Measure Theory

Prof. D Larman

Autum 2011

Introduction

In this course we first seek to define the measure of a set eg. the length, area, volume, probability of a set. We also seek to improve on the riemann integral by defining the lebesgue integral.

If λ denotes the "length" of a set in $\mathbb{R}$., clearly we would expect λ[0,1]=1. But what about the length of $[0,1]\backslash\mathbb{Q}$ where $\mathbb{Q}$ is the set of rationals? Or the set $\bigcup_{i=0}^{\infty}[\frac{1}{2^{i+1}}+\frac{1}{2^i}]$? Since $\mathbb{Q}$ is quite "small" we might expect $\lambda([0,1]\backslash\mathbb{Q})$=1. Also we might expect $\lambda(\bigcup_{i=0}^{\infty}[\frac{1}{2^{i+1}}+\frac{1}{2^i}])=\sum_{i=0}^{\infty}\lambda([\frac{1}{2^{i+1}}+\frac{1}{2^i}]$. Both expectations are true!

If we take the function $f(x)=\begin{cases} 1 & \text{for} \quad x \textit{ irrational} \\ 0 & \text{for} \quad x \textit{ rational} \end{cases}$

then you will know from analysis 2 that $(\mathbf{R})\int\limits_{0}^{-1} f(x)\,dx=1$ and $(\mathbf{R})\int\limits_{-0}^{1} f(x)\,dx=1$

however the vast majority of x in [0,1] are irrational and and so we might expect the integral to be 1. When we have defined the labesgue integral we will find $(\mathbf{L})\int_0^1 f(x)\,dx=1$

1 Measures

We will work within a set Ω. For example $\Omega = \mathbb{R}$, $\Omega = \mathbb{R}^n$, $\Omega = \{\text{sequence of heads \& tails}\}$. Families of subsets of Ω will be denoted by $\mathcal{F}$, $\mathcal{G}$ etc.

Definition 1 (Algebra of sets)**.** *A family $\mathcal{F}$ of subsets of Ω is called an Algebra if it satisfies:*

(i) $\phi, \Omega \in \mathcal{F}$

(ii) If $A \in \mathcal{F}$ *then* $A^c = \Omega \backslash A \in \mathcal{F}$

(iii) If A,$B \in \mathcal{F}$ *then* $A \cup B \in \mathcal{F}$

Example 1. *If Ω=[0,1] and $\mathcal{F}$ is the family of all subsets of [0,1] which can be expressed as a finite union of intervals (which can be open, closed half open, empty) then $\mathcal{F}$ is an algebra.*

Definition 2 (σ-Algebra of sets)**.** *A family $\mathcal{F}$ of subsets of Ω is called a σ-Algebra if it satisfies:*

(i) $\phi, \Omega \in \mathcal{F}$

(ii) If $A \in \mathcal{F}$ *then* $A^c = \Omega \backslash A \in \mathcal{F}$

(iii) If $A_1, A_2, \ldots$ *is a sequence of sets in* $\mathcal{F}$ *then* $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

Example 2. *For any Ω.*

$\mathcal{F}$=$\{\phi, \Omega\}$ *is a σ-algebra.*

$\mathcal{F}$=$\{$*all subsets of* $\Omega\}$ *is a σ-algebra.*

Remark: althouth example 1 is an algebra, it is not a σ-algebra (try to prove it). Notice that a σ-algebra is an algebra.

