Tutorial: Geometry of Gaussian Random Fields

Note

This tutorial aims at simulating Gaussian random fields and analyzing their geometry.

1 Introduction

Let ϕ be a real-valued stationary Gaussian Random Field (GRF) on $\mathbb{R}^2$. $\forall p \in \mathbb{R}^2$, $\phi(p)$ defines a random variable. The family $(\phi(p), p \in \mathbb{R}^2)$ consists of identically distributed random variables on a probability space $\Omega, \mathcal{F}, P$ that satisfies for any subset of $\{p_1, ..., p_n \in \mathbb{R}^2\}$ and $\alpha_k \in \mathbb{R}$ that the random variable $\sum_{k=1}^{n} \alpha_k \phi(p_k)$ is normally distributed. The reader could find more informations in [4, 1, 2].

Thus, ϕ is completely characterized by its mean and its covariance:

$$m = \mathbb{E}(\phi(p)) \qquad = \mathbb{E}(\phi(0)) \qquad , p \in \mathbb{R}^2 \tag{1}$$

$$C(p) = \mathbb{E}(\phi(0)\phi(p)) - m^2 \qquad = \mathbb{E}(\phi(q)\phi(p+q)) - m^2 \qquad , p, q \in \mathbb{R}^2 \tag{2}$$

In this tutorial, we will assume $m = 0$. The goal is to construct ϕ for a given covariance C.

2 Simulation

For more details on algorithms simulating GRFs in $\mathbb{R}^d$, see [3].

2.1 Gaussian white noise random field

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- Use the matlab function randn to generate a white noise on $\mathbb{R}^2$. Choose an $n \times n$ size. Display it on the screen (see for example Fig. 1).
- Modify the variance and observe the results (imagesc is sufficient for diplaying the images.).

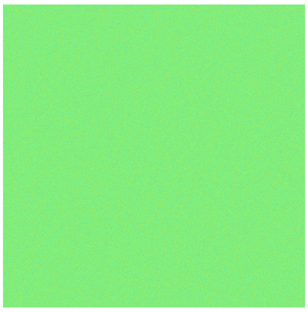


Figure 1: Gaussian White Noise, with $m = 128$ and $\sigma = 30$ for display purposes.

2.2 Gaussian Random Field

Let W be an independent centered white noise random fields. $\mathcal{F}$ is the Fourier Transform. The gaussian random field ϕ is defined by:

$$p \in \mathbb{R}^2, \phi(p) = \mathcal{F}^{-1}(\hat{\phi})(p)$$

and

$$\hat{\phi}(k) = \sqrt{\mathcal{F}(C)(k)\mathcal{F}(W)}$$

- Generate a 2D Gaussian covariance function , see Fig. 2a (we recall the definition, $u^2 = x^2 + y^2$):

  $$C(u) = e^{-\frac{u^2}{2\sigma^2}}$$

  The use of the matlab meshlab function is indicated.

- Generate the Gaussian random field (see Fig. 2b). Test with different values of σ.

