Final Report for Electrodynamics (II): Knots in Electromagnet Fields

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Abstract

In physics, the study of conservation laws is important in any field. In this report, we review the conservation of helicity in both magnetohydrodynamics (MHD) and topology perspectives. In barotropic and incompressible MHD, we show the conservation of helicity and field lines under the deformation by fluids. The closed magnetic field lines can be treated as knots, and the magnetic helicity conservation can be understood with the aid of the topological invariant. Additionally, we review Rañada's approach, which allows us to construct electromagnetic field solutions (electromagnetic Hopfions) by means of the Hopf fibration.

Contents

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1 Introduction

In the late 1800s, it was established that charged particles radiate if they are not in constant motion. This is now a well-known phenomenon in electrodynamics. However, the stability of the atoms seems to violate this behavior. Some physicists attempted to use an invariant quantity under time evolution in topology to explain this phenomenon: the linking number in knot theory. They believed the atoms can be treated as a non-trivial knot and remain stable due to the topology invariant. Although nowadays we understand the stability of the atoms through quantum mechanics, they still achieved some interesting results and applications, such as the connection between helicity and linking number, and the ball lightning phenomenon.

Since then, the development of the modern mathematical formalism for electrodynamics, together with the advance of other areas of physics and mathematics (as in differential geometry and quantum field theory), as allowed physicist to develop the notion of knots in electromagnetism. That is, constructions involving a non-trivial configuration of the field lines of the electric and magnetic fields. Hence, nowadays there are several ways to describe linked and knotted electromagnetic fields (cf., e.g., [1] for a very comprehensive review). This kind of knotted solutions, even though extremely complicated to realize experimentally, have generated a lot of interest in the physics community, bringing together concepts from knot theory, topology and several areas of physics. Indeed, currently, electromagnetic knots and Hopf solitons have evolved from mathematical curiosities to a very active area of research in plasma physics (for example in spheromaks and magnetohydrodynamics), quantum optics and topological systems, to name a few.

In this report we will attempt to provide a more general idea of knots in electromagnetism, going beyond limiting ourselves to a single formulation. The report is arranged as follows: In Sec. 2, we review properties of magnetohydrodynamics and derive the frozen-in field solutions and the conservation of magnetic helicity for high conductivity and incompressible limit. In section 3, we introduce the basic knowledge of knot theory and connect topological invariant quantities in knot theory to the magnetic flux and magnetic helicity. Lastly, in Section 4, we provide an overlook of the construction developed by Rañada and Trueba for electromagnetic knots. This approach is almost canonical, allowing for a complete physical construction of electromagnetic knot solutions from simple, but deep, mathematical concepts.

(In the following Einstein's summation convention will be used. The distinction between covariant and contravariant indices is a matter of convenience, since the metric tensor is trivial, i.e., the identity matrix.)

2 Magnetic Helicity in Magnetohydrodynamics (MHD)

The dynamics of a conducting fluid is governed by both the Maxwell equations and fluid mechanics. For plasma, magnetohydrodynamics (MHD) is applicable. The corresponding equations are as follows[4],

[4] Most of the content in this section comes from Chapter 2 to Chapter 6 of Webb's book[5]

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \\
&\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot \left[\rho \mathbf{u}\mathbf{u} + \left(p + \frac{B^2}{2\mu_0} I - \frac{\mathbf{B}\mathbf{B}}{\mu_0} \right) \right] = -\rho \nabla \phi, \\
&\frac{\partial \rho S}{\partial t} + \nabla \cdot (\rho \mathbf{u} S) = 0, \\
&\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) + \mathbf{u} \nabla \cdot \mathbf{B} = 0.
\end{aligned}
\tag{2.1}
$$

An ideal and incompressible fluid has many good mathematical properties. The development of topology and Lie algebra is deeply related to fluid mechanics and Euler equations, which leads to the invocation of knots (Sec. 3.1), Hopf fibration (Sec. 4.2) and all the relevant mathematics in the following sections. The incompressible limit is introduced in Sec. 2.2.

2.1 Magnetic Flux and Field Line Conservation

Flux Conservation

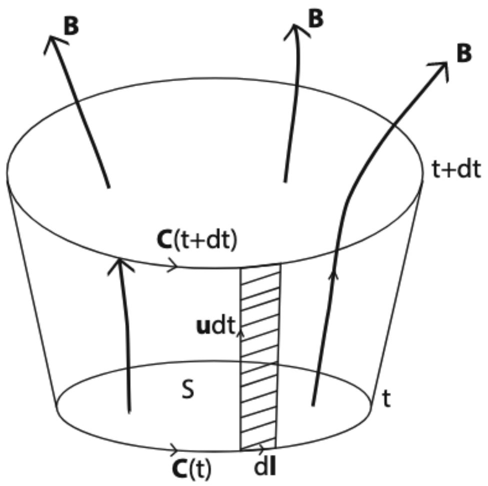


Figure 1: Magnetic flux in a volume spanned by a moving boundary. This picture is from the book: Webb G., Magnetohydrodynamics and Fluid Dynamics: Action Principles and Conservation Laws.

Start with Gauss's theorem,

$$
\int_V (\nabla \cdot \mathbf{B}) d^3x = \int \mathbf{B}(t) \cdot d\mathbf{S}_2 - \int \mathbf{B}(t) \cdot d\mathbf{S}_1 + \int \mathbf{B}(t) \cdot dl \times \mathbf{u} \Delta t, \tag{2.2}
$$

where $S_1 = S(t)$, and $S_2 = S(t + \Delta t)$. Calculate the rate of change of flux through S using Taylor expansion,

$$
\begin{aligned}
\frac{\Delta}{\Delta t} \left(\int \mathbf{B} \cdot d\mathbf{S} \right) &= \frac{1}{\Delta t} \left(\int \mathbf{B}(t + \Delta t) \cdot d\mathbf{S}_2 - \int \mathbf{B}(t) \cdot d\mathbf{S}_1 \right) \\
&\approx \frac{1}{\Delta t} \left(\int \mathbf{B}(t) \cdot (d\mathbf{S}_2 - d\mathbf{S}_1) \right) + \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}_2.
\end{aligned}
\tag{2.3}
$$

Substitute Eq. 2.2 into Eq. 2.3,

$$\begin{aligned}
\frac{d\Phi_B}{dt} &= \lim_{\Delta t\to 0}\frac{\Delta}{\Delta t}\left(\int \mathbf{B}\cdot d\mathbf{S}\right)\\
&= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\int_V(\nabla\cdot\mathbf{B})d^3x\right) - \int \mathbf{B}(t)\cdot dl\times\mathbf{u} + \int\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{S}_2\\
&= \int(\nabla\cdot\mathbf{B})\mathbf{u}\cdot d\mathbf{S} - \int\mathbf{B}(t)\cdot dl\times\mathbf{u} + \int\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{S}.
\end{aligned} \tag{2.4}$$

The rate of change of magnetic flux moving with the flow is

$$\frac{d\Phi_B}{dt} = \int_S[\frac{\partial\mathbf{B}}{\partial t} + \mathbf{u}(\nabla\cdot\mathbf{B}) - \nabla\times(\mathbf{u}\times\mathbf{B})]\cdot d\mathbf{S} = 0. \tag{2.5}$$

The Faraday's equation(the last equation of Eq. 2.1) makes the conservation of magnetic flux moving with the flow in Eq. 2.5 is zero. Thus, the magnetic flux is conserved even for the most general magnetohydrodynamical system.

Affine Parameter and Line Preservation

A fluid motion is called **line preserving** if two fluid particles initially on the same line remain on the same line at a later time throughout their motion. Define the motion along the field line as $\mathbf{X} = \mathbf{x}(\lambda, t)$, where λ is an affine parameter. To understand the meaning of it, we can start with the concept of geodesic. An trajectory $\mathbf{X}$ on a manifold T is

$$\mathbf{X}(u^1(x,y,z), u^2(x,y,z)) = \mathbf{X}(\lambda), (x,y,z)\in T, \tag{2.6}$$

where u^1 and u^2 are the coordinates at a specific point of the manifold, and λ is the parameterization of a curve on the manifold T. The accelaration along the curve is

$$\begin{aligned}
\frac{d^2\mathbf{X}}{d\lambda^2} &= \left(\frac{d^2\mathbf{X}}{d\lambda^2}\right)_{tangent} + \left(\frac{d^2\mathbf{X}}{d\lambda^2}\right)_{normal}\\
&= \left(\frac{d^2u^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{du^\nu}{d\lambda}\frac{du^\rho}{d\lambda}\right)\frac{\partial\mathbf{X}}{\partial u^\mu} + L_{\nu\rho}\frac{du^\nu}{d\lambda}\frac{du^\rho}{d\lambda}\hat{n},
\end{aligned} \tag{2.7}$$

where $\Gamma^\mu_{\nu\rho}$ is the Christoffel symbol and $L_{\nu\rho}$ is called the second fundamental form. A curve is a **geodesic** if it satisfies the geodesic equation

$$\frac{d^2u^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}(\mathbf{x})\frac{du^\nu}{d\lambda}\frac{du^\rho}{d\lambda} = 0, \tag{2.8}$$

which is a special case of the covariant directional derivative. An **affine parameter** λ is the parameterization of a geodesic. For every λ, the tangent vector $\dot{x}^\mu$ is parallel transported along the curve[5]. In Fig. 2, consider two close points $\mathbf{X}_A$ and $\mathbf{X}_B$. The change rate of the distance $\delta\mathbf{X}$ is

$$\frac{d}{dt}\delta\mathbf{X} = \frac{(\nabla\mathbf{u})\,\delta\mathbf{X}\delta t}{\delta t} = (\nabla\mathbf{u})\,\delta\mathbf{X}, \tag{2.9}$$

