Multivariate Analysis
MATH 3109
UCL

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Contents

1 Mulivarible Calculus

1.1 Notation

$X \in \mathbb{R}^n$, $X = \{x_1, x_2, \dots, x_n\}$ where $x_i \in \mathbb{R}$ $\mathbb{R}^n$ is a vector space
length norm:

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

If $Y, X \in \mathbb{R}^n$ and $Y = \{y_1, y_2, \dots, y_n\}$ then

$$X \cdot Y = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

Standard Basis:

$$e_j = (0, \dots, \underset{\text{j-1}}{0}, \underset{\text{j}}{1}, \underset{\text{j+1}}{0}, \dots)$$

Properties of norm

$$|x| \geq 0$$

$$|x| = 0 \Leftrightarrow x = \vec{0}$$

$$|\lambda x| = |\lambda| \cdot |x|, \quad x \in \mathbb{R}^n, \quad \lambda \in \mathbb{R}$$

linear Transformation

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

(i) $T(x + y) = T(x) + T(y)$

(ii) $T(\lambda x) = \lambda T(x)$

Matrix Representation of T with respect to the standard basis:

$$T(e_i) = \sum_{j=1}^{m} a_{i,j}e_j \quad \text{where} \quad [T]_\epsilon^\epsilon = A = (a_i, j)_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$$

Given: $T : \mathbb{R}^n \to \mathbb{R}^m, S : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^m \to \mathbb{R}^k$

(i) $[UT]_{kxm} = [U]_{kxm}[T]_{mxn}$

(ii) $[T + S] = [T] + [S]$

(iii) $\lambda[T] = [\lambda T]$

$T : \mathbb{R}^n \to \mathbb{R}^m, X \in \mathbb{R}^n, Y \in \mathbb{R}^m, X = (x^1, \dots, x^n), Y = (y^1, \dots, y^m)$

$$\begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^m \end{pmatrix} = [T] \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}$$

1.2 Functions & Continuity

$f : \mathbb{R}^n \to \mathbb{R}^m$ vector valued function
$f : A \to \mathbb{R}^m$ where $A \subset \mathbb{R}^n$
then f has components which are scalar fields.
$f^i : A \to \mathbb{R}$

$$f(x) = (f^1(x), \ldots, f^m(x))$$

$\Pi^i : \mathbb{R}^m \to \mathbb{R}$

$$\Pi^i((x)^1, \ldots, (x)^m)$$

Π^i is a linear transformation for i=1,...,m

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "f"] \arrow[dr, "f^i"'] & \mathbb{R}^m \arrow[d, "\Pi^i"] \\
 & \mathbb{R}
\end{tikzcd}$$

Definition 1.1. *$f : \mathbb{R}^n \to \mathbb{R}^m$ then $\lim_{x \to a}(f(x)) = b$ means:*

$$\forall \epsilon > 0, \exists \delta > 0 \ st, \ 0 < |x - a| < \delta \Rightarrow |f(x) - b| < \epsilon$$

Definition 1.2. *f is called continuious at a if:*

$$\lim_{x \to a}(f(x)) = f(a)$$

f is called continuious on the set A if it is continuious at a, $\forall a \in A$

Theorem 1.1 (Combination Theorm)**.** *Assume*

$$\lim_{x \to a}(f(x)) = b, \quad lim_{x \to a}(g(x)) = c$$

then:

(i) $\lim_{x \to a}(f(x) + g(x)) = b + c$

(ii) $\lim_{x \to a}(\lambda f(x)) = \lambda b$

(iii) $\lim_{x \to a}(f(x) \cdot g(x)) = b \cdot c$

(iv) $\lim_{x \to a} |f(x)| = |b|$

Proof. of (iii)

$$\begin{aligned}
f(x) \cdot g(x) - b \cdot c &= f(x) \cdot g(x) - b \cdot g(x) + b \cdot g(x) - b \cdot c \\
&= g(x)\dot{(}f(x) - b) + b \cdot (g(x) - c) \\
|f(x) \cdot g(x) - b \cdot c| &= |g(x)\dot{(}f(x) - b) + b \cdot (g(x) - c)| \\
&\leq |g(x)\dot{(}f(x) - b)| + |b \cdot (g(x) - c)|
\end{aligned}$$

Cauchy-Schwartz: $|x^1y^1 + \dots + x^ny^n| \leq \sqrt{(x^1)^2 + \dots + (x^n)^2} \cdot \sqrt{(y^1)^2 + \dots + (y^n)^2}$

$$|f(x) \cdot g(x) - b \cdot c| \leq |g(x)\dot{(}f(x) - b)| + |b \cdot (g(x) - c)| \leq |g(x)| \cdot |f(x) - b| + |b| \cdot |g(x) - c|$$

Since $lim_{x \to a}(g(x)) = c$, g is a bounded neighbourhood of a, i.e:

$$\exists M \geq 0,\ \exists \delta > 0\ st,\ |g(x)| \leq M\ for\ |x - a| < \delta$$

□

Remark. *We have:*

(i) $f : \mathbb{R}^n \to \mathbb{R}^m$ *is continuous iff:* $f^i : \mathbb{R}^n \to \mathbb{R}$ *is continuous for* $i = 1, \dots, m$

(ii) Polynomial functions in n-variables, $f(x^1, \dots, x^n)$*, are continuous*

(iii) Rational functions, $R(x) = \frac{P(x)}{Q(x)}$*, are continuous where defined, ie:* $Q(x) \neq 0$ *and* P*,* Q *are polynomials in n-variables.*

Theorem 1.2. *Linear transformations are continuous.*

Proof. $T : \mathbb{R}^n \to \mathbb{R}^m$ let $a \in \mathbb{R}^n$ to show:

$$\lim_{h \to 0} T(a + h) = T(a)$$

where $h = (h^1, \dots, h^n)$

$$\begin{aligned} |T(a + h) - T(a)| &= |T(h)| = |T(h^1e_1 + \dots + h^ne_n)| = |h^1T(e_1) + \dots + h^nT(e_n)| \\ &\leq |h^1||T(e_1)| + \dots |h^n||T(e_n)| \leq |h|(T(e_1) + \dots T(e_n)) \end{aligned}$$

$$So: \quad |T(a + h) - T(a)| \leq M|h| \quad where \quad M = \sum_{i=1}^{n} |T(e_i)|$$

$$So\,given \quad \epsilon > 0, \quad choose \quad \delta = \frac{\epsilon}{M} \quad such\ that \quad |h| < \delta \Rightarrow |T(a + h) - T(a)| < \epsilon$$

□

Example 1.1. $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$, $\quad (x, y) = (0, 0)$ *assume* $\quad \lim_{(x,y) \to (0,0)} f(x, y) = L$

$$\forall \epsilon > 0, \quad \exists \delta > 0 \quad such\ that \quad 0 < |(x, y)| < \delta \Rightarrow |f(x, y) - L| < \epsilon$$

Plug $(x, 0)$ *into* f*:*

$$f(x, 0) \ = \frac{x^2 - 0}{x^2 - 0} = 1$$

Plug $(0, y)$ *into* f*:*

$$f(0, y) \ = \frac{0 - y^2}{0 + y^2} = -1$$

$$If\ |x| < \delta \quad |f(x,0)| < \delta \Rightarrow |f(x,0) - L| < \epsilon \quad ie \quad |1 - L| < \epsilon$$
$$If\ |y| < \delta \quad |f(0,y)| < \delta \Rightarrow |f(0,y) - L| < \epsilon \quad ie \quad |-1 - L| < \epsilon$$
$$\Rightarrow \epsilon = \frac{1}{2} \quad contradiction!$$

Now consider $y = mx, m \in \mathbb{R}$

$$f(x, mx) = \frac{x^2 - (mx)^2}{x^2 + (mx)^2} = \frac{1 - m^2}{1 + m^2}$$
$$\lim_{x \to 0}(\lim_{y \to 0} f(x,y)) = \lim_{x \to 0} 1 = 1$$
$$\lim_{y \to 0}(\lim_{x \to 0} f(x,y)) = \lim_{y \to 0} -1 = -1$$

However checking along straight lines is not enough to prove continuity.

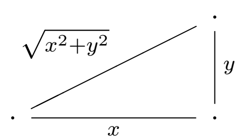
Example 1.2.

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & if\ (x,y) \neq (0,0) \\ 0 & if\ (x,y) = (0,0) \end{cases}$$

Show f is continuous at (0,0)

$$\forall \epsilon > 0, \quad \exists \delta > 0$$
$$\left| \frac{xy}{\sqrt{x^2+y^2}} \right| \leq \frac{|x| \cdot |y|}{\sqrt{x^2+y^2}} \leq \frac{\sqrt{x^2+y^2} \cdot \sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} = |(x,y)|$$

Since:



Note. *if the total degree of the neumerator is higher than the denominator in a rational function, then the limit should be 0.*

Theorem 1.3. *If f is continuous at a and g is continuous at $f(a)$ then $g \circ f$ is continuous at a.*

1.3 Partial Derivitives

Definition 1.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$, $a \in \mathbb{R}$ Define:*

$$D_i f(a) = \lim_{h \to 0} \frac{f(a^1, \ldots, a^{i-1}, a^i + h, a^{i+1}, \ldots, a^n)}{h}$$

Example 1.3. *if $f : \mathbb{R}^2 \to \mathbb{R}$*

$$\left.\frac{df}{dx}\right|_{(a,b)} = D_1 f(a,b)$$
$$\left.\frac{df}{dy}\right|_{(a,b)} = D_2 f(a,b)$$
$$and\ in\ \mathbb{R}^3\ we\ use\ \frac{df}{dx}, \frac{df}{dy}\ and\ \frac{df}{dz}\ etc.$$

Example 1.4.

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \textit{if } (x,y) \neq (0,0) \\ 1 & \textit{if } (x,y) = (0,0) \end{cases}$$

$$D_1 f(0,0) = \left.\frac{df}{dx}\right|_{(0,0)} = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} \frac{\frac{x^2-0}{x^2-0}-1}{x} = 0$$

$$D_2 f(0,0) = \left.\frac{df}{dy}\right|_{(0,0)} = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0} \frac{\frac{0-y^2}{0+y^2}-1}{y} = \lim_{y\to 0} \frac{-2}{y} = \pm\infty$$

1.4 Total Derivitive

In 1 dimention we write the following for the derivitive of $f : \mathbb{R} \to \mathbb{R}$

$$f^{'}(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

we try to write it in higher dimentions $f : \mathbb{R}^n \to \mathbb{R}^m$ in this form

$$\begin{aligned} \lim_{h\to 0} \left[\frac{f(a+h)-f(a)}{h} - f^{'}(a) \right] &= \lim_{h\to 0} \left[\frac{f(a+h)-f(a)-h\cdot f^{'}(a)}{h} \right] \\ &= \lim_{h\to 0} \frac{|f(a+h)-f(a)-h\cdot f^{'}(a)|}{|h|} = 0 \end{aligned}$$

For $f : \mathbb{R}^n \to \mathbb{R}^m$ consider the tangent line at a: $y = f(a) + f^{'}(a)(x-a)$
call $x - a = h$ then we have:

$$y = f(a) + f^{'}(a)(h)$$

this is an Affine transformation, not a linear map.
Look at the map:

$$\lambda : h \to hf^{'}(a), \quad h \in \mathbb{R}$$

This is a linear map.

$$\lambda(h_1 + h_2) = (h_1 + h_2)f^{'}(a) = h_1 f^{'}(a) + h_2 f^{'}(a) = \lambda(h_1) + \lambda(h_2)$$

$$\lambda(\alpha \cdot h) = (\alpha h)f^{'}(a) = \alpha(hf^{'}(a)) = \alpha \cdot \lambda(h)$$

$$\lim_{h\to 0} \frac{|f(a+h)-f(a)-\lambda(h)|}{|h|} = 0$$

Definition 1.4 (Total Derivitive)**.** $f : \mathbb{R}^n \to \mathbb{R}^m$ *or* ($f : A \to \mathbb{R}^m$, $A \subset \mathbb{R}^n$, A *is open*) *is differentiable at a* ($a \in A$) *if we can find a linear transformation* $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ *st:*

$$\lim_{h\to 0} \frac{|f(a+h)-f(a)-\lambda(h)|}{|h|} = 0$$

The linear transformation λ *is called the total derivitive of* f *at* a *and denoted Df(a) st*

$$Df(a) = \lambda(h)$$

Example 1.5. *$f:\mathbb{R}^n \to \mathbb{R}^m$, $f(x)=k$, $k \in \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$ with the 0 linear transformation* $0:\mathbb{R}^n \to \mathbb{R}^m$, $0(h)=0$

$$\lim_{h\to 0} \frac{|f(a+h)-f(a)-0(h)|}{|h|} = \lim_{h\to 0} \frac{|k-k-0|}{|h|} = 0$$

Example 1.6. *If $f:\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformaton, it is differentiable at $a \in \mathbb{R}^n$ with linear transformation $Df(a)=f$*

$$\lim_{h\to 0} \frac{|f(a+h)-f(a)-f(h)|}{|h|} = \lim_{h\to 0} \frac{|f(a+h-a-h)|}{|h|} = 0$$

Theorem 1.4 (Uniqueness of Total Derivitive)**.** *If f is differentiable at a then there exists a unique linear transformation, $\lambda:\mathbb{R}^n \to \mathbb{R}^m$, such that*

$$\lim_{h\to 0} \frac{|f(a+h)-f(a)-\lambda(h)|}{|h|} = 0$$

Proof. suppose $\mu:\mathbb{R}^n \to \mathbb{R}^m$ is another linear transformation such that:

$$\lim_{h\to 0} \frac{|f(a+h)-f(a)-\mu(h)|}{|h|} = 0$$

deduce that $\lambda=\mu$ $i.e$ $\forall h \in \mathbb{R}^n$

$$\lambda(h)=\mu(h)$$

$$\begin{aligned}\frac{|\lambda(h)-\mu(h)|}{|h|} &= \frac{|\lambda(h)+f(a)-f(a+h)+f(a+h)-f(a)-\mu(h)|}{|h|} \\ &\leq \frac{|f(a+h)-f(a)-\lambda(h)|}{|h|} + \frac{|f(a+h)-f(a)-\mu(h)|}{|h|}\end{aligned}$$

Conclude that:

$$\lim_{h\to 0} \frac{|\lambda(h)-\mu(h)|}{|h|} \leq 0+0=0 \quad (*)$$

Let h=0 $\lambda=0=\mu$ since λ,μ are linear. Now fix $h \in \mathbb{R}^n$, $h \neq 0$ and let $t \in \mathbb{R}$ such that $th \in \mathbb{R}^n$ then replace h with th in (*):

$$\begin{aligned}\lim_{t\to 0} \frac{|\lambda(th)-\mu(th)|}{|th|} &= \lim_{t\to 0} \frac{|t\lambda(h)-t\mu(h)|}{|t||h|} \\ &= \lim_{t\to 0} \frac{|t|}{|t|}\frac{|\lambda(h)-\mu(h)|}{|h|} = \frac{|\lambda(h)-\mu(h)|}{|h|} = 0\end{aligned}$$

$$\Rightarrow |\lambda(h)-\mu(h)|=0 \Rightarrow \lambda(h)=\mu(h)$$

deduce that $\lambda=\mu$ $i.e$ $\forall h \in \mathbb{R}^n$

$$\lambda(h)=\mu(h)$$

$\square$

Definition 1.5 (Jacobian Matrix). *$f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$ and it is derivitive at a $Df(a) : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. Then the matrix representation of $Df(a)$ is $f^{'}(a) \in \mathbb{M}_{mxn}$ and is called the Jacobian Matrix of f at a.*

Example 1.7. $f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x,y) = (x^2y, x+5) \quad x, y \in \mathbb{R}$
Show that $Df(1,2)(h^1,h^2) = (4h^1 + h^2, h^1)$:

$$\begin{aligned}
&f((1,2) + (h^1,h^2)) - f(1,2) - Df(1,2)(h^1,h^2)\\
&= f(1+h^1, 2+h^2) - f(1,2) - (4h^1 + h^2, h^1)\\
&= ((1+h^1)^2(2+h^2), (1+h^1+5)) - (2,6) - (4h^1 + h^2, h^1)\\
&= (2 + h^2 + 2(h^1)^2 + (h^1)^2h^2 + 2h^1h^2 + 4h^1 - 2 - 4h^1 - h^2, 6 + h^1 - 6 - h^1)
\end{aligned}$$

Take length:

$$|(2(h^1)^2 + (h^1)^2h^2 + 2h^1h^2, 0)| \le 2|h|^2 + |h|^2|h| + 2|h||h| = 4|h|^2 + |h|^3$$

So:

$$\begin{aligned}
&\lim_{h\to 0} \frac{|f((1,2) + (h^1,h^2)) - f(1,2) - Df(1,2)(h^1,h^2)|}{|h|}\\
&\le \lim_{h\to 0} \frac{4|h|^2 + |h|^3}{|h|} = \lim_{h\to 0} 4|h| + |h|^2 = 0
\end{aligned}$$

Definition 1.6. *$f^{'}(a)$ is the matrix representation of $Df(a)$*

$$Df(a)(h)^t = \begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^m \end{pmatrix} = f^{'}(a) \begin{pmatrix} h^1 \\ h^2 \\ \vdots \\ h^n \end{pmatrix}$$

$$f^{'}(a) = \begin{pmatrix} D_1f^1(a) & D_2f^1(a) & \cdots & D_nf^1(a) \\ D_1f^2(a) & D_2f^2(a) & \cdots & D_nf^2(a) \\ \vdots & \vdots & \ddots & \vdots \\ D_1f^m(a) & D_2f^m(a) & \cdots & D_nf^m(a) \end{pmatrix}$$

Example 1.8. *With this new information we can tackle example 1.7:*
$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x,y) = (x^2y, x+5) \quad x, y \in \mathbb{R}$
Show that $Df(1,2)(h^1,h^2) = (4h^1 + h^2, h^1)$:

$$\frac{df^1}{dx} = 2xy, \quad \frac{df^1}{dy} = x^2, \quad \frac{df^2}{dx} = 1, \quad \frac{df^2}{dy} = 0$$

$$f^{'}(1,2) = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}$$

$$f^{'}(1,2) \begin{pmatrix} h^1 \\ h^2 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} h^1 \\ h^2 \end{pmatrix} = \begin{pmatrix} 4h^1 + h^2 \\ h^2 \end{pmatrix}$$

Remark. *Having directional derivitives in all directions $u \neq 0$ is not enough to guarantee $df(a)$ exists.*

Theorem 1.5. *If f is differentiable at a then f is continuous at a.*

Proof.

$$\begin{aligned}\lim_{h\to 0}|f(a+h)-f(a)| &= \lim_{h\to 0}|f(a+h)-f(a)-Df(a)+Df(a)| \\ &\leq \lim_{h\to 0}\frac{|f(a+h)-f(a)-Df(a)(h)|}{|h|}\cdot|h| + \lim_{h\to 0}|Df(a)(h)| \\ &= 0\end{aligned}$$

since $Df(a)$ is a linear transformation $Df(a)$ is continuous so:

$$\lim_{h\to 0}|Df(a)(h)| = |Df(a)(0)| = 0$$

. $\square$

1.5 The Chain Rule

Theorem 1.6 (Chain Rule)**.** *if $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a and $f : \mathbb{R}^m \to \mathbb{R}^k$ is differentiable at $f(a)$ then $g \circ f : \mathbb{R}^n \to \mathbb{R}^k$ is differentiable at a and*

$$D(g\circ f)(a) = Dg(f(a)) \circ Df(a)$$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "Df(a)"] \arrow[dr, "D(g\circ f)(a)"'] & \mathbb{R}^m \arrow[d, "Dg(f(a))"] \\ & \mathbb{R}^k \end{tikzcd}$$

$$(g\circ f)'(a) = g'(f(a))\cdot f'(a), \quad \textit{where } \cdot \textit{ represents matrix multiplication}$$

Proof. if $b = f(a)$ and we let $Df(a) = \lambda$ and $Dg(f(a)) = \mu$ then if we define:

$$\varphi(x) = f(x) - f(a) - \lambda(x-a) \tag{1}$$
$$\psi(y) = g(y) - g(b) - \mu(y-b) \tag{2}$$
$$\rho(x) = g\circ f(x) - g\circ f(a) - \mu\circ\lambda(x-a) \tag{3}$$

Then:

$$\lim_{h\to 0}\frac{|f(a+h)-f(a)-Df(a)(h)|}{|h|} = \lim_{x\to a}\frac{|\varphi(x)|}{|x-a|} = 0 \tag{4}$$
$$\lim_{h\to 0}\frac{|g(b+h)-g(b)-Dg(b)(h)|}{|h|} = \lim_{y\to b}\frac{|\psi(y)|}{|y-b|} = 0 \tag{5}$$

We must show:

$$\lim_{h\to 0}\frac{|g\circ f(x) - g\circ f(a) - \mu\circ\lambda(x-a)|}{|h|} = \lim_{x\to b}\frac{|\rho(x)|}{|x-b|} = 0$$

Now:

$$\begin{aligned}\rho(x) &= g(f(x)) - g(b) - \mu(\lambda(x-a)) \\ &= g(f(x)) - g(b) - \mu(f(x)-f(a)-\varphi(x)) && \text{by (1)} \\ &= [g(f(x)) - g(b) - \mu(f(x)-f(a))] + \mu(\varphi(x)) \\ &= \psi(f(x)) + \mu(\varphi(x)) && \text{by (2)}\end{aligned}$$

Thus we must Prove

$$\lim_{x \to a} \frac{|\psi(f(x))|}{|x-a|} = 0 \tag{6}$$

$$\lim_{x \to a} \frac{|\mu(\varphi(x))|}{|x-a|} = 0 \tag{7}$$

It follows from (4) that for some $\delta > 0$ we have

$$|\psi(f(x))| < \epsilon |f(x) - b| \quad if \ |f(x) - b| < \delta$$

which is true if $|x - a| < \delta_1$ for a suitable δ_1. We also have that if T is a linear transformation then $\exists M \geq 0 \ such \ that \ |T(x)| < M|x|$. So then:

$$\begin{aligned} |\psi(f(x))| &< \epsilon |f(x) - b| \\ &= \epsilon |\varphi(x) + \lambda(x-a)| \\ &\leq \epsilon |\varphi(x)| + \epsilon M |x-a| \end{aligned}$$

So

$$\lim_{x \to a} \frac{|\psi(f(x))|}{|x-a|} \leq \lim_{x \to a} \frac{\epsilon|\varphi(x)|}{|x-a|} + \lim_{x \to a} \frac{\epsilon M|x-a|}{|x-a|} = \epsilon M \to 0$$

Also

$$\lim_{x \to a} \frac{|\mu\varphi(x)|}{|x-a|} \leq \lim_{x \to a} \frac{M|\varphi(x)|}{|x-a|} = 0$$

□

Theorem 1.7. *Define* $s : \mathbb{R}^2 \to \mathbb{R} \quad s(x, y) = x + y$ *then* s *is differentiable and* $Ds = s$

Proof. S is linear i.e

$$\begin{aligned} s((x, y) + (x', y')) &= s(x + x', y + y') = s(x, y) + s(x', y') \\ s(\lambda(x, y)) &= \lambda s(x, y) \end{aligned}$$

So

$$\lim_{h \to 0} \frac{|s(a + h) - s(a) - s(h)|}{|h|} = 0$$

□

Theorem 1.8. *Define* $p : \mathbb{R}^2 \to \mathbb{R}, \quad p(x, y) = xy$*, then* p *is differentiable and:*
$Dp(a, b) : \mathbb{R}^2 \to \mathbb{R}$ *is linear with* $Dp(a, b)(h, k) = ak + bh$ *and* $p' = (b, a)$

Proof. use of derivitive

$$\begin{aligned} p((a, b) + (h, k)) - p(a, b) - Dp(a, b)(h, k) &= p(a + h, b + k) - p(a, b) - (ak + bh) \\ &= (a + h)(b + k) - ab - (ak + bh) = hk \end{aligned}$$

$$\frac{|p((a, b) + (h, k)) - p(a, b) - Dp(a, b)(h, k)|}{|(h, k)|} = \frac{|hk|}{\sqrt{h^2 + k^2}} \leq \frac{\sqrt{h^2 + k^2}\sqrt{h^2 + k^2}}{\sqrt{h^2 + k^2}} = \sqrt{h^2 + k^2} \to 0$$

□

Remark. *To check some $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear we listed two properties:*

$$T(x + y) = T(x) + T(y)$$
$$T(\lambda x) = \lambda T(x)$$

we can instead just check:

$$T(\lambda x + y) = \lambda T(x) + T(y)$$

Corollary 1.1. *$f, g : \mathbb{R}^n \to \mathbb{R}$ differentiable at $a \in \mathbb{R}^n$*

(i) $D(f + g)(a) = Df(a) + Dg(a)$

(ii) Product rule: $D(f \cdot g)(a) = g(a) \cdot Df(a) + f(a) \cdot Dg(a)$

(iii) Quotient rule: if $g(a) \neq 0,\ D(\frac{f}{g})(a) = \frac{1}{g(a)^2} \cdot (g(a) \cdot Df(a) - f(a) \cdot Dg(a))$

Proof. For (i):
We can consder the function s from theorem 1.7, $s : \mathbb{R}^2 \to \mathbb{R} \quad s(x, y) = x + y$, but acting on f and g ie $s(f, g) = f + g$ and $Ds = s$

$$D(f + g)(a) = Ds(f(a), g(a)) \circ D(f, g)(a) = s \circ (Df(a), Dg(a)) = Df(a) + Dg(a)$$

For (ii):
We can consder the function p from theorem 1.8, $p : \mathbb{R}^2 \to \mathbb{R} \quad p(x, y) = xy$, but acting on f and g ie $p(f, g) = fg$ with $Dp(f, g)(h, k) = fk + gh$

$$D(f \cdot g)(a) = Dp(f, g)(a) \cdot D(f, g)(a) = Dp(f(a), g(a)) \cdot (Df(a), Dg(a)) = f(a) \cdot Dg(a) + g(a) \cdot Df(a)$$

(iii) follows from (ii) $\square$

1.6 Mixed Derivitives

$f : \mathbb{R}^n \to \mathbb{R}, a \in \mathbb{R}$

$$D_i = lim_{h \to 0} \frac{f(a^1, \dots, a^{i-1}, a^i + h, a^{i+1}, \dots, a^n) - f(a)}{h}$$

if $D_i f(x)$ exists for al a in some open set U then we get a function $U \xrightarrow{D_i} \mathbb{R},\ x \to D_i f(x)$ then we can talk about partial derivitives of $D_i f$ eg $D_j(D_i f(x)) = D_{ij} f(x)$
If $D_i f(x)$ exists $\forall x \in U$ this is a function of x and we can consider $D_j(D_i f(x)) = D_{ji} f(x)$
In general $i \neq j$ eg $f(x, y) = x^3 y^5$:

$$D_1 f(x, y) = 3x^2 y^5 \qquad D_2 f(x, y) = 5x^3 y^4$$
$$D_{2,1} f(x, y) = 15x^2 y^4 \qquad D_{1,2} f(x, y) = 15x^2 y^4$$

Theorem 1.9. *If $D_{i,j}$ and $D_{j,i}$ are continuous on an open set containing a then*

$$D_{i,j} = D_{j,i}$$

Proof. from homework 5:
First we repeat the well-known proof that, if $g : U \to \mathbb{R}$ is continuous and $g(p) > 0$, then there exists a neighborhood V of $p(p \in V \subset U,\ V\ open)$ with

$$q \in V \Rightarrow g(q) > 0$$

Take $\epsilon = g(p)$ in the defnition of continuity of g. There there exists a V open with $p \in V$ and

$$q \in V \Rightarrow |g(q) - g(p)| < g(p)$$

Since

$$g(p) - g(q) \leq |g(q) - g(p)| < g(p) \Rightarrow -g(q) < 0 \Leftrightarrow g(q) > 0$$

we get the result. The set V can be taken to contain a closed rectangle $[a, b] \times [c, d]$.
We apply the result to $g = D_{1,2}f - D_{2,1}f$. Assume (by contradiction) that $g(p)$ is not always 0. Then there exists a point p with $g(p) \neq 0$. We can assume that $g(p) > 0$, otherwise consider $-g$. The function g is given to be continuous. We have (using Fubini twice)

$$0 < \int_{[a,b]\times[c,d]} (D_{1,2}f(x, y) - D_{2,1}f(x, y))dA$$

$$= \int_a^b \left(\int_c^d D_{1,2}f(x, y)dy \right) dx - \int_a^b \left(\int_c^d D_{2,1}f(x, y)dx \right) dy$$

$$= \int_a^b (D_1f(x, d) - D_1f(x, c))\, dx - \int_c^d (D_2f(b, y) - D_2f(a, y))\, dy$$

$$= (f(b, d) - f(a, d) - f(b, c) + f(a, c)) - (f(b, d) - f(b, c) - f(a, d) + f(a, c)) = 0$$

using the fundamental theorem of calculus 6 times. This is a contradiction, so the mixed partial derivatives are equal on the rectangle. □

Theorem 1.10. *$A \subset \mathbb{R}$ If the max or min of $f : A \to \mathbb{R}$ occur at a point a in the interior of A and $D_if(x)$ exists then $Df(a) = 0$*

Proof. Consider $h(x) = f(a^1, \ldots, a^{i-1}, x^i, a^{i+1}, \ldots a^n)$ x in an open interval arround a^i.
Since f has a max or min at a, h has a max or min at a^i

$$\frac{dh}{dx}(a^i) = D_if(a)$$

By analysis 2:

$$\frac{dh}{dx}(a^i) = 0 \Rightarrow Df(a) = 0$$

□

Note. *The converse of Theorem 1.10 is not true, even in one dimension.*

1.7 Jacobian

For $f : \mathbb{R}^n \to \mathbb{R}^m$ with total derivitive $Df(a) : \mathbb{R}^n \to \mathbb{R}^m$ a linear map. Then the Jacobian $f^{'}(a) \in \mathbb{M}_{mxn}$ is the unique representation of $Df(a)$ in the standard basis.

