

CHAPTER 11 INFINITE SEQUENCES AND SERIES

11.1 SEQUENCES

1. $a_1 = \frac{1-1}{1^2} = 0, a_2 = \frac{1-2}{2^2} = -\frac{1}{4}, a_3 = \frac{1-3}{3^2} = -\frac{2}{9}, a_4 = \frac{1-4}{4^2} = -\frac{3}{16}$
2. $a_1 = \frac{1}{1!} = 1, a_2 = \frac{1}{2!} = \frac{1}{2}, a_3 = \frac{1}{3!} = \frac{1}{6}, a_4 = \frac{1}{4!} = \frac{1}{24}$
3. $a_1 = \frac{(-1)^2}{2-1} = 1, a_2 = \frac{(-1)^3}{4-1} = -\frac{1}{3}, a_3 = \frac{(-1)^4}{6-1} = \frac{1}{5}, a_4 = \frac{(-1)^5}{8-1} = -\frac{1}{7}$
4. $a_1 = 2 + (-1)^1 = 1, a_2 = 2 + (-1)^2 = 3, a_3 = 2 + (-1)^3 = 1, a_4 = 2 + (-1)^4 = 3$
5. $a_1 = \frac{2}{2^2} = \frac{1}{2}, a_2 = \frac{2^2}{2^3} = \frac{1}{2}, a_3 = \frac{2^3}{2^4} = \frac{1}{2}, a_4 = \frac{2^4}{2^5} = \frac{1}{2}$
6. $a_1 = \frac{2-1}{2} = \frac{1}{2}, a_2 = \frac{2^2-1}{2^2} = \frac{3}{4}, a_3 = \frac{2^3-1}{2^3} = \frac{7}{8}, a_4 = \frac{2^4-1}{2^4} = \frac{15}{16}$
7. $a_1 = 1, a_2 = 1 + \frac{1}{2} = \frac{3}{2}, a_3 = \frac{3}{2} + \frac{1}{2^2} = \frac{7}{4}, a_4 = \frac{7}{4} + \frac{1}{2^3} = \frac{15}{8}, a_5 = \frac{15}{8} + \frac{1}{2^4} = \frac{31}{16}, a_6 = \frac{63}{32},$
 $a_7 = \frac{127}{64}, a_8 = \frac{255}{128}, a_9 = \frac{511}{256}, a_{10} = \frac{1023}{512}$
8. $a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{(\frac{1}{2})}{3} = \frac{1}{6}, a_4 = \frac{(\frac{1}{6})}{4} = \frac{1}{24}, a_5 = \frac{(\frac{1}{24})}{5} = \frac{1}{120}, a_6 = \frac{1}{720}, a_7 = \frac{1}{5040}, a_8 = \frac{1}{40,320},$
 $a_9 = \frac{1}{362,880}, a_{10} = \frac{1}{3,628,800}$
9. $a_1 = 2, a_2 = \frac{(-1)^2(2)}{2} = 1, a_3 = \frac{(-1)^3(1)}{2} = -\frac{1}{2}, a_4 = \frac{(-1)^4(-\frac{1}{2})}{2} = -\frac{1}{4}, a_5 = \frac{(-1)^5(-\frac{1}{4})}{2} = \frac{1}{8},$
 $a_6 = \frac{1}{16}, a_7 = -\frac{1}{32}, a_8 = -\frac{1}{64}, a_9 = \frac{1}{128}, a_{10} = \frac{1}{256}$
10. $a_1 = -2, a_2 = \frac{1 \cdot (-2)}{2} = -1, a_3 = \frac{2 \cdot (-1)}{3} = -\frac{2}{3}, a_4 = \frac{3 \cdot (-\frac{2}{3})}{4} = -\frac{1}{2}, a_5 = \frac{4 \cdot (-\frac{1}{2})}{5} = -\frac{2}{5}, a_6 = -\frac{1}{3},$
 $a_7 = -\frac{2}{7}, a_8 = -\frac{1}{4}, a_9 = -\frac{2}{9}, a_{10} = -\frac{1}{5}$
11. $a_1 = 1, a_2 = 1, a_3 = 1 + 1 = 2, a_4 = 2 + 1 = 3, a_5 = 3 + 2 = 5, a_6 = 8, a_7 = 13, a_8 = 21, a_9 = 34, a_{10} = 55$
12. $a_1 = 2, a_2 = -1, a_3 = -\frac{1}{2}, a_4 = \frac{(-\frac{1}{2})}{(-1)} = \frac{1}{2}, a_5 = \frac{(\frac{1}{2})}{(-\frac{1}{2})} = -1, a_6 = -2, a_7 = 2, a_8 = -1, a_9 = -\frac{1}{2}, a_{10} = \frac{1}{2}$
13. $a_n = (-1)^{n+1}, n = 1, 2, \dots$
14. $a_n = (-1)^n, n = 1, 2, \dots$
15. $a_n = (-1)^{n+1}n^2, n = 1, 2, \dots$
16. $a_n = \frac{(-1)^{n+1}}{n^2}, n = 1, 2, \dots$
17. $a_n = n^2 - 1, n = 1, 2, \dots$
18. $a_n = n - 4, n = 1, 2, \dots$
19. $a_n = 4n - 3, n = 1, 2, \dots$
20. $a_n = 4n - 2, n = 1, 2, \dots$
21. $a_n = \frac{1 + (-1)^{n+1}}{2}, n = 1, 2, \dots$
22. $a_n = \frac{n - \frac{1}{2} + (-1)^n(\frac{1}{2})}{2} = \lfloor \frac{n}{2} \rfloor, n = 1, 2, \dots$
23. $\lim_{n \rightarrow \infty} 2 + (0.1)^n = 2 \Rightarrow \text{converges}$ (Theorem 5, #4)

$$24. \lim_{n \rightarrow \infty} \frac{n+(-1)^n}{n} = \lim_{n \rightarrow \infty} 1 + \frac{(-1)^n}{n} = 1 \Rightarrow \text{converges}$$

$$25. \lim_{n \rightarrow \infty} \frac{1-2n}{1+2n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n}\right)-2}{\left(\frac{1}{n}\right)+2} = \lim_{n \rightarrow \infty} \frac{-2}{2} = -1 \Rightarrow \text{converges}$$

$$26. \lim_{n \rightarrow \infty} \frac{2n+1}{1-3\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{2\sqrt{n} + \left(\frac{1}{\sqrt{n}}\right)}{\left(\frac{1}{\sqrt{n}} - 3\right)} = -\infty \Rightarrow \text{diverges}$$

$$27. \lim_{n \rightarrow \infty} \frac{1-5n^4}{n^4+8n^3} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n^4}\right)-5}{1+\left(\frac{8}{n}\right)} = -5 \Rightarrow \text{converges}$$

$$28. \lim_{n \rightarrow \infty} \frac{n+3}{n^2+5n+6} = \lim_{n \rightarrow \infty} \frac{n+3}{(n+3)(n+2)} = \lim_{n \rightarrow \infty} \frac{1}{n+2} = 0 \Rightarrow \text{converges}$$

$$29. \lim_{n \rightarrow \infty} \frac{n^2-2n+1}{n-1} = \lim_{n \rightarrow \infty} \frac{(n-1)(n-1)}{n-1} = \lim_{n \rightarrow \infty} (n-1) = \infty \Rightarrow \text{diverges}$$

$$30. \lim_{n \rightarrow \infty} \frac{1-n^3}{70-4n^2} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n^2}\right)-n}{\left(\frac{70}{n^2}\right)-4} = \infty \Rightarrow \text{diverges}$$

$$31. \lim_{n \rightarrow \infty} (1+(-1)^n) \text{ does not exist} \Rightarrow \text{diverges} \qquad 32. \lim_{n \rightarrow \infty} (-1)^n \left(1 - \frac{1}{n}\right) \text{ does not exist} \Rightarrow \text{diverges}$$

$$33. \lim_{n \rightarrow \infty} \left(\frac{n+1}{2n}\right) \left(1 - \frac{1}{n}\right) = \lim_{n \rightarrow \infty} \left(\frac{1}{2} + \frac{1}{2n}\right) \left(1 - \frac{1}{n}\right) = \frac{1}{2} \Rightarrow \text{converges}$$

$$34. \lim_{n \rightarrow \infty} \left(2 - \frac{1}{2^n}\right) \left(3 + \frac{1}{2^n}\right) = 6 \Rightarrow \text{converges} \qquad 35. \lim_{n \rightarrow \infty} \frac{(-1)^{n+1}}{2n-1} = 0 \Rightarrow \text{converges}$$

$$36. \lim_{n \rightarrow \infty} \left(-\frac{1}{2}\right)^n = \lim_{n \rightarrow \infty} \frac{(-1)^n}{2^n} = 0 \Rightarrow \text{converges}$$

$$37. \lim_{n \rightarrow \infty} \sqrt{\frac{2n}{n+1}} = \sqrt{\lim_{n \rightarrow \infty} \frac{2n}{n+1}} = \sqrt{\lim_{n \rightarrow \infty} \left(\frac{2}{1+\frac{1}{n}}\right)} = \sqrt{2} \Rightarrow \text{converges}$$

$$38. \lim_{n \rightarrow \infty} \frac{1}{(0.9)^n} = \lim_{n \rightarrow \infty} \left(\frac{10}{9}\right)^n = \infty \Rightarrow \text{diverges}$$

$$39. \lim_{n \rightarrow \infty} \sin\left(\frac{\pi}{2} + \frac{1}{n}\right) = \sin\left(\lim_{n \rightarrow \infty} \left(\frac{\pi}{2} + \frac{1}{n}\right)\right) = \sin \frac{\pi}{2} = 1 \Rightarrow \text{converges}$$

$$40. \lim_{n \rightarrow \infty} n\pi \cos(n\pi) = \lim_{n \rightarrow \infty} (n\pi)(-1)^n \text{ does not exist} \Rightarrow \text{diverges}$$

$$41. \lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0 \text{ because } -\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \Rightarrow \text{converges by the Sandwich Theorem for sequences}$$

$$42. \lim_{n \rightarrow \infty} \frac{\sin^2 n}{2^n} = 0 \text{ because } 0 \leq \frac{\sin^2 n}{2^n} \leq \frac{1}{2^n} \Rightarrow \text{converges by the Sandwich Theorem for sequences}$$

$$43. \lim_{n \rightarrow \infty} \frac{n}{2^n} = \lim_{n \rightarrow \infty} \frac{1}{2^n \ln 2} = 0 \Rightarrow \text{converges (using l'Hôpital's rule)}$$

$$44. \lim_{n \rightarrow \infty} \frac{3^n}{n^3} = \lim_{n \rightarrow \infty} \frac{3^n \ln 3}{3n^2} = \lim_{n \rightarrow \infty} \frac{3^n (\ln 3)^2}{6n} = \lim_{n \rightarrow \infty} \frac{3^n (\ln 3)^3}{6} = \infty \Rightarrow \text{diverges (using l'Hôpital's rule)}$$

$$45. \lim_{n \rightarrow \infty} \frac{\ln(n+1)}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{2\sqrt{n}}\right)} = \lim_{n \rightarrow \infty} \frac{2\sqrt{n}}{n+1} = \lim_{n \rightarrow \infty} \frac{\left(\frac{2}{\sqrt{n}}\right)}{1+\left(\frac{1}{n}\right)} = 0 \Rightarrow \text{converges}$$

46. $\lim_{n \rightarrow \infty} \frac{\ln n}{\ln 2n} = \lim_{n \rightarrow \infty} \frac{(\frac{1}{2})}{(\frac{2}{n})} = 1 \Rightarrow \text{converges}$
47. $\lim_{n \rightarrow \infty} 8^{1/n} = 1 \Rightarrow \text{converges}$ (Theorem 5, #3)
48. $\lim_{n \rightarrow \infty} (0.03)^{1/n} = 1 \Rightarrow \text{converges}$ (Theorem 5, #3)
49. $\lim_{n \rightarrow \infty} \left(1 + \frac{7}{n}\right)^n = e^7 \Rightarrow \text{converges}$ (Theorem 5, #5)
50. $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n = \lim_{n \rightarrow \infty} \left[1 + \frac{(-1)}{n}\right]^n = e^{-1} \Rightarrow \text{converges}$ (Theorem 5, #5)
51. $\lim_{n \rightarrow \infty} \sqrt[n]{10n} = \lim_{n \rightarrow \infty} 10^{1/n} \cdot n^{1/n} = 1 \cdot 1 = 1 \Rightarrow \text{converges}$ (Theorem 5, #3 and #2)
52. $\lim_{n \rightarrow \infty} \sqrt[n]{n^2} = \lim_{n \rightarrow \infty} (\sqrt[n]{n})^2 = 1^2 = 1 \Rightarrow \text{converges}$ (Theorem 5, #2)
53. $\lim_{n \rightarrow \infty} \left(\frac{3}{n}\right)^{1/n} = \frac{\lim_{n \rightarrow \infty} 3^{1/n}}{\lim_{n \rightarrow \infty} n^{1/n}} = \frac{1}{1} = 1 \Rightarrow \text{converges}$ (Theorem 5, #3 and #2)
54. $\lim_{n \rightarrow \infty} (n+4)^{1/(n+4)} = \lim_{x \rightarrow \infty} x^{1/x} = 1 \Rightarrow \text{converges}$; (let $x = n+4$, then use Theorem 5, #2)
55. $\lim_{n \rightarrow \infty} \frac{\ln n}{n^{1/n}} = \frac{\lim_{n \rightarrow \infty} \ln n}{\lim_{n \rightarrow \infty} n^{1/n}} = \frac{\infty}{1} = \infty \Rightarrow \text{diverges}$ (Theorem 5, #2)
56. $\lim_{n \rightarrow \infty} [\ln n - \ln(n+1)] = \lim_{n \rightarrow \infty} \ln\left(\frac{n}{n+1}\right) = \ln\left(\lim_{n \rightarrow \infty} \frac{n}{n+1}\right) = \ln 1 = 0 \Rightarrow \text{converges}$
57. $\lim_{n \rightarrow \infty} \sqrt[n]{4^n n} = \lim_{n \rightarrow \infty} 4 \sqrt[n]{n} = 4 \cdot 1 = 4 \Rightarrow \text{converges}$ (Theorem 5, #2)
58. $\lim_{n \rightarrow \infty} \sqrt[n]{3^{2n+1}} = \lim_{n \rightarrow \infty} 3^{2+(1/n)} = \lim_{n \rightarrow \infty} 3^2 \cdot 3^{1/n} = 9 \cdot 1 = 9 \Rightarrow \text{converges}$ (Theorem 5, #3)
59. $\lim_{n \rightarrow \infty} \frac{n!}{n^n} = \lim_{n \rightarrow \infty} \frac{1 \cdot 2 \cdot 3 \cdots (n-1)(n)}{n \cdot n \cdot n \cdots n \cdot n} \leq \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right) = 0$ and $\frac{n!}{n^n} \geq 0 \Rightarrow \lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0 \Rightarrow \text{converges}$
60. $\lim_{n \rightarrow \infty} \frac{(-4)^n}{n!} = 0 \Rightarrow \text{converges}$ (Theorem 5, #6)
61. $\lim_{n \rightarrow \infty} \frac{n!}{10^{6n}} = \lim_{n \rightarrow \infty} \frac{1}{\left(\frac{(10^6)^n}{n!}\right)} = \infty \Rightarrow \text{diverges}$ (Theorem 5, #6)
62. $\lim_{n \rightarrow \infty} \frac{n!}{2^n 3^n} = \lim_{n \rightarrow \infty} \frac{1}{\left(\frac{6^n}{n!}\right)} = \infty \Rightarrow \text{diverges}$ (Theorem 5, #6)
63. $\lim_{n \rightarrow \infty} \left(\frac{1}{n}\right)^{1/(\ln n)} = \lim_{n \rightarrow \infty} \exp\left(\frac{1}{\ln n} \ln\left(\frac{1}{n}\right)\right) = \lim_{n \rightarrow \infty} \exp\left(\frac{\ln 1 - \ln n}{\ln n}\right) = e^{-1} \Rightarrow \text{converges}$
64. $\lim_{n \rightarrow \infty} \ln\left(1 + \frac{1}{n}\right)^n = \ln\left(\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n\right) = \ln e = 1 \Rightarrow \text{converges}$ (Theorem 5, #5)
65. $\lim_{n \rightarrow \infty} \left(\frac{3n+1}{3n-1}\right)^n = \lim_{n \rightarrow \infty} \exp\left(n \ln\left(\frac{3n+1}{3n-1}\right)\right) = \lim_{n \rightarrow \infty} \exp\left(\frac{\ln(3n+1) - \ln(3n-1)}{\frac{1}{n}}\right)$

$$= \lim_{n \rightarrow \infty} \exp \left(\frac{\frac{3}{3n+1} - \frac{3}{3n-1}}{\left(-\frac{1}{n^2}\right)} \right) = \lim_{n \rightarrow \infty} \exp \left(\frac{6n^2}{(3n+1)(3n-1)} \right) = \exp \left(\frac{6}{9} \right) = e^{2/3} \Rightarrow \text{converges}$$

$$\begin{aligned} 66. \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n &= \lim_{n \rightarrow \infty} \exp \left(n \ln \left(\frac{n}{n+1} \right) \right) = \lim_{n \rightarrow \infty} \exp \left(\frac{\ln n - \ln(n+1)}{\left(\frac{1}{n}\right)} \right) = \lim_{n \rightarrow \infty} \exp \left(\frac{\frac{1}{n} - \frac{1}{n+1}}{\left(-\frac{1}{n^2}\right)} \right) \\ &= \lim_{n \rightarrow \infty} \exp \left(-\frac{n^2}{n(n+1)} \right) = e^{-1} \Rightarrow \text{converges} \end{aligned}$$

$$\begin{aligned} 67. \lim_{n \rightarrow \infty} \left(\frac{x^n}{2n+1} \right)^{1/n} &= \lim_{n \rightarrow \infty} x \left(\frac{1}{2n+1} \right)^{1/n} = x \lim_{n \rightarrow \infty} \exp \left(\frac{1}{n} \ln \left(\frac{1}{2n+1} \right) \right) = x \lim_{n \rightarrow \infty} \exp \left(\frac{-\ln(2n+1)}{n} \right) \\ &= x \lim_{n \rightarrow \infty} \exp \left(\frac{-2}{2n+1} \right) = xe^0 = x, x > 0 \Rightarrow \text{converges} \end{aligned}$$

$$\begin{aligned} 68. \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n^2} \right)^n &= \lim_{n \rightarrow \infty} \exp \left(n \ln \left(1 - \frac{1}{n^2} \right) \right) = \lim_{n \rightarrow \infty} \exp \left(\frac{\ln \left(1 - \frac{1}{n^2} \right)}{\left(\frac{1}{n}\right)} \right) = \lim_{n \rightarrow \infty} \exp \left[\frac{\left(\frac{2}{n^3}\right) / \left(1 - \frac{1}{n^2}\right)}{\left(-\frac{1}{n^2}\right)} \right] \\ &= \lim_{n \rightarrow \infty} \exp \left(\frac{-2n}{n^2-1} \right) = e^0 = 1 \Rightarrow \text{converges} \end{aligned}$$

$$69. \lim_{n \rightarrow \infty} \frac{3^n \cdot 6^n}{2^{-n} \cdot n!} = \lim_{n \rightarrow \infty} \frac{36^n}{n!} = 0 \Rightarrow \text{converges} \quad (\text{Theorem 5, \#6})$$

$$\begin{aligned} 70. \lim_{n \rightarrow \infty} \frac{\left(\frac{10}{11}\right)^n}{\left(\frac{9}{10}\right)^n + \left(\frac{11}{12}\right)^n} &= \lim_{n \rightarrow \infty} \frac{\left(\frac{12}{11}\right)^n \left(\frac{10}{11}\right)^n}{\left(\frac{12}{11}\right)^n \left(\frac{9}{10}\right)^n + \left(\frac{12}{11}\right)^n \left(\frac{11}{12}\right)^n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{120}{121}\right)^n}{\left(\frac{108}{110}\right)^n + 1} = 0 \Rightarrow \text{converges} \\ &(\text{Theorem 5, \#4}) \end{aligned}$$

$$71. \lim_{n \rightarrow \infty} \tanh n = \lim_{n \rightarrow \infty} \frac{e^n - e^{-n}}{e^n + e^{-n}} = \lim_{n \rightarrow \infty} \frac{e^{2n} - 1}{e^{2n} + 1} = \lim_{n \rightarrow \infty} \frac{2e^{2n}}{2e^{2n}} = \lim_{n \rightarrow \infty} 1 = 1 \Rightarrow \text{converges}$$

$$72. \lim_{n \rightarrow \infty} \sinh(\ln n) = \lim_{n \rightarrow \infty} \frac{e^{\ln n} - e^{-\ln n}}{2} = \lim_{n \rightarrow \infty} \frac{n - \left(\frac{1}{n}\right)}{2} = \infty \Rightarrow \text{diverges}$$

$$73. \lim_{n \rightarrow \infty} \frac{n^2 \sin \left(\frac{1}{n}\right)}{2n-1} = \lim_{n \rightarrow \infty} \frac{\sin \left(\frac{1}{n}\right)}{\left(\frac{2}{n} - \frac{1}{n^2}\right)} = \lim_{n \rightarrow \infty} \frac{-\left(\cos \left(\frac{1}{n}\right)\right) \left(\frac{1}{n^2}\right)}{\left(-\frac{2}{n^2} + \frac{2}{n^3}\right)} = \lim_{n \rightarrow \infty} \frac{-\cos \left(\frac{1}{n}\right)}{-2 + \left(\frac{2}{n}\right)} = \frac{1}{2} \Rightarrow \text{converges}$$

$$74. \lim_{n \rightarrow \infty} n \left(1 - \cos \frac{1}{n} \right) = \lim_{n \rightarrow \infty} \frac{(1 - \cos \frac{1}{n})}{\left(\frac{1}{n}\right)} = \lim_{n \rightarrow \infty} \frac{\left[\sin \left(\frac{1}{n}\right)\right] \left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n \rightarrow \infty} \sin \left(\frac{1}{n}\right) = 0 \Rightarrow \text{converges}$$

$$75. \lim_{n \rightarrow \infty} \tan^{-1} n = \frac{\pi}{2} \Rightarrow \text{converges}$$

$$76. \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \tan^{-1} n = 0 \cdot \frac{\pi}{2} = 0 \Rightarrow \text{converges}$$

$$77. \lim_{n \rightarrow \infty} \left(\frac{1}{3} \right)^n + \frac{1}{\sqrt{2^n}} = \lim_{n \rightarrow \infty} \left(\left(\frac{1}{3} \right)^n + \left(\frac{1}{\sqrt{2}} \right)^n \right) = 0 \Rightarrow \text{converges} \quad (\text{Theorem 5, \#4})$$

$$78. \lim_{n \rightarrow \infty} \sqrt[n]{n^2 + n} = \lim_{n \rightarrow \infty} \exp \left[\frac{\ln(n^2 + n)}{n} \right] = \lim_{n \rightarrow \infty} \exp \left(\frac{2n+1}{n^2+n} \right) = e^0 = 1 \Rightarrow \text{converges}$$

$$79. \lim_{n \rightarrow \infty} \frac{(\ln n)^{200}}{n} = \lim_{n \rightarrow \infty} \frac{200(\ln n)^{199}}{n} = \lim_{n \rightarrow \infty} \frac{200 \cdot 199 (\ln n)^{198}}{n} = \dots = \lim_{n \rightarrow \infty} \frac{200!}{n} = 0 \Rightarrow \text{converges}$$