[5] I learned the term affine parameter from geodesic equations, but unfortunately, after some studies, the field-lines satisfying field line preservation are not necessary to be geodesics.

which is the same equation as Eq. 2.18. Thus, once the vector field $\mathbf{B}/\rho$ is parallel to $\delta\mathbf{X}$ at some time t_0, i.e., $\frac{\mathbf{B}}{\rho}(\mathbf{x}, \mathbf{t_0}) = c_0\delta\mathbf{X}(t_0)$, two infinitely close fluid particle parameterized by A and B will stay on the same line of force for all time, and it is natural to write down $\mathbf{B}$ satisfies the relation $\mathbf{X}(\lambda) \times \mathbf{B} = 0$. With this condition, the field line preservation condition can be derived

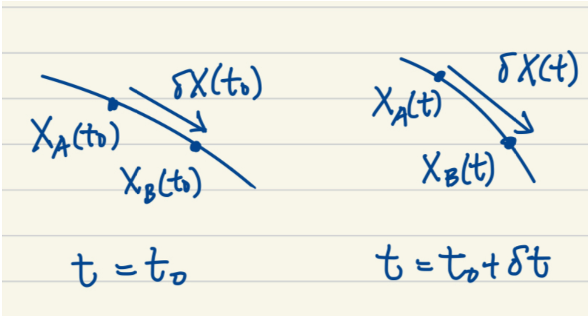


Figure 2: Illustration of two points $\mathbf{X}_A$ and $\mathbf{X}_B$ at different time t_0 and t, respectively.

$$\frac{d}{dt}(\mathbf{X}_\lambda \times \mathbf{B}) = \lambda\mathbf{B} \times \left(\frac{\partial\mathbf{B}}{\partial t} + [\mathbf{u}, \mathbf{B}]\right) = 0, \tag{2.10}$$

where $[\mathbf{u}, \mathbf{B}]^i = (\mathbf{u} \cdot \nabla\mathbf{B} - \mathbf{B} \cdot \nabla\mathbf{u})^i$. After some mathematical massage,

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_u\right)(\mathbf{B} \cdot d\mathbf{S}) = [\frac{\partial\mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) + \mathbf{u}(\nabla \cdot \mathbf{B})] \cdot d\mathbf{S} = 0. \tag{2.11}$$

This implies that field line preserving is equivalent to $\mathbf{B} \cdot d\mathbf{S}$ Lie dragged with the flow. In general $\mathbf{B}$ can be any vector field; for our case $\mathbf{B}$ is the magnetic field, and the right-hand side of Eq. 2.11 is zero because of the Faraday's equation. Two **remarks**: First, in Eq. 2.11, a flux preserving motion gurantees that field line preserving holds, but a flied line preserving motion doesn't imply the Faraday's equation is satisfied; secondly, a **frozen-in field** is an example of field line preserving. In Sec. 2.2, we will see that in the incompressible limit, the Faraday's equation is reduced to the EOM for frozen-in fields, who has a category called **knotted fields**.

2.2 Incompressible limit

As mentioned at the beginning of Sec. 2, the properties of fluid is also part of the plasma physics. For those who are not familiar with fluid mechanics, it's necessary to build up the intuitions such that one knows to what extent the knotted field can be deformed. The equation of states for an ideal gas $p = p(\rho, S)$ can be written as

$$\frac{dp}{dt} = \frac{\partial p}{\partial\rho}\frac{d\rho}{dt} + \frac{\partial p}{\partial S}\frac{dS}{dt}. \tag{2.12}$$

With the condition $\frac{dS}{dt} = 0$[6] and the continuity equation $\frac{d\rho}{dt} + \rho\nabla \cdot \mathbf{u} = 0$,

$$\frac{dp}{dt} = \frac{\partial p}{\partial\rho}\frac{d\rho}{dt} \equiv a^2\frac{d\rho}{dt} = a^2\rho\nabla \cdot \mathbf{u}, \tag{2.13}$$

[6]This is called the "isentropic flow".

where $a^2 = \frac{\partial p}{\partial \rho}$[7] is the square of sound velocity. The **incompressible limit** $a^2 \to \infty$, or equivalently, the velocity of disturbance $v_d \ll 1$, means that $\nabla \cdot \mathbf{u}$ is asked to approach zero

$$\nabla \cdot \mathbf{u} \to 0 \tag{2.14}$$

in order to keep the right-hand side of Eq. 2.13 finite. If it was not the case, the dramatic change of thermodynamic variables might cause unexpected effects[8]. Another perspective to understand the incompressible limit is from the bulk modulus B, which is related to the sound velocity a by

$$B = a^2 \rho = -V \frac{\partial p}{\partial V}, \tag{2.15}$$

Because of mass conservation, $a^2 \to \infty$ implies that an infinitesimal volume change requires an relatively enormous change of pressure, and the fluid is therefore incompressible. Eq. 2.1 reduces to

$$\begin{aligned}
&\nabla \cdot \mathbf{u} = 0, \\
&\nabla \cdot \mathbf{B} = 0, \\
&\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho_0} \nabla \left(p + \frac{B^2}{2\mu_0} \right) + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{\mu_0 \rho_0}, \\
&\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) = 0.
\end{aligned} \tag{2.16}$$

Expand the term $\nabla \times (\mathbf{u} \times \mathbf{B})$ in Eq. 2.16 as

$$\nabla \times (\mathbf{u} \times \mathbf{B}) = -(\mathbf{u} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{u} - \mathbf{B} (\nabla \cdot \mathbf{u}), \tag{2.17}$$

where the right-hand side terms are the effects of edvection, stretching, and compression[9], respectively. Combine the Faraday's equation of Eq. 2.16 with continuity equation, such that it can be written as

$$\frac{d}{dt} \left(\frac{\mathbf{B}}{\rho} \right) - \left(\frac{\mathbf{B}}{\rho} \right) \cdot \nabla \mathbf{u} = 0, \tag{2.18}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla)$. The solution for Eq. 2.16 is a field "**frozen**" in the fluid, or called the **Cauchy solution** for the magnetic field,

$$B_i(\mathbf{X}, t) = \left(det^{-1} G \right) B_j(\mathbf{x}, 0) \frac{\partial \mathbf{X}_i}{\partial \mathbf{x}_j}, \tag{2.19}$$

where the magnetic field at $(\mathbf{X}, t)$ is determined by the convection the field from $(\mathbf{x}, 0)$ and the deformation tensor $\frac{\partial \mathbf{X}_i}{\partial \mathbf{x}_j}$[7], and $det^{-1} G = \frac{\rho(\mathbf{X},t)}{\rho(\mathbf{x},0)}$[8]. For those fields described by Eq. 2.19, their knots and links in the field-line structure are conserved.

[7] There's a sign difference in Blundell's book[11]. The sign change could come from the default direction of pressure.

[8] For instance, turbulence, or anything that will make the system lose its good properties like quasi-equilibrium. This is my personal intuition on physics

[9] The compression term will be dropped, since we are discussing the incompressible limit.

Physical Meaning of the Ideal MHD condition

Sometimes the unknowns are more than the equations, and constitutive relations need to be invoked. The choice of constitutive relation depends on the problems. One of the choices is importing microscopic physics, i.e., the Ohm's law,

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{2.20}$$

If $\sigma \to \infty$, to keep the current $\mathbf{J}$ finite, the condition

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} \tag{2.21}$$

called an **ideal MHD condition** is required. It is called "ideal" for the zero resistivity. Plugged into Faraday's law $\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$, this condition gives an EOM for the magnetic field,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \tag{2.22}$$

Here are two **remarks**. First, Eq. 2.22 is exactly the same as what we get in Eq. 2.16, which implies that the Ohm's law as a constitutive relation is consistent with the physics we have discussed so far. Secondly, plasma is often considered a perfect conductor, so through Eq. 2.21 we know that the actual dynamical field in astrophysics (or plasma physics) is the magnetic field.

2.3 Helicity Conservation in Fluids

Start from the Euler's equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{\nabla p}{\rho} - \nabla \Phi. \tag{2.23}$$

With the first thermodynamic law

$$-\frac{1}{\rho}\nabla p = T\nabla S - \nabla h \tag{2.24}$$

and the identity

$$\mathbf{u} \cdot \nabla \mathbf{u} = -\mathbf{u} \times \omega + \frac{1}{2}\nabla |\mathbf{u}|^2, \tag{2.25}$$

we get an equivalent Euler's equation

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \omega = T\nabla S - \nabla \left(h + \Phi + \frac{1}{2}|\mathbf{u}|^2 \right). \tag{2.26}$$

Taking the curl of Eq. 2.26,

$$\frac{\partial \omega}{\partial t} - \nabla \times (\mathbf{u} \times \omega) = \nabla T \times \nabla S. \tag{2.27}$$

Combine the two equations,

$$\frac{\partial(\mathbf{u}\cdot\omega)}{\partial t} + \nabla\cdot\left[(\mathbf{u}\cdot\omega)\mathbf{u} + \left(h + \Phi + \frac{1}{2}|\mathbf{u}|^2\right)\omega\right] = \omega\cdot T\nabla S + \mathbf{u}\cdot\nabla T\times\nabla S. \tag{2.28}$$

For barotropic and ideal fluids, that is, the equation of state is $p = p(\rho)$, $\nabla S = 0$, we get the continuity equation for magnetic helicity.