Theorem 1.11. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a then $D_j f^i(a)$ exists $\forall i = 1, \ldots, m$ $\forall j = 1, \ldots, n$ and the jacobian matrix is*

$$f^{'}(a) = (D_j f^i(a))_{j=1,\ldots,n}^{i=1,\ldots,m}$$

$$f^{'}(a) = \begin{pmatrix} D_1 f^1(a) & D_2 f^1(a) & \cdots & D_n f^1(a) \\ D_1 f^2(a) & D_2 f^2(a) & \cdots & D_n f^2(a) \\ \vdots & \vdots & \ddots & \vdots \\ D_1 f^m(a) & D_2 f^m(a) & \cdots & D_n f^m(a) \end{pmatrix}$$

Where $f(x) = (f^1(x), \ldots, f^m(x))$, $f^i : \mathbb{R}^n \to \mathbb{R}$

Proof. Case $m = 1$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "h"] \arrow[dr, "f \circ h"'] & \mathbb{R}^n \arrow[d, "f"] \\ & \mathbb{R} \end{tikzcd}$$

$$h(t) = (a^1, \ldots, a^{i-1}, t, a^{i+1}, \ldots, a^n) \qquad \left.\frac{d(f \circ h)}{dt}\right|_{t=a^i} = D_i f(a)$$

$$\lim_{t \to a^i} \frac{(f \circ h)(t) - (f \circ h)(a^i)}{t - a^i} = \lim_{t \to a^i} \frac{f(a^1, \ldots, a^{i-1}, t, a^{i+1}, \ldots, a^n) - f(a^1, \ldots, a^n)}{t - a^i}$$

h is differentiable because its components are differentiable ie component h^i is either constant a^j where $j \neq i$ or t when $j = i$

$$\begin{aligned} Dh(t) &= (Dh^1(t), \ldots, Dh^n(t)) \\ &= (0, \ldots, 1, \ldots, 0) \end{aligned}$$

$$h^{'}(a^i) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}_{(m \times 1)}$$

Case $m > 1$
$f : \mathbb{R}^n \to \mathbb{R}^m$

$$f(x) = (f^1(x), \ldots f^m(x))$$
$$Df(a) = (Df^1(a), \ldots Df^a)$$
$$f^{'}(a) = \begin{pmatrix} (f^1)^{'}(a) \\ \vdots \\ (f^m)^{'}(a) \end{pmatrix}_{(m \times n)}$$

$$f^{'}(a) = \begin{pmatrix} D_1f^1(a) & D_2f^1(a) & \cdots & D_nf^1(a) \\ D_1f^2(a) & D_2f^2(a) & \cdots & D_nf^2(a) \\ \vdots & \vdots & \ddots & \vdots \\ D_1f^m(a) & D_2f^m(a) & \cdots & D_nf^m(a) \end{pmatrix}$$

□

Remark. *Abuse of notation since if $f : \mathbb{R} \to \mathbb{R}$*

$$\textit{this is a number} \to \frac{dg(t_0)}{dt}| = g'(t_0) \leftarrow \textit{this is the } 1 \times 1 \textit{ jacobian matrix}$$

Example 1.9.

$$G(x, y) = \begin{cases} \frac{x^2y}{x^4+y^2} & \textit{if } (x, y) \neq (0, 0) \\ 0 & \textit{if } (x, y) = (0, 0) \end{cases}$$

Fix a vector $u \in \mathbb{R}^2$, $u = (u^1, u^2) \neq (0, 0)$, $u^2 \neq 0$ then the directional derivitive D_u with $h \in \mathbb{R}$ is:

$$\begin{aligned} D_uG(0,0) &= \lim_{h\to 0} \frac{G((0,0)+hu) - G(0,0)}{h} = \lim_{h\to 0} \frac{G(hu^1, hu^2) - 0}{h} \\ &= \lim_{h\to 0} \frac{(hu^1)^2(hu^2)}{(hu^1)^4 + (hu^2)^2} \cdot \frac{1}{h} = \lim_{h\to 0} \frac{h^3(u^1)^2u^2}{h(h^4(u^1)^4 + h^2(u^2)^2)} \\ &= \lim_{h\to 0} \frac{(u^1)^2u^2}{h^2(u^1)^4 + (u^2)^2} = \frac{(u^1)^2}{u^2} \end{aligned}$$

$u^2 = 0$

$$D_uG(0,0) = \lim_{h\to 0} \frac{G(hu^1, h\cdot 0)}{h} = \lim_{h\to 0} \frac{(\frac{(hu^1)^2 0}{(hu^1)^4+0^2})}{h} = 0$$

Theorem 1.12. *$f : \mathbb{R}^n \to \mathbb{R}^m$ if $D_jf^i(x)$ exist $\forall x \in u, U$ open, $a \in U, \forall i = 1, \ldots, m$, and $j = 1, \ldots, n$ and if $D_jf^i(x)$ continuous at a ie*

$$\lim_{x\to a}(D_jf^i(x)) = D_jf^i(a)$$

then $Df(a)$ exists and f is differentiable at a

Proof. As in the proof of theorem 1.11 It suffices to consider the case $m = 1$, so that $f : \mathbb{R}^n \to \mathbb{R}$. Then

$$\begin{aligned} f(a+h) - f(a) = {} & f(a^1 + h^1, a^2, \ldots, a^n) & & - f(a^1, \ldots, a^n) \\ & + f(a^1 + h^1, a^2 + h^2, a^3, \ldots, a^n) & & - f(a^1 + h^1, a^2, \ldots, a^n) \\ & + \ldots & & - \ldots \\ & + f(a^1 + h^1, \ldots, a^n + h^n) & & - f(a^1 + h^1, \ldots, a^{n-1} + h^{n-1}, a^n) \end{aligned}$$

Recal from theorem 1.11 that D_1f is the derivitive of the function h defined by $h(x) = (x, a^2, \ldots, a^n)$. Applying the mean-value theorem to h we obtain

$$f(a^1 + h^1, a^2, \ldots, a^n) - f(a^1, \ldots, a^n) = h^1 \cdot D_1f(b_1, a^2, \ldots, a^n)$$

for some b_1 between a^1 and a^1+h^1. Similarly the ith term in the sum equals

$$h^i \cdot D_i f(a^1+h^1,\ldots,a^{i-1}+h^{i-1},b_i,a^{i+1},\ldots,a^n) = h^i D_i f(c_i) \quad \text{for some } c_i$$

Then

$$\begin{aligned}
\lim_{h\to 0}\frac{|f(a+h)-f(a)-\sum_{i=1}^n D_i f(a)\cdot h^i|}{|h|} &= \lim_{h\to 0}\frac{|\sum_{i=1}^n [D_i f(c_i)-D_i f(a)]\cdot h^i|}{|h|} \\
&\leq \lim_{h\to 0}|\sum_{i=1}^n [D_i f(c_i)-D_i f(a)]|\cdot\frac{|h^i|}{|h|} \\
&\leq \lim_{h\to 0}|\sum_{i=1}^n [D_i f(c_i)-D_i f(a)]| \\
&= 0
\end{aligned}$$

Since $D_i f$ is continuous at a and as $h \to 0, c^i \to a^i$. □

Definition 1.7. *If $f:\mathbb{R}^n \to \mathbb{R}^m$ has partial derivitives $D_j f^i \quad \forall x \in U$, U open, $a \in U$ and $D_j f^i$ is continuous at a then we say f is continuously differentiable at a.*

Example 1.10. *$f:\mathbb{R}^2 \to \mathbb{R}$, with $x:\mathbb{R}\to\mathbb{R}, y:\mathbb{R}\to\mathbb{R}$.*

$$\textit{Define} \quad g:\mathbb{R}\to\mathbb{R} \quad g(t) = f(x(t),y(t))$$

$$\begin{aligned}
\frac{dg(t_0)}{dt} &= (g^{'}(t_0)) = f^{'}(x(t_0),y(t_0))\cdot\begin{pmatrix} x^{'}(t_0) \\ y^{'}(t_0)\end{pmatrix} \\
&= \frac{df}{dx}(x(t_0),y(t_0))\cdot\frac{dx}{dt}(t_0) + \frac{df}{dy}(x(t_0),y(t_0))\cdot\frac{dy}{dt}(t_0) \\
&= \frac{df}{dx}\cdot\frac{dx}{dt} + \frac{df}{dy}\cdot\frac{dy}{dt}
\end{aligned}$$

1.8 Inverse Function Theorem

Lemma 1.13. *Let $A \subset \mathbb{R}^n$ be a rectangle with interior A^0 and let $g: A \to \mathbb{R}^n$ be continuously differentiable. If there exist a constant $M > 0$ such that*

$$|D_j g^i(x)| \leq M, \quad x \in A^0, \quad i,j = 1,\ldots,n.$$

then

$$|g(x)-g(y)| \leq n^2 M|x-y|, \quad x,y \in A.$$

Proof. Fix $i = 1,\ldots,n$. Then

$$\begin{aligned}
g^i(y)-g^i(x) &= g^i(y^1,y^2,\ldots,y^n) - g^i(x^1,x^2,\ldots,x^n) \\
&= g^i(y^1,y^2,\ldots,y^n) - g^i(x^1,y^2,\ldots,y^n) + g^i(x^1,y^2,\ldots,y^n) - g^i(x^1,x^2,\ldots,y^n) \\
&\quad + g^i(x^1,x^2,\ldots,y^n) - \cdots + g^i(x^1,x^2,\ldots,y^n) - g^i(x^1,x^2,\ldots,x^n) \\
&= \sum_{j=1}^n (g^i(x^1,x^2,\ldots,x^{j-1},y^j,\ldots,y^n) - g^i(x^1,x^2,\ldots,x^{j-1},x^j,y^{j+1},\ldots,y^n) \\
&= \sum_{j=1}^n (y^j - x^j) D_j g^i(z_j^i)
\end{aligned}$$

where z^i_j is between y^j and x^j, and we used the mean-value theorem in the interval between y_j and x_j and in the j variable. Using the triangle inequality and $|z^j| \le |z|$, we get

$$|g^i(y) - g^i(x)| \le \sum_{j=1}^n |y^i - x^i| M \le \sum_{j=1}^n |y - x| M = nM|y - x|.$$

Since $|z| \le \sum_i |z^i|$, finally we get

$$|g(x) - g(y)| \le \sum_{i=1}^n |g^i(y) - g^i(x)| \le \sum_{i=1}^n nM|y - x| = n^2 M|y - x|.$$

□

Remark. *It is clear that the dimension of the target space enters only in the last line of the calculation. If $g : \mathbb{R}^n \to \mathbb{R}^m$, then we get as upper bound $nmM|x - y|$. The inequality is actually not optimal: one can use the Cauchy-Schwarz inequality twice to get a bound $n^{1/2}m^{1/2}M|x-y|$ for $g : \mathbb{R}^n \to \mathbb{R}^m$.*

Theorem 1.14 (Inverse Function Theorem)**.** *Theorem Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on an open set containing a and assume $\det f^{'}(a) \neq 0$. Then there exists an open set V containing a and an open set W containing $f(a)$ such that $f : V \to W$ is bijective with $f^{-1} : W \to V$ continuously differentiable and which satisfies:*

$$(f^{-1})^{'}(y) = [f^{'}(f^{-1}(y))]^{-1}, \quad y \in W.$$

Proof. Step 1:
We reduce proving the theorem to the case where actually $f^{'}(a) = I_{nn}$. Call $\lambda = Df(a)$. This is a linear transformation with nonsingular matrix representation $f^{'}(a)$, as $\det f^{'}(a) \neq 0$. Therefore, λ is invertible. The inverse λ^{-1} is also a linear transformation, so $D(\lambda^{-1})(y) = \lambda^{-1}$ for $y \in \mathbb{R}^n$. Both λ and its inverse are continuous as linear transformations. Consider the function $h = \lambda^{-1} \circ f$ defined on an open set comtaining a.
Then:

$$Dh(a) = D\lambda^{-1}(f(a)) \circ Df(a) = \lambda^{-1} \circ Df(a) = \lambda^{-1} \circ \lambda = Id,$$

by using the chain rule. Here Id is the identity transformation. This gives $h^{'}(a) = I_{n\times n}$, which has determinant $1 \neq 0$. Let A be the matrix representation of λ^{-1}. (which gives that A^{-1} is the matrix representation of $\lambda = D\lambda$). This is an $n \times n$ matrix with constant entries, i.e. not depending on y. Moreover, h is continuously differentiable, as

$$(D_j h^i(x)) = h^{'}(x) = [\lambda^{-1} \circ Df(x)] = A \cdot f^{'}(x) = A(D_j f^i(x)),$$

with entries depending continuously on x. Therefore, h satisfies the conditions of the inverse function theorem. Suppose that we can prove the conclusion of it for h, i.e. that there exists an open set V containing a and $\tilde{W}$ open containing $h(a) = \lambda^{-1}(f(a))$ such that $h : V \to \tilde{W}$ is bijective with continuously differentiable inverse h^{-1}. Even more, assume that we have prove the formula for the derivative of the inverse of h:

$$(h^{-1})'(z) = [h'(h^{-1}(z))]^{-1}.$$

Define $W = \lambda(\tilde{W}) = (\lambda^{-1})^{-1}(\tilde{W})$. This is the inverse image of $\tilde{W}$ by λ^{-1}, which is continuous, so it is an open set. Since λ is bijective, $f = \lambda \circ h$ is bijective on V with image $\lambda(\tilde{W}) = W$. Moreover,

$$f^{-1} = h^{-1} \circ \lambda^{-1},$$

which is continuously differentiable as the composition of two such maps. By the chain rule for Jacobian

$$(f^{-1})'(y) = (h^{-1})'(\lambda^{-1}(y))\cdot(\lambda^{-1})'(y) = [h'(h^{-1}(\lambda^{-1}(y)))]^{-1}A = [h'((\lambda\circ h)^{-1}(y))]^{-1}A = [h'(f^{-1}(y))]^{-1}A$$

$$= [A^{-1}h'(f^{-1}(y))]^{-1} = [\lambda' h'(f^{-1}(y))]^{-1} = [(\lambda\circ h)'(f^{-1}(y))]^{-1} = [f'(f^{-1}(y))]^{-1}.$$

All these imply that it is enough to work with $h = \lambda^{-1}\circ f$. The main property we will use is that $h'(a) = I_{n\times n}$. For simplicity in our notation we call this function f so we can assume that

$$f'(a) = I_{n\times n}.$$

This also means that $\lambda = Df(a) = Id$.
Step 2: The function f cannot take the value $f(a)$ arbitrarily close to a. Suppose that there is a sequence $h_n \in \mathbb{R}^n$ such that $h_n \to 0$ and $f(a+h_n) = f(a)$. We plug the sequence into the definition of the derivative at a and use that $Df(a) = Id$ to get

$$0 = \lim_{h_n\to 0}\frac{|f(a+h_n) - f(a) - Df(a)(h_n)|}{|h_n|} = \lim_{h_n\to 0}\frac{|-h_n|}{|h_n|} = 1$$

So this is a contradiction. Therefore, we can find a closed rectangle U containing a such that

$$f(x) \neq f(a), \quad \forall x \in U\backslash\{a\}.$$

Step 3: The determinant is a polynomial expression in the entries of a matrix. If the matrix entries depend continuously on x, the same is true for the determinant of the matrix. So $\det f'(x)$ is a continuous function on an open set containing a. Since $\det f'(x) \neq 0$, by the inertia principle, there exists a small enough (rectangular) neighbourhood of a, which we call U again, such that

$$\det f'(x) \neq 0, \quad x \in U \tag{1}$$

Moreover the partial derivatives $D_j f^i(x)$ are continuous and $D_j f^i(a) = \delta_{ij}$, as $Df(a) = Id$. So, for x close enough to a we have

$$|D_j f^i(x) - \delta_{ij}| < \frac{1}{2n^2}, \quad i,j = 1,\dots,n, \quad x \in U \tag{2}$$

We assumed again that the neighbourhood is U
Step 4: Constructing a contraction map and showing that f is injective in appropriate small meighbourhood. Now we define the function

$$g(x) = f(x) - x$$

and apply the Lemma to this function for the closed rectangle U. We notice that $D_j g^i(x) = D_j f^i(x) - \delta_{ij}$, as we know the partial derivatives of the identity function x. We deduce that

$$|g(x_1) - g(x_2)| \leq n^2\frac{1}{2n^2}|x_1 - x_2| = \frac{1}{2}|x_1 - x_2| \tag{3}$$

The choice of the neighbourhood in (2) so that the constant $1/(2n^2)$ appears on the right is motivated with the desire to get g as a contraction map (with constant $1/2$) as we see in (3). Now the triangle inequality in the form $|a| - |b| \leq |a-b|$ gives

$$|x_1 - x_2| - |f(x_1) - f(x_2)| \leq |(x_1 - x_2) - (f(x_1) - f(x_2))| = |-g(x_1) + g(x_2)| < \frac{1}{2}|x_1 - x_2|$$

$$\Rightarrow |x_1 - x_2| - \frac{1}{2}|x_1 - x_2| < |f(x_1) - f(x_2)| \Rightarrow \frac{1}{2}|x_1 - x_2| < |f(x_1) - f(x_2)| \tag{4}$$

Here x_1, x_2 are in U. We immediately see that on U the function f is injective:

$$f(x_1) = f(x_2) \Rightarrow |x_1 - x_2| = 0 \Rightarrow x_1 = x_2.$$

We still have not determined the neighbourhoods W of $f(a)$ and V of a.
Step 5: Determination of the minimum distance of $f(a)$ to the image of the boundary of U and definition of W.
We have assumed that on the closed rectangle U we have $f(x) \neq f(a)$ for $x \neq a$. This is definitely true on the boundary of U, denoted ∂U, which is a closed and bounded set, i.e. compact. The function $m(x) = |f(x) - f(a)|$ is continuous on a neighbourhood of ∂U and nonzero on it. It achieves a minimum value on ∂U (an advanced argument from Real Analysis is that the image of a compact set is compact, so that $m(\partial U)$ is compact, which means closed and bounded. Such a set has a maximum and minimum). The minimum value cannot be zero, say

$$\min_{x \in \partial U} m(x) = \min_{x \in \partial U} |f(x) - f(a)| > 0.$$

Now define

$$W = \{y \in \mathbb{R}^n, |y - f(a)| < \delta/2\}.$$

Step 6: Comparison of $|y - f(x)|$ with $|y - f(a)|$ for $x \in \partial U$, and $y \in W$. We have

$$|f(x) - f(a)| \geq \delta, \quad |y - f(a)| \leq \delta/2 \Rightarrow -|y - f(x)| + \delta \leq -|y - f(x)| + |f(x) - f(a)| \leq |y - f(a)| < \delta/2$$

$$\Rightarrow \delta/2 = \delta - \delta/2 < |y - f(x)| \Rightarrow |y - f(a)| < \delta/2 < |y - f(x)|.$$

Step 7: Show that for $y_0 \in W$ there exists a unique $x_0 \in U^0$ such that $f(x_0) = y_0$. The uniqueness is obvious from the fact that f is injective on U. The construction of such an x_0 is tricky. We define another function on U by

$$g(x) = |f(x) - y_0|^2 = \sum_{i=1}^{n} (f^i(x) - y_0^i)^2.$$

This function in continuously differentiable, as it is a sum of the squares of the components. On the compact set U the function g achieves its minimum, say at x_0, i.e. $g(x_0) \leq g(x)$ for $x \in U$. We claim that x_0 is the desired point with $f(x_0) = y_0$. First we see that x_0 is in the interior of the set U. On the boundary of U the function $g(x)$ has values $> \delta/2$, by Step 6, while $g(a) < \delta/2$. So the minimum is not achieved on the boundary of U. Therefore, it is achieved in an interior point. This point has to be a critical point of g, i.e. $D_j g(x_0) = 0$, $j = 1, \ldots, n$. We calculate them to be

$$2 \sum_{i=1}^{n} (f^i(x_0) - y_0^i) D_j f^i(x_0) = 0, \quad j = 1, \ldots, n.$$

This is a homogeneous system of linear equations with unknowns $f^i(x_0) - y_0^i$ and coefficients $D_j f^i(x_0)$. The determinant of the coefficients of the system is nonzero, as $x_0 \in U$. The system has a unique solution, and this solution is the zero vector, i.e

$$0 = f^i(x_0) - y_0^i, \quad i = 1, \ldots, n \Rightarrow f(x_0) = y_0.$$

Step 8: We define V and Show that $f : V \to W$ is bijective and continuous. We define $V = U^0 \cap f^{-1}(W)$. Clearly $f : V \to W$ is bijective. Moreover, V is open as the intersection of the open set U^0 and the open set $f^{-1}(W)$, which is open as the inverse image of an open set W by the continuous function f. We now rewrite (4) as

$$|x_1 - x_2| < 2|f(x_1) - f(x_2)| \Leftrightarrow |f^{-1}(y_1) - f^{-1}(y_2)| < 2|y_1 - y_2| \tag{5}$$

with $y_1 = f(x_1)$ and $y_2 = f(x_2),\ y_i \in W$. This shows that f^{-1} is a Lipschitz function with constant 2, so that it is continuous. Alternatively, choose $\delta = \epsilon/2$ in the definition of continuity.

