

# Measure Theory

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Autum 2011

## Introduction

In this course we first seek to define the measure of a set eg. the length, area, volume, probability of a set. We also seek to improve on the riemann integral by defining the lebesgue integral.

If  $\lambda$  denotes the "length" of a set in  $\mathbb{R}$ ., clearly we would expect  $\lambda[0,1]=1$ . But what about the length of  $[0,1]\setminus\mathbb{Q}$  where  $\mathbb{Q}$  is the set of rationals? Or the set  $\bigcup_{i=0}^{\infty}[\frac{1}{2^{i+1}} + \frac{1}{2^i}]$ ? Since  $\mathbb{Q}$  is quite "small" we might expect  $\lambda([0,1]\setminus\mathbb{Q})=1$ . Also we might expect  $\lambda(\bigcup_{i=0}^{\infty}[\frac{1}{2^{i+1}} + \frac{1}{2^i}]) = \sum_{i=0}^{\infty} \lambda([\frac{1}{2^{i+1}} + \frac{1}{2^i}])$ . Both expectations are true!

If we take the function  $f(x) = \begin{cases} 1 & \text{for } x \text{ irrational} \\ 0 & \text{for } x \text{ rational} \end{cases}$

then you will know from analysis 2 that  $(\mathbf{R}) \int_0^{-1} f(x) dx = 1$  and  $(\mathbf{R}) \int_{-0}^1 f(x) dx = 1$

however the vast majority of  $x$  in  $[0,1]$  are irrational and so we might expect the integral to be 1. When we have defined the lebesgue integral we will find  $(\mathbf{L}) \int_0^1 f(x) dx = 1$

# 1 Measures

We will work within a set  $\Omega$ . For example  $\Omega = \mathbb{R}$ ,  $\Omega = \mathbb{R}^n$ ,  $\Omega = \{\text{sequence of heads \& tails}\}$ . Families of subsets of  $\Omega$  will be denoted by  $\mathcal{F}$ ,  $\mathcal{G}$  etc.

**Definition 1** (Algebra of sets). *A family  $\mathcal{F}$  of subsets of  $\Omega$  is called an Algebra if it satisfies:*

$$(i) \phi, \Omega \in \mathcal{F}$$

$$(ii) \text{ If } A \in \mathcal{F} \text{ then } A^c = \Omega \setminus A \in \mathcal{F}$$

$$(iii) \text{ If } A, B \in \mathcal{F} \text{ then } A \cup B \in \mathcal{F}$$

**Example 1.** *If  $\Omega = [0, 1]$  and  $\mathcal{F}$  is the family of all subsets of  $[0, 1]$  which can be expressed as a finite union of intervals (which can be open, closed half open, empty) then  $\mathcal{F}$  is an algebra.*

**Definition 2** ( $\sigma$ -Algebra of sets). *A family  $\mathcal{F}$  of subsets of  $\Omega$  is called a  $\sigma$ -Algebra if it satisfies:*

$$(i) \phi, \Omega \in \mathcal{F}$$

$$(ii) \text{ If } A \in \mathcal{F} \text{ then } A^c = \Omega \setminus A \in \mathcal{F}$$

$$(iii) \text{ If } A_1, A_2, \dots \text{ is a sequence of sets in } \mathcal{F} \text{ then } \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$$

**Example 2.** *For any  $\Omega$ .*

$\mathcal{F} = \{\phi, \Omega\}$  *is a  $\sigma$ -algebra.*

$\mathcal{F} = \{\text{all subsets of } \Omega\}$  *is a  $\sigma$ -algebra.*

**Remark:** although example 1 is an algebra, it is not a  $\sigma$ -algebra (try to prove it). Notice that a  $\sigma$ -algebra is an algebra.