$$80. \lim_{n \rightarrow \infty} \frac{(\ln n)^5}{\sqrt{n}} = \lim_{n \rightarrow \infty} \left[\frac{\left(\frac{5(\ln n)^4}{n}\right)}{\left(\frac{1}{2\sqrt{n}}\right)} \right] = \lim_{n \rightarrow \infty} \frac{10(\ln n)^4}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{80(\ln n)^3}{\sqrt{n}} = \dots = \lim_{n \rightarrow \infty} \frac{3840}{\sqrt{n}} = 0 \Rightarrow \text{converges}$$

$$81. \lim_{n \rightarrow \infty} (n - \sqrt{n^2 - n}) = \lim_{n \rightarrow \infty} (n - \sqrt{n^2 - n}) \left(\frac{n + \sqrt{n^2 - n}}{n + \sqrt{n^2 - n}} \right) = \lim_{n \rightarrow \infty} \frac{n}{n + \sqrt{n^2 - n}} = \lim_{n \rightarrow \infty} \frac{1}{1 + \sqrt{1 - \frac{1}{n}}} \\ = \frac{1}{2} \Rightarrow \text{converges}$$

$$82. \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n^2 - 1} - \sqrt{n^2 + n}} = \lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{n^2 - 1} - \sqrt{n^2 + n}} \right) \left(\frac{\sqrt{n^2 - 1} + \sqrt{n^2 + n}}{\sqrt{n^2 - 1} + \sqrt{n^2 + n}} \right) = \lim_{n \rightarrow \infty} \frac{\sqrt{n^2 - 1} + \sqrt{n^2 + n}}{-1 - n} \\ = \lim_{n \rightarrow \infty} \frac{\sqrt{1 - \frac{1}{n^2}} + \sqrt{1 + \frac{1}{n}}}{(-\frac{1}{n} - 1)} = -2 \Rightarrow \text{converges}$$

$$83. \lim_{n \rightarrow \infty} \frac{1}{n} \int_1^n \frac{1}{x} dx = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \Rightarrow \text{converges} \quad (\text{Theorem 5, \#1})$$

$$84. \lim_{n \rightarrow \infty} \int_1^n \frac{1}{x^p} dx = \lim_{n \rightarrow \infty} \left[\frac{1}{1-p} \frac{1}{x^{p-1}} \right]_1^n = \lim_{n \rightarrow \infty} \frac{1}{1-p} \left(\frac{1}{n^{p-1}} - 1 \right) = \frac{1}{p-1} \text{ if } p > 1 \Rightarrow \text{converges}$$

$$85. 1, 1, 2, 4, 8, 16, 32, \dots = 1, 2^0, 2^1, 2^2, 2^3, 2^4, 2^5, \dots \Rightarrow x_1 = 1 \text{ and } x_n = 2^{n-2} \text{ for } n \geq 2$$

$$86. (a) 1^2 - 2(1)^2 = -1, 3^2 - 2(2)^2 = 1; \text{ let } f(a, b) = (a + 2b)^2 - 2(a + b)^2 = a^2 + 4ab + 4b^2 - 2a^2 - 4ab - 2b^2 \\ = 2b^2 - a^2; a^2 - 2b^2 = -1 \Rightarrow f(a, b) = 2b^2 - a^2 = 1; a^2 - 2b^2 = 1 \Rightarrow f(a, b) = 2b^2 - a^2 = -1$$

$$(b) r_n^2 - 2 = \left(\frac{a+2b}{a+b} \right)^2 - 2 = \frac{a^2 + 4ab + 4b^2 - 2a^2 - 4ab - 2b^2}{(a+b)^2} = \frac{-(a^2 - 2b^2)}{(a+b)^2} = \frac{\pm 1}{y_n^2} \Rightarrow r_n = \sqrt{2 \pm \left(\frac{1}{y_n} \right)^2}$$

In the first and second fractions, $y_n \geq n$. Let $\frac{a}{b}$ represent the $(n-1)$ th fraction where $\frac{a}{b} \geq 1$ and $b \geq n-1$ for n a positive integer ≥ 3 . Now the n th fraction is $\frac{a+2b}{a+b}$ and $a+b \geq 2b \geq 2n-2 \geq n \Rightarrow y_n \geq n$. Thus,

$$\lim_{n \rightarrow \infty} r_n = \sqrt{2}.$$

$$87. (a) f(x) = x^2 - 2; \text{ the sequence converges to } 1.414213562 \approx \sqrt{2}$$

$$(b) f(x) = \tan(x) - 1; \text{ the sequence converges to } 0.7853981635 \approx \frac{\pi}{4}$$

$$(c) f(x) = e^x; \text{ the sequence } 1, 0, -1, -2, -3, -4, -5, \dots \text{ diverges}$$

$$88. (a) \lim_{n \rightarrow \infty} n f\left(\frac{1}{n}\right) = \lim_{\Delta x \rightarrow 0^+} \frac{f(\Delta x)}{\Delta x} = \lim_{\Delta x \rightarrow 0^+} \frac{f(0+\Delta x) - f(0)}{\Delta x} = f'(0), \text{ where } \Delta x = \frac{1}{n}$$

$$(b) \lim_{n \rightarrow \infty} n \tan^{-1}\left(\frac{1}{n}\right) = f'(0) = \frac{1}{1+0^2} = 1, f(x) = \tan^{-1} x$$

$$(c) \lim_{n \rightarrow \infty} n(e^{1/n} - 1) = f'(0) = e^0 = 1, f(x) = e^x - 1$$

$$(d) \lim_{n \rightarrow \infty} n \ln\left(1 + \frac{2}{n}\right) = f'(0) = \frac{2}{1+2(0)} = 2, f(x) = \ln(1 + 2x)$$

$$89. (a) \text{ If } a = 2n + 1, \text{ then } b = \lfloor \frac{a^2}{2} \rfloor = \lfloor \frac{4n^2 + 4n + 1}{2} \rfloor = \lfloor 2n^2 + 2n + \frac{1}{2} \rfloor = 2n^2 + 2n, c = \lceil \frac{a^2}{2} \rceil = \lceil 2n^2 + 2n + \frac{1}{2} \rceil \\ = 2n^2 + 2n + 1 \text{ and } a^2 + b^2 = (2n + 1)^2 + (2n^2 + 2n)^2 = 4n^2 + 4n + 1 + 4n^4 + 8n^3 + 4n^2 \\ = 4n^4 + 8n^3 + 8n^2 + 4n + 1 = (2n^2 + 2n + 1)^2 = c^2.$$

$$(b) \lim_{a \rightarrow \infty} \frac{\lfloor \frac{a^2}{2} \rfloor}{\lceil \frac{a^2}{2} \rceil} = \lim_{a \rightarrow \infty} \frac{2n^2 + 2n}{2n^2 + 2n + 1} = 1 \text{ or } \lim_{a \rightarrow \infty} \frac{\lfloor \frac{a^2}{2} \rfloor}{\lceil \frac{a^2}{2} \rceil} = \lim_{a \rightarrow \infty} \sin \theta = \lim_{\theta \rightarrow \pi/2} \sin \theta = 1$$

$$90. (a) \lim_{n \rightarrow \infty} (2n\pi)^{1/(2n)} = \lim_{n \rightarrow \infty} \exp\left(\frac{\ln 2n\pi}{2n}\right) = \lim_{n \rightarrow \infty} \exp\left(\frac{\left(\frac{2\pi}{2n\pi}\right)}{2}\right) = \lim_{n \rightarrow \infty} \exp\left(\frac{1}{2n}\right) = e^0 = 1;$$

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2n\pi}, \text{ Stirling's approximation} \Rightarrow \sqrt[n]{n!} \approx \left(\frac{n}{e}\right) (2n\pi)^{1/(2n)} \approx \frac{n}{e} \text{ for large values of } n$$

(b)

n	$\sqrt[n]{n!}$	$\frac{n}{e}$
40	15.76852702	14.71517765
50	19.48325423	18.39397206
60	23.19189561	22.07276647

91. (a) $\lim_{n \rightarrow \infty} \frac{\ln n}{n^c} = \lim_{n \rightarrow \infty} \frac{(\frac{1}{n})}{cn^{c-1}} = \lim_{n \rightarrow \infty} \frac{1}{cn^c} = 0$

(b) For all $\epsilon > 0$, there exists an $N = e^{-(\ln \epsilon)/c}$ such that $n > e^{-(\ln \epsilon)/c} \Rightarrow \ln n > -\frac{\ln \epsilon}{c} \Rightarrow \ln n^c > \ln \left(\frac{1}{\epsilon}\right) \Rightarrow n^c > \frac{1}{\epsilon} \Rightarrow \frac{1}{n^c} < \epsilon \Rightarrow \left|\frac{1}{n^c} - 0\right| < \epsilon \Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n^c} = 0$

92. Let $\{a_n\}$ and $\{b_n\}$ be sequences both converging to L . Define $\{c_n\}$ by $c_{2n} = b_n$ and $c_{2n-1} = a_n$, where $n = 1, 2, 3, \dots$. For all $\epsilon > 0$ there exists N_1 such that when $n > N_1$ then $|a_n - L| < \epsilon$ and there exists N_2 such that when $n > N_2$ then $|b_n - L| < \epsilon$. If $n > 1 + 2\max\{N_1, N_2\}$, then $|c_n - L| < \epsilon$, so $\{c_n\}$ converges to L .

93. $\lim_{n \rightarrow \infty} n^{1/n} = \lim_{n \rightarrow \infty} \exp\left(\frac{1}{n} \ln n\right) = \lim_{n \rightarrow \infty} \exp\left(\frac{1}{n}\right) = e^0 = 1$

94. $\lim_{n \rightarrow \infty} x^{1/n} = \lim_{n \rightarrow \infty} \exp\left(\frac{1}{n} \ln x\right) = e^0 = 1$, because x remains fixed while n gets large

95. Assume the hypotheses of the theorem and let ϵ be a positive number. For all ϵ there exists a N_1 such that when $n > N_1$ then $|a_n - L| < \epsilon \Rightarrow -\epsilon < a_n - L < \epsilon \Rightarrow L - \epsilon < a_n$, and there exists a N_2 such that when $n > N_2$ then $|c_n - L| < \epsilon \Rightarrow -\epsilon < c_n - L < \epsilon \Rightarrow c_n < L + \epsilon$. If $n > \max\{N_1, N_2\}$, then $L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon \Rightarrow |b_n - L| < \epsilon \Rightarrow \lim_{n \rightarrow \infty} b_n = L$.

96. Let $\epsilon > 0$. We have f continuous at $L \Rightarrow$ there exists δ so that $|x - L| < \delta \Rightarrow |f(x) - f(L)| < \epsilon$. Also, $a_n \rightarrow L \Rightarrow$ there exists N so that for $n > N$ $|a_n - L| < \delta$. Thus for $n > N$, $|f(a_n) - f(L)| < \epsilon \Rightarrow f(a_n) \rightarrow f(L)$.

97. $a_{n+1} \geq a_n \Rightarrow \frac{3(n+1)+1}{(n+1)+1} > \frac{3n+1}{n+1} \Rightarrow \frac{3n+4}{n+2} > \frac{3n+1}{n+1} \Rightarrow 3n^2 + 3n + 4n + 4 > 3n^2 + 6n + n + 2 \Rightarrow 4 > 2$; the steps are reversible so the sequence is nondecreasing; $\frac{3n+1}{n+1} < 3 \Rightarrow 3n + 1 < 3n + 3 \Rightarrow 1 < 3$; the steps are reversible so the sequence is bounded above by 3

98. $a_{n+1} \geq a_n \Rightarrow \frac{(2(n+1)+3)!}{((n+1)+1)!} > \frac{(2n+3)!}{(n+1)!} \Rightarrow \frac{(2n+5)!}{(n+2)!} > \frac{(2n+3)!}{(n+1)!} \Rightarrow \frac{(2n+5)!}{(2n+3)!} > \frac{(n+2)!}{(n+1)!} \Rightarrow (2n+5)(2n+4) > n+2$; the steps are reversible so the sequence is nondecreasing; the sequence is not bounded since $\frac{(2n+3)!}{(n+1)!} = (2n+3)(2n+2)\cdots(n+2)$ can become as large as we please

99. $a_{n+1} \leq a_n \Rightarrow \frac{2^{n+1}3^{n+1}}{(n+1)!} \leq \frac{2^n 3^n}{n!} \Rightarrow \frac{2^{n+1}3^{n+1}}{2^n 3^n} \leq \frac{(n+1)!}{n!} \Rightarrow 2 \cdot 3 \leq n+1$ which is true for $n \geq 5$; the steps are reversible so the sequence is decreasing after a_5 , but it is not nondecreasing for all its terms; $a_1 = 6$, $a_2 = 18$, $a_3 = 36$, $a_4 = 54$, $a_5 = \frac{324}{5} = 64.8 \Rightarrow$ the sequence is bounded from above by 64.8

100. $a_{n+1} \geq a_n \Rightarrow 2 - \frac{2}{n+1} - \frac{1}{2^{n+1}} \geq 2 - \frac{2}{n} - \frac{1}{2^n} \Rightarrow \frac{2}{n} - \frac{2}{n+1} \geq \frac{1}{2^{n+1}} - \frac{1}{2^n} \Rightarrow \frac{2}{n(n+1)} \geq -\frac{1}{2^{n+1}}$; the steps are reversible so the sequence is nondecreasing; $2 - \frac{2}{n} - \frac{1}{2^n} \leq 2 \Rightarrow$ the sequence is bounded from above

101. $a_n = 1 - \frac{1}{n}$ converges because $\frac{1}{n} \rightarrow 0$ by Example 1; also it is a nondecreasing sequence bounded above by 1

102. $a_n = n - \frac{1}{n}$ diverges because $n \rightarrow \infty$ and $\frac{1}{n} \rightarrow 0$ by Example 1, so the sequence is unbounded

103. $a_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$ and $0 < \frac{1}{2^n} < \frac{1}{n}$; since $\frac{1}{n} \rightarrow 0$ (by Example 1) $\Rightarrow \frac{1}{2^n} \rightarrow 0$, the sequence converges; also it is a nondecreasing sequence bounded above by 1

104. $a_n = \frac{2^n - 1}{3^n} = \left(\frac{2}{3}\right)^n - \frac{1}{3^n}$; the sequence converges to 0 by Theorem 5, #4

105. $a_n = ((-1)^n + 1) \left(\frac{n+1}{n}\right)$ diverges because $a_n = 0$ for n odd, while for n even $a_n = 2 \left(1 + \frac{1}{n}\right)$ converges to 2; it diverges by definition of divergence

106. $x_n = \max \{\cos 1, \cos 2, \cos 3, \dots, \cos n\}$ and $x_{n+1} = \max \{\cos 1, \cos 2, \cos 3, \dots, \cos(n+1)\} \geq x_n$ with $x_n \leq 1$ so the sequence is nondecreasing and bounded above by 1 \Rightarrow the sequence converges.

107. If $\{a_n\}$ is nonincreasing with lower bound M , then $\{-a_n\}$ is a nondecreasing sequence with upper bound $-M$. By Theorem 1, $\{-a_n\}$ converges and hence $\{a_n\}$ converges. If $\{a_n\}$ has no lower bound, then $\{-a_n\}$ has no upper bound and therefore diverges. Hence, $\{a_n\}$ also diverges.

108. $a_n \geq a_{n+1} \Leftrightarrow \frac{n+1}{n} \geq \frac{(n+1)+1}{n+1} \Leftrightarrow n^2 + 2n + 1 \geq n^2 + 2n \Leftrightarrow 1 \geq 0$ and $\frac{n+1}{n} \geq 1$; thus the sequence is nonincreasing and bounded below by 1 \Rightarrow it converges

109. $a_n \geq a_{n+1} \Leftrightarrow \frac{1+\sqrt{2n}}{\sqrt{n}} \geq \frac{1+\sqrt{2(n+1)}}{\sqrt{n+1}} \Leftrightarrow \sqrt{n+1} + \sqrt{2n^2+2n} \geq \sqrt{n} + \sqrt{2n^2+2n} \Leftrightarrow \sqrt{n+1} \geq \sqrt{n}$ and $\frac{1+\sqrt{2n}}{\sqrt{n}} \geq \sqrt{2}$; thus the sequence is nonincreasing and bounded below by $\sqrt{2} \Rightarrow$ it converges

110. $a_n \geq a_{n+1} \Leftrightarrow \frac{1-4^n}{2^n} \geq \frac{1-4^{n+1}}{2^{n+1}} \Leftrightarrow 2^{n+1} - 2^{n+1}4^n \geq 2^n - 2^n4^{n+1} \Leftrightarrow 2^{n+1} - 2^n \geq 2^{n+1}4^n - 2^n4^{n+1}$
 $\Leftrightarrow 2 - 1 \geq 2 \cdot 4^n - 4^{n+1} \Leftrightarrow 1 \geq 4^n(2 - 4) \Leftrightarrow 1 \geq (-2) \cdot 4^n$; thus the sequence is nonincreasing. However,
 $a_n = \frac{1}{2^n} - \frac{4^n}{2^n} = \frac{1}{2^n} - 2^n$ which is not bounded below so the sequence diverges

111. $\frac{4^{n+1}+3^n}{4^n} = 4 + \left(\frac{3}{4}\right)^n$ so $a_n \geq a_{n+1} \Leftrightarrow 4 + \left(\frac{3}{4}\right)^n \geq 4 + \left(\frac{3}{4}\right)^{n+1} \Leftrightarrow \left(\frac{3}{4}\right)^n \geq \left(\frac{3}{4}\right)^{n+1} \Leftrightarrow 1 \geq \frac{3}{4}$ and
 $4 + \left(\frac{3}{4}\right)^n \geq 4$; thus the sequence is nonincreasing and bounded below by 4 \Rightarrow it converges

112. $a_1 = 1, a_2 = 2 - 3, a_3 = 2(2 - 3) - 3 = 2^2 - (2^2 - 1) \cdot 3, a_4 = 2(2^2 - (2^2 - 1) \cdot 3) - 3 = 2^3 - (2^3 - 1) \cdot 3,$
 $a_5 = 2[2^3 - (2^3 - 1) \cdot 3] - 3 = 2^4 - (2^4 - 1) \cdot 3, \dots, a_n = 2^{n-1} - (2^{n-1} - 1) \cdot 3 = 2^{n-1} - 3 \cdot 2^{n-1} + 3$
 $= 2^{n-1}(1 - 3) + 3 = -2^{n-1} + 3; a_n \geq a_{n+1} \Leftrightarrow -2^n + 3 \geq -2^{n+1} + 3 \Leftrightarrow -2^n \geq -2^{n+1} \Leftrightarrow 1 \leq 2$
 so the sequence is nonincreasing but not bounded below and therefore diverges

113. Let $0 < M < 1$ and let N be an integer greater than $\frac{M}{1-M}$. Then $n > N \Rightarrow n > \frac{M}{1-M} \Rightarrow n - nM > M$
 $\Rightarrow n > M + nM \Rightarrow n > M(n+1) \Rightarrow \frac{n}{n+1} > M$.

114. Since M_1 is a least upper bound and M_2 is an upper bound, $M_1 \leq M_2$. Since M_2 is a least upper bound and M_1 is an upper bound, $M_2 \leq M_1$. We conclude that $M_1 = M_2$ so the least upper bound is unique.

115. The sequence $a_n = 1 + \frac{(-1)^n}{2}$ is the sequence $\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \dots$. This sequence is bounded above by $\frac{3}{2}$, but it clearly does not converge, by definition of convergence.

116. Let L be the limit of the convergent sequence $\{a_n\}$. Then by definition of convergence, for $\frac{\epsilon}{2}$ there corresponds an N such that for all m and $n, m > N \Rightarrow |a_m - L| < \frac{\epsilon}{2}$ and $n > N \Rightarrow |a_n - L| < \frac{\epsilon}{2}$. Now
 $|a_m - a_n| = |a_m - L + L - a_n| \leq |a_m - L| + |L - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ whenever $m > N$ and $n > N$.

117. Given an $\epsilon > 0$, by definition of convergence there corresponds an N such that for all $n > N$,
 $|L_1 - a_n| < \epsilon$ and $|L_2 - a_n| < \epsilon$. Now $|L_2 - L_1| = |L_2 - a_n + a_n - L_1| \leq |L_2 - a_n| + |a_n - L_1| < \epsilon + \epsilon = 2\epsilon$.
 $|L_2 - L_1| < 2\epsilon$ says that the difference between two fixed values is smaller than any positive number 2ϵ .
 The only nonnegative number smaller than every positive number is 0, so $|L_1 - L_2| = 0$ or $L_1 = L_2$.