Theorem 1 (De Morgan's Laws)**.** *If A_α, $\alpha \in I$ is a family of sets inΩ then*

(i) $(\cup_{\alpha \in I} A_\alpha)^c = \cap_{\alpha \in I} A_\alpha^c$

(ii) $(\cap_{\alpha \in I} A_\alpha)^c = \cup_{\alpha \in I} A_\alpha^c$

From the defintion of an algebra or a σ-algebra we can deduce the following properties:

Algebra

(i) $A_i, i = 1, 2, \ldots, n \in \mathcal{F} \implies \bigcup_{i=1}^{n} A_i \in \mathcal{F}$ (induction)

(ii) $A_i, i = 1, 2, \ldots, n \in \mathcal{F} \implies \bigcap_{i=1}^{n} A_i \in \mathcal{F}$ (By De Morgan (ii))

(iii) $A, B \in \mathcal{F} \implies A \backslash B \in \mathcal{F}$ (Since $A \backslash B = A \cap B^c$)

σ-Algebra

(i) $A_1, A_2, \cdots \in \mathcal{F}$ then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F} (\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} (A_i^c)^c = (\bigcup_{i=1}^{\infty} A_i^c)^c \in \mathcal{F})$

Proposition 1. *For any family of subsets A of Ω, there is a smallest σ-algebra $\sigma(A)$ containing A.*

Proof. Just note that there is a σ-algebra containing A, namely {all subsets of A}. Consider all σ-algebras containing A and let $\sigma(A)$ be their intersection. i.e. B$\in \sigma(A)$ iff B belongs to every σ-algebra containing A. We certainly have A$\subset \sigma(A)$ and if $\mathcal{F}$ is a σ-algebra containing A then $\sigma(A) \subset \mathcal{F}$. It remains to show that $\sigma(A)$ is a σ-algebra.

(i) $\phi, \Omega \in \sigma(A)$ since they belong to every σ-algebra containing A.

(ii) If A$\in \sigma(A)$ and $\mathcal{F}$ is a σ-algebra containing A, then A$\in \mathcal{F}$ and so $A^c \in \mathcal{F}$.
So $A^c \in \sigma(A)$

(iii) If $\{A_i\}_{i=1}^{\infty} \in \sigma(A)$ and $\mathcal{F}$ is a σ-algebra containing A then $\{A_i\}_{i=1}^{\infty} \in \mathcal{F}$ & so $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$. Hence $\bigcup_{i=1}^{\infty} A_i \in \sigma(A)$.

□

The most important σ-algebra is the:

Definition 3 (Borel σ-algebra)**.** *This is the σ-algebra on $\mathbb{R}$ generated by the family of open intervals in $\mathbb{R}$.*

Definition 4 (Borel Set)**.** *A Borel Set is any set which belongs to the Borel σ-algebra eg. $\phi, \mathbb{R}$, any open interval, any closed interval ($[a,b] = \bigcap_{i=1}^{\infty}(a - \frac{1}{i}, b + \frac{1}{i})$).*
Most reasonable sets are Borel:
$[a,b) = \bigcap_{n=1}^{\infty}(a - \frac{1}{n}, b), \{a\} = \bigcap_{n=1}^{\infty}(a - \frac{1}{n}, a + \frac{1}{n}, \mathbb{Q} = \bigcup_{n=1}^{\infty} r_n, I(irrationals) = \mathbb{Q}^c.$

Proposition 2. *Open sets are Borel.*

Proof. If G is open and $g \in G$, we can choose an I_g with rational end points such that $g \in I_g \subset G$. Since there are only countably many open intervals with rational end points, we may arrange the intervals I_g, $g \in G$ as a sequence of open intervals $\{I_n\}_{n=1}^{\infty}$. Then $G = \bigcup_{n=1}^{\infty} I_n$ and so G is a Borel set. □

Corollary 1. *Closed sets are Borel sets.*

Proof. They are complements of open sets □

Note. *Two different collections of sets can give rise to the same σ-algebra.*

Example 3. *Let*

$$I = \textit{collection of open intervals in } \mathbb{R} \textit{ and}$$
$$\theta = \textit{collection of open sets in } \mathbb{R}.$$

