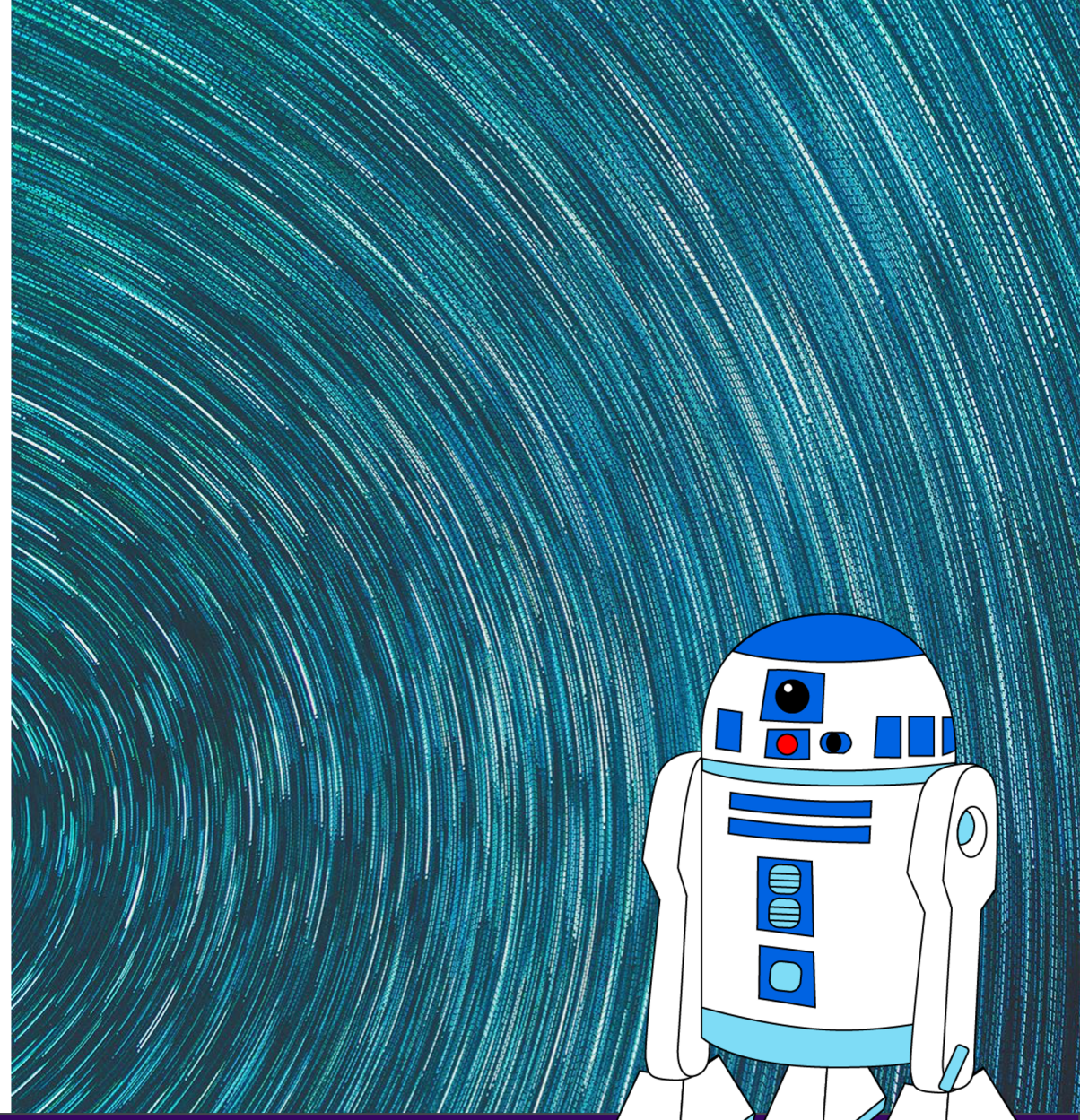
CIS 521:
ARTIFICIAL INTELLIGENCE

Constraint Satisfaction Problems

Professor Chris Callison-Burch



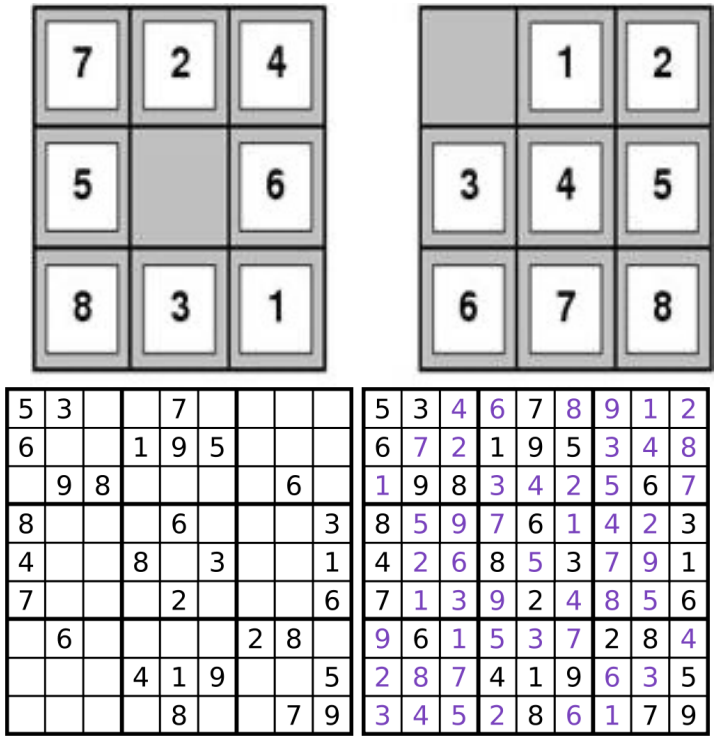
Penn Engineering
University of Pennsylvania

What is Search For?

- **Assumptions about the world: a single agent, deterministic actions, fully observed state, discrete state space**

- **Planning: sequences of actions**
  - The path to the goal is the important thing
  - Paths have various costs, depths
  - Heuristics give problem-specific guidance

- **Identification: assignments to variables**
  - The goal itself is important, not the path
  - All paths at the same depth (for some formulations)
  - CSPs are specialized for identification problems



Big idea

- **Represent the *constraints* that solutions must satisfy in a uniform *declarative* language**
- **Find solutions by *GENERAL PURPOSE* search algorithms with no changes from problem to problem**
  - No hand-built transition functions
  - No hand-built heuristics
- **Just specify the problem in a formal declarative language, and a general-purpose algorithm does everything else!**

Constraint Satisfaction Problems

A CSP consists of:

Finite set of variables $X_1, X_2, \ldots, X_n$

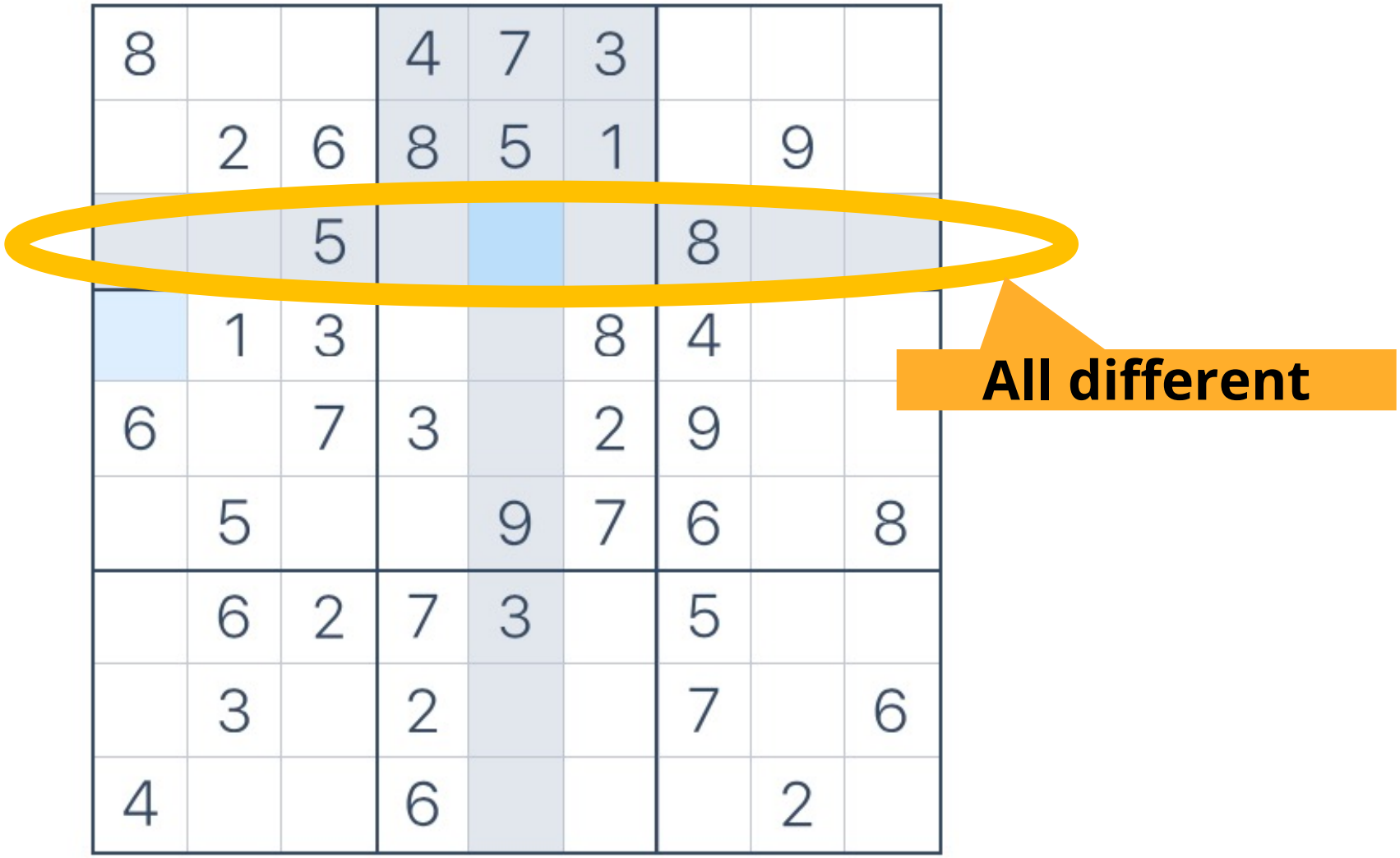
Nonempty ***domain*** *of possible values* for each variable $D_1, D_2, \ldots D_n$ *where* $D_i = \{v_1, \ldots, v_k\}$