118. Let $k(n)$ and $i(n)$ be two order-preserving functions whose domains are the set of positive integers and whose ranges are a subset of the positive integers. Consider the two subsequences $a_{k(n)}$ and $a_{i(n)}$, where $a_{k(n)} \rightarrow L_1$, $a_{i(n)} \rightarrow L_2$ and $L_1 \neq L_2$. Thus $|a_{k(n)} - a_{i(n)}| \rightarrow |L_1 - L_2| > 0$. So there does not exist N such that for all $m, n > N \Rightarrow |a_m - a_n| < \epsilon$. So by Exercise 116, the sequence $\{a_n\}$ is not convergent and hence diverges.
119. $a_{2k} \rightarrow L \Leftrightarrow$ given an $\epsilon > 0$ there corresponds an N_1 such that $[2k > N_1 \Rightarrow |a_{2k} - L| < \epsilon]$. Similarly, $a_{2k+1} \rightarrow L \Leftrightarrow [2k+1 > N_2 \Rightarrow |a_{2k+1} - L| < \epsilon]$. Let $N = \max\{N_1, N_2\}$. Then $n > N \Rightarrow |a_n - L| < \epsilon$ whether n is even or odd, and hence $a_n \rightarrow L$.
120. Assume $a_n \rightarrow 0$. This implies that given an $\epsilon > 0$ there corresponds an N such that $n > N \Rightarrow |a_n - 0| < \epsilon \Rightarrow |a_n| < \epsilon \Rightarrow ||a_n|| < \epsilon \Rightarrow ||a_n| - 0| < \epsilon \Rightarrow |a_n| \rightarrow 0$. On the other hand, assume $|a_n| \rightarrow 0$. This implies that given an $\epsilon > 0$ there corresponds an N such that for $n > N$, $||a_n| - 0| < \epsilon \Rightarrow ||a_n|| < \epsilon \Rightarrow |a_n| < \epsilon \Rightarrow |a_n - 0| < \epsilon \Rightarrow a_n \rightarrow 0$.
121. $|\sqrt[n]{0.5} - 1| < 10^{-3} \Rightarrow -\frac{1}{1000} < (\frac{1}{2})^{1/n} - 1 < \frac{1}{1000} \Rightarrow (\frac{999}{1000})^n < \frac{1}{2} < (\frac{1001}{1000})^n \Rightarrow n > \frac{\ln(\frac{1}{2})}{\ln(\frac{999}{1000})} \Rightarrow n > 692.8 \Rightarrow N = 692; a_n = (\frac{1}{2})^{1/n}$ and $\lim_{n \rightarrow \infty} a_n = 1$
122. $|\sqrt[n]{n} - 1| < 10^{-3} \Rightarrow -\frac{1}{1000} < n^{1/n} - 1 < \frac{1}{1000} \Rightarrow (\frac{999}{1000})^n < n < (\frac{1001}{1000})^n \Rightarrow n > 9123 \Rightarrow N = 9123; a_n = \sqrt[n]{n} = n^{1/n}$ and $\lim_{n \rightarrow \infty} a_n = 1$
123. $(0.9)^n < 10^{-3} \Rightarrow n \ln(0.9) < -3 \ln 10 \Rightarrow n > \frac{-3 \ln 10}{\ln(0.9)} \approx 65.54 \Rightarrow N = 65; a_n = (\frac{9}{10})^n$ and $\lim_{n \rightarrow \infty} a_n = 0$
124. $\frac{2^n}{n!} < 10^{-7} \Rightarrow n! > 2^n 10^7$ and by calculator experimentation, $n > 14 \Rightarrow N = 14; a_n = \frac{2^n}{n!}$ and $\lim_{n \rightarrow \infty} a_n = 0$
125. (a) $f(x) = x^2 - a \Rightarrow f'(x) = 2x \Rightarrow x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} \Rightarrow x_{n+1} = \frac{2x_n^2 - (x_n^2 - a)}{2x_n} = \frac{x_n^2 + a}{2x_n} = \frac{(x_n + \frac{a}{x_n})}{2}$
 (b) $x_1 = 2, x_2 = 1.75, x_3 = 1.732142857, x_4 = 1.73205081, x_5 = 1.732050808$; we are finding the positive number where $x^2 - 3 = 0$; that is, where $x^2 = 3, x > 0$, or where $x = \sqrt{3}$.
126. $x_1 = 1.5, x_2 = 1.416666667, x_3 = 1.414215686, x_4 = 1.414213562, x_5 = 1.414213562$; we are finding the positive number $x^2 - 2 = 0$; that is, where $x^2 = 2, x > 0$, or where $x = \sqrt{2}$.
127. $x_1 = 1, x_2 = 1 + \cos(1) = 1.540302306, x_3 = 1.540302306 + \cos(1 + \cos(1)) = 1.570791601, x_4 = 1.570791601 + \cos(1.570791601) = 1.570796327 = \frac{\pi}{2}$ to 9 decimal places. After a few steps, the arc (x_{n-1}) and line segment $\cos(x_{n-1})$ are nearly the same as the quarter circle.
128. (a) $S_1 = 6.815, S_2 = 6.4061, S_3 = 6.021734, S_4 = 5.66042996, S_5 = 5.320804162, S_6 = 5.001555913, S_7 = 4.701462558, S_8 = 4.419374804, S_9 = 4.154212316, S_{10} = 3.904959577, S_{11} = 3.670662003, S_{12} = 3.450422282$ so it will take Ford about 12 years to catch up
 (b) $x \approx 11.8$

129-140. Example CAS Commands:

Maple:

with(Student[Calculus1]);

f := x -> sin(x);

a := 0;

b := Pi;


```

plot( f(x), x=a..b, title="#23(a) (Section 5.1)" );
N := [ 100, 200, 1000 ];          # (b)
for n in N do
  Xlist := [ a+1.*(b-a)/n*i $ i=0..n ];
  Ylist := map( f, Xlist );
end do;
for n in N do                      # (c)
  Avg[n] := evalf(add(y,y=Ylist)/nops(Ylist));
end do;
avg := FunctionAverage( f(x), x=a..b, output=value );
evalf( avg );
FunctionAverage(f(x),x=a..b,output=plot);  # (d)
fsolve( f(x)=avg, x=0.5 );
fsolve( f(x)=avg, x=2.5 );
fsolve( f(x)=Avg[1000], x=0.5 );
fsolve( f(x)=Avg[1000], x=2.5 );

```

Mathematica: (sequence functions may vary):

```

Clear[a, n]
a[n_]:= n1/n
first25= Table[N[a[n]],{n, 1, 25}]
Limit[a[n], n → 8]

```

The last command (Limit) will not always work in Mathematica. You could also explore the limit by enlarging your table to more than the first 25 values.

If you know the limit (1 in the above example), to determine how far to go to have all further terms within 0.01 of the limit, do the following.

```

Clear[minN, lim]
lim= 1
Do[{diff=Abs[a[n] - lim], If[diff < .01, {minN= n, Abort[]}]}, {n, 2, 1000}]
minN

```

For sequences that are given recursively, the following code is suggested. The portion of the command `a[n_]:=a[n]` stores the elements of the sequence and helps to streamline computation.

```

Clear[a, n]
a[1]= 1;
a[n_]:= a[n]= a[n - 1] + (1/5)(n-1)
first25= Table[N[a[n]], {n, 1, 25}]

```

The limit command does not work in this case, but the limit can be observed as 1.25.

```

Clear[minN, lim]
lim= 1.25
Do[{diff=Abs[a[n] - lim], If[diff < .01, {minN= n, Abort[]}]}, {n, 2, 1000}]
minN

```

141. Example CAS Commands:

Maple:

```

with( Student[Calculus1] );
A := n->(1+r/m)*A(n-1) + b;
A(0) := A0;
A(0) := 1000; r := 0.02015; m := 12; b := 50;          # (a)
pts1 := [seq( [n,A(n)], n=0..99 )];
plot( pts1, style=point, title="#141(a) (Section 11.1)");

```

```

A(60);
The sequence { A[n] } is not unbounded;
limit( A[n], n=infinity ) = infinity.
A(0) := 5000; r := 0.0589; m := 12; b := -50;          # (b)
pts1 := [seq( [n,A(n)], n=0..99 )]:
plot( pts1, style=point, title="#141(b) (Section 11.1)");
A(60);
pts1 := [seq( [n,A(n)], n=0..199 )]:
plot( pts1, style=point, title="#141(b) (Section 11.1)");
# This sequence is not bounded, and diverges to -infinity:
limit( A[n], n=infinity ) = -infinity.
A(0) := 5000; r := 0.045; m := 4; b := 0;             # (c)
for n from 1 while A(n)<20000 do end do; n;

```

It takes 31 years (124 quarters) for the investment to grow to \$20,000 when the interest rate is 4.5%, compounded quarterly.

```

r := 0.0625;
for n from 1 while A(n)<20000 do end do; n;

```

When the interest rate increases to 6.25% (compounded quarterly), it takes only 22.5 years for the balance to reach \$20,000.

```

B := k -> (1+r/m)^k * (A(0)+m*b/r) - m*b/r;          # (d)
A(0) := 1000.; r := 0.02015; m := 12; b := 50;
for k from 0 to 49 do
  printf( "%5d %9.2f %9.2f %9.2f\n", k, A(k), B(k), B(k)-A(k) );
end do;
A(0) := 'A(0)'; r := 'r'; m := 'm'; b := 'b'; n := 'n';
eval( AA(n+1) - ((1+r/m)*AA(n) + b), AA=B );
simplify( % );

```

142. Example CAS Commands:

Maple:

```

r := 3/4.;          # (a)
for k in $1..9 do
  A := k/10.;
  L := [0,A];
  for n from 1 to 99 do
    A := r*A*(1-A);
    L := L, [n,A];
  end do;
  pt[r,k/10] := [L];
end do;
plot( [seq( pt[r,a], a=[$1..9]/10 )], style=point, title="#142(a) (Section 11.1) );
R1 := [1.1, 1.2, 1.5, 2.5, 2.8, 2.9];          # (b)
for r in R1 do
  for k in $1..9 do
    A := k/10.;
    L := [0,A];
    for n from 1 to 99 do
      A := r*A*(1-A);
      L := L, [n,A];
    end do;
  end do;
end do;

