

# MCMC Algorithm Derivation

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Given  $XZ, k$ , the log likelihood and the composite likelihood are

$$\begin{aligned}
 \ell(\tau_k^{XZ^2}, \mu_k^{XZ}) &= -\frac{1}{2} \sum_{i \in N_{XZ}^k} \left[ \log(s_{k,i}^{XZ^2} + \tau_k^{XZ^2}) + \frac{(y_{k,i}^{XZ} - \mu_k^{XZ})^2}{s_{k,i}^{XZ^2} + \tau_k^{XZ^2}} \right] \\
 &= \sum_{i \in N_{XZ}^k} \left[ \log \frac{1}{\sqrt{s_{k,i}^{XZ^2} + \tau_k^{XZ^2}}} - \frac{(y_{k,i}^{XZ} - \mu_k^{XZ})^2}{2(s_{k,i}^{XZ^2} + \tau_k^{XZ^2})} \right] \\
 L(\tau_k^{XZ^2}, \mu_k^{XZ}) &= \left[ \prod_{i \in N_{XZ}^k} \left( \frac{1}{\sqrt{s_{k,i}^{XZ^2} + \tau_k^{XZ^2}}} \right) \right] \cdot \exp \left( \sum_{i \in N_{x_2}^k} -\frac{(y_{k,i}^{XZ} - \mu_k^{XZ})^2}{2(s_{k,i}^{XZ^2} + \tau_k^{XZ^2})} \right) \\
 &= \left[ \prod_{i \in N_{XZ}^k} \left( \frac{1}{\sqrt{s_{k,i}^{XZ^2} + \tau_k^{XZ^2}}} \right) \right] \cdot \exp \left( \sum_{i \in N_{XZ}^k} -\frac{(y_{k,i}^{XZ} - \bar{y}_k^{XZ})^2}{2(s_{k,i}^{XZ^2} + \tau_k^{XZ^2})} - \sum_{i \in N_{kz}^k} \frac{(\bar{y}_k^{XZ} - \mu_k^{XZ})^2}{2(s_{k,i}^{XZ^2} + \tau_k^{XZ^2})} \right) \\
 &= \left[ \prod_{i \in N_{XZ}^k} \left( \frac{1}{\sqrt{s_{k,i}^{XZ^2} + \tau_k^{XZ^2}}} \right) \right] \cdot \exp \left( - \sum_{i \in N_{XZ}^k} \frac{(y_{k,i}^{XZ} - \bar{y}_k^{XZ})^2}{2(s_{k,i}^{XZ^2} + \tau_k^{XZ^2})} \right) \exp \left( - \sum_{i \in N_{XZ}^k} \frac{(\bar{y}_k^{XZ} - \mu_k^{XZ})^2}{2(s_{k,i}^{XZ^2} + \tau_k^{XZ^2})} \right) \\
 &= \left[ \prod_{i \in N_{XZ}^k} \frac{1}{\sqrt{s_{k,i}^{XZ^2} + \tau_k^{XZ^2}}} \right] \cdot \exp \left[ \sum_{i \in N_{kz}^k} -\frac{(y_{k,i}^{XZ} - \bar{y}_k^{XZ})^2}{2(s_{k,i}^{XZ^2} + \tau_k^{XZ^2})} \right] \cdot \exp \left[ (\bar{y}_k^{XZ} - \mu_k^{XZ})^2 \sum_{i \in N_{XZ}^k} \frac{1}{2(s_{k,i}^{XZ^2} + \tau_k^{XZ^2})} \right]
 \end{aligned}$$

where

$$\begin{aligned}
 n_k^{xz} &= \sum_{i \in N_{XZ}^k} \mathbb{I}_{i \in N_{XZ}^k} \\
 \bar{y}_k^{xz} &= \frac{1}{n_{xz}^k} \sum_{i \in N_{XZ}^k} y_{k,i}^{xz}
 \end{aligned}$$

Let

$$\sum_{i \in N_{XZ}^k} \frac{1}{(s_{k,i}^{XZ^2} + \tau_k^{XZ^2})} = \frac{n_{xz}^k}{\sigma_k^{XZ^2}}$$