Then $I \subset \theta$ so $I \subset \sigma(\theta)$. $\sigma(I)$ is the smallest σ-algebra containing I so $\sigma(I) \subset \sigma(\theta)$.
Open sets are Borel sets so $\theta \subset \sigma(I)$. $\sigma(\theta)$ is the smallest σ-algebra containing θ so $\sigma(\theta) \subset \sigma(I)$. Hence $\sigma(\theta) = \sigma(I)$

Definition 5. *If $\mathcal{F}$ is a σ-algebra on a set Ω, then a* measure *on $\mathcal{F}$ is a function, μ such that:*

$$\mu : \mathcal{F} \to [0, \infty]$$

satisfying:

1. $\mu(\emptyset) = 0$

2. *If $E_1, E_2, ... \in \mathcal{F}$ and $E_i \cap E_j = \emptyset, i \neq j$, then*

Example 4. *Let $\Omega =$ any set, $\mathcal{F} = \{$all subsets of $\Omega\}$. Fix $x \in \Omega$, then for $E \in \mathcal{F}$ define*

$$\delta_x(E) = \begin{cases} 0, & \text{if } x \notin E \\ 1, & \text{if } x \in E \end{cases}$$

We claim that δ_x is a measure on $\mathcal{F}$.

Proof. We prove the properties of measures one-by-one.

1. $\delta_x(\emptyset) = 0$
2. If $E_1, E_2, ... \in \mathcal{F}$ and $E_i \cap E_j = \emptyset, i \neq j$, then,

   either $x \notin \bigcup_{i=1}^{\infty} E_i$ and hence $x \notin E_i$ for all i so

$$\delta_x(\bigcup_{i=1}^{\infty}) = 0 = \sum_{i=1}^{\infty} \delta_x(E_i)$$

   or $x \in \bigcup_{i=1}^{\infty} E_i$ so $x \in$ exactly oneE_j and $\delta_x(E_i) = 0, \text{for} i \neq j$. Then

$$\delta_x(\bigcup_{i=1}^{\infty} E_i) = 1 = \delta_x(E_j) = \sum_{i=1}^{\infty} \delta_x(E_i)$$

$\square$

Note. *If $c \in [0, \infty]$, then $c\delta_x$ is also a measure. ($\infty \cdot 0 = 0$)*

Example 5. *We define the discrete counting measure, γ, by*

$$\gamma(E) = \sum_{x \in E} 1 = \textit{number of elements in } E$$

Proposition 3. Properties of Measures

1. *If $A, B \in \mathcal{F}$ and $A \subset B$, then*
$$\mu(A) \leq \mu(B)$$
2. *If $A, B \in \mathcal{F}$, $A \subset B$ and $\mu(A) < \infty$, then*
$$\mu(B) - \mu(A) = \mu(B \setminus A)$$
3. σ-subadditivty*. If $E_1, E_2, ... \in \mathcal{F}$, then*
$$\mu(\bigcup_{i=1}^{\infty}) \leq \sum_{i=1}^{\infty} \mu(E_i)$$
4. Continuity of measures*. If $E_1, E_2, ... \in \mathcal{F}$ and $E_1 \subset E_2 \subset ...$, then*
$$\mu(\bigcup_{i=1}^{\infty} E_i) = \lim_{n \to \infty} \mu(E_n)$$
5. *If $E_1, E_2, ... \in \mathcal{F}$, $E_1 \supset E_2 \supset ...$ and $\mu(E_1) < \infty$, then*

Proof.