```

```

end do;
pt[r,k/10] := [L];
end do;
t := sprintf("#142(b) (Section 11.1)\nr = %f", r);
P[r] := plot( [seq( pt[r,a], a=[($1..9)/10] )], style=point, title=t );
end do;
display( [seq(P[r], r=R1)], insequence=true );
R2 := [3.05, 3.1, 3.2, 3.3, 3.35, 3.4];          # (c)
for r in R2 do
  for k in $1..9 do
    A := k/10.;
    L := [0,A];
    for n from 1 to 99 do
      A := r*A*(1-A);
      L := L, [n,A];
    end do;
    pt[r,k/10] := [L];
  end do;
  t := sprintf("#142(c) (Section 11.1)\nr = %f", r);
  P[r] := plot( [seq( pt[r,a], a=[($1..9)/10] )], style=point, title=t );
end do;
display( [seq(P[r], r=R2)], insequence=true );
R3 := [3.46, 3.47, 3.48, 3.49, 3.5, 3.51, 3.52, 3.53, 3.542, 3.544, 3.546, 3.548];    # (d)
for r in R3 do
  for k in $1..9 do
    A := k/10.;
    L := [0,A];
    for n from 1 to 199 do
      A := r*A*(1-A);
      L := L, [n,A];
    end do;
    pt[r,k/10] := [L];
  end do;
  t := sprintf("#142(d) (Section 11.1)\nr = %f", r);
  P[r] := plot( [seq( pt[r,a], a=[($1..9)/10] )], style=point, title=t );
end do;
display( [seq(P[r], r=R3)], insequence=true );
R4 := [3.5695];          # (e)
for r in R4 do
  for k in $1..9 do
    A := k/10.;
    L := [0,A];
    for n from 1 to 299 do
      A := r*A*(1-A);
      L := L, [n,A];
    end do;
    pt[r,k/10] := [L];
  end do;
  t := sprintf("#142(e) (Section 11.1)\nr = %f", r);

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P[r] := plot( [seq( pt[r,a], a=[($1..9)/10] )], style=point, title=t );
end do:
display( [seq(P[r], r=R4)], insequence=true );
R5 := [3.65];                                     # (f)
for r in R5 do
  for k in $1..9 do
    A := k/10.;
    L := [0,A];
    for n from 1 to 299 do
      A := r*A*(1-A);
      L := L, [n,A];
    end do;
    pt[r,k/10] := [L];
  end do:
  t := sprintf("#142(f) (Section 11.1)\nr = %f", r);
  P[r] := plot( [seq( pt[r,a], a=[($1..9)/10] )], style=point, title=t );
end do:
display( [seq(P[r], r=R5)], insequence=true );
R6 := [3.65, 3.75];                               # (g)
for r in R6 do
  for a in [0.300, 0.301, 0.600, 0.601 ] do
    A := a;
    L := [0,a];
    for n from 1 to 299 do
      A := r*A*(1-A);
      L := L, [n,A];
    end do;
    pt[r,a] := [L];
  end do:
  t := sprintf("#142(g) (Section 11.1)\nr = %f", r);
  P[r] := plot( [seq( pt[r,a], a=[0.300, 0.301, 0.600, 0.601] )], style=point, title=t );
end do:
display( [seq(P[r], r=R6)], insequence=true );

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11.2 INFINITE SERIES

1. $s_n = \frac{a(1-r^n)}{(1-r)} = \frac{2(1-(\frac{1}{3})^n)}{1-(\frac{1}{3})} \Rightarrow \lim_{n \rightarrow \infty} s_n = \frac{2}{1-(\frac{1}{3})} = 3$
2. $s_n = \frac{a(1-r^n)}{(1-r)} = \frac{(\frac{9}{100})(1-(\frac{1}{100})^n)}{1-(\frac{1}{100})} \Rightarrow \lim_{n \rightarrow \infty} s_n = \frac{(\frac{9}{100})}{1-(\frac{1}{100})} = \frac{1}{11}$
3. $s_n = \frac{a(1-r^n)}{(1-r)} = \frac{1-(-\frac{1}{2})^n}{1-(-\frac{1}{2})} \Rightarrow \lim_{n \rightarrow \infty} s_n = \frac{1}{(\frac{3}{2})} = \frac{2}{3}$
4. $s_n = \frac{1-(-2)^n}{1-(-2)}$, a geometric series where $|r| > 1 \Rightarrow$ divergence
5. $\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2} \Rightarrow s_n = (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \dots + (\frac{1}{n+1} - \frac{1}{n+2}) = \frac{1}{2} - \frac{1}{n+2} \Rightarrow \lim_{n \rightarrow \infty} s_n = \frac{1}{2}$

$$6. \frac{5}{n(n+1)} = \frac{5}{n} - \frac{5}{n+1} \Rightarrow s_n = \left(5 - \frac{5}{2}\right) + \left(\frac{5}{2} - \frac{5}{3}\right) + \left(\frac{5}{3} - \frac{5}{4}\right) + \dots + \left(\frac{5}{n-1} - \frac{5}{n}\right) + \left(\frac{5}{n} - \frac{5}{n+1}\right) = 5 - \frac{5}{n+1}$$

$$\Rightarrow \lim_{n \rightarrow \infty} s_n = 5$$

$$7. 1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \dots, \text{ the sum of this geometric series is } \frac{1}{1 - (-\frac{1}{4})} = \frac{1}{1 + (\frac{1}{4})} = \frac{4}{5}$$

$$8. \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots, \text{ the sum of this geometric series is } \frac{(\frac{1}{16})}{1 - (\frac{1}{4})} = \frac{1}{12}$$

$$9. \frac{7}{4} + \frac{7}{16} + \frac{7}{64} + \dots, \text{ the sum of this geometric series is } \frac{(\frac{7}{4})}{1 - (\frac{1}{4})} = \frac{7}{3}$$

$$10. 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \dots, \text{ the sum of this geometric series is } \frac{5}{1 - (-\frac{1}{4})} = 4$$

$$11. (5 + 1) + \left(\frac{5}{2} + \frac{1}{3}\right) + \left(\frac{5}{4} + \frac{1}{9}\right) + \left(\frac{5}{8} + \frac{1}{27}\right) + \dots, \text{ is the sum of two geometric series; the sum is}$$

$$\frac{5}{1 - (\frac{1}{2})} + \frac{1}{1 - (\frac{1}{3})} = 10 + \frac{3}{2} = \frac{23}{2}$$

$$12. (5 - 1) + \left(\frac{5}{2} - \frac{1}{3}\right) + \left(\frac{5}{4} - \frac{1}{9}\right) + \left(\frac{5}{8} - \frac{1}{27}\right) + \dots, \text{ is the difference of two geometric series; the sum is}$$

$$\frac{5}{1 - (\frac{1}{2})} - \frac{1}{1 - (\frac{1}{3})} = 10 - \frac{3}{2} = \frac{17}{2}$$

$$13. (1 + 1) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{4} + \frac{1}{25}\right) + \left(\frac{1}{8} - \frac{1}{125}\right) + \dots, \text{ is the sum of two geometric series; the sum is}$$

$$\frac{1}{1 - (\frac{1}{2})} + \frac{1}{1 + (\frac{1}{5})} = 2 + \frac{5}{6} = \frac{17}{6}$$

$$14. 2 + \frac{4}{5} + \frac{8}{25} + \frac{16}{125} + \dots = 2 \left(1 + \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \dots\right); \text{ the sum of this geometric series is } 2 \left(\frac{1}{1 - (\frac{2}{5})}\right) = \frac{10}{3}$$

$$15. \frac{4}{(4n-3)(4n+1)} = \frac{1}{4n-3} - \frac{1}{4n+1} \Rightarrow s_n = \left(1 - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{13}\right) + \dots + \left(\frac{1}{4n-7} - \frac{1}{4n-3}\right)$$

$$+ \left(\frac{1}{4n-3} - \frac{1}{4n+1}\right) = 1 - \frac{1}{4n+1} \Rightarrow \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{4n+1}\right) = 1$$

$$16. \frac{6}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1} = \frac{A(2n+1) + B(2n-1)}{(2n-1)(2n+1)} \Rightarrow A(2n+1) + B(2n-1) = 6$$

$$\Rightarrow (2A + 2B)n + (A - B) = 6 \Rightarrow \begin{cases} 2A + 2B = 0 \\ A - B = 6 \end{cases} \Rightarrow \begin{cases} A + B = 0 \\ A - B = 6 \end{cases} \Rightarrow 2A = 6 \Rightarrow A = 3 \text{ and } B = -3. \text{ Hence,}$$

$$\sum_{n=1}^k \frac{6}{(2n-1)(2n+1)} = 3 \sum_{n=1}^k \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = 3 \left(\frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \dots - \frac{1}{2(k-1)+1} + \frac{1}{2k-1} - \frac{1}{2k+1}\right)$$

$$= 3 \left(1 - \frac{1}{2k+1}\right) \Rightarrow \text{the sum is } \lim_{k \rightarrow \infty} 3 \left(1 - \frac{1}{2k+1}\right) = 3$$

$$17. \frac{40n}{(2n-1)^2(2n+1)^2} = \frac{A}{(2n-1)} + \frac{B}{(2n-1)^2} + \frac{C}{(2n+1)} + \frac{D}{(2n+1)^2}$$

$$= \frac{A(2n-1)(2n+1)^2 + B(2n+1)^2 + C(2n+1)(2n-1)^2 + D(2n-1)^2}{(2n-1)^2(2n+1)^2}$$

$$\Rightarrow A(2n-1)(2n+1)^2 + B(2n+1)^2 + C(2n+1)(2n-1)^2 + D(2n-1)^2 = 40n$$

$$\Rightarrow A(8n^3 + 4n^2 - 2n - 1) + B(4n^2 + 4n + 1) + C(8n^3 - 4n^2 - 2n + 1) + D(4n^2 - 4n + 1) = 40n$$

$$\Rightarrow (8A + 8C)n^3 + (4A + 4B - 4C + 4D)n^2 + (-2A + 4B - 2C - 4D)n + (-A + B + C + D) = 40n$$

$$\Rightarrow \begin{cases} 8A + 8C = 0 \\ 4A + 4B - 4C + 4D = 0 \\ -2A + 4B - 2C - 4D = 40 \\ -A + B + C + D = 0 \end{cases} \Rightarrow \begin{cases} 8A + 8C = 0 \\ A + B - C + D = 0 \\ -A + 2B - C - 2D = 20 \\ -A + B + C + D = 0 \end{cases} \Rightarrow \begin{cases} B + D = 0 \\ 2B - 2D = 20 \end{cases} \Rightarrow 4B = 20 \Rightarrow B = 5$$

$$\text{and } D = -5 \Rightarrow \begin{cases} A + C = 0 \\ -A + 5 + C - 5 = 0 \end{cases} \Rightarrow C = 0 \text{ and } A = 0. \text{ Hence, } \sum_{n=1}^k \left[\frac{40n}{(2n-1)^2(2n+1)^2} \right]$$

$$\begin{aligned}
&= 5 \sum_{n=1}^k \left[\frac{1}{(2n-1)^2} - \frac{1}{(2n+1)^2} \right] = 5 \left(\frac{1}{1} - \frac{1}{9} + \frac{1}{9} - \frac{1}{25} + \frac{1}{25} - \dots - \frac{1}{(2k-1)^2} + \frac{1}{(2k-1)^2} - \frac{1}{(2k+1)^2} \right) \\
&= 5 \left(1 - \frac{1}{(2k+1)^2} \right) \Rightarrow \text{the sum is } \lim_{n \rightarrow \infty} 5 \left(1 - \frac{1}{(2k+1)^2} \right) = 5
\end{aligned}$$

$$\begin{aligned}
18. \quad \frac{2n+1}{n^2(n+1)^2} &= \frac{1}{n^2} - \frac{1}{(n+1)^2} \Rightarrow s_n = \left(1 - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{9} \right) + \left(\frac{1}{9} - \frac{1}{16} \right) + \dots + \left[\frac{1}{(n-1)^2} - \frac{1}{n^2} \right] + \left[\frac{1}{n^2} - \frac{1}{(n+1)^2} \right] \\
&\Rightarrow \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left[1 - \frac{1}{(n+1)^2} \right] = 1
\end{aligned}$$

$$\begin{aligned}
19. \quad s_n &= \left(1 - \frac{1}{\sqrt{2}} \right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} \right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} \right) + \dots + \left(\frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{n}} \right) + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right) = 1 - \frac{1}{\sqrt{n+1}} \\
&\Rightarrow \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{\sqrt{n+1}} \right) = 1
\end{aligned}$$

$$\begin{aligned}
20. \quad s_n &= \left(\frac{1}{2} - \frac{1}{2^{1/2}} \right) + \left(\frac{1}{2^{1/2}} - \frac{1}{2^{1/3}} \right) + \left(\frac{1}{2^{1/3}} - \frac{1}{2^{1/4}} \right) + \dots + \left(\frac{1}{2^{1/(n-1)}} - \frac{1}{2^{1/n}} \right) + \left(\frac{1}{2^{1/n}} - \frac{1}{2^{1/(n+1)}} \right) = \frac{1}{2} - \frac{1}{2^{1/(n+1)}} \\
&\Rightarrow \lim_{n \rightarrow \infty} s_n = \frac{1}{2} - \frac{1}{1} = -\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
21. \quad s_n &= \left(\frac{1}{\ln 3} - \frac{1}{\ln 2} \right) + \left(\frac{1}{\ln 4} - \frac{1}{\ln 3} \right) + \left(\frac{1}{\ln 5} - \frac{1}{\ln 4} \right) + \dots + \left(\frac{1}{\ln(n+1)} - \frac{1}{\ln n} \right) + \left(\frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)} \right) \\
&= -\frac{1}{\ln 2} + \frac{1}{\ln(n+2)} \Rightarrow \lim_{n \rightarrow \infty} s_n = -\frac{1}{\ln 2}
\end{aligned}$$

$$\begin{aligned}
22. \quad s_n &= [\tan^{-1}(1) - \tan^{-1}(2)] + [\tan^{-1}(2) - \tan^{-1}(3)] + \dots + [\tan^{-1}(n-1) - \tan^{-1}(n)] \\
&\quad + [\tan^{-1}(n) - \tan^{-1}(n+1)] = \tan^{-1}(1) - \tan^{-1}(n+1) \Rightarrow \lim_{n \rightarrow \infty} s_n = \tan^{-1}(1) - \frac{\pi}{2} = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}
\end{aligned}$$

$$23. \text{ convergent geometric series with sum } \frac{1}{1 - \left(\frac{1}{\sqrt{2}}\right)} = \frac{\sqrt{2}}{\sqrt{2}-1} = 2 + \sqrt{2}$$

$$24. \text{ divergent geometric series with } |r| = \sqrt{2} > 1 \qquad 25. \text{ convergent geometric series with sum } \frac{\left(\frac{3}{2}\right)}{1 - \left(-\frac{1}{2}\right)} = 1$$

$$26. \lim_{n \rightarrow \infty} (-1)^{n+1} n \neq 0 \Rightarrow \text{diverges} \qquad 27. \lim_{n \rightarrow \infty} \cos(n\pi) = \lim_{n \rightarrow \infty} (-1)^n \neq 0 \Rightarrow \text{diverges}$$

$$28. \cos(n\pi) = (-1)^n \Rightarrow \text{convergent geometric series with sum } \frac{1}{1 - \left(-\frac{1}{3}\right)} = \frac{5}{6}$$

$$29. \text{ convergent geometric series with sum } \frac{1}{1 - \left(\frac{1}{e^2}\right)} = \frac{e^2}{e^2-1}$$

$$30. \lim_{n \rightarrow \infty} \ln \frac{1}{n} = -\infty \neq 0 \Rightarrow \text{diverges}$$

$$31. \text{ convergent geometric series with sum } \frac{2}{1 - \left(\frac{1}{10}\right)} - 2 = \frac{20}{9} - \frac{18}{9} = \frac{2}{9}$$

$$32. \text{ convergent geometric series with sum } \frac{1}{1 - \left(\frac{1}{x}\right)} = \frac{x}{x-1}$$

$$33. \text{ difference of two geometric series with sum } \frac{1}{1 - \left(\frac{2}{3}\right)} - \frac{1}{1 - \left(\frac{1}{3}\right)} = 3 - \frac{3}{2} = \frac{3}{2}$$

$$34. \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n} \right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{-1}{n} \right)^n = e^{-1} \neq 0 \Rightarrow \text{diverges}$$

$$35. \lim_{n \rightarrow \infty} \frac{n!}{1000^n} = \infty \neq 0 \Rightarrow \text{diverges}$$

$$36. \lim_{n \rightarrow \infty} \frac{n^n}{n!} = \lim_{n \rightarrow \infty} \frac{n \cdot n \cdots n}{1 \cdot 2 \cdots n} > \lim_{n \rightarrow \infty} n = \infty \Rightarrow \text{diverges}$$

$$37. \sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} [\ln(n) - \ln(n+1)] \Rightarrow s_n = [\ln(1) - \ln(2)] + [\ln(2) - \ln(3)] + [\ln(3) - \ln(4)] + \dots \\ + [\ln(n-1) - \ln(n)] + [\ln(n) - \ln(n+1)] = \ln(1) - \ln(n+1) = -\ln(n+1) \Rightarrow \lim_{n \rightarrow \infty} s_n = -\infty, \Rightarrow \text{diverges}$$

$$38. \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \ln\left(\frac{n}{2n+1}\right) = \ln\left(\frac{1}{2}\right) \neq 0 \Rightarrow \text{diverges}$$

$$39. \text{convergent geometric series with sum } \frac{1}{1 - \left(\frac{e}{\pi}\right)} = \frac{\pi}{\pi - e}$$

$$40. \text{divergent geometric series with } |r| = \frac{e^\pi}{\pi^e} \approx \frac{23.141}{22.459} > 1$$

$$41. \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n; a = 1, r = -x; \text{converges to } \frac{1}{1 - (-x)} = \frac{1}{1+x} \text{ for } |x| < 1$$

$$42. \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-x^2)^n; a = 1, r = -x^2; \text{converges to } \frac{1}{1+x^2} \text{ for } |x| < 1$$

$$43. a = 3, r = \frac{x-1}{2}; \text{converges to } \frac{3}{1 - \left(\frac{x-1}{2}\right)} = \frac{6}{3-x} \text{ for } -1 < \frac{x-1}{2} < 1 \text{ or } -1 < x < 3$$

$$44. \sum_{n=0}^{\infty} \frac{(-1)^n}{2} \left(\frac{1}{3+\sin x}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2} \left(\frac{-1}{3+\sin x}\right)^n; a = \frac{1}{2}, r = \frac{-1}{3+\sin x}; \text{converges to } \frac{\left(\frac{1}{2}\right)}{1 - \left(\frac{-1}{3+\sin x}\right)} \\ = \frac{3+\sin x}{2(4+\sin x)} = \frac{3+\sin x}{8+2\sin x} \text{ for all } x \text{ (since } \frac{1}{4} \leq \frac{1}{3+\sin x} \leq \frac{1}{2} \text{ for all } x)$$

$$45. a = 1, r = 2x; \text{converges to } \frac{1}{1-2x} \text{ for } |2x| < 1 \text{ or } |x| < \frac{1}{2}$$

$$46. a = 1, r = -\frac{1}{x^2}; \text{converges to } \frac{1}{1 - \left(\frac{-1}{x^2}\right)} = \frac{x^2}{x^2+1} \text{ for } \left|\frac{1}{x^2}\right| < 1 \text{ or } |x| > 1.$$

$$47. a = 1, r = -(x+1)^n; \text{converges to } \frac{1}{1+(x+1)} = \frac{1}{2+x} \text{ for } |x+1| < 1 \text{ or } -2 < x < 0$$

$$48. a = 1, r = \frac{3-x}{2}; \text{converges to } \frac{1}{1 - \left(\frac{3-x}{2}\right)} = \frac{2}{x-1} \text{ for } \left|\frac{3-x}{2}\right| < 1 \text{ or } 1 < x < 5$$

$$49. a = 1, r = \sin x; \text{converges to } \frac{1}{1 - \sin x} \text{ for } x \neq (2k+1)\frac{\pi}{2}, k \text{ an integer}$$

$$50. a = 1, r = \ln x; \text{converges to } \frac{1}{1 - \ln x} \text{ for } |\ln x| < 1 \text{ or } e^{-1} < x < e$$

$$51. 0.\overline{23} = \sum_{n=0}^{\infty} \frac{23}{100} \left(\frac{1}{10^2}\right)^n = \frac{\left(\frac{23}{100}\right)}{1 - \left(\frac{1}{100}\right)} = \frac{23}{99}$$

$$52. 0.\overline{234} = \sum_{n=0}^{\infty} \frac{234}{1000} \left(\frac{1}{10^3}\right)^n = \frac{\left(\frac{234}{1000}\right)}{1 - \left(\frac{1}{1000}\right)} = \frac{234}{999}$$

$$53. 0.\overline{7} = \sum_{n=0}^{\infty} \frac{7}{10} \left(\frac{1}{10}\right)^n = \frac{\left(\frac{7}{10}\right)}{1 - \left(\frac{1}{10}\right)} = \frac{7}{9}$$

$$54. 0.\overline{d} = \sum_{n=0}^{\infty} \frac{d}{10} \left(\frac{1}{10}\right)^n = \frac{\left(\frac{d}{10}\right)}{1 - \left(\frac{1}{10}\right)} = \frac{d}{9}$$

$$55. 0.0\overline{6} = \sum_{n=0}^{\infty} \left(\frac{1}{10}\right) \left(\frac{6}{10}\right) \left(\frac{1}{10}\right)^n = \frac{\left(\frac{6}{100}\right)}{1 - \left(\frac{1}{10}\right)} = \frac{6}{90} = \frac{1}{15}$$

$$56. 1.\overline{414} = 1 + \sum_{n=0}^{\infty} \frac{414}{1000} \left(\frac{1}{10^3}\right)^n = 1 + \frac{\left(\frac{414}{1000}\right)}{1 - \left(\frac{1}{1000}\right)} = 1 + \frac{414}{999} = \frac{1413}{999}$$

$$57. 1.24\overline{123} = \frac{124}{100} + \sum_{n=0}^{\infty} \frac{123}{10^3} \left(\frac{1}{10^3}\right)^n = \frac{124}{100} + \frac{\left(\frac{123}{10^3}\right)}{1 - \left(\frac{1}{10^3}\right)} = \frac{124}{100} + \frac{123}{10^3 - 10^2} = \frac{124}{100} + \frac{123}{99,900} = \frac{123,999}{99,900} = \frac{41,333}{33,300}$$

$$58. 3.\overline{142857} = 3 + \sum_{n=0}^{\infty} \frac{142,857}{10^6} \left(\frac{1}{10^6}\right)^n = 3 + \frac{\left(\frac{142,857}{10^6}\right)}{1 - \left(\frac{1}{10^6}\right)} = 3 + \frac{142,857}{10^6 - 1} = \frac{3,142,854}{999,999} = \frac{116,402}{37,037}$$

$$59. (a) \sum_{n=-2}^{\infty} \frac{1}{(n+4)(n+5)}$$

$$(b) \sum_{n=0}^{\infty} \frac{1}{(n+2)(n+3)}$$

$$(c) \sum_{n=5}^{\infty} \frac{1}{(n-3)(n-2)}$$

$$60. (a) \sum_{n=-1}^{\infty} \frac{5}{(n+2)(n+3)}$$

$$(b) \sum_{n=3}^{\infty} \frac{5}{(n-2)(n-1)}$$

$$(c) \sum_{n=20}^{\infty} \frac{5}{(n-19)(n-18)}$$

