ΔOAB.



$$V = \int_0^6 \int_0^{(6-x)/2} \left(\frac{6 - x - 2y}{3}\right) dy\, dx$$

$$= \frac{1}{3} \int_0^6 \left((6-x)y - y^2\right)\Big|_{y=0}^{(6-x)/2} dx$$

$$= \frac{1}{3} \int_0^6 \left[(6-x)\left(\frac{6-x}{2}\right) - \left(\frac{6-x}{2}\right)^2\right] dx$$

$$= \frac{1}{3} \int_0^6 \left((6-x)^2 - (6-x)^2\right) dx$$

$$= \frac{1}{3}\int_0^6 \left(\left(\frac{6-x}{2}\right)^2 - \left(\frac{6-x}{4}\right)^2\right) dx$$

$$= \frac{1}{3}\int_0^6 \frac{(6-x)^2}{4}\, dx = \frac{1}{12}\int_0^6 (6-x)^2\, dx \qquad \boxed{\begin{array}{l} u = 6-x \\ du = -dx \end{array}}$$

$$V = \frac{1}{12}\int_6^0 u^2(-du) \underset{\substack{\uparrow \\ \text{Switching} \\ \text{the limits}}}{=} \frac{1}{12}\int_0^6 u^2\, du$$

$$= \frac{1}{12}\left(\frac{u^3}{3}\right)\Big|_0^6 = \frac{1}{12 \times 3}(6)^3 = 6 \qquad // \text{Ans.}$$

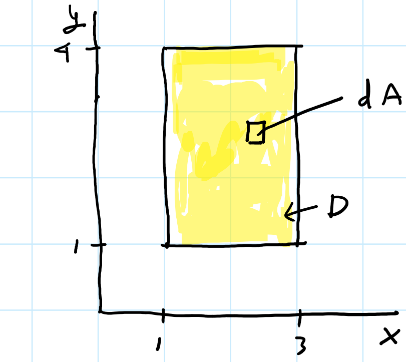
2. If $f(x,y) \equiv 1$ over the region R then $\iint_R f(x,y)\,dA$ is reduced to $\iint_R dA$ which is the area of R.

3. If $f(x,y) = \rho(x,y)$ which is the **mass density** of the lamina R then the mass m of the lamina is

$$m = \iint_R \rho(x,y)\, dA$$

Ex2: Find the mass of the lamina D defined by $1 \leq x \leq 3$, $1 \leq y \leq 4$ with the mass density $\rho(x,y) = ky^2$ (k is a +ve constant).

Solution



$$dm = \rho(x,y)\, dA$$
$$= (ky^2)\, dA$$

$$\therefore \quad m = \iint_D dm$$

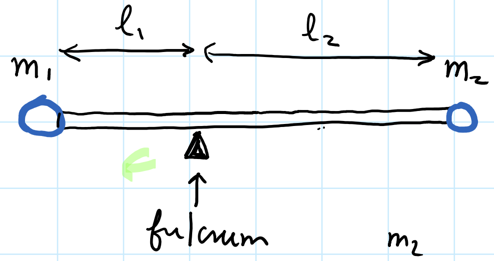
$$= \int_1^4 \left(\int_1^3 ky^2\, dx\right) dy$$

$$m = k \left(\underbrace{\int_1^3 dx}\right)\left(\int_1^4 y^2\, dy\right)$$

$$= k\,(3-1)\left(\frac{y^3}{3}\right)\Big|_1^4 = k\left(\frac{2}{3}\right)(4^3 - 1^3)$$

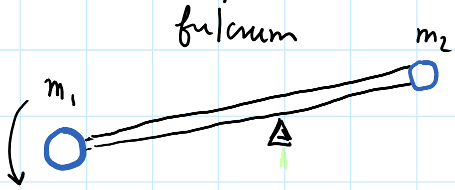
$$= \frac{2k}{3}(64-1) = \frac{2k}{3}(63) = \frac{2k}{\cancel{3}}(\cancel{9}^{\,3} \times 7) = 42k \qquad // \text{Ans.}$$

4. Moments, Centers of Mass (CM) and Centroid

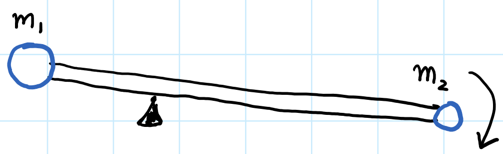


($m_1 > m_2$)