**Theorem 1** (De Morgan's Laws). *If  $A_\alpha$ ,  $\alpha \in I$  is a family of sets in  $\Omega$  then*

$$(i) (\bigcup_{\alpha \in I} A_\alpha)^c = \bigcap_{\alpha \in I} A_\alpha^c$$

$$(ii) (\bigcap_{\alpha \in I} A_\alpha)^c = \bigcup_{\alpha \in I} A_\alpha^c$$

From the definition of an algebra or a  $\sigma$ -algebra we can deduce the following properties:

## Algebra

$$(i) A_i, i = 1, 2, \dots, n \in \mathcal{F} \implies \bigcup_{i=1}^n A_i \in \mathcal{F} \text{ (induction)}$$

$$(ii) A_i, i = 1, 2, \dots, n \in \mathcal{F} \implies \bigcap_{i=1}^n A_i \in \mathcal{F} \text{ (By De Morgan (ii))}$$

$$(iii) A, B \in \mathcal{F} \implies A \setminus B \in \mathcal{F} \text{ (Since } A \setminus B = A \cap B^c)$$

## $\sigma$ -Algebra

$$(i) A_1, A_2, \dots \in \mathcal{F} \text{ then } \bigcap_{i=1}^{\infty} A_i \in \mathcal{F} (\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} (A_i^c)^c = (\bigcup_{i=1}^{\infty} A_i^c)^c \in \mathcal{F})$$

**Proposition 1.** *For any family of subsets  $A$  of  $\Omega$ , there is a smallest  $\sigma$ -algebra  $\sigma(A)$  containing  $A$ .*

*Proof.* Just note that there is a  $\sigma$ -algebra containing  $A$ , namely  $\{\text{all subsets of } A\}$ . Consider all  $\sigma$ -algebras containing  $A$  and let  $\sigma(A)$  be their intersection. i.e.  $B \in \sigma(A)$  iff  $B$  belongs to every  $\sigma$ -algebra containing  $A$ . We certainly have  $A \subset \sigma(A)$  and if  $\mathcal{F}$  is a  $\sigma$ -algebra containing  $A$  then  $\sigma(A) \subset \mathcal{F}$ . It remains to show that  $\sigma(A)$  is a  $\sigma$ -algebra.

- (i)  $\phi, \Omega \in \sigma(A)$  since they belong to every  $\sigma$ -algebra containing  $A$ .
- (ii) If  $A \in \sigma(A)$  and  $\mathcal{F}$  is a  $\sigma$ -algebra containing  $A$ , then  $A \in \mathcal{F}$  and so  $A^c \in \mathcal{F}$ .  
So  $A^c \in \sigma(A)$
- (iii) If  $\{A_i\}_{i=1}^{\infty} \in \sigma(A)$  and  $\mathcal{F}$  is a  $\sigma$ -algebra containing  $A$  then  $\{A_i\}_{i=1}^{\infty} \in \mathcal{F}$  & so  $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ . Hence  $\bigcup_{i=1}^{\infty} A_i \in \sigma(A)$ .

□

The most important  $\sigma$ -algebra is the:

**Definition 3** (Borel  $\sigma$ -algebra). *This is the  $\sigma$ -algebra on  $\mathbb{R}$  generated by the family of open intervals in  $\mathbb{R}$ .*

**Definition 4** (Borel Set). *A Borel Set is any set which belongs to the Borel  $\sigma$ -algebra eg.  $\phi, \mathbb{R}$ , any open interval, any closed interval  $([a, b] = \bigcap_{i=1}^{\infty} (a - \frac{1}{i}, b + \frac{1}{i}))$ .*

*Most reasonable sets are Borel:*

$$[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b), \{a\} = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, a + \frac{1}{n}), \mathbb{Q} = \bigcup_{n=1}^{\infty} r_n, I(\text{irrationals}) = \mathbb{Q}^c.$$

**Proposition 2.** *Open sets are Borel.*

*Proof.* If  $G$  is open and  $g \in G$ , we can choose an  $I_g$  with rational end points such that  $g \in I_g \subset G$ . Since there are only countably many open intervals with rational end points, we may arrange the intervals  $I_g, g \in G$  as a sequence of open intervals  $\{I_n\}_{n=1}^{\infty}$ . Then  $G = \bigcup_{n=1}^{\infty} I_n$  and so  $G$  is a Borel set. □