Finite set of constraints $C_1, C_2, \ldots, C_m$

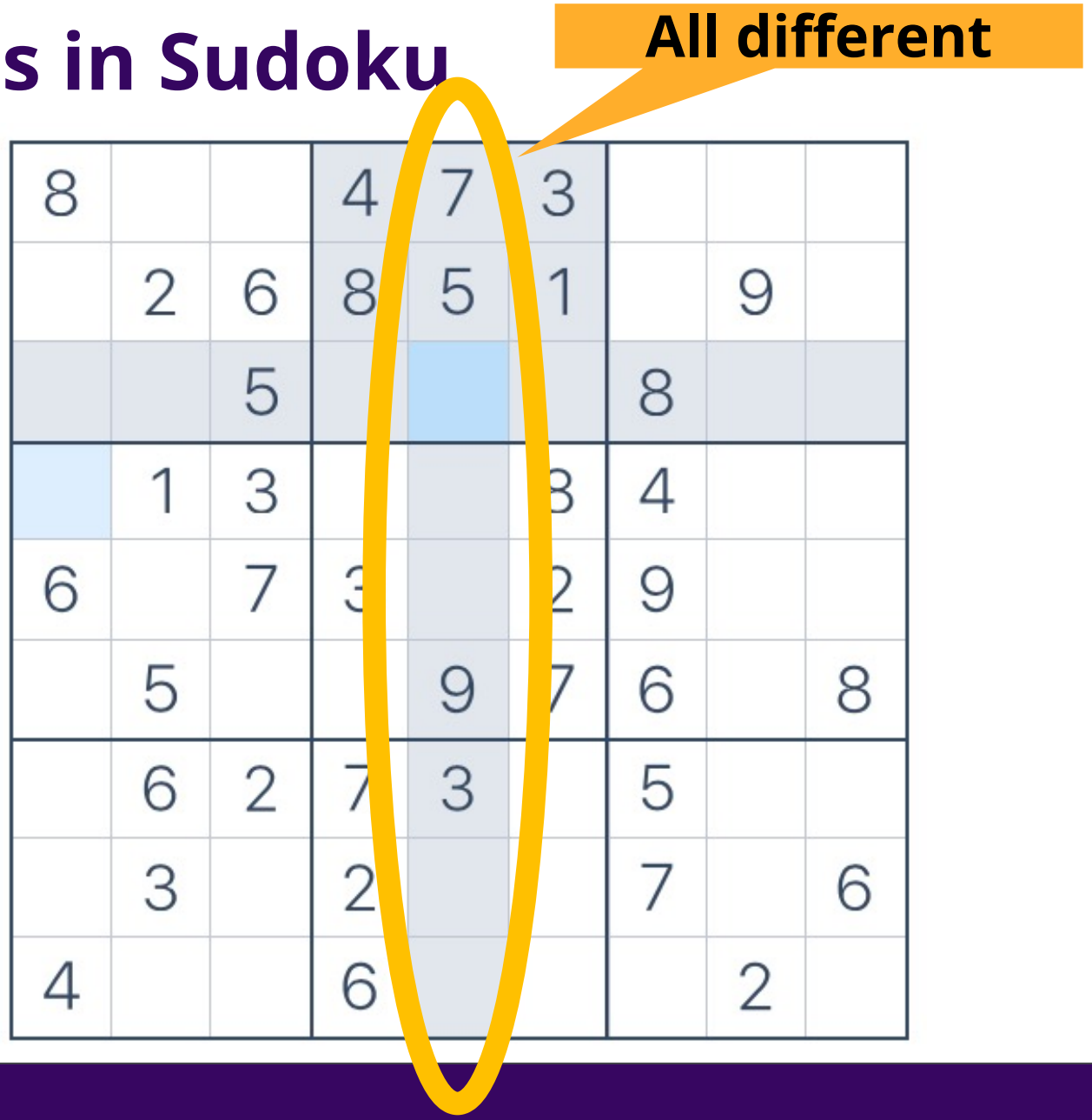
- Each *constraint* C_i limits the values that variables can take, e.g., $X_1 \neq X_2$ A *state* is defined as an *assignment* of values to some or all variables.

- **A *consistent* assignment does not violate the constraints.**
- **Example problem: Sudoku**

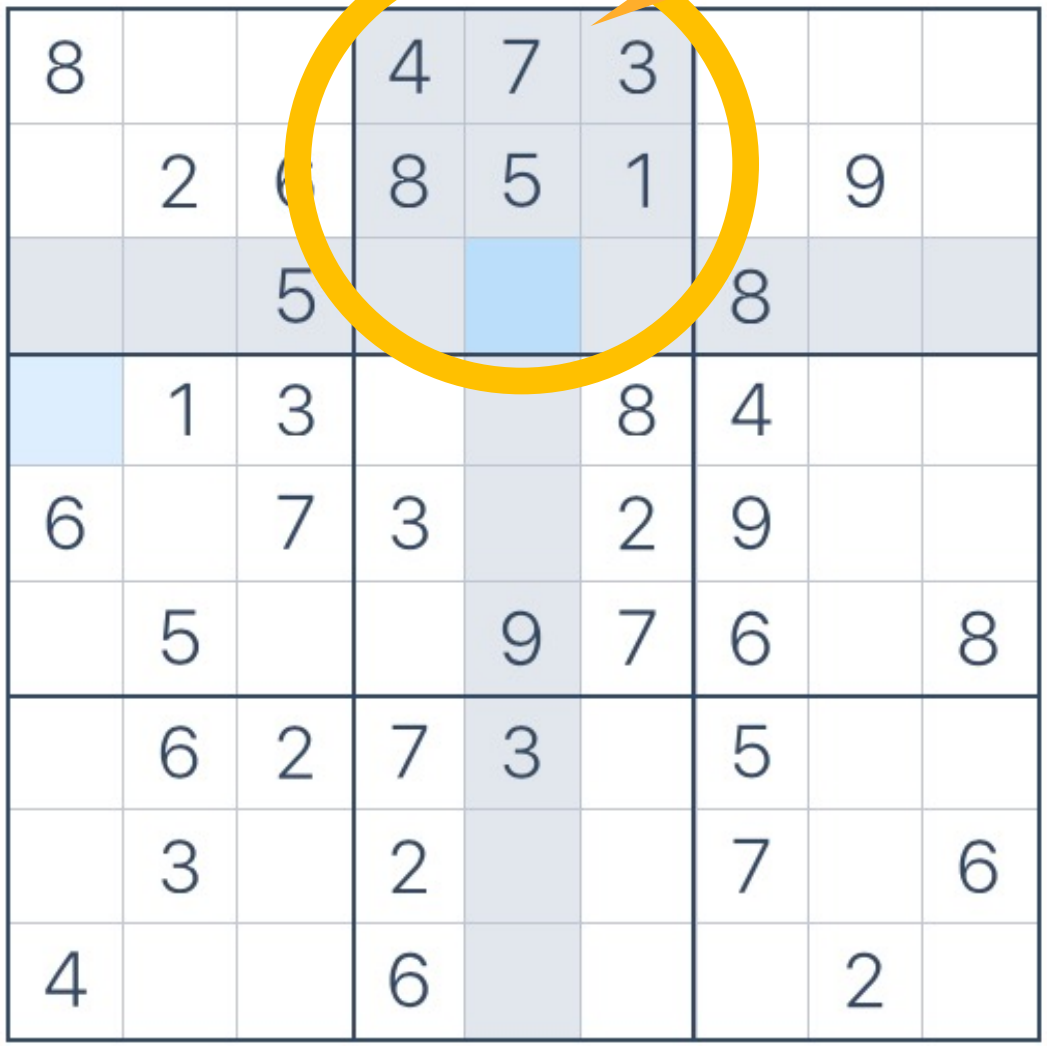
Constraints in Sudoku



Constraints in Sudoku



Constraints in Sudoku



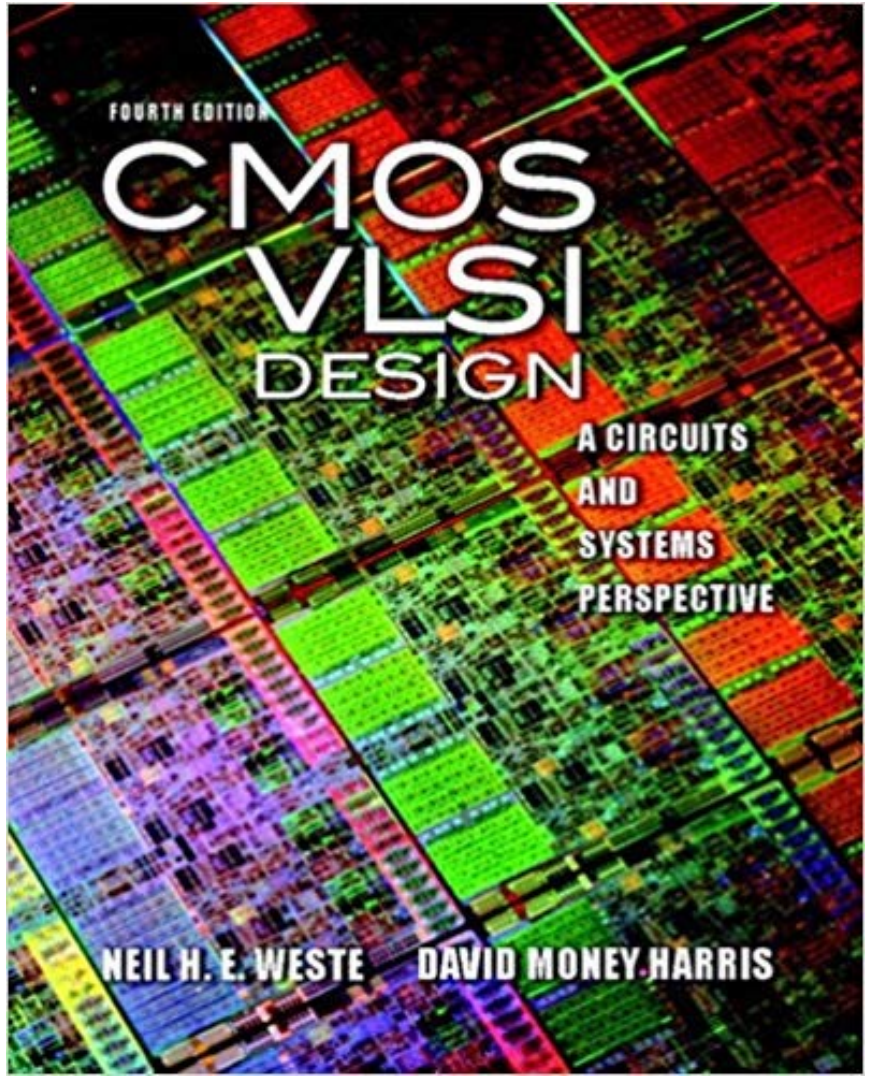
Constraint satisfaction problems

- An assignment is *complete* when every variable is assigned a value.
- A *solution* to a CSP is a *complete, consistent* assignment.
- Solutions to CSPs can be found by a completely *general purpose* algorithm, given only the formal specification of the CSP.

- Beyond our scope: CSPs that require a solution that maximizes an *objective function*.