Then the composite likelihood and marginal likelihood can also

$$\begin{aligned}
L(\tau_k^{XZ^2}, \mu^{XZ}) &= \left[ \prod_{i \in N_{XZ}^k} \frac{1}{\sqrt{s_{k,i}^{XZ^2} + \tau_k^{XZ^2}}} \right] \exp \left[ \sum_{i \in N_{XZ}^k} -\frac{(y_{k,i}^{XZ} - \bar{y}_k^{XZ})^2}{2(s_{k,i}^{XZ^2} + \tau_k^{XZ^2})} \right] \exp \left[ -\frac{n_{xz}^k}{2\sigma_k^{xz^2}} (\bar{y}_{xz}^k - \mu_k^{xz})^2 \right] \\
L(\tau_k^{XZ^2}) &= \int_{-\infty}^{\infty} L(\tau_k^{XZ^2}, \mu_k^{XZ}) d\mu_k^{XZ} \\
&= \left[ \prod_{i \in N_{XZ}^k} \frac{1}{\sqrt{s_{k,i}^{XZ^2} + \tau_k^{XZ^2}}} \right] \exp \left[ \sum_{i \in N_{XZ}^k} -\frac{(y_{k,i}^{XZ} - \bar{y}_k^{XZ})^2}{2(s_{k,i}^{XZ^2} + \tau_k^{XZ^2})} \right] \sqrt{\frac{2\pi\sigma_k^{xz^2}}{n_{xz}^k}} \int_{-\infty}^{\infty} \sqrt{\frac{n_{xz}^k}{2\pi\sigma_k^{xz^2}}} \exp \left[ -\frac{n_{xz}^k}{2\sigma_k^{xz^2}} (\mu_k^{xz} - \bar{y}_{xz}^k)^2 \right] d\mu_k^{xz} \\
&= \left[ \prod_{i \in N_{XZ}^k} \frac{1}{\sqrt{s_{k,i}^{XZ^2} + \tau_k^{XZ^2}}} \right] \exp \left[ \sum_{i \in N_{XZ}^k} -\frac{(y_{k,i}^{XZ} - \bar{y}_k^{XZ})^2}{2(s_{k,i}^{XZ^2} + \tau_k^{XZ^2})} \right] \sqrt{\frac{2\pi\sigma_k^{xz^2}}{n_{xz}^k}}
\end{aligned}$$

Conditioning on  $\tau_k^{XZ^2}$  and  $y$ , the posterior distribution of  $\mu_k^{XZ}$  follows

$$\begin{aligned}
p(\mu_k^{XZ} | \tau_k^{XZ^2}, y) &= \frac{P(\mu_k^{XZ}, \tau_k^{XZ^2}, y)}{P(\tau_k^{XZ^2}, y)} \\
&= \frac{p(y | \tau_k^{XZ^2}, \mu_k^{XZ}) \cdot P(\tau_k^{XZ^2}, \mu_k^{XZ})}{p(\tau_k^{XZ^2}, y)} \\
&= \frac{P(y | \tau_k^{XZ^2}, \mu_k^{XZ}) P(\tau_k^{XZ^2}, \mu_k^{XZ})}{P(y | \tau_k^{XZ^2}) P(\tau_k^{XZ^2})} \\
&\propto \frac{L(\tau_k^{XZ^2}, \mu_k^{XZ})}{L(\tau_k^{XZ^2})} \\
&= \sqrt{\frac{n_{XZ}^k}{2\pi\sigma_k^{xz^2}}} \exp \left[ -\frac{n_{XZ}^k}{2\sigma_k^{xz^2}} (\bar{y}_{XZ}^k - \mu_k^{XZ})^2 \right]
\end{aligned}$$