**Corollary 1.** *Closed sets are Borel sets.*

*Proof.* They are complements of open sets □

**Note.** *Two different collections of sets can give rise to the same  $\sigma$ -algebra.*

**Example 3.** *Let*

$$\begin{aligned} I &= \text{collection of open intervals in } \mathbb{R} \text{ and} \\ \theta &= \text{collection of open sets in } \mathbb{R}. \end{aligned}$$

*Then  $I \subset \theta$  so  $I \subset \sigma(I)$ .  $\sigma(I)$  is the smallest  $\sigma$ -algebra containing  $I$  so  $\sigma(I) \subset \sigma(\theta)$ . Open sets are Borel sets so  $\theta \subset \sigma(I)$ .  $\sigma(\theta)$  is the smallest  $\sigma$ -algebra containing  $\theta$  so  $\sigma(\theta) \subset \sigma(I)$ . Hence  $\sigma(\theta) = \sigma(I)$*

**Definition 5.** *If  $\mathcal{F}$  is a  $\sigma$ -algebra on a set  $\Omega$ , then a measure on  $\mathcal{F}$  is a function,  $\mu$  such that:*

$$\mu : \mathcal{F} \rightarrow [0, \infty]$$

*satisfying:*

1.  $\mu(\emptyset) = 0$
2. If  $E_1, E_2, \dots \in \mathcal{F}$  and  $E_i \cap E_j = \emptyset, i \neq j$ , then

**Example 4.** Let  $\Omega = \text{any set}$ ,  $\mathcal{F} = \{\text{all subsets of } \Omega\}$ . Fix  $x \in \Omega$ , then for  $E \in \mathcal{F}$  define

$$\delta_x(E) = \begin{cases} 0, & \text{if } x \notin E \\ 1, & \text{if } x \in E \end{cases}$$

We claim that  $\delta_x$  is a measure on  $\mathcal{F}$ .

*Proof.* We prove the properties of measures one-by-one.

$$1. \delta_x(\emptyset) = 0$$

2. If  $E_1, E_2, \dots \in \mathcal{F}$  and  $E_i \cap E_j = \emptyset, i \neq j$ , then,

either  $x \notin \bigcup_{i=1}^{\infty} E_i$  and hence  $x \notin E_i$  for all  $i$  so

$$\delta_x\left(\bigcup_{i=1}^{\infty} E_i\right) = 0 = \sum_{i=1}^{\infty} \delta_x(E_i)$$

or  $x \in \bigcup_{i=1}^{\infty} E_i$  so  $x \in$  exactly one  $E_j$  and  $\delta_x(E_i) = 0, \text{ for } i \neq j$ . Then

$$\delta_x\left(\bigcup_{i=1}^{\infty} E_i\right) = 1 = \delta_x(E_j) = \sum_{i=1}^{\infty} \delta_x(E_i)$$

□

**Note.** If  $c \in [0, \infty]$ , then  $c\delta_x$  is also a measure. ( $\infty \cdot 0 = 0$ )

**Example 5.** We define the discrete counting measure,  $\gamma$ , by

$$\gamma(E) = \sum_{x \in E} 1 = \text{number of elements in } E$$

**Proposition 3.** Properties of Measures

1. If  $A, B \in \mathcal{F}$  and  $A \subset B$ , then

$$\mu(A) \leq \mu(B)$$

2. If  $A, B \in \mathcal{F}$ ,  $A \subset B$  and  $\mu(A) < \infty$ , then

$$\mu(B) - \mu(A) = \mu(B \setminus A)$$

3.  $\sigma$ -subadditivity. If  $E_1, E_2, \dots \in \mathcal{F}$ , then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

4. Continuity of measures. If  $E_1, E_2, \dots \in \mathcal{F}$  and  $E_1 \subset E_2 \subset \dots$ , then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n \rightarrow \infty} \mu(E_n)$$

5. If  $E_1, E_2, \dots \in \mathcal{F}$ ,  $E_1 \supset E_2 \supset \dots$  and  $\mu(E_1) < \infty$ , then

*Proof.*

1.  $\mu(B) = \mu(A) + \mu(B \setminus A)$  and  $\mu(B \setminus A) \geq 0$ , so  $\mu(B) \geq \mu(A)$ .
2. Rearrange 1. and  $\mu(A) < \infty$  so the sum makes sense.
3. Let

$$F_i = E_i \setminus \bigcup_{i < j} E_j$$

then  $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$ ,  $F_1, F_2, \dots \in \mathcal{F}$ ,  $F_i \cap F_j = \emptyset, i \neq j$  and  $F_i \subset E_i$  for all  $i$ .  
So