$$\frac{\partial h_f}{\partial t} + \nabla\cdot\left[h_f\mathbf{u} + \left(h + \Phi + \frac{1}{2}|\mathbf{u}|^2\right)\omega\right] = 0, \quad h_f = \mathbf{u}\cdot\omega. \tag{2.29}$$

h_f is the helicity density and $\omega = \nabla\times\mathbf{u}$ is the vorticity.[10]

2.4 Magnetic Helicity Conservation law

From Eq. 2.29, we define the magnetic helicity density

$$h_m = \mathbf{A}\cdot\mathbf{B}. \tag{2.30}$$

With Faraday's equation, Eq. 2.29 can be rederived as the continuity equation for magnetic helicity,

$$\frac{\partial h_m}{\partial t} + \nabla\cdot[\mathbf{u}h_m + \mathbf{B}(\phi_E - \mathbf{u}\cdot\mathbf{A})] = 0. \tag{2.31}$$

The total magnetic helicity and its total time derivative are

$$\begin{aligned}
H_M &= \int_V \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\rho}\right)\rho d^3x, \\
\frac{dH_M}{dt} &= \int_V \frac{d}{dt}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{\rho}\right)\rho d^3x \\
&= \int_V \frac{\mathbf{B}}{\rho}\cdot\nabla\left(\mathbf{A}\cdot\mathbf{u} - \phi_E\right)\rho d^3x \\
&= \int_V \nabla\cdot[\mathbf{B}\left(\mathbf{A}\cdot\mathbf{u} - \phi_E\right)]d^3x \\
&= \int_{\partial V} [\mathbf{B}\cdot\mathbf{n}\left(\mathbf{A}\cdot\mathbf{u} - \phi_E\right)]d\mathbf{S}.
\end{aligned} \tag{2.32}$$

where we use the continuity equation $\frac{d}{dt}(\rho d^3x) = 0$. To make $\frac{dH_M}{dt} = 0$, one can ask $\mathbf{B}\cdot\mathbf{n} = 0$ at every point of the boundary ∂V, but this is not generically true. An alternative way is to choose a proper gauge

$$\Lambda = \int^t (\phi_E - \mathbf{u}\cdot\mathbf{A})dt', \tag{2.33}$$

The vector potential $\tilde{\mathbf{A}}$ after the gauge transformation $\tilde{\mathbf{A}} = \mathbf{A} + \nabla\Lambda$ forms a **Lie dragged with the flow**[11] $\tilde{\mathbf{A}}\cdot d\mathbf{x}$, and the conservation law of magnetic helicity Eq. 2.31 is reduced to a simpler form

[10] Kelvin's theorem and Ertel's theorem are two important statements regarding the vorticity of ideal fluids. But the importance of these two theorems is more subtle. For now let us skip them.

[11] Lie dragged with the flow = **advected invariant** = the Lagrangian derivative of some geometrical object is zero.

$$\frac{\partial \tilde{h}}{\partial t} + \nabla \cdot (\tilde{h}\mathbf{u}) = 0. \tag{2.34}$$

It turns out that $\mathbf{B} \cdot \mathbf{n} = 0$ is not a necessary condition for the conservation of magnetic helicity. **Remark**: When the magnetic vector potential $1 - form\ \alpha = \tilde{A} \cdot d\mathbf{x}$ is an advected invariant, the magnetic helicity

$$I_h = \int_V \tilde{A} \cdot (\nabla \times \tilde{A}) d^3x \propto l \tag{2.35}$$

is the **Hopf invariant of the magnetic field**[12], where l is the **linking number**. Because of the concept of continuous deformation in topology, knots can be found in various field theories, including electromagnetism. Some of the phenomena could be explained or designed through knotted electromagnetism, including tokamaks, plasma fireballs, or ball lightning[10].

2.5 Taylor's relaxation hypothesis

Taylor's relaxation hypothesis is that, "in a high conductivity plasma, the total magnetic helicity to lowest order is conserved during turbulent magnetic reconnection"[13]. Eq. 2.31 can be generalized to the cases of finite conductivity

$$\frac{\partial h_m}{\partial t} + \nabla \cdot [\mathbf{u}h_m + \mathbf{B}(\phi_E - \mathbf{u} \cdot \mathbf{A}) + \frac{\mathbf{J} \times \mathbf{A}}{\sigma}] = -2\frac{\mathbf{J} \cdot \mathbf{B}}{\sigma}. \tag{2.36}$$

Notice that when $\sigma \to \infty$, Eq. 2.36 is reduced to the helicity continuity equation Eq. 2.31. Together with the Poynting's theorem

$$\frac{\partial w_B}{\partial t} + \nabla \cdot \left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_0}\right) = \mathbf{J} \cdot \mathbf{E} = \mathbf{J} \cdot \left(-\mathbf{u} \times \mathbf{B} + \frac{\mathbf{J}}{\sigma}\right), \tag{2.37}$$

one can find the change rate of energy and helicity,

$$\begin{aligned} \frac{dH_M}{dt} &\sim -2 \int \frac{\mathbf{J} \cdot \mathbf{B}}{\sigma} d^3\mathbf{x} = -2\mu_0 \eta \int_V \mathbf{J} \cdot \mathbf{B} d^3\mathbf{x}, \\ \frac{dW_B}{dt} &\sim \int \frac{\mathbf{J}^2}{\sigma} d^3\mathbf{x} = -\mu_0 \eta \int_V \mathbf{J}^2 d^3\mathbf{x}. \end{aligned} \tag{2.38}$$

The decay rate of energy and helicity in momentum space is written as

$$\frac{\dot{H}_k}{\dot{W}_k} \sim 2\sqrt{\eta/\omega}\mu_0. \tag{2.39}$$

In high conductivity limit, the plasma diffusivity $\eta = \frac{1}{\mu_0 \sigma} \to 0$.

A Cute Example: Ball lightning

When a lightning ionizes the air, it creates a nearly unbounded space of charged fluid in which electromagnetic knots could survive. M. Berry suggested Rañada and Trueba to model ball lightning using electromagnetic knots. With Taylor's relaxation hypothesis, we are allowed

[12] Moffatt and Ricca, 1992

[13] Quoted from the book[5].

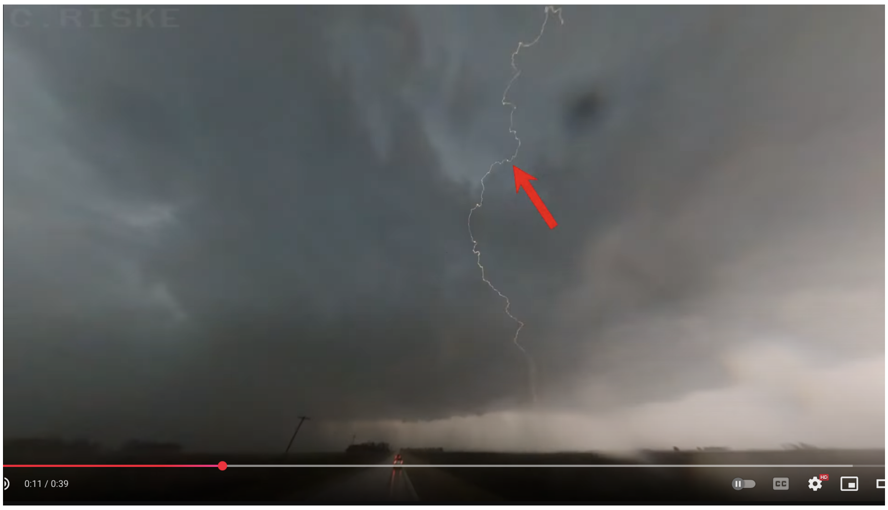


Figure 3: A ball lightning moving with air even after the normal lightning disappears. (Video Link)

to describe the dissipation of ball lightning using the linking number n. The characteristic decay time[10] of the ball lightning is

$$\tau_l = (l^2+1)\tau_0 = (l^2+1)\frac{L_0^5 a}{20\pi^2\sigma^*\beta^4}, \tag{2.40}$$

where l is the linking number. The time dependence of the temperature is

$$T(t) = T(0)(1+\frac{t}{\tau_l})^{-\frac{2}{5}}. \tag{2.41}$$

2.6 A Brief Summary

In Sec. 2, we start from the basic properties of magnetohydrodynamics (MHD) to see the deformation of fields which are advected by the fluids. In a non-viscous and incompressible fluid, we derive the frozen-in field solutions. One class of this solutions is the knotted fields. We also derive the conservation of magnetic helicity and briefly mention the gauge choice for the MHD problems. This section provides the physical motivation and intuition for the study of linking number (Hopf invariant), knot theory, and differential geometry in the following sections. Within the context, we also imply the close relation among fluid mechanics(hydrodynamics), geodesics, and topology[14]. In this report, we skipped the concept of **field line action principle**[5], which may be important for characterizing the magnetic field braiding[15]. After derivation, you'll see the action depends on the magnetic helicity $\mathbf{A}\cdot\mathbf{B}$[16].

[14] To see more details please refer to the ref.[6].
[15] A. R. Yeates, and G. Hornig
[16] We skip the parts on cross helicity and non-local conservation law, which might also be important for characterizing the topology.

3 From Knot Theory to Helicity

In Sec 2, we already saw the magnetic flux and helicity conservation based on different conditions in MHD. In this section, we discuss the magnetic flux and helicity in topology direction; magnetic flux and helicity are related to the topological invariant quantities in knot theory called linking number and self-linking number, respectively. These quantities are based on the diffeomorphism, which is a map that plays a similar role to the dynamics of the ordered system in physics.