Conditioning on  $\mu_k^{XZ}$  and  $y$ , the postier distribution of  $\tau_k^{XZ^2}$  follows

$$\begin{aligned}
p(\tau_k^{XZ^2} | \mu_k^{XZ}, y) &\propto P(y | \tau_k^{XZ^2}, \mu_k^{XZ}) P(\tau_k^{XZ^2}, \mu_k^{XZ}) \\
&\propto L(\tau_k^{XZ^2}, \mu_k^{XZ}) (\tau_k^{XZ})^{-2} \\
&\propto \left[ \prod_{i \in N_{XZ}^k} \left( \frac{1}{\sqrt{s_{k,i}^{XZ^2} + \tau_k^{XZ^2}}} \right) \right] \cdot \exp \left( \sum_{i \in N_{XZ}^k} -\frac{(y_{k,i}^{XZ} - \mu_k^{XZ})^2}{2(s_{k,i}^{XZ^2} + \tau_k^{XZ^2})} \right) (\tau_k^{XZ})^{-2}
\end{aligned}$$

where

$$\begin{aligned}
P(\tau_k^{XZ^2}, \mu_k^{XZ}) &\propto (\tau_k^{XZ^2})^{-1} \\
\frac{P(\tau_k^{XZ^2}, \mu_k^{XZ})}{P(\tau_k^{XZ^2})} &\propto 1
\end{aligned}$$

We use Gibbs Sampler to sample  $\mu_k^{XZ}$  and  $\tau_k^{XZ^2}$  from the posterior distribution

Input: data  $y$

Output: A realization of length  $N$  from a Markov Chain for  $(\tau_k^{XZ}, \mu_k^{XZ})$

for  $k \in \{1, 2\}$

for  $XZ \in \{AB, AC, BC\}$

for  $t = 1 \dots N$ :

$$\text{sample } \mu_k^{XZ(p)} \sim N\left(\bar{y}_{XZ}^k, \frac{(\sigma_k^{XZ})^2}{n_{xz}^k}\right) \quad \text{where} \quad \frac{(\sigma_k^{XZ})^2}{n_{xz}^k} = \frac{1}{\sum_{i \in N_{XZ}^k} \frac{1}{(s_{k,i}^{XZ} + \tau_k^{XZ^2})}} \quad \text{and} \quad \bar{y}_{xz}^k =$$

$$\frac{1}{n_{xz}^k} \sum_{i \in N_{XZ}^k} y_{k,i}^{xz}$$

$$\text{sample } \tau_k^{XZ(p)} \sim p\left(\tau_k^{XZ^2} | y, \mu_k^{XZ(p)}\right)$$

$$\tau_k^{XZ^2(t)} = \tau_k^{XZ^2(p)}$$

$$\mu_k^{XZ(t)} = \mu_k^{XZ(p)}$$

We use Metropolis-Hasting algorithm to sample  $\tau_k^{XZ}$  from  $p(\tau_k^{XZ^2} | y, \mu_k^{XZ})$ .

Input: data  $y$

Output: A realization of length  $N$  from a Markov Chain for  $\tau_k^{XZ}$  from  $p(\tau_k^{XZ^2} | y)$

Propose  $q(\tau_{k,t}^{XZ^2} | \tau_{k,t-1}^{XZ^2}) = N(\tau_{k,t-1}^{XZ^2}, 0.1)$

$$\tau_{k,0}^{XZ^2} = 0.3$$

for  $t$  from 1 to  $N$ :

$$\text{sample } \tau_{k,*}^{XZ^2} \sim N(\tau_{k,t-1}^{XZ^2}, 0.1)$$

$$a = \frac{P(\tau_{k,*}^{XZ^2} | y, \mu_k^{XZ}) / q(\tau_{k,*}^{XZ^2} | \tau_{k,t-1}^{XZ^2})}{P(\tau_{k,t-1}^{XZ^2} | y, \mu_k^{XZ}) / q(\tau_{k,t-1}^{XZ^2} | \tau_{k,*}^{XZ^2})}$$

if  $\text{runif}(1) < \min(a, 1)$ :

$$\tau_{k,t}^{XZ^2} = \tau_{k,*}^{XZ^2}$$

else:

$$\tau_{k,t}^{XZ^2} = \tau_{k,t-1}^{XZ^2}$$