1. $\mu(B) = \mu(A) + \mu(B \setminus A)$ and $\mu(B \setminus A) \geq 0$, so $\mu(B) \geq \mu(A)$.
2. Rearrange 1. and $\mu(A) < \infty$ so the sum makes sense.
3. Let
$$F_i = E_i \setminus \bigcup_{i<j} E_j$$
then $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$, $F_1, F_2, ... \in \mathcal{F}$, $F_i \cap F_j = \emptyset, i \neq j$ and $F_i \subset E_i$ for all i.
So
$$\mu(\bigcup_{i=1}^{\infty} E_i) = \mu(\bigcup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} \mu(F_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$$
4. Let
$$F_i = E_i \setminus \bigcup_{i<j} E_j$$
then
$$\begin{aligned}\mu(\bigcup_{i=1}^{\infty} E_i) = \mu(\bigcup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} \mu(F_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(F_i) &= \lim_{n\to\infty} \mu(\bigcup_{i=1}^{n} F_i) \\ &= \lim_{n\to\infty} \mu(E_n)\end{aligned}$$
5.
$$\mu(E_1) = \mu(E_1 \setminus \bigcap_{i=1}^{\infty} E_i) + \mu(\bigcap_{i=1}^{\infty} E_i)$$
So
$$\begin{aligned}\mu(\bigcap_{i=1}^{\infty} E_i) &= \mu(E_1) - \mu(E_1 \setminus \bigcap_{i=1}^{\infty} E_i) \\ &= \mu(E_1) - \mu(\bigcup_{i=1}^{\infty} E_1 \setminus E_i) \\ &= \mu(E_1) - \lim_{n\to\infty} \mu(E_1 \setminus E_n) \\ &= \mu(E_1) - \mu(E_1) + \lim_{n\to\infty} \mu(E_n)\end{aligned}$$

□

λ, *Lebesgue measure*, will be our means of defining a concept for length, area, volume etc. of a set.
On $\mathbb{R}$ we clearly desire $\lambda_1((a,b)) = b - a$.
On $\mathbb{R}^2$ we clearly desire $\lambda_2((a,b) \times (c,d)) = (b-a)(d-c)$. And so on.

On $\mathbb{R}$, if a set A is contained in $\bigcup_{i=1}^{\infty}(a_i, b_i)$ we must have by σ-subadditivity:
$$\lambda(A) \leq \sum_{i=1}^{\infty} \lambda(a_i, b_i) \leq \sum_{i=1}^{\infty} (b_i - a_i)$$
which motivates us to define:
$$\lambda^*(A) = \inf\{\sum_{i=1}^{\infty}(b_i - a_i) : A \subset \bigcup_{i=1}^{\infty}(a_i, b_i)\}$$

2 Outer Measure

Definition 6. *An* outer measure *on a set* Ω *is a function:*

$$\mu^* : \{\textit{All subsets of } \Omega\} \to [0, \infty]$$

such that:

1. $\mu^*(\emptyset) = 0$
2. Monotonicity. *If* $A \subset B$, *then* $\mu^*(A) \leq \mu^*(B)$.
3. σ-subadditivity. $\mu(\bigcup_{i=1}^{\infty} B_i) \leq \sum_{i=1}^{\infty} \mu^*(B_i)$

Definition 7. *Let* $\mathcal{A}$ *be a family of subsets of* Ω. *Now define a function* $\phi : \mathcal{A} \to [0, \infty]$. *Let* B *be an arbitrary subset of* Ω *and define:*

$$\mu^*(A) = \inf\{\sum_{i=1}^{\infty} \phi(A_i) : B \subset \bigcup_{i=1}^{\infty} A_i, \ \textit{and } A_1, A_2, ... \in \mathcal{A}\}$$

Further, define $\mu^*(\emptyset) = 0$ *and* $\mu^*(B) = \infty$ *if no such* A_i *(i.e. no such cover) exist(s).*

Lemma 2. μ^* *is an outer measure.*

Proof.

