Measure Theory

Prof. D Larman

Introduction. In this course we first seek to define the measure of a set eg. the length, area, volume, probability of a set. We also seek to improve on the riemann integral by defining the lebesgue integral.
If λ denotes the "length" of a set in $\mathbb{R}$., clearly we would expect λ[0,1]=1. But what about the length of [0,1]\\$\mathbb{Q}$ where $\mathbb{Q}$ is the set of rationals? Or the set $\bigcup_{i=0}^{\infty}[\frac{1}{2^{i+1}}+\frac{1}{2^i}]$? Since $\mathbb{Q}$ is quite "small" we might expect $\lambda([0,1]\backslash\mathbb{Q})$=1. Also we might expect
$\lambda(\bigcup_{i=0}^{\infty}[\frac{1}{2^{i+1}}+\frac{1}{2^i}])$=$\sum_{i=0}^{\infty}\lambda([\frac{1}{2^{i+1}}+\frac{1}{2^i}]$. Both expectations are true!

If we take the function $f(x) = \begin{cases} 1 & \text{for} \quad x \textit{ irrational} \\ 0 & \text{for} \quad x \textit{ rational} \end{cases}$

then you will know from analysis 2 that $(\mathbf{R})\int\limits_{0}^{-1} f(x)\,dx = 1$ and $(\mathbf{R})\int\limits_{-0}^{1} f(x)\,dx = 1$
however the vast majority of x in [0,1] are irrational and and so we might expect the integral to be 1. When we have defined the labesgue integral we will find $(\mathbf{L})\int_0^1 f(x)\,dx = 1$

1 Measures

We will work within a set Ω. For example $\Omega = \mathbb{R}$, $\Omega = \mathbb{R}^n$, $\Omega = \{sequence of heads \& tails\}$. Families of subsets of Ω will be denoted by $\mathcal{F}$, $\mathcal{G}$ etc.

Definition 1. *Algebra of sets:*
A family $\mathcal{F}$ of subsets of Ω is called an Algebra if it satisfies:

(i) $\phi, \Omega \in \mathcal{F}$

(ii) If $A \in \mathcal{F}$ *then* $A^c = \Omega \backslash A \in \mathcal{F}$

(iii) If $A,B \in \mathcal{F}$ *then* $A \cup B \in \mathcal{F}$

Example 1. *If Ω=[0,1] and $\mathcal{F}$ is the family of all subsets of [0,1] which can be expressed as a finite union of intervals (which can be open, closed half open, empty) then $\mathcal{F}$ is an algebra.*

Definition 2. *σ-Algebra of sets:*
A family $\mathcal{F}$ of subsets of Ω is called a σ-Algebra if it satisfies:

(i) $\phi, \Omega \in \mathcal{F}$

(ii) If $A \in \mathcal{F}$ *then* $A^c = \Omega \backslash A \in \mathcal{F}$

(iii) If $A_1, A_2, \ldots$ *is a sequence of sets in* $\mathcal{F}$ *then* $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

Example 2. *For any Ω.*

$\mathcal{F}$=$\{\phi, \Omega\}$ *is a σ-algebra.*

$\mathcal{F}$=$\{$*all subsets of* $\Omega\}$ *is a σ-algebra.*

Remark: althouth example 1 is an algebra, it is not a σ-algebra (try to prove it). Notice that a σ-algebra is an algebra.

Theorem 1. *De Morgan's Laws If A_α, $\alpha \in I$ is a family of sets inΩ then*
$(\cup_{\alpha\in I} A_\alpha)^c = \cap_{\alpha\in I} A_\alpha^c$
$(\cap_{\alpha\in I} A_\alpha)^c = \cup_{\alpha\in I} A_\alpha^c$

From the definition of an algebra or a σ-algebra we can deduce the following properties:

Algebra

(i) $A_i, i = 1, 2, \dots, n \in \mathcal{F} \implies \bigcup_{i=1}^n A_i \in \mathcal{F}$ (induction)

(ii) $A_i, i = 1, 2, \dots, n \in \mathcal{F} \implies \bigcap_{i=1}^n A_i \in \mathcal{F}$ (By De Morgan (ii))

(iii) A,B $\in \mathcal{F} \implies A\backslash B \in \mathcal{F}$ (Since $A\backslash B = A \cap B^c$)

σ-Algebra

(i) $A_1, A_2, \dots \in \mathcal{F}$ then $\bigcap_{i=1}^\infty A_i \in \mathcal{F}$ ($\bigcap_{i=1}^\infty A_i = \bigcap_{i=1}^\infty (A_i^c)^c = (\bigcup_{i=1}^\infty A_i^c)^c \in \mathcal{F}$)

Proposition 1. *(σ-algebra generated by A)*
For any family of subsets A of Ω, there is a smallest σ-algebra $\sigma(A)$ containing A.
i.e. $\sigma(A) \supset A$ and if $\mathcal{F}$ is any σ-algebra containing A then $\sigma(A) \subset \mathcal{F}$. We call $\sigma(A)$ the σ-algebra generated by A.

Proof. Just note that there is a σ-algebra containing A, namely {all subsets of A}.
Consider all σ-algebras containing A and let $\sigma(A)$ be their intersection. i.e. B$\in \sigma(A)$ iff B belongs to every σ-algebra containing A. We certainly have A$\subset \sigma(A)$ and if $\mathcal{F}$ is a σ-algebra containing A then $\sigma(A) \subset \mathcal{F}$. It remains to show that $\sigma(A)$ is a σ-algebra.

(i) $\phi, \Omega \in \sigma(A)$ since they belong to every σ-algebra containing A.

(ii) If A$\in \sigma(A)$ and $\mathcal{F}$ is a σ-algebra containing A, then A$\in \mathcal{F}$ and so $A^c \in \mathcal{F}$.
So $A^c \in \sigma(A)$

(iii) If $\{A_i\}_{i=1}^\infty \in \sigma(A)$ and $\mathcal{F}$ is a σ-algebra containing A then $\{A_i\}_{i=1}^\infty \in \mathcal{F}$ & so $\bigcup_{i=1}^\infty A_i \in \mathcal{F}$. Hence $\bigcup_{i=1}^\infty A_i \in \sigma(A)$.

□

The most important σ-algebra is the:

Definition 3. *Borel σ-algebra:*
This is the σ-algebra on $\mathbb{R}$ generated by the family of open intervals in $\mathbb{R}$.

Definition 4. *Borel Set:*
A Borel Set is any set which belongs to the Borel σ-algebra eg.
$\phi, \mathbb{R}$, any open interval, any closed interval ($[a, b] = \bigcap_{i=1}^\infty (a - \frac{1}{i}, b + \frac{1}{i})$).
Most reasonable sets are Borel:
$[a, b) = \bigcap_{n=1}^\infty (a - \frac{1}{n}, b)$, $\{a\} = \bigcap_{n=1}^\infty (a - \frac{1}{n}, a + \frac{1}{n})$, $\mathbb{Q} = \bigcup_{n=1}^\infty r_n$, $I(irrationals) = \mathbb{Q}^c$.