3.1 Introduction to the knot theory

Knot theory[17, 3, 18] in geometric topology is the study of knots and links, inspired by the dairy string with two connected ends. It shows us how to classify the knots and transform among them. For the continuity of the report, please read the Appendix A for the additional topology and differential geometry terminology from this section.

Definition: Knot

A knot K is an embedding of the S^1 in $\mathbb{R}^3$, or the projection of the knot on $\mathbb{R}^2$ with crossing order, as shown in Fig. 4a.

Definition: Link

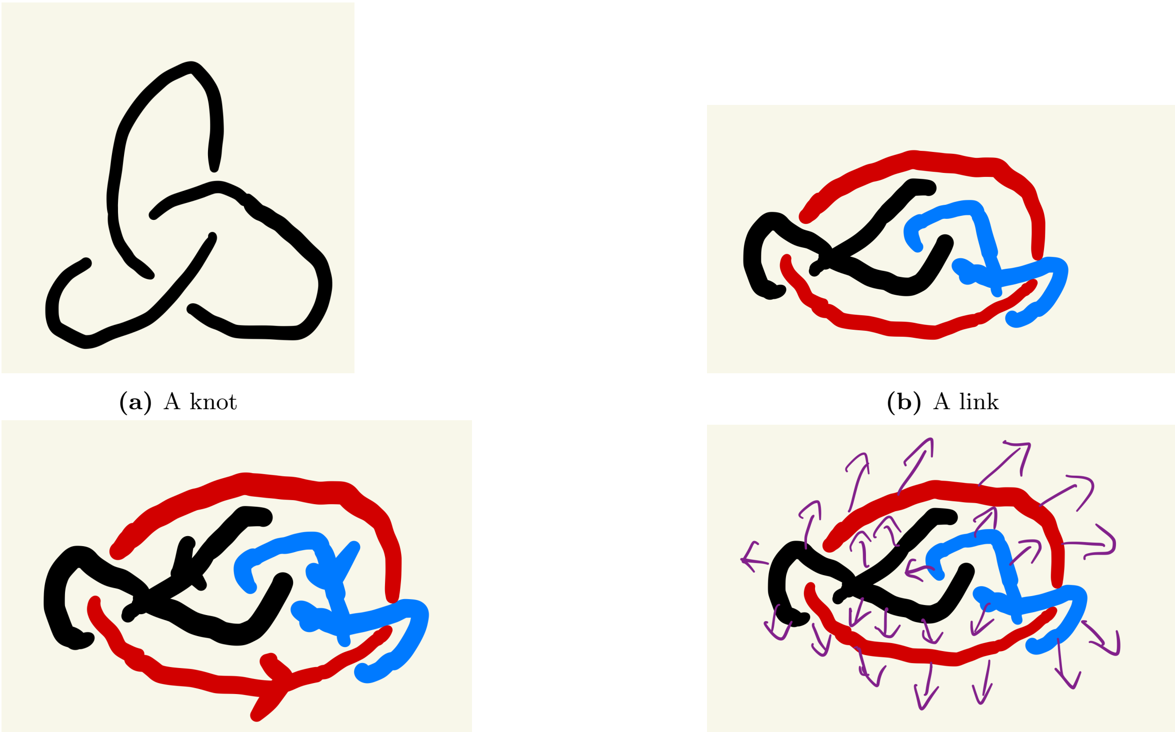
A link L of the components m consists of m disjoint knots, as shown in Fig. 4b. It is common to consider a link with orientation and call it an oriented link, as shown in Fig. 4c. In addition, considering a link L equipped with a vector field v, and all points $p \in L$ satisfy $v_p \notin T_pL$, and call this link a framed link, as shown in Fig. 4d.

Definition: Two oriented links L_1 and L_2 in $\mathbb{R}^3$ are equivalent or (ambient) isotopic if there is an orientation-preserving diffeomorphism[17] $\{h_t : \mathbb{R}^3 \to \mathbb{R}^3 \text{ for } t \in [0,1]\}$ with identity h_0 and $h_1(L_1) = L_2$. Similarly, two framed and oriented links are (regular) isotopic if there is a framing and orientation-preserving diffeomorphism. Fig. 5 shows two examples of non-isotopic links or knots.

To simplify the problem, two oriented links are in the same isotopy class if and only if they can be transformed by a sequence of **Reidemeister moves** to each other. The Reidemeister moves have four kinds of moves, as shown in Fig. 6. The Type 0 Reidemeister move is simply bending the links without changing any crossings. The other three Reidemeister moves only modify part of the link: Type I twists part of the link to create or eliminate a self-kink, Type II creates or annihilates a pair of crossings, and Type III shifts the position of a pair of crossings. The link diagrams connected by the Reidemeister moves are ambient isotopic. However, Type I breaks for framed links will be excluded from the regular isotopy and replaced by Type I'.

Now is the time to explore the topology invariant quantities – the linking number and self-linking number (writhe).

[17]Isotopy is usually defined by homotopy, as detailed in the Appendix. However, in knot theory, we usually use diffeomorphisms[3] or homeomorphisms[17] to define isotopic, which are more strict than homotopy.



(a) A knot

(b) A link

(c) A oriented link

(d) A framed link

Figure 4: Example of a knot and a link.

Definition: Linking number

$$L(L_1, L_2) = \frac{1}{2} \sum_{p_{12}} \mathrm{sign}(p_{12}) \tag{3.1}$$

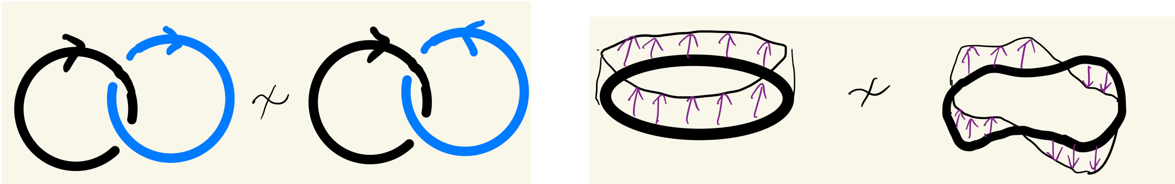
where p_{12} is the crossing point between oriented links L_1 and L_2 with signs as shown in Fig 7.

Definition: Self-linking number (writhe)

$$w(L) = \sum_{p} \mathrm{sign}(p) = \sum_{i \neq j} L(K_i, K_j) + \sum_{i} w(K_i) \tag{3.2}$$

where L is an oriented link with m components $L = \bigcup_{i=1}^{m} K_i$ and p is the crossing point in the oriented link L with signs as shown in Fig 7.

It is straightforward to see that the linking number is invariant under all Reidemeister moves, so it is a topological invariant for the oriented link. However, the self-linking number, or writhe, is the sum of all crossing points, which is invariant under all Reidemeister moves if only considering the case of Type I', so it is a topological invariant for framed links.



(a) Two Hopf links with different orientations are not (ambient) isotopic.

(b) Two unknots equipped with different vector fields are not (regular) isotopic.

Figure 5: Example of non-isotopic knots or links.

3.2 Connection between magnetic flux and linking number

Here we follow Baez's reference[3] to argue the connection between flux and linking number, but note that the argument is not rigorous. First, due to the $\nabla \times \mathbf{A} = \mathbf{B}$ and $\nabla \cdot \mathbf{B} = 0$ with magnetic field B and magnetic vector potential A, it's common to consider a divergence-free 2-form magnetic field B and 1-form magnetic vector potential A that satisfies $B = dA$ and $dB = 0$ in differential form language. The magnetic flux Φ_B due to the Stokes–Cartan theorem can be represented in both

$$\Phi_B = \int_S B = \int_{K'} A. \tag{3.3}$$

Next, if we imagine the incoming and outcoming flux to the surface, the $\int_S B$ is nothing but the intersection number $I(K, S)$ that sums over all of the intersection points p between surface S and knot K with signs as shown in Fig. 8a.

$$\int_S B = I(K, S) = \sum_{p \in K \bigcap S} \mathrm{sign}(p). \tag{3.4}$$

Finally, if we sketch the boundary loop K', we can connect the linking number and the intersection number due to the relation between two crossings and one intersection point p with the correct sign as shown in Fig. 8.

$$\sum_{p \in K \cap S} \mathrm{sign}(p) = L(K, K'). \tag{3.5}$$

This is true for all Reidemeister moves. Therefore, as long as knots K do not break, change the orientation, or cross the given boundary K' during time evolution, the magnetic flux is conserved.

Last but not least, the Gauss linking integral also characterizes a similar relationship[3, 22, 5]

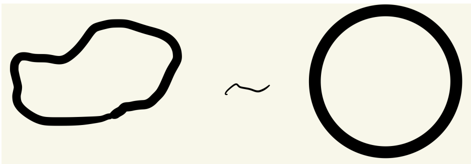
$$L(K, K') = \oint_K \oint_{K'} \frac{1}{4\pi} \frac{(d\mathbf{l} \times d\mathbf{l}') \cdot (\mathbf{l} - \mathbf{l}')}{|\mathbf{l} - \mathbf{l}'|^3}. \tag{3.6}$$

Gauss linking integral can easily be derived from the Biot-Savart law

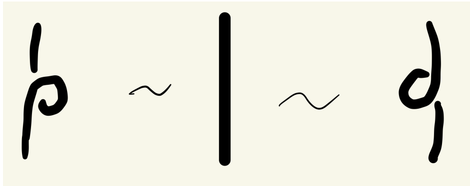
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint_{K'} \frac{d\mathbf{l}' \times (\mathbf{r} - \mathbf{l}')}{|\mathbf{r} - \mathbf{l}'|^3} \tag{3.7}$$

and Ampère's law

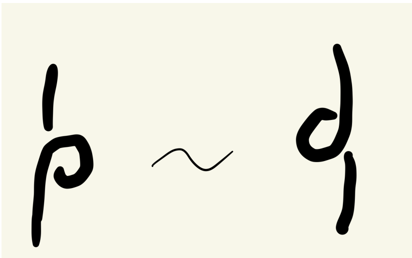
$$\mu_0 I_{enc} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S} = \int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \oint_{K=\partial S} \mathbf{B}(\mathbf{l}) \cdot d\mathbf{l} \tag{3.8}$$



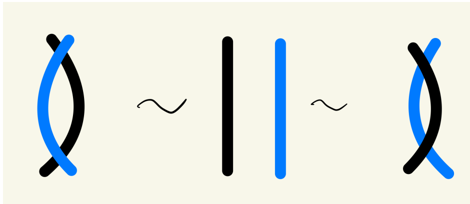
(a) Type 0: Isotopy for both oriented links and framed links.