Step 9: Show that f^{-1} is differentiable. Let $\mu = Df(x_1)$. Since $f^{-1} \circ f = Id$, the chain rule gives the only possible choice for $Df^{-1}(y_1) = \mu^{-1}$. Here $f(x_1) = y_1$ and later $f(x) = y$. By the definition of the derivative we have

$$f(x) - f(x_1) = \mu(x - x_1) + \phi(x - x_1), \quad \lim_{x \to x_1} \frac{|\phi(x - x_1)|}{|x - x_1|} = 0.$$

We apply to the equation the linear transformation μ^{-1} to get

$$\mu^{-1}(y - y_1) = x - x_1 + \mu^{-1}(\phi(x - x_1)) \Rightarrow x - x_1 - \mu^{-1}(y - y_1) = \mu^{-1}(\phi(x - x_1))$$
$$\Rightarrow f^{-1}(y) - f^{-1}(y_1) - \mu^{-1}(y - y_1) = -\mu^{-1}(\phi(x - x_1)).$$

By the definition of the derivative of f^{-1} at y_1 we need to show that

$$\lim_{y \to Y_1} \frac{|-\mu^{-1}(\phi(x - x_1))|}{|y - y_1|} = 0 \tag{6}$$

Since μ^{-1} is a linear transformation, we have seen that it is a bounded linear operator, i.e. there exists a constant $\tilde{M}$ with

$$|\mu^{-1}(y)| \leq \tilde{M}|y|, \quad \forall y \in \mathbb{R}^n.$$

Since

$$\frac{|-\mu^{-1}(\phi(x - x_1))|}{|y - y_1|} \leq \frac{\tilde{M}|\phi(x - x_1)|}{|y - y_1|}$$

by the sandwich theorem it is enough to prove that

$$\lim_{y \to Y_1} \frac{|\phi(x - x_1)|}{|y - y_1|} = 0$$

We have

$$\frac{|\phi(x - x_1)|}{|y - y_1|} = \frac{|\phi(x - x_1)|}{|x - x_1|} \frac{|x - x_1|}{|y - y_1|} \leq \frac{|\phi(x - x_1)|}{|x - x_1|} \cdot 2,$$

by (5). Moreover, $y \to y_1$ iff $x \to x_1$ as f is continuous at x_1 and f^{-1} is continuous at y_1.We know that

$$\lim_{x \to x_1} \frac{|(\phi(x - x_1))|}{|x - x_1|} = 0$$

This suffices to prove (6)

Step 10: The partial derivatives $D_j(f^{-1})^i(y)$ are continuous. We know that the Jacobian of $f^{-1}(y)$ is

$$(f^{-1})'(y) = (D_j(f^{-1})^i(y)) = [f'(f^{-1}(y))]^{-1} = (D_j f^i(x))^{-1}.$$

The inverse of the matrix $(D_j f^i(x))$ can be calculated as a quotient of two $n \times n$ determinants with entries among $D_j f^i(x)$. The denominator is the determinant of the Jacobian at x, which is nonzero for $x \in U$. The whole expression depends continuously on $x \in V$. As f^{-1} is continuous, the inverse matrix depends continuously on $y \in W$. The individual entries are the partial derivatives of f^{-1}. $\square$

Example 1.11. $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$(z, w) = f(x, y) = (xy, x^2 + y^2), \quad z = xy,\ w = x^2 + y^2$$

$$f'(x, y) = \begin{pmatrix} y & x \\ 2x & 2y \end{pmatrix}$$

$$det\, f'(x, y) = 2y^2 - 2x^2 = 2(y + x)(y - x)$$

$$det\, f'(x, y) \neq 0 \Leftrightarrow x \neq \pm y$$

Solving:

$$y = \frac{z}{x}$$

$$\therefore\ w = x^2 + \frac{z^2}{x^2}$$

$$\therefore\ wx^2 = x^4 + z^2$$

$$\Rightarrow x^4 - wx^2 + z^2 = 0$$

Let $t = x^2 \therefore\ t^2 - wt + z^2$ *So*

$$t = \frac{w \pm \sqrt{w^2 - 4z^2}}{2}$$

$$x = \pm\sqrt{\frac{w \pm \sqrt{w^2 - 4z^2}}{2}}$$

And

$$y = \frac{z}{\pm\sqrt{\frac{w \pm \sqrt{w^2 - 4z^2}}{2}}}$$

You should be able to differentiate if $w^2 - 4z^2 \neq 0\ \Leftrightarrow if\ y \neq \pm x$

$$\begin{bmatrix} \frac{dx}{dz} & \frac{dx}{dw} \\ \frac{dy}{dz} & \frac{dy}{dw} \end{bmatrix} = (f^{-1})'(z, w) = \begin{pmatrix} y & x \\ 2x & 2y \end{pmatrix}^{-1} = \frac{1}{2(y^2 - x^2)} \begin{pmatrix} 2y & -x \\ -2x & y \end{pmatrix}$$

$$\Rightarrow \frac{dx}{dz} = \frac{2y}{2(y^2 - x^2)}$$

$$\Rightarrow \frac{dx}{dw} = \frac{-x}{2(y^2 - x^2)}$$

When we have $z = xy,\ w = x^2 + y^2$ *along the lines* $y = \pm x$ *the circle meets the hyperbola tangentially so we cannot invert.*

1.9 Implicit Function Theorem

Example 1.12.

$$x^2 + y^2 = 1, \quad y = g(x)$$

$$2x + 2y\frac{dy}{dx} = 0, \quad \frac{dy}{dx} = \frac{dg}{dx} = \frac{-x}{y}, \quad y \neq 0$$

Example 1.13.

$$y^2 + xz + z^2 - e^z - 4 = 0 \quad \textit{(impossible to solve for z)}$$

$$\textit{set } F(x,y,z) = y^2 + xz + z^2 - e^z - 4, \quad F(x,y,g(x,y)) = 0$$

$$\begin{tikzcd}
& F \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
X \arrow[d, dash] & Y \arrow[d, dash] & Z \arrow[dll, dash] \arrow[dl, dash] \\
X \arrow[r, dash] & Y &
\end{tikzcd}$$

Differentiate in x:

$$\begin{aligned}
\frac{d}{dx}F(x,y,g(x,y)) &= \frac{dF}{dx}\frac{dx}{dx} + \frac{dF}{dy}\frac{dy}{dx} + \frac{dF}{dz}\frac{dz}{dx} \\
&= \frac{dF}{dx} + \frac{dF}{dz}\frac{dg}{dx} = 0 \\
\frac{dg}{dx} &= -\frac{\frac{dF}{dx}}{\frac{dF}{dz}} = -\frac{z}{x^2 + 2z - e^z} \\
\frac{dF}{dy} = 0 \underset{rule}{\overset{chain}{\Rightarrow}} \frac{dF}{dy} + \frac{dF}{dz}\frac{dg}{dy} \Rightarrow \frac{dg}{dy} &= -\frac{\frac{dF}{dy}}{\frac{dF}{dz}} = -\frac{2y}{x^2 + 2z - e^z}
\end{aligned}$$

the point $(0, e, 2)$ *satisfies* $F(x,y,z) = 0$

$$e^2 + 0 \cdot 2 + 2^2 - e^2 - 4 = 0$$

$$\begin{aligned}
\left.\frac{dg}{dx}\right|_{(0,e)} &= -\frac{z}{x^2 + 2z - e^z} = -\frac{2}{0 + 2 \cdot 2 - e^2} \\
\left.\frac{dg}{dy}\right|_{(0,e)} &= -\frac{2y}{x^2 + 2z - e^z} = -\frac{2e}{0 + 2 \cdot 2 - e^2}
\end{aligned}$$

valid for $\frac{dF}{dz} \neq 0$

General situation: m equations with m unknowns $y^1, \ldots, y^m$

$$\begin{aligned}
f^1(x^1, \ldots, x^n, y^1, \ldots, y^m) &= 0 \qquad \text{depends on n parameters: } x^1, \ldots, x^n \\
f^2(x^1, \ldots, x^n, y^1, \ldots, y^m) &= 0 \qquad \text{Try to solve for: } y^1, \ldots, y^m \\
\vdots \qquad \vdots \qquad \vdots & \\
f^m(x^1, \ldots, x^n, y^1, \ldots, y^m) &= 0
\end{aligned}$$

$$x = (x^1, \ldots, x^n), \qquad y = (y^1, \ldots, y^m)$$

So we have:

$$\begin{aligned} f^1(x,y) &= 0 \\ f^2(x,y) &= 0 \\ &\vdots \\ f^m(x,y) &= 0 \end{aligned}$$

Define $f(x,y) = (f^1(x,y),\ldots,f^m(x,y)) = \underbrace{0}_{vector} = \underbrace{(0,\ldots,0)}_{m}$

Let $a \in \mathbb{R}^n,\ b \in \mathbb{R}^m$ such that $f(a,b) = 0$ when we can find for each $(x^1,\ldots,x^n)$ near $a = (a^1,\ldots,a^n)$ a unique $y = (y^1,\ldots,y^m)$ near $b = (b^1,\ldots,b^m)$ such that: $f(x,y) = 0$,
$f(x^1,\ldots,x^n,y^1,\ldots,y^m) = 0$

Theorem 1.15 (Implicit Function Theorem)**.** *$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ continuously differentiable on an open set containing $(a,b),\ a \in \mathbb{R}^n,\ b \in \mathbb{R}^m$. moreover $f(a,b) = 0$*
consider the matrix

$$M = (D_{j+n}f^i(a,b))_{j=1,;m}^{i=1,\ldots,m}$$

assume $detM \neq 0$. Then there exist two open sets $A \subset \mathbb{R}^n,\ b \subset \mathbb{R}^m,\ a \in A,\ b \in B$. such that $\forall x \in A, \exists$ unique $g(x) \in B$ such that $f(x,g(x)) = 0$ Moreover $g : A \to B$ is differentiable.

Proof. Increase the dimension of the target. Define $F : \underbrace{U}_{\in \mathbb{R}^n \times \mathbb{R}^m} \to \mathbb{R}^n \times \mathbb{R}^m$

$$F(x^1,\ldots,x^n,y^1,\ldots,y^m) = (x^1,\ldots,x^n,f^1(x,y),\ldots,f^m(x,y))$$

$$F(x,y) = (x,f(x,y))$$

F is continuously differentiable because $x^1,\ldots,x^n$ are continuously differentiable and $f^1(x,y),\ldots,f^m(x,y)$ are continuously differentiable (because $f(x,y)$ is continuously differentiable)

$$F(a,b) = (a,f(a,b)) = (a,0)$$

$$F'(a,b) = \left(\begin{array}{cccc|ccc} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ \hline \frac{df^1}{dx^1} & \frac{df^1}{dx^2} & \cdots\frac{df^1}{dx^n} & \frac{df^1}{dy^1} & \cdots & \frac{df^1}{dy^m} & \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ \frac{df^m}{dx^1} & \frac{df^m}{dx^2} & \cdots\frac{df^m}{dx^n} & \frac{df^m}{dy^1} & \cdots & \frac{df^m}{dy^m} & \end{array}\right)$$

$$F'(a,b) = \left(\begin{array}{c|c} I_{n\times n} & 0_{n\times m} \\ \hline \begin{array}{c}\text{* Some } m\times n \\ \text{matrix}\end{array} & M_{m\times m} \end{array}\right)$$

$\det M \neq 0$ (reducing from top left entry).
By the inverse function theorem, $\exists$ an open set W containing $F(a,b) = (a,0)$ and an open set containing $(a,0)$ which I can take to be a rectangle $A \times B,\ a \in A,\ b \in B,\ A$ open in $\mathbb{R}^n,\ B$ open in $\mathbb{R}^m$.

$$F : A \times B \to W \text{ is bijective}$$

$$\exists h = F^{-1} : W \to A \times B \text{ such that } F \cdot h = id$$

h is continuously differentiable.

$$\boxed{\begin{array}{c} F(x^1, \ldots, x^n, y^1, \ldots, y^m) = (x^1, \ldots, x^n, f^1(x,y), \ldots, f^m(x,y)) \\ F(x,y) = (x, f(x,y)) \\ F \text{ is continuously differentiable because } x^1, \ldots x^n \text{ are continuously differentiable} \end{array}}$$

h must have the form: $h(x,y) = (x, k(x,y))$ for some function $k : W \to B,\ B \subset \mathbb{R}^m$, k continuously differentiable.

$$F(h(x,y) = (x, f(x, k(x,y))) = (x,y)$$

$$f(x, k(x,y)) = y$$

Set $y = 0$

$$f(x, k(x,0)) = 0$$

The solution is $g(x) = k(x,0)$ (solution to $f(x,y) = 0$). □

Theorem 1.16. *Let $g : \mathbb{R}^n \to \mathbb{R}^p$ be a continuously differentiable function in an open set containing a and assume that $p \leq n$. If $g(a) = 0$ and the rank of the $p \times n$ matrix*

$$(D_j g^i(a))_{i=1,\ldots,p\ j=1,\ldots,n}$$

be equal to p. Then there exists an open set $A \subset \mathbb{R}^n$ and a differentiable function $h : A \to \mathbb{R}^n$ which is bijective onto an open set V and h^{-1} is differentiable and

$$(g \circ h)(x^1, x^2, \ldots, x^n) = (x^{n-p+1}, x^{n-p+2}, \ldots, x^n)$$

Proof. We can consider the function g as $g : \mathbb{R}^{n-p} \times \mathbb{R}^P \to \mathbb{R}^p$. The 'easy' case is as follows:
If the $p \times n$ matrix above is such that the last p columns give a matrix M with $det(M) \neq 0$, then we are exactly in the situation of the Implicit Function Theorem as worked out above. The notation has only slightly changed: $x^{n-p+1} = y^1,\ x^{n-p+2} = y^2, \ldots, x^n = y^p,\ p = m,\ g = f$. We have found h with $h(x,y) = (x, k(x,y))$ and

$$(f \circ h)(x,y) = f(h(x,y)) = f(x, k(x,y)) = y,$$

and in our notation

$$(g \circ h)(x^1, x^2, \ldots, x^n) = (x^{n-p+1}, x^{n-p+2}, \ldots, x^n).$$

In general we cannot assume that the last columns of the matrix give nonzero determinant. We know from Linear Algebra that there will be some p columns with this property. Let these columns be $j^1, j^2, \ldots, j^p$ with

$$M = (D_{j_k} g^i(a))_{i=1,\ldots,p\ k=1,\ldots,p}, \qquad det(M) \neq 0.$$

We rearrange the variables as follows: Let $m : \mathbb{R}^n \to \mathbb{R}^n$ be defined by (put the variables with superscript $j_k,\ k = 1, 2, \ldots, p$ in the last entries and order in whatever way you want the other variables)

$$m(x^1, x^2, \ldots, x^n) = (\ldots, x^{j_1}; x^{j_2}, \ldots, x^{j_p}).$$

Then $g \circ m$ is a function of the type discussed theorem 1.15, so we can find a function $s : A \to \mathbb{R}^n$ which is bijective onto an open set V and s^{-1} is differentiable and

$$((g \circ m) \circ s)(x^1, x^2, \ldots, x^n) = (x^{n-p+1}; x^{n-p+2}, \ldots, x^n).$$

Then use $h = m \circ s$. □

Example 1.14.

$$f:\mathbb{R}^2\to\mathbb{R}^2,\quad f(x,y)=(xy,x^2+y^2)=(z,w)$$

$$\begin{pmatrix}\frac{dx}{dz} & \frac{dx}{dw}\\ \frac{dy}{dz} & \frac{dy}{dw}\end{pmatrix}=\begin{pmatrix}\frac{dz}{dx} & \frac{dz}{dy}\\ \frac{dw}{dx} & \frac{dw}{dy}\end{pmatrix}^{-1}$$

$$\begin{aligned}
&z=xy, && y=\frac{z}{w}\\
&w=x^2+y^2=x^2+\frac{z^2}{x^2}\\
&wx^2=x^4+z^2\\
&x^4-wx^2+z^2 && (*)
\end{aligned}$$

$$x=g(z,w)$$

Use implicit differentiation on $(*)$ *with respect to* z*:*

$$4x^3\frac{dx}{dz}-w\cdot 2x\frac{dx}{dz}+2z=0$$

$$\frac{dx}{dz}(4x^3-2xw)=-2z$$

$$\frac{dx}{dz}=\frac{-2z}{4x^3-2xw}=\frac{-z}{x(2x^2-w)}=\frac{-y}{2x^2-w}$$

Valid for

$$2x^2-w\neq 0$$

$$2x^2-(x^2+y^2)\neq 0$$

$$x^2-y^2\neq 0$$

$$\Leftrightarrow f'(x,y)\neq 0$$

$$\left.\begin{array}{ll}
f(x,y)=0 & f(a,b)=0\\
f(x,g(x))=0 & \textit{solve implicitly for } y\\
x\in\mathbb{R}^n,\ y\in\mathbb{R}^m & g:\mathbb{R}^n\to\mathbb{R}^m\\
f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^m &
\end{array}\right\}\quad \textit{Set up of implicit function theorem}$$

$$i=1,\dots,m\quad f^i(x^1,\dots,x^n,g^1(x^1,\dots,x^n),\dots,g^m(x^1,\dots,x^n))=0$$

how to compute D_jg^i?

$$D_jg^i(\dots)=0$$

$$D_1f^i\cancel{\frac{dx^1}{dx^j}}+\cancel{\dots}+D_jf^i\underbrace{\frac{dx^j}{dx^j}}_{=1}+\cancel{\dots}+D_nf^i\cancel{\frac{dx^n}{dx^j}}+D_{n+1}f^i\frac{dg^1}{dx^j}+\dots+D_{n+m}f^i\frac{dg^m}{dx^j}=0$$

$$\underbrace{D_{n+1}f^i\frac{dg^1}{dx^j}+\dots+D_{n+m}f^i\frac{dg^m}{dx^j}}_{m\ \textit{unknowns}}=-D_jf^i\frac{dx^j}{dx^j}$$

Check det of coefficents is $\neq 0$

$$\begin{bmatrix} D_{n+1}f^1 & \dots & D_{n+m}f^1\\ \vdots & & \vdots\\ D_{n+1}f^m & \dots & D_{n+m}f^m\end{bmatrix}=M$$

2 Integration

2.1 Multiple integrals

$f : A \to \mathbb{R}$, A is a rectangle in $\mathbb{R}^n$ $A = [a_1, b_1] \times \cdots \times [a_n, b_n]$
Recall a partition $\mathcal{P}$ of $[a, b]$ is a collection of of points: $t_0, \ldots, t_k$ with $a = t_0 < t_1 < \cdots < t_k = b$
A Partition of a rectangle $[a_1, b_1] \times \cdots \times [a_k, b_k]$ is a collection $\mathcal{P} = (P_1, \ldots, P_n)$ where P_i is a partition of $[a_i, b_i]$, $i = 1, \ldots, n$ Subrectangles $[s_{j-1}, s_j] \times [t_{m-1}, t_m]$ Let f be bounded on the rectangle $[a_1, b_1] \times \cdots \times [a_k, b_k]$

Definition 2.1. *Let f be bounded on the rectangle $[a_1, b_1] \times \cdots \times [a_k, b_k]$ and let S be subrectangle of the partition $\mathcal{P}$*

$$m_S(f) = \inf_{x \in S} f(x), \qquad M_S(f) = \sup_{x \in S} f(x)$$

Lower Riemann sum:

$$\mathcal{L}(f, \mathcal{P}) = \sum_S m_S(f).v(S)$$

where $v(s)$ is the volume of the subrectangle

$$S = [s_{l-1}, s_l] \times [t_{j-1}, t_j] \times \cdots \times [r_{k-1}, r_k]$$

$$v(S) = (s_{l-1} - s_l) \cdot (t_{j-1}, t_j) \cdots (r_{k-1}, r_k)$$

Upper Riemann sum:

$$\mathcal{U}(f, \mathcal{P}) = \sum_S M_S(f).v(S)$$

$$\mathcal{L}(f, \mathcal{P}) \leq \mathcal{U}(f, \mathcal{P})$$

Refinement: A refinement $\mathcal{P}'$ of the partition $\mathcal{P}$ is as follows. Given S a subrectangle of $\mathcal{P}'$, I can find a subrectangle T of $\mathcal{P}$ such that $S \subset T$ and $T = \cup_{S \subset T} S$, S for $\mathcal{P}'$

Lemma 2.1. *if $\mathcal{P}'$ is a refinement of $\mathcal{P}$, then:*

$$\mathcal{L}(f, \mathcal{P}) \leq \mathcal{L}(f, \mathcal{P}') \tag{1}$$

$$\mathcal{U}(f, \mathcal{P}) \geq \mathcal{U}(f, \mathcal{P}') \tag{2}$$

Proof. of (1)
Let S be a subrectangle of $\mathcal{P}'$ and T a subrectangle of $\mathcal{P}$ such that $S \subset T$ and

$$m_S(f) \geq m_T(f)$$

$$m_S(f) v(S) \geq m_T(f) v(S)$$

now sum over all $S \subset T,\ S\ for\ \mathcal{P}'$

$$\sum_{S \subset T} m_S(f) v(S) \geq \sum_{S \subset T} m_T(f) v(S) = m_T(f) v(T)$$

$$\mathcal{L}(f, \mathcal{P}') = \sum_T \sum_{S \subset T} m_S(f) v(S) \geq \sum_T m_T(f) v(T) = \mathcal{L}(f, \mathcal{P})$$

□

Lemma 2.2. *For any two partitions $\mathcal{P}$ and $\mathcal{P}'$ we have:*

$$\mathcal{L}(f,\mathcal{P}) \leq \mathcal{U}(f,\mathcal{P}')$$

Proof. Take $\mathcal{P}''$ a refinement of $\mathcal{P}$ and $\mathcal{P}'$:

$$\mathcal{L}(f,\mathcal{P}) \leq \mathcal{L}(f,\mathcal{P}'') \leq \mathcal{U}(f,\mathcal{P}'') \leq \mathcal{U}(f,\mathcal{P}')$$

$\square$

Definition 2.2.
The lower Riemann integral

$$\int\limits_{A-} f = \sup_{\mathcal{P}} \mathcal{L}(f,\mathcal{P}), \quad (\mathcal{P} \textit{ partition of rectangle } A)$$

The upper Riemann integral

$$\bar{\int\limits_{A}} f = \inf_{\mathcal{P}} \mathcal{U}(f,\mathcal{P})$$

f is called integrable if

$$\int\limits_{A-} f = \bar{\int\limits_{A}} f \quad \textit{and} \quad \int\limits_{A} f = \int\limits_{A-} f = \bar{\int\limits_{A}} f$$

Theorem 2.3 (Riemann's Integrability Criterion)**.** *f is integrable over the rectangle $A \Leftrightarrow \forall \epsilon > 0$, $\exists$ a partition $\mathcal{P}$ of A such that*

$$\mathcal{U}(f,\mathcal{P}) - \mathcal{L}(f,\mathcal{P}) < \epsilon$$

Proof. $(\Rightarrow)$

$$\begin{aligned}
&\inf_{\mathcal{P}}(\mathcal{U}(f,\mathcal{P}) - \mathcal{L}(f,\mathcal{P})) = 0 \\
\Leftrightarrow &\inf_{\mathcal{P}} \mathcal{U}(f,\mathcal{P}) - \sup_{\mathcal{P}} \mathcal{L}(f,\mathcal{P}) = 0 \\
\Leftrightarrow &\int\limits_{A-} f = \bar{\int\limits_{A}} f
\end{aligned}$$

$(\Leftarrow)$

Assume $\int\limits_{A-} f = \bar{\int\limits_{A}} f$, fix $\epsilon > 0$

$$\text{Since} \int\limits_{A-} f = \sup_{\mathcal{P}} \mathcal{L}(f,\mathcal{P}), \quad \text{so} \quad \exists \mathcal{P}' \; s.t \int\limits_{A-} f - \frac{\epsilon}{2} < \mathcal{L}(f,\mathcal{P}')$$

$$\text{Since} \bar{\int\limits_{A}} f = \inf_{\mathcal{P}} \mathcal{U}(f,\mathcal{P}), \quad \text{so} \quad \exists \mathcal{P}' \; s.t \bar{\int\limits_{A}} f + \frac{\epsilon}{2} > \mathcal{U}(f,\mathcal{P}')$$

Take $\mathcal{P}''$ a common refinement of $\mathcal{P}$ and $\mathcal{P}'$

$$\overline{\int_A} f + \frac{\epsilon}{2} \; > \mathcal{U}(f, \mathcal{P}'') \geq \mathcal{L}(f, \mathcal{P}'') \; > \underline{\int_A} f - \frac{\epsilon}{2}$$

So

$$\mathcal{U}(f, \mathcal{P}'') - \mathcal{L}(f, \mathcal{P}'') < \left(\cancel{\overline{\int_A}} f + \frac{\epsilon}{2} \right) - \left(\cancel{\underline{\int_A}} f - \frac{\epsilon}{2} \right) = \epsilon$$

$\square$

Example 2.1. *Non-Riemann integrable function $f : \mathbb{R}^2 \to \mathbb{R}^2$*

$$f(x, y) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

$$\begin{aligned} m_S(f) &= 0 \qquad & M_S(f) &= 1 \\ \mathcal{L}(f, \mathcal{P}) &= 0 \qquad & \mathcal{U}(f, \mathcal{P}) &= 1 \end{aligned}$$

Definition 2.3. *If $C \subset \mathbb{R}^n$, define the characteristic function of C to be*

$$X_C(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases}$$

If f is bounded on $\bar{C}$ and C is contained in a rectangle A, we define

$$\int_C f = \int_A f X_C$$

$f : [a, b] \times [c, d] \to \mathbb{R}$
Fix x and consider $g_x : [c, d] \to \mathbb{R}$

$$g_x(y) = f(x, y)$$

$$I(x) = \int_c^d g_x dy = \int_c^d f(x, y) dy$$

$$\int_a^b I(x) dx = \int_a^b \left(\int_c^d f(x, y) dy \right) dx \tag{1}$$

Fix y and define $h_y : [a, b] \to \mathbb{R}$

$$h_y(x) = f(x, y)$$

$$J(y) = \int_a^b h_y dx = \int_a^b f(x, y) dx$$

$$\int_c^d J(y) dy = \int_c^d \left(\int_a^b f(x, y) dx \right) dy \tag{2}$$

(1) = (2)

2.2 Fubini's theorem

Theorem 2.4 (Fubini)**.** *Let A be a rectangle in $\mathbb{R}^n$ and let B be a rectangle in $\mathbb{R}^m$. $f : A \times B \to \mathbb{R}$ is intergrable. define:*

$$g_x : B \to \mathbb{R} \quad by \quad g_x = f(x,y), \qquad \forall y \in B, \ \forall x \in A$$

and let:

$$\left.\begin{aligned} \mathfrak{L}(x) &= \int\limits_{B-} g_x = \int\limits_{B-} f(x,y)dy \\ \mathfrak{U}(x) &= \overline{\int\limits_{B}} g_x = \overline{\int\limits_{B}} f(x,y)dy \end{aligned}\right\} \qquad exists \ \forall x \in A$$

Then $\mathfrak{L}(x)$ and $\mathfrak{U}$ are intergrable over A, and:

$$\int\limits_A \mathfrak{L}(x)dx = \int\limits_A \left(\int\limits_{B-} f(x,y)dy \right) dx = \int\limits_A \left(\overline{\int\limits_B} f(x,y)dy \right) dx = \int\limits_A \mathfrak{U}(x)dx = \int\limits_{A\times B} f$$

Proof. Let $\mathcal{P}_A$ be a partition of A, $\mathcal{P}_B$ be a partition of B. Let S_A a subrectangle of A, S_B a subrectangle of B. Then the rectangles $S_A \times S_B$ give a partition $\mathcal{P}$ of $A \times B$.
We will prove:

$$\mathcal{L}(f,\mathcal{P}) \underset{(1)}{\leq} \mathcal{L}(\mathfrak{L},\mathcal{P}_A) \underset{(2)}{\leq} \mathcal{U}(\mathfrak{L},\mathcal{P}_A) \underset{(3)}{\leq} \mathcal{U}(\mathfrak{U},\mathcal{P}_A) \underset{(4)}{\leq} \mathcal{U}(f,\mathcal{P})$$