Applications

- **Map coloring**
- **Scheduling problems**
  - Job shop scheduling
  - Scheduling the Hubble Space Telescope
- **Floor planning for VLSI**
- **Sudoku**
- **…**

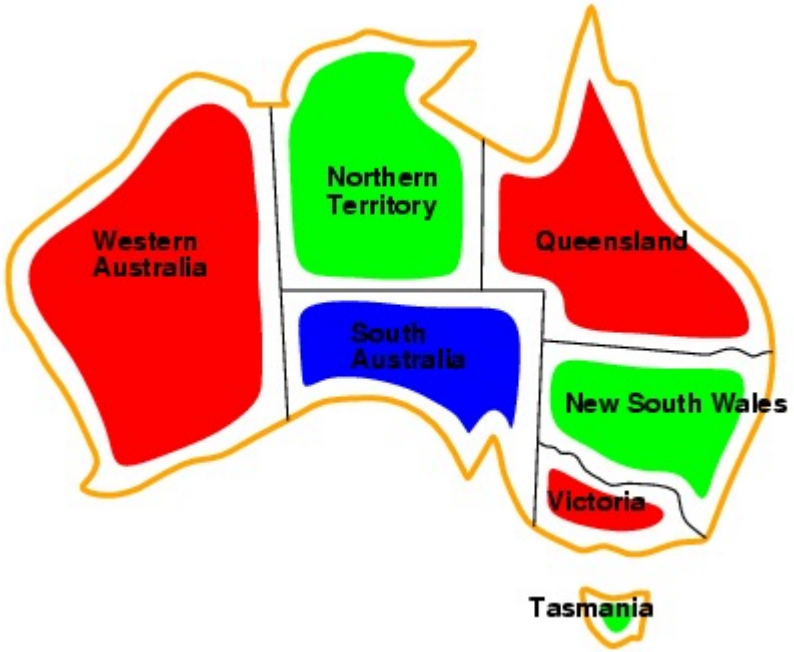


Example: Map-coloring

- **Variables:** ***WA, NT, Q, NSW, V, SA, T***
- **Domains:** D_i **= {red,green,blue}**
- **Constraints: adjacent regions must have different colors**

  e.g., WA ≠ NT
  - So (WA,NT) must be in {(red,green),(red,blue),(green,red), ...}

Example: Map-coloring



Solutions: **complete** and **consistent** **assignments**

e.g., WA = red, NT = green,Q = red, NSW = green,
V = red, SA = blue, T = green

Benefits of CSP

- **Clean specification of many problems, generic goal, successor function & heuristics**

  Just represent problem as a CSP & solve with general package

- **CSP "knows" which variables violate a constraint**

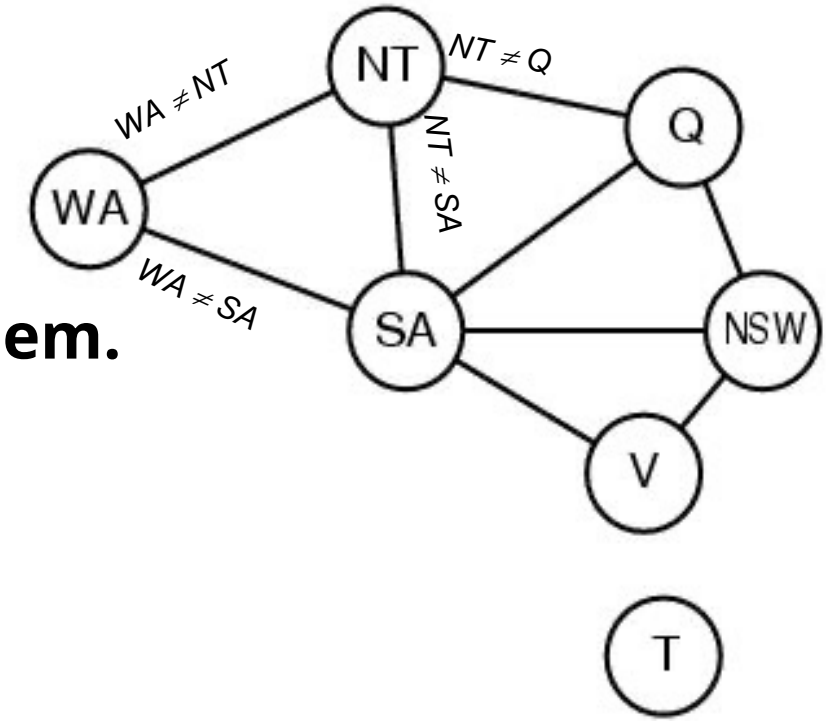
  And hence where to focus the search

- ***CSPs:* Automatically prune off all branches that violate constraints**

  (State space search could do this only by *hand-building constraints into the successor function)*

CSP Representations

- *Constraint graph:*
  - *nodes* are variables
  - *arcs* are (binary) constraints
- *Standard representation pattern:*
  - variables with values
- *Constraint graph* simplifies search.
  - e.g. Tasmania is an independent subproblem.
- *This problem: A binary CSP:*
  - each constraint relates two variables



Varieties of CSPs

- ***Discrete variables***

  finite domains:

  - *n* variables, domain size *d* → $O(d^n)$ complete assignments
  - e.g., Boolean CSPs, includes Boolean satisfiability (NP-complete)

  infinite domains:

  - integers, strings, etc.
  - e.g., job scheduling, variables are start/end days for each job
  - need a constraint language, e.g., $StartJob_1 + 5 \leq StartJob_3$

- ***Continuous variables***

  e.g., start/end times for Hubble Space Telescope observations

  linear constraints solvable in polynomial time by linear programming

Varieties of constraints

- ***Unary*** **constraints involve a single variable,**
  e.g., SA ≠ green

- ***Binary*** **constraints involve pairs of variables,**
  e.g., SA ≠ WA

- ***Higher-order*** **constraints involve 3 or more variables**
  e.g., crypt-arithmetic column constraints

- ***Preference*** **(soft constraints) e.g.** ***red*** **is better than** ***green*** **can be represented by a cost for each variable assignment**
  Constrained optimization problems.

Idea 1: CSP as a search problem

- **A CSP can easily be expressed as a search problem**

  Initial State: the empty assignment {}.

  Successor function: Assign value to any unassigned variable *provided that there is not a constraint conflict*.

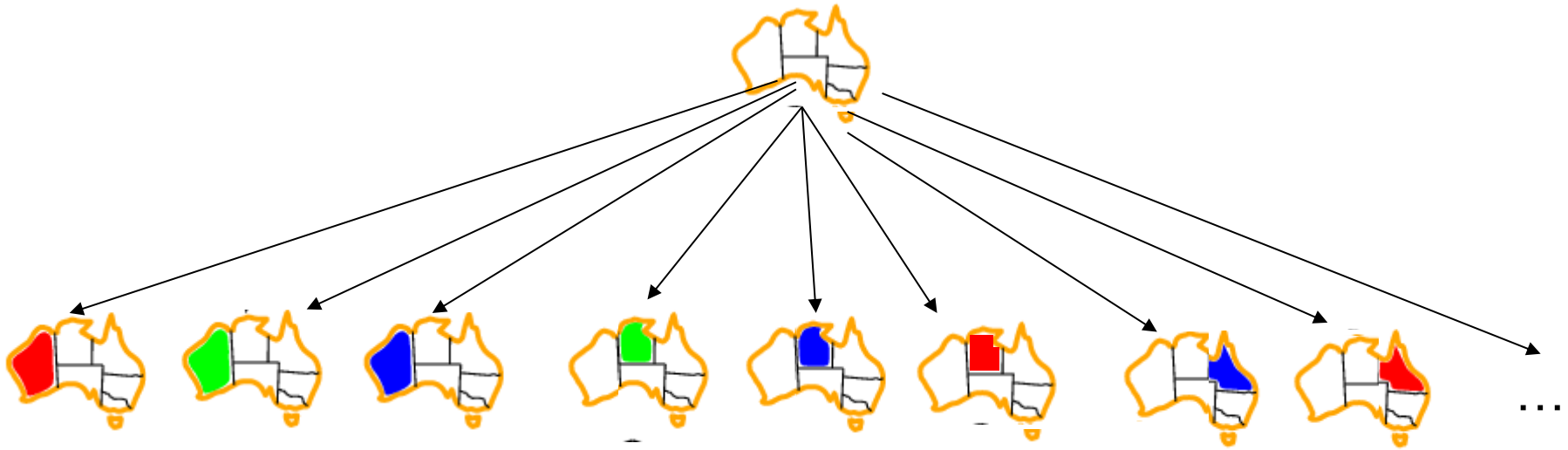
  Goal test: the current assignment is complete.

  Path cost: a constant cost for every step.

- **Solution is always found at depth *n*, for *n* variables**

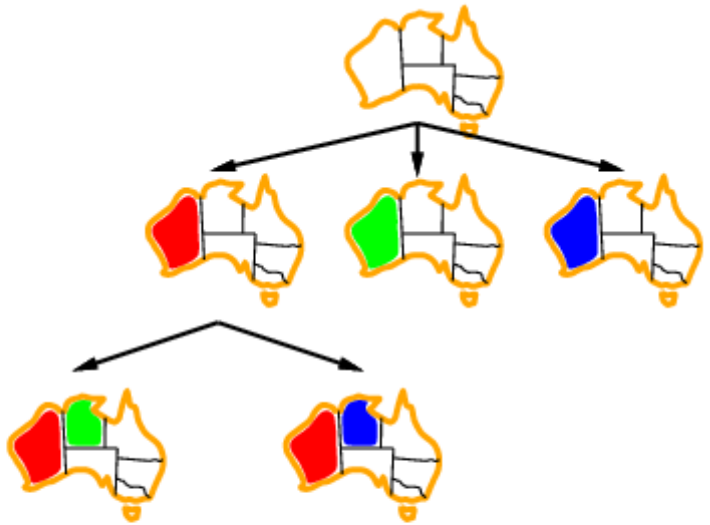
  Hence Depth First Search can be used

Search and branching factor



- n variables of domain size d
- Branching factor at the root is n*d
- Branching factor at next level is (n-1)*d
- Tree has $n!*d^n$ leaves

Search and branching factor



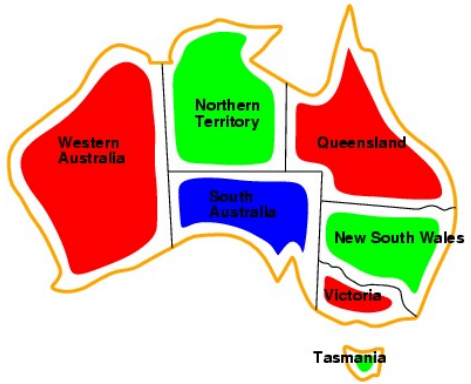
- **The variable assignments are *commutative***
  Eg *[step 1:* ***WA = red****; step 2:* ***NT = green*** *]*
  equivalent to *[step 1:* ***NT = green****; step 2:* ***WA = red*** *]*
  Therefore, a *tree search,* not a *graph search*

- **Only need to consider assignments to a single variable at each node**
  $b = d$ and there are d^n leaves *(n variables, domain size d)*