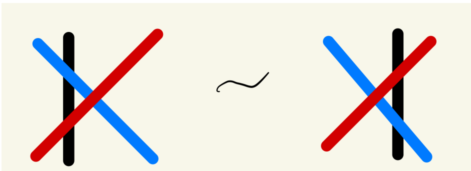
(b) Type I: Isotopy for oriented links but not for framed links.



(c) Type I': Isotopy for both oriented links and framed links.



(d) Type II: Isotopy for both oriented links and framed links.



(e) Type III: Isotopy for both oriented links and framed links.

Figure 6: Reidemeister moves.

because I_{enc} is nothing but the net current or the number of currents crossing the surface S, which is $L(K,K')I$. After rewriting $\oint_K \mathbf{B}(\mathbf{l}) \cdot d\mathbf{l}$ to $\int_{K'} \star B$ and $\nabla \times \mathbf{B} = \mathbf{J}$ to $d \star B = \star J$, if we change $\star J \to B$ and $\star B \to A$, then everything matches the magnetic flux and the linking number[18].

3.3 Connection between magnetic helicity and self-linking number

To see the connection between magnetic helicity and self-linking number, let's consider a picture[3]: all kinds of divergence-free fields can be covered by a series of solid tori $T_i \backsimeq S_i^1 \times D_i^2$ as shown in Fig. 9. Based on this picture, we let T_i cover the B_i field, and separate the A and B fields by different T_i

$$A = \sum_i A_i, \;\; B = \sum_i B_i, \;\; \text{and } dA_i = B_i \tag{3.9}$$

Furthermore, each B_i in T_i can be represented in

$$B_i = f_i(r_i, \theta_i) dr_i \wedge d\theta_i \tag{3.10}$$

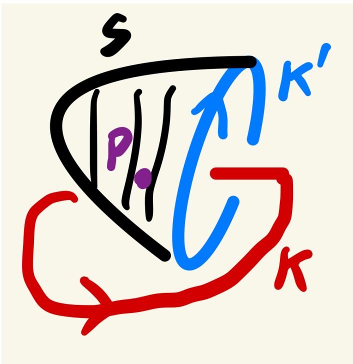
with

$$\int_{D^2} B_i = \int_{D^2} f_i(r_i, \theta_i) dr_i d\theta_i = 1 \tag{3.11}$$

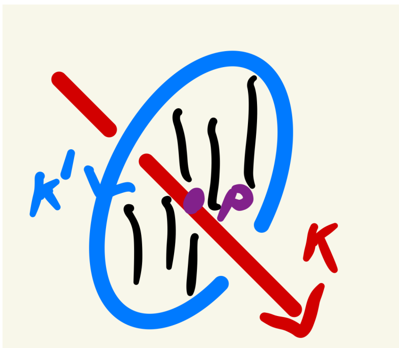
[18] It is ambiguous to connect a concrete relationship from the reference[3].

(a) sign(p)=1. **(b)** sign(p)=-1.

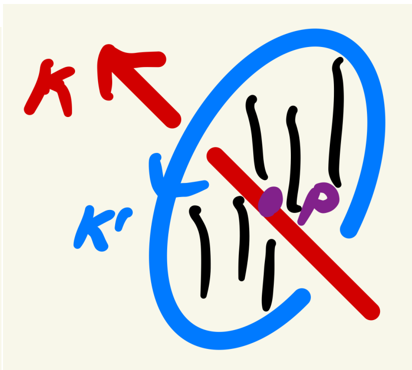
Figure 7: Signs of the orientation. For the linking number $L(L_1, L_2)$, the crossing point p_{12} only considers the points where the blue line and the black line belong to different links. As for the self-linking number $w(L)$, the crossing point p considers all of the points regardless of whether the blue line and the black line belong to the same knot.



(a) Intersection point $p \in K \bigcap S$.



(b) $\text{sign}(p) = L(K, K') = 1$.



(c) $\text{sign}(p) = L(K, K') = -1$.

Figure 8: Signs of the intersection point.

where (t_i, r_i, θ_i) is the T_i coordinate for S^1 and D^2 respectively. We can also separate the magnetic helicity h_m by different T_i

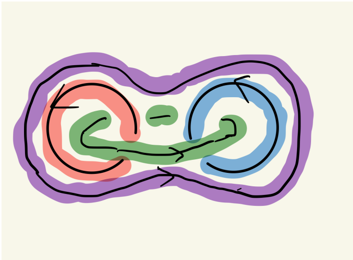
$$h_m = \int_{\mathbb{R}^3} A \wedge B = \sum_{i \neq j} \int_{\mathbb{R}^3} A_j \wedge B_i + \sum_i \int_{\mathbb{R}^3} A_i \wedge B_i. \tag{3.12}$$

Since the B_i is now the local fields, for simplicity, we can change the integration region from $\mathbb{R}^3$ to T_i and expand the inside A_j

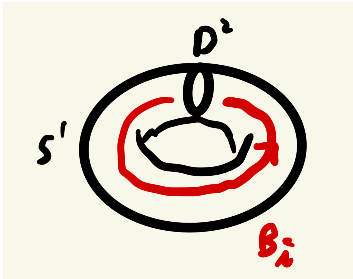
$$A_j = (A_j)_t dt_i + (A_j)_r dr_i + (A_j)_\theta d\theta_i. \tag{3.13}$$

Thus, the first term becomes

$$\begin{aligned} \int_{\mathbb{R}^3} A_j \wedge B_i &= \int_{T_i} (A_j)_t f_i(r_i, \theta_i) dt_i \wedge dr_i \wedge d\theta_i \\ &= \oint_{S_i^1} (A_j)_t dt \int_{D_i^2} f_i(r_i, \theta_i) dr d\theta \\ &= L(K_i, K_j). \end{aligned} \tag{3.14}$$



(a) B field is covered by a series of solid tori (coloring sections).



(b) Torus $T_i \backsimeq S_i^1 \times D_i^2$ and local B_i field.

Figure 9

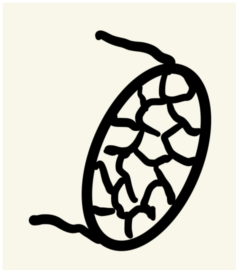


Figure 10: The cross-section of subdivided T_i.

As for the second term, let's consider the picture: each T_i can be subdivided into n tori $T_{i,\alpha}$ as shown in Fig. 10. Again, we can separate the A_i and B_i fields by different $T_{i,\alpha}$

$$\int_{T_i} A_i \wedge B_i = \sum_{\alpha \neq \beta} \int_{T_i} A_{i,\beta} \wedge B_{i,\alpha} + \sum_{\alpha} \int_{T_i} A_{i,\alpha} \wedge B_{i,\alpha}. \tag{3.15}$$

The first term is again the linking number $L(K_{i,\alpha}, K_{i,\beta})$, but the second term is zero if we subdivide the T_i to infinity since the second term only contains n terms which are infinitely smaller than the n^2 number of the first term. As a result, we can follow the Eq. 3.2

$$w(K_i) = \sum_{\alpha \neq \beta} L(K_{i,\alpha}, K_{i,\beta}), \tag{3.16}$$

and the magnetic helicity is the self-linking number of the link L that contains all knots $\{K_i\}$

$$h_m = \sum_{i \neq j} L(K_i, K_j) + \sum_i w(K_i) = w(L). \tag{3.17}$$

In summary, if the knots K_i do not break, change the orientation or framing during time evolution, the magnetic helicity is conserved.

4 Electromagnetic knots: Rañada's approach

Now that we have a clearer picture of what knots are and how their study is relevant in Electrodynamics, let us for now provide an intuitive definition of electromagnetic knots:

Definition: Electromagnetic knot [2]:

Electromagnetic knots are defined as "standard classical electromagnetic waves in empty space, characterized by two integers n_m, and n_e, such that all the force lines are **closed** and any pair of magnetic (electric) lines are linked with multiplicities n_m (n_e).

That is, "any pair of magnetic lines or any pair of electric lines are a link". Incidentally, we must point out that the self-linking of the knotted configurations described in this section is null.

The previous the definition is provided by Rañada and Trueba in their seminal 1995 work. Due to the simplicity and almost pedagogical nature of this work, we will devote this section of the report to explain in detail how this construction works. Indeed, Rañada's work on electromagnetic knots is a prime example of how, at times, the mathematical formalism brings about the physics in the most natural way.

We note that, in keeping the original paper's convention, we will make use of natural units throughout this discussion.