Since f is integrable over $A \times B$, given $\epsilon > 0$ Riemann's integrability criterion given a partition $\mathcal{P}$ of $A \times B$, such that: $\mathcal{U}(f,\mathcal{P}) - \mathcal{L}(f,\mathcal{P}) < \epsilon$. Then $\mathcal{P}$ defines $\mathcal{P}_A$, $\mathcal{P}_B$ partitions of A, B respectively. By the inequality above: $\mathcal{U}(\mathfrak{L},\mathcal{P}_A) - \mathcal{L}(\mathfrak{L},\mathcal{P}_A) < \epsilon$. By reimann's integrability criterion, $\mathcal{L}$ is integrable over A, since:

$$\sup_{\mathcal{P}} \mathcal{L}(f,\mathcal{P}) = \inf_{\mathcal{P}} \mathcal{U}(f,\mathcal{P}) = \int\limits_{A\times B} f \quad \Rightarrow \quad \int\limits_A \mathfrak{L}(x)dx = \sup_{\mathcal{P}_A} \mathcal{L}(\mathfrak{L},\mathcal{P}_A) = \inf_{\mathcal{P}_A} \mathcal{U}(\mathfrak{L},\mathcal{P}_A) = \int\limits_{A\times B} f$$

Works simularly with $\mathfrak{U}(x)$.
Side remark:

$$\begin{gathered} \mathcal{L}(f,\mathcal{P}) \leq \mathcal{L}(\mathfrak{L},\mathcal{P}_A) \\ \sup_{\mathcal{P}} \mathcal{L}(f,\mathcal{P}) \leq \sup_{\mathcal{P}_A} \mathcal{L}(\mathfrak{L},\mathcal{P}_A) \\ \mathcal{U}(\mathfrak{L},\mathcal{P}_A) \leq \mathcal{U}(f,\mathcal{P}) \\ \inf_{\mathcal{P}_A} \mathcal{U}(\mathfrak{L},\mathcal{P}_A) \leq \inf_{\mathcal{P}} \mathcal{L}(f,\mathcal{P}) \end{gathered}$$

(2) $\quad \mathcal{L}(\mathfrak{L}, \mathcal{P}_A) \leq \mathcal{U}(\mathfrak{L}, \mathcal{P}_A)$

always true for a function $\mathfrak{L}$, partition $\mathcal{P}_A$

(3) $\quad \mathcal{U}(\mathfrak{L}, \mathcal{P}_A) \leq \mathcal{U}(\mathfrak{U}, \mathcal{P}_A)$

$$\mathfrak{L}(x) = \int\limits_{B\,-} f(x,y)dy, \ \mathfrak{U}(x) = \overline{\int\limits_{B}} f(x,y)dy \Rightarrow \mathfrak{L}(x) \leq \mathfrak{U}(x)$$

$$\Rightarrow \mathcal{U}(\mathfrak{L}(x), \mathcal{P}_A) \leq \mathcal{U}(\mathfrak{U}(x), \mathcal{P}_A)$$

(4) is proved simularly to (1) so we only prove (1).

$$\mathcal{L}(f, \mathcal{P}) = \sum_S m_s(f)v(S) = \sum_{S_A,\, S_B} m_{S_A \times S_B}(f)v(S_A \times S_B) = \sum_{S_A} \left(\sum_{S_B} m_{S_A \times S_B}(f)v(S_B) \right) v(S_A)$$

Now, if $x \in S_A$, then clearly $m_{S_A \times S_B}(f) \leq m_{S_B}(g_x)$. Consequently, for $x \in S_A$ we have

$$\sum_{S_B} m_{S_A \times S_B}(f) \cdot v(S_B) \leq \sum_{S_B} m_{S_B}(g_x) \cdot v(S_B) \leq \int\limits_{B\,-} g_x = \mathfrak{L}(x).$$

Therefore

$$\sum_{S_A} \left(\sum_{S_B} m_{S_A \times S_B}(f)v(S_B) \right) v(S_A) \leq \mathcal{L}(\mathfrak{L}, \mathcal{P}_A).$$

When the function is reimenn integral $\square$

2.3 Change of variables

Theorem 2.5. *Let $A \in \mathbb{R}^n$ be open, $g : A \to \mathbb{R}^n$ be injective and continuously differentiable with $det\ g'(x) \neq 0,\ \forall x \in A$. Let $f : g(A) \to \mathbb{R}$ be integrable.Then we have change of variables formula:*

$$\int\limits_{g(A)} f = \int\limits_A (f \circ g) \cdot |det\ g'(x)| dx$$

3 Calculus on Manifolds

3.1 Manifolds

Definition 3.1 (C^∞)**.** *A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called a C^∞ function if all partial derivitives of all orders of all components exists and are continuous*

Definition 3.2 (Diffeomorphism)**.** *Let $U,\ V$ be open sets in $\mathbb{R}^n$. A C^∞ function $h : U \to V$ bijective and $h^{-1} : V \to U$, also a C^∞ function, is called a diffeomorphism from U to V*

Definition 3.3. *A set M is a K-dim manifold in $\mathbb{R}^n$ if the following condition (M) holds. For every $x \in M$:*
(M): There exisits two open sets U, V of $\mathbb{R}^n$, $x \in U$ and a diffeomorphsim $h : U \to V$ such that:

$$h(U \cap M) = \{y \in V \ s.t. \ y^{k+1} = y^{k+2} = \cdots = y^n = 0\}$$

Remark. *Reminder of linear algebra*
$T : \mathbb{R}^n \to \mathbb{R}^p$ Linear transformation, $rank(T) = dim(T(\mathbb{R}^n)) \leq p$

$$[T] : \quad rank = \text{max number of linearly independant rows or columns.}$$

$rank(T) \leq min(n, p)$

$$[T] = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{p1} & & a_{pn} \end{pmatrix}, \quad [T] \in M_{p \times n}$$

determinant of minors: If r is the max size of an $r \times r$ minor with non-zero determinant, then rank(T) $= r$.

Theorem 3.1. *Let $A \to \mathbb{R}^n$ be open and let $g : A \to \mathbb{R}^p$ be a differentiable function such that $g'(x)$ has rank p on the set $g^{-1}(0)$. Then $g^{-1}(0)$ is an $n-p$ dimensional manifold in $\mathbb{R}^n$.*

Proof. It follows directly from theorem 1.16. Let $x \in g^{-1}(0) = M$. We take $V = A$ in theorem1.16 so that we can find a diffeomorphism $H : V \to U$, where U is open in $\mathbb{R}^n$ and

$$g \circ H(x^1, x^2, \dots, x^n) = (x^{n-p+1}, x^{n-p+2}, \dots, x^n)$$

Let $h = H^{-1} : U \to V$. We need to show that

$$h(U \cap M) = \{y \in V, \ y^{n-p+1} = y^{n-p+2} = \cdots = y^n = 0\}.$$

Let $y \in U \cap M$. Then $y \in g^{-1}(0)$, i.e. $g(y) = 0$. Since

$$h(g^1(0)) = H^{-1}(g^{-1}(0)) = (g \circ H)^{-1}(0) = \{y \in V, \ y^{n-p+1} = y^{n-p+2} = \cdots = y^n = 0\},$$

clearly we have for $y \in g^{-1}(0)$ that $h(y)$ has its last p coordinates zero. The converse is also obvious: if $z \in \{y \in V, \ y^{n-p+1} = y^{n-p+2} = \cdots = y^n = 0\}$ then set $y = H(z) \in U$ and $g(y) = g(H(z)) = (z^{n-p+1}, \dots, z^n) = (0, \dots, 0) \Rightarrow y \in g^{-1}(0) = M$ and $z = h(y) \in h(U \cap M)$. $\square$

Example 3.1.

$$S^2 = \{(x, y, z) \in \mathbb{R}^3; \ x^2 + y^2 + z^2 = 1\} \quad \text{2-dim manifold in } \mathbb{R}^3$$

$g : \mathbb{R}^3 \to \mathbb{R}$

$$g(x, y, z) = x^2 + y^2 + z^2 - 1 \qquad S^2 = g^{-1}(0)$$

$$g'(x, y, z) = (\frac{dg}{dx}, \frac{dg}{dy}, \frac{dg}{dz}) = (2x, 2y, 2z)$$

Rank can be 0 *or* 1

$$\underbrace{2}_{(n-p)} = \underbrace{3}_{(n)} - \underbrace{1}_{(p)} \qquad \text{Aim to show } rank(g') = 1 \text{ on } M = g^{-1}(0)$$

$$\begin{aligned} rank \ g' = 0 &\Leftrightarrow 2x = 2y = 2z = 0 \\ &\Leftrightarrow (x, y, z) = (0, 0, 0) \end{aligned}$$

but $(0, 0, 0) \notin g^{-1}(0)$ because $g(0, 0, 0) = 0^2 + 0^2 + 0^2 - 1 = -1$

Example 3.2. *The Sphere*

$$S^n = \{(x^1, \dots, x^{n+1});\ \ (x^1)^2 + \cdots + (x^{n+1})^2 = 1\} \quad \text{is an } n\text{-dim manifold in } \mathbb{R}^{n+1}$$

$g : \mathbb{R}^{n+1} \to \mathbb{R}^p$

$$g(x^1, \dots, x^{n+1}) = (x^1)^2 + \cdots + (x^{n+1})^2 - 1 \qquad S^n = g^{-1}(0)$$

$$g'(x^1, \dots, x^{n+1}) = (\frac{dg}{dx^1}, \dots, \frac{dg}{dx^{n+1}}) = \underbrace{(2x^1, \dots, 2x^{n+1})}_{1\times(n+1)\ matix}$$

$$rank\ g' = 0 \Leftrightarrow 2x^1 = \cdots = 2x^{n+1} = 0$$
$$\Leftrightarrow (x^1, \dots, x^{n+1}) = (0, \dots, 0)$$

but $(0, \dots, 0) \notin S^n$

Example 3.3. *Hyperbolic space*

$$\mathbb{H}^n = \{(x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1},\ \ (x^1)^2 - [(x^2)^2 + \cdots + (x^{n+1})^2] = 1\}$$

n-dim hyperbolic space $g : \mathbb{R}^{n+1} \to \mathbb{R}$

$$g(x^1, \dots, x^{n+1}) = (x^1)^2 - [(x^2)^2 + \cdots + (x^{n+1})^2] - 1$$

$$\mathbb{H}^n = g^{-1}(0) \Rightarrow (g : \mathbb{R}^{n+1} \to \mathbb{R}) \to (g : A \to \mathbb{R}), \quad A = \{x \in \mathbb{R}^{n+1},\ x^1 > 0\}.$$
$$g'(x^1, \dots, x^{n+1}) = (2x^1, -2x^2, \dots, -2x^{n+1})$$

$$rank\ g' = 0 \Leftrightarrow x^1 = \cdots = x^{n+1} = 0$$
$$\Leftrightarrow (x^1, \dots, x^{n+1}) = (0, \dots, 0)$$

but $(0, \dots, 0) \notin \mathbb{H}^n$ *so* $rank(g') = 1$ *on* $g^{-1}(0)$ *so by theorem 3.1* $g^{-1}(0)$ *is an* $(n+1) - 1$ *dim manifold in* $\mathbb{R}^{n+1}$.

Example 3.4. *Ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad a, b, c > 0$$

$$g(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

$$g'(x, y, z) = (\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}) = 0 \Leftrightarrow x = y = z = 0$$

but $(0, 0, 0)$ *does not belong to the ellipsoid.*

Example 3.5. *The graph of a differentiable function*
$f : U \to \mathbb{R}, \quad U \subseteq \mathbb{R}^2$

$$M = \{(x, y, z) \in \mathbb{R}^3,\ z = f(x, y)\} \quad (mangepatch)$$

2-dim manifold in $\mathbb{R}3$

$$g(x, y, z) = f(x, y) - z$$

$$g'(x, y, z) = (\frac{df}{dx}, \frac{df}{dy}, -1) \neq 0 \Rightarrow rank(g') = 1$$

The following theorem gives the coordinate definition of a manifold M.

Theorem 3.2. *A subset M of $\mathbb{R}^n$ is a k-dimensional manifold iff for every point $x \in M$ the following holds:*
(C) There exists an open set $U \in \mathbb{R}^n$, $x \in U$ and an open set $W \subset \mathbb{R}^k$ and an injective differentiable map $f : W \to \mathbb{R}^n$ such that

(i) $f(W) = U \cap M$

(ii) *rank* $f'(y) = k \quad \forall y \in W$

(iii) $f^{-1} : f(W) \to W$ *is continuous.*

Proof. Lets assume that M is a manifold according to the deffnition (M). We choose the function $h : U \to V$ as in the definition. We define the set W and the function f as follows:

$$W = \{a \in \mathbb{R}^k,\ (a, 0) \in h(U \cap M)\}, \quad f : W \to \mathbb{R}^n, \quad f(a) = h^{-1}(a, 0).$$

Here $(a, 0)$ is the vector with the last $n-k$ coordinates equal to 0. Obviously $f(W) = U \cap M$, since

$$a \in W \Leftrightarrow (a, 0) \in h(U \cap M) \Leftrightarrow h^{-1}(a, 0) \in U \cap M \Leftrightarrow f(a) \in U \cap M.$$

We prove that W is open. For $a \in W$, we have:

$$(a, 0) \in h(U \cap M) \Leftrightarrow h^{-1}(a, 0) \in U \cap M \Rightarrow h^{-1}(a, 0) \in U.$$

Since h^{-1} is continuous, if b is sufficiently close to a, so that $(a, 0)$ and $(b, 0)$ are sufficiently close, we can deduce that $h^{-1}(b, 0)$ is close enough to $h^{-1}(a, 0)$. Because U open, if $h^{-1}(a, 0) \in U$, then also $h^{-1}(b, 0) \in U$. This gives $(b, 0) \in h(U)$. Because $h(U \cap M)$ consists exactly of the points with last $n - k$ components equal to 0, $(b, 0) \in h(U \cap M) \Leftrightarrow b \in W$. We immediately see from the definition of f and W that f^{-1} is continuous (it maps $h^{-1}(a, 0)$ to a while h is continuous).
We prove that the rank of $f'(y)$ is k on W. For this we introduce another function

$$H : U \to \mathbb{R}^k, \quad H(z) = (h^1(z), \ldots, h^k(z)),$$

i.e. H has the same first k coordinates as h (and ignores the last $n - k$). We have

$$H(f(y)) = H(h^{-1}(y, 0)) = y, \quad y \in W.$$

Therefore, $H'(f(y)) \cdot f'(y) = I_{k \times k}$ or, in terms of linear transformations:

$$DH(f(y)) \circ Df(y) = Id_{\mathbb{R}^k}.$$

Because the composition is injective, $Df(y)$ is injective and the nullity plus rank theorem for $Df(y) : \mathbb{R}^k \to \mathbb{R}^n$ gives that the rank of $Df(y)$ is k.

The converse: Suppose that $f : W \to \mathbb{R}^n$ satisfies condition (C). We have $f'(y) \in M_{n \times k}$. By rearranging the coordinates in $\mathbb{R}^n$, we can assume that the rank of the first k rows of $f'(a)$ is k. This means

$$det(D_j f^i(a))_{i,j=1,\ldots,k} \neq 0.$$

We define

$$g : W \times \mathbb{R}^{n-k} \to \mathbb{R}^n, \quad g(a, b) = f(a) + (0, b),$$

where $(0,b)$ has the first k coordinates 0. We have

$$g^i(a,b) = f^i(a), \quad i \leq k, \quad g^i(a,b) = f^i(a) + b^i, \quad i > k.$$

We compute its Jacobian matrix. For $i \leq k$

$$\begin{aligned} D_j g^i(a,b) = D_j f^i(a) \Rightarrow D_j g^i(a,b) &= D_j f^i(a), \quad j \leq k, \\ \textit{and} \quad D_j g^i(a,b) &= 0, \quad j > k. \end{aligned}$$

For $i > k$, however, we have

$$D_j g^i(a,b) = D_j f^i(a) + D_j b^i \Rightarrow D_j g^i(a,b) = \delta_{ij}, \quad j > k$$

while

$$D_j g^i(a,b) = D_j f^i(a), \quad j \leq k.$$

The Jacobian matrix is therefore in block form

$$g'(a,b) = \left(\begin{array}{c|c} D_j f^i(a)_{i,j=1,\ldots,k} & 0 \\ \hline D_j f^i(a)^{i=k+1,\ldots,n}_{j=1,\ldots,k} & I_{(n-k)\times(n-k)} \end{array} \right)$$

The calculation of the determinant in block form (which can be considered as successive expansion on the last column) gives that $det\ g'(a,b) \neq 0$. By the inverse function theorem, there exists an open set V_1 with $(a,0) \in V_1$ and an open set V_2 containing $g(a,0) = f(a)$, such that $g : V_1 \to V_2$ has a differentiable inverse $h : V_2 \to V_1$. Then, since $f(W) = U \cap M$, we have for $(x,0) \in V_1,\ g(x,0) \in M \Leftrightarrow f(x) \in M$: This gives

$$V_2 \cap M = \{g(x,0),\ (x,0) \in V_1\}.$$

$$h(V_2 \cap M) = g^{-1}(V_2 \cap M) = g^{-1}(\{g(x,0),\ (x,0) \in V_1\}) = V_1 \cap (\mathbb{R}^k \times \{0\}).$$

□

Example 3.6. *2-dim torus.*

$$\begin{aligned} (x-2)^2 = z^2 = 1 \\ (r-2)^2 + z^2 = 1 \end{aligned}$$

$$\begin{aligned} z &= \sin\phi & x &= r\cos\theta \\ r - 2 &= \cos\phi & y &= r\sin\theta \end{aligned}$$

$$f(\theta,\phi) = ((2+\cos\phi)\cos\theta,\ (2+\cos\phi)\sin\theta,\ \sin\phi) \quad \theta,\phi \in (-\pi,\pi)$$

For theorem 3.2, take $U = \mathbb{R}^3$

$$\begin{aligned} f(W) &= U \cap M \\ f(W) &= M \end{aligned}$$

$f : \underset{\subseteq \mathbb{R}^2}{W} \to \mathbb{R}^3$

$$f'(\theta,\phi) = \begin{pmatrix} (2+\cos\phi)(-\sin\theta) & -\sin\phi\cos\theta \\ (2+\cos\phi)\cos\phi & -\sin\phi\sin\theta \\ 0 & \cos\phi \end{pmatrix}$$

$$2 \times 2 \ minor: \quad \begin{vmatrix} (2+\cos\phi)(-\sin\theta) & -\sin\phi\cos\theta \\ (2+\cos\phi)\cos\phi & -\sin\phi\sin\theta \end{vmatrix}$$
$$= (2+\cos\phi)(-\sin\phi) \begin{vmatrix} -\sin\theta & \cos\theta \\ \cos\phi & \sin\theta \end{vmatrix}$$
$$= (2+\cos\phi)\sin\phi \neq 0 \ iff \ \sin\phi \neq 0 \Leftrightarrow \phi \neq 0$$

$\therefore$ rank $f' = 2$ whenever $\phi \neq 0$
When $\phi = 0$

$$f'(\theta,\phi) = \begin{pmatrix} -3\sin\theta & 0 \\ 3\cos\theta & 0 \\ 0 & 1 \end{pmatrix}$$

if $\theta = 0$ use:

$$\begin{vmatrix} 3\cos\theta & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3$$

if $\theta = 0$ use:

$$\begin{vmatrix} 3\sin\theta & 0 \\ 0 & 1 \end{vmatrix} = 3\sin\theta \neq 0 \quad on\ \theta \in (-\infty,\infty),\ \theta \neq 0$$

Example 3.7. *Any nice surface of revolution is a 2-dim manifold in $\mathbb{R}^3$*

$$\gamma(t) = (r(t), z(t)) \quad t \in (a,b)$$

γ does not have any self intersections $r(t) > 0$. If γ is differentiable and

$$\gamma'(t) = (r'(t), z'(t)) \neq 0 \quad \forall t \in (a,b)$$

then when we rotate it arround the z-axis we get the surface

$$f(t,\theta) = (r(t)\cos\theta,\ r(t)\sin\theta,\ z(t)) \quad t \in (a,b),\ \theta \in (-\pi,\pi)$$

$$f'(t,\theta) = \begin{pmatrix} r'\cos\theta & -r\sin\theta \\ r'\sin\theta & r\cos\theta \\ z' & 0 \end{pmatrix}$$

$$\begin{vmatrix} r'\cos\theta & -r\sin\theta \\ r'\sin\theta & r\cos\theta \end{vmatrix} = r \cdot r'$$

$$r > 0,\ r' \neq 0 \Rightarrow rank = 2$$

We also need the definition of manifold with boundary. While a k-dimensional manifold in $\mathbb{R}^n$ looks like a k-dim slice of $\mathbb{R}^n$, according to condition (M), for a manifold with boundary in $\mathbb{R}^n$, the part close to the boundary looks likes a half-slice of dimension k. To make this precise we define the half-space

$$\mathbb{H}^k = \{x \in \mathbb{R}^k,\ x^k \geq 0\}.$$

Then

$$h(U \cap M) = \{y \in V :\ y^k \geq 0,\ y^{k+1} = y^{k+2} = \cdots = y^n = 0\}$$

is the substitute for condition (M). More precisely:

Definition 3.4. *A subset M of $\mathbb{R}^n$ is a k-dimensional manifold with boundary if for every point x of M either condition (M) holds or (exclusive) the following condition holds:*
(M') There is an open set U of $\mathbb{R}^n$ containing x, an open set V contained in $\mathbb{R}^n$ and a diffeomorphism $h : U \to V$ such that

$$h(U \cap M) = V \cap (\mathbb{H}^k \times \{0\}) = \{y \in V : \ y^k \geq 0, \ y^{k+1} = y^{k+2} = \cdots = y^n = 0\}.$$

Moreover, $h^k(x) = 0$.The set of points where condition (M') holds is called the boundary of M and is denoted by ∂M.

3.2 Dual Space

Definition 3.5. *Let $g^i : \mathbb{R}^n \to \mathbb{R}$ be a linear map, such a map is called a linear functional. The set of all linear functionals from $\mathbb{R}^n \to \mathbb{R}$ is called the dual space of $\mathbb{R}^n$, denoted $(\mathbb{R}^n)^*$*
let $g^1, \dots, g^m$ be linear functionals $g^i : \mathbb{R}^n \to \mathbb{R}$, then I can combine them to get a map $g : \mathbb{R}^n \to \mathbb{R}^m$ by $g(x) = (g^1(x), \dots, g^m(x))$
$g : \mathbb{R}^n \to \mathbb{R}^m$ is linear such for $x, y \in \mathbb{R}^n, \ \lambda \in \mathbb{R}$

$$g(\lambda x + y) = \lambda g(x) + g(y)$$

this can be seen by

$$\begin{aligned} g(\lambda x + y) &= (g^1(\lambda x + y), \dots, g^m(\lambda x + y) \\ &= (\lambda g(x)^1 + g^1(y), \dots, \lambda g(x)^m + g^m(y)) \\ &= \lambda(g^1(x), \dots, g^m(x)) + (g^1(y), \dots, g^m(y)) \end{aligned}$$

$[g^i]$ is the matrix representation of g^i
$[g^i] = (g^i_1, \dots, g^i_n)$

$$[g]_{mxn} = \begin{pmatrix} g^1_1 & \cdots & g^1_n \\ \vdots & & \vdots \\ g^m_1 & \cdots & g^m_n \end{pmatrix}$$

Theorem 3.3. *$f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a iff f^i are differentiable at a, $i = 1, \dots, m$ and $Df(a) = (Df^1, \dots, Df^m(a))$*

Proof. assume f is differentiable at a we take the linear function $\Pi^i(x^1, \dots, x^m) = x^i$ and compose it with f we get

$$f^i = \Pi^i \circ f$$

this is differentiable by chain rule since f and Π^i are differentiable $\forall i = 1, \dots, m$

$$\Rightarrow Df^i = D\Pi^i(a) \cdot Df(a)$$

$D\Pi^i = \Pi^i$

$$\Rightarrow Df^i = \Pi^i(a) \cdot Df(a)$$

Now assume the all f^i are differentiable at a $\forall i = 1, \dots, m$

$$\begin{aligned} &f(a+h) - f(a) - (Df^1(a)(h), \dots, Df^m(a)(h)) \\ &= (f^1(a+h), \dots, f^m(a+h)) - (f^1, \dots, f^m) - (Df^1 * (a)(h), \dots, Df^m(a)(h)) \\ &= (f^1(a+h) - f^1(a) - df^1(a), \dots, f^m(a+h) - f^m(a) - df^m(a)) \end{aligned}$$

So

$$\frac{|f(a+h) - f(a) - (Df^1(a)(h), \dots, Df^m(a)(h))|}{|h|}$$
$$\leq \frac{|f^1(a+h) - f^1(a) - df^1(a)|}{|h|}, \dots, \frac{|f^m(a+h) - f^m(a) - df^m(a)|}{|h|} \to 0$$

□

Remark. *If $T, S : \mathbb{R}^n \to \mathbb{R}^m$ are linear then $(T+S) : \mathbb{R}^n \to \mathbb{R}^m, (T+S)(x) = T(x) + S(x)$ is linear.*
If $\lambda \in \mathbb{R}$ then $(\lambda T) : \mathbb{R}^n \to \mathbb{R}^m, (\lambda T)(x) = \lambda \cdot T(x)$ is also linear.