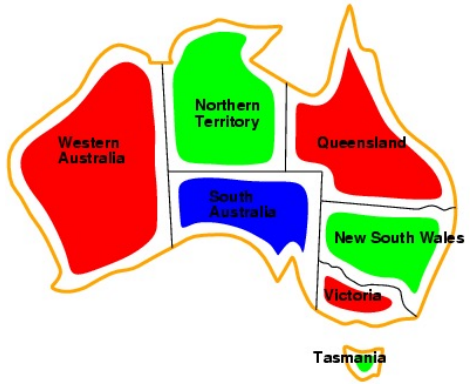
Search and *Backtracking*

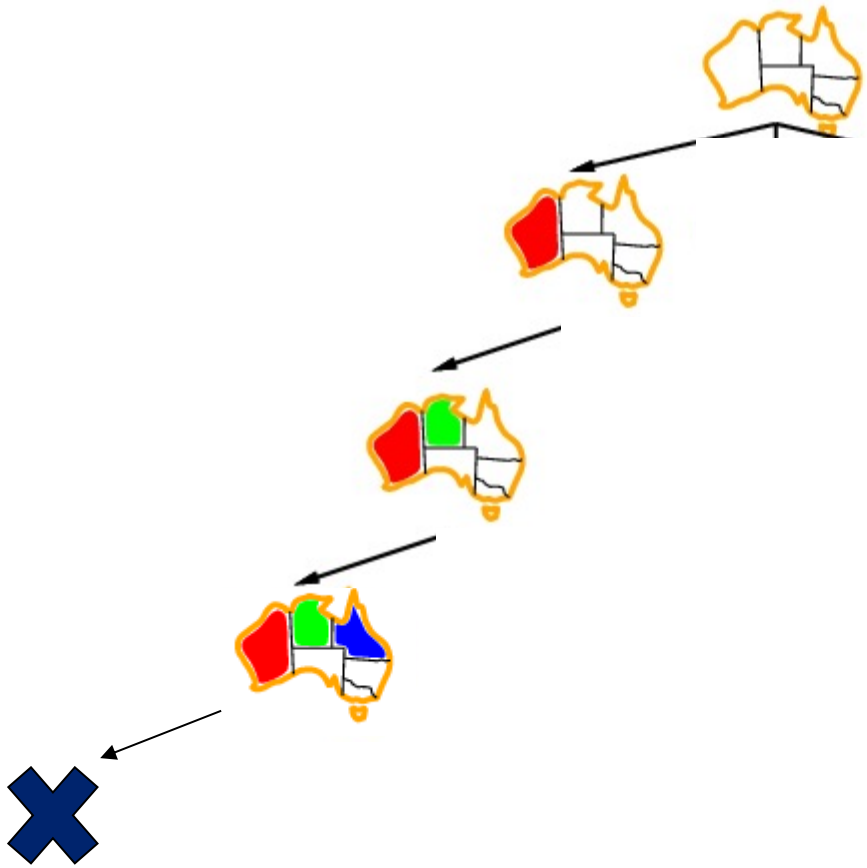
- **Depth-first search for CSPs with single-variable assignments is called *backtracking* search**
- **The term backtracking search is used for a depth-first search that chooses values for one variable at a time and backtracks when a variable has no legal values left to assign.**
- **Backtracking search is the basic *uninformed* algorithm for CSPs**

Backtracking example



Backtracking example





Idea 2: Improving backtracking efficiency

- ***General-purpose*** **methods &** ***general-purpose*** **heuristics can give huge gains in speed,** ***on average***
- **Heuristics:**

  Q: Which variable should be assigned next?

  1. **Most constrain*ed* variable**
  2. **(if ties:) Most constrain*ing* variable**

  Q: In what order should that variable's values be tried?

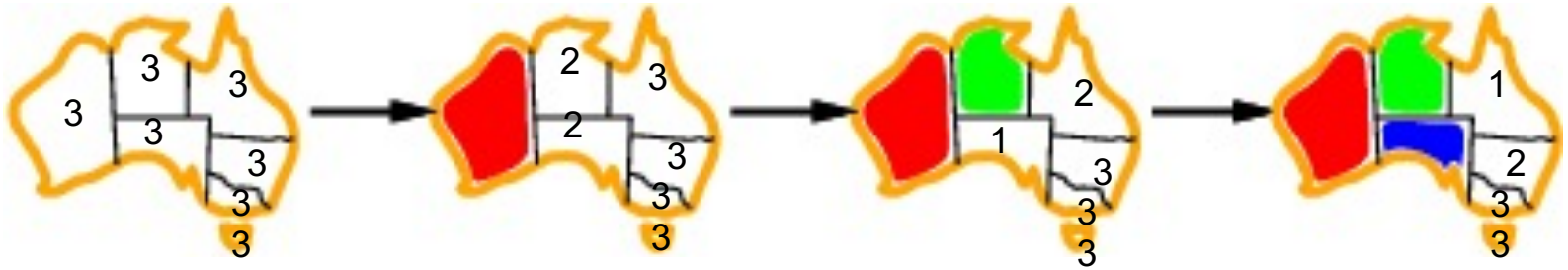
  3. **Least constraining *value***

  Q: Can we detect inevitable failure early?

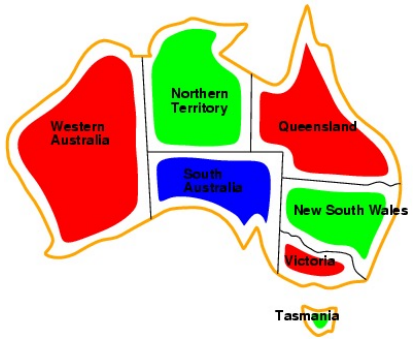
  4. **Forward checking**

Heuristic 1: Most constrain*ed* variable

- Choose a variable with the *fewest legal values*

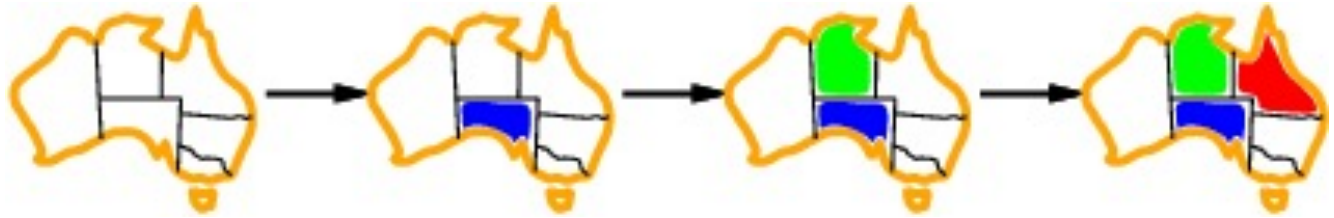


- a.k.a. *minimum remaining values (MRV)* heuristic

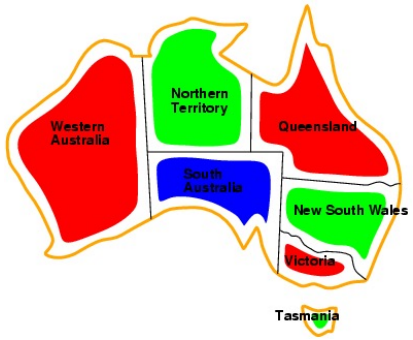


Heuristic 2: Most constrain*ing* variable

- **Tie-breaker among most constrained variables**
- **Choose the variable with the *most constraints on remaining variables***



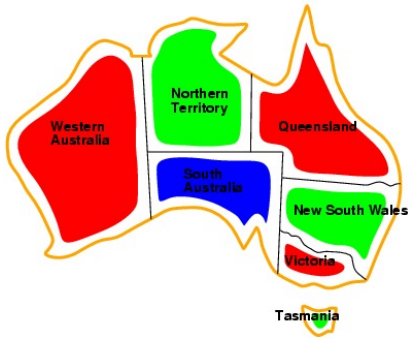
These two heuristics together lead to immediate solution of our example problem



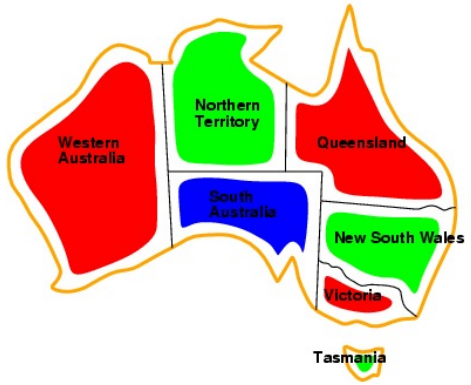
Heuristic 3: Least constraining *value*

- **Given a variable, *choose the least constraining value:***
  the one that rules out the fewest values in the remaining variables

Note: demonstrated here independent of the other heuristics



Heuristic 4: Forward checking

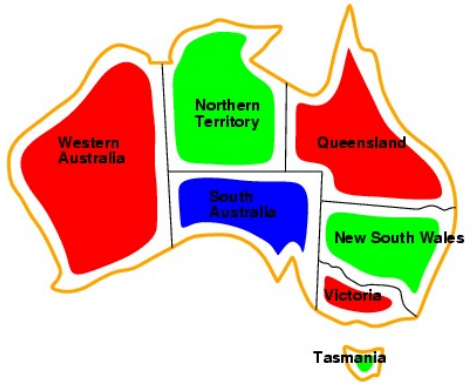


- **Idea**:
  - Keep track of *remaining* legal values for *unassigned* variables
  - Terminate search when any unassigned variable has no remaining legal values

New data structure

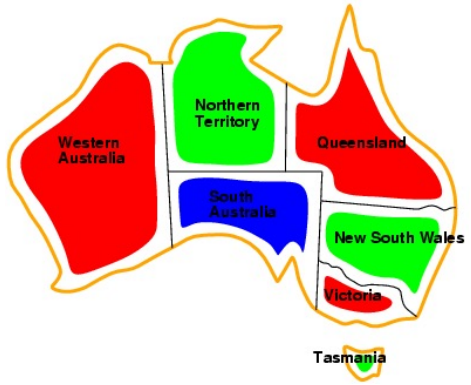
(A first step towards Arc Consistency & AC-3)

Forward checking

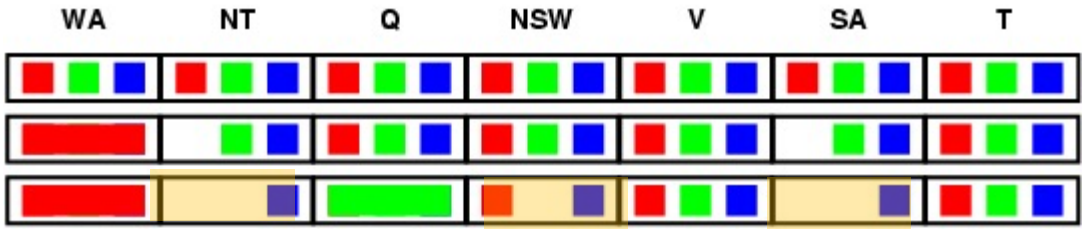


- **Idea**:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any unassigned variable has no remaining legal values

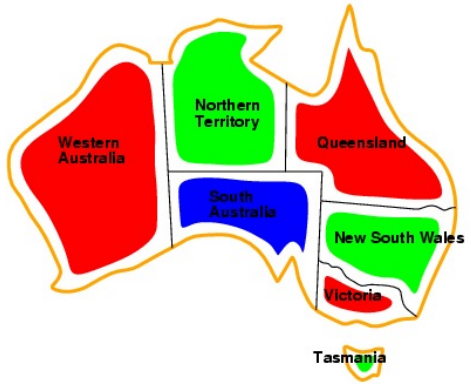
Forward checking



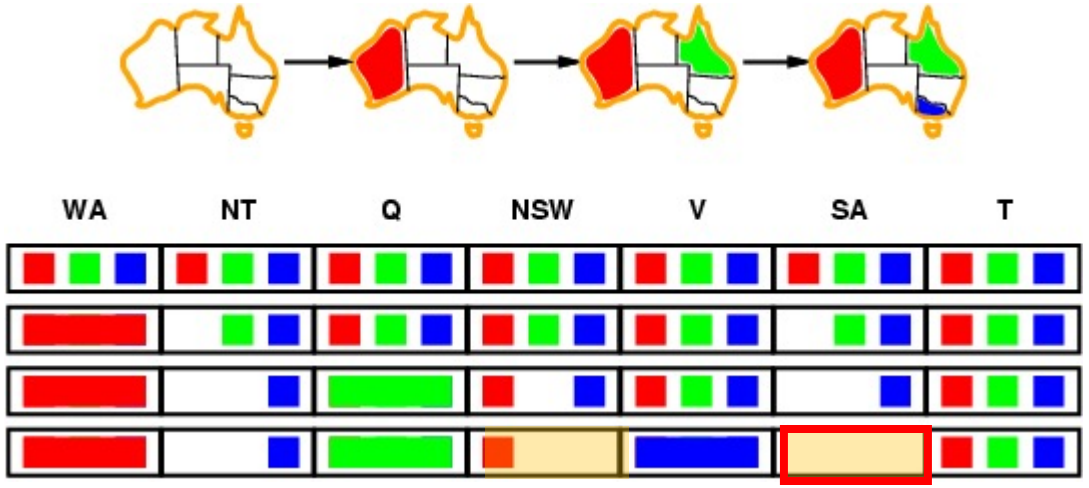
- **Idea**:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any unassigned variable has no remaining legal values