1. from definition
2. from definition
3. Consider a set B_j and cover B_j by sets $A_i^{(j)}$ in $\mathcal{A}$ such that

$$B_j \subset \bigcup_{i=1}^{\infty} A_i^{(j)} \text{ and } \sum_{i=1}^{\infty} \phi(A_i^{(j)}) \leq \mu^*(B_j) + \frac{\epsilon}{2^j}$$

then

$$\mu^*(\bigcup_{i=1}^{\infty} B_j) \leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \phi(A_i^{(j)}) \leq \sum_{i=1}^{\infty} \mu^*(B_j) + \epsilon$$

□

Definition 8. *If* μ^* *is an outer measure on a set* Ω, *we say that a set* $A \subset \Omega$ *is* μ^* measurable *if, for any* $T \subset \Omega$*:*

$$\mu^*(T \cap A) + \mu^*(T \setminus A) = \mu^*(T)$$

Theorem 3. *If* μ^* *is an outer measure on* Ω, *then the family of* μ^* *measurable sets,* $\mathcal{F}(\mu^*)$, *is a* σ*-algebra and* μ^* *is a measure on* $\mathcal{F}(\mu^*)$.

Proof. We first show that $\mathcal{F}(\mu^*)$ is an algebra:

1. If $A \in \mathcal{F}(\mu^*)$, then $\Omega \setminus A \in \mathcal{F}(\mu^*)$

   Proof. For any $T \subset \Omega$

$$\begin{aligned} \mu^*(T) &= \mu^*(T \cap A) + \mu^*(T \setminus A) \\ &= \mu^*(T \setminus (\Omega \setminus A)) + \mu^*(T \cap (\Omega \setminus A)) \end{aligned}$$

2. $\emptyset \in \mathcal{F}(\mu^*)$ and $\Omega \in \mathcal{F}(\mu^*)$

Proof. For any $T \subset \Omega$

$$\begin{aligned}\mu^*(T) &= \mu^*(T \cap \emptyset) + \mu^*(T \setminus \emptyset)\\ &= 0 + \mu^*(T)\\ &= \mu^*(T)\end{aligned}$$

By 1., $\Omega \in \mathcal{F}(\mu^*)$. □

3. If $A, B \in \mathcal{F}(\mu^*)$, then $A \cup B \in \mathcal{F}(\mu^*)$

Proof. Let $A, B \in \mathcal{F}(\mu^*)$ and let $T \subset \Omega$ be an arbitrary set. A is μ^* measurable, so

$$\mu^*(T) = \mu^*(T \cap A) + \mu^*(T \setminus A)$$

We now test the measurability of B with $T \cap A$

$$\begin{aligned}\mu^*(T \cap A) &= \mu^*(T \cap A \cap B) + \mu^*((T \cap A) \setminus B)\\ \text{so } \mu^*(T) &= \mu^*(T \cap A \cap B) + \mu^*((T \cap A) \setminus B) + \mu^*(T \setminus A)\\ &\geq \mu^*(T \cap A \cap B) + \mu^*(((T \cap A) \setminus B) \cup (T \setminus A))\\ &\geq \mu^*(T \cap (A \cap B)) + \mu^*(T \setminus (A \cap B))\end{aligned}$$

Since $(((T \cap A) \setminus B) \cup (T \setminus A)) \supset (T \setminus (A \cap B))$ (monotonicity).
Now, by the subadditivity of outer measures,

$$\mu^*(T) \leq \mu^*(T \cap (A \cap B)) + \mu^*(T \setminus (A \cap B))$$

and hence

$$\mu^*(T) = \mu^*(T \cap (A \cap B)) + \mu^*(T \setminus (A \cap B))$$

So $A \cap B$ is μ^* measurable, for $A, B \in \mathcal{F}(\mu^*)$. By *De Morgan's Laws*, $A \cup B = \Omega \setminus ((\Omega \setminus A) \cap (\Omega \setminus B))$ so, by 2., $A \cup B \in \mathcal{F}(\mu^*)$. □

So $\mathcal{F}(\mu^*)$ is an algebra. We must now prove that $\mathcal{F}(\mu^*)$ is a σ-algebra.

Let $F_1, ..., F_n \in \mathcal{F}(\mu^*)$ be disjoint sets, then, since $\mathcal{F}(\mu^*)$ is an algebra, $\bigcup_{i=1}^n F_i$ and $\bigcap_{i=1}^n F_i \in \mathcal{F}(\mu^*)$.
We claim $\mu^*(T \cap \bigcup_{i=1}^n F_i) = \sum_{i=1}^n \mu^*(T \cap F_i)$ for all n.