Definition 3.6. *A linear functional f is a linear transformation $f : V \to \mathbb{R}$*

$$f(\lambda x + y) = \lambda f(x) + f(y) \quad \forall x, y \in V,\ \forall \lambda \in \mathbb{R}$$

Definition 3.7 (Dual Space)**.**

$$V^* = \{f : V \to \mathbb{R} : f \text{ are linear functionals}\}$$

Definition 3.8. *If $f, g \in V^*$ and $\lambda \in \mathbb{R}$ then*

$$f + g,\ \lambda f : V \to \mathbb{R}$$

with:

$$(f+g)(x) = f(x) + g(x)$$
$$(\lambda f) = \lambda f(x)$$

Proposition 3.1.

$$\dim(V^*) = \dim(V)$$

Proof. We have $\{v_1, \dots, v_n\}$ a basis of V, $\forall i = 1, \dots, n$ define $\varphi_i : V \to \mathbb{R}$ as follows:
Given $x \in V$

$$x = x^1 v_1 + \cdots + x^n v_n \quad (uniquely) \quad x^i \in \mathbb{R}$$
$$\varphi_i(x) = x^i$$

φ_i is a linear functional. If $y \in V$

$$y = y^1 v_1 + \cdots + y^n v_n \quad v^i \in \mathbb{R}$$

for $\lambda \in \mathbb{R}$

$$\lambda x + y = (\lambda x^1 + y^1) v_1 + \cdots + (\lambda x^n + y^n) v_n$$
$$\varphi_i(\lambda x + y) = \lambda x^i + y^i = \lambda \varphi_i(x) + \varphi_i(y)$$
$$\varphi_i(v_j) = \delta_{ij}$$

Is $\{\varphi_1, \dots, \varphi_n\}$ a basis for V^*?
Spanning:
Given $f \in V^*$, define $a^i \in \mathbb{R}$

$$f(v_i) = a^i$$

we will show

$$f = a^1\varphi_1 + \cdots + a^n\varphi_n$$

if f and $a^1\varphi_1 + \cdots + a^n\varphi_n$ agree on the basis $\{v_1, \ldots, v_n\}$ then its true. For $K = 1, \ldots, n$

$$f(v_k) = a^k$$

$$(a^1\varphi_1, \ldots, a^n\varphi_n)(v_k) = a^1\varphi_1(v_k) + \cdots + a^k\varphi_k(v_k) + \cdots + a^n\varphi_n(v_k) = a^k \cdot 1$$

Linear Independence:

$$b^1\varphi_1 + \cdots + b^n\varphi_n = 0 \overset{?}{\Rightarrow} b^k = 0 \quad \forall k$$

Apply to a basis vector v_k

$$(b^1\varphi_1 + \cdots + b^n\varphi_n)(v_k) = b^1 \cdot 0 + \cdots + b^{k-1} \cdot 0 + b^k \cdot 1 + b^{k+1} \cdot 0 + \cdots + b^n \cdot 0 = b^k$$

$\square$

Hence if $\{v_1, \ldots, v_n\}$ a basis of V, then $\{\varphi_1, \ldots, \varphi_n\}$ a basis of V^*

3.3 Multilinear Algebra

Definition 3.9. *Let V be a vector space over $\mathbb{R}$, define*

$$V^k = \underbrace{V \times \cdots \times V}_{k\ times}$$

To be

$$V^k = \{(v_1, \ldots, v_k) : v_i \in V\}.$$

This is a vector space with opperations

$$(v_1, \ldots, v_k) + (w_1, \ldots, w_k) = (v_1 + w_1, \ldots, v_k + w_k)$$

$$\lambda(v_1, \ldots, v_k) = (\lambda v_1, \ldots, \lambda v_k)$$

Definition 3.10. *$\mathcal{T} : V^k \to \mathbb{R}$ is called multilinear if $\forall i = 1, \ldots, k$*

$$\begin{aligned} \mathcal{T}(v_1, \ldots, v_{i-1}, v_i + v_i', v_{i+1}, \ldots, v_k) &= \mathcal{T}(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_k) \\ &\quad + \mathcal{T}(v_1, \ldots, v_{i-1}, v_i', v_{i+1}, \ldots, v_k) \\ \mathcal{T}(v_1, \ldots, v_{i-1}, \lambda v_i, v_{i+1}, \ldots, v_k) &= \lambda\mathcal{T}(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_k) \end{aligned}$$

$\mathcal{T}$ is not linear on V^k but linear on each of its components. A $\mathcal{T}$ like this is called a k-tensor on V.

Definition 3.11.

$$\mathcal{J}^k(V) = \{\mathcal{T} : V^k \to \mathbb{R} : k - multilinear\}$$

Note. *if $k = 2$ then $\mathcal{T}$ is called bilinear, and*

$$\begin{aligned} \mathcal{T}(v_1 + v_2, w) &= \mathcal{T}(v_1, w) + \mathcal{T}(v_2, w) \\ \mathcal{T}(\lambda v, w) &= \lambda\mathcal{T}(v, w) \\ \mathcal{T}(v, w_1 + w_2) &= \mathcal{T}(v, w_1) + \mathcal{T}(v, w_2) \\ \mathcal{T}(v, \lambda w) &= \lambda\mathcal{T}(v, w) \end{aligned}$$

also $\mathcal{T}$ is called Symmetric if

$$\mathcal{T}(v, w) = \mathcal{T}(w, v)$$

$\mathcal{T}$ is called positive definite if

$$\mathcal{T}(v, w) \geq 0 \quad \forall\, v, w$$

Definition 3.12 (Symmetric k-tensor)**.** *$\mathcal{T}$ is a Symmetric k-tensor if $\forall v_i, \ldots, v_k \in V$*

$$\mathcal{T}(v_1, \ldots, v_i, v_{i+1}, \ldots, v_j, \ldots, v_k) = \mathcal{T}(v_1, \ldots, v_j, v_{i+1}, \ldots, v_i, \ldots, v_k)$$

Definition 3.13 (Alternating k-tensor)**.** *$\mathcal{T}$ is a Alternating k-tensor if $\forall v_i, \ldots, v_k \in \mathcal{V}$*

$$\mathcal{T}(v_1, \ldots, v_i, v_{i+1}, \ldots, v_j, \ldots, v_k) = -\mathcal{T}(v_1, \ldots, v_j, v_{i+1}, \ldots, v_i, \ldots, v_k)$$

Example 3.8. $V = \mathbb{R}^2,\ V^2 = \mathbb{R}^2 \times \mathbb{R}^2$

$$\mathcal{T}(v_1, v_2) = v_1^1 v_2^2 - v_2^1 v_1^2 \qquad \boxed{\textit{determinant}}$$

$$v_1 = (v_1^1, v_1^2) \qquad v_2 = (v_2^1, v_2^2)$$

$$\begin{vmatrix} \lambda v_1 + v_1' \\ v_2 \end{vmatrix} = \lambda \begin{vmatrix} v_1 \\ v_2 \end{vmatrix} + \begin{vmatrix} v_1' \\ v_2 \end{vmatrix}$$

$$\begin{vmatrix} v_1 \\ \lambda v_2 + v_2' \end{vmatrix} = \lambda \begin{vmatrix} v_1 \\ v_2 \end{vmatrix} + \begin{vmatrix} v_1 \\ v_2' \end{vmatrix}$$

$$\begin{vmatrix} v_1 \\ v_2 \end{vmatrix} = - \begin{vmatrix} v_2 \\ v_1 \end{vmatrix}$$

det *on $k \times k$-matrices as a function of k vectors in $\mathbb{R}^k$ is an aternating k-tensor.*

Definition 3.14. *If $\mathcal{T}, \mathcal{S} \in \mathcal{J}^k(V)$, we define:*

$$(\mathcal{T} + \mathcal{S})(v_1, \ldots, v_k) = \mathcal{T}(v_1, \ldots, v_k) + \mathcal{S}(v_1, \ldots, v_k)$$

Similarly if $\lambda \in \mathbb{R},\ \lambda\mathcal{T} \in \mathcal{J}^k(V)$

$$(\lambda\mathcal{T})(v_1, \ldots, v_k) = \lambda\mathcal{T}(v_1, \ldots, v_k) \quad \forall v_i \in v$$

Definition 3.15. *Let $\mathcal{T} \in \mathcal{J}^k(V),\ \mathcal{S} \in \mathcal{J}^l(V), \quad k, l \in \mathbb{N}, \quad \mathcal{T} : V^k \to \mathbb{R},\ \mathcal{S} : V^l \to \mathbb{R}$. Define $\mathcal{T} \otimes \mathcal{S} \in \mathcal{J}^{k+l}$:*

$$\mathcal{T} \otimes \mathcal{S}(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+l}) = \mathcal{T}(v_1, \ldots, v_k) \cdot \mathcal{S}(v_k, \ldots, v_{k+l})$$

$$\mathcal{T}, \mathcal{S} \in \mathcal{J}^{k+l} \Rightarrow \mathcal{T} \otimes \mathcal{S} \neq \mathcal{S} \otimes \mathcal{T} \quad \textit{in general}$$

Properties.

1. $\mathcal{T} \otimes \mathcal{S} \in \mathcal{J}^{k+l}$
2. $(\mathcal{S}_1 + \mathcal{S}_2) \otimes \mathcal{T} = \mathcal{S}_1 \otimes \mathcal{T} + \mathcal{S}_2 \otimes \mathcal{T}$
3. $\mathcal{S} \otimes (\mathcal{T}_1 + \mathcal{T}_2) = \mathcal{S} \otimes \mathcal{T}_1 + \mathcal{S} \otimes \mathcal{T}_2$

4. $(\lambda\mathcal{S})\otimes\mathcal{T} = \lambda(\mathcal{S}\otimes\mathcal{T}) = \mathcal{S}\otimes(\lambda\mathcal{T})$

5. $(\mathcal{S}\otimes\mathcal{T})\otimes\mathcal{U} = \mathcal{S}\otimes(\mathcal{T}\otimes\mathcal{U})$

6. $\mathcal{J}^1(V) = V^*$

Theorem 3.4. *Let $i_1,\ldots,i_k \in \{1,\ldots,n\}$, V has a basis $\{v_1,\ldots,v_n\}$ $\dim V = n$. Let $\{\varphi_1,\ldots,\varphi_n\}$ be the basis basis of V^*, $\varphi_i(v_i) = \delta_{ij}$. Then*

$$\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k} \quad \textit{where} \quad \{i_1,\ldots,i_k\}\subseteq\{1,\ldots,n\}$$

form basis for $\mathcal{J}^k(V)$

$$\dim(\mathcal{J}^k(V)) = n^k$$

Proof. Clearly $\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}\in\mathcal{J}^k(V)$ since $\varphi_{i_j}\in V^* = \mathcal{J}_1(V)$. The set spans $\mathcal{J}^k(V)$ and is $L.I.$ Let $\mathcal{T}\in\mathcal{J}^k(V)$. need to write

$$\mathcal{T} = \sum_{\substack{i_1=1,\ldots,n\\ \vdots \\ i_k=1,\ldots,n}} a^{i_1 i_2\cdots i_k}\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}$$

Plug $(v_{j_1},\ldots,v_{j_k})$ into the suspected identity.

$$\begin{aligned}
\mathcal{T}(v_{j_1},\ldots,v_{j_k}) &= \sum_{i_1,\ldots,i_k} a^{i_1 i_2\cdots i_k}\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}(v_{j_1},\ldots,v_{j_k})\\
&= \sum_{i_1,\ldots,i_k} a^{i_1 i_2\cdots i_k}\varphi_{i_1}(v_{j_1})\cdots\varphi_{i_k}(v_{j_k})\\
&= \sum_{i_1,\ldots,i_k} a^{i_1 i_2\cdots i_k}\delta_{i_1 j_1}\cdots\delta_{i_k j_k}\\
&= a^{j_1\cdots j_k}
\end{aligned}$$

Define

$$a^{i_1 i_2\cdots i_k} = \mathcal{T}(v_{j_1},\ldots,v_{j_k})$$

Let $w_1,\ldots,w_k\in V$

$$w_1 = \sum_{j=1}^n a^{1_j}v_j \qquad \ldots \qquad w_k = \sum_{j=1}^n a^{k_j}v_j$$

$$\begin{aligned}
\mathcal{T}(w_1,\ldots,w_k) &= \mathcal{T}\left(\sum_{j_1} a^{1_{j_1}}v_{j_1},\ldots,\sum_{j_k} a^{k_{j_k}}v_{j_k}\right)\\
&= \sum_{j_1,\ldots,j_k=1}^n a^{1_{j_1}}\cdots a^{k_{j_k}}\cdot\mathcal{T}(v_{j_1},\ldots,v_{j_k})\\
&= \sum_{j_1,\ldots,j_k} a^{1_{j_1}}\cdots a^{k_{j_k}}\cdot a^{i_1 i_2\cdots i_k}
\end{aligned}$$

$$\begin{aligned}\sum_{i_1,\ldots,i_k} a^{i_1\cdots i_k}\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}(w_1,\ldots,w_k) &= \sum_{i_1,\ldots,i_k} a^{i_1\cdots i_k}\varphi_{i_1}(w_1)\otimes\cdots\otimes\varphi_{i_k}(w_k)\\ &= \sum_{j_1,\ldots,j_k} a^{i_1 i_2\cdots i_k}\cdot a^{1_{i_1}}\cdots a^{k_{i_k}}\end{aligned}$$

Relabel:

$$\begin{gathered} i_1 \mapsto j_1 \\ \vdots \\ i_k \mapsto j_k \end{gathered}$$

So it Spans. Now check L.I.

$$\sum_{i_1,\ldots,i_k=1}^{n} a^{i_1\cdots i_k}\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}=0$$

Plus $v_{j_1},\ldots,v_{j_k}$ in to it

$$\begin{aligned}\sum_{i_1,\ldots,i_k=1}^{n} a^{i_1\cdots i_k}\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}(v_{j_1},\ldots,v_{j_k}) &= \sum_{i_1,\ldots,i_k} a^{i_1\cdots i_k}\delta_{i_1 j_1}\cdots\delta_{i_k j_k}\\ &= a^{j_1\cdots j_k}=0\end{aligned}$$

put in all possible combinations of basis vectors

$$\Rightarrow \text{all coeffecitents } \underline{\textit{Zero}}$$

$\square$

3.4 Alternating Tensors

Remark. *for $f:\mathbb{R}\to\mathbb{R}$. If f is even:*

$$f(-x)=f(x)$$

if f is odd:

$$f(-x)=-f(x)$$

Every function can be writen as

$$f=\underset{even}{f_1}+\underset{odd}{f_2}$$

Where

$$f_1=\frac{f(x)+f(-x)}{2}\qquad f_2=\frac{f(x)-f(-x)}{2}$$

or σ is a bijection on $\mathbb{R}\to\mathbb{R}$, $\sigma^2=id$.

$$\begin{gathered} x\overset{\sigma}{\mapsto}-x \\ \frac{f(x)+f(\sigma x)}{2}\end{gathered}$$

Let S_k be that symmetric group on k letters.

$$S_k\overset{hom}{\to}\{\pm1\}\quad \textit{multiplicitave group}$$

$$\sigma \mapsto \begin{cases} +1 & if\ \sigma\ even \\ -1 & if\ \sigma\ odd \end{cases}$$

$$\sigma \mapsto sign(\sigma).$$

Definition 3.16. *If $\mathcal{T} \in \mathcal{J}^k(V)$*

$$\operatorname{Alt}(\mathcal{T})(w_1, \dots, w_k) = \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) \mathcal{T}(w_{\sigma(1)}, \dots, w_{\sigma(k)}).$$

Example 3.9. $k = 2$

$$\operatorname{Alt}(\mathcal{T})(w_1, w_2) = \frac{1}{2!}(\mathcal{T}(w_1, w_2) - \mathcal{T}(w_2, w_1))$$

Definition 3.17. *The set of alternating k-tensors is denoted by $\Lambda^k(V)$, it is a subspace of $\mathcal{J}^k(V)$*

Theorem 3.5.

(a) if $\mathcal{T} \in \mathcal{J}^k(V)$, $\operatorname{Alt}(\mathcal{T}) \in \mathcal{J}^k(V)$ and $\operatorname{Alt}(T)$ is alternating

(b) if $\mathcal{W}$ is alternating, $\operatorname{Alt}(\mathcal{W}) = \mathcal{W}$

(c) $\operatorname{Alt}(\operatorname{Alt}(\mathcal{T})) = \operatorname{Alt}(\mathcal{T})$

Proof. (c) follows from (b), use $\mathcal{W} = \operatorname{Alt}(\mathcal{T})$ which is alternating by (a)

$$\operatorname{Alt}(\mathcal{T}) = \mathcal{W} = \operatorname{Alt}(\mathcal{W}) = \operatorname{Alt}(\operatorname{Alt}(\mathcal{T}))$$

Proof of (a):
Show $\operatorname{Alt}(\mathcal{T}) \in \mathcal{J}^k(V)$. I will show it is alternating.

$$\operatorname{Alt}(\mathcal{T})(w_1, \dots, w_i, \dots, w_j, \dots, w_k) = -\operatorname{Alt}(\mathcal{T})(w_1, \dots, w_j, \dots, w_i, \dots, w_k)$$

$$\begin{aligned} i &\mapsto j \\ j &\mapsto i \\ k &\mapsto k \quad \text{if } k \neq i, j \end{aligned}$$

$$\left.\begin{aligned} S_k &\to S_k \\ \sigma &\mapsto \sigma(ij) = \sigma^1 \\ even &\mapsto odd \\ odd &\mapsto even \end{aligned}\right\} \qquad \text{Bijection} \tag{3}$$

$$\begin{aligned} \operatorname{Alt}(\mathcal{T})(w_1, \dots, \overset{i^{th}}{w_j}, \dots, \overset{j^{th}}{w_i}, \dots, w_k) &= \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) \operatorname{Alt}(\mathcal{T})(w_{\sigma(1)}, \dots, \overset{i^{th}}{w_{\sigma(j)}}, \dots, \overset{j^{th}}{w_{\sigma(i)}}, \dots, w_{\sigma(k)}) \\ &= \frac{1}{k!} \sum_{\sigma' \in S_k} sgn(\sigma') \operatorname{Alt}(\mathcal{T})(w_{\sigma'(1)}, \dots, w_{\sigma'(i)}, \dots, w_{\sigma'(j)}, \dots, w_{\sigma'(k)}) \\ &= -\operatorname{Alt}(\mathcal{T})(w_1, \dots, w_k) \end{aligned}$$

Proof of (b):
Let ω be alternating.

$$
\begin{aligned}
\omega(w_1,\dots,\overset{i^{th}}{w_j},\dots,\overset{j^{th}}{w_i},\dots,w_k) &= -\omega(w_1,\dots,\overset{i^{th}}{w_i},\dots,\overset{j^{th}}{w_j},\dots,w_k) \\
&= \omega(w_{\sigma(1)},\dots,w_{\sigma(k)}) \\
&= sgn(\sigma)\omega(w_1,\dots,w_k) \quad \sigma \in S_k \\
&= \mathrm{Alt}(\omega)(w_1,\dots,w_k) \\
&= \frac{1}{k!}\sum_{\sigma\in S_k} sgn(\sigma)\omega(w_{\sigma(1)},\dots,w_{\sigma(k)}) \\
&= \frac{1}{k!}\sum_{\sigma\in S_k} sgn(\sigma)^2\omega(w_1,\dots,w_k) \\
&= \frac{\cancel{1}}{\cancel{k!}}\cancel{|S_k|}\omega(w_1,\dots,w_k) \\
\therefore\ \mathrm{Alt}(\omega) &= \omega
\end{aligned}
$$

□

Remark. *If $\omega \in \Lambda^k(V),\ \eta \in \Lambda^l(V)$ then*

$$\omega \otimes \eta \in \mathcal{J}^{k+l}(V)$$

Definition 3.18.

$$\omega \wedge \eta = \frac{(k+l)!}{k!\,l!}\,\mathrm{Alt}(\omega\otimes\eta) \qquad \in \Lambda^{k+l}(V)$$

Properties. *if $\omega,\ \omega_1,\ \omega_2 \in \Lambda^k(V),\ \eta,\ \eta_1,\ \eta_2 \in \Lambda^l(V)$*

- $(\omega_1+\omega_2)\wedge\eta = \omega_1\wedge\eta + \omega_2\wedge\eta$
- $\omega\wedge(\eta_1+\eta_2) = \omega\wedge\eta_1 + \omega\wedge\eta_2$
- $(\alpha\omega)\wedge\eta = \alpha(\omega\wedge\eta) = \omega\wedge(\alpha\eta) \quad \alpha\in\mathbb{R}$
- $\omega\wedge\eta = (-1)^{k\cdot l}\eta\wedge\omega$

Definition 3.19. *Let V, W be vector spaces $f : V \to W$ be a linear transformation. If $\mathcal{T}$ is a linear functional on W, $\mathcal{T} : W \to \mathbb{R}$ then*

$$\mathcal{T}\circ f$$

is a linear functional on V.

$$
\begin{tikzcd}
V \arrow[r, "f"] \arrow[dr, "\mathcal{T}\circ f"'] & W \arrow[d, "\mathcal{T}"] \\
& \mathbb{R}
\end{tikzcd}
$$

Notation.

$$f^*(\mathcal{T}) = \mathcal{T}\circ f$$

$f^*(\mathcal{T})$ is called the pullback of $\mathcal{T}$ by f.

$$f^* : W^* \to V^*$$

by $f^*(\mathcal{T}) = \mathcal{T}\circ f$

Definition 3.20 (Pullback of Tensors)**.** *If $\mathcal{T}$ is a k-tensor on W ie $\mathcal{T} \in \mathcal{J}^k(W)$ we define the pullback $f^* \in \mathcal{J}^k(V)$ by:*

$$f^*(\mathcal{T})(v_1, \ldots, v_k) = \mathcal{T}(f(v_1), \ldots, f(v_k))$$

This is a k-tensor on V.

Need to show linearity in the i^{th} entry. Let $v_i, v_i' \in V,\ \lambda \in \mathbb{R}$.

$$\begin{aligned}
f^*(\mathcal{T})(v_1, \ldots, \lambda v_i + v_i', \ldots, v_k) &= \mathcal{T}(f(v_1), \ldots, f(\lambda v_i + v_i'), \ldots, f(v_k)) \\
&= \mathcal{T}(f(v_1), \ldots, f(\lambda v_i) + f(v_i'), \ldots, f(v_k)) \\
&= \lambda \mathcal{T}(f(v_1), \ldots, f(v_i), \ldots, f(v_k)) + \mathcal{T}(f(v_1), \ldots, f(v_i'), \ldots, f(v_k)) \\
&= \lambda f^*(\mathcal{T})(v_1, \ldots, v_i, \ldots, v_k) + f^*(\mathcal{T})(v_1, \ldots, v_i', \ldots, v_k)
\end{aligned}$$

□

Properties.

(a)

$$f^*(\mathcal{T} \otimes \mathcal{S}) = f^*(\mathcal{T}) \otimes f^*(\mathcal{S}) \quad \mathcal{T} \in \mathcal{J}^k(W),\ \mathcal{S} \in \mathcal{J}^l(W)$$

(b)

$$f^*(\mathcal{T} \wedge \mathcal{S}) = f^*(\mathcal{T}) \wedge f^*(\mathcal{S}) \quad \mathcal{T} \in \Lambda^k(W),\ \mathcal{S} \in \Lambda^l(W)$$

Theorem 3.6.

(a) if $\mathcal{S} \in \mathcal{J}^k(V),\ \mathcal{T} \in \mathcal{J}^l(V)$ and $\mathrm{Alt}(\mathcal{S}) = 0$ then

$$\mathrm{Alt}(\mathcal{S} \otimes \mathcal{T}) = \mathrm{Alt}(\mathcal{T} \otimes \mathcal{S}) = 0$$

(b) if $\omega \in \Lambda^k(V),\ \eta \in \Lambda^l(V),\ \vartheta \in \Lambda^m(V)$ then

$$\mathrm{Alt}(\mathrm{Alt}(\omega \otimes \eta) \otimes \vartheta) = \mathrm{Alt}(\omega \otimes \eta \otimes \vartheta) = \mathrm{Alt}(\omega \otimes \mathrm{Alt}(\eta \otimes \vartheta))$$

(c) if $\omega \in \Lambda^k(V),\ \eta \in \Lambda^l(V),\ \vartheta \in \Lambda^m(V)$ then

$$(\omega \wedge \eta) \wedge \vartheta = \omega \wedge (\vartheta \wedge \eta) = \frac{(k+l+m)!}{k!\,l!\,m!}\,\mathrm{Alt}(\omega \otimes \eta \otimes \vartheta)$$

Proof.
Proof of (a):

$$\mathrm{Alt}(\mathcal{S} \otimes \mathcal{T}) = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} sgn(\sigma)(\mathcal{S} \otimes \mathcal{T})(w_{\sigma(1)}, \ldots, w_{\sigma(k)}, w_{\sigma(k+1)}, \ldots, w_{\sigma(k+l)})$$

Let G be the subgroup of S_{k+l} such that

$$G = \left\{ \sigma \in S_{k+l} : \begin{array}{r} \sigma(k+1) = k+1 \\ \vdots \\ \sigma(k+l) = k+l \end{array} \right\}$$

The contribution of these to the sum is:

$$\begin{aligned}
&\frac{1}{(k+l)!}\sum_{\sigma\in G} sgn(\sigma)\mathcal{S}(w_{\sigma(1)},\ldots,w_{\sigma(k)})\cdot\mathcal{T}(w_{k+1},\ldots,w_{k+l})\\
&=\frac{1}{(k+l)!}k!\ \mathrm{Alt}(\mathcal{S})(w_1,\ldots,w_k)\cdot\mathcal{T}(w_{k+1},\ldots,w_{k+l})=0
\end{aligned}$$

Let $G\sigma_0$ be a coset of G in $S_{k+l},\ \sigma_0\neq 0$

$$G\sigma_0=\{\sigma'\cdot\sigma_0:\sigma'\in G\}$$

define

$$(z_1,\ldots,z_{k+l})=(w_{\sigma_0(1)},\ldots,w_{\sigma_0(k+l)})$$

The contribution of these elements is

$$\begin{aligned}
&\frac{1}{(k+l)!}\sum_{\sigma'\in G} sgn(\sigma'\cdot\sigma_0)\mathcal{S}(z_{\sigma'(1)},\ldots,z_{\sigma'(k)})\cdot\mathcal{T}(z_{\sigma'(k+1)},\ldots,z_{\sigma'(k+l)})\\
&=\frac{1}{(k+l)!}\sum_{\sigma'\in G} sgn(\sigma')sgn(\sigma_0)\mathcal{S}(z_{\sigma'(1)},\ldots,z_{\sigma'(k)})\cdot\mathcal{T}(z_{k+1},\ldots,z_{k+l})\\
&=\frac{1}{(k+l)!}sgn(\sigma_0)\mathcal{T}(z_{k+1},\ldots,z_{k+l})k!\,\mathrm{Alt}(\mathcal{S})(z_1,\ldots,z_k)=0
\end{aligned}$$

Proof of (b):

$$\mathrm{Alt}(\omega\otimes\eta)-\omega\otimes\eta=\mathcal{S}$$

$$\begin{aligned}
\mathrm{Alt}(\mathcal{S})&=\mathrm{Alt}(\mathrm{Alt}(\omega\otimes\eta)-\omega\otimes\eta)=\mathrm{Alt}(\mathrm{Alt}(\omega\otimes\eta))-\mathrm{Alt}(\omega\otimes\eta)\\
&=\mathrm{Alt}(\omega\otimes\eta)-\mathrm{Alt}(\omega\otimes\eta)=0
\end{aligned}$$

Apply (a) with $\mathcal{S}$

$$alt(\mathcal{S}\otimes\vartheta)=0$$

$$\mathrm{Alt}([\mathrm{Alt}(\omega\otimes\eta)-\omega\otimes\eta]\otimes\vartheta)=0$$

$$\mathrm{Alt}(\mathrm{Alt}(\omega\otimes\eta)\otimes\vartheta)-\mathrm{Alt}(\omega\otimes\eta\otimes\vartheta)=0$$

Proof of (c):

$$\begin{aligned}
(\omega\wedge\eta)\wedge\vartheta&=\frac{((k+l)+m)!}{(k+l)!\,m!}\,\mathrm{Alt}((\omega\wedge\eta)\otimes\vartheta)\\
&=\frac{((k+l)+m)!}{(k+l)!\,m!}\,\mathrm{Alt}(\frac{(k+l)!}{k!\,l!}\,\mathrm{Alt}(\omega\otimes\eta)\otimes\vartheta)\\
&=\frac{((k+l)+m)!}{\cancel{(k+l)!}\,m!}\frac{\cancel{(k+l)!}}{k!\,l!}\,\mathrm{Alt}(\mathrm{Alt}(\omega\otimes\eta)\otimes\vartheta)\\
&=\frac{(k+l+m)!}{k!\,l!\,m!}\,\mathrm{Alt}(\omega\otimes\eta\otimes\vartheta)
\end{aligned}$$

$\square$

Theorem 3.7. *Let $dim\, V = n$, then the following is a basis for $\Lambda^k(V)$*

$$\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}, \quad 1 \le i_1 < \cdots < i_k \le n.$$

therefore

$$dim\, \Lambda^k(V) = \binom{n}{k}$$

Since choosing a subset of k items from a set of n items and reordering.