4.1 Geometric aspects of electromagnetic knots

In this section we will establish the mathematical formalism necessary for Rañada and Trueba's approach to construct electromagnetic knots. While the discussion is certainly formal, it provides a beautiful example of how differential geometry can help us materialize very concrete physical phenomena. We will attempt to keep the discussion as self-contained as possible, relegating some of the more technical details to the annex.

We will begin by providing a simple definition of a **bundle**.

Definition: *Bundle*[3]

A bundle is the triple (E, M, π), where E and M are manifolds, referred to respectively as the **total space** and the **base space**; and a surjective **projection** $\pi : E \longrightarrow M$.

Sometimes, the fiber bundle is directly identified with its defining projection π. While more technical definitions of a fiber bundle exist, this one will suffice to capture the general idea.

Perhaps one of the most relevant examples of a non-trivial fiber bundle, and one that will be central in our discussion, is the **Hopf fibration**. This is a special case of a fiber bundle in which the total space is $E = S^3$, the four-dimensional sphere, and the base space is $M = S^2$. A Hopf fibration is therefore implemented by the map $\phi : S^3 \to S^2$. This is also sometimes referred to as the **Hopf map**.

An important observation is that the set of smooth maps ϕ can be classified in homotopy classes, labeled by the **Hopf invariant** (or Hopf index); a topological invariant $h \in \mathbb{Z}$.

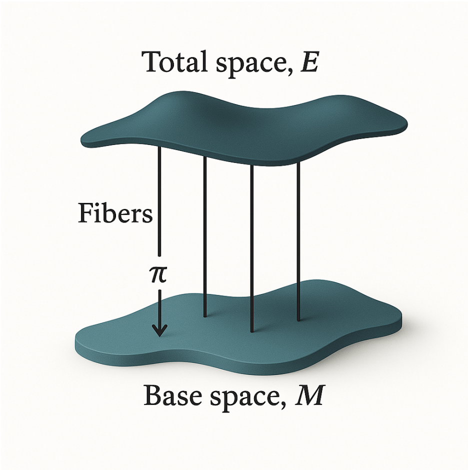


Figure 11: A schematic visualization of a fiber bundle.

In 1947, John H. C. Whitehead proposed an integral expression for the Hopf invariant which, in our case, can be expressed as[13]:

$$h = \int_{S^3} \alpha \wedge \omega \in \mathbb{Z} \tag{4.1}$$

where ω is the pullback of the area 2-form in S^2. Cohomology guarantees the existence of a 1-form α such that $\omega = d\alpha$.

As we have previously explained, the linking number gives us how many times two disjoint closed curves wind around each other. Crucially, the Hopf invariant is related to the linking number by the relation

$$h = \mu^2 l \tag{4.2}$$

where l is the linking number, and μ is the **multiplicity**, i.e., the number of connected components of a fiber.

It must be noted that in Rañada's paper the linking numbers n_e and n_m is identified with the **Hopf indices**. Also, quoting [2]: "the sign of n_m indicates the axiality of the curling of the magnetic lines". Indeed, as we saw in the previous part of this report, the sign of the linking is related to the way the curves curl around an axis.

4.2 Electromagnetic fields from Hopf fibrations in Rañada's proposal

In our attempt to characterize electromagnetic knots, we build on the aforementioned concept of the Hopf fibration. In fact, a concrete way to realize the defining projection of the Hopf fibration $\phi : S^3 \to S^2$ is as follows:

By definition, the stereographic projection allows us to establish the following diffeomorphisms:

$$\begin{cases} \mathbb{R}^3 \cup \{\infty\} \cong S^3 & \text{(Compactified } \mathbb{R}^3\text{)} \\ \mathbb{C} \cup \{\infty\} \cong S^2 & \text{(Riemann sphere)} \end{cases} \tag{4.3}$$

In order for the projections ϕ to uniquely implement the stereographic projection, they must be taken to be asymptotically isotropic **complex scalar fields** $\phi = \phi(\mathbf{r})$. [4]

Theorem: Let us then take an arbitrary 1-form (in $d = 3$)

$$\alpha = -A^i dx_i \tag{4.4}$$

such that $\omega = d\alpha$, where $\mathbf{A}$ is a rotational field that we define to be $\mathbf{B} = \nabla \times \mathbf{A}$. Then, if for the pullback of the area 2-form in S^2

$$\omega = \frac{1}{2} F_{ij}\, dx^i \wedge dx^j \tag{4.5}$$

we take

$$F_{ij} = \frac{1}{2\pi i} \frac{\partial_i \phi^* \partial_j \phi - \partial_i \phi \partial_j \phi^*}{(1 + \phi^* \phi)^2} \tag{4.6}$$

the vector field $\mathbf{B}$ is given by

$$B_i = -\frac{1}{2} \varepsilon_{ijk} F_{jk} \tag{4.7}$$

and equation 4.1 can be written as

$$h = \int_{\mathbb{R}^3} d^3r\ \mathbf{A} \cdot \mathbf{B} \tag{4.8}$$

The proof of this theorem is beyond the purpose of this report, but can be easily found in the literature (cf. e.g., [2],[4]). However, the general idea should be clear: the formalism of differential forms allows us to construct what we will identify with the electromagnetic tensor and the magnetic field. This in turn provides us with a new, more tractable expression, for the Hopf invariant in terms of physical quantities. Moreover, as it will soon become clear, the Hopf invariant carries distinct physical meaning.

Definition: The integral 4.28 is a topological invariant (the Hopf invariant) that we term the **helicity** of the vector $\mathbf{B}$.

Corollary: *The vector field* $\boldsymbol{B}$ *is divergenceless* ($\nabla \cdot \mathbf{B} = 0$).

重點 》 By now it should be clear that the vector field $\mathbf{B}$ behaves very much like the magnetic field. **Every value of the magnetic field corresponds to a point in space of the complex fiber projections**. Crucially, these projections are complex scalar fields with "only one value at infinity". While the asymptotic behavior of these complex fields is far from trivial, it is integral for our setup to be mathematically well-defined.[4]

4.3 Electromagnetic knots construction

Now that the more mathematical aspects of the framework have been laid out, we proceed to explain how electromagnetic knots are constructed in this setup.

As we saw in the previous section, the Hopf invariant allows us to define a tensor, given by equation (4.6). This tensor takes the complex projections ϕ of the Hopf fibration as an input, from which we can construct a vector

$$B_i = -\frac{1}{2}\varepsilon_{ijk}F_{jk}\,, \quad \text{such that} \quad \begin{cases} \mathbf{B} = \nabla \times \mathbf{A} \\ \nabla \cdot \mathbf{B} = \mathbf{0} \end{cases} \tag{4.9}$$

The fact that this vector has all the properties of the magnetic field is far from a coincidence. Indeed, our electromagnetic knots can be defined through this construction. Let us proceed:

Firstly we consider two different projections of the Hopf fibration

$$\phi,\, \theta : S^3 \to S^2 \tag{4.10}$$

from which we can define the tensors

$$f_{ij}(\phi) := \sqrt{a}\, F_{ij}(\phi)\,, \quad f_{ij}(\theta) := \sqrt{a}\, F_{ij}(\theta) \tag{4.11}$$

where $a \in \mathbb{R}^+$ is dimensionless in natural units. [19]

This, in turn, allows us to immediately build the electric and magnetic vectors

$$B_i(\mathbf{r}) = -\frac{1}{2}\varepsilon_{ijk}\, f_{jk}(\phi)\,, \quad E_i(\mathbf{r}) = \frac{1}{2}\varepsilon_{ijk}\, f_{jk}(\theta) \tag{4.12}$$

Note that this construction implies that the magnetic field satisfies $\nabla \cdot \mathbf{E} = 0$, as would be expected of a vacuum solution of Maxwell's equations. Moreover, in the same manner that the magnetic field is obtained from the vector potential as $\mathbf{B} = \nabla \times \mathbf{A}$, the electric field can also be derived from an electric vector potential, that is, $\mathbf{E} = \nabla \times \mathbf{C}$.

However, this discussion, no matter how simple or elegant, has been purely mathematical in nature. In order for it to have physical meaning the Hopf fibration construction must satisfy Maxwell's equations. While we have already determined that the electromagnetic fields are indeed divergenceless, we still have to address the remaining Maxwell equations

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \quad \nabla \times \mathbf{B} = \partial_t \mathbf{E} \tag{4.13}$$

In order for these equations to be satisfied the Cauchy problem for the electromagnetic field must be well-defined. To accomplish this we must establish the **Cauchy data**, given by $\mathbf{E}(\mathbf{x}, t = 0)$ and $\mathbf{E}(\mathbf{x}, t = 0)$, which we identify with the electromagnetic fields as given by equation (4.12).

There is however an additional, non-trivial, subtlety: In our construction, we are identifying the tensor components $f_{ij}(\phi)$ with the spatial components of the electromagnetic tensor $F_{\mu\nu}$, and the $f_{ij}(\theta)$ with those of the Hodge dual $*F_{\mu\nu}$. However, it can be shown [2] that, if we wish to express the electromagnetic tensor as $F_{\mu\nu} = f_{\mu\nu}(\phi)$ and its dual as $*F_{\mu\nu} = f_{\mu\nu}(\theta)$, the fibers ϕ and θ must be orthogonal , i.e.,

$$\Big[\nabla \phi^*(\mathbf{r}) \times \nabla \phi(\mathbf{r})\Big] \cdot \Big[\nabla \theta^*(\mathbf{r}) \times \nabla \theta(\mathbf{r})\Big] = 0 \tag{4.14}$$

[19] As the paper point out, the units of a are $\hbar c a$ if the units are re-introduced.