Proof. Let $n = 1$, then trivially $\mu^*(T \cap F_1) = \mu^*(T \cap F_1)$.
Assume our claim holds for some $n \geq 1$, then consider

$$\begin{aligned}\mu^*(T \cap \bigcup_{i=1}^{n+1} F_i) &= \mu^*((T \cap \bigcup_{i=1}^{n+1} F_i) \cap F_{n+1}) + \mu^*((T \cap \bigcup_{i=1}^{n+1} F_i) \setminus F_{n+1})\\ &= \mu^*(T \cap F_{n+1}) + \mu^*(T \cap \bigcup_{i=1}^{n} F_i)\\ &= \mu^*(T \cap F_{n+1}) + \sum_{i=1}^{n} \mu^*(T \cap F_i)\\ &= \sum_{i=1}^{n+1} \mu^*(T \cap F_i)\end{aligned}$$

Now let $E_1, E_2, ... \in \mathcal{F}(\mu^*)$ be arbitrary sets and define

$$F_i = E_i \setminus \bigcup_{j<i} E_j$$

So that $F_i \cap F_j = \emptyset, i \neq j$ and, since $\mathcal{F}(\mu^*)$ is an algebra, $F_i \in \mathcal{F}(\mu^*)$ for all i. Also note that

$$\bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{n} F_i \text{ for all } n \text{, so } \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$$

Now let $T \subset \Omega$ be any set and recall $\bigcup_{i=1}^{n} F_i \in \mathcal{F}(\mu^*)$, so

$$\begin{aligned}
\mu^*(T) &= \mu^*(T \cap \bigcup_{i=1}^{n} F_i) + \mu^*(T \setminus \bigcup_{i=1}^{n} F_i) \\
&\geq \mu^*(T \cap \bigcup_{i=1}^{n} F_i) + \mu^*(T \setminus \bigcup_{i=1}^{\infty} F_i) \\
&= \sum_{i=1}^{n} \mu^*(T \cap F_i) + \mu^*(T \setminus \bigcup_{i=1}^{\infty} F_i) \\
&\xrightarrow{\text{as } n \to \infty} \sum_{i=1}^{\infty} \mu^*(T \cap F_i) + \mu^*(T \setminus \bigcup_{i=1}^{\infty} F_i) \\
&\geq \mu^*(T \cap \bigcup_{i=1}^{\infty} F_i) + \mu^*(T \setminus \bigcup_{i=1}^{\infty} F_i)
\end{aligned}$$

But μ^* is subadditive so

$$\mu^*(T) \leq \mu^*(T \cap \bigcup_{i=1}^{\infty} F_i) + \mu^*(T \setminus \bigcup_{i=1}^{\infty} F_i)$$

Consequently

$$\begin{aligned}
\mu^*(T) &= \mu^*(T \cap \bigcup_{i=1}^{\infty} F_i) + \mu^*(T \setminus \bigcup_{i=1}^{\infty} F_i) \\
&= \mu^*(T \cap \bigcup_{i=1}^{\infty} E_i) + \mu^*(T \setminus \bigcup_{i=1}^{\infty} E_i)
\end{aligned}$$

So $\bigcup_{i=1}^{\infty} E_i \in \mathcal{F}(\mu^*)$ and $\mathcal{F}(\mu^*)$ is a σ-algebra. □

Definition 9. *If we restrict μ^* to $\mathcal{F}(\mu^*)$, then we replace μ^* by μ and simply say "the measure μ".*

3 Lebesgue Measure

Definition 10. *The* Lebesgue outer measure on $\mathbb{R}$ *is defined as*

$$\lambda^*(A) = \inf\{\sum_{i=1}^{\infty}(b_i - a_i) : A \subset \bigcup_{i=1}^{\infty}(a_i, b_i)\}$$