Proof. $\mathcal{T} \in \Lambda^k(V)$ then $\mathrm{Alt}(\mathcal{T}) = \mathcal{T}$. Since

$$\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$$

is a basis for $\mathcal{J}^k(V)$ we have

$$\mathcal{T} = \sum_{i_1,\ldots,i_k=1,\ldots,n} a^{i_1\cdots i_k} \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$$

Apply Alt on both sides

$$\mathcal{T} = \sum_{i_1,\ldots,i_k} a^{i_1\cdots i_k}\, \mathrm{Alt}(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})$$

$\mathrm{Alt}(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})$ is a multiple of $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$. Since

$$\varphi_{i_j} \wedge \varphi_{i_s} = -\varphi_{i_s} \wedge \varphi_{i_j}$$

you can reorder to

$$\sum_{i_1 < \cdots < i_k}$$

So $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$ with $i_1 < \cdots < i_k$ span $\Lambda^k(V)$. It is easy to see that they are L.I. ☐

Example 3.10. *$dim\, V = 3$, $k = 1$*

$$dim\, \Lambda^1(V) = \binom{3}{1} = 3$$

$$\Lambda^1(V) = \mathcal{J}^1(V) = V^*$$

if $\{v_i\}_{i=1,2,3}$ is a basis of V, then the dual basis $\varphi_1, \varphi_2, \varphi_3$ is a basis of $\Lambda^1(V)$

Example 3.11. *$dim\, V = 3$, $k = 2$*

$$dim\, \Lambda^2(V) = \binom{3}{2} = 3$$

$$\Lambda^1(V) = \mathcal{J}^1(V) = V^*$$

basis is

$$\varphi_1 \wedge \varphi_2,\ \varphi_1 \wedge \varphi_3 \text{ and } \varphi_2 \wedge \varphi_3$$

$$
\begin{aligned}
(\varphi_1 \wedge \varphi_2)(w_1, w_2) &= \frac{(1+1)!}{1!\,1!}\,\mathrm{Alt}(\varphi_1 \otimes \varphi_2)(w_1, w_2) \\
&= 2!\frac{1}{2!}(\varphi_1 \otimes \varphi_2(w_1, w_2) - \varphi_1 \otimes \varphi_2(w_2, w_1)) \\
&= \varphi_1(w_1) \cdot \varphi_2(w_2) - \varphi_1(w_2) \cdot \varphi_2(w_1) \\
&= \varphi_1 \otimes \varphi_2(w_1, w_2) - \varphi_2 \otimes \varphi_1(w_1, w_2)
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ \varphi_1 \wedge \varphi_2 &= \varphi_1 \otimes \varphi_2 - \varphi_2 \otimes \varphi_1 \\
\varphi_1 \wedge \varphi_3 &= \varphi_1 \otimes \varphi_3 - \varphi_3 \otimes \varphi_1 \\
\varphi_2 \wedge \varphi_3 &= \varphi_2 \otimes \varphi_3 - \varphi_3 \otimes \varphi_2
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ \varphi_1 \wedge \varphi_2 &= \varphi_1 \otimes \varphi_2 - \varphi_2 \otimes \varphi_1 = -\varphi_2 \wedge \varphi_1 \\
\varphi_1 \wedge \varphi_3 &= -\varphi_3 \wedge \varphi_1 \\
\varphi_2 \wedge \varphi_3 &= -\varphi_3 \wedge \varphi_2
\end{aligned}
$$

$$
(\varphi_1 \wedge \varphi_1)(w_1, w_2) = \varphi_1(w_1)\varphi_1(w_2) - \varphi_1(w_1)\varphi_1(w_2) = 0
$$

$$
\therefore\ \boxed{\varphi_1 \wedge \varphi_1 = \varphi_2 \wedge \varphi_2 = \varphi_3 \wedge \varphi_3 = 0}
$$

Example 3.12. $dim\, V = 3$, $k = 3$

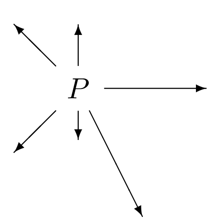
$$
dim\, \Lambda^3(V) = \binom{3}{3} = 1 \quad basis:\ \varphi_1 \wedge \varphi_2 \wedge \varphi_3
$$

$$
\begin{aligned}
(\varphi_1 \wedge \varphi_2 \wedge \varphi_3)(w_1, w_2, w_3) &= 3!\,\mathrm{Alt}(\varphi_1 \otimes \varphi_2 \otimes \varphi_3) \\
&= \sum_{\sigma \in S_3} sgn(\sigma)\varphi_1 \otimes \varphi_2 \otimes \varphi_3(w_{\sigma(1)}, w_{\sigma(2)}, w_{\sigma(3)}) \\
&= \varphi_1(w_1)\varphi_2(w_2)\varphi_3(w_3) - \varphi_1(w_2)\varphi_2(w_1)\varphi_3(w_3) \\
&\quad - \varphi_1(w_3)\varphi_2(w_2)\varphi_3(w_1) - \varphi_1(w_1)\varphi_2(w_3)\varphi_3(w_2) \\
&\quad + \varphi_1(w_2)\varphi_2(w_3)\varphi_3(w_1) + \varphi_1(w_3)\varphi_2(w_1)\varphi_3(w_2)
\end{aligned}
$$

$$
\begin{aligned}
\varphi_1 \wedge \varphi_2 \wedge \varphi_3 = \varphi_1 \otimes \varphi_2 \otimes \varphi_3 &- \varphi_2 \otimes \varphi_1 \otimes \varphi_3 - \varphi_2 \otimes \varphi_2 \otimes \varphi_1 \\
&- \varphi_1 \otimes \varphi_3 \otimes \varphi_2 + \varphi_3 \otimes \varphi_1 \otimes \varphi_2 + \varphi_2 \otimes \varphi_3 \otimes \varphi_1
\end{aligned}
$$

3.5 Differential Forms

Definition 3.21 (Tangent Space)**.**



$$
\mathbb{R}^n_p = \{(p, v) : v \in \mathbb{R}^n\} \quad \textit{This is the tangent space at } p
$$

$$(p, v) + (p, w) = (p, v + w)$$
$$\lambda(p, v) = (p, \lambda v)$$

with these opperations $\mathbb{R}^n_p$ is a vector space. The opperation $(p, v) + (q, w)$ makes no sense if $p \neq q$.

Notation. $v_p = (p, v)$

on $\mathbb{R}^n_p$ we have:

$$\langle (p, v), (p, w) \rangle = \langle v, w \rangle$$

Definition 3.22 (Vector Field). *A vector field in $\mathbb{R}^n$ is a function*

$$F : p \to F(p) \in \mathbb{R}^n_p$$

$$F(p) = (p, v)$$
$$v = (F^1(p), \ldots, F^n(p))$$

Properties.

- *if the components F^i, $i \in \{1, \ldots, n\}$ are continuous, the vector field is continuous*

$$F^i : p \to F^i(p)$$

- *if the components are differentiable then the vector function is differentiable.*
- *if F, G are vector fields in $\mathbb{R}^n$, then $F + G$ is also a vector field in $\mathbb{R}^n$*

$$(F + G)(p) = F(p) + G(p)$$

- *λF is a vector field $\forall \lambda \in \mathbb{R}$ and*

$$(\lambda F)(p) = \lambda \cdot F(p)$$

- *if $f : \mathbb{R}^n \to \mathbb{R}$ is a function (continuous and differentiable) then $f \cdot F$ is a new vector field on $\mathbb{R}^n$.*

$$(f \cdot F)(p) = f(p) \cdot F(p)$$

Definition 3.23 (Divergence). *If F is a vector field then its divergence is defined to be:*

$$(\operatorname{div} F)(p) = \sum_{i=1}^{n} D_i F^i(p) \in \mathbb{R}$$

So $\operatorname{div} F : \mathbb{R}^n \to \mathbb{R}$.

Notation.

$$\operatorname{div} F = \nabla \cdot F$$

Definition 3.24 (k-form). *Given $p \in \mathbb{R}^n$, let $\omega(p) \in \Lambda^k(\mathbb{R}^k_p)$*

$$\omega(p) = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \cdots i_k}(p) \varphi_{i_1}(p) \wedge \cdots \wedge \varphi_{i_k}(p)$$

it is defined by $\binom{n}{k}$ functions $p \to \omega_{i_1 \cdots i_k}(p)$, $i_1 < \cdots < i_k$

Properties.

- *if these functions are continuous, then the k-form is continuous.*
- *if these functions are differentiable, then ω is a differential k-form.*
- *if ω and η are differentiable k-forms on $\mathbb{R}^n$, $\omega + \eta$ is a differentiable k-form on $\mathbb{R}^n$*

$$(\omega + \eta)(p) = \omega(p) + \eta(p)$$

- *if $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function then $f \cdot \omega$ is a differentiable k-form.*

$$(f \cdot \omega)(p) = f(p)\omega(p)$$

- *if ω is a differentiable k-form and η is a differentiable l-form, then $\omega \wedge \eta$ is a differentiable $(k + l)$- form.*

$$(\omega \wedge \eta)(p) = \omega(p) \wedge \eta(p)$$

Definition 3.25. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable, then*

$$Df(p) : \mathbb{R}^n \to \mathbb{R}$$

is a linear map

$$Df(p) \in (\mathbb{R}^n_p)^* = \mathcal{J}^1(\mathbb{R}^n_p) = \Lambda^1(\mathbb{R}^n_p)$$

Note. *we define the following 1-form*

$$df(p) \in \Lambda^1(\mathbb{R}^n_p)$$

$$df(p)(v_p) = Df(p)(v), \quad v_p = (v, p)$$

Let $f = \pi^i$, the projection into the i-component.

$$\pi^i(x^1, \dots, x^n) = x^i$$

is a linear map, sometimes denoted $x^i(x) = x^i$

$$d\pi^i(p)(v_p) = D\pi^i(p)(v) = \pi^i(p)(v) = \pi^i(v) = v^i$$

But this is the same as $\varphi_i(v)$

$$\therefore \ d\pi^i = \varphi_i = dx^i$$

A differentiable k-form on $\mathbb{R}^n$ will look like

$$\omega(p) = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \cdots i_k}(p) dx^{i_1}(p) \wedge \cdots \wedge dx^{i_k}(p)$$

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

Example 3.13. $\mathbb{R}^2 : (x^1, x^2) = (x, y)$ *ie* $n = 2$

$$\begin{aligned} k &= 0 \qquad \omega = f(x,y) \\ k &= 1 \qquad \omega = f(x,y)dx + g(x,y)dy \\ k &= 2 \qquad \omega = f(x,y,z)dx \wedge dy \end{aligned}$$

Example 3.14. $\mathbb{R}^3 : (x^1, x^2, x^3) = (x, y, z)$

$$\begin{aligned} k &= 0 \qquad \omega = f(x,y,z) \\ k &= 1 \qquad \omega = f(x,y,z)dx + g(x,y,z)dy + h(x,y,z)dz \\ k &= 2 \qquad \omega = f(x,y,z)dx \wedge dy + g(x,y,z)dx \wedge dz + h(x,y,z)dy \wedge dz \\ k &= 3 \qquad \omega = f(x,y,z)dx \wedge dy \wedge dz \end{aligned}$$

$$dx \wedge dx = dy \wedge dy = dz \wedge dz = 0$$

$$\begin{aligned} dx \wedge dy &= -dy \wedge dx \\ dx \wedge dz &= -dz \wedge dx \\ dy \wedge dz &= -dz \wedge dy \end{aligned}$$

Theorem 3.8. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable, then the 1-form df is:*

$$df = D_1 f dx^1 + D_2 f dx^2 + \cdots + D_n f dx^n$$

Proof. $df(p) \in \Lambda^1(\mathbb{R}^n_p)$

$$df(p)(V_p) = df(p)(v) = (D_1 f(p), \ldots, D_n f(p)) \cdot \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} = \sum_{i=1}^{n} D_i f(p) v^i$$

Calculate:

$$\begin{aligned} (D_1 f dx^1 + \cdots + D_n f dx^n)(p)(V_p) &= [D_1 f(p) dx^1(p) + \cdots + D_n f(p) dx^n(p)](V_p) \\ &= D_1 f(p) dx^1(p)(V_p) + \cdots + D_n f(p) dx^n(p)(V_p) \\ &= D_1 f(p) v^1 + \cdots + D_n f(p) v^n \\ &= \sum_{i=1}^{n} D_i f(p) v^i \end{aligned}$$

□

Example 3.15. *for* $\mathbb{R}^3$

$$df = \frac{df}{dx}dx + \frac{df}{dy}dy + \frac{df}{dz}dz$$

Definition 3.26 (The Opperator d on k-forms)**.**

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \quad \textit{(k-form)}$$

$$d\omega = \sum_{i_1 < \cdots < i_k} \sum_{\alpha=1}^{n} D_i \omega_{i_1 \cdots i_k} x^{\alpha} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \quad \textit{(k+1)-form}$$

Example 3.16. $n = 3,\ k = 1$

$$\omega = f\,dx + g\,dy + h\,dz$$

$$\begin{aligned}
d\omega &= \frac{df}{dx}\cancel{dx \wedge dx} + \frac{df}{dy}dy \wedge dx + \frac{df}{dz}dz \wedge dx \\
&\quad + \frac{dg}{dx}dx \wedge dy + \frac{dg}{dy}\cancel{dy \wedge dy} + \frac{df}{dz}dz \wedge dy \\
&\quad + \frac{dh}{dx}dx \wedge dz + \frac{dh}{dy}dy \wedge dz + \frac{dh}{dz}\cancel{dz \wedge dz} \\
&= \left(\frac{dg}{dx} - \frac{df}{dy}\right) dx \wedge dy + \left(\frac{dh}{dy} - \frac{dg}{dz}\right) dy \wedge dz + \left(\frac{dh}{dx} - \frac{df}{dz}\right) dx \wedge dz
\end{aligned}$$

$$\begin{vmatrix} i & j & k \\ dx & dy & dz \\ f & g & h \end{vmatrix} = \left(\frac{dh}{dy} - \frac{dg}{dz}\right)\mathbf{i} - \left(\frac{dh}{dx} - \frac{df}{dz}\right)\mathbf{j} + \left(\frac{dg}{dx} - \frac{df}{dy}\right)\mathbf{z}$$

$$\mathbf{i} \leftrightarrow dy \wedge dz \quad \mathbf{j} \leftrightarrow dz \wedge dx \quad \mathbf{k} \leftrightarrow dx \wedge dy$$

Example 3.17. $n = 3,\ k = 2$

$$\omega = f\,dy \wedge dz + g\,dz \wedge dx + h\,dx \wedge dy$$

$$\begin{aligned}
d\omega &= \frac{df^1}{dx}dx \wedge dy \wedge dz + \frac{df^1}{dy}dy \wedge dy \wedge dz + \frac{df^1}{dz}dz \wedge dy \wedge dz \\
&\quad + \frac{df^2}{dx}dx \wedge dz \wedge dx + \frac{df^2}{dy}dy \wedge dz \wedge dx + \frac{df^2}{dz}dz \wedge dz \wedge dx \\
&\quad + \frac{df^3}{dx}dx \wedge dx \wedge dy + \frac{df^3}{dy}dy \wedge dx \wedge dy + \frac{df^3}{dz}dz \wedge dx \wedge dy \\
&= \left(\frac{df^1}{dx} + \frac{df^2}{dy} + \frac{df^3}{dz}\right) dx \wedge dy \wedge dz
\end{aligned}$$

Example 3.18. *2-form* $\leftrightarrow F = (f^1, f^2, f^3)$

$$\operatorname{div} F = \frac{df^1}{dx} + \frac{df^2}{dy} + \frac{df^3}{dz} \leftrightarrow d\omega$$

Example 3.19. $n = 3,\ k = 3$

$$\omega = f\,dx \wedge dy \wedge dz$$

$$d\omega = 0 \qquad \text{4-form on } \mathbb{R}^3 \quad \binom{3}{4} = 0$$

Example 3.20. $n = 2,\ k = 1$

$$\omega = f\,dx + g\,dy$$

$$\begin{aligned}
d\omega &= \frac{df}{dx}dx \wedge dx + \frac{df}{dy}dy \wedge dx + \frac{dg}{dx}dx \wedge dy + \frac{dg}{dy}dy \wedge dy \\
&= \left(\frac{dg}{dx} - \frac{df}{dy}\right) dx \wedge dy
\end{aligned}$$

Example 3.21. $n = 2,\ k = 2$

$$\omega = f\,dx \wedge dy$$

$$d\omega = 0$$

Example 3.22. $n = 1,\ k = 1$

$$\omega = f\,dx$$

$$d\omega = 0$$

Theorem 3.9.

(a) $d(\omega + \eta) = d(\omega) + d(\eta)$

(b) if ω *is a k-form,* η *is an l-form,* $\omega \wedge \eta$ *is a* $(k+l)$*- form.*

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge (d\eta)$$

(c) $d(d\omega) = 0$

Proof.
(c):

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

$$\begin{aligned} d(d\omega) &= d\left(\sum_{i_1 < \cdots < i_k} \sum_{i=1}^{n} D_i \omega_{i_1 \cdots i_k} x^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \right) \\ &= \sum_{i_1 < \cdots < i_k} \sum_{i=1}^{n} \sum_{j=1}^{n} D_j \left(D_i \omega_{i_1 \cdots i_k} \right) dx^j \wedge dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \end{aligned}$$

if $i = j$ then $dx^i \wedge dx^j = 0$, if $i \neq j:\ (i,j),(j,i)$

$$D_j D_i \omega_{i_1 \cdots i_k} dx^j \wedge dx^i - D_i D_j \omega_{i_1 \cdots i_k} \overset{\textit{swapped in } i,j}{dx^j \wedge dx^i}$$

for functions which have continuous mixed partial derivitives we have proved

$$D_i D_j \omega_{i_1 \cdots i_k} = D_j D_i \omega_{i_1 \cdots i_k}$$

$$\Rightarrow \text{property } (c)$$

(b):
Take

$$\omega \wedge \eta = \sum_{i_1 < \cdots < i_k} \sum_{j_1 < \cdots < j_c} \omega_{i_1 \cdots i_k} \eta_{j_1 \cdots j_c} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge dx^{j_c}$$

$$d(\omega \wedge \eta) = \sum_{i_1 < \cdots < i_k} \sum_{j_1 < \cdots < j_c} \sum_{\alpha=1}^{n} D_\alpha (\omega_{i_1 \cdots i_k} \eta_{j_1 \cdots j_c}) dx^\alpha \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge dx^{j_c}$$

$$= \sum_{i_1<\cdots<i_k} \sum_{j_1<\cdots<j_c} \sum_{\alpha=1}^{n} ((D_\alpha \omega_{i_1\cdots i_k})\eta_{j_1\cdots j_c} + \omega_{i_1\cdots i_k}(D_\alpha \eta_{j_1\cdots j_c}))dx^\alpha \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge dx^{j_c}$$

$$\begin{aligned} &= \sum_{i_1<\cdots<i_k} \sum_{j_1<\cdots<j_c} \sum_{\alpha=1}^{n} (D_\alpha \omega_{i_1\cdots i_k})\eta_{j_1\cdots j_c} dx^\alpha \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge dx^{j_c} \\ &+ \sum_{i_1<\cdots<i_k} \sum_{j_1<\cdots<j_c} \sum_{\alpha=1}^{n} \omega_{i_1\cdots i_k}(D_\alpha \eta_{j_1\cdots j_c}) \overset{*}{dx^\alpha} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge dx^{j_c} \end{aligned}$$

* shift the dx^α term down to one place infront of the dx^{j_1} term.

$$\begin{aligned} &= \left(\sum_{i_1<\cdots<i_k} \sum_{\alpha=1}^{n} D_\alpha \omega_{i_1\cdots i_k} dx^\alpha \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \right) \wedge \left(\sum_{j_1<\cdots<j_c} \eta_{j_1\cdots j_c} dx^{j_1} \wedge dx^{j_c} \right) \\ &+ \left(\sum_{i_1<\cdots<i_k} \omega_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \right) \wedge (-1)^k \left(\sum_{j_1<\cdots<j_c} \sum_{\alpha=1}^{n} D_\alpha \eta_{j_1\cdots j_c} dx^\alpha \wedge dx^{j_1} \wedge dx^{j_c} \right) \end{aligned}$$

$$= d(\omega) \wedge \eta + (-1)^k \omega \wedge d(\eta)$$

□

3.6 Closed and Exact forms

Definition 3.27. *Let ω be a k-form, ω is called closed if*

$$d\omega = 0$$

Definition 3.28. *Let ω be a k-form, ω is called exact if $\exists$ a (k-1)-form η such that:*

$$d\eta = \omega$$

Proposition 3.2. *If ω is exact then it is closed.*

Proof.

$$d(\omega) = d(d(\eta)) = 0$$

□

Example 3.23. $n = 2,\ k = 1$

$$\omega = p(x,y)dx + q(x,y)dy$$

$$d\omega = \left(-\frac{dp}{dy} + \frac{dq}{dx} \right) dx \wedge dy$$

ω is closed if $\frac{dp}{dy} = \frac{dq}{dx}$

Example 3.24. *when ω is an exact 1- form we have a $\eta = f$ 0-form such that*

$$\omega = d\eta = df = \frac{df}{dx}dx + \frac{df}{dy}dy$$

$$grad(f) = \frac{df}{dx}\mathbf{i} + \frac{df}{dy}\mathbf{j}$$

Definition 3.29. *For a vector field*

$$F = P\mathbf{i} + Q\mathbf{j}$$

if

$$\frac{dP}{dy} = \frac{dQ}{dx}$$

we call it a conservative field. Also F is a conservative field if

$$F = grad(f)$$

Example 3.25.

$$\omega = xy^2dx + ydy$$

$$\begin{aligned} d\omega &= \frac{d(xy^2)}{dy}dy \wedge dx + \frac{d(y)}{dx}dx \wedge dy \\ &= -2xydy \wedge dx \neq 0 \end{aligned}$$

$$\Rightarrow \textit{not closed and not exact}$$

Example 3.26.

$$\omega = xy^2dx + x^2ydy$$

$$d\omega = 2xydy \wedge dx + 2xydx \wedge dy = 0$$

$$\Rightarrow \textit{closed}$$

Is it exact?

$$\exists f = \frac{x^2y^2}{2} + h$$

$$\frac{df}{dx}dx + \frac{df}{dy}dy = xy^2dx + x^2ydy = \omega$$

so it is also exact.