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} \mu(F_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

4. Let

$$F_i = E_i \setminus \bigcup_{i < j} E_j$$

then

$$\begin{aligned} \mu\left(\bigcup_{i=1}^{\infty} E_i\right) &= \mu\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} \mu(F_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(F_i) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{i=1}^n F_i\right) \\ &= \lim_{n \rightarrow \infty} \mu(E_n) \end{aligned}$$

- 5.

$$\mu(E_1) = \mu\left(E_1 \setminus \bigcap_{i=1}^{\infty} E_i\right) + \mu\left(\bigcap_{i=1}^{\infty} E_i\right)$$

So

$$\begin{aligned} \mu\left(\bigcap_{i=1}^{\infty} E_i\right) &= \mu(E_1) - \mu\left(E_1 \setminus \bigcap_{i=1}^{\infty} E_i\right) \\ &= \mu(E_1) - \mu\left(\bigcup_{i=1}^{\infty} E_1 \setminus E_i\right) \\ &= \mu(E_1) - \lim_{n \rightarrow \infty} \mu(E_1 \setminus E_n) \\ &= \mu(E_1) - \mu(E_1) + \lim_{n \rightarrow \infty} \mu(E_n) \end{aligned}$$

□

$\lambda$ , *Lebesgue measure*, will be our means of defining a concept for length, area, volume etc. of a set.

On  $\mathbb{R}$  we clearly desire  $\lambda_1((a, b)) = b - a$ .

On  $\mathbb{R}^2$  we clearly desire  $\lambda_2((a, b) \times (c, d)) = (b - a)(d - c)$ . And so on.

On  $\mathbb{R}$ , if a set  $A$  is contained in  $\bigcup_{i=1}^{\infty} (a_i, b_i)$  we must have by  $\sigma$ -subadditivity:

$$\lambda(A) \leq \sum_{i=1}^{\infty} \lambda(a_i, b_i) \leq \sum_{i=1}^{\infty} (b_i - a_i)$$

which motivates us to define:

$$\lambda^*(A) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) : A \subset \bigcup_{i=1}^{\infty} (a_i, b_i) \right\}$$

We don't need to check that this satisfies the properties of a measure (which is the Riemann integral).

## 2 Outer Measure

**Definition 6.** An outer measure on a set  $\Omega$  is a function:

$$\mu^* : \{\text{All subsets of } \Omega\} \rightarrow [0, \infty]$$

such that:

1.  $\mu^*(\emptyset) = 0$
2. Monotonicity. If  $A \subset B$ , then  $\mu^*(A) \leq \mu^*(B)$ .
3.  $\sigma$ -subadditivity.  $\mu^*(\bigcup_{i=1}^{\infty} B_i) \leq \sum_{i=1}^{\infty} \mu^*(B_i)$

**Definition 7.** Let  $\mathcal{A}$  be a family of subsets of  $\Omega$ . Now define a function  $\phi : \mathcal{A} \rightarrow [0, \infty]$ . Let  $B$  be an arbitrary subset of  $\Omega$  and define:

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \phi(A_i) : B \subset \bigcup_{i=1}^{\infty} A_i, \text{ and } A_1, A_2, \dots \in \mathcal{A} \right\}$$

Further, define  $\mu^*(\emptyset) = 0$  and  $\mu^*(B) = \infty$  if no such  $A_i$  (i.e. no such cover) exist(s).