Forward checking

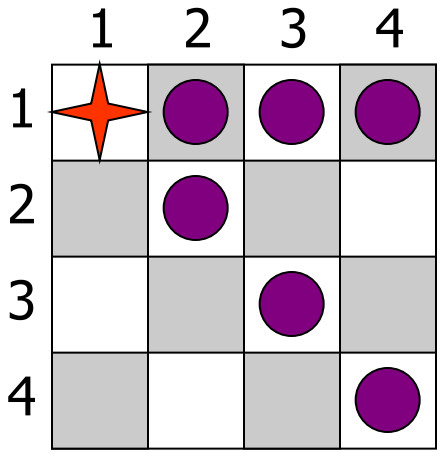


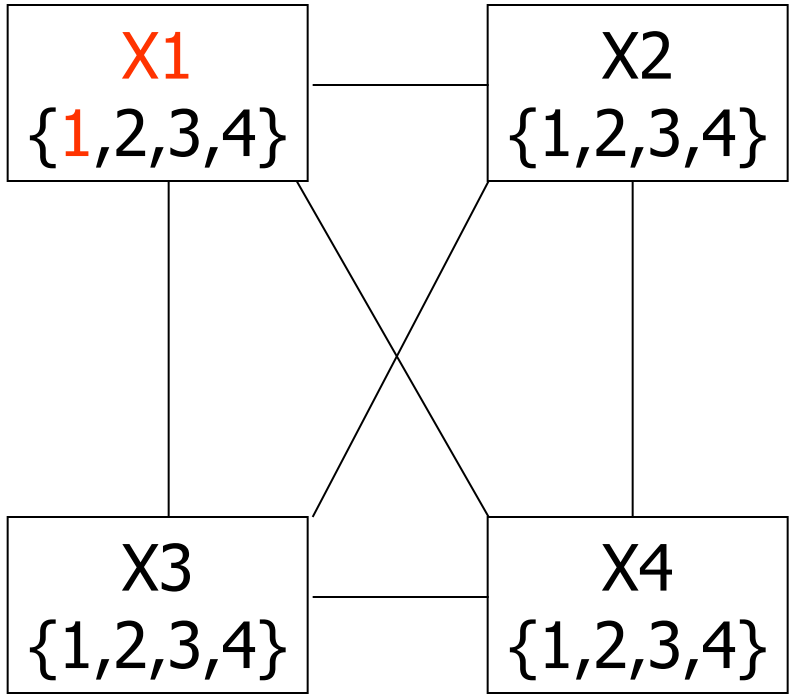
- **Idea**:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any unassigned variable has no remaining legal values



Terminate! No possible value for SA

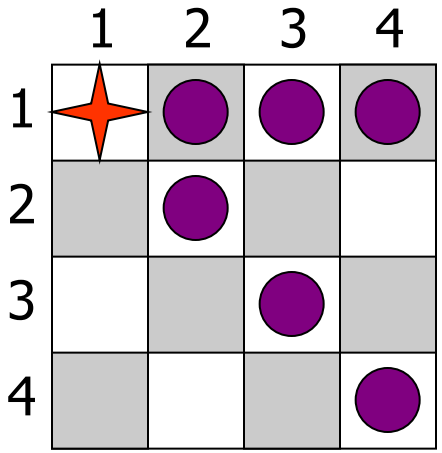
Example: 4-Queens Problem

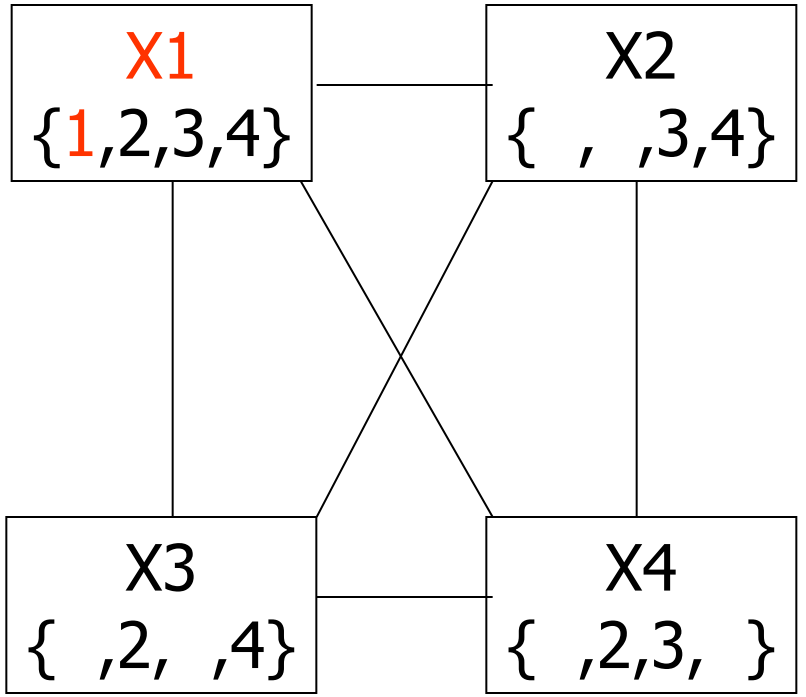




Assign value to unassigned variable

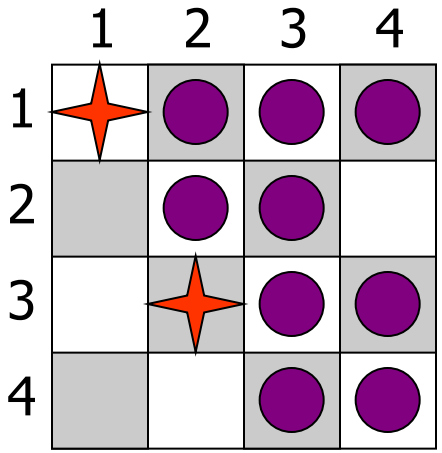
Example: 4-Queens Problem

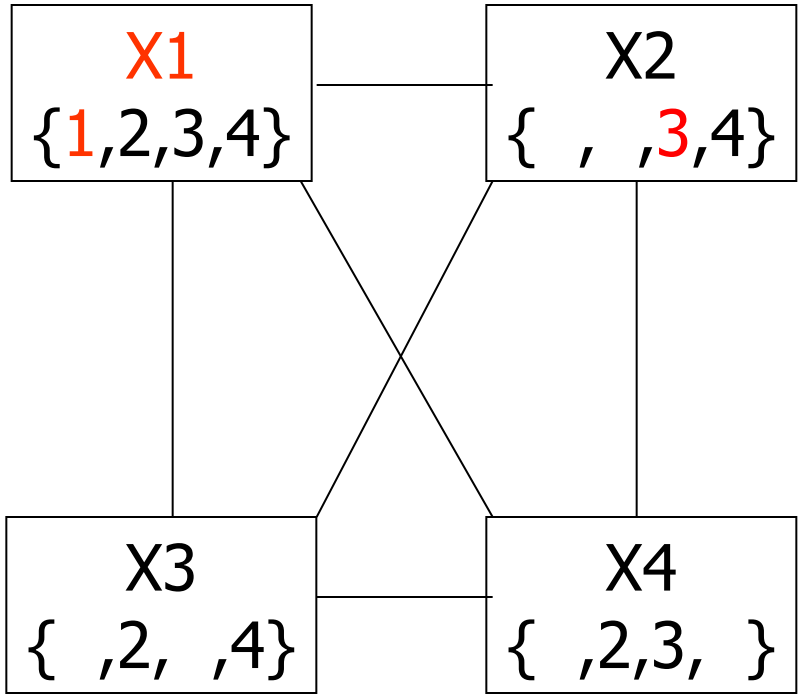




Forward check!

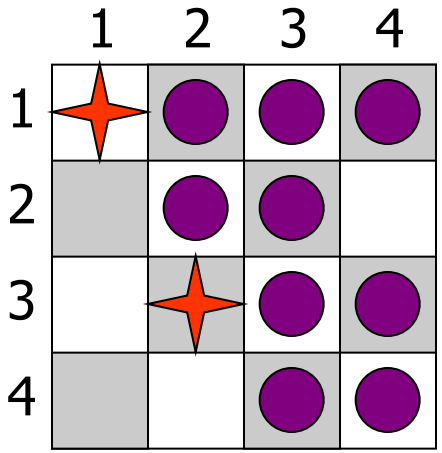
Example: 4-Queens Problem

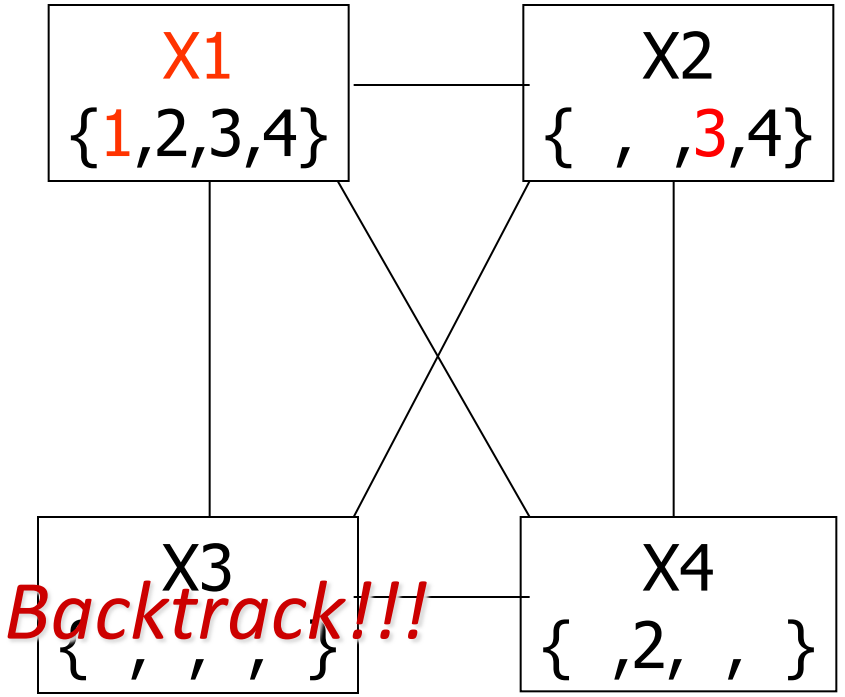




Assign value to unassigned variable

Example: 4-Queens Problem

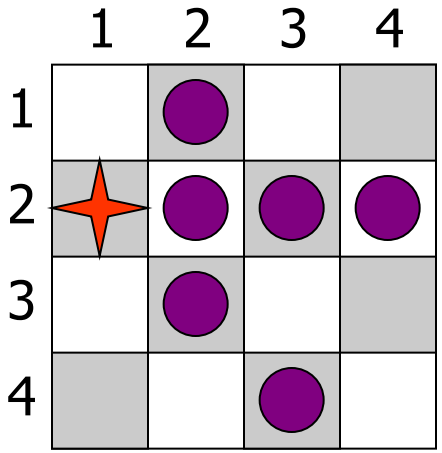


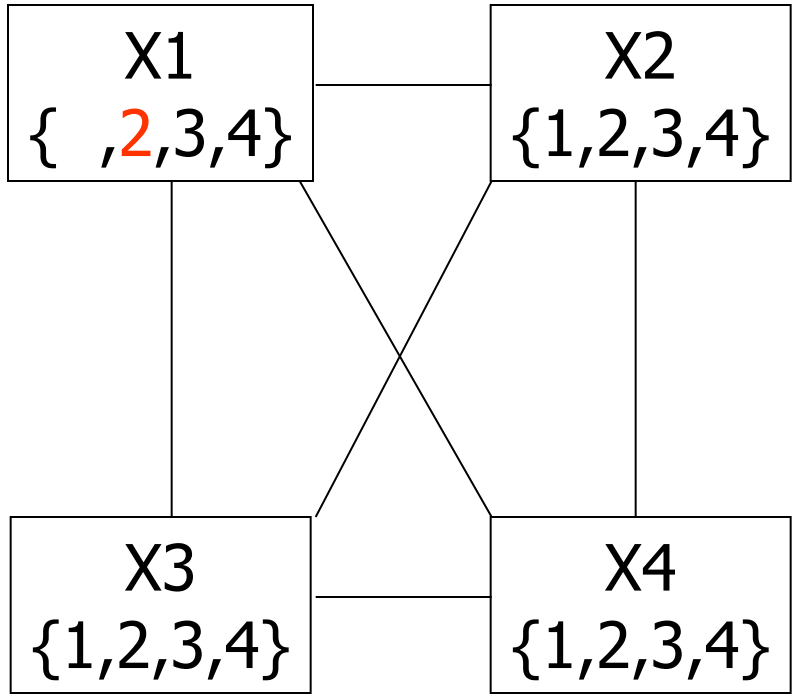


Forward check!