the rod is in balance



the rod has tendency to rotates in counter-clockwise direction

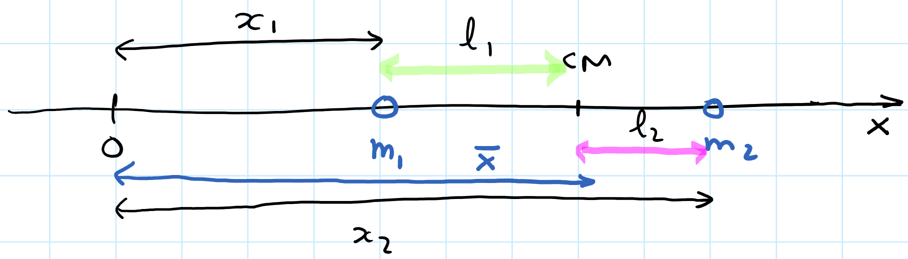


the rod has tendency to rotate in the clockwise direction

The rod is in balance when we have

$$\boxed{m_1 l_1 = m_2 l_2} \qquad (1)$$

When the rod is in balance, the position of the fulcrum is at the Center of Mass of the rod.



Using (1),

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

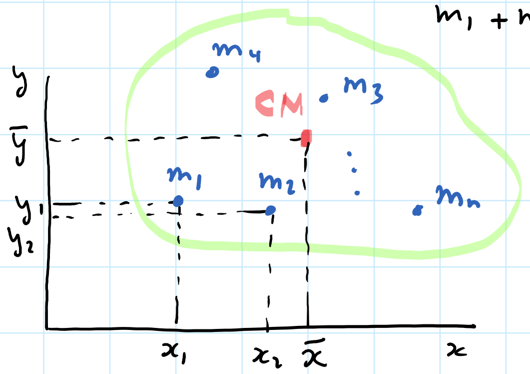
$$m_1\bar{x} - m_1 x_1 = m_2 x_2 - m_2 \bar{x}$$

$$(m_1 + m_2)\bar{x} = m_1 x_1 + m_2 x_2$$

$$\therefore \quad \boxed{\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}} \qquad (2)$$

If the system consists of n point masses $m_1, m_2, \dots, m_n$ located at $x_1, x_2, \dots, x_n$ on the x-axis then the CM of the system is located at $\bar{x}$ where located at $x_1, x_2, \dots, x_n$ on the x-axis then the CM of the system is located at $\bar{x}$ where

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \dots + m_n x_n}{m_1 + m_2 + \dots + m_n} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \qquad (3)$$



If these point masses $m_1, m_2, \dots, m_n$ are located at $(x_1,y_1), (x_2,y_2), \dots, (x_n,y_n)$ then the CM of the system is located at $(\bar{x},\bar{y})$

where

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \qquad (4a)$$

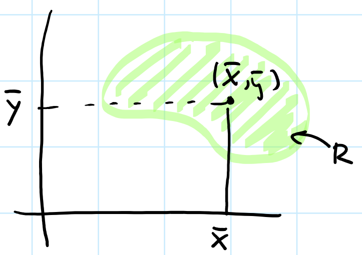
$$\bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i} \qquad (4b)$$

If a lamina has the mass density $\rho(x,y)$ continuous distributed on R then the summations in (4a) and (4b) become double integrals.

$$\sum m_i \longrightarrow \iint_R dm$$

$$\sum m_i x_i \longrightarrow \iint_R x\, dm$$

$$\sum m_i y_i \longrightarrow \iint_R y\, dm$$



$$\bar{x} = \frac{\underbrace{\iint_R x\, \rho(x,y)\, dA}_{=M_y}}{\iint_R \rho(x,y)\, dA} \quad (5a) \quad = \frac{M_y}{m}$$

$$\bar{y} = \frac{\underbrace{\iint_R y\, \rho(x,y)\, dA}_{=M_x}}{\iint_R \rho(x,y)\, dA} \quad (5b) \quad = \frac{M_x}{m}$$

If $\rho(x,y) = \rho$ a constant then the CM is called the

If $\rho(x,y) = \rho_0$ a constant then the CM is called the centroid of the lamina. In this case (5) becomes

$$\bar{x} = \frac{\iint_R x \, dA}{\iint_R dA} \qquad (6a)$$

$$\bar{y} = \frac{\iint_R y \, dA}{\iint_R dA} \qquad (6b)$$

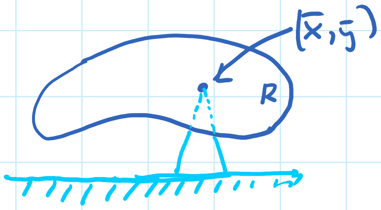
We note that $\iint_R dA$ is the area of the lamina R.

The quantity $M_y = \iint_R x \, \rho(x,y) \, dA$ is called the moment of the lamina R about the y-axis. Similarly,

$$M_x = \iint_R y \, \rho(x,y) \, dA$$

is called the moment of the lamina R about the x-axis

The physical meaning of the CM of a lamina is this - the entire mass of the lamina is concentrated at its CM. Hence, the lamina has its balance horizontally when supported at its CM.



The lamina is in balance if it is supported at the CM.

See you on Wednesday next week.