$$61. (a) \text{ one example is } \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \frac{\left(\frac{1}{2}\right)}{1 - \left(\frac{1}{2}\right)} = 1$$

$$(b) \text{ one example is } -\frac{3}{2} - \frac{3}{4} - \frac{3}{8} - \frac{3}{16} - \dots = \frac{\left(-\frac{3}{2}\right)}{1 - \left(\frac{1}{2}\right)} = -3$$

$$(c) \text{ one example is } 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \dots; \text{ the series } \frac{k}{2} + \frac{k}{4} + \frac{k}{8} + \dots = \frac{\left(\frac{k}{2}\right)}{1 - \left(\frac{1}{2}\right)} = k \text{ where } k \text{ is any positive or negative number.}$$

$$62. \text{ The series } \sum_{n=0}^{\infty} k\left(\frac{1}{2}\right)^{n+1} \text{ is a geometric series whose sum is } \frac{\left(\frac{k}{2}\right)}{1 - \left(\frac{1}{2}\right)} = k \text{ where } k \text{ can be any positive or negative number.}$$

$$63. \text{ Let } a_n = b_n = \left(\frac{1}{2}\right)^n. \text{ Then } \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1, \text{ while } \sum_{n=1}^{\infty} \left(\frac{a_n}{b_n}\right) = \sum_{n=1}^{\infty} (1) \text{ diverges.}$$

$$64. \text{ Let } a_n = b_n = \left(\frac{1}{2}\right)^n. \text{ Then } \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1, \text{ while } \sum_{n=1}^{\infty} (a_n b_n) = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{3} \neq AB.$$

$$65. \text{ Let } a_n = \left(\frac{1}{4}\right)^n \text{ and } b_n = \left(\frac{1}{2}\right)^n. \text{ Then } A = \sum_{n=1}^{\infty} a_n = \frac{1}{3}, B = \sum_{n=1}^{\infty} b_n = 1 \text{ and } \sum_{n=1}^{\infty} \left(\frac{a_n}{b_n}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1 \neq \frac{A}{B}.$$

$$66. \text{ Yes: } \sum \left(\frac{1}{a_n}\right) \text{ diverges. The reasoning: } \sum a_n \text{ converges} \Rightarrow a_n \rightarrow 0 \Rightarrow \frac{1}{a_n} \rightarrow \infty \Rightarrow \sum \left(\frac{1}{a_n}\right) \text{ diverges by the } n\text{th-Term Test.}$$

$$67. \text{ Since the sum of a finite number of terms is finite, adding or subtracting a finite number of terms from a series that diverges does not change the divergence of the series.}$$

$$68. \text{ Let } A_n = a_1 + a_2 + \dots + a_n \text{ and } \lim_{n \rightarrow \infty} A_n = A. \text{ Assume } \sum (a_n + b_n) \text{ converges to } S. \text{ Let } \\ S_n = (a_1 + b_1) + (a_2 + b_2) + \dots + (a_n + b_n) \Rightarrow S_n = (a_1 + a_2 + \dots + a_n) + (b_1 + b_2 + \dots + b_n) \\ \Rightarrow b_1 + b_2 + \dots + b_n = S_n - A_n \Rightarrow \lim_{n \rightarrow \infty} (b_1 + b_2 + \dots + b_n) = S - A \Rightarrow \sum b_n \text{ converges. This} \\ \text{contradicts the assumption that } \sum b_n \text{ diverges; therefore, } \sum (a_n + b_n) \text{ diverges.}$$

69. (a) $\frac{2}{1-r} = 5 \Rightarrow \frac{2}{5} = 1-r \Rightarrow r = \frac{3}{5}; 2 + 2\left(\frac{3}{5}\right) + 2\left(\frac{3}{5}\right)^2 + \dots$

(b) $\frac{\left(\frac{13}{2}\right)}{1-r} = 5 \Rightarrow \frac{13}{10} = 1-r \Rightarrow r = -\frac{3}{10}; \frac{13}{2} - \frac{13}{2}\left(\frac{3}{10}\right) + \frac{13}{2}\left(\frac{3}{10}\right)^2 - \frac{13}{2}\left(\frac{3}{10}\right)^3 + \dots$

70. $1 + e^b + e^{2b} + \dots = \frac{1}{1-e^b} = 9 \Rightarrow \frac{1}{9} = 1-e^b \Rightarrow e^b = \frac{8}{9} \Rightarrow b = \ln\left(\frac{8}{9}\right)$

71. $s_n = 1 + 2r + r^2 + 2r^3 + r^4 + 2r^5 + \dots + r^{2n} + 2r^{2n+1}, n = 0, 1, \dots$

$\Rightarrow s_n = (1 + r^2 + r^4 + \dots + r^{2n}) + (2r + 2r^3 + 2r^5 + \dots + 2r^{2n+1}) \Rightarrow \lim_{n \rightarrow \infty} s_n = \frac{1}{1-r^2} + \frac{2r}{1-r^2}$
 $= \frac{1+2r}{1-r^2}, \text{ if } |r^2| < 1 \text{ or } |r| < 1$

72. $L - s_n = \frac{a}{1-r} - \frac{a(1-r^n)}{1-r} = \frac{ar^n}{1-r}$

73. distance $= 4 + 2\left[\left(4\right)\left(\frac{3}{4}\right) + \left(4\right)\left(\frac{3}{4}\right)^2 + \dots\right] = 4 + 2\left(\frac{3}{1-\left(\frac{3}{4}\right)}\right) = 28 \text{ m}$

74. time $= \sqrt{\frac{4}{4.9}} + 2\sqrt{\left(\frac{4}{4.9}\right)\left(\frac{3}{4}\right)} + 2\sqrt{\left(\frac{4}{4.9}\right)\left(\frac{3}{4}\right)^2} + 2\sqrt{\left(\frac{4}{4.9}\right)\left(\frac{3}{4}\right)^3} + \dots = \sqrt{\frac{4}{4.9}} + 2\sqrt{\frac{4}{4.9}}\left[\sqrt{\frac{3}{4}} + \sqrt{\left(\frac{3}{4}\right)^2} + \dots\right]$
 $= \frac{2}{\sqrt{4.9}} + \left(\frac{4}{\sqrt{4.9}}\right)\left[\frac{\sqrt{\frac{3}{4}}}{1-\sqrt{\frac{3}{4}}}\right] = \frac{2}{\sqrt{4.9}} + \left(\frac{4}{\sqrt{4.9}}\right)\left(\frac{\sqrt{3}}{2-\sqrt{3}}\right) = \frac{(4-2\sqrt{3})+4\sqrt{3}}{\sqrt{4.9}(2-\sqrt{3})} = \frac{4+2\sqrt{3}}{\sqrt{4.9}(2-\sqrt{3})} \approx 12.58 \text{ sec}$

75. area $= 2^2 + \left(\sqrt{2}\right)^2 + (1)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \dots = 4 + 2 + 1 + \frac{1}{2} + \dots = \frac{4}{1-\frac{1}{2}} = 8 \text{ m}^2$

76. area $= 2\left[\frac{\pi\left(\frac{1}{2}\right)^2}{2}\right] + 4\left[\frac{\pi\left(\frac{1}{4}\right)^2}{2}\right] + 8\left[\frac{\pi\left(\frac{1}{8}\right)^2}{2}\right] + \dots = \pi\left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots\right) = \pi\left(\frac{\left(\frac{1}{4}\right)}{1-\left(\frac{1}{2}\right)}\right) = \frac{\pi}{2}$

77. (a) $L_1 = 3, L_2 = 3\left(\frac{4}{3}\right), L_3 = 3\left(\frac{4}{3}\right)^2, \dots, L_n = 3\left(\frac{4}{3}\right)^{n-1} \Rightarrow \lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} 3\left(\frac{4}{3}\right)^{n-1} = \infty$

(b) Using the fact that the area of an equilateral triangle of side length s is $\frac{\sqrt{3}}{4}s^2$, we see that $A_1 = \frac{\sqrt{3}}{4}$,

$A_2 = A_1 + 3\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{3}\right)^2 = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}, A_3 = A_2 + 3(4)\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{3^2}\right)^2 = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} + \frac{\sqrt{3}}{27},$

$A_4 = A_3 + 3(4)^2\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{3^3}\right)^2, A_5 = A_4 + 3(4)^3\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{3^4}\right)^2, \dots,$

$A_n = \frac{\sqrt{3}}{4} + \sum_{k=2}^n 3(4)^{k-2}\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{3^k}\right)^{n-1} = \frac{\sqrt{3}}{4} + \sum_{k=2}^n 3\sqrt{3}(4)^{n-3}\left(\frac{1}{9}\right)^{n-1} = \frac{\sqrt{3}}{4} + 3\sqrt{3}\left(\sum_{k=2}^n \frac{4^{n-3}}{9^{n-1}}\right)$

$= \frac{\sqrt{3}}{4} + 3\sqrt{3}\left(\frac{\frac{1}{36}}{1-\frac{4}{9}}\right) = \frac{\sqrt{3}}{4} + 3\sqrt{3}\left(\frac{1}{20}\right) = \frac{2\sqrt{3}}{5}$

78. Each term of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ represents the area of one of the squares shown in the figure, and all of the squares lie inside the rectangle of width 1 and length $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$. Since the squares do not fill the rectangle completely, and the area of the rectangle is 2, we have $\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$.

11.3 THE INTEGRAL TEST

1. converges; a geometric series with $r = \frac{1}{10} < 1$

2. converges; a geometric series with $r = \frac{1}{e} < 1$

3. diverges; by the n th-Term Test for Divergence, $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$
4. diverges by the Integral Test; $\int_1^n \frac{5}{x+1} dx = 5 \ln(n+1) - 5 \ln 2 \Rightarrow \int_1^\infty \frac{5}{x+1} dx \rightarrow \infty$
5. diverges; $\sum_{n=1}^\infty \frac{3}{\sqrt{n}} = 3 \sum_{n=1}^\infty \frac{1}{\sqrt{n}}$, which is a divergent p -series ($p = \frac{1}{2}$)
6. converges; $\sum_{n=1}^\infty \frac{-2}{n\sqrt{n}} = -2 \sum_{n=1}^\infty \frac{1}{n^{3/2}}$, which is a convergent p -series ($p = \frac{3}{2}$)
7. converges; a geometric series with $r = \frac{1}{8} < 1$
8. diverges; $\sum_{n=1}^\infty \frac{-8}{n} = -8 \sum_{n=1}^\infty \frac{1}{n}$ and since $\sum_{n=1}^\infty \frac{1}{n}$ diverges, $-8 \sum_{n=1}^\infty \frac{1}{n}$ diverges
9. diverges by the Integral Test: $\int_2^n \frac{\ln x}{x} dx = \frac{1}{2} (\ln^2 n - \ln^2 2) \Rightarrow \int_2^\infty \frac{\ln x}{x} dx \rightarrow \infty$
10. diverges by the Integral Test: $\int_2^\infty \frac{\ln x}{\sqrt{x}} dx$; $\left[\begin{array}{l} t = \ln x \\ dt = \frac{dx}{x} \\ dx = e^t dt \end{array} \right] \rightarrow \int_{\ln 2}^\infty t e^{t/2} dt = \lim_{b \rightarrow \infty} [2t e^{t/2} - 4e^{t/2}]_{\ln 2}^b$
 $= \lim_{b \rightarrow \infty} [2e^{b/2}(b - 2) - 2e^{(\ln 2)/2}(\ln 2 - 2)] = \infty$
11. converges; a geometric series with $r = \frac{2}{3} < 1$
12. diverges; $\lim_{n \rightarrow \infty} \frac{5^n}{4^{n+3}} = \lim_{n \rightarrow \infty} \frac{5^n \ln 5}{4^n \ln 4} = \lim_{n \rightarrow \infty} \left(\frac{\ln 5}{\ln 4} \right) \left(\frac{5}{4} \right)^n = \infty \neq 0$
13. diverges; $\sum_{n=0}^\infty \frac{-2}{n+1} = -2 \sum_{n=0}^\infty \frac{1}{n+1}$, which diverges by the Integral Test
14. diverges by the Integral Test: $\int_1^n \frac{dx}{2x-1} = \frac{1}{2} \ln(2n-1) \rightarrow \infty$ as $n \rightarrow \infty$
15. diverges; $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{2^n}{n+1} = \lim_{n \rightarrow \infty} \frac{2^n \ln 2}{1} = \infty \neq 0$
16. diverges by the Integral Test: $\int_1^n \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$; $\left[\begin{array}{l} u = \sqrt{x} + 1 \\ du = \frac{dx}{\sqrt{x}} \end{array} \right] \rightarrow \int_2^{\sqrt{n}+1} \frac{du}{u} = \ln(\sqrt{n}+1) - \ln 2$
 $\rightarrow \infty$ as $n \rightarrow \infty$
17. diverges; $\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\ln n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{2\sqrt{n}} \right)}{\left(\frac{1}{n} \right)} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{2} = \infty \neq 0$
18. diverges; $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e \neq 0$
19. diverges; a geometric series with $r = \frac{1}{\ln 2} \approx 1.44 > 1$
20. converges; a geometric series with $r = \frac{1}{\ln 3} \approx 0.91 < 1$

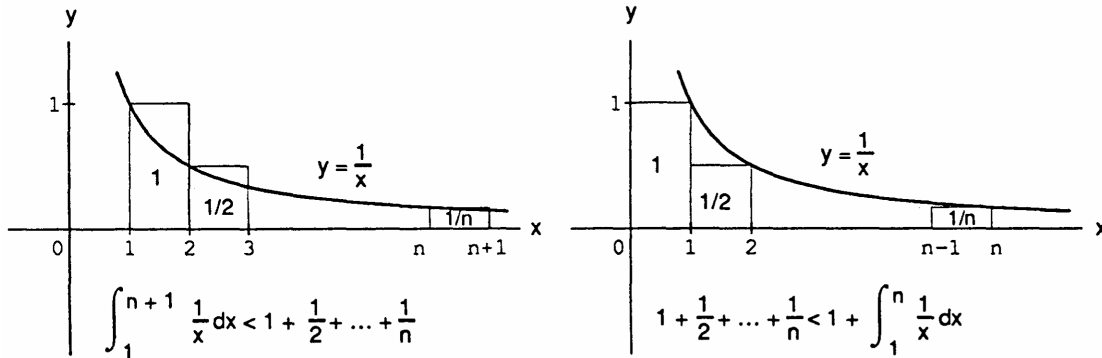
21. converges by the Integral Test: $\int_3^\infty \frac{\left(\frac{1}{x}\right)}{(\ln x) \sqrt{(\ln x)^2 - 1}} dx; \left[\begin{array}{l} u = \ln x \\ du = \frac{1}{x} dx \end{array} \right] \rightarrow \int_{\ln 3}^\infty \frac{1}{u \sqrt{u^2 - 1}} du$
 $= \lim_{b \rightarrow \infty} [\sec^{-1} |u|]_{\ln 3}^b = \lim_{b \rightarrow \infty} [\sec^{-1} b - \sec^{-1} (\ln 3)] = \lim_{b \rightarrow \infty} [\cos^{-1} \left(\frac{1}{b}\right) - \sec^{-1} (\ln 3)]$
 $= \cos^{-1} (0) - \sec^{-1} (\ln 3) = \frac{\pi}{2} - \sec^{-1} (\ln 3) \approx 1.1439$
22. converges by the Integral Test: $\int_1^\infty \frac{1}{x(1+\ln^2 x)} dx = \int_1^\infty \frac{\left(\frac{1}{x}\right)}{1+(\ln x)^2} dx; \left[\begin{array}{l} u = \ln x \\ du = \frac{1}{x} dx \end{array} \right] \rightarrow \int_0^\infty \frac{1}{1+u^2} du$
 $= \lim_{b \rightarrow \infty} [\tan^{-1} u]_0^b = \lim_{b \rightarrow \infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$
23. diverges by the nth-Term Test for divergence; $\lim_{n \rightarrow \infty} n \sin \left(\frac{1}{n}\right) = \lim_{n \rightarrow \infty} \frac{\sin \left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \neq 0$
24. diverges by the nth-Term Test for divergence; $\lim_{n \rightarrow \infty} n \tan \left(\frac{1}{n}\right) = \lim_{n \rightarrow \infty} \frac{\tan \left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \rightarrow \infty} \frac{\left(-\frac{1}{n^2}\right) \sec^2 \left(\frac{1}{n}\right)}{\left(-\frac{1}{n^2}\right)}$
 $= \lim_{n \rightarrow \infty} \sec^2 \left(\frac{1}{n}\right) = \sec^2 0 = 1 \neq 0$
25. converges by the Integral Test: $\int_1^\infty \frac{e^x}{1+e^{2x}} dx; \left[\begin{array}{l} u = e^x \\ du = e^x dx \end{array} \right] \rightarrow \int_e^\infty \frac{1}{1+u^2} du = \lim_{b \rightarrow \infty} [\tan^{-1} u]_e^b$
 $= \lim_{b \rightarrow \infty} (\tan^{-1} b - \tan^{-1} e) = \frac{\pi}{2} - \tan^{-1} e \approx 0.35$
26. converges by the Integral Test: $\int_1^\infty \frac{2}{1+e^x} dx; \left[\begin{array}{l} u = e^x \\ du = e^x dx \\ dx = \frac{1}{u} du \end{array} \right] \rightarrow \int_e^\infty \frac{2}{u(1+u)} du = \int_e^\infty \left(\frac{2}{u} - \frac{2}{u+1}\right) du$
 $= \lim_{b \rightarrow \infty} \left[2 \ln \frac{u}{u+1}\right]_e^b = \lim_{b \rightarrow \infty} 2 \ln \left(\frac{b}{b+1}\right) - 2 \ln \left(\frac{e}{e+1}\right) = 2 \ln 1 - 2 \ln \left(\frac{e}{e+1}\right) = -2 \ln \left(\frac{e}{e+1}\right) \approx 0.63$
27. converges by the Integral Test: $\int_1^\infty \frac{8 \tan^{-1} x}{1+x^2} dx; \left[\begin{array}{l} u = \tan^{-1} x \\ du = \frac{dx}{1+x^2} \end{array} \right] \rightarrow \int_{\pi/4}^{\pi/2} 8u du = [4u^2]_{\pi/4}^{\pi/2} = 4 \left(\frac{\pi^2}{4} - \frac{\pi^2}{16}\right) = \frac{3\pi^2}{4}$
28. diverges by the Integral Test: $\int_1^\infty \frac{x}{x^2+1} dx; \left[\begin{array}{l} u = x^2+1 \\ du = 2x dx \end{array} \right] \rightarrow \frac{1}{2} \int_2^\infty \frac{du}{u} = \lim_{b \rightarrow \infty} \left[\frac{1}{2} \ln u\right]_2^b$
 $= \lim_{b \rightarrow \infty} \frac{1}{2} (\ln b - \ln 2) = \infty$
29. converges by the Integral Test: $\int_1^\infty \operatorname{sech} x dx = 2 \lim_{b \rightarrow \infty} \int_1^b \frac{e^x}{1+(e^x)^2} dx = 2 \lim_{b \rightarrow \infty} [\tan^{-1} e^x]_1^b$
 $= 2 \lim_{b \rightarrow \infty} (\tan^{-1} e^b - \tan^{-1} e) = \pi - 2 \tan^{-1} e \approx 0.71$
30. converges by the Integral Test: $\int_1^\infty \operatorname{sech}^2 x dx = \lim_{b \rightarrow \infty} \int_1^b \operatorname{sech}^2 x dx = \lim_{b \rightarrow \infty} [\tanh x]_1^b = \lim_{b \rightarrow \infty} (\tanh b - \tanh 1)$
 $= 1 - \tanh 1 \approx 0.76$
31. $\int_1^\infty \left(\frac{a}{x+2} - \frac{1}{x+4}\right) dx = \lim_{b \rightarrow \infty} [a \ln |x+2| - \ln |x+4|]_1^b = \lim_{b \rightarrow \infty} \ln \frac{(b+2)^a}{b+4} - \ln \left(\frac{3^a}{5}\right);$
 $\lim_{b \rightarrow \infty} \frac{(b+2)^a}{b+4} = a \lim_{b \rightarrow \infty} (b+2)^{a-1} = \begin{cases} \infty, & a > 1 \\ 1, & a = 1 \end{cases} \Rightarrow \text{the series converges to } \ln \left(\frac{5}{3}\right) \text{ if } a = 1 \text{ and diverges to } \infty \text{ if } a > 1.$ If $a < 1$, the terms of the series eventually become negative and the Integral Test does not apply. From that point on, however, the series behaves like a negative multiple of the harmonic series, and so it diverges.

$$32. \int_3^\infty \left(\frac{1}{x-1} - \frac{2a}{x+1} \right) dx = \lim_{b \rightarrow \infty} \left[\ln \left| \frac{x-1}{(x+1)^{2a}} \right| \right]_3^b = \lim_{b \rightarrow \infty} \ln \frac{b-1}{(b+1)^{2a}} - \ln \left(\frac{2}{4^{2a}} \right); \lim_{b \rightarrow \infty} \frac{b-1}{(b+1)^{2a}}$$

$$= \lim_{b \rightarrow \infty} \frac{1}{2a(b+1)^{2a-1}} = \begin{cases} 1, & a = \frac{1}{2} \\ \infty, & a < \frac{1}{2} \end{cases} \Rightarrow \text{the series converges to } \ln \left(\frac{4}{2} \right) = \ln 2 \text{ if } a = \frac{1}{2} \text{ and diverges to } \infty \text{ if}$$

if $a < \frac{1}{2}$. If $a > \frac{1}{2}$, the terms of the series eventually become negative and the Integral Test does not apply. From that point on, however, the series behaves like a negative multiple of the harmonic series, and so it diverges.

33. (a)



(b) There are $(13)(365)(24)(60)(60)(10^9)$ seconds in 13 billion years; by part (a) $s_n \leq 1 + \ln n$ where $n = (13)(365)(24)(60)(60)(10^9) \Rightarrow s_n \leq 1 + \ln((13)(365)(24)(60)(60)(10^9))$
 $= 1 + \ln(13) + \ln(365) + \ln(24) + 2 \ln(60) + 9 \ln(10) \approx 41.55$

34. No, because $\sum_{n=1}^{\infty} \frac{1}{nx} = \frac{1}{x} \sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

35. Yes. If $\sum_{n=1}^{\infty} a_n$ is a divergent series of positive numbers, then $\left(\frac{1}{2}\right) \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{a_n}{2}\right)$ also diverges and $\frac{a_n}{2} < a_n$.

There is no "smallest" divergent series of positive numbers: for any divergent series $\sum_{n=1}^{\infty} a_n$ of positive numbers $\sum_{n=1}^{\infty} \left(\frac{a_n}{2}\right)$ has smaller terms and still diverges.

36. No, if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive numbers, then $2 \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 2a_n$ also converges, and $2a_n \geq a_n$.

There is no "largest" convergent series of positive numbers.

37. Let $A_n = \sum_{k=1}^n a_k$ and $B_n = \sum_{k=1}^n 2^k a_{(2^k)}$, where $\{a_k\}$ is a nonincreasing sequence of positive terms converging to

0. Note that $\{A_n\}$ and $\{B_n\}$ are nondecreasing sequences of positive terms. Now,

$$B_n = 2a_2 + 4a_4 + 8a_8 + \dots + 2^n a_{(2^n)} = 2a_2 + (2a_4 + 2a_4) + (2a_8 + 2a_8 + 2a_8 + 2a_8) + \dots$$

$$+ \underbrace{(2a_{(2^n)} + 2a_{(2^n)} + \dots + 2a_{(2^n)})}_{2^{n-1} \text{ terms}} \leq 2a_1 + 2a_2 + (2a_3 + 2a_4) + (2a_5 + 2a_6 + 2a_7 + 2a_8) + \dots$$

$$+ (2a_{(2^{n-1})} + 2a_{(2^{n-1}+1)} + \dots + 2a_{(2^n)}) = 2A_{(2^n)} \leq 2 \sum_{k=1}^{\infty} a_k. \text{ Therefore if } \sum a_k \text{ converges,}$$