Example: 4-Queens Problem

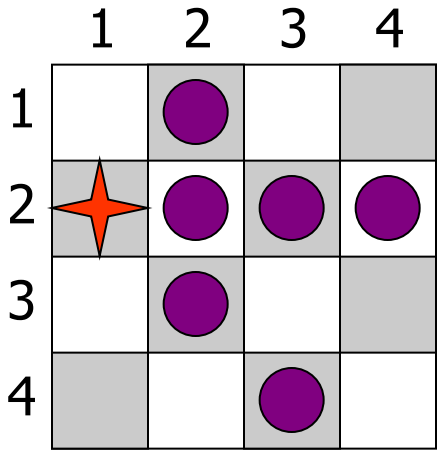
Picking up a little later after two steps of backtracking....

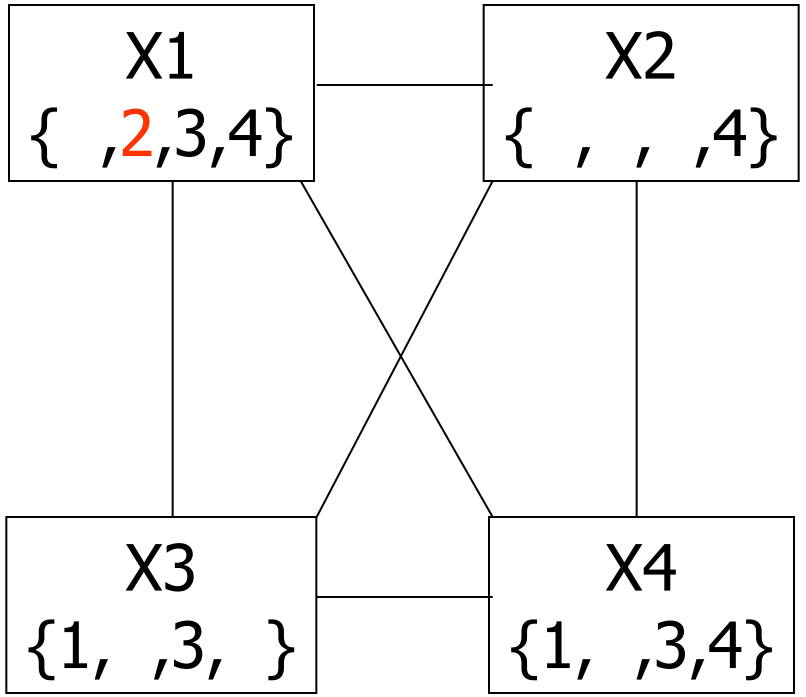




Assign value to unassigned variable

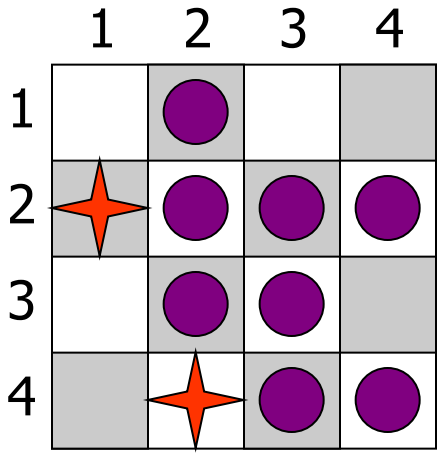
Example: 4-Queens Problem

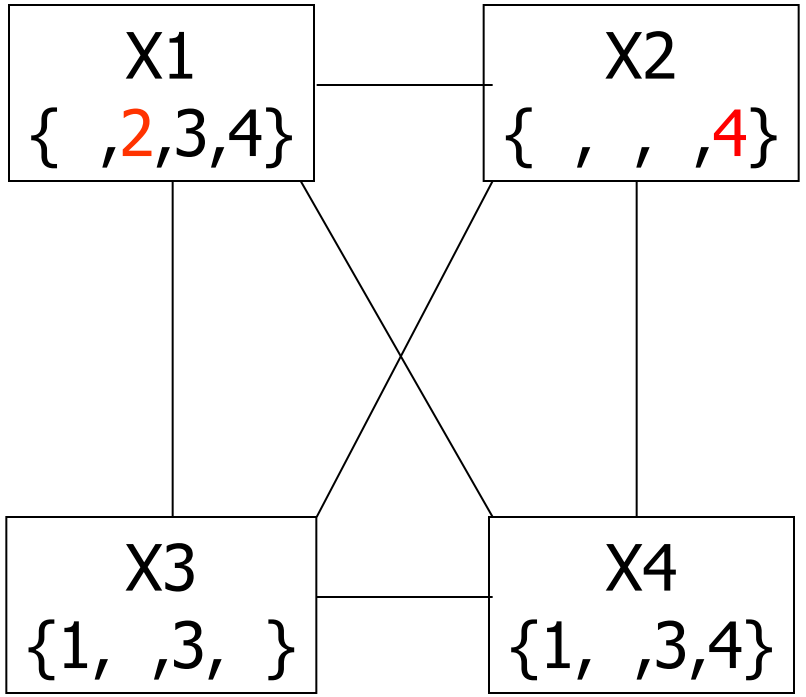




Forward check!

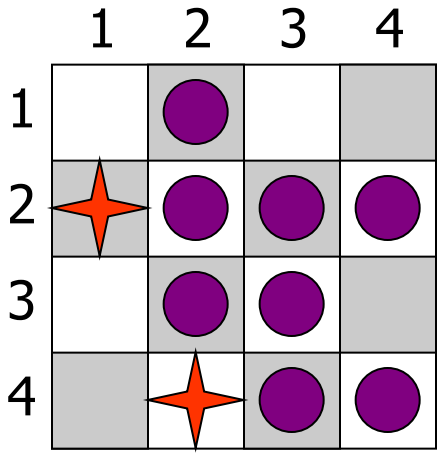
Example: 4-Queens Problem

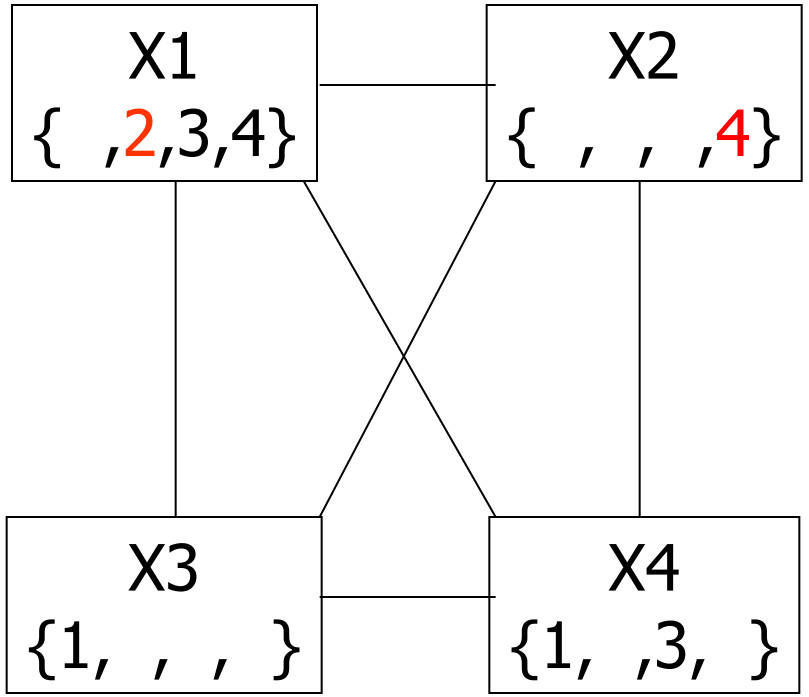




Assign value to unassigned variable

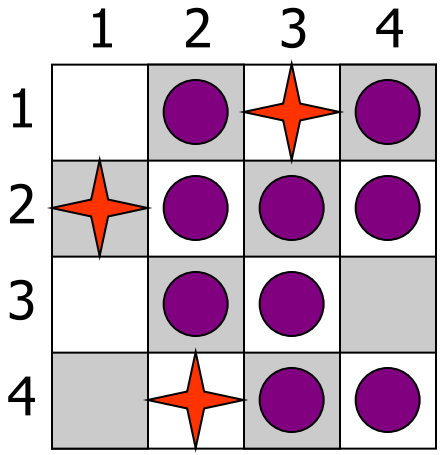
Example: 4-Queens Problem

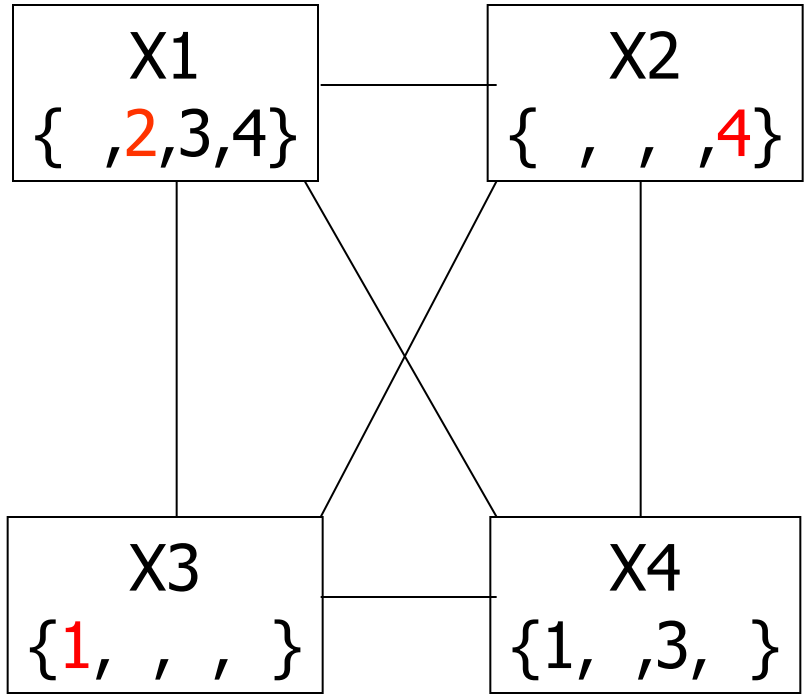




Forward check!