**Lemma 2.**  $\mu^*$  is an outer measure.

*Proof.*

1. from definition
2. from definition
3. Consider a set  $B_j$  and cover  $B_j$  by sets  $A_i^{(j)}$  in  $\mathcal{A}$  such that

$$B_j \subset \bigcup_{i=1}^{\infty} A_i^{(j)} \text{ and } \sum_{i=1}^{\infty} \phi(A_i^{(j)}) \leq \mu^*(B_j) + \frac{\epsilon}{2^j}$$

then

$$\mu^*\left(\bigcup_{j=1}^{\infty} B_j\right) \leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \phi(A_i^{(j)}) \leq \sum_{j=1}^{\infty} \mu^*(B_j) + \epsilon$$

□

**Definition 8.** If  $\mu^*$  is an outer measure on a set  $\Omega$ , we say that a set  $A \subset \Omega$  is  $\mu^*$  measurable if, for any  $T \subset \Omega$ :

$$\mu^*(T \cap A) + \mu^*(T \setminus A) = \mu^*(T)$$

**Theorem 3.** If  $\mu^*$  is an outer measure on  $\Omega$ , then the family of  $\mu^*$  measurable sets,  $\mathcal{F}(\mu^*)$ , is a  $\sigma$ -algebra and  $\mu^*$  is a measure on  $\mathcal{F}(\mu^*)$ .

*Proof.* We first show that  $\mathcal{F}(\mu^*)$  is an algebra:

1. If  $A \in \mathcal{F}(\mu^*)$ , then  $\Omega \setminus A \in \mathcal{F}(\mu^*)$

*Proof.* For any  $T \subset \Omega$

$$\begin{aligned} \mu^*(T) &= \mu^*(T \cap A) + \mu^*(T \setminus A) \\ &= \mu^*(T \setminus (\Omega \setminus A)) + \mu^*(T \cap (\Omega \setminus A)) \end{aligned}$$

2.  $\emptyset \in \mathcal{F}(\mu^*)$  and  $\Omega \in \mathcal{F}(\mu^*)$

*Proof.* For any  $T \subset \Omega$

$$\begin{aligned}\mu^*(T) &= \mu^*(T \cap \emptyset) + \mu^*(T \setminus \emptyset) \\ &= 0 + \mu^*(T) \\ &= \mu^*(T)\end{aligned}$$

By 1.,  $\Omega \in \mathcal{F}(\mu^*)$ . □

3. If  $A, B \in \mathcal{F}(\mu^*)$ , then  $A \cup B \in \mathcal{F}(\mu^*)$

*Proof.* Let  $A, B \in \mathcal{F}(\mu^*)$  and let  $T \subset \Omega$  be an arbitrary set.  $A$  is  $\mu^*$  measurable, so

$$\mu^*(T) = \mu^*(T \cap A) + \mu^*(T \setminus A)$$

We now test the measurability of  $B$  with  $T \cap A$

$$\begin{aligned}\mu^*(T \cap A) &= \mu^*(T \cap A \cap B) + \mu^*((T \cap A) \setminus B) \\ \text{so } \mu^*(T) &= \mu^*(T \cap A \cap B) + \mu^*((T \cap A) \setminus B) + \mu^*(T \setminus A) \\ &\geq \mu^*(T \cap A \cap B) + \mu^*((T \cap A) \setminus B \cup (T \setminus A)) \\ &\geq \mu^*(T \cap (A \cap B)) + \mu^*(T \setminus (A \cap B))\end{aligned}$$

Since  $((T \cap A) \setminus B) \cup (T \setminus A) \supset (T \setminus (A \cap B))$  (monotonicity).

Now, by the subadditivity of outer measures,

$$\mu^*(T) \leq \mu^*(T \cap (A \cap B)) + \mu^*(T \setminus (A \cap B))$$

and hence

$$\mu^*(T) = \mu^*(T \cap (A \cap B)) + \mu^*(T \setminus (A \cap B))$$

So  $A \cap B$  is  $\mu^*$  measurable, for  $A, B \in \mathcal{F}(\mu^*)$ . By *De Morgan's Laws*,  $A \cup B = \Omega \setminus ((\Omega \setminus A) \cap (\Omega \setminus B))$  so, by 2.,  $A \cup B \in \mathcal{F}(\mu^*)$ . □

So  $\mathcal{F}(\mu^*)$  is an algebra. We must now prove that  $\mathcal{F}(\mu^*)$  is a  $\sigma$ -algebra.