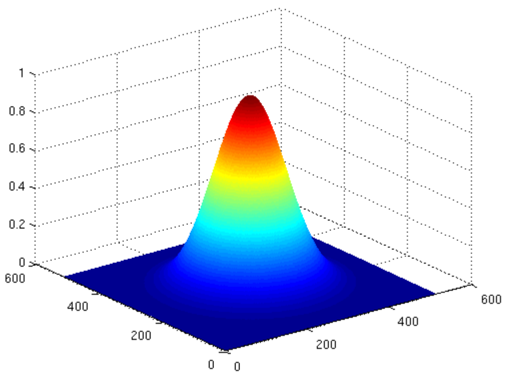
3 Geometry

3.1 Excursion set

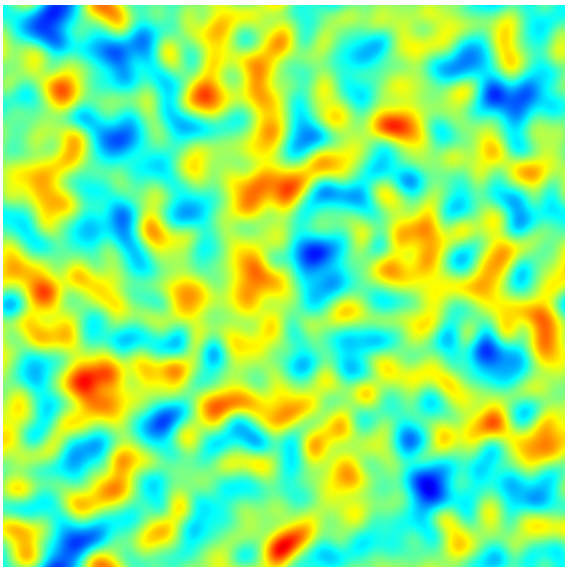
Let $Y(x)$, $x \in \mathbb{R}^2$, be a stationary real-valued random field. An excursion set, denoted by $E_h(Y, S)$, of Y inside a compact subset $D \subset \mathbb{R}^2$ above a level h, is defined as:

$$E_h(Y, D) = \{p \in D : Y(p) \geq h\}$$

In $D \subset \mathbb{R}^2$, an excursion set is a binary image.



(a) Generation of a 2D Gaussian function.



(b) Generation of a Gaussian random field.

Figure 2: Covariance and Gaussian random field examples.

- Represent, for each level h, the area A, the perimeter P and the Euler number χ of $E_h(Y, D)$ (see Fig. 3).
- Matlab functions bwperim(levelset), bwarea and bweuler(levelset) can be useful. Use default values for the neighborhood.

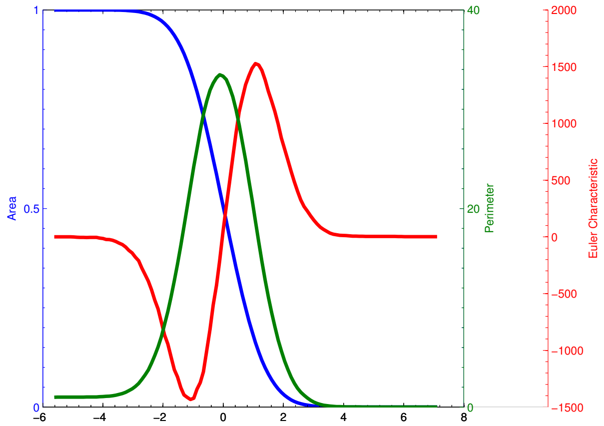


Figure 3: Example of computation of Area, Perimeter and Euler number of level sets E_h.

3.2 Analytical values

We define three values, area A, perimeter C (Contour length) and Euler number χ, as functions of the level h, with:

$$A(h) = \mathcal{L}_2(D)\rho_0(h) \tag{3}$$

$$C(h) = 2\left(\mathcal{L}_1(D)\rho_0(h) + \frac{\pi}{2}\mathcal{L}_2(D)\rho_1(h)\right) \tag{4}$$

$$\chi(h) = \mathcal{L}_0(D)\rho_0(h) + \mathcal{L}_1(D)\rho_1(h) + \mathcal{L}_2(D)\rho_2(h) \tag{5}$$

with $\mathcal{L}_0(D) = \chi(D) = 1$, $\mathcal{L}_1(D)$ is half the boundary length of the rectangle D and $\mathcal{L}_2(D)$ is its area.

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- First, compute the following values. The matlab function erf can be useful.

$$\rho_0(h) = \int_h^\infty \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \quad (6)$$

$$\rho_1(h) = \frac{\sqrt{\lambda}}{2\pi} e^{-h^2/2} \quad (7)$$

$$\rho_2(h) = \frac{\lambda}{(2\pi)^{\frac{2}{3}}} e^{-h^2/2} h \quad (8)$$

$$\lambda = \frac{1}{\sigma^2} \quad (9)$$

- Then, compute the analytical values of A, C and χ.
- Compare the analytical values to the empirical ones.

References

[1] Robert J. Adler. *The Geometry of Random Fields*. Wiley, Chichester, UK, June 1981.

[2] Ola S. Ahmad. *Stochastic representation and analysis of rough surface topography by random fields and integral geometry – Application to the UHMWPE cup involved in total hip arthroplasty*. PhD thesis, École Nationale Supérieure des Mines de Saint-Etienne, 2013.

[3] Annika Lang and Jürgen Potthoff. Fast simulation of gaussian random fields. *Monte Carlo Methods and Applications*, 17(3):195–214, 2011.

[4] K.J. Worsley. The geometry of random fields. *Chance*, 9(1):27–40, 1997.