Proposition 3.3. *If $n > 2,\ k = 1$*

$$\omega = \omega_1 dx^1 + \cdots + \omega_n dx^n$$

is closed. Is it exact? i.e does there exist f st

$$\omega = df = D_1 f dx^1 + \cdots + d_n f dx^n$$

$\omega_i = D_i f$ and can assume $f(0) = 0$ you can recover the f by integration in one variable t. $f : \mathbb{R}^n \to \mathbb{R}$.

$$f(x) - \cancel{f(0)} = \int_0^1 \frac{d}{dt}[f(tx)]dt$$

$$
\begin{aligned}
f(x) &= \int_0^1 \sum_{\alpha=1}^{n} D_\alpha f(tx) \frac{d}{dt}(tx^\alpha) dt \\
&= \int_0^1 \sum_{\alpha=1}^{n} D_\alpha f(tx) x^\alpha dt \\
&= \int_0^1 \sum_{\alpha=1}^{n} \omega_\alpha(tx) x^\alpha dt
\end{aligned}
$$

Definition 3.30 (Star-Shaped region)**.** *A is a star-shaped region with respect to 0 (or p) if $\forall t \in [0,1]$, $\forall x \in A$*

$$t \cdot x \in A$$

i.e.

$$p + t(x-p) \in A$$

Lemma 3.10 (Poincaré Lemma)**.** *If A is star-shaped with respect to 0 and ω is a closed form on A then ω is an exact form on A.*

Proof. For any $l$form ω, i will define an (l-1)-form $I(\omega)$ such that

- $$I(\lambda\omega_1 + \omega_2) = \lambda I(\omega_1) + I(\omega_2)$$
- $$I(0) = 0$$
- $$\underbrace{d(\underbrace{I(\omega)}_{l-1})}_{l} + \underbrace{I(\underbrace{d(\omega)}_{l+1})}_{l} = \omega$$

Then if ω is closed, $d\omega = 0$ so $I(d\omega) = 0$ so we get

$$d(I(\omega)) = \omega \Rightarrow \omega \textit{ is exact}$$

$$\omega = \sum_{i_1 < \cdots < i_l} \omega_{i_1 \cdots i_l} dx^{i_1} \wedge \cdots \wedge dx^{i_l}$$

$$I(\omega) = \sum_{i_1 < \cdots < i_l} \sum_{\alpha=1}^{l} (-1)^{\alpha-1} \int_0^1 t^{l-1} \omega_{i_1 \cdots i_l}(tx) x^{i_\alpha} dt\, dx^{i_1} \wedge \cdots \wedge \hat{dx^\alpha} \wedge \cdots \wedge dx^{i_l}$$

$$I(0) = 0 \qquad \text{trivial to show}$$

$$I(\lambda\omega_1 + \omega_2) = \lambda I(\omega_1) + I(\omega_2) \qquad \text{trivial to show}$$

□

Definition 3.31. *The set $I^k = [0,1]^k$ is called the standard k-cube. A continuous map $C : I^k \to A$, where A is open in $\mathbb{R}^n$ is called a singular k-cube.*

Example 3.27. $k = 1$, $C : [0,1] \to A$ *(curve)*

Example 3.28. $K = 2$, $C : [0,1]^2 \to A$ *parameterisation of a curve/surface.*

Example 3.29. $k = 0$, $[0,1]^0 = \{0\}$ *A singular 0-cube,* $\{0\} \to A$ *maps to a point on some surface.*

Example 3.30. $k = 1$, $\partial(I^1) = +1, -0$ *The motivation for this is* $\int_0^1 f'(x)dx = f(1) - f(0)$

Example 3.31. $k = 2$ $I^2 = [0,1]^2$

$$\begin{tikzcd}
(0,1) \arrow[d, "-\gamma_4"'] & (1,1) \arrow[l, "-\gamma_3"'] \\
(0,0) \arrow[r, "\gamma_1"'] & (1,0) \arrow[u, "\gamma_2"']
\end{tikzcd}$$

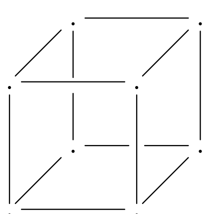
$$\begin{tikzcd}
\cdot \arrow[r, "\gamma_3"] & \cdot \\
\cdot \arrow[u, "\gamma_4"] \arrow[r, "\gamma_1"'] & \cdot \arrow[u, "\gamma_2"']
\end{tikzcd}$$

$$\partial(I^2) = \gamma_1 + \gamma_2 - \gamma_3 - \gamma_4$$

$$\begin{aligned}
&\gamma_1 \qquad I^2_{(2,0)} = \{(x,0), 0 \le x \le 1\} \\
&\gamma_2 \qquad I^2_{(1,1)} = \{(1,y), 0 \le y \le 1\} \\
&\gamma_3 \qquad I^2_{(2,1)} = \{(x,1), 0 \le x \le 1\} \\
&\gamma_4 \qquad I^2_{(1,0)} = \{(0,y), 0 \le y \le 1\}
\end{aligned}$$

Note. $I^n_{(a,b)}$ *means: fix* a^{th} *variable and set* a^{th} *variable to* b.

Example 3.32. $k = 3$



$$\begin{aligned}
top: &\qquad I^3_{(3,1)} = \{(x,y,1), 0 \le x \le 1,\ 0 \le y \le 1\} \\
base: &\qquad I^3_{(3,0)} = \{(x,y,0), 0 \le x \le 1,\ 0 \le y \le 1\} \\
front: &\qquad I^3_{(1,1)} = \{(1,y,z), 0 \le y \le 1,\ 0 \le z \le 1\} \\
back: &\qquad I^3_{(1,0)} = \{(0,y,z), 0 \le y \le 1,\ 0 \le z \le 1\} \\
left: &\qquad I^3_{(2,0)} = \{(x,0,z), 0 \le x \le 1,\ 0 \le z \le 1\} \\
right: &\qquad I^3_{(2,1)} = \{(x,1,z), 0 \le x \le 1,\ 0 \le z \le 1\}
\end{aligned}$$

$$\therefore \partial I^3 = I^3_{(3,1)} - I^3_{(3,0)} + I^3_{(1,1)} - I^3_{(1,0)} + I^3_{(2,0)} - I^3_{(2,1)}$$

Definition 3.32. *Give an n-cube* $I^n = [0,1]^n$*, we define the various faces of it to be*

$$\begin{aligned}
I^n_{(i,0)} &= \{(x^1, x^2, \dots, x^{i-1}, 0, x^{i+1}, \dots, x^n), 0 \le x^j \le 1\} \\
I^n_{(i,1)} &= \{(x^1, x^2, \dots, x^{i-1}, 1, x^{i+1}, \dots, x^n), 0 \le x^j \le 1\}
\end{aligned}$$

We define the boundry of I^n to be

$$\partial I^n = \sum_{i=1}^{n}\sum_{\alpha=0}^{1}(-1)^{i+\alpha} I^n_{(i,\alpha)}$$

We form forma sums of singular n-cubes with integer coefficents. (This is the construction of a certain abelian group, or $\mathbb{Z}$-module.)

Example 3.33.

$$C_1 : I^1 \to A$$
$$C_2 : I^1 \to A$$

$$3C_1 + (-5)C_2$$

These are singular n-chains

Definition 3.33. *A singular n-chain, $C \in A$, is a finite linear combination of singular n-cubes with integer coefficents.*

$$C = \sum_{j=1}^{m} m_j C_j \quad m_j \in \mathbb{Z},\ C_j : I^n \to A$$

Definition 3.34. *If C is a singular n-cube, $C : I \to A$. Then*

$$\partial C = \sum_{i=1}^{n}\sum_{\alpha=0}^{1}(-1)^{i+\alpha} C(I^n_{(i,\alpha)})$$

for a singular n-chain

$$C = \sum_{j=1}^{m} m_j C_j$$

where C_j are singular n-cubes

$$\partial C = \sum_{j=1}^{m} m_j \partial(C_j)$$

In $\mathbb{R}^k$ we will define integration of a k-form on a k-cube and a k-form on a k-cube face.

Let ω be a k-form on I^k

$$\forall p \in I^k : \omega(p) \in \Lambda^k(\mathbb{R}^k_p)$$

$$\omega = f(x^1, \dots, x^k) dx^1 \wedge \cdots \wedge dx^k$$

$$\begin{aligned}\int\limits_{I^k} \omega &= \int\limits_{I^k} f(x^1, \dots, x^k) dx^1 \wedge \cdots \wedge dx^k \\ &= \int\limits_{[0,1]^k} f(x^1, \dots, x^k) dx^1 \wedge \cdots \wedge dx^k \quad \textit{(Riemann Integral)}\end{aligned}$$

By Fubini

$$= \int_0^1 \left(\int_0^1 \left(\cdots \left(\int_0^1 f(x^1, \ldots, x^k) dx^1 \right) dx^2 \right) \cdots \right) dx^k$$

this can be evaluated

On $\mathbb{R}^k$, let η be a k-1 form on $I^k_{(i,\alpha)}$. The basis of k-1 forms in $\mathbb{R}^k$ is

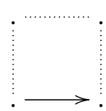
$$dx^1 \wedge \cdots \wedge \hat{dx^j} \wedge \cdots \wedge dx^k \quad j = 1, \ldots k$$

Assume η is given by:

$$\eta = g(x^1, \ldots, x^k) dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^k$$

$$\int\limits_{I^k_{(i,\alpha)}} \eta = \begin{cases} \int\limits_{[0,1]^k} g(x^1, \ldots, x^{j-1}, \alpha, x^{j+1}, \ldots, x^k) dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^k & if\, i = j \\ 0 & if\, i \neq j \end{cases}$$

Example 3.34.



$$\int\limits_{I^k_{(2,0)}} dy = 0$$

$$\int\limits_{I^k_{(1,0)}} dx = 0$$

If

$$\eta = \sum_{j=1}^{n} g_j(x^1, \ldots, x^k) dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^k$$

then

$$\int\limits_{I^k_{(i,\alpha)}} \eta = \sum_{j=1}^{n} \int\limits_{I^k_{(i,\alpha)}} g_j(x^1, \ldots, x^k) dx^1 \wedge \cdots \wedge \hat{dx^j} \wedge \cdots \wedge dx^k$$

If ω is a 0-form, then ω is a function $f(x^1, \ldots, x^k)$, 0-cube is the point $\{0\}$. Then

$$\int\limits_{I^0} \omega = f(0, \ldots, 0)$$

If

$$C = \sum_{j=1}^{m} m_j C_j$$

where C_j are all k-cubes standard, then

$$\int\limits_C \omega = \sum_{j=1}^{m} m_j \int\limits_{C_j} \omega$$

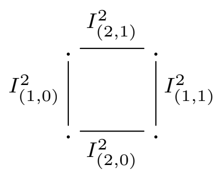
If

$$C = \sum_{j=1}^{m} m_j C_j$$

where C_j are all k-1 cubes, then

$$\int\limits_C \eta = \sum_{j=1}^{m} m_j \int\limits_{C_j} \eta$$

$$I^1 = {}_0\vdash\!\!\!-\!\!\!-\!\!\!\dashv_1 \qquad \int\limits_{I^1} \omega, \quad \int\limits_{5I^1} \omega = 5 \int\limits_{I^1} \omega$$



$$\int\limits_{\partial I^2} \overbrace{\eta}^{1-form} = + \int\limits_{I^2_{(2,0)}} \eta + \int\limits_{I^2_{(1,1)}} \eta - \int\limits_{I^2_{(2,1)}} \eta - \int\limits_{I^2_{(1,0)}} \eta$$

Lemma 3.11 (poincaré lemma)**.** *If A is star-shaped with respect to 0 and ω is a closed l-form on A, then ω is exact*

$$\omega = \sum_{i_1 < \cdots < i_l} \omega_{i_1 \cdots i_l} dx^{i_1} \wedge \cdots \wedge dx^{i_l}$$

$$I(\underbrace{\omega}_{l-form}) = \sum_{i_1 < \cdots < i_l} \sum_{\alpha}^{l} (-1)^{\alpha-1} \int_0^1 t^{l-1} \omega_{i_1 \cdots i_l}(tx) dt x^{i_\alpha} dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_\alpha}} \wedge \cdots \wedge dx^{i_l}$$

$$I(0) = 0$$
$$I(\omega_1 + \omega_2) = I(\omega_1) + I(\omega_2)$$

$$\boxed{\text{if } \omega \text{ is closed, } d\omega = 0}$$

$$dI(\omega) + I(d\omega) = \omega \quad (unproved)$$

If ω is closed $d\omega = 0$, so

$$dI(\omega) + \cancel{I(0)} = \omega \qquad \boxed{\omega \; is \; exact}$$

Because I is a linear opperator, it suffices to prove it for

$$\omega = f(x^1, \ldots, x^n) dx^{i_1} \wedge \cdots \wedge dx^{i_l}$$

$$d\omega = \sum_{\beta=1}^{n} D_\beta[f(x^1, \ldots, x^n)]dx^\beta \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_l}$$

$$
\begin{aligned}
I(\omega) &= \sum_{\alpha}^{l}(-1)^{\alpha-1}\int_0^1 t^{l-1}f(tx)dt\, x^{i_\alpha}dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_\alpha}} \wedge \cdots \wedge dx^{i_l} \\
d(I(\omega)) &= \sum_{\beta=1}^{n}\sum_{\alpha}^{l}(-1)^{\alpha-1}\int_0^1 t^{l-1}D_\beta(f(tx)x^{i_\alpha})dt\, dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_\alpha}} \wedge \cdots \wedge dx^{i_l} \\
&= \sum_{\beta=1}^{n}\sum_{\alpha}^{l}(-1)^{\alpha-1}\int_0^1 t^{l-1}\left(\delta_{\beta_1, i_\alpha}f(tx) + x^{i_\alpha}(D_\beta f)(tx)\cdot t\right)dt\, dx^\beta \wedge dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_\alpha}} \wedge \cdots \wedge dx^{i_l} \\
&= \sum_{\alpha}^{l}(-1)^{\alpha-1}\int_0^1 t^{l-1}f(tx)dt\, dx^{i_\alpha} \wedge dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_\alpha}} \wedge \cdots \wedge dx^{i_l} \\
&\quad + \sum_{\beta=1}^{n}\sum_{\alpha}^{l}(-1)^{\alpha-1}\int_0^1 t^l x^{i_\alpha}(D_\beta f)(tx)dt\, dx^\beta \wedge dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_\alpha}} \wedge \cdots \wedge dx^{i_l} \\
&= l\int_0^1 t^{l-1}f(tx)dt\, dx^{i_1} \wedge \cdots \wedge \underbrace{dx^{i_\alpha}}_{\textit{put back}} \wedge \cdots \wedge dx^{i_l} \\
&\quad + \sum_{\beta=1}^{n}\sum_{\alpha}^{l}(-1)^{\alpha-1}\int_0^1 t^l x^{i_\alpha}(D_\beta f)(tx)dt\, dx^\beta \wedge dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_\alpha}} \wedge \cdots \wedge dx^{i_l} \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
I(d\omega) &= \sum_{j \in \{\beta, i_1, \ldots, i_l\}}\sum_{\beta}^{n}(-1)^{j-1}\int_0^1 t^l D_\beta f(tx)dt\, x^j\left(dx^\beta \wedge dx^{i_1} \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^{i_l}\right) \\
&= \sum_{\beta}^{n}\int_0^1 t^l(D_\beta f)(tx)dt\, x^\beta dx^{i_1} \wedge \cdots \wedge dx^j \wedge \cdots \wedge dx^{i_l} \\
&\quad + \sum_{\alpha=1}^{l}\sum_{\beta}^{n}(-1)^\alpha\int_0^1 t^l(D_\beta f)(tx)dt\, x^\alpha\, dx^\beta \wedge dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_\alpha}} \wedge \cdots \wedge dx^{i_l} \qquad (2)
\end{aligned}
$$

(1) cancels (2).

$$
\begin{aligned}
dI(\omega) + I(d\omega) &= l \int_0^1 t^{l-1} f(tx) dt\, dx^{i_1} \wedge \cdots \wedge dx^{i_\alpha} \wedge \cdots \wedge dx^{i_l} \\
&\quad + \sum_{\beta}^{n} (-1)^{j-1} \int_0^1 t^l (D_\beta f)(tx) dt\, x^\beta dx^{i_1} \wedge \cdots \wedge dx^j \wedge \cdots \wedge dx^{i_l} \\
&= \left(\int_0^1 \left[l t^{l-1} f(tx) + \sum_{\beta}^{n} t^l x^\beta (D_\beta f)(tx) \right] dt \right) dx^{i_1} \wedge \cdots \wedge dx^{i_l} \\
&= \int_0^1 \frac{d}{dt} \left(t^l f(tx) \right) dt\, dx^{i_1} \cdots \wedge \wedge dx^{i_l} \\
&= t^l f(tx) \Big|_{t=0}^{t=1} dx^{i_1} \cdots \wedge \wedge dx^{i_l} \\
&= f(1 \cdot x) dx^{i_1} \wedge \cdots \wedge dx^{i_l} - 0 \\
&= \omega
\end{aligned}
$$

3.7 Stoke's Theorem

Theorem 3.12 (Stoke's Theorem)**.**

$$\int\limits_{\partial C} \omega = \int\limits_{C} d\omega$$

where ω is a (k-1)-form, $d\omega$ is a k-form, ∂C is a (k-1) singular chain and C is a k-singular chain.

Proof of stokes theorem on $\mathbb{R}^k$ for ω k-1 form. $C = I^k$ standard k-cube

$$\int\limits_{\partial I^k} \omega = \int\limits_{I^k} d\omega$$

we know $\int\limits_C \eta$ is linear in η , ie

$$\int\limits_C \lambda \eta_1 + \eta_2 = \lambda \int\limits_C \eta_1 + \int\limits_C \eta_2$$

therefore it suffices to prove it for:

$$
\begin{aligned}
\omega &= f(x^1, \ldots, x^k) dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^k \\
d\omega &= \sum_{\beta=1}^{k} D_\beta f(x^1, \ldots, x^k) dx^\beta \wedge dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^k \\
&= D_j f(x^1, \ldots, x^k) dx^j \wedge dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^k \\
&= (-1)^{j-1} D_j f(x^1, \ldots, x^k) dx^1 \wedge \cdots \wedge dx^j \wedge \cdots \wedge dx^k
\end{aligned}
$$

$$\begin{aligned}
\int_{I^k} d\omega &= (-1)^{j-1} \int_{I^k} D_j f dx^1 \wedge \cdots \wedge dx^k \\
&= (-1)^{j-1} \int_{[0,1]^k} D_j f dx^1 \cdots dx^k \\
&= (-1)^{j-1} \int_0^1 \cdots \int_0^1 \left(\int_0^1 D_j f dx^j \right) dx^1 \cdots \widehat{dx^j} \cdots dx^k \\
&= (-1)^{j-1} \int_0^1 \cdots \int_0^1 f(x^1, \dots, x^k) \Big|_{x^j=0}^{x^j=1} dx^1 \cdots \widehat{dx^j} \cdots dx^k \\
&= (-1)^{j-1} \int_0^1 \cdots \int_0^1 f(x^1, \dots, x^{j-1}, 1, x^{j+1}, \dots, x^k) dx^1 \cdots \widehat{dx^j} \cdots dx^k \\
&\quad - (-1)^{j-1} \int_0^1 \cdots \int_0^1 f(x^1, \dots, x^{j-1}, 0, x^{j+1}, \dots, x^k) dx^1 \cdots \widehat{dx^j} \cdots dx^k
\end{aligned}$$

$$\begin{aligned}
\int_{\partial I^k} \omega &= \sum_{i=1}^k \sum_{\alpha=0,1} (-1)^{i+\alpha} \int_{I^k_{(i,\alpha)}} \overbrace{\omega}^{k-1\, form} \\
only\, i = j\, remaining \quad &= \sum_{\alpha=0,1} (-1)^{j+\alpha} \int_{[0,1]^{k-1}} f(x^1, \dots, x^{j-1}, \alpha, x^{j+1}, \dots, x^k) dx^1 \cdots \widehat{dx^j} \cdots dx^k \\
&= (-1)^{j+1} \int_{[0,1]^{k-1}} f(x^1, \dots, x^{j-1}, 1, x^{j+1}, \dots, x^k) dx^1 \cdots \widehat{dx^j} \cdots dx^k \\
&\quad + (-1)^j \int_{[0,1]^{k-1}} f(x^1, \dots, x^{j-1}, 0, x^{j+1}, \dots, x^k) dx^1 \cdots \widehat{dx^j} \cdots dx^k
\end{aligned}$$

$\square$

Definition 3.35 (Pullback $C^*(\omega)$)**.** *If ω is a k-form on A containing a singular k-cube C, ($C : I^k \to A$), then:*

$$\int_C \omega = \int_{I^k} C^*(\omega)$$

How do we define the pullback of a k-form on C?

Remark. *If S is linear, $S^*(T) = T \cdot S$ linear*

$$\begin{array}{ccc} V & \xrightarrow{S} & W \\ & & \downarrow T \\ & & \mathbb{R} \end{array} \qquad V, W \text{ vector spaces}, \ T \text{ linear functional}$$

Recall the Pullback of Tensors: $T \in \mathcal{J}^k(W)$, then $S \in \mathcal{J}^k(W)$

$$S^*(T)(v_1, \ldots, v_k) = T(S(v_1), \ldots, S(v_k)), \quad v_i \in V$$

Let ω be a k-form on $\mathbb{R}^m$ and let $f : \mathbb{R}^n \to \mathbb{R}^m$ $f^*(\omega)(p) \in \Lambda^k(\mathbb{R}^n_p) \quad \forall p \in \mathbb{R}^n$

$$f^*(\omega)(p)(\underbrace{v_1, \ldots, v_k}_{v_i \in \mathbb{R}^n_p}) = \underbrace{\omega(f(p))}_{\in \Lambda^k(\mathbb{R}^m_{f(p)})}(Df(v_1), \ldots, Df(v_k))$$

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable, $p \in \mathbb{R}^n$, $Df(p) : \mathbb{R}^n \to \mathbb{R}^n$ linear map. It helps us the push-forward of $\mathbb{R}^k_p$ to $\mathbb{R}^m_{f(p)}$ If $v_p \in \mathbb{R}^n_p,\ v_p = (p, v),\ v \in \mathbb{R}^n$, then $F_*(v_p) \in \mathbb{R}^m_{f(p)}$ defined by

$$f_*(v_p) = (f(p), Df(p)(v))$$

Proposition 3.4. *$F_*(v_p) : \mathbb{R}^n_p \to \mathbb{R}^m_{f(p)}$ is linear.*

Proof. If $v_p, w_p \in \mathbb{R}^n_p,\ \lambda \in \mathbb{R}$

$$\begin{aligned} f_*(\lambda v_p + w_p) &= f_*(\lambda(p, v) + (p, w)) = f_*((p, \lambda, v, w)) = (f(p), Df(p)(\lambda v + w)) \\ &= (f(p), \lambda Df(p)(v) + Df(p)(w)) = \lambda(f(p), Df(p)(v)) + (f(p), Df(p)(w)) \\ &= \lambda f_*(v_p) + \lambda f_*(w_p) \end{aligned}$$

□

Definition 3.36. *If $T \in \mathcal{J}^k(\mathbb{R}^m_{f(p)})$ then $f^*(T) \in \mathcal{J}^k(\mathbb{R}^k_p)$ will be defined by*

$$f^*(\omega)(p)(v_1, \ldots, v_k) = \omega(f(p))(f_*(v_1), \ldots, f_*(v_k)) \quad \forall p \in \mathbb{R}^n,\ v_i \in \mathbb{R}^n_p$$

Theorem 3.13. *$f : \mathbb{R}^n \to \mathbb{R}^m$ differentiable*

(i) $f^*(dx^i) = \sum_{j=1}^n D_j f^i dx^j$

(ii) $f^*(\lambda w_1 + w_2) = \lambda f^*(w_1) + f^*(w_2)$

(iii) $f^*(g \cdot \omega) = (g \circ f) f^*(\omega)$

(iv) $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$

Proof.
Proof of (i):

$$f^*(dx^i) = \sum_{j=1}^n D_j f^i dx^j$$

Take $p \in \mathbb{R}^n,\ f^*(dx^i)(p) \in \Lambda^1(\mathbb{R}^m_p)$

$$f^*(dx^i)(p)(v_p) = dx^i(f(p))(f_*(v_p)) = dx^i(f(p))(f(p), Df(p)(v))$$

dx^i picks up the i^{th} component of the vector:

$$(f(p), Df(p)(v))^i = \sum_{j=1}^n D_j f^i(p) v_j$$

$f : \mathbb{R}^n \to \mathbb{R}^m$

$$Df(p)(v) = f'(p) \cdot v = \left(\begin{array}{ccc} D_1 f^1 & \dots & D_n f^1 \\ \vdots & & \vdots \\ \hline & i^th\ row & \\ \hline & & \\ \vdots & & \vdots \\ D_1 f^m & \dots & D_n f^m \end{array} \right) \cdot \begin{pmatrix} v_1 \\ \vdots \\ \vdots \\ \vdots \\ v_n \end{pmatrix} = \sum_{j=1}^{n} D_j f^i v_j$$

compute:

$$\left(\sum_{j=1}^{n} D_j f^i dx^i \right) (p)(v_p) = \sum_{j=1}^{n} D_j f^i(p) \underbrace{dx^j}_{\textit{Picks up } j^{th} \textit{ component}}(p)(v_p) = \sum_{j=1}^{n} D_j f^i(p) v_j$$

Proof of (iii):

$$F^*(g \cdot \omega) = g \circ f \cdot f^*(\omega)$$

$f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^m \to \mathbb{R}$, $p \in \mathbb{R}^n$, $v_1, \dots, v_k \in \mathbb{R}^n_p$

$$\begin{aligned} F^*(g \cdot \omega)(p)(v_1, \dots, v_k) &= (g\omega)(f(p))(f_*(v_1), \dots, f_*(v_k)) \\ &= g(f(p)) \cdot \omega(f(p))(f_*(v_1), \dots, f_*(v_k)) \end{aligned}$$

Compute:

$$g \circ f \cdot f^*(\omega)(p)(v_1, \dots, v_k) = (g \circ f)(p) \omega(f(p))(f_*(v_1), \dots, f_*(v_k))$$

□

Example 3.35. *ω a 1-form in $\mathbb{R}^3$*

$$\omega = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz$$

$f : [0, 1] \to \mathbb{R}^3$ (parameterises curve in $\mathbb{R}^3$) $f^*(\omega)$ 1-form on $[0, 1]$
Let v_t be a tangent vector on $\mathbb{R}^1_t$, $v_t = (t, v)$

$$\begin{aligned} f^*(\omega)(t)(v_t) &= \omega(f(t))(f_*(v_t)) \\ &= (Pdx + Qdy + Rdz)(f(t))(f_*(v_t)) \\ &= P(f(t))dx(f(t))(f_*(v_t)) + Q(f(t))dy(f(t))(f_*(v_t)) + R(f(t))dz(f(t))(f_*(v_t)) \end{aligned}$$

$$\boxed{\begin{aligned} f^*(v_t) &= (f(t), Df(t)(v)) \\ &= (f(t), (Df^1(t)(v), Df^2(t)(v), Df^3(t)(v))) \\ f &= f^1 + f^2 + f^3 \end{aligned}}$$

$$= P(f(t))Df^1(t)(v) + Q(f(t))Df^2(t)(v) + R(f(t))Df^3(t)(v)$$

$$\Rightarrow f^*(\omega) = (P \circ f)\frac{df^1}{dt}dt + (Q \circ f)\frac{df^2}{dt}dt + (R \circ f)\frac{df^3}{dt}dt$$

$$\begin{aligned} f^*(\omega) &= f^*(Pdx + Qdy + Rdz) \\ &= (P \circ f)f^*(dx) + (Q \circ f)f^*(dy) + (R \circ f)f^*(dz) \\ &= (P \circ f)Df^1dt + (Q \circ f)Df^2dt + (R \circ f)Df^3dt \end{aligned}$$