then $\{B_n\}$ is bounded above $\Rightarrow \sum 2^k a_{(2^k)}$ converges. Conversely,

$$A_n = a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \dots + a_n < a_1 + 2a_2 + 4a_4 + \dots + 2^n a_{(2^n)} = a_1 + B_n < a_1 + \sum_{k=1}^{\infty} 2^k a_{(2^k)}.$$

Therefore, if $\sum_{k=1}^{\infty} 2^k a_{(2^k)}$ converges, then $\{A_n\}$ is bounded above and hence converges.

$$38. (a) a_{(2^n)} = \frac{1}{2^n \ln(2^n)} = \frac{1}{2^n \cdot n(\ln 2)} \Rightarrow \sum_{n=2}^{\infty} 2^n a_{(2^n)} = \sum_{n=2}^{\infty} 2^n \frac{1}{2^n \cdot n(\ln 2)} = \frac{1}{\ln 2} \sum_{n=2}^{\infty} \frac{1}{n}, \text{ which diverges}$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ diverges.}$$

$$(b) a_{(2^n)} = \frac{1}{2^{np}} \Rightarrow \sum_{n=1}^{\infty} 2^n a_{(2^n)} = \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2^{np}} = \sum_{n=1}^{\infty} \frac{1}{(2^n)^{p-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n, \text{ a geometric series that}$$

converges if $\frac{1}{2^{p-1}} < 1$ or $p > 1$, but diverges if $p \leq 1$.

$$39. (a) \int_2^{\infty} \frac{dx}{x(\ln x)^p}; \left[\begin{array}{l} u = \ln x \\ du = \frac{dx}{x} \end{array} \right] \rightarrow \int_{\ln 2}^{\infty} u^{-p} du = \lim_{b \rightarrow \infty} \left[\frac{u^{-p+1}}{-p+1} \right]_{\ln 2}^b = \lim_{b \rightarrow \infty} \left(\frac{1}{1-p} \right) [b^{-p+1} - (\ln 2)^{-p+1}]$$

$$= \begin{cases} \frac{1}{p-1} (\ln 2)^{-p+1}, & p > 1 \\ \infty, & p < 1 \end{cases} \Rightarrow \text{the improper integral converges if } p > 1 \text{ and diverges}$$

if $p < 1$. For $p = 1$: $\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{b \rightarrow \infty} [\ln(\ln x)]_2^b = \lim_{b \rightarrow \infty} [\ln(\ln b) - \ln(\ln 2)] = \infty$, so the improper integral diverges if $p = 1$.

$$(b) \text{ Since the series and the integral converge or diverge together, } \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges if and only if } p > 1.$$

$$40. (a) p = 1 \Rightarrow \text{the series diverges}$$

$$(b) p = 1.01 \Rightarrow \text{the series converges}$$

$$(c) \sum_{n=2}^{\infty} \frac{1}{n(\ln n^3)} = \frac{1}{3} \sum_{n=2}^{\infty} \frac{1}{n(\ln n)}; p = 1 \Rightarrow \text{the series diverges}$$

$$(d) p = 3 \Rightarrow \text{the series converges}$$

$$41. (a) \text{ From Fig. 11.8 in the text with } f(x) = \frac{1}{x} \text{ and } a_k = \frac{1}{k}, \text{ we have } \int_1^{n+1} \frac{1}{x} dx \leq 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

$$\leq 1 + \int_1^n f(x) dx \Rightarrow \ln(n+1) \leq 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \leq 1 + \ln n \Rightarrow 0 \leq \ln(n+1) - \ln n$$

$$\leq \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right) - \ln n \leq 1. \text{ Therefore the sequence } \left\{\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right) - \ln n\right\} \text{ is bounded above}$$

by 1 and below by 0.

$$(b) \text{ From the graph in Fig. 11.8(a) with } f(x) = \frac{1}{x}, \frac{1}{n+1} < \int_n^{n+1} \frac{1}{x} dx = \ln(n+1) - \ln n$$

$$\Rightarrow 0 > \frac{1}{n+1} - [\ln(n+1) - \ln n] = \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n+1} - \ln(n+1)\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n\right).$$

If we define $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n$, then $0 > a_{n+1} - a_n \Rightarrow a_{n+1} < a_n \Rightarrow \{a_n\}$ is a decreasing sequence of nonnegative terms.

$$42. e^{-x^2} \leq e^{-x} \text{ for } x \geq 1, \text{ and } \int_1^{\infty} e^{-x} dx = \lim_{b \rightarrow \infty} [-e^{-x}]_1^b = \lim_{b \rightarrow \infty} (-e^{-b} + e^{-1}) = e^{-1} \Rightarrow \int_1^{\infty} e^{-x^2} dx \text{ converges by}$$

the Comparison Test for improper integrals $\Rightarrow \sum_{n=0}^{\infty} e^{-n^2} = 1 + \sum_{n=1}^{\infty} e^{-n^2}$ converges by the Integral Test.

11.4 COMPARISON TESTS

1. diverges by the Limit Comparison Test (part 1) when compared with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, a divergent p-series:

$$\lim_{n \rightarrow \infty} \frac{\left(\frac{1}{2\sqrt{n} + \sqrt[3]{n}}\right)}{\left(\frac{1}{\sqrt{n}}\right)} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{2\sqrt{n} + \sqrt[3]{n}} = \lim_{n \rightarrow \infty} \left(\frac{1}{2 + n^{-1/6}}\right) = \frac{1}{2}$$

2. diverges by the Direct Comparison Test since $n + n + n > n + \sqrt{n} + 0 \Rightarrow \frac{3}{n + \sqrt{n}} > \frac{1}{n}$, which is the n th term of the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$ or use Limit Comparison Test with $b_n = \frac{1}{n}$
3. converges by the Direct Comparison Test; $\frac{\sin^2 n}{2^n} \leq \frac{1}{2^n}$, which is the n th term of a convergent geometric series
4. converges by the Direct Comparison Test; $\frac{1 + \cos n}{n^2} \leq \frac{2}{n^2}$ and the p -series $\sum \frac{1}{n^2}$ converges
5. diverges since $\lim_{n \rightarrow \infty} \frac{2n}{3n-1} = \frac{2}{3} \neq 0$
6. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^{3/2}}$, the n th term of a convergent p -series:

$$\lim_{n \rightarrow \infty} \frac{\left(\frac{n+1}{n^2 \sqrt{n}}\right)}{\left(\frac{1}{n^{3/2}}\right)} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right) = 1$$
7. converges by the Direct Comparison Test; $\left(\frac{n}{3n+1}\right)^n < \left(\frac{n}{3n}\right)^n = \left(\frac{1}{3}\right)^n$, the n th term of a convergent geometric series
8. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^{3/2}}$, the n th term of a convergent p -series:

$$\lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n^{3/2}}\right)}{\left(\frac{1}{\sqrt{n^3+2}}\right)} = \lim_{n \rightarrow \infty} \sqrt{\frac{n^3+2}{n^3}} = \lim_{n \rightarrow \infty} \sqrt{1 + \frac{2}{n^3}} = 1$$
9. diverges by the Direct Comparison Test; $n > \ln n \Rightarrow \ln n > \ln \ln n \Rightarrow \frac{1}{n} < \frac{1}{\ln n} < \frac{1}{\ln(\ln n)}$ and $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges
10. diverges by the Limit Comparison Test (part 3) when compared with $\sum_{n=2}^{\infty} \frac{1}{n}$, a divergent p -series:

$$\lim_{n \rightarrow \infty} \frac{\left(\frac{1}{(\ln n)^2}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \rightarrow \infty} \frac{n}{(\ln n)^2} = \lim_{n \rightarrow \infty} \frac{1}{2(\ln n)\left(\frac{1}{n}\right)} = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{n}{\ln n} = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{1}{\left(\frac{1}{n}\right)} = \frac{1}{2} \lim_{n \rightarrow \infty} n = \infty$$
11. converges by the Limit Comparison Test (part 2) when compared with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, a convergent p -series:

$$\lim_{n \rightarrow \infty} \frac{\left[\frac{(\ln n)^2}{n^3}\right]}{\left(\frac{1}{n^2}\right)} = \lim_{n \rightarrow \infty} \frac{(\ln n)^2}{n} = \lim_{n \rightarrow \infty} \frac{2(\ln n)\left(\frac{1}{n}\right)}{1} = 2 \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$$
12. converges by the Limit Comparison Test (part 2) when compared with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, a convergent p -series:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\left[\frac{(\ln n)^3}{n^3}\right]}{\left(\frac{1}{n^2}\right)} &= \lim_{n \rightarrow \infty} \frac{(\ln n)^3}{n} = \lim_{n \rightarrow \infty} \frac{3(\ln n)^2 \left(\frac{1}{n}\right)}{1} = 3 \lim_{n \rightarrow \infty} \frac{(\ln n)^2}{n} = 3 \lim_{n \rightarrow \infty} \frac{2(\ln n)\left(\frac{1}{n}\right)}{1} = 6 \lim_{n \rightarrow \infty} \frac{\ln n}{n} \\ &= 6 \cdot 0 = 0 \end{aligned}$$
13. diverges by the Limit Comparison Test (part 3) with $\frac{1}{n}$, the n th term of the divergent harmonic series:

$$\lim_{n \rightarrow \infty} \frac{\left[\frac{1}{\sqrt{n} \ln n}\right]}{\left(\frac{1}{n}\right)} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\ln n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{2\sqrt{n}}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{2} = \infty$$

14. converges by the Limit Comparison Test (part 2) with $\frac{1}{n^{5/4}}$, the n th term of a convergent p -series:

$$\lim_{n \rightarrow \infty} \frac{\left(\frac{(\ln n)^2}{n^{3/2}}\right)}{\left(\frac{1}{n^{5/4}}\right)} = \lim_{n \rightarrow \infty} \frac{(\ln n)^2}{n^{1/4}} = \lim_{n \rightarrow \infty} \frac{\left(\frac{2 \ln n}{n}\right)}{\left(\frac{1}{4n^{3/4}}\right)} = 8 \lim_{n \rightarrow \infty} \frac{\ln n}{n^{1/4}} = 8 \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{4n^{3/4}}\right)} = 32 \lim_{n \rightarrow \infty} \frac{1}{n^{1/4}} = 32 \cdot 0 = 0$$

15. diverges by the Limit Comparison Test (part 3) with $\frac{1}{n}$, the n th term of the divergent harmonic series:

$$\lim_{n \rightarrow \infty} \frac{\left(\frac{1}{1 + \ln n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \rightarrow \infty} \frac{n}{1 + \ln n} = \lim_{n \rightarrow \infty} \frac{1}{\left(\frac{1}{n}\right)} = \lim_{n \rightarrow \infty} n = \infty$$

16. diverges by the Limit Comparison Test (part 3) with $\frac{1}{n}$, the n th term of the divergent harmonic series:

$$\lim_{n \rightarrow \infty} \frac{\left(\frac{1}{(1 + \ln n)^2}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \rightarrow \infty} \frac{n}{(1 + \ln n)^2} = \lim_{n \rightarrow \infty} \frac{1}{\left[\frac{2(1 + \ln n)}{n}\right]} = \lim_{n \rightarrow \infty} \frac{n}{2(1 + \ln n)} = \lim_{n \rightarrow \infty} \frac{1}{\left(\frac{2}{n}\right)} = \lim_{n \rightarrow \infty} \frac{n}{2} = \infty$$

17. diverges by the Integral Test: $\int_2^{\infty} \frac{\ln(x+1)}{x+1} dx = \int_{\ln 3}^{\infty} u du = \lim_{b \rightarrow \infty} \left[\frac{1}{2} u^2\right]_{\ln 3}^b = \lim_{b \rightarrow \infty} \frac{1}{2} (b^2 - \ln^2 3) = \infty$

18. diverges by the Limit Comparison Test (part 3) with $\frac{1}{n}$, the n th term of the divergent harmonic series:

$$\lim_{n \rightarrow \infty} \frac{\left(\frac{1}{1 + \ln^2 n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \rightarrow \infty} \frac{n}{1 + \ln^2 n} = \lim_{n \rightarrow \infty} \frac{1}{\left(\frac{2 \ln n}{n}\right)} = \lim_{n \rightarrow \infty} \frac{n}{2 \ln n} = \lim_{n \rightarrow \infty} \frac{1}{\left(\frac{2}{n}\right)} = \lim_{n \rightarrow \infty} \frac{n}{2} = \infty$$

19. converges by the Direct Comparison Test with $\frac{1}{n^{3/2}}$, the n th term of a convergent p -series: $n^2 - 1 > n$ for $n \geq 2 \Rightarrow n^2 (n^2 - 1) > n^3 \Rightarrow n\sqrt{n^2 - 1} > n^{3/2} \Rightarrow \frac{1}{n^{3/2}} > \frac{1}{n\sqrt{n^2 - 1}}$ or use Limit Comparison Test with $\frac{1}{n^2}$.

20. converges by the Direct Comparison Test with $\frac{1}{n^{3/2}}$, the n th term of a convergent p -series: $n^2 + 1 > n^2 \Rightarrow n^2 + 1 > \sqrt{n} n^{3/2} \Rightarrow \frac{n^2 + 1}{\sqrt{n}} > n^{3/2} \Rightarrow \frac{\sqrt{n}}{n^2 + 1} < \frac{1}{n^{3/2}}$ or use Limit Comparison Test with $\frac{1}{n^{3/2}}$.

21. converges because $\sum_{n=1}^{\infty} \frac{1-n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n2^n} + \sum_{n=1}^{\infty} \frac{-1}{2^n}$ which is the sum of two convergent series:

$\sum_{n=1}^{\infty} \frac{1}{n2^n}$ converges by the Direct Comparison Test since $\frac{1}{n2^n} < \frac{1}{2^n}$, and $\sum_{n=1}^{\infty} \frac{-1}{2^n}$ is a convergent geometric series

22. converges by the Direct Comparison Test: $\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{n2^n} + \frac{1}{n^2}\right)$ and $\frac{1}{n2^n} + \frac{1}{n^2} \leq \frac{1}{2^n} + \frac{1}{n^2}$, the sum of the n th terms of a convergent geometric series and a convergent p -series

23. converges by the Direct Comparison Test: $\frac{1}{3^{n-1} + 1} < \frac{1}{3^{n-1}}$, which is the n th term of a convergent geometric series

24. diverges; $\lim_{n \rightarrow \infty} \left(\frac{3^{n-1} + 1}{3^n}\right) = \lim_{n \rightarrow \infty} \left(\frac{1}{3} + \frac{1}{3^n}\right) = \frac{1}{3} \neq 0$

25. diverges by the Limit Comparison Test (part 1) with $\frac{1}{n}$, the n th term of the divergent harmonic series:

$$\lim_{n \rightarrow \infty} \frac{\left(\frac{\sin \frac{1}{n}}{\left(\frac{1}{n}\right)}\right)}{\left(\frac{1}{n}\right)} = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

26. diverges by the Limit Comparison Test (part 1) with $\frac{1}{n}$, the n th term of the divergent harmonic series:

$$\lim_{n \rightarrow \infty} \frac{\left(\frac{\tan \frac{1}{n}}{\left(\frac{1}{n}\right)}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \rightarrow \infty} \left(\frac{1}{\cos \frac{1}{n}}\right) \left(\frac{\sin \frac{1}{n}}{\left(\frac{1}{n}\right)}\right) = \lim_{x \rightarrow 0} \left(\frac{1}{\cos x}\right) \left(\frac{\sin x}{x}\right) = 1 \cdot 1 = 1$$

27. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$, the n th term of a convergent p -series:

$$\lim_{n \rightarrow \infty} \frac{\left(\frac{10n+1}{n(n+1)(n+2)}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n \rightarrow \infty} \frac{10n^2+n}{n^2+3n+2} = \lim_{n \rightarrow \infty} \frac{20n+1}{2n+3} = \lim_{n \rightarrow \infty} \frac{20}{2} = 10$$

28. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$, the n th term of a convergent p -series:

$$\lim_{n \rightarrow \infty} \frac{\left(\frac{5n^3-3n}{n^2(n-2)(n^2+5)}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n \rightarrow \infty} \frac{5n^3-3n}{n^3-2n^2+5n-10} = \lim_{n \rightarrow \infty} \frac{15n^2-3}{3n^2-4n+5} = \lim_{n \rightarrow \infty} \frac{30n}{6n-4} = 5$$

29. converges by the Direct Comparison Test: $\frac{\tan^{-1} n}{n^{1.1}} < \frac{\pi}{2n^{1.1}}$ and $\sum_{n=1}^{\infty} \frac{\pi}{2n^{1.1}} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ is the product of a convergent p -series and a nonzero constant

30. converges by the Direct Comparison Test: $\sec^{-1} n < \frac{\pi}{2} \Rightarrow \frac{\sec^{-1} n}{n^{1.3}} < \frac{(\frac{\pi}{2})}{n^{1.3}}$ and $\sum_{n=1}^{\infty} \frac{(\frac{\pi}{2})}{n^{1.3}} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^{1.3}}$ is the product of a convergent p -series and a nonzero constant

31. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$: $\lim_{n \rightarrow \infty} \frac{\left(\frac{\coth n}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n \rightarrow \infty} \coth n = \lim_{n \rightarrow \infty} \frac{e^n + e^{-n}}{e^n - e^{-n}} = \lim_{n \rightarrow \infty} \frac{1 + e^{-2n}}{1 - e^{-2n}} = 1$

32. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$: $\lim_{n \rightarrow \infty} \frac{\left(\frac{\tanh n}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n \rightarrow \infty} \tanh n = \lim_{n \rightarrow \infty} \frac{e^n - e^{-n}}{e^n + e^{-n}} = \lim_{n \rightarrow \infty} \frac{1 - e^{-2n}}{1 + e^{-2n}} = 1$

33. diverges by the Limit Comparison Test (part 1) with $\frac{1}{n}$: $\lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n\sqrt[n]{n}}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{n}} = 1$.

34. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$: $\lim_{n \rightarrow \infty} \frac{\left(\frac{\sqrt[n]{n}}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$

35. $\frac{1}{1+2+3+\dots+n} = \frac{1}{\left(\frac{n(n+1)}{2}\right)} = \frac{2}{n(n+1)}$. The series converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$:

$$\lim_{n \rightarrow \infty} \frac{\left(\frac{2}{n(n+1)}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n \rightarrow \infty} \frac{2n^2}{n^2+n} = \lim_{n \rightarrow \infty} \frac{4n}{2n+1} = \lim_{n \rightarrow \infty} \frac{4}{2} = 2.$$

36. $\frac{1}{1+2^2+3^2+\dots+n^2} = \frac{1}{\frac{n(n+1)(2n+1)}{6}} = \frac{6}{n(n+1)(2n+1)} \leq \frac{6}{n^3} \Rightarrow$ the series converges by the Direct Comparison Test

37. (a) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$, then there exists an integer N such that for all $n > N$, $\left|\frac{a_n}{b_n} - 0\right| < 1 \Rightarrow -1 < \frac{a_n}{b_n} < 1 \Rightarrow a_n < b_n$. Thus, if $\sum b_n$ converges, then $\sum a_n$ converges by the Direct Comparison Test.

(b) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$, then there exists an integer N such that for all $n > N$, $\frac{a_n}{b_n} > 1 \Rightarrow a_n > b_n$. Thus, if $\sum b_n$ diverges, then $\sum a_n$ diverges by the Direct Comparison Test.

38. Yes, $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges by the Direct Comparison Test because $\frac{a_n}{n} < a_n$

39. $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty \Rightarrow$ there exists an integer N such that for all $n > N$, $\frac{a_n}{b_n} > 1 \Rightarrow a_n > b_n$. If $\sum a_n$ converges, then $\sum b_n$ converges by the Direct Comparison Test

40. $\sum a_n$ converges $\Rightarrow \lim_{n \rightarrow \infty} a_n = 0 \Rightarrow$ there exists an integer N such that for all $n > N$, $0 \leq a_n < 1 \Rightarrow a_n^2 < a_n \Rightarrow \sum a_n^2$ converges by the Direct Comparison Test

41. Example CAS commands:

Maple:

```
a := n -> 1./n^3/sin(n)^2;
s := k -> sum( a(n), n=1..k );           # (a)]
limit( s(k), k=infinity );
pts := [seq( [k,s(k)], k=1..100 )];      # (b)
plot( pts, style=point, title="#41(b) (Section 11.4)" );
pts := [seq( [k,s(k)], k=1..200 )];      # (c)
plot( pts, style=point, title="#41(c) (Section 11.4)" );
pts := [seq( [k,s(k)], k=1..400 )];      # (d)
plot( pts, style=point, title="#41(d) (Section 11.4)" );
evalf( 355/113 );
```

Mathematica:

```
Clear[a, n, s, k, p]
a[n_]:= 1 / ( n^3 Sin[n]^2 )
s[k_]= Sum[ a[n], {n, 1, k}]
points[p_]:= Table[{k, N[s[k]]}, {k, 1, p}]
points[100]
ListPlot[points[100]]
points[200]
ListPlot[points[200]]
points[400]
ListPlot[points[400], PlotRange -> All]
```

To investigate what is happening around $k = 355$, you could do the following.

```
N[355/113]
N[ $\pi$  - 355/113]
Sin[355]/N
a[355]/N
N[s[354]]
N[s[355]]
N[s[356]]
```

11.5 THE RATIO AND ROOT TESTS

1. converges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\left[\frac{(n+1)\sqrt{2}}{2^{n+1}} \right]}{\left[\frac{n\sqrt{2}}{2^n} \right]} = \lim_{n \rightarrow \infty} \frac{(n+1)^{\sqrt{2}}}{2^{n+1}} \cdot \frac{2^n}{n^{\sqrt{2}}}$
 $= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{\sqrt{2}} \left(\frac{1}{2}\right) = \frac{1}{2} < 1$
2. converges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{(n+1)^2}{e^{n+1}}\right)}{\left(\frac{n^2}{e^n}\right)} = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{e^{n+1}} \cdot \frac{e^n}{n^2} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^2 \left(\frac{1}{e}\right) = \frac{1}{e} < 1$
3. diverges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{(n+1)!}{e^{n+1}}\right)}{\left(\frac{n!}{e^n}\right)} = \lim_{n \rightarrow \infty} \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} = \lim_{n \rightarrow \infty} \frac{n+1}{e} = \infty$
4. diverges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{(n+1)!}{10^{n+1}}\right)}{\left(\frac{n!}{10^n}\right)} = \lim_{n \rightarrow \infty} \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} = \lim_{n \rightarrow \infty} \frac{n+1}{10} = \infty$
5. converges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{(n+1)^{10}}{10^{n+1}}\right)}{\left(\frac{n^{10}}{10^n}\right)} = \lim_{n \rightarrow \infty} \frac{(n+1)^{10}}{10^{n+1}} \cdot \frac{10^n}{n^{10}} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{10} \left(\frac{1}{10}\right)$
 $= \frac{1}{10} < 1$
6. diverges; $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{n-2}{n}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{-2}{n}\right)^n = e^{-2} \neq 0$
7. converges by the Direct Comparison Test: $\frac{2+(-1)^n}{(1.25)^n} = \left(\frac{4}{5}\right)^n [2 + (-1)^n] \leq \left(\frac{4}{5}\right)^n (3)$ which is the n^{th} term of a convergent geometric series
8. converges; a geometric series with $|r| = \left| -\frac{2}{3} \right| < 1$
9. diverges; $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(1 - \frac{3}{n}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{-3}{n}\right)^n = e^{-3} \approx 0.05 \neq 0$
10. diverges; $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{3n}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{(-1/3)}{n}\right)^n = e^{-1/3} \approx 0.72 \neq 0$
11. converges by the Direct Comparison Test: $\frac{\ln n}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}$ for $n \geq 2$, the n^{th} term of a convergent p -series.
12. converges by the n^{th} -Root Test: $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{(\ln n)^n}{n^n}} = \lim_{n \rightarrow \infty} \frac{((\ln n)^n)^{1/n}}{(n^n)^{1/n}} = \lim_{n \rightarrow \infty} \frac{\ln n}{n}$
 $= \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n}\right)}{1} = 0 < 1$
13. diverges by the Direct Comparison Test: $\frac{1}{n} - \frac{1}{n^2} = \frac{n-1}{n^2} > \frac{1}{2} \left(\frac{1}{n}\right)$ for $n > 2$ or by the Limit Comparison Test (part 1) with $\frac{1}{n}$.
14. converges by the n^{th} -Root Test: $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{1}{n} - \frac{1}{n^2}\right)^n} = \lim_{n \rightarrow \infty} \left(\left(\frac{1}{n} - \frac{1}{n^2}\right)^n\right)^{1/n}$
 $= \lim_{n \rightarrow \infty} \left(\frac{1}{n} - \frac{1}{n^2}\right) = 0 < 1$

15. diverges by the Direct Comparison Test: $\frac{\ln n}{n} > \frac{1}{n}$ for $n \geq 3$
16. converges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(n+1) \ln(n+1)}{2^{n+1}} \cdot \frac{2^n}{n \ln(n)} = \frac{1}{2} < 1$
17. converges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(n+2)(n+3)}{(n+1)!} \cdot \frac{n!}{(n+1)(n+2)} = 0 < 1$
18. converges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(n+1)^3}{e^{n+1}} \cdot \frac{e^n}{n^3} = \frac{1}{e} < 1$
19. converges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(n+4)!}{3!(n+1)!3^{n+1}} \cdot \frac{3!n!3^n}{(n+3)!} = \lim_{n \rightarrow \infty} \frac{n+4}{3(n+1)} = \frac{1}{3} < 1$
20. converges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(n+1)2^{n+1}(n+2)!}{3^{n+1}(n+1)!} \cdot \frac{3^n n!}{n2^n(n+1)!}$
 $= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right) \left(\frac{2}{3}\right) \left(\frac{n+2}{n+1}\right) = \frac{2}{3} < 1$
21. converges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(n+1)!}{(2n+3)!} \cdot \frac{(2n+1)!}{n!} = \lim_{n \rightarrow \infty} \frac{n+1}{(2n+3)(2n+2)} = 0 < 1$
22. converges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^n = \lim_{n \rightarrow \infty} \frac{1}{\left(\frac{n+1}{n}\right)^n}$
 $= \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} < 1$
23. converges by the Root Test: $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n}{(\ln n)^n}} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n}}{\ln n} = \lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0 < 1$
24. converges by the Root Test: $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n}{(\ln n)^{n/2}}} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n}}{\sqrt{\ln n}} = \frac{\lim_{n \rightarrow \infty} \sqrt[n]{n}}{\lim_{n \rightarrow \infty} \sqrt{\ln n}} = 0 < 1$
 $\left(\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1\right)$
25. converges by the Direct Comparison Test: $\frac{n! \ln n}{n(n+2)!} = \frac{\ln n}{n(n+1)(n+2)} < \frac{n}{n(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} < \frac{1}{n^2}$
 which is the n th-term of a convergent p -series
26. diverges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{3^{n+1}}{(n+1)^3 2^{n+1}} \cdot \frac{n^3 2^n}{3^n} = \lim_{n \rightarrow \infty} \frac{n^3}{(n+1)^3} \left(\frac{3}{2}\right) = \frac{3}{2} > 1$
27. converges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1+\sin n}{n}\right) a_n}{a_n} = 0 < 1$
28. converges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1+\tan^{-1} n}{n}\right) a_n}{a_n} = \lim_{n \rightarrow \infty} \frac{1+\tan^{-1} n}{n} = 0$ since the numerator approaches $1 + \frac{\pi}{2}$ while the denominator tends to ∞
29. diverges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{3n-1}{2n+1}\right) a_n}{a_n} = \lim_{n \rightarrow \infty} \frac{3n-1}{2n+1} = \frac{3}{2} > 1$
30. diverges; $a_{n+1} = \frac{n}{n+1} a_n \Rightarrow a_{n+1} = \left(\frac{n}{n+1}\right) \left(\frac{n-1}{n}\right) a_{n-1} \Rightarrow a_{n+1} = \left(\frac{n}{n+1}\right) \left(\frac{n-1}{n}\right) \left(\frac{n-2}{n-1}\right) a_{n-2}$
 $\Rightarrow a_{n+1} = \left(\frac{n}{n+1}\right) \left(\frac{n-1}{n}\right) \left(\frac{n-2}{n-1}\right) \cdots \left(\frac{1}{2}\right) a_1 \Rightarrow a_{n+1} = \frac{a_1}{n+1} \Rightarrow a_{n+1} = \frac{3}{n+1}$, which is a constant times the general term of the diverging harmonic series
31. converges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{2}{n}\right) a_n}{a_n} = \lim_{n \rightarrow \infty} \frac{2}{n} = 0 < 1$

32. converges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{n}{2}\right)^{a_n}}{a_n} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n}}{n} = \frac{1}{2} < 1$

33. converges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1+\ln n}{n}\right)^{a_n}}{a_n} = \lim_{n \rightarrow \infty} \frac{1+\ln n}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 < 1$

34. $\frac{n+\ln n}{n+10} > 0$ and $a_1 = \frac{1}{2} \Rightarrow a_n > 0$; $\ln n > 10$ for $n > e^{10} \Rightarrow n + \ln n > n + 10 \Rightarrow \frac{n+\ln n}{n+10} > 1$
 $\Rightarrow a_{n+1} = \frac{n+\ln n}{n+10} a_n > a_n$; thus $a_{n+1} > a_n \geq \frac{1}{2} \Rightarrow \lim_{n \rightarrow \infty} a_n \neq 0$, so the series diverges by the nth-Term Test

35. diverges by the nth-Term Test: $a_1 = \frac{1}{3}$, $a_2 = \sqrt{\frac{1}{3}}$, $a_3 = \sqrt[3]{\sqrt{\frac{1}{3}}} = \sqrt[6]{\frac{1}{3}}$, $a_4 = \sqrt[4]{\sqrt[3]{\sqrt{\frac{1}{3}}}} = \sqrt[4!]{\frac{1}{3}}$, \dots ,
 $a_n = \sqrt[n!]{\frac{1}{3}} \Rightarrow \lim_{n \rightarrow \infty} a_n = 1$ because $\left\{\sqrt[n!]{\frac{1}{3}}\right\}$ is a subsequence of $\left\{\sqrt[n]{\frac{1}{3}}\right\}$ whose limit is 1 by Table 8.1

36. converges by the Direct Comparison Test: $a_1 = \frac{1}{2}$, $a_2 = \left(\frac{1}{2}\right)^2$, $a_3 = \left(\left(\frac{1}{2}\right)^2\right)^3 = \left(\frac{1}{2}\right)^6$, $a_4 = \left(\left(\frac{1}{2}\right)^6\right)^4 = \left(\frac{1}{2}\right)^{24}$, \dots
 $\Rightarrow a_n = \left(\frac{1}{2}\right)^{n!} < \left(\frac{1}{2}\right)^n$ which is the nth-term of a convergent geometric series

37. converges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\frac{2^{n+1}(n+1)!(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{2^n n! n!}}{\frac{2(n+1)(n+1)}{(2n+2)(2n+1)}}$
 $= \lim_{n \rightarrow \infty} \frac{n+1}{2n+1} = \frac{1}{2} < 1$

38. diverges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(3n+3)!}{(n+1)!(n+2)!(n+3)!} \cdot \frac{n!(n+1)!(n+2)!}{(3n)!}$
 $= \lim_{n \rightarrow \infty} \frac{(3n+3)(3+2)(3n+1)}{(n+1)(n+2)(n+3)} = \lim_{n \rightarrow \infty} 3 \left(\frac{3n+1}{n+2}\right) \left(\frac{3n+1}{n+3}\right) = 3 \cdot 3 \cdot 3 = 27 > 1$

39. diverges by the Root Test: $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{(n!)^n}{(n^n)^2}} = \lim_{n \rightarrow \infty} \frac{n!}{n^2} = \infty > 1$

40. converges by the Root Test: $\lim_{n \rightarrow \infty} \sqrt[n]{\frac{(n!)^n}{n^{n^2}}} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{(n!)^n}{(n^n)^n}} = \lim_{n \rightarrow \infty} \frac{n!}{n^n} = \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right) \left(\frac{2}{n}\right) \left(\frac{3}{n}\right) \cdots \left(\frac{n-1}{n}\right) \left(\frac{n}{n}\right)$
 $\leq \lim_{n \rightarrow \infty} \frac{1}{n} = 0 < 1$

41. converges by the Root Test: $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n}{2^{n^2}}} = \lim_{n \rightarrow \infty} \frac{n}{2^n} = \lim_{n \rightarrow \infty} \frac{1}{2^n \ln 2} = 0 < 1$

42. diverges by the Root Test: $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n}{(2^n)^2}} = \lim_{n \rightarrow \infty} \frac{n}{4} = \infty > 1$

43. converges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{1 \cdot 3 \cdots (2n-1)(2n+1)}{4^{n+1} 2^{n+1} (n+1)!} \cdot \frac{4^n 2^n n!}{1 \cdot 3 \cdots (2n-1)}$
 $= \lim_{n \rightarrow \infty} \frac{2n+1}{(4 \cdot 2)(n+1)} = \frac{1}{4} < 1$

44. converges by the Ratio Test: $a_n = \frac{1 \cdot 3 \cdots (2n-1)}{(2 \cdot 4 \cdots 2n)(3^n+1)} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n-1)(2n)}{(2 \cdot 4 \cdots 2n)^2 (3^n+1)} = \frac{(2n)!}{(2^n n!)^2 (3^n+1)}$
 $\Rightarrow \lim_{n \rightarrow \infty} \frac{(2n+2)!}{[2^{n+1}(n+1)!]^2 (3^{n+1}+1)} \cdot \frac{(2^n n!)^2 (3^n+1)}{(2n)!} = \lim_{n \rightarrow \infty} \frac{(2n+1)(2n+2)(3^n+1)}{2^2(n+1)^2 (3^{n+1}+1)}$
 $= \lim_{n \rightarrow \infty} \left(\frac{4n^2+6n+2}{4n^2+8n+4}\right) \left(\frac{1+3^{-n}}{3+3^{-n}}\right) = 1 \cdot \frac{1}{3} = \frac{1}{3} < 1$

45. Ratio: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{1}{(n+1)^p} \cdot \frac{n^p}{1} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^p = 1^p = 1 \Rightarrow$ no conclusion
 Root: $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n^p}} = \lim_{n \rightarrow \infty} \frac{1}{(\sqrt[n]{n})^p} = \frac{1}{(1)^p} = 1 \Rightarrow$ no conclusion

$$46. \text{ Ratio: } \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{1}{(\ln(n+1))^p} \cdot \frac{(\ln n)^p}{1} = \left[\lim_{n \rightarrow \infty} \frac{\ln n}{\ln(n+1)} \right]^p = \left[\lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n+1}\right)} \right]^p = \left(\lim_{n \rightarrow \infty} \frac{n+1}{n} \right)^p = (1)^p = 1 \Rightarrow \text{no conclusion}$$

$$\text{Root: } \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{(\ln n)^p}} = \frac{1}{\left(\lim_{n \rightarrow \infty} (\ln n)^{1/n} \right)^p}; \text{ let } f(n) = (\ln n)^{1/n}, \text{ then } \ln f(n) = \frac{\ln(\ln n)}{n}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \ln f(n) = \lim_{n \rightarrow \infty} \frac{\ln(\ln n)}{n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n \ln n}\right)}{1} = \lim_{n \rightarrow \infty} \frac{1}{n \ln n} = 0 \Rightarrow \lim_{n \rightarrow \infty} (\ln n)^{1/n} = 1$$

$$= \lim_{n \rightarrow \infty} e^{\ln f(n)} = e^0 = 1; \text{ therefore } \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \frac{1}{(1)^p} = \frac{1}{1^p} = 1 \Rightarrow \text{no conclusion}$$

$$47. a_n \leq \frac{n}{2^n} \text{ for every } n \text{ and the series } \sum_{n=1}^{\infty} \frac{n}{2^n} \text{ converges by the Ratio Test since } \lim_{n \rightarrow \infty} \frac{(n+1)}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2} < 1$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges by the Direct Comparison Test}$$

11.6 ALTERNATING SERIES, ABSOLUTE AND CONDITIONAL CONVERGENCE

- converges absolutely \Rightarrow converges by the Absolute Convergence Test since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a convergent p-series
- converges absolutely \Rightarrow converges by the Absolute Convergence Test since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is a convergent p-series
- diverges by the nth-Term Test since for $n > 10 \Rightarrow \frac{n}{10} > 1 \Rightarrow \lim_{n \rightarrow \infty} \left(\frac{n}{10}\right)^n \neq 0 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{10}\right)^n$ diverges
- diverges by the nth-Term Test since $\lim_{n \rightarrow \infty} \frac{10^n}{n^{10}} = \lim_{n \rightarrow \infty} \frac{10^n (\ln 10)^{10}}{10!} = \infty$ (after 10 applications of L'Hôpital's rule)
- converges by the Alternating Series Test because $f(x) = \ln x$ is an increasing function of $x \Rightarrow \frac{1}{\ln x}$ is decreasing $\Rightarrow u_n \geq u_{n+1}$ for $n \geq 1$; also $u_n \geq 0$ for $n \geq 1$ and $\lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0$
- converges by the Alternating Series Test since $f(x) = \frac{\ln x}{x} \Rightarrow f'(x) = \frac{1 - \ln x}{x^2} < 0$ when $x > e \Rightarrow f(x)$ is decreasing $\Rightarrow u_n \geq u_{n+1}$; also $u_n \geq 0$ for $n \geq 1$ and $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n}\right)}{1} = 0$
- diverges by the nth-Term Test since $\lim_{n \rightarrow \infty} \frac{\ln n}{\ln n^2} = \lim_{n \rightarrow \infty} \frac{\ln n}{2 \ln n} = \lim_{n \rightarrow \infty} \frac{1}{2} = \frac{1}{2} \neq 0$
- converges by the Alternating Series Test since $f(x) = \ln(1 + x^{-1}) \Rightarrow f'(x) = \frac{-1}{x(x+1)} < 0$ for $x > 0 \Rightarrow f(x)$ is decreasing $\Rightarrow u_n \geq u_{n+1}$; also $u_n \geq 0$ for $n \geq 1$ and $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \ln\left(1 + \frac{1}{n}\right) = \ln\left(\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)\right) = \ln 1 = 0$
- converges by the Alternating Series Test since $f(x) = \frac{\sqrt{x}+1}{x+1} \Rightarrow f'(x) = \frac{1-x-2\sqrt{x}}{2\sqrt{x}(x+1)^2} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow u_n \geq u_{n+1}$; also $u_n \geq 0$ for $n \geq 1$ and $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{\sqrt{n}+1}{n+1} = 0$

10. diverges by the nth-Term Test since $\lim_{n \rightarrow \infty} \frac{3\sqrt[n]{n+1}}{\sqrt[n]{n+1}} = \lim_{n \rightarrow \infty} \frac{3\sqrt[n]{1+\frac{1}{n}}}{1+\left(\frac{1}{\sqrt[n]{n}}\right)} = 3 \neq 0$
11. converges absolutely since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$ a convergent geometric series
12. converges absolutely by the Direct Comparison Test since $\left| \frac{(-1)^{n+1}(0.1)^n}{n} \right| = \frac{1}{(10)^n n} < \left(\frac{1}{10}\right)^n$ which is the nth term of a convergent geometric series
13. converges conditionally since $\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} > 0$ and $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0 \Rightarrow$ convergence; but $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a divergent p-series
14. converges conditionally since $\frac{1}{1+\sqrt{n}} > \frac{1}{1+\sqrt{n+1}} > 0$ and $\lim_{n \rightarrow \infty} \frac{1}{1+\sqrt{n}} = 0 \Rightarrow$ convergence; but $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ is a divergent series since $\frac{1}{1+\sqrt{n}} \geq \frac{1}{2\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a divergent p-series
15. converges absolutely since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n^5+1}$ and $\frac{n}{n^5+1} < \frac{1}{n^2}$ which is the nth-term of a converging p-series
16. diverges by the nth-Term Test since $\lim_{n \rightarrow \infty} \frac{n!}{2^n} = \infty$
17. converges conditionally since $\frac{1}{n+3} > \frac{1}{(n+1)+3} > 0$ and $\lim_{n \rightarrow \infty} \frac{1}{n+3} = 0 \Rightarrow$ convergence; but $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n+3}$ diverges because $\frac{1}{n+3} \geq \frac{1}{4n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series
18. converges absolutely because the series $\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right|$ converges by the Direct Comparison Test since $\left| \frac{\sin n}{n^2} \right| \leq \frac{1}{n^2}$
19. diverges by the nth-Term Test since $\lim_{n \rightarrow \infty} \frac{3+n}{5+n} = 1 \neq 0$
20. converges conditionally since $f(x) = \ln x$ is an increasing function of $x \Rightarrow \frac{1}{3 \ln x} = \frac{1}{\ln(x^3)}$ is decreasing
 $\Rightarrow \frac{1}{3 \ln n} > \frac{1}{3 \ln(n+1)} > 0$ for $n \geq 2$ and $\lim_{n \rightarrow \infty} \frac{1}{3 \ln n} = 0 \Rightarrow$ convergence; but $\sum_{n=2}^{\infty} |a_n| = \sum_{n=2}^{\infty} \frac{1}{\ln(n^3)}$
 $= \sum_{n=2}^{\infty} \frac{1}{3 \ln n}$ diverges because $\frac{1}{3 \ln n} > \frac{1}{3n}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges
21. converges conditionally since $f(x) = \frac{1}{x^2} + \frac{1}{x} \Rightarrow f'(x) = -\left(\frac{2}{x^3} + \frac{1}{x^2}\right) < 0 \Rightarrow f(x)$ is decreasing and hence
 $u_n > u_{n+1} > 0$ for $n \geq 1$ and $\lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{1}{n}\right) = 0 \Rightarrow$ convergence; but $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1+n}{n^2}$
 $= \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n}$ is the sum of a convergent and divergent series, and hence diverges
22. converges absolutely by the Direct Comparison Test since $\left| \frac{(-2)^{n+1}}{n+5^n} \right| = \frac{2^{n+1}}{n+5^n} < 2 \left(\frac{2}{5}\right)^n$ which is the nth term of a convergent geometric series

23. converges absolutely by the Ratio Test: $\lim_{n \rightarrow \infty} \left(\frac{u_{n+1}}{u_n} \right) = \lim_{n \rightarrow \infty} \left[\frac{(n+1)^2 \left(\frac{2}{3} \right)^{n+1}}{n^2 \left(\frac{2}{3} \right)^n} \right] = \frac{2}{3} < 1$
24. diverges by the nth-Term Test since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} 10^{1/n} = 1 \neq 0$
25. converges absolutely by the Integral Test since $\int_1^\infty (\tan^{-1} x) \left(\frac{1}{1+x^2} \right) dx = \lim_{b \rightarrow \infty} \left[\frac{(\tan^{-1} x)^2}{2} \right]_1^b$
 $= \lim_{b \rightarrow \infty} \left[(\tan^{-1} b)^2 - (\tan^{-1} 1)^2 \right] = \frac{1}{2} \left[\left(\frac{\pi}{2} \right)^2 - \left(\frac{\pi}{4} \right)^2 \right] = \frac{3\pi^2}{32}$
26. converges conditionally since $f(x) = \frac{1}{x \ln x} \Rightarrow f'(x) = -\frac{[\ln(x)+1]}{(x \ln x)^2} < 0 \Rightarrow f(x)$ is decreasing
 $\Rightarrow u_n > u_{n+1} > 0$ for $n \geq 2$ and $\lim_{n \rightarrow \infty} \frac{1}{n \ln n} = 0 \Rightarrow$ convergence; but by the Integral Test,
 $\int_2^\infty \frac{dx}{x \ln x} = \lim_{b \rightarrow \infty} \int_2^b \left(\frac{\frac{1}{x}}{\ln x} \right) dx = \lim_{b \rightarrow \infty} [\ln(\ln x)]_2^b = \lim_{b \rightarrow \infty} [\ln(\ln b) - \ln(\ln 2)] = \infty$
 $\Rightarrow \sum_{n=1}^\infty |a_n| = \sum_{n=1}^\infty \frac{1}{n \ln n}$ diverges
27. diverges by the nth-Term Test since $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$
28. converges conditionally since $f(x) = \frac{\ln x}{x - \ln x} \Rightarrow f'(x) = \frac{\left(\frac{1}{x} \right)(x - \ln x) - (\ln x) \left(1 - \frac{1}{x} \right)}{(x - \ln x)^2}$
 $= \frac{1 - \left(\frac{\ln x}{x} \right) - \ln x + \left(\frac{\ln x}{x} \right)}{(x - \ln x)^2} = \frac{1 - \ln x}{(x - \ln x)^2} < 0 \Rightarrow u_n \geq u_{n+1} > 0$ when $n > e$ and $\lim_{n \rightarrow \infty} \frac{\ln n}{n - \ln n}$
 $= \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n} \right)}{1 - \left(\frac{1}{n} \right)} = 0 \Rightarrow$ convergence; but $n - \ln n < n \Rightarrow \frac{1}{n - \ln n} > \frac{1}{n} \Rightarrow \frac{\ln n}{n - \ln n} > \frac{1}{n}$ so that
 $\sum_{n=1}^\infty |a_n| = \sum_{n=1}^\infty \frac{\ln n}{n - \ln n}$ diverges by the Direct Comparison Test
29. converges absolutely by the Ratio Test: $\lim_{n \rightarrow \infty} \left(\frac{u_{n+1}}{u_n} \right) = \lim_{n \rightarrow \infty} \frac{(100)^{n+1}}{(n+1)!} \cdot \frac{n!}{(100)^n} = \lim_{n \rightarrow \infty} \frac{100}{n+1} = 0 < 1$
30. converges absolutely since $\sum_{n=1}^\infty |a_n| = \sum_{n=1}^\infty \left(\frac{1}{5} \right)^n$ is a convergent geometric series
31. converges absolutely by the Direct Comparison Test since $\sum_{n=1}^\infty |a_n| = \sum_{n=1}^\infty \frac{1}{n^2 + 2n + 1}$ and
 $\frac{1}{n^2 + 2n + 1} < \frac{1}{n^2}$ which is the nth-term of a convergent p-series
32. converges absolutely since $\sum_{n=1}^\infty |a_n| = \sum_{n=1}^\infty \left(\frac{\ln n}{\ln n^2} \right)^n = \sum_{n=1}^\infty \left(\frac{\ln n}{2 \ln n} \right)^n = \sum_{n=1}^\infty \left(\frac{1}{2} \right)^n$ is a convergent geometric series
33. converges absolutely since $\sum_{n=1}^\infty |a_n| = \sum_{n=1}^\infty \left| \frac{(-1)^n}{n\sqrt{n}} \right| = \sum_{n=1}^\infty \frac{1}{n^{3/2}}$ is a convergent p-series
34. converges conditionally since $\sum_{n=1}^\infty \frac{\cos n\pi}{n} = \sum_{n=1}^\infty \frac{(-1)^n}{n}$ is the convergent alternating harmonic series, but
 $\sum_{n=1}^\infty |a_n| = \sum_{n=1}^\infty \frac{1}{n}$ diverges

35. converges absolutely by the Root Test: $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \left(\frac{(n+1)^n}{(2n)^n} \right)^{1/n} = \lim_{n \rightarrow \infty} \frac{n+1}{2n} = \frac{1}{2} < 1$

36. converges absolutely by the Ratio Test: $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$

37. diverges by the nth-Term Test since $\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{(2n)!}{2^n n! n} = \lim_{n \rightarrow \infty} \frac{(n+1)(n+2) \cdots (2n)}{2^n} = \lim_{n \rightarrow \infty} \frac{(n+1)(n+2) \cdots (n+(n-1))}{2^{n-1}} > \lim_{n \rightarrow \infty} \left(\frac{n+1}{2} \right)^{n-1} = \infty \neq 0$

38. converges absolutely by the Ratio Test: $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)!(n+1)! 3^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{n! n! 3^n} = \lim_{n \rightarrow \infty} \frac{(n+1)^2 3}{(2n+2)(2n+3)} = \frac{3}{4} < 1$

39. converges conditionally since $\frac{\sqrt{n+1}-\sqrt{n}}{1} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$ and $\left\{ \frac{1}{\sqrt{n+1}+\sqrt{n}} \right\}$ is a decreasing sequence of positive terms which converges to 0 $\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$ converges; but

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ diverges by the Limit Comparison Test (part 1) with $\frac{1}{\sqrt{n}}$; a divergent p-series:
 $\lim_{n \rightarrow \infty} \left(\frac{\frac{1}{\sqrt{n+1}+\sqrt{n}}}{\frac{1}{\sqrt{n}}} \right) = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}$

40. diverges by the nth-Term Test since $\lim_{n \rightarrow \infty} \left(\sqrt{n^2+n} - n \right) = \lim_{n \rightarrow \infty} \left(\sqrt{n^2+n} - n \right) \cdot \left(\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} \right) = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+n}+n} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2} \neq 0$

41. diverges by the nth-Term Test since $\lim_{n \rightarrow \infty} \left(\sqrt{n+\sqrt{n}} - \sqrt{n} \right) = \lim_{n \rightarrow \infty} \left[\left(\sqrt{n+\sqrt{n}} - \sqrt{n} \right) \left(\frac{\sqrt{n+\sqrt{n}}+\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n}} \right) \right] = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+\frac{1}{\sqrt{n}}}+1} = \frac{1}{2} \neq 0$

42. converges conditionally since $\left\{ \frac{1}{\sqrt{n}+\sqrt{n+1}} \right\}$ is a decreasing sequence of positive terms converging to 0

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$ converges; but $\lim_{n \rightarrow \infty} \frac{\left(\frac{1}{\sqrt{n}+\sqrt{n+1}} \right)}{\left(\frac{1}{\sqrt{n}} \right)} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n+1}} = \lim_{n \rightarrow \infty} \frac{1}{1+\sqrt{1+\frac{1}{n}}} = \frac{1}{2}$

so that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$ diverges by the Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is a divergent p-series

43. converges absolutely by the Direct Comparison Test since $\operatorname{sech}(n) = \frac{2}{e^n + e^{-n}} = \frac{2e^n}{e^{2n} + 1} < \frac{2e^n}{e^{2n}} = \frac{2}{e^n}$ which is the nth term of a convergent geometric series

44. converges absolutely by the Limit Comparison Test (part 1): $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{2}{e^n - e^{-n}}$

Apply the Limit Comparison Test with $\frac{1}{e^n}$, the n-th term of a convergent geometric series:

$\lim_{n \rightarrow \infty} \left(\frac{\frac{2}{e^n - e^{-n}}}{\frac{1}{e^n}} \right) = \lim_{n \rightarrow \infty} \frac{2e^n}{e^n - e^{-n}} = \lim_{n \rightarrow \infty} \frac{2}{1 - e^{-2n}} = 2$

45. $|\text{error}| < \left| (-1)^6 \left(\frac{1}{5} \right) \right| = 0.2$

46. $|\text{error}| < \left| (-1)^6 \left(\frac{1}{10^5} \right) \right| = 0.00001$

$$47. |\text{error}| < \left| (-1)^6 \frac{(0.01)^5}{5} \right| = 2 \times 10^{-11}$$

$$48. |\text{error}| < |(-1)^4 t^4| = t^4 < 1$$

$$49. \frac{1}{(2n)!} < \frac{5}{10^6} \Rightarrow (2n)! > \frac{10^6}{5} = 200,000 \Rightarrow n \geq 5 \Rightarrow 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} \approx 0.54030$$

$$50. \frac{1}{n!} < \frac{5}{10^6} \Rightarrow \frac{10^6}{5} < n! \Rightarrow n \geq 9 \Rightarrow 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} \approx 0.367881944$$

$$51. (a) a_n \geq a_{n+1} \text{ fails since } \frac{1}{3} < \frac{1}{2}$$

(b) Since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left[\left(\frac{1}{3}\right)^n + \left(\frac{1}{2}\right)^n \right] = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is the sum of two absolutely convergent series, we can rearrange the terms of the original series to find its sum:

$$\left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right) = \frac{\left(\frac{1}{3}\right)}{1 - \left(\frac{1}{3}\right)} - \frac{\left(\frac{1}{2}\right)}{1 - \left(\frac{1}{2}\right)} = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$52. s_{20} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{19} - \frac{1}{20} \approx 0.6687714032 \Rightarrow s_{20} + \frac{1}{2} \cdot \frac{1}{21} \approx 0.692580927$$

53. The unused terms are $\sum_{j=n+1}^{\infty} (-1)^{j+1} a_j = (-1)^{n+1} (a_{n+1} - a_{n+2}) + (-1)^{n+3} (a_{n+3} - a_{n+4}) + \dots$