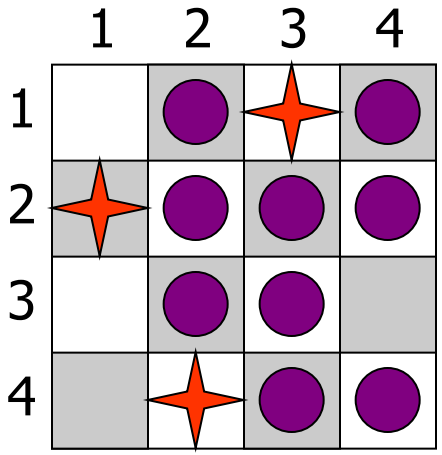
Example: 4-Queens Problem

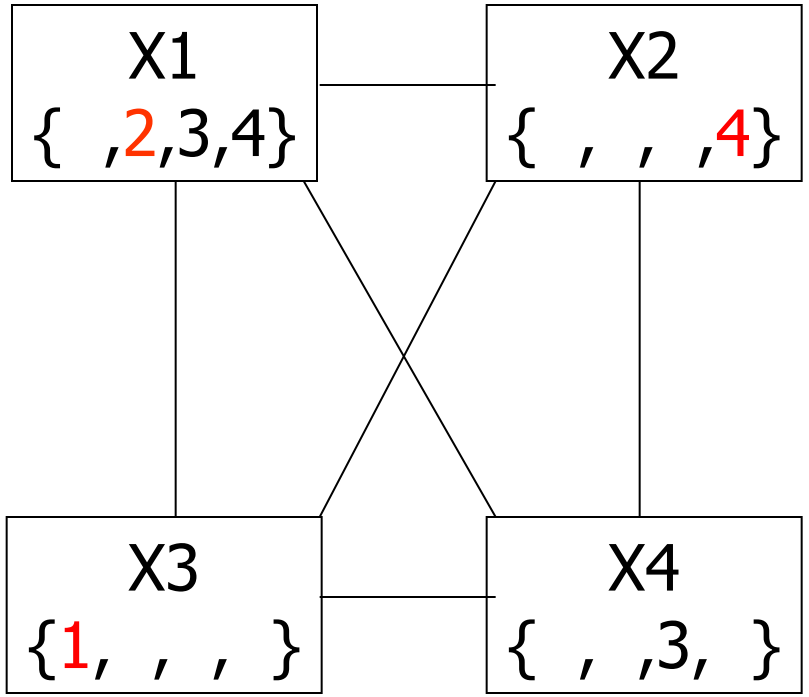




Assign value to unassigned variable

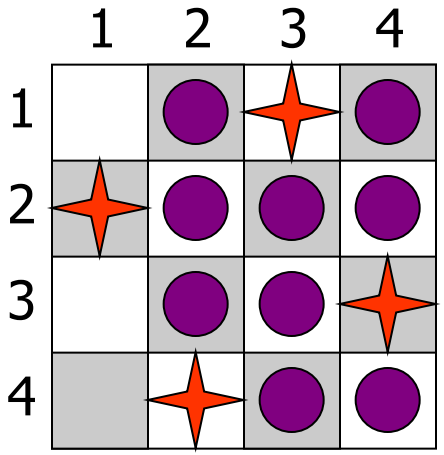
Example: 4-Queens Problem

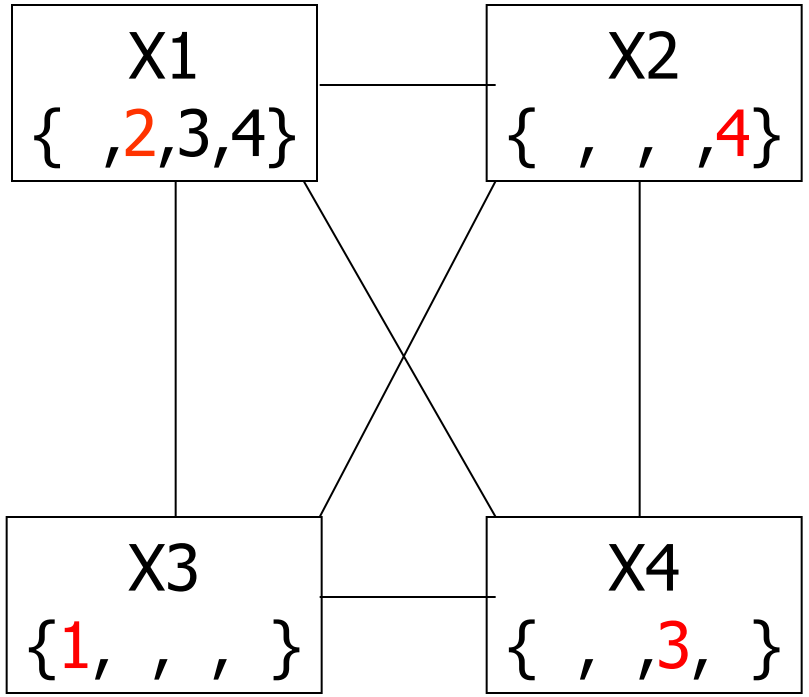




Forward check!

Example: 4-Queens Problem

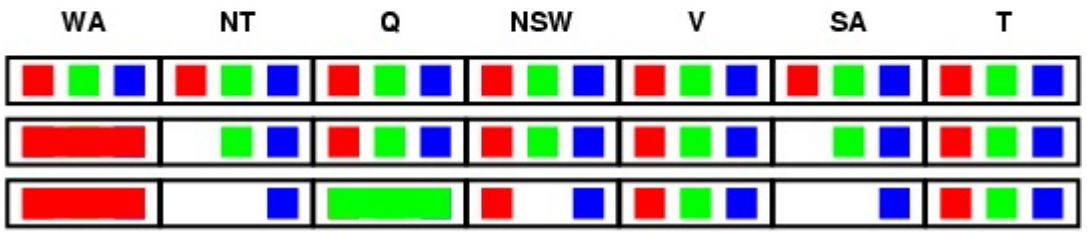




Assign value to unassigned variable

Towards Constraint propagation

- **Forward checking** propagates information from *assigned* to *unassigned* variables, but doesn't provide early detection for all failures:



- NT and SA cannot both be blue!
- **Constraint propagation** goes beyond forward checking & repeatedly enforces constraints locally

Arc Consistency, Constraint Propagation & AC-3

Idea 3 *(big* idea)*: Inference* in CSPs

- **CSP solvers combine search *and inference***

  Search
  - assigning a value to a variable

  Constraint propagation (inference)
  - Eliminates possible values for a variable if the value would violate local consistency

  Can do inference first, or intertwine it with search
  - You'll investigate this in the Sudoku homework

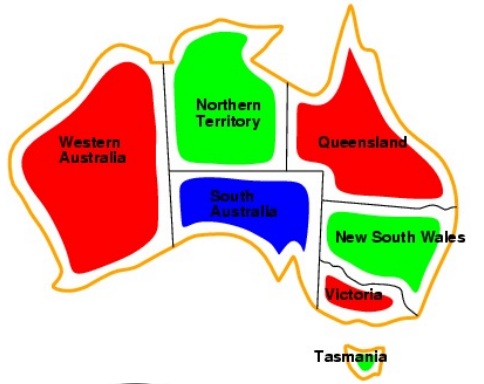
- **Local consistency**

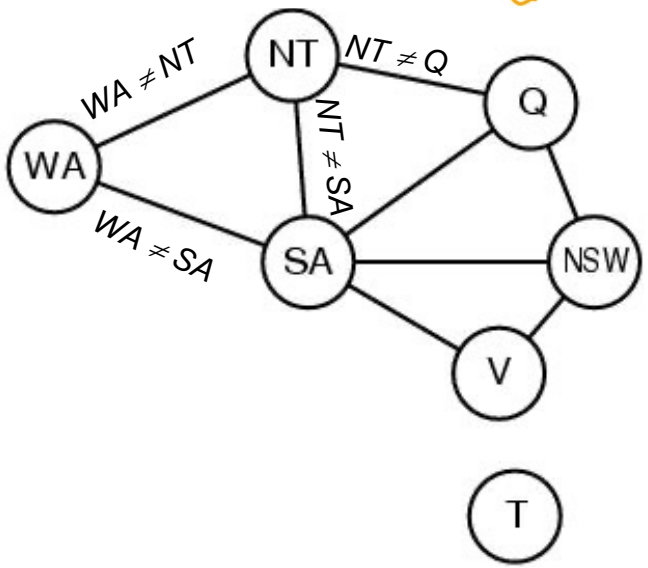
  Node consistency: satisfies unary constraints
  - This is trivial!

  Arc consistency: satisfies binary constraints
  - (X_i is arc-consistent w.r.t. X_j if for every value v in D_i, there is some value w in D_j that satisfies the binary constraint on the arc between X_i *and* X_j)

CSP Representations

- ***Constraint graph:***
  nodes are variables
  edges are constraints

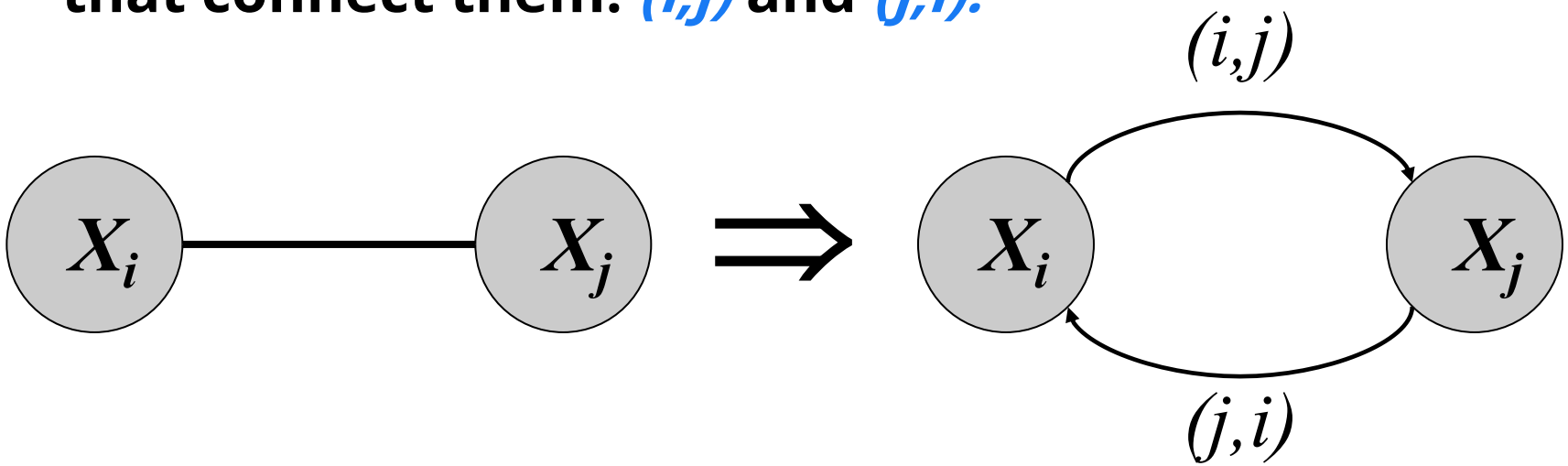




Edges to Arcs: From Constraint Graph to Directed Graph

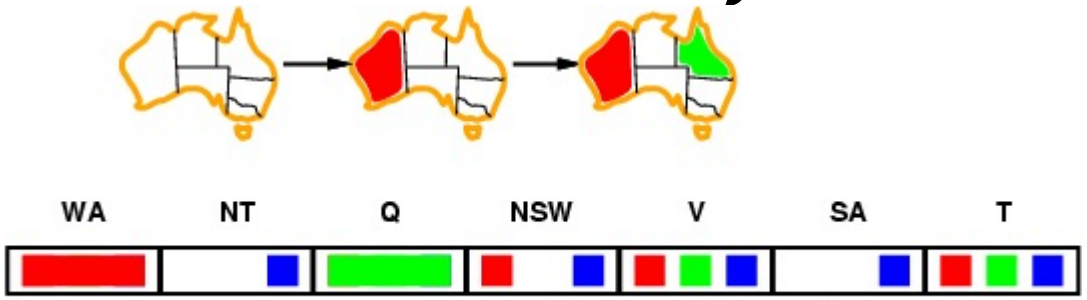
- Given a pair of nodes X_i and X_j connected by a constraint ***edge***, we represent this not by a single undirected edge, but a ***pair of directed arcs***.

