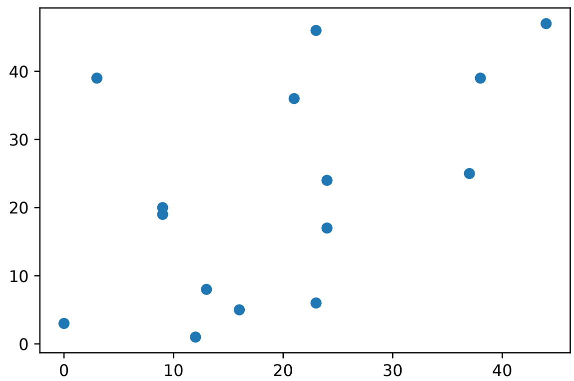
Tutorial: Voronoï Diagrams and Delaunay Triangulation

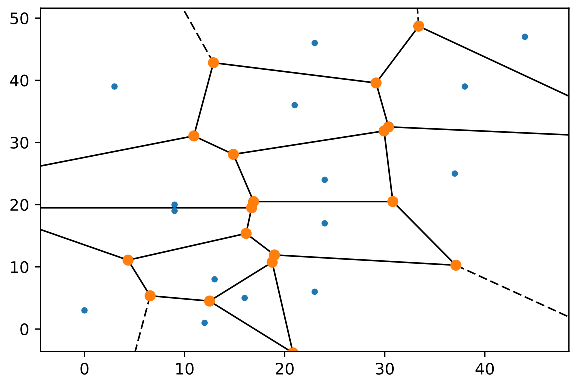
This tutorial aims to spatially characterize a spatial point pattern by using some tools of computational geometry: the Voronoi diagram, the Delaunay triangulation and the Minimum Spanning Tree (MST), illustrated in Fig.1.
For biomedical issues, this point pattern analysis can help the biologists to classify different populations of cells.

Figure 1: Random point pattern and some geometrical structures used to characterize it.

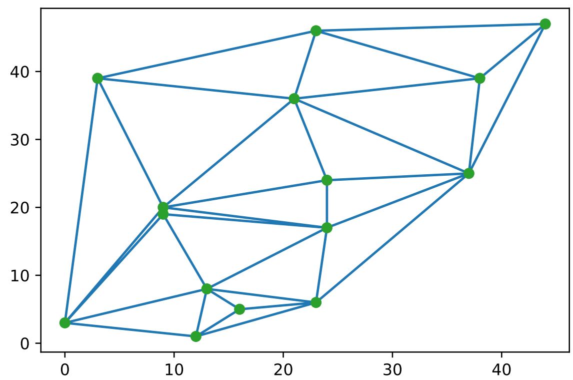
(a) Initial points



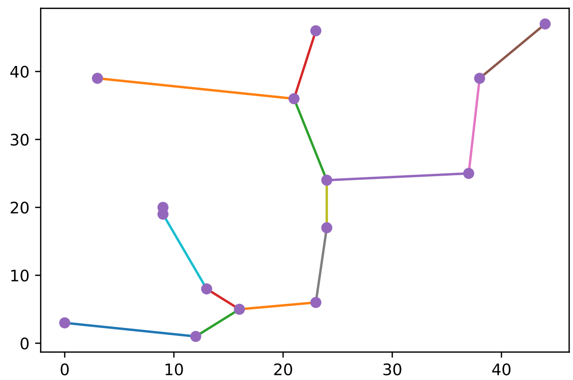
(b) Voronoi diagram



(c) Delaunay triangulation



(d) Minimum spanning tree of the Delaunay triangulation



1 Voronoi and Delaunay

A voronoi diagram, in 2D, is defined as a partition of the plane into cells R_k according to a distance function d and a set of seeds (germs) P_k.

$$R_k = \{x \in X \mid d(x, P_k) \leq d(x, P_j) \text{ for all } j \neq k\}$$

The Delaunay graph is the dual graph that links the germs of the neighborhing Voronoi cells.