 $= (-1)^{n+1} [(a_{n+1} - a_{n+2}) + (a_{n+3} - a_{n+4}) + \dots]$. Each grouped term is positive, so the remainder has the same sign as $(-1)^{n+1}$, which is the sign of the first unused term.

$$54. s_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} = \sum_{k=1}^n \frac{1}{k(k+1)} = \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$ which are the first $2n$ terms of the first series, hence the two series are the same. Yes, for

$$s_n = \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

$\Rightarrow \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right) = 1 \Rightarrow$ both series converge to 1. The sum of the first $2n+1$ terms of the first series is $\left(1 - \frac{1}{n+1}\right) + \frac{1}{n+1} = 1$. Their sum is $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right) = 1$.

55. Theorem 16 states that $\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges. But this is equivalent to $\sum_{n=1}^{\infty} a_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} |a_n|$ diverges.

56. $|a_1 + a_2 + \dots + a_n| \leq |a_1| + |a_2| + \dots + |a_n|$ for all n ; then $\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges and these

$$\text{imply that } \left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n|$$

57. (a) $\sum_{n=1}^{\infty} |a_n + b_n|$ converges by the Direct Comparison Test since $|a_n + b_n| \leq |a_n| + |b_n|$ and hence

$$\sum_{n=1}^{\infty} (a_n + b_n) \text{ converges absolutely}$$

(b) $\sum_{n=1}^{\infty} |b_n|$ converges $\Rightarrow \sum_{n=1}^{\infty} -b_n$ converges absolutely; since $\sum_{n=1}^{\infty} a_n$ converges absolutely and

$$\sum_{n=1}^{\infty} -b_n \text{ converges absolutely, we have } \sum_{n=1}^{\infty} [a_n + (-b_n)] = \sum_{n=1}^{\infty} (a_n - b_n) \text{ converges absolutely by part (a)}$$

(c) $\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow |k| \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} |ka_n|$ converges $\Rightarrow \sum_{n=1}^{\infty} ka_n$ converges absolutely

58. If $a_n = b_n = (-1)^n \frac{1}{\sqrt{n}}$, then $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges, but $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

59. $s_1 = -\frac{1}{2}$, $s_2 = -\frac{1}{2} + 1 = \frac{1}{2}$,

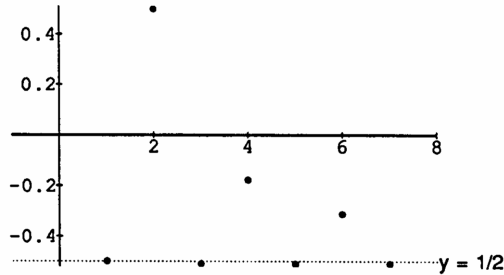
$$s_3 = -\frac{1}{2} + 1 - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16} - \frac{1}{18} - \frac{1}{20} - \frac{1}{22} \approx -0.5099,$$

$$s_4 = s_3 + \frac{1}{3} \approx -0.1766,$$

$$s_5 = s_4 - \frac{1}{24} - \frac{1}{26} - \frac{1}{28} - \frac{1}{30} - \frac{1}{32} - \frac{1}{34} - \frac{1}{36} - \frac{1}{38} - \frac{1}{40} - \frac{1}{42} - \frac{1}{44} \approx -0.512,$$

$$s_6 = s_5 + \frac{1}{5} \approx -0.312,$$

$$s_7 = s_6 - \frac{1}{46} - \frac{1}{48} - \frac{1}{50} - \frac{1}{52} - \frac{1}{54} - \frac{1}{56} - \frac{1}{58} - \frac{1}{60} - \frac{1}{62} - \frac{1}{64} - \frac{1}{66} \approx -0.51106$$



60. (a) Since $\sum |a_n|$ converges, say to M , for $\epsilon > 0$ there is an integer N_1 such that $\left| \sum_{n=1}^{N_1-1} |a_n| - M \right| < \frac{\epsilon}{2}$

$$\Leftrightarrow \left| \sum_{n=1}^{N_1-1} |a_n| - \left(\sum_{n=1}^{N_1-1} |a_n| + \sum_{n=N_1}^{\infty} |a_n| \right) \right| < \frac{\epsilon}{2} \Leftrightarrow \left| - \sum_{n=N_1}^{\infty} |a_n| \right| < \frac{\epsilon}{2} \Leftrightarrow \sum_{n=N_1}^{\infty} |a_n| < \frac{\epsilon}{2}. \text{ Also, } \sum a_n$$

converges to $L \Leftrightarrow$ for $\epsilon > 0$ there is an integer N_2 (which we can choose greater than or equal to N_1) such that $|s_{N_2} - L| < \frac{\epsilon}{2}$. Therefore, $\sum_{n=N_1}^{\infty} |a_n| < \frac{\epsilon}{2}$ and $|s_{N_2} - L| < \frac{\epsilon}{2}$.

(b) The series $\sum_{n=1}^{\infty} |a_n|$ converges absolutely, say to M . Thus, there exists N_1 such that $\left| \sum_{n=1}^k |a_n| - M \right| < \epsilon$ whenever $k > N_1$. Now all of the terms in the sequence $\{|b_n|\}$ appear in $\{|a_n|\}$. Sum together all of the terms in $\{|b_n|\}$, in order, until you include all of the terms $\{|a_n|\}_{n=1}^{N_1}$, and let N_2 be the largest index in the sum $\sum_{n=1}^{N_2} |b_n|$ so obtained. Then $\left| \sum_{n=1}^{N_2} |b_n| - M \right| < \epsilon$ as well $\Rightarrow \sum_{n=1}^{\infty} |b_n|$ converges to M .

61. (a) If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges and $\frac{1}{2} \sum_{n=1}^{\infty} a_n + \frac{1}{2} \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{a_n + |a_n|}{2}$

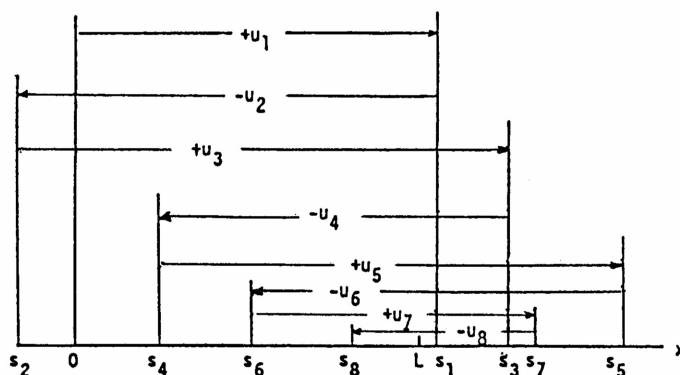
$$\text{converges where } b_n = \frac{a_n + |a_n|}{2} = \begin{cases} a_n, & \text{if } a_n \geq 0 \\ 0, & \text{if } a_n < 0 \end{cases}.$$

(b) If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges and $\frac{1}{2} \sum_{n=1}^{\infty} a_n - \frac{1}{2} \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{a_n - |a_n|}{2}$

$$\text{converges where } c_n = \frac{a_n - |a_n|}{2} = \begin{cases} 0, & \text{if } a_n \geq 0 \\ a_n, & \text{if } a_n < 0 \end{cases}.$$

62. The terms in this conditionally convergent series were not added in the order given.

63. Here is an example figure when $N = 5$. Notice that $u_3 > u_2 > u_1$ and $u_3 > u_5 > u_4$, but $u_n \geq u_{n+1}$ for $n \geq 5$.



11.7 POWER SERIES

- $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{x^n} \right| < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} (-1)^n$, a divergent series; when $x = 1$ we have $\sum_{n=1}^{\infty} 1$, a divergent series
 - the radius is 1; the interval of convergence is $-1 < x < 1$
 - the interval of absolute convergence is $-1 < x < 1$
 - there are no values for which the series converges conditionally
- $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(x+5)^{n+1}}{(x+5)^n} \right| < 1 \Rightarrow |x+5| < 1 \Rightarrow -6 < x < -4$; when $x = -6$ we have $\sum_{n=1}^{\infty} (-1)^n$, a divergent series; when $x = -4$ we have $\sum_{n=1}^{\infty} 1$, a divergent series
 - the radius is 1; the interval of convergence is $-6 < x < -4$
 - the interval of absolute convergence is $-6 < x < -4$
 - there are no values for which the series converges conditionally
- $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(4x+1)^{n+1}}{(4x+1)^n} \right| < 1 \Rightarrow |4x+1| < 1 \Rightarrow -1 < 4x+1 < 1 \Rightarrow -\frac{1}{2} < x < 0$; when $x = -\frac{1}{2}$ we have $\sum_{n=1}^{\infty} (-1)^n(-1)^n = \sum_{n=1}^{\infty} (-1)^{2n} = \sum_{n=1}^{\infty} 1^n$, a divergent series; when $x = 0$ we have $\sum_{n=1}^{\infty} (-1)^n(1)^n = \sum_{n=1}^{\infty} (-1)^n$, a divergent series
 - the radius is $\frac{1}{4}$; the interval of convergence is $-\frac{1}{2} < x < 0$
 - the interval of absolute convergence is $-\frac{1}{2} < x < 0$
 - there are no values for which the series converges conditionally
- $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(3x-2)^{n+1}}{n+1} \cdot \frac{n}{(3x-2)^n} \right| < 1 \Rightarrow |3x-2| \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right) < 1 \Rightarrow |3x-2| < 1$
 $\Rightarrow -1 < 3x-2 < 1 \Rightarrow \frac{1}{3} < x < 1$; when $x = \frac{1}{3}$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is the alternating harmonic series and is conditionally convergent; when $x = 1$ we have $\sum_{n=1}^{\infty} \frac{1}{n}$, the divergent harmonic series
 - the radius is $\frac{1}{3}$; the interval of convergence is $\frac{1}{3} \leq x < 1$
 - the interval of absolute convergence is $\frac{1}{3} < x < 1$

(c) the series converges conditionally at $x = \frac{1}{3}$

5. $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{10^{n+1}} \cdot \frac{10^n}{(x-2)^n} \right| < 1 \Rightarrow \frac{|x-2|}{10} < 1 \Rightarrow |x-2| < 10 \Rightarrow -10 < x-2 < 10$
 $\Rightarrow -8 < x < 12$; when $x = -8$ we have $\sum_{n=1}^{\infty} (-1)^n$, a divergent series; when $x = 12$ we have $\sum_{n=1}^{\infty} 1$, a divergent series

- (a) the radius is 10; the interval of convergence is $-8 < x < 12$
 (b) the interval of absolute convergence is $-8 < x < 12$
 (c) there are no values for which the series converges conditionally

6. $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(2x)^{n+1}}{(2x)^n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} |2x| < 1 \Rightarrow |2x| < 1 \Rightarrow -\frac{1}{2} < x < \frac{1}{2}$; when $x = -\frac{1}{2}$ we have $\sum_{n=1}^{\infty} (-1)^n$, a divergent series; when $x = \frac{1}{2}$ we have $\sum_{n=1}^{\infty} 1$, a divergent series

- (a) the radius is $\frac{1}{2}$; the interval of convergence is $-\frac{1}{2} < x < \frac{1}{2}$
 (b) the interval of absolute convergence is $-\frac{1}{2} < x < \frac{1}{2}$
 (c) there are no values for which the series converges conditionally

7. $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(n+1)x^{n+1}}{(n+3)} \cdot \frac{(n+2)}{nx^n} \right| < 1 \Rightarrow |x| \lim_{n \rightarrow \infty} \frac{(n+1)(n+2)}{(n+3)(n)} < 1 \Rightarrow |x| < 1$
 $\Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$, a divergent series by the n th-term Test; when $x = 1$ we

have $\sum_{n=1}^{\infty} \frac{n}{n+2}$, a divergent series

- (a) the radius is 1; the interval of convergence is $-1 < x < 1$
 (b) the interval of absolute convergence is $-1 < x < 1$
 (c) there are no values for which the series converges conditionally

8. $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(x+2)^{n+1}}{n+1} \cdot \frac{n}{(x+2)^n} \right| < 1 \Rightarrow |x+2| \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right) < 1 \Rightarrow |x+2| < 1$
 $\Rightarrow -1 < x+2 < 1 \Rightarrow -3 < x < -1$; when $x = -3$ we have $\sum_{n=1}^{\infty} \frac{1}{n}$, a divergent series; when $x = -1$ we have

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, a convergent series

- (a) the radius is 1; the interval of convergence is $-3 < x \leq -1$
 (b) the interval of absolute convergence is $-3 < x < -1$
 (c) the series converges conditionally at $x = -1$

9. $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)\sqrt{n+1} 3^{n+1}} \cdot \frac{n\sqrt{n} 3^n}{x^n} \right| < 1 \Rightarrow \frac{|x|}{3} \left(\lim_{n \rightarrow \infty} \frac{n}{n+1} \right) \left(\sqrt{\lim_{n \rightarrow \infty} \frac{n}{n+1}} \right) < 1$
 $\Rightarrow \frac{|x|}{3} (1)(1) < 1 \Rightarrow |x| < 3 \Rightarrow -3 < x < 3$; when $x = -3$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$, an absolutely convergent series;

when $x = 3$ we have $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, a convergent p -series

- (a) the radius is 3; the interval of convergence is $-3 \leq x \leq 3$
 (b) the interval of absolute convergence is $-3 < x < 3$
 (c) there are no values for which the series converges conditionally

10. $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(x-1)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(x-1)^n} \right| < 1 \Rightarrow |x-1| \sqrt{\lim_{n \rightarrow \infty} \frac{n}{n+1}} < 1 \Rightarrow |x-1| < 1$

$\Rightarrow -1 < x - 1 < 1 \Rightarrow 0 < x < 2$; when $x = 0$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}$, a conditionally convergent series; when $x = 2$

we have $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, a divergent series

- (a) the radius is 1; the interval of convergence is $0 \leq x < 2$
- (b) the interval of absolute convergence is $0 < x < 2$
- (c) the series converges conditionally at $x = 0$

$$11. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| < 1 \Rightarrow |x| \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} \right) < 1 \text{ for all } x$$

- (a) the radius is ∞ ; the series converges for all x
- (b) the series converges absolutely for all x
- (c) there are no values for which the series converges conditionally

$$12. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{3^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n x^n} \right| < 1 \Rightarrow 3|x| \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} \right) < 1 \text{ for all } x$$

- (a) the radius is ∞ ; the series converges for all x
- (b) the series converges absolutely for all x
- (c) there are no values for which the series converges conditionally

$$13. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{x^{2n+3}}{(n+1)!} \cdot \frac{n!}{x^{2n+1}} \right| < 1 \Rightarrow x^2 \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} \right) < 1 \text{ for all } x$$

- (a) the radius is ∞ ; the series converges for all x
- (b) the series converges absolutely for all x
- (c) there are no values for which the series converges conditionally

$$14. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(2x+3)^{2n+3}}{(n+1)!} \cdot \frac{n!}{(2x+3)^{2n+1}} \right| < 1 \Rightarrow (2x+3)^2 \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} \right) < 1 \text{ for all } x$$

- (a) the radius is ∞ ; the series converges for all x
- (b) the series converges absolutely for all x
- (c) there are no values for which the series converges conditionally

$$15. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{\sqrt{(n+1)^2+3}} \cdot \frac{\sqrt{n^2+3}}{x^n} \right| < 1 \Rightarrow |x| \sqrt{\lim_{n \rightarrow \infty} \frac{n^2+3}{n^2+2n+4}} < 1 \Rightarrow |x| < 1$$

$\Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+3}}$, a conditionally convergent series; when $x = 1$ we have

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3}}$, a divergent series

- (a) the radius is 1; the interval of convergence is $-1 \leq x < 1$
- (b) the interval of absolute convergence is $-1 < x < 1$
- (c) the series converges conditionally at $x = -1$

$$16. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{\sqrt{(n+1)^2+3}} \cdot \frac{\sqrt{n^2+3}}{x^n} \right| < 1 \Rightarrow |x| \sqrt{\lim_{n \rightarrow \infty} \frac{n^2+3}{n^2+2n+4}} < 1 \Rightarrow |x| < 1$$

$\Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3}}$, a divergent series; when $x = 1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+3}}$,

a conditionally convergent series

- (a) the radius is 1; the interval of convergence is $-1 < x \leq 1$
- (b) the interval of absolute convergence is $-1 < x < 1$
- (c) the series converges conditionally at $x = 1$

$$17. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x+3)^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n(x+3)^n} \right| < 1 \Rightarrow \frac{|x+3|}{5} \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right) < 1 \Rightarrow \frac{|x+3|}{5} < 1$$

$$\Rightarrow |x+3| < 5 \Rightarrow -5 < x+3 < 5 \Rightarrow -8 < x < 2; \text{ when } x = -8 \text{ we have } \sum_{n=1}^{\infty} \frac{n(-5)^n}{5^n} = \sum_{n=1}^{\infty} (-1)^n n, \text{ a divergent}$$

series; when $x = 2$ we have $\sum_{n=1}^{\infty} \frac{n5^n}{5^n} = \sum_{n=1}^{\infty} n$, a divergent series

- (a) the radius is 5; the interval of convergence is $-8 < x < 2$
- (b) the interval of absolute convergence is $-8 < x < 2$
- (c) there are no values for which the series converges conditionally

$$18. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(n+1)x^{n+1}}{4^{n+1}(n^2+2n+2)} \cdot \frac{4^n(n^2+1)}{nx^n} \right| < 1 \Rightarrow \frac{|x|}{4} \lim_{n \rightarrow \infty} \left| \frac{(n+1)(n^2+1)}{n(n^2+2n+2)} \right| < 1 \Rightarrow |x| < 4$$

$$\Rightarrow -4 < x < 4; \text{ when } x = -4 \text{ we have } \sum_{n=1}^{\infty} \frac{n(-1)^n}{n^2+1}, \text{ a conditionally convergent series; when } x = 4 \text{ we have}$$

$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$, a divergent series

- (a) the radius is 4; the interval of convergence is $-4 \leq x < 4$
- (b) the interval of absolute convergence is $-4 < x < 4$
- (c) the series converges conditionally at $x = -4$

$$19. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{\sqrt{n+1}x^{n+1}}{3^{n+1}} \cdot \frac{3^n}{\sqrt{n}x^n} \right| < 1 \Rightarrow \frac{|x|}{3} \sqrt{n} \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right) < 1 \Rightarrow \frac{|x|}{3} < 1 \Rightarrow |x| < 3$$

$$\Rightarrow -3 < x < 3; \text{ when } x = -3 \text{ we have } \sum_{n=1}^{\infty} (-1)^n \sqrt{n}, \text{ a divergent series; when } x = 3 \text{ we have}$$

$\sum_{n=1}^{\infty} \sqrt{n}$, a divergent series

- (a) the radius is 3; the interval of convergence is $-3 < x < 3$
- (b) the interval of absolute convergence is $-3 < x < 3$
- (c) there are no values for which the series converges conditionally

$$20. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{\sqrt[n+1]{n+1} (2x+5)^{n+1}}{\sqrt[n]{n} (2x+5)^n} \right| < 1 \Rightarrow |2x+5| \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}} \right) < 1$$

$$\Rightarrow |2x+5| \left(\frac{\lim_{n \rightarrow \infty} \sqrt[n]{n}}{\lim_{n \rightarrow \infty} \sqrt[n]{n}} \right) < 1 \Rightarrow |2x+5| < 1 \Rightarrow -1 < 2x+5 < 1 \Rightarrow -3 < x < -2; \text{ when } x = -3 \text{ we have}$$

$\sum_{n=1}^{\infty} (-1)^n \sqrt[n]{n}$, a divergent series since $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$; when $x = -2$ we have $\sum_{n=1}^{\infty} \sqrt[n]{n}$, a divergent series

- (a) the radius is $\frac{1}{2}$; the interval of convergence is $-3 < x < -2$
- (b) the interval of absolute convergence is $-3 < x < -2$
- (c) there are no values for which the series converges conditionally

$$21. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{\left(1 + \frac{1}{n+1}\right)^{n+1} x^{n+1}}{\left(1 + \frac{1}{n}\right)^n x^n} \right| < 1 \Rightarrow |x| \left(\frac{\lim_{t \rightarrow \infty} \left(1 + \frac{1}{t}\right)^t}{\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n} \right) < 1 \Rightarrow |x| \left(\frac{e}{e} \right) < 1 \Rightarrow |x| < 1$$

$$\Rightarrow -1 < x < 1; \text{ when } x = -1 \text{ we have } \sum_{n=1}^{\infty} (-1)^n \left(1 + \frac{1}{n}\right)^n, \text{ a divergent series by the } n\text{th-Term Test since}$$

$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \neq 0$; when $x = 1$ we have $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$, a divergent series

- (a) the radius is 1; the interval of convergence is $-1 < x < 1$
- (b) the interval of absolute convergence is $-1 < x < 1$
- (c) there are no values for which the series converges conditionally

22. $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{\ln(n+1)x^{n+1}}{x^n \ln n} \right| < 1 \Rightarrow |x| \lim_{n \rightarrow \infty} \left| \frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)} \right| < 1 \Rightarrow |x| \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right) < 1 \Rightarrow |x| < 1$
 $\Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} (-1)^n \ln n$, a divergent series by the nth-Term Test since
 $\lim_{n \rightarrow \infty} \ln n \neq 0$; when $x = 1$ we have $\sum_{n=1}^{\infty} \ln n$, a divergent series
 (a) the radius is 1; the interval of convergence is $-1 < x < 1$
 (b) the interval of absolute convergence is $-1 < x < 1$
 (c) there are no values for which the series converges conditionally
23. $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(n+1)^{n+1} x^{n+1}}{n^n x^n} \right| < 1 \Rightarrow |x| \left(\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \right) \left(\lim_{n \rightarrow \infty} (n+1) \right) < 1$
 $\Rightarrow e|x| \lim_{n \rightarrow \infty} (n+1) < 1 \Rightarrow$ only $x = 0$ satisfies this inequality
 (a) the radius is 0; the series converges only for $x = 0$
 (b) the series converges absolutely only for $x = 0$
 (c) there are no values for which the series converges conditionally
24. $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(n+1)!(x-4)^{n+1}}{n!(x-4)^n} \right| < 1 \Rightarrow |x-4| \lim_{n \rightarrow \infty} (n+1) < 1 \Rightarrow$ only $x = 4$ satisfies this inequality
 (a) the radius is 0; the series converges only for $x = 4$
 (b) the series converges absolutely only for $x = 4$
 (c) there are no values for which the series converges conditionally
25. $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(x+2)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{(x+2)^n} \right| < 1 \Rightarrow \frac{|x+2|}{2} \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right) < 1 \Rightarrow \frac{|x+2|}{2} < 1 \Rightarrow |x+2| < 2$
 $\Rightarrow -2 < x+2 < 2 \Rightarrow -4 < x < 0$; when $x = -4$ we have $\sum_{n=1}^{\infty} \frac{-1}{n}$, a divergent series; when $x = 0$ we have
 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, the alternating harmonic series which converges conditionally
 (a) the radius is 2; the interval of convergence is $-4 < x \leq 0$
 (b) the interval of absolute convergence is $-4 < x < 0$
 (c) the series converges conditionally at $x = 0$
26. $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(-2)^{n+1}(n+2)(x-1)^{n+1}}{(-2)^n(n+1)(x-1)^n} \right| < 1 \Rightarrow 2|x-1| \lim_{n \rightarrow \infty} \left(\frac{n+2}{n+1}\right) < 1 \Rightarrow 2|x-1| < 1$
 $\Rightarrow |x-1| < \frac{1}{2} \Rightarrow -\frac{1}{2} < x-1 < \frac{1}{2} \Rightarrow \frac{1}{2} < x < \frac{3}{2}$; when $x = \frac{1}{2}$ we have $\sum_{n=1}^{\infty} (n+1)$, a divergent series; when $x = \frac{3}{2}$
 we have $\sum_{n=1}^{\infty} (-1)^n(n+1)$, a divergent series
 (a) the radius is $\frac{1}{2}$; the interval of convergence is $\frac{1}{2} < x < \frac{3}{2}$
 (b) the interval of absolute convergence is $\frac{1}{2} < x < \frac{3}{2}$
 (c) there are no values for which the series converges conditionally
27. $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)(\ln(n+1))^2} \cdot \frac{n(\ln n)^2}{x^n} \right| < 1 \Rightarrow |x| \left(\lim_{n \rightarrow \infty} \frac{n}{n+1} \right) \left(\lim_{n \rightarrow \infty} \frac{\ln n}{\ln(n+1)} \right)^2 < 1$
 $\Rightarrow |x| \left(1\right) \left(\lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n+1}\right)}\right)^2 < 1 \Rightarrow |x| \left(\lim_{n \rightarrow \infty} \frac{n+1}{n}\right)^2 < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$; when $x = -1$ we have
 $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$ which converges absolutely; when $x = 1$ we have $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^2}$ which converges
 (a) the radius is 1; the interval of convergence is $-1 \leq x \leq 1$

- (b) the interval of absolute convergence is $-1 \leq x \leq 1$
 (c) there are no values for which the series converges conditionally

$$28. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1) \ln(n+1)} \cdot \frac{n \ln(n)}{x^n} \right| < 1 \Rightarrow |x| \left(\lim_{n \rightarrow \infty} \frac{n}{n+1} \right) \left(\lim_{n \rightarrow \infty} \frac{\ln(n)}{\ln(n+1)} \right) < 1$$

$$\Rightarrow |x|(1)(1) < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1; \text{ when } x = -1 \text{ we have } \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}, \text{ a convergent alternating series;}$$

when $x = 1$ we have $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ which diverges by Exercise 38, Section 11.3

- (a) the radius is 1; the interval of convergence is $-1 \leq x < 1$
 (b) the interval of absolute convergence is $-1 < x < 1$
 (c) the series converges conditionally at $x = -1$

$$29. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(4x-5)^{2n+3}}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{(4x-5)^{2n+1}} \right| < 1 \Rightarrow (4x-5)^2 \left(\lim_{n \rightarrow \infty} \frac{n}{n+1} \right)^{3/2} < 1 \Rightarrow (4x-5)^2 < 1$$

$$\Rightarrow |4x-5| < 1 \Rightarrow -1 < 4x-5 < 1 \Rightarrow 1 < x < \frac{3}{2}; \text{ when } x = 1 \text{ we have } \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n^{3/2}} = \sum_{n=1}^{\infty} \frac{-1}{n^{3/2}} \text{ which is}$$

absolutely convergent; when $x = \frac{3}{2}$ we have $\sum_{n=1}^{\infty} \frac{(1)^{2n+1}}{n^{3/2}}$, a convergent p-series

- (a) the radius is $\frac{1}{4}$; the interval of convergence is $1 \leq x \leq \frac{3}{2}$
 (b) the interval of absolute convergence is $1 \leq x \leq \frac{3}{2}$
 (c) there are no values for which the series converges conditionally

$$30. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(3x+1)^{n+2}}{2n+4} \cdot \frac{2n+2}{(3x+1)^{n+1}} \right| < 1 \Rightarrow |3x+1| \lim_{n \rightarrow \infty} \left(\frac{2n+2}{2n+4} \right) < 1 \Rightarrow |3x+1| < 1$$

$$\Rightarrow -1 < 3x+1 < 1 \Rightarrow -\frac{2}{3} < x < 0; \text{ when } x = -\frac{2}{3} \text{ we have } \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}, \text{ a conditionally convergent series;}$$

when $x = 0$ we have $\sum_{n=1}^{\infty} \frac{(1)^{n+1}}{2n+1} = \sum_{n=1}^{\infty} \frac{1}{2n+1}$, a divergent series

- (a) the radius is $\frac{1}{3}$; the interval of convergence is $-\frac{2}{3} \leq x < 0$
 (b) the interval of absolute convergence is $-\frac{2}{3} < x < 0$
 (c) the series converges conditionally at $x = -\frac{2}{3}$

$$31. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(x+\pi)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(x+\pi)^n} \right| < 1 \Rightarrow |x+\pi| \lim_{n \rightarrow \infty} \left| \sqrt{\frac{n}{n+1}} \right| < 1$$

$$\Rightarrow |x+\pi| \sqrt{\lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)} < 1 \Rightarrow |x+\pi| < 1 \Rightarrow -1 < x+\pi < 1 \Rightarrow -1-\pi < x < 1-\pi;$$

$$\text{when } x = -1-\pi \text{ we have } \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}, \text{ a conditionally convergent series; when } x = 1-\pi \text{ we have}$$

$\sum_{n=1}^{\infty} \frac{1^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, a divergent p-series

- (a) the radius is 1; the interval of convergence is $(-1-\pi) \leq x < (1-\pi)$
 (b) the interval of absolute convergence is $-1-\pi < x < 1-\pi$
 (c) the series converges conditionally at $x = -1-\pi$

$$32. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(x-\sqrt{2})^{2n+3}}{2^{n+1}} \cdot \frac{2^n}{(x-\sqrt{2})^{2n+1}} \right| < 1 \Rightarrow \frac{(x-\sqrt{2})^2}{2} \lim_{n \rightarrow \infty} |1| < 1$$

$$\Rightarrow \frac{(x-\sqrt{2})^2}{2} < 1 \Rightarrow (x-\sqrt{2})^2 < 2 \Rightarrow |x-\sqrt{2}| < \sqrt{2} \Rightarrow -\sqrt{2} < x-\sqrt{2} < \sqrt{2} \Rightarrow 0 < x < 2\sqrt{2}; \text{ when}$$

$$x = 0 \text{ we have } \sum_{n=1}^{\infty} \frac{(-\sqrt{2})^{2n+1}}{2^n} = -\sum_{n=1}^{\infty} \frac{2^{n+1/2}}{2^n} = -\sum_{n=1}^{\infty} \sqrt{2} \text{ which diverges since } \lim_{n \rightarrow \infty} a_n \neq 0; \text{ when } x = 2\sqrt{2}$$

we have $\sum_{n=1}^{\infty} \frac{(\sqrt{2})^{2n+1}}{2^n} = \sum_{n=1}^{\infty} \frac{2^{n+1/2}}{2^n} = \sum_{n=1}^{\infty} \sqrt{2}$, a divergent series

(a) the radius is $\sqrt{2}$; the interval of convergence is $0 < x < 2\sqrt{2}$

(b) the interval of absolute convergence is $0 < x < 2\sqrt{2}$

(c) there are no values for which the series converges conditionally

$$33. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(x-1)^{2n+2}}{4^{n+1}} \cdot \frac{4^n}{(x-1)^{2n}} \right| < 1 \Rightarrow \frac{(x-1)^2}{4} \lim_{n \rightarrow \infty} |1| < 1 \Rightarrow (x-1)^2 < 4 \Rightarrow |x-1| < 2$$

$$\Rightarrow -2 < x-1 < 2 \Rightarrow -1 < x < 3; \text{ at } x = -1 \text{ we have } \sum_{n=0}^{\infty} \frac{(-2)^{2n}}{4^n} = \sum_{n=0}^{\infty} \frac{4^n}{4^n} = \sum_{n=0}^{\infty} 1, \text{ which diverges; at } x = 3$$

we have $\sum_{n=0}^{\infty} \frac{2^{2n}}{4^n} = \sum_{n=0}^{\infty} \frac{4^n}{4^n} = \sum_{n=0}^{\infty} 1$, a divergent series; the interval of convergence is $-1 < x < 3$; the series

$\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4^n} = \sum_{n=0}^{\infty} \left(\left(\frac{x-1}{2} \right)^2 \right)^n$ is a convergent geometric series when $-1 < x < 3$ and the sum is

$$\frac{1}{1 - \left(\frac{x-1}{2} \right)^2} = \frac{1}{\left[\frac{4 - (x-1)^2}{4} \right]} = \frac{4}{4 - x^2 + 2x - 1} = \frac{4}{3 + 2x - x^2}$$

$$34. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(x+1)^{2n+2}}{9^{n+1}} \cdot \frac{9^n}{(x+1)^{2n}} \right| < 1 \Rightarrow \frac{(x+1)^2}{9} \lim_{n \rightarrow \infty} |1| < 1 \Rightarrow (x+1)^2 < 9 \Rightarrow |x+1| < 3$$

$$\Rightarrow -3 < x+1 < 3 \Rightarrow -4 < x < 2; \text{ when } x = -4 \text{ we have } \sum_{n=0}^{\infty} \frac{(-3)^{2n}}{9^n} = \sum_{n=0}^{\infty} 1 \text{ which diverges; at } x = 2 \text{ we have}$$

$\sum_{n=0}^{\infty} \frac{3^{2n}}{9