  For a connected pair of nodes X_i and X_j, there are ***two*** arcs that connect them: ***(i,j)*** and ***(j,i).***



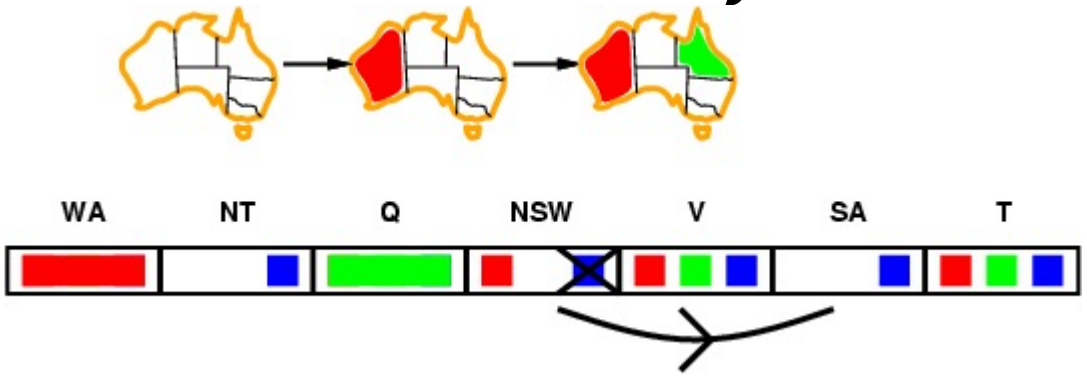
Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff
  for every value x of X there is some allowed y



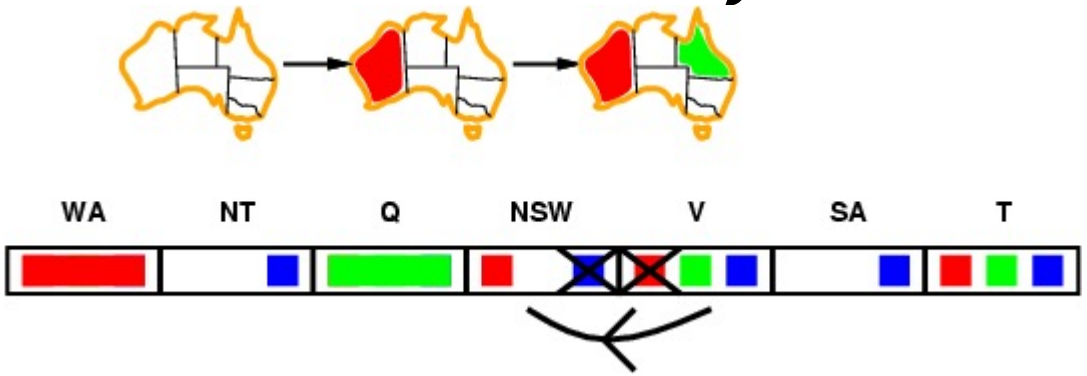
Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff
  for every value x of X there is some allowed y



Arc consistency

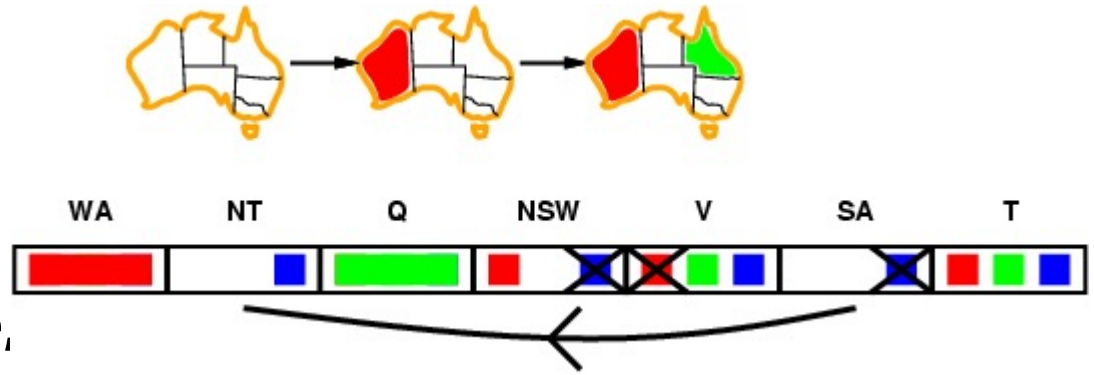
- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff
  for every value x of X there is some allowed y



- If X loses a value, recheck neighbors of X

Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff
  for every value x of X there is some allowed y



- If X loses a value,
- Detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment

Arc Consistency

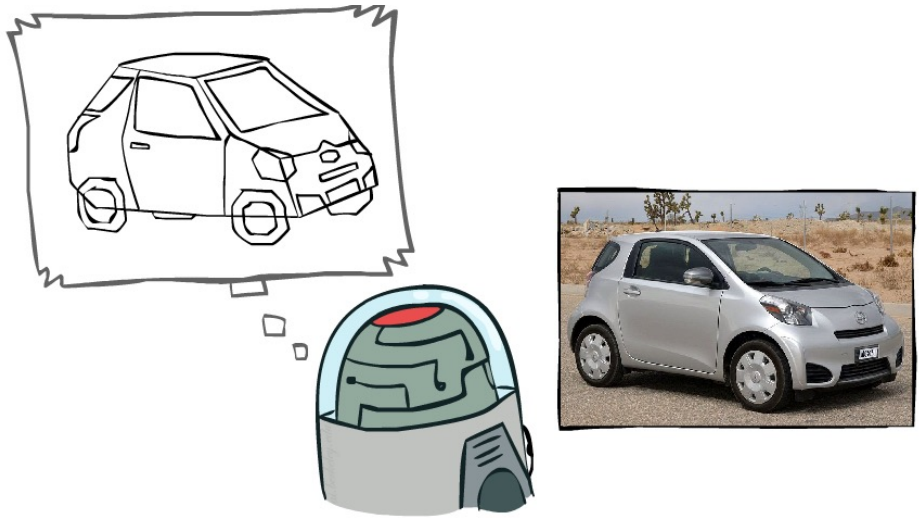
An arc *(i,j)* *is* arc consistent if and only if every value *v* on X_i is consistent with some label on Y_j.

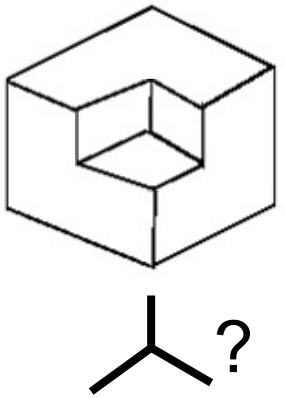
To make an arc *(i,j)* arc consistent,
for each value *v* on X_i,
if there is no label on Y_j consistent with *v*
then remove *v* from X_i

- **Given *d* values, checking arc (i,j) takes $O(d^2)$ time worst case**

Example: The Waltz Algorithm

- The Waltz algorithm is for interpreting line drawings of solid polyhedra as 3D objects
- An early example of an AI computation posed as a CSP

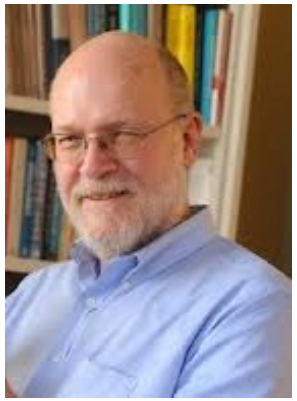




- **Approach:**
  - Each intersection is a variable
  - Adjacent intersections impose constraints on each other
  - Solutions are physically realizable 3D interpretations

Slide credit: Dan Klein and Pieter Abbeel
http://ai.berkeley.edu

Replacing Search:
Constraint Propagation Invented…



Dave Waltz's insight:

- **By *iterating* over the graph, the arc-consistency *constraints* can be *propagated* along arcs of the graph.**
- ***Search*: Use constraints to *add* labels to find *one solution***
- ***Constraint Propagation*: Use constraints to *eliminate labels* to simultaneously find *all solutions***

The Waltz/Mackworth Constraint Propagation Algorithm

1. **Assign *every* node in the constraint graph a set of *all* possible values**
2. **Repeat until there is no change in the set of values associated with any node:**
    3. For each node i:
        4. For each neighboring node j in the picture:
            5. Remove any value from i which is not arc consistent with j.

Inefficiencies: Towards AC-3

1. At each iteration, we only need to examine those X_i *where at least one neighbor of X_i has lost a value* in the previous iteration.
2. If X_i loses a value only because of arc inconsistencies with Y_j, we *don't need to check Y_j* on the next iteration.
3. Removing a value on X_i can only make Y_j arc-inconsistent with respect to X_i itself. Thus, we only need to check that *(j,i)* is still arc-consistent.

These insights lead a much better algorithm…

AC-3

function AC-3(*csp*) return the CSP, possibly with reduced domains

inputs: *csp*, a binary csp with variables $\{X_1, X_2, \ldots, X_n\}$

local variables: *queue*, a queue of arcs initially the arcs in *csp*

while queue is not empty do

$(X_i, X_j) \leftarrow$ *queue*.pop()

if REMOVE-INCONSISTENT-VALUES(X_i, X_j) then

for each X_k in NEIGHBORS[X_i] – {X_j} do
add (X_k, X_i) to queue

function REMOVE-INCONSISTENT-VALUES(X_i, X_j) return *true* iff we remove a value

removed $\leftarrow$ *false*

for each *x* in DOMAIN[X_i] do

if no value *y* in DOMAIN[X_j] allows (x,y) to satisfy the constraints between X_i and X_j

then delete x from DOMAIN[X_i]; *removed* $\leftarrow$ *true*

return *removed*

Keep track of what arcs we need to process

Add back arcs to neighbors whenever a node had values removed

AC-3: Worst Case Complexity Analysis

- **All nodes can be connected to *every* other node,**
  so each of ***n*** nodes must be compared against ***n-1*** other nodes,
  so total # of arcs is ***2*n*(n-1), i.e. $O(n^2)$***
- **If there are *d* values, checking arc (i,j) takes *$O(d^2)$* time**
- **Each arc (i,j) can only be inserted into the queue *d* times**
- **Worst case complexity: *$O(n^2d^3)$***

(For *planar* constraint graphs, the number of arcs can only be *linear in N and* the time complexity is only *$O(nd^3)$*)