The proof that this is indeed a necessary and sufficient condition can be found in Rañada's paper "*Topological electromagnetism*" [4].

Crucially, this relation guarantees that the "duality" between the scalar fields ϕ and θ is preserved over time, allowing us to obtain the electromagnetic field from the initial conditions (4.12). Indeed, the time dependent fields $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ may be propagated from the Cauchy data via Fourier transform. The explicit expressions for the time dependent electric and magnetic fields can be found in [14]. Therefore, specifying the Cauchy data is enough to determine the fields at any point in time and will by construction, immediately satisfy Maxwell's equations.

A slightly more technical fact is that this condition ensures that the Faraday tensor $F_{\mu\nu}$ and its dual $*F_{\mu\nu}$ are always pullbacks of the $2-$form in S^2 and the helicities are constants of motion. As a consequence, it follows that $\mathbf{E} \cdot \mathbf{B} = 0$ is preserved under time evolution. This condition is a well-known relativistic invariant and consistent with radiation propagating in a vacuum. As we will later see, the orthogonality condition (4.14) is in fact responsible for the **topological invariance of the helicities**.

4.4 Knot construction

So far we have been able to build electromagnetic fields from the projections (the complex scalar fields) of the Hopf fibration. However, we have so far conspicuously avoided providing a concrete characterization As one would imagine, these maps are far from arbitrary, and are responsible for giving rise to the knotted structures.

Needless to say that finding suitable projections ϕ (or θ) is not trivial. An example of such a map with helicity $h = 1$ is

$$\phi_H = \frac{2(x + iy)}{2z + i(r^2 - 1)} := \frac{\beta}{\gamma} \tag{4.15}$$

where $r^2 = x^2 + y^2 + z^2$. This relationship is sometimes referred to as the **Hopf map**. Interestingly, we may use equation (4.15) to construct additional complex fields with other helicities. Indeed, let us define

$$\phi^{(n,k)} = \frac{\beta^n}{\gamma^k} \tag{4.16}$$

where $n, k \in \mathbb{N}/n < 2k$. Then, the Hopf invariant of such a map is $h = nq$, where q is the maximum common divisor of n and k.

Now, for our discussion we focus our attention on those maps with the functional form

$$\phi^{(n)} = (\phi_H)^n \tag{4.17}$$

By construction they have linking number $l = 1$ and multiplicity $\mu = n$, so that their Hopf index is $h = n^2$.

Now, in order to build the ϕ and θ fields from the Hopf map we introduce a the dimensionless scale $(X, Y, Z) = \lambda(x, y, z)$, where the parameter λ is arbitrary and has units of inverse length. We then define

$$\phi(\mathbf{r}) = \phi_H(X, Y, Z), \quad \theta(\mathbf{r}) = \phi_H(Y, Z, X) \tag{4.18}$$

where the arguments of each map reflect their orthogonality, and by definition both maps have helicity 1. From equation (4.12), it is uncomplicated to write down the initial conditions for the electromagnetic field $\mathbf{E}^{(1)}(\mathbf{r},0)$ and $\mathbf{B}^{(1)}(\mathbf{r},0)$:

$$\mathbf{E}^{(1)}(\mathbf{r},0)=\frac{4\lambda^2a^{1/2}}{\pi(1+R^2)^3}\begin{pmatrix}X^2-Y^2-Z^2+1,\\ 2(XY-Z),\\ 2(XZ+Y)\end{pmatrix},\quad \mathbf{B}^{(1)}(\mathbf{r},0)=\frac{4\lambda^2a^{1/2}}{\pi(1+R^2)^3}\begin{pmatrix}2(Y-XZ),\\ -2(X+YZ),\\ X^2+Y^2-Z^2-1\end{pmatrix} \tag{4.19}$$

where $R^2=X^2+Y^2+Z^2$. Although not particularly enlightening, it is worth noting that the expression for the $\mathbf{E}^{(1)}$ and $\mathbf{B}^{(1)}$ is remarkably simple. As previously pointed out, both fields are orthogonal.

By making use of the previous geometric description, we may then define the magnetic and electric helicities from their respective fields and potentials

$$h_m=\int_{\mathbb{R}^3}d^3r\;\mathbf{A}\cdot\mathbf{B},\quad h_e=\int_{\mathbb{R}^3}d^3r\;\mathbf{C}\cdot\mathbf{E} \tag{4.20}$$

Then, if we build up the general electromagnetic fields as

$$\mathbf{B}^{(m)}(\mathbf{r},0)=m\mathbf{B}^{(1)},\quad \mathbf{E}^{(n)}(\mathbf{r},0)=n\mathbf{E}^{(1)} \tag{4.21}$$

we find that the respective helicities of these field configurations is

$$h_m=am^2,\quad h_e=an^2 \tag{4.22}$$

where a is the normalization constant that we introduced in equation (4.11). Their linking number is still $l=1$.

It is worth noting that, while the multiplicities m and n scale the strength of the magnetic and electric fields respectively, the Hopf index is proportional to the helicity. In this way, m^2 and n^2 quantify how many times magnetic/electric field lines are linked. We therefore define an **electromagnetic knot** as the electromagnetic configurations $K_{m^2n^2}$ where, in general, $m,n\in\mathbb{N}$. However, it is also possible that either m or n are null, leaving either the electric or the magnetic fields unlinked.

4.5 Invariance of the Hopf index

From a more general standpoint, solutions characterized by a non-null Hopf invariant, are sometimes broadly referred to as **"Hopf solitons"**, or "**Hopfions**" [16]. The $K_{m^2n^2}$ knots discussed in this report correspond to the simplest topologically non-trivial solution, in which the helicity of the base Hopf map (4.15) is $h=1$. These solutions, built from the Hopf fibration are more specifically referred to as the **electromagnetic Hopfion** [1]. [20] Needless to say that more complex Hopfions exist, and their study is particularly relevant when discussing topological solitons.

[20] Other sources reserve the term "Hopfion" altogether for soliton solutions in which the Hopf index is the unity, cf. e.g., [15]. This shows that the nomenclature is not as consistent as one would hope.

It is worth noting that fact that the Hopf index is a topological invariant preserves the topology of the knotted configurations under time evolution. Interestingly, this can be shown to follow as a direct consequence of the relativistic invariance of electrodynamics. Let us recall that there are well-known two relativistic invariants related to the electromagnetic field. These are the **null field conditions**

$$\mathbf{E} \cdot \mathbf{B} = 0, \quad E^2 - B^2 = 0 \tag{4.23}$$

We previously saw that the orthogonality of the electric and magnetic fields follows from the orthogonality condition of the fibers, equation (4.14). The second condition can be verified explicitly from (4.19), and is guaranteed to hold on grounds of the theory being Lorentz invariant.

A less trivial fact is that the null field conditions allow us to express the helicities of the electromagnetic Hopfion in terms of the $\mathbf{E}$ and $\mathbf{B}$ fields. Indeed, it can be shown [1] that the time evolution of the electric and magnetic helicities can be expressed in terms of the fields as

$$\frac{dh_m}{dt} = -\int d^3r \, \mathbf{E} \cdot \mathbf{B} = 0 \tag{4.24}$$

$$\frac{dh_e}{dt} = \int d^3r \, \mathbf{E} \cdot \mathbf{B} = 0 \tag{4.25}$$

This shows that the helicities are constant upon time evolution and therefore a true invariant of the construction. This expression also highlights the high degree of symmetry between the $\mathbf{E}$ and the $\mathbf{B}$ fields in this formulation. As a consequence, the helicities

$$h_m = h_m(t = 0) = \int d^3r \, \mathbf{A}(\mathbf{r}, t = 0) \cdot \mathbf{B}(\mathbf{r}, t = 0) \tag{4.26}$$

$$h_e = h_e(t = 0) = \int d^3r \, \mathbf{C}(\mathbf{r}, t = 0) \cdot \mathbf{E}(\mathbf{r}, t = 0) \tag{4.27}$$

are fully determined from the Cauchy data.

4.6 A note on gauge freedom

As it was very appropriately pointed out by one of the authors, the gauge choice for the magnetic vector potential is far from trivial. Indeed, the Hopf invariant

$$h = \int_{\mathbb{R}^3} d^3r \, \mathbf{A} \cdot \mathbf{B} \tag{4.28}$$

is, a priori, not gauge invariant. Under a gauge transformation $\mathbf{A} \to \mathbf{A} + \nabla\chi$ we find an additional boundary contribution

$$h \longrightarrow h + \int_{\partial\mathbb{R}^3} d\sigma^2 \, \chi \, (\mathbf{B} \cdot \mathbf{n}) \tag{4.29}$$

Gauge invariance is nevertheless intuitively true for localized (well-behaved) fields. This is indeed the case for Rañada's construction, e.g., $\mathbf{B} \propto r^{-6}$ as can be construed from (4.12).

Ultimately, this arises as a consequence of us demanding asymptotic isotropy of the $\phi(\mathbf{r})$ fields.

In Rañada's work[2], the Coulomb gauge is chosen as a matter of convenience. This is clear from electric and magnetic potentials associated to equation (4.19), which can be chosen to be

$$\mathbf{C}(\mathbf{r},0) = D \begin{pmatrix} 1 \\ -Z \\ Y \end{pmatrix}, \quad \mathbf{A}(\mathbf{r},0) = D \begin{pmatrix} Y \\ -X \\ -1 \end{pmatrix} \tag{4.30}$$

where D is just a constant that we take to be $D = \frac{2\lambda\sqrt{a}}{\pi(1+R^2)^2}$. This is obviously consistent with the Coulomb gauge, $\nabla \cdot \mathbf{C} = \nabla \cdot \mathbf{A} = 0$.