Proof. 1. If $A \subset \bigcup_{i=1}^{\infty}(a_i, b_i)$, then $A \subset \bigcup_{i=1}^{\infty}[a_i, b_i]$ and so

$$\inf\{\sum_{i=1}^{\infty}(b_i - a_i) : A \subset \bigcup_{i=1}^{\infty}[a_i, b_i]\} \leq \inf\{\sum_{i=1}^{\infty}(b_i - a_i) : A \subset \bigcup_{i=1}^{\infty}(a_i, b_i)\}$$

2. Let $\epsilon > 0$. If $A \subset \bigcup_{i=1}^{\infty}[a_i, b_i]$, then $A \subset \bigcup_{i=1}^{\infty}(a_i - \frac{\epsilon}{2^i}, b_i + \frac{\epsilon}{2^i})$ and

$$\sum_{i=1}^{\infty}((b_i + \frac{\epsilon}{2^i}) - (a_i - \frac{\epsilon}{2^i})) = 2\epsilon + \sum_{i=1}^{\infty}(b_i - a_i)$$

So

$$\inf\{\sum_{i=1}^{\infty}(b_i - a_i) : A \subset \bigcup_{i=1}^{\infty}(a_i, b_i)\} \leq 2\epsilon + \inf\{\sum_{i=1}^{\infty}(b_i - a_i) : A \subset \bigcup_{i=1}^{\infty}[a_i, b_i]\}$$

Combining 1. and 2. yields equality. □

Lemma 4. *If* $[a, b] \subset \bigcup_{i=1}^{\infty}(a_i, b_i)$*, then* $b - a \leq \sum_{i=1}^{\infty}(b_i - a_i)$

Proof. By Heine-Borel theorem, if a closed interval is contained in an union of open intervals, then there exists a finite subcover of the closed interval. In our case there exists a finite n such that

$$[a, b] \subset \bigcup_{i=1}^{n}(a_i, b_i)$$

So we need only show that for such an $[a, b] \subset \mathbb{R}$, $b - a \leq \sum_{i=1}^{n}(b_i - a_i)$.
Result holds for $n = 1$. Assume result holds for some finite $n \geq 1$. For the case $n + 1$, we may assume $a_{n+1} \leq a_i$ for all i and $a_{n+1} < a$.

1. If $b_{n+1} > b$, then

$$b - a \leq b_{n+1} - a_{n+1} \leq \sum_{i=1}^{n+1}(b_i - a_i)$$

2. If $b_{n+1} < b$ (and $b_{n+1} > a$), then $[b_{n+1}, b]$ is covered by $\bigcup_{i=1}^{n}(a_i, b_i)$, so by inductive hypothesis

$$\begin{aligned} b - a &= (b - b_{n+1}) + (b_{n+1} - a) \\ &\leq \sum_{i=1}^{n}(b_i - a_i) + (b_{n+1} - a_{n+1}) \\ &= \sum_{i=1}^{n+1}(b_i - a_i) \end{aligned}$$

1. and 2. prove our claim inductively for $n + 1$, so claim holds inductively for all n and our lemma is proved. □

Lemma 5. $\lambda^*(a, b) = \lambda^*[a, b] = b - a$

Proof. Note, by Definition 10

Now $[a, b] \subset (a - \epsilon, b + \epsilon)$ for all $\epsilon > 0$ so

$$\lambda^*[a, b] \leq b - a + 2\epsilon$$

and by Lemma 4 we may deduce

$$\lambda^*[a, b] = b - a$$

Furthermore

$$b - a - 2\epsilon \leq \lambda^*[a + \epsilon, b - \epsilon] \leq \lambda^*(a, b) \leq \lambda^*[a, b] = b - a$$

So $\lambda^*(a, b) = b - a$ also. $\square$