Let  $F_1, \dots, F_n \in \mathcal{F}(\mu^*)$  be disjoint sets, then, since  $\mathcal{F}(\mu^*)$  is an algebra,  $\bigcup_{i=1}^n F_i$  and  $\bigcap_{i=1}^n F_i \in \mathcal{F}(\mu^*)$ .

We claim  $\mu^*(T \cap \bigcup_{i=1}^n F_i) = \sum_{i=1}^n \mu^*(T \cap F_i)$  for all  $n$ .

*Proof.* Let  $n = 1$ , then trivially  $\mu^*(T \cap F_1) = \mu^*(T \cap F_1)$ .

Assume our claim holds for some  $n \geq 1$ , then consider

$$\begin{aligned}\mu^*(T \cap \bigcup_{i=1}^{n+1} F_i) &= \mu^*((T \cap \bigcup_{i=1}^{n+1} F_i) \cap F_{n+1}) + \mu^*((T \cap \bigcup_{i=1}^{n+1} F_i) \setminus F_{n+1}) \\ &= \mu^*(T \cap F_{n+1}) + \mu^*(T \cap \bigcup_{i=1}^n F_i) \\ &= \mu^*(T \cap F_{n+1}) + \sum_{i=1}^n \mu^*(T \cap F_i) \\ &= \sum_{i=1}^{n+1} \mu^*(T \cap F_i)\end{aligned}$$

Now let  $E_1, E_2, \dots \in \mathcal{F}(\mu^*)$  be arbitrary sets and define

$$F_i = E_i \setminus \bigcup_{j < i} E_j$$

So that  $F_i \cap F_j = \emptyset, i \neq j$  and, since  $\mathcal{F}(\mu^*)$  is an algebra,  $F_i \in \mathcal{F}(\mu^*)$  for all  $i$ . Also note that

$$\bigcup_{i=1}^n E_i = \bigcup_{i=1}^n F_i \text{ for all } n, \text{ so } \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$$

Now let  $T \subset \Omega$  be any set and recall  $\bigcup_{i=1}^n F_i \in \mathcal{F}(\mu^*)$ , so

$$\begin{aligned} \mu^*(T) &= \mu^*(T \cap \bigcup_{i=1}^n F_i) + \mu^*(T \setminus \bigcup_{i=1}^n F_i) \\ &\geq \mu^*(T \cap \bigcup_{i=1}^n F_i) + \mu^*(T \setminus \bigcup_{i=1}^{\infty} F_i) \\ &= \sum_{i=1}^n \mu^*(T \cap F_i) + \mu^*(T \setminus \bigcup_{i=1}^{\infty} F_i) \\ &\xrightarrow{\text{as } n \rightarrow \infty} \sum_{i=1}^{\infty} \mu^*(T \cap F_i) + \mu^*(T \setminus \bigcup_{i=1}^{\infty} F_i) \\ &\geq \mu^*(T \cap \bigcup_{i=1}^{\infty} F_i) + \mu^*(T \setminus \bigcup_{i=1}^{\infty} F_i) \end{aligned}$$

But  $\mu^*$  is subadditive so

$$\mu^*(T) \leq \mu^*(T \cap \bigcup_{i=1}^{\infty} F_i) + \mu^*(T \setminus \bigcup_{i=1}^{\infty} F_i)$$

Consequently

$$\begin{aligned} \mu^*(T) &= \mu^*(T \cap \bigcup_{i=1}^{\infty} F_i) + \mu^*(T \setminus \bigcup_{i=1}^{\infty} F_i) \\ &= \mu^*(T \cap \bigcup_{i=1}^{\infty} E_i) + \mu^*(T \setminus \bigcup_{i=1}^{\infty} E_i) \end{aligned}$$

So  $\bigcup_{i=1}^{\infty} E_i \in \mathcal{F}(\mu^*)$  and  $\mathcal{F}(\mu^*)$  is a  $\sigma$ -algebra. □

**Definition 9.** If we restrict  $\mu^*$  to  $\mathcal{F}(\mu^*)$ , then we replace  $\mu^*$  by  $\mu$  and simply say “the measure  $\mu$ ”.

