

Hessian Matrix and J matrix

Given the treatment XZ and the outcome k, the hessian matrix (H) is

$$H_k = \begin{bmatrix} \frac{\partial^2 \ell}{\partial (\mu_k^{xz})^2} & \frac{\partial^2 \ell}{\partial \mu_k^{xz} \partial \tau_k^{xz^2}} \\ \frac{\partial^2 \ell}{\partial \mu_k^{xz} \partial \tau_k^{xz^2}} & \frac{\partial^2 \ell}{\partial (\tau_k^{xz^2})^2} \end{bmatrix}$$

The J matrix is

$$\begin{aligned} J_R &= \begin{bmatrix} \frac{\partial \ell}{\partial \mu_k^{xz}} \\ \frac{\partial \ell}{\partial \tau_k^{xz^2}} \end{bmatrix} \begin{bmatrix} \frac{\partial \ell}{\partial \mu_k^{xz}} & \frac{\partial \ell}{\partial \tau_k^{xz^2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial \ell}{\partial \mu_k^{xz}} \cdot \frac{\partial \ell}{\partial \mu_k^{xz}} & \frac{\partial \ell}{\partial \mu_k^{xz}} \cdot \frac{\partial \ell}{\partial \tau_k^{xz^2}} \\ \frac{\partial \ell}{\partial \tau_k^{xz^2}} \cdot \frac{\partial \ell}{\partial \mu_k^{xz}} & \frac{\partial \ell}{\partial \tau_k^{xz^2}} \cdot \frac{\partial \ell}{\partial \tau_k^{xz^2}} \end{bmatrix} \end{aligned}$$

where the var (\cdot) and cov (\cdot) are calculated based on the generated sample and the first and second derivatives are

$$\begin{aligned} \ell(\tau_k^{XZ^2}, \mu_k^{XZ}) &= -\frac{1}{2} \sum_{i \in N_{XZ}^k} \left[\log(s_{k,i}^{XZ^2} + \tau_k^{XZ^2}) + \frac{(y_{k,i}^{XZ} - \mu_k^{XZ})^2}{s_{k,i}^{XZ^2} + \tau_k^{XZ^2}} \right] \\ \frac{\partial \ell}{\partial \tau_k^{xz^2}} &= -\frac{1}{2} \sum_{i \in N_{xz}^k} \left[\frac{1}{\tau_k^{xz^2} + s_{k,i}^{xz^2}} - \frac{(y_{k,i}^{xz} - \mu_k^{xz})^2}{(s_{k,i}^{xz^2} + \tau_k^{xz^2})^2} \right] \\ \frac{\partial \ell}{\partial \mu_k^{xz}} &= -\frac{1}{2} \sum_{i \in N_{xz}^k} \left[\frac{2(y_{k,i}^{xz} - \mu_k^{xz})}{s_{k,i}^2 + \tau_k^{xz^2}} (-1) \right] = \sum_{i \in N_{xz}^k} \left[\frac{(y_{k,i}^{xz} - \mu_k^{xz})}{s_{k,i}^{xz^2} + \tau_k^{xz^2}} \right] \\ \frac{\partial^2 \ell}{\partial (\tau_k^{xz^2})^2} &= -\frac{1}{2} \sum_{i \in N_{xz}^k} \left[-\frac{1}{(\tau_k^{xz^2} + s_{k,i}^{xz^2})^2} + \frac{2(y_{k,i}^{xz} - \mu_k^{xz})^2}{(s_{k,i}^{xz^2} + \tau_k^{xz^2})^3} \right] \\ \frac{\partial^2 \ell}{\partial (\mu_k^{xz})^2} &= \sum_{i \in N_{xz}^k} \left[\frac{-1}{s_{k,i}^{xz^2} + \tau_k^{xz^2}} \right] \\ \frac{\partial^2 \ell}{\partial \mu_k^{xz} \partial \tau_k^{xz^2}} &= \sum_{i \in N_{xz}^k} \left[-\frac{(y_{k,i}^{xz} - \mu_k^{xz})}{(s_{k,i}^{xz^2} + \tau_k^{xz^2})^2} \right] \end{aligned}$$