1.1 Random tesselation

Follow these instructions to generate a random tesselation:

1. Generate a simple random point process, using a uniform distribution. The result is an array of size N by 2.

2. Compute the Delaunay triangulation.

Use the MATLAB® functions delaunayTriangulation and voronoiDiagram.

Use the python functions Delaunay and Voronoi from scipy.spatial.

Notice that these structures will be used to loop over vertices, edges or regions. They contain attributes that make this process easy.

1.2 Characterization of the Voronoi diagram

This basic approach characterizes the set of the cells. With the help of the Voronoi diagram, it is possible to make the two following measurements, Area Disorder (AD) and Round Factor Homogeneity (RFH), defined by:

$$AD = 1 - \frac{1}{1 + \frac{\sigma(A)}{\mu(A)}} \quad (1)$$

$$RFH = 1 - \frac{\sigma(RF)}{\mu(RF)} \quad (2)$$

where A and RF are calculated on the regions R_k of the Voronoi diagram. $\mu(A)$ and $\sigma(A)$ are the mean and standard deviation of the areas of the Voronoi cells.

The circularity (RF) of a polygon can be defined as the ratio between its area and the area of the disk of an equivalent perimeter. $\mu(RF)$ and $\sigma(RF)$ are thus the mean and standard deviation of RF.

- Code these measurements with the following prototypes:

```matlab
function ad = AD(V, R)
% computes AD (area disorder) parameters
% V: Vertices of the Voronoi diagram
% R: Regions of the Voronoi diagram
```

```python
def AD(vor):
# takes a voronoi diagram to compute area disorder
```

In order to evaluate the area of each Voronoi cell, transform each cell to a polygon.

- RFH can be computed with (almost) the same algorithm.
- Represent the couple (ad, rfh) in a graph, which gives a characterization of the Voronoi diagram.

See polyarea for evaluating the area of a polygon.

See shapely.geometry.Polygon for evaluating the area of a polygon.

1.3 Characterization of the Delaunay graph

If L denotes the set of the edge lengths of the Delaunay triangulation, the mean and the standard deviation of L can also give informations on the graph.

- Compute and display in a graph the point of coordinates $(\mu(L), \sigma(L))$, with μ representing the mean and σ the standard deviation.

It is advised to use import the following module:

```python
import networkx as nx
```

Then, create a function that transforms the simplices of the Delaunay triangulation into a networkx graph, with the following prototype:

```python
def triToNx(tri):
    """
    Convert a triangulation into a NX graph.
    tri : Delaunay triangulation from scipy.spatial

    Returns
    G : networkx graph
    """
```

This function will loop over all simplices of the triangulation, and add an edge in the graph with the computed distance between vertices.

2 Minimum spanning tree

Definition from Wikipedia: a minimum spanning tree (MST) or minimum weight spanning tree is a subset of the edges of a connected, edge-weighted undirected graph that connects all the vertices together, without any cycles and with the minimum possible total edge weight. That is, it is a spanning tree whose sum of edge weights is as small as possible.

One of the methods to compute the MST is the Kruskal algorithm.

It is implemented in Matlab via the function minspantree (introduced in MATLAB® 2015b) or graphminspantree.

The networkx module has a function in order to compute the minimum spanning tree. You will then be able to get all edges weights of this MST.

- Compute the MST.
- Compute $(\mu(L^*), \sigma(L^*))$ where L^* denotes the set of the edge lengths of the MST.

3 Characterization of various point patterns

1. Generate n conditional Poisson point processes of 100 points each. For each realization, calculate the parameters (AD, RFH), $(\mu(L), \sigma(L))$ and $(\mu(L^*), \sigma(L^*))$. Display these n points in a 2D diagram in order to analyze the robustness of the quantification.

2. Generate 3 different point processes with regular, uniform and Gaussian dispersion. Display the different diagrams. Which one is the most discriminant?