### 3 Lebesgue Measure

**Definition 10.** The Lebesgue outer measure on  $\mathbb{R}$  is defined as

$$\lambda^*(A) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) : A \subset \bigcup_{i=1}^{\infty} (a_i, b_i) \right\}$$



*Proof.* 1. If  $A \subset \bigcup_{i=1}^{\infty} (a_i, b_i)$ , then  $A \subset \bigcup_{i=1}^{\infty} [a_i, b_i]$  and so

$$\inf\left\{\sum_{i=1}^{\infty} (b_i - a_i) : A \subset \bigcup_{i=1}^{\infty} [a_i, b_i]\right\} \leq \inf\left\{\sum_{i=1}^{\infty} (b_i - a_i) : A \subset \bigcup_{i=1}^{\infty} (a_i, b_i)\right\}$$

2. Let  $\epsilon > 0$ . If  $A \subset \bigcup_{i=1}^{\infty} [a_i, b_i]$ , then  $A \subset \bigcup_{i=1}^{\infty} (a_i - \frac{\epsilon}{2^i}, b_i + \frac{\epsilon}{2^i})$  and

$$\sum_{i=1}^{\infty} ((b_i + \frac{\epsilon}{2^i}) - (a_i - \frac{\epsilon}{2^i})) = 2\epsilon + \sum_{i=1}^{\infty} (b_i - a_i)$$

So

$$\inf\left\{\sum_{i=1}^{\infty} (b_i - a_i) : A \subset \bigcup_{i=1}^{\infty} (a_i, b_i)\right\} \leq 2\epsilon + \inf\left\{\sum_{i=1}^{\infty} (b_i - a_i) : A \subset \bigcup_{i=1}^{\infty} [a_i, b_i]\right\}$$

Combining 1. and 2. yields equality. □

**Lemma 4.** If  $[a, b] \subset \bigcup_{i=1}^{\infty} (a_i, b_i)$ , then  $b - a \leq \sum_{i=1}^{\infty} (b_i - a_i)$

*Proof.* By Heine-Borel theorem, if a closed interval is contained in an union of open intervals, then there exists a finite subcover of the closed interval. In our case there exists a finite  $n$  such that

$$[a, b] \subset \bigcup_{i=1}^n (a_i, b_i)$$

So we need only show that for such an  $[a, b] \subset \mathbb{R}$ ,  $b - a \leq \sum_{i=1}^n (b_i - a_i)$ .

Result holds for  $n = 1$ . Assume result holds for some finite  $n \geq 1$ . For the case  $n + 1$ , we may assume  $a_{n+1} \leq a_i$  for all  $i$  and  $a_{n+1} < a$ .

1. If  $b_{n+1} > b$ , then

$$b - a \leq b_{n+1} - a_{n+1} \leq \sum_{i=1}^{n+1} (b_i - a_i)$$

2. If  $b_{n+1} < b$  (and  $b_{n+1} > a$ ), then  $[b_{n+1}, b]$  is covered by  $\bigcup_{i=1}^n (a_i, b_i)$ , so by inductive hypothesis

$$\begin{aligned} b - a &= (b - b_{n+1}) + (b_{n+1} - a) \\ &\leq \sum_{i=1}^n (b_i - a_i) + (b_{n+1} - a_{n+1}) \\ &= \sum_{i=1}^{n+1} (b_i - a_i) \end{aligned}$$

1. and 2. prove our claim inductively for  $n + 1$ , so claim holds inductively for all  $n$  and our lemma is proved. □

**Lemma 5.**  $\lambda^*(a, b) = \lambda^*[a, b] = b - a$

*Proof.* Note, by Definition 10

Now  $[a, b] \subset (a - \epsilon, b + \epsilon)$  for all  $\epsilon > 0$  so

$$\lambda^*[a, b] \leq b - a + 2\epsilon$$

and by Lemma 4 we may deduce

$$\lambda^*[a, b] = b - a$$

Furthermore

$$b - a - 2\epsilon \leq \lambda^*[a + \epsilon, b - \epsilon] \leq \lambda^*(a, b) \leq \lambda^*[a, b] = b - a$$

So  $\lambda^*(a, b) = b - a$  also.

□