Definition 3.37. *If $C : I^k \to A$ is a singular k-cube in A and ω is a k-form on A, then:*

$$\int\limits_C \omega = \int\limits_{I^k} C^*(\omega)$$

Example 3.36. *ω 1-form on $\mathbb{R}^2$,*

$$\omega = xdy$$

$C : [0,1] \to \mathbb{R}^2$,

$$C(t) = (a\cos(2\pi t), b\sin(2\pi t)), \quad a, b > 0.$$

$$\begin{aligned} \int\limits_C xdy = \int\limits_{[0,1]} C^*(xdy) &= \int_0^1 (x \cdot C)(t)\frac{dC^2}{dt}dt = \int_0^1 a\cos(2\pi t) \cdot b2\pi\cos(2\pi t)dt \\ &= \int_0^1 ab2\pi \cdot \cos^2(2\pi t)dt = ab\pi \int_0^1 1 + \cos(4\pi t)dt = \pi ab \end{aligned}$$

Stokes Theorem:

$$\int\limits_C \omega = \int\limits_{\tilde{C}} d\omega = \int\limits_{\tilde{C}} d(xdy) = \int\limits_{\tilde{C}} dx \wedge dy \quad \leftarrow \textit{area of reigon parameterised by } \tilde{C}$$

Call $\tilde{C}$ the inside of the ellipse (2-cube)

$$\tilde{C}(u,t) = (au\cos(2\pi t), bu\sin(2\pi t)), \quad t \in [0,1],\ u \in [0,1]$$

$$\partial\tilde{C} = C$$

Definition 3.38. *If $C : I^k \to A$ is a singular k-cube in A and ω is a k-form on A then,*

$$\int\limits_C \omega = \int\limits_{I^k} C^*(\omega)$$

If C is a singular k-chain, ie

$$C = \sum_{j=1}^m m_j C_j \quad m_j \in \mathbb{Z},\ C_j \textit{ singular } k-\textit{cubes}$$

then

$$\int\limits_C \omega = \sum_{j=1}^m m_j \int\limits_{C_j} \omega = \sum_{j=1}^m m_j \int\limits_{I^k} C_j^*(\omega)$$

Theorem 3.14 (Stoke's Theorem for Singular k-chains in $\mathbb{R}^k$)**.** *If ω is a (k-1)-form on $\mathbb{R}^k$, $d\omega$ is a k-form on $\mathbb{R}^k$, C a k-singular chain, ∂C a (k-1)-singular chain. Then:*

$$\int\limits_{\partial C} \omega = \int\limits_{C} d\omega$$

$C = \sum_{j=1}^{m} m_j C_j \quad m_j \in \mathbb{Z},\ C_j \ singular\ k-cube,\ C_j : I^k \to \mathbb{R}^k$

$$\partial C_j = C_j(\partial I^k) = \sum_{i=1}^{k} \sum_{\alpha=0,1} (-1)^{i+\alpha} C_j(I^k_{(i,\alpha)})$$

Proof.

$$\sum_{j=1}^{m}\sum_{i=1}^{k}\sum_{\alpha=0,1} m_j(-1)^{i+\alpha} \int\limits_{C_j(I^k_{(i,\alpha)})} \overset{Def^n}{=} \sum_{j=1}^{m}\sum_{i=1}^{k}\sum_{\alpha=0,1} \left(m_j(-1)^{i+\alpha} \int\limits_{I^k_{(i,\alpha)}} C_j^*(\omega) \right) \tag{1}$$

Now compute

$$\int\limits_{C} d\omega = \sum_{j=1}^{m} m_j \int_{C_j} d\omega \overset{Def^n}{=} \sum_{j=1}^{m} m_j \int_{I^k} C_j^*(d\omega) = \sum_{j=1}^{m} m_j \int_{I^k} d(C_j^*(\omega))$$

since

$$\boxed{d(C_j^*(\omega)) = C_j^*(d\omega)}$$

Now apply Stoke's theorem for standard k-cube

$$= \sum_{j=1}^{m} m_j \int_{\partial I^k} C_j^*(\omega) = \sum_{j=1}^{m} m_j \sum_{i=1}^{k}\sum_{\alpha=0,1} (-1)^{i+\alpha} \int\limits_{I^k_{(i,\alpha)}} C_j^*(\omega). \tag{2}$$

$$\therefore\ (1) = (2)$$

□

3.8 Classical Stoke's Theorem in $\mathbb{R}^2$

$$\int\limits_{\gamma} P(x,y)dx + Q(x,y)dy = \iint_D \left(-\frac{dP}{dy} + \frac{dQ}{dx} \right) dxdy$$

s^1, s^2 are axes for the 2-cube

$$C(s^1, s^2) = (C^1(s^1,s^2), C^2(s^1,s^2))$$

$\gamma : [0,1] \to \mathbb{R}^2$

$$\gamma(t) = (\gamma^1(t), \gamma^2(t))$$

$$\partial C \overset{Def^n}{=} C(\partial I^2) = \gamma$$

$$\int_{\gamma} Pdx + Qdy = \int_{C(\partial I^2)} Pdx + Qdy \overset{Def^n}{=} \int_{\partial I^2} C^*(Pdx + Qdy)$$

$$= \int_{\partial I^2} P(C^1(s^1,s^2), C^2(s^1,s^2))C^*(dx) + Q(C^1(s^1,s^2), C^2(s^1,s^2))C^*(dy)$$

$$= \int_{\partial I^2} P\frac{d\gamma^1}{dt}dt + Q\frac{d\gamma^2}{dt}dt = \int_{\partial I^2} \left[P(\gamma^1(t),\gamma^2(t))\frac{d\gamma^1}{dt} + Q(\gamma^1(t),\gamma^2(t))\frac{d\gamma^2}{dt}\right] dt$$

$$\int_{\substack{\gamma \\ =\partial C}} Pdx + Qdy \underset{in\ \mathbb{R}^2}{\overset{Stoke's}{=}} \int_C \underbrace{d(Pdx + Qdy)}_{d\omega}$$

where γ is the the singular 1-cube which is the boundry of $C(I^k)$

$$\therefore \int_{\partial C} \omega = \int_C d\omega$$

$$\int_C d(Pdx + Qdy) = \int_C \cancel{P_x dx \wedge dx} + P_y dy \wedge dx + Q_x dx \wedge dy + \cancel{Q_y dy \wedge dy}$$

$$= \int_C -\frac{dP}{dy}dx \wedge dy + \frac{dQ}{dy}dx \wedge dy = \int_C \left(-\frac{dP}{dy} + \frac{dQ}{dy}\right) dx \wedge dy$$

$$\overset{def^n}{=} \int_{I^2} C^* \left(-\frac{dP}{dy} + \frac{dQ}{dy}\right) dx \wedge dy$$

$$= \int_{I^2} \left[-\frac{dP}{dy}(C^1,C^2) + \frac{dQ}{dy}(C^1,C^2)\right] C^*(dx \wedge dy) \qquad (*)$$

Proposition 3.5 (What is $C^*(dx \wedge dy)$?)**.**

$$\begin{aligned} C^*(dx \wedge dy) &= C^*(dx) \wedge C^*(dy) \\ &= (\tfrac{dC^1}{ds^1}ds^1 + \tfrac{dC^1}{ds^2}ds^2) \wedge (\tfrac{dC^2}{ds^1}ds^1 + \tfrac{dC^2}{ds^2}ds^2) \\ &= \tfrac{dC^1}{ds^1}\tfrac{dC^2}{ds^2}ds^1 \wedge ds^2 + \tfrac{dC^1}{ds^2}\tfrac{dC^2}{ds^1}ds^2 \wedge ds^1 \\ &= \left(\tfrac{dC^1}{ds^1}\tfrac{dC^2}{ds^2} - \tfrac{dC^1}{ds^2}\tfrac{dC^2}{ds^1}\right) ds^1 \wedge ds^2 \qquad = det\left[C'(s^1,s^2)\right] ds^1 \wedge ds^2 \end{aligned}$$

where

$$C'(s^1,s^2) = \begin{bmatrix} \frac{dC^1}{ds^1} & \frac{dC^1}{ds^2} \\ \frac{dC^2}{ds^1} & \frac{dC^2}{ds^2} \end{bmatrix}$$

$$(*) = \int_{I^2} \left[-\frac{dP}{dy}(C^1,C^2) + \frac{dQ}{dy}(C^1,C^2)\right] det\left(C'(s^1,s^2)\right) ds^1 \wedge ds^2$$

$$\therefore \textit{ ordinary double integral}$$

Recall the change of variables formula for n-dim integrals (2 in this case):

> $A \subseteq \mathbb{R}^n$, $g : A \to \mathbb{R}^n$ injective and differentiable $det[g'(x)] \neq 0\ \forall x \in A$. If $f : g(A) \to \mathbb{R}$ is integrable
> $$\int_{g(A)} f = \int_{A} (f \circ g)|det\ g'|$$

Assuming that anticlockwise orientation it can be shown that $det(C'(s^1, s^2)) > 0$

$$\int_{I^2} \left(-\frac{dP}{dy}(C^1, C^2) + \frac{dQ}{dy}(C^1, C^2) \right) det\left(C'(s^1, s^2)\right) ds^1 ds^2 = \int_{D} \left(-\frac{dP}{dy}(x, y) + \frac{dQ}{dy}(x, y) \right) dxdy$$

Theorem 3.15 (Gauss or Divergence Theorem)**.** *Solid T in $\mathbb{R}^3$ with boundry surface S and a vector function $F = (F^1, F^2, F^3)$*

$$\begin{aligned} S_x \equiv&\ \textit{Tangent plane to the solid at point } x \in S \\ n(x) \equiv&\ \textit{Outward unit normal vector} \end{aligned}$$

$$\int_{S} \langle \vec{F}, \vec{\hat{n}} \rangle dA = \iiint_{T} (div\, \vec{F}) dxdydz$$

S_x has dim 2 (tangent plane at x)

$$dim\, \Lambda^2(S_x) = 1$$

$v, w \in S_x$

$$\omega(v, w) = (v \times w) \cdot n = \langle v \times w, n \rangle \quad \text{Scalar triple product}$$

$$\omega(v, w) = \begin{vmatrix} v^1 & v^2 & v^3 \\ w^1 & w^2 & w^3 \\ n^1 & n^2 & n^3 \end{vmatrix} = \begin{vmatrix} n^1 & n^2 & n^3 \\ v^1 & v^2 & v^3 \\ w^1 & w^2 & w^3 \end{vmatrix}$$

Choose $\bar{a}, \bar{b} \in S_x$ such that $\bar{a}, \bar{b}, \bar{n}$ are an orthonormal system, right-handed.

Notation (Orthonormal System)**.** *Call $\omega(v, w) = dA(v, w)$ or $\omega = dA$ where $\omega(\bar{a}, \bar{b}) = 1$*

Theorem 3.16.

$$dA = n^1 dy \wedge dz + n^2 dz \wedge dx + n^3 dx \wedge dy$$

Proof.

$$dA(v, w) = \begin{vmatrix} n^1 & n^2 & n^3 \\ v^1 & v^2 & v^3 \\ w^1 & w^2 & w^3 \end{vmatrix} = n^1(v^2 w^3 - w^2 v^3) + n^2(-v^1 w^3 + v^3 w^1) + n^3(v^1 w^2 - v^2 w^1)$$

$$
\begin{aligned}
(dy \wedge dz)(v, w) &= (dy \otimes dz - dz \otimes dy)(v, w) \\
&= dy(v)dz(w) - dz(v)dy(w) \\
&= v^2 w^3 - w^2 v^3 \\
(dx \wedge dz)(v, w) &= -v^1 w^3 + v^3 w^1 \\
(dx \wedge dy)(v, w) &= v^1 w^2 - v^2 w^1
\end{aligned}
$$

□

Theorem 3.17.

$$
\begin{aligned}
n^1 dA &= dy \wedge dz \\
n^2 dA &= dz \wedge dx \\
n^3 dA &= dx \wedge dy
\end{aligned}
$$

Proof.

$$(*) \qquad (dy \wedge dz)(v, w) = v^2 w^3 - w^2 v^3 \qquad where\ v, w \in S_x$$

Since v and w are perpendicular to n, then $v \times w = \lambda n$, $\lambda \in \mathbb{R}$.

$$n^1 dA(v, w) = n^1(\lambda n, n) = n^1 \lambda \qquad since\ |n| = 1$$

$$\langle v \times w, i \rangle = \langle \lambda n, i \rangle = \lambda n^1$$

$$\langle v \times w, i \rangle = \begin{vmatrix} i & j & k \\ v^1 & v^2 & v^3 \\ w^1 & w^2 & w^3 \end{vmatrix} \cdot i = v^2 w^3 - w^2 v^3 = (dy \wedge dz)(v, w) \quad by\ (*)$$

$$\therefore\ n^1 dA = dy \wedge dz$$

Similarly for other equations. □

Proof of Divergence Theorem. Given $\bar{F} = (F^1, F^2, F^3) = F^1 \underline{i} + F^2 \underline{j} + F^3 \underline{k}$

$$div(\bar{F}) = \frac{dF^1}{dx} + \frac{dF^2}{Dy} + \frac{dF^3}{dz}$$

To $\bar{F}$ assign the 2-form:

$$\eta = F^1 dy \wedge dz + F^2 dz \wedge dx + F^1 dx \wedge dy \quad (2 - form)$$

Calculate:

$$d\eta = \frac{dF^1}{dx} dx \wedge dy \wedge dz + \frac{dF^2}{dy} dy \wedge dz \wedge dx + \frac{dF^3}{dz} dz \wedge dx \wedge dy$$

$$\underset{3-form}{d\eta} = \left(\frac{dF^1}{dx} + \frac{dF^2}{dy} + \frac{dF^3}{dz} \right) dx \wedge dy \wedge dz$$

$$\int_T d\eta = \int_T (div\ F)\, dx \wedge dy \wedge dz \overset{*}{=} \iiint_T f div\ F \cdot dxdydz$$

* Change of variables for I^3 to T. By Stoke's theorem

$$\int_T d\eta = \int_{\partial T} \eta = \int_S F^1 dy \wedge dz + F^2 dz \wedge dx + F^3 dx \wedge dy$$
$$= \int_S F^1 n^1 dA + F^2 n^2 dA + F^3 n^3 dA$$
$$= \int_S (F^1 n^1 + F^2 n^2 + F^3 n^3) dA$$
$$= \int_S \bar{F} \cdot \bar{n} dA$$

□

Recall.

Definition 3.39. *A set M is a K-dim manifold in $\mathbb{R}^n$ if the following condition (M) holds. For every $x \in M$:*
(M): There exisits two open sets U, V of $\mathbb{R}^n, z\ x \in U$ and a diffeomorphsim $h : U \to V$ such that:

$$h(U \cap M) = \{y \in V \ s.t.\ y^{k+1} = y^{k+2} = \cdots = y^n = 0\}$$

Definition 3.40. *A subset M of $\mathbb{R}^n$ is a k-dimensional manifold iff for every point $x \in M$ the following holds:*
(C) There exists an open set $U \in \mathbb{R}^n,\ x \in U$ and an open set $W \subset \mathbb{R}^k$ and an injective differentiable map $f : W \to \mathbb{R}^n$ such that

(i) $f(W) = U \cap M$

(ii) *rank* $f'(y) = k \quad \forall y \in W$

(iii) $f^{-1} : f(W) \to W$ *is continuous.*

Definition 3.41. *A subset M of $\mathbb{R}^n$ is a k-dimensional manifold with boundary if $\forall x \in M$ either (M) holds or (exclusive) (M') holds*

(M') $\exists$ open set U of $\mathbb{R}^n$ containing x, an open set V contained in $\mathbb{R}^n$ and a diffeomorphism $h : U \to V$ such that

$$h(U \cap M) = V \cap (\mathbb{H}^k \times \{0\}) = \{y \in V :\ y^k \geq 0,\ y^{k+1} = y^{k+2} = \cdots = y^n = 0\}.$$

Moreover, $h^k(x) = 0$. The set of points where condition (M') holds is called the boundary of M and is denoted by ∂M.

Definition 3.42. $\forall v \in \mathbb{R}^k$

$$(a, v) \to (f(a), Df(a)(v)) \in \mathbb{R}^n_{f(a)} = \mathbb{R}^n_x$$

(a, v) is pushed forward to give a vector in $\mathbb{R}^n_x$. $f : W \to U \cap M$, $v_a = (a, v)$. Let $a \in W$ such that $f(a) = x$ then

$$f_*(v_a) = (x, Df(a)(v)) \in \mathbb{R}^n_x$$

$\mathbb{R}^k_a = \{(a, v) : v \in \mathbb{R}^k\}$ a vector space.

Definition 3.43. *The tangent space of M at x is defined to be $M_x = f_*(\mathbb{R}^k_a)$ dim $M_x = k$, given $x = f(a)$, f is a chart.*

Definition 3.44. *A Vector field on M is a function F on$M$ such that $\forall x \in M$:*

$$F(x) \in M_x.$$

Let $x = f(a)$, $f : W \to U$ $f(W) = U \cap M$. Let $G(a) \in \mathbb{R}^k_a$ such that:

$$f_*(G(a)) = F(f(a)) = F(x)$$

$G(a)$ is unique, since $f_* : \mathbb{R}^k_a \to M_x$ is injective

Definition 3.45. *A vector field on M is called continuous (or differentiable) if $\forall x \in M$ the vector field G on W is continuous (or differentiable).*

Definition 3.46. *ω is a (differential) p-form on M if $\forall x \in M$*

$$\omega(x) \in \Lambda^p(M_x)$$

Then $f^*(\omega)$ is (differentiable) p-form on W.

If $f^*(\omega)$ is differential then ω is differential on $W \subseteq \mathbb{R}^k$.

Definition 3.47. *If ω is a p-form on M which k-dim in $\mathbb{R}^n$, $x \in M$.*

$$\omega(x) = \sum_{i_1 < \cdots < i_p} \omega_{i_1, \cdots, i_p}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

ω is continuous if $f^*(\omega)$ is continuous on W
ω is differentiable if $f^*(\omega)$ is differentiable on W

We have difficulty with $D_j(\omega_{i_1 \cdots i_p})$ since $\omega_{i_1 \cdots i_p}$ is not defined on an open set $U \ni x$

Theorem 3.18. *Given a differential p-form on M which is k-dim in $\mathbb{R}^n$, there exists a unique differential (p+1)-form $d\omega$ on M such that $\forall x \in M$*

$$d(f^*(\omega)) = f^*(d\omega)$$

and $f : W \to U \cap M$ is a chart. $d(\omega) \in \Lambda^{p+1}(M_x)$, $v_i \in M_x$, $d\omega(x)(v_1, \ldots, v_{p+1})$. Since f_* is a bijection, $f_* : \mathbb{R}^k_a \to M_x$, $\exists$ unique vectors $w_1, \ldots, w_{p+1} \in \mathbb{R}^k_a$ such that

$$f_*(w_i) = v_i$$

$$d\omega(x)(v_1, \ldots, v_{p+1}) = d \underbrace{f^*(\omega)(a)}_{\in \Lambda^{p+1}(\mathbb{R}^k_a)} (w_1, \ldots, w_{p+1})$$

Aim. *To understand Stoke's theorem for M k-dim manifold in $\mathbb{R}^n$ with boundry ∂M*

$$\int_{\partial M} \omega + \int_M d\omega$$

where ω is a differential (k-1)-form on M.

Definition 3.48 (Orientation on Vector spaces).

$$Bases : \mathcal{F} = \{v_1, \dots, v_n\}$$
$$\mathcal{B} = \{w_1, \dots, w_n\}$$

We say $\mathcal{F}$ and $\mathcal{B}$ define the same orientation if $det[Id]^{\mathcal{B}}_{\mathcal{F}} > 0$ and oppisite oientation if $det[Id]^{\mathcal{B}}_{\mathcal{F}} < 0$. Also we have

$$det[Id]^{\mathcal{B}}_{\mathcal{F}} = \left[det[Id]^{\mathcal{F}}_{\mathcal{B}}\right]^{-1}$$

$\mathcal{F} \sim \mathcal{B}$ iff they have the same orientation. This is an equlivalence relation.
Standard orientation on $\mathbb{R}^n$: $\mathcal{F} = [e_1, \dots, e_n]$

Example 3.37. *$\{e_1, e_2\}$ has oppisite orientation to $\{e_2, e_1\}$*

This standard orientation is denoted

$$\mu = [e_1, \dots, e_n]$$

When $f(a) = x$ on $\mathbb{R}^k_a$ we have the standard basis $[(e_1)_a, \dots, (e_k)_a]$.
Basis for M_x : $[f_*((e_1)_a), \dots, f_*((e_k)_a)]$

$$\mu_x = [f_*((e_1)_a), \dots, f_*((e_k)_a)]$$

If $b \in W$, then

$$\mu_{f(b)} = [f_*((e_1)_b), \dots, f_*((e_k)_b)]$$

If we have $z = f(c)$ and $z = f(d)$ we assign two orientations at z

$$[f_*((e_1)_c), \dots, f_*((e_k)_c)] = [g_*((e_1)_d), \dots, g_*((e_k)_d)\}$$

If the two orientations are equal, ie $det(Id) > 0$ on these two bases, then we say f and g define consistent orientations at point z. Hopefully this is true on $f(\omega) \cap g(\omega')$ then we call the two orientations consistent.

If there exists consistent orientation on all of M, we say that M is orientable and the manifold is oriented once we fix orientation. If S is a surface and the manifold is oriented once we fix orientation. If S is a surface in $\mathbb{R}^3$ which is orientable, let

$$\mu_x = [v_1, v_2] \quad x \in S \ (2 - manifold)$$

Draw the line perpendicular to S_x at x. Pick a unit vector in $n(x)$ such that $[n(x), v_1, v_2]$ is that standard orientation in $\mathbb{R}^3$, then $n(x)$ is the outer unit normal.
M is a k-dim manifold with boundry in $\mathbb{R}^3$. $(\partial M)_x$ has a basis

$$[f_*((e_1)_a), \dots, f_*((e_{k-1})_a)]$$

then let $v_0 \in \mathbb{R}^k_a$ such that $f_*(v_0)$ is perpencular at B, then $|f_x(v_0)| = 1$ and $n(x) = f_x(v_0)$

Definition 3.49 (Integrals)**.** *Let C be a singular p-cube on m k-dim manifold. $C : I^k \to M$. Let ω be a p-form on M. We define:*

$$\int_C \omega \underset{pullback}{=} \int_{I^k} C^*(\omega)$$

If C is a k-cube in M a k-manifold and $I^k \subseteq W$, $f : W \to U \cap M$ is the chart and $C(x) = f(x)$ $\forall x \in I^k$. if f is perserving orientation, then we say C is orientation preserving singular k-cube on M. If ω is a k-form on M with $\omega(y) = 0$, $\forall y \in C(I^k)$ then we define

$$\int_M \omega \underset{Def^n}{=} \int_C \omega$$

$$\boxed{d(f^*(\omega)) = f^*(d\omega) \quad f_1^*(\omega)\ a\ k-1-form\ on\ \mathbb{R}^k}$$

can partition W in to sections W_i then

$$\int_M \omega = \int_{f(W_1)} \omega + \cdots + \int_{f(W_i)} \omega + \dots$$

use partitions of unity to define $\int\limits_M \underset{k-form}{\omega}$, $\int\limits_{\partial M} \underset{(k-1)-form}{\eta}$

Theorem 3.19. *Let M be a compact, oriented k-manifold with boundry ∂M and ω be a differential (k-1)-form on M, then:*

$$\int_{\partial M} \omega = \int_M d\omega$$

Proof.

Recall (Classical Stoke's theorem).

- *M is a oriented 2-dim manifold with boundary*
- *F differentiable vector field on M*

$$\int_{\partial M} \bar{F} \cdot \bar{T} ds = \int_M curl\vec{F} \cdot \underline{\vec{n}} dA$$

Let M be a compact orientated, 2-dim manifold with bounry ∂M in $\mathbb{R}^3$. Let T be a vector field on ∂M such that $ds(M) = 1$ where ds is the length element of ∂M, $\vec{F}$ be a differentiable vector fied on an open set containing M, $\vec{n}$ be the outer unit normal on M. Then

$$\int_{\partial M} \vec{F} \cdot \vec{T} ds = \int_M curl\vec{F} \cdot \underline{\vec{n}} dA$$

if

$$F = (F^1, F^2, F^3) = F^1\underline{i} + F^2\underline{j} + F^3\underline{k}$$

we define 1-form

$$\omega = F^1dx + F^2dy + F^3dz$$

then we calculate

$$\begin{aligned} d\omega &= \frac{dF^1}{dy}dy \wedge dx + \frac{dF^1}{dz}dz \wedge dx + \frac{dF^2}{dx}dx \wedge dy + \frac{dF^2}{dz}dz \wedge dy + \frac{dF^3}{dx}dx \wedge dz + \frac{dF^3}{dy}dy \wedge dz \\ &= G^1dy \wedge dz + G^2dz \wedge dx + G^3dx \wedge dy \end{aligned}$$

$$G^1\underline{i} + G^2\underline{j} + G^3\underline{k} = curl(F)$$

We Know

$$n^1 dA = dy \wedge dz$$
$$n^2 dA = dz \wedge dx$$
$$n^3 dA = dx \wedge dy$$

$$\begin{aligned}\int\limits_M G^1 dy \wedge dz + G^2 dz \wedge dx + G^3 dx \wedge dy &= \int\limits_M \left(G^1 n^1 + G^2 n^2 + G^3 n^3\right) dA \\ &= \int\limits_M \vec{G} \cdot \vec{n} dA \overset{Def^n}{\underset{of\, G}{=\!=}} \int\limits_M curl(F) \cdot \vec{n}\end{aligned}$$

According to Stoke's general theorem

$$\int\limits_{\partial M} \omega = \int\limits_M d\omega = \int\limits_M curl(\vec{F}) \cdot \vec{n} dA$$

Since $ds(T) = 1$, we can prove as in that

$$dx = T^1 ds$$
$$dy = T^2 ds$$
$$dz = T^3 ds$$

$$\begin{aligned}\int\limits_{\partial M} \omega &= \int\limits_{\partial M} F^1 dx + F^2 dy + F^3 dz \\ &= \int\limits_{\partial M} F^1 T^1 ds + F^2 T^2 ds + F^3 T^3 ds \\ &= \int\limits_{\partial M} \vec{F} \cdot \vec{T} ds\end{aligned}$$

$\square$