Professor Spinrath pointed out that the gauge invariance of the helicity should be true for $U(1)$, but not necessarily for more general gauge groups. This matches the notion that Rañada's paper is consistent with the quantization of the electromagnetic field, and not inconsistent with the idea that the boundary conditions imposed on the theory are reflected on its symmetry groups.

Notably, this gauge dependence is explicitly manifest in some specific contexts, as in **Magnetohydrodynamics**. For a detailed exposition of this subject, we refer the reader to [5], chapter 6.

5 Summary and Conclusions

In this report, we have tried to provide a general, broader perspective of knotted configurations in electrodynamics, exploring some of the different intersections of topology, knot theory, and electrodynamics that underpin this specific area of research. Thus, instead of limiting ourselves to a single approach, we have worked out an approach that provides a more holistic description. In doing so, we have hopefully been able to give an idea of how rich the topic really is from both the physical and mathematical standpoint.

We also want to point out that, while the exposition has mainly focused on the underlying theoretical concepts, the experimental and phenomenological aspects are no less interesting. Indeed, the experimental realization of electromagnetic configurations with non-trivial helicities is notoriously complicated, and only partial progress has been made in this regard, with research still ongoing. While we have limited ourselves to a more formal exposition of the subject in order to keep the report self-contained, future work should address the experimental progress made in this topic.

In addition, during the elaboration of this report, we ran into a number of topics that could not be addressed in the present work. Some of these are turbulence, choice of torus(Lagrangian map), gauge choice, fluid relabeling symmetry, knots description for topological charge[5]. Moreover, in the discussion of the electromagnetic Hopfion, some of the technical details have been omitted or sketched out at places for the sake of brevity and conciseness. In order to bridge this limitation, relevant references containing the mathematical proofs have been provided wherever suitable. The relationship of these solutions to the quantization of the electromagnetic field has also been omitted in order to avoid introducing additional formalism and in an attempt to keep within the scope of ”classical” electrodynamics. The discussion is, however, as simple as it is fascinating.

A Appendix Section

In this appendix section, we will present the definition or proper statement of the terminology that is mentioned in the main body, also including some related terminology. Note that some of the definitions or statements might have some slight differences in different references[19, 20, 23, 21], which causes the summarizing difficulty, but the idea might be correct.

A.1 Algebra

- **Isomorphism**: An invertible morphism. Morphisms can be thought of as structure-preserving maps. Given two spaces M and N we write $M \cong N$.
- **Alternating multilinear map** (Alt): A map from Alt : $f \to \mathrm{Alt}(f)$ is defined by antisymmetrizing the multilinear map $f : V^n \mapsto U$ from the tensor product of vector space V to vector space U

$$\mathrm{Alt}(f)(x_1 \dots x_{i_n}) = \sum_i \mathrm{sgn}(I) f(x_1 \dots x_{i_n}),\ I = (i_1 \dots i_n) \tag{A.1}$$

  where sign sgn(I) is $+1$ for even permutation and -1 for odd permutation. The multilinear map with $\mathrm{Alt}(f) = f$ is called the alternating multilinear form.

A.2 Topological terminology

- **Homeomorphism** (**topology isomorphism**): A **bijective**, continuous map f between topological spaces X and Y

$$f : X \mapsto Y, \tag{A.2}$$

  such X and Y are called **homeomorphic**.
- **Homotopy**: A continuous map between two maps $f, g : X \mapsto Y$

$$h_t : X \mapsto Y,\ \text{for } t \in [0, 1] \tag{A.3}$$

  such that $h_0 = f$ and $h_1 = g$. Such f and g are called **homotopic**. Note that homeomorphisms are a special case of homotopy equivalence.
- **Embedding**: An **injective** and homeomorphic map between a topological space Y and its subspace X

$$f : X \mapsto Y, \tag{A.4}$$

  and called X is **embedded** to Y.
- **Isotopy**: A homotopy h_t that is an embedding for each fixed t. Such $h_0 = f$ and $h_1 = g$ are called **isotopic** ($\sim$).
- **Diffeomorphism**: A continuous, **differentiable** and **bijective** map

$$f : X \mapsto Y \tag{A.5}$$

  with smooth map $f^{-1} : Y \mapsto X$. Such X and Y are called **diffeomorphic** ($\backsimeq$). In other words, a diffeomorphism is a smooth homeomorphism whose inverse is also smooth.
- **Manifold**: A topological space that locally looks like Euclidean space (informal statement).

A.3 Differential geometry terminology

- **(Differential) k-form** ω on a manifold M is a smooth assignment at each point p **alternating multilinear map** from tangent space $(T_pM)^k$ (a vector space that its element at p is the tangent vector of M at p) with the basis $(\partial_{j_1}, \dots, \partial_{j_k})_p$ to the real numbers $\mathbb{R}$

$$\omega_p : (T_pM)^k \to \mathbb{R},\ \omega_p \in (T_p^*M)^k \tag{A.6}$$

where T_p^*M is the cotangent space with the basis $(dx^{i_1}, \dots, dx^{i_k})_p$ and satisfying

$$(dx^{i_1}, \dots, dx^{i_k})_p(\partial_{j_1}, \dots, \partial_{j_k})_p = \delta^i_j. \tag{A.7}$$

Such k-from ω can be represented in

$$\omega(p) = \sum_I \mathrm{sgn}(I) f_{i_1,\dots,i_k}(p)(dx^{i_1})_p \dots (dx^{i_k})_p,\ I = (i_1, \dots, i_k). \tag{A.8}$$

or simply

$$\omega = f_{i_1,\dots,i_k}(dx^{i_1}) \wedge \dots \wedge (dx^{i_k}) = f_I dx^I \tag{A.9}$$

where $\wedge$ is the wedge product symbol that handles the antisymmetry and I is the multi-index notation.

- **Exterior product** (**wedge product**) between k-form $\alpha = f_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} = f_K dx^K$ and p-form $\beta = g_{j_1 \dots j_p} dx^{j_1} \wedge \dots \wedge dx^{j_p} = g_P dx^P$ is

$$\alpha \wedge \beta = f_K g_P dx^{K+P} = (-1)^{kp} \beta \wedge \alpha. \tag{A.10}$$

Exterior product can also be defined by the **alternating multilinear map** of the tensor product of a k-form α and a p-form β

$$\alpha \wedge \beta = \mathrm{Alt}(\alpha \otimes \beta). \tag{A.11}$$

- **Exterior derivative** (d): A map from k-form $\omega = f_K dx^K$ to (k+1)-form $d\omega$

$$d : \Omega^k(M) \to \Omega^{k+1}(M) \tag{A.12}$$

and is defined by

$$d\omega = df_K \wedge dx^K = (\sum_i^k \frac{\partial f_K}{\partial x^i} dx^i) \wedge dx^K \tag{A.13}$$

and thus have $d^2\omega = 0$.

Some comparisons in $\mathbb{R}^3$:

  - Gradient $d : \Omega^0(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3)$.
  - Curl $d : \Omega^1(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$.
  - Divergence $d : \Omega^2(\mathbb{R}^3) \to \Omega^3(\mathbb{R}^3)$.
  - $d^2\omega = \nabla \times (\nabla\omega) = 0$ for 0-form ω.
  - $d^2\omega = \nabla \cdot (\nabla \times \omega) = 0$ for 1-form ω.

- **Stokes–Cartan theorem** (**generalized Stokes theorem**): The integral of the exterior derivative of k-form ω over a n-dimensional manifold M is equal to the integral of the k-form ω over the (n-1)-dimensional boundary ∂M

$$\int_M dw = \int_{\partial M} w. \tag{A.14}$$

Some comparisons in $\mathbb{R}^3$:

  - Gradient theorem: $\int_a^b dw = w(b) - w(a)$ for 0-form ω.
  - Curl theorem: $\int_S dw = \oint_C \omega$ for 1-form ω.
  - Divergence theorem: $\int_V dw = \oint_S \omega$ for 2-form ω.

- **Hodge star operator** ($\star$): A linear **isomorphism** map from the vector space of k-forms $\Omega^k(V)$ with oriented n-dimensional smooth manifold V to (n-k)-forms

$$\star : \Omega^k(V) \to \Omega^{n-k}(V) \tag{A.15}$$

defined by k-form α, β

$$\alpha \wedge \star\beta =< \alpha, \beta > \mathrm{vol}_g, \ \forall \alpha, \beta \in \Omega^k(V) \tag{A.16}$$

where vol_g is the volume form (top-dimensional differential form) with Riemannian metric g endows $\star^2 = (-1)^{k(n-k)}$. As a result, the Hodge dual $\star\omega$ of k-form $\omega = f_{i_1 \dots i_k} dx^{i_1} \wedge \dots dx^{i_k} = f_K dx^K$ is

$$\star\omega = \sqrt{|\det(g_{ij})|} f_K \epsilon_I g^{j_1 i_1} \dots g^{j_k i_k} dx^{i_{k+1}} \dots dx^{i_n} \tag{A.17}$$

where $I = (i_1 \dots i_n)$ and ϵ is Levi-Civita symbol.

- **Pullback** (*): The pullback of $f : N \to A$ by $\phi : M \to N$ is a smooth map from $M \to A$ and defined by

$$(\phi^* f) = f \circ \phi. \tag{A.18}$$

An example of a pullback is that of the Euclidean metric onto the 2-sphere, given by $d\Omega^2 = d\theta^2 + \sin^2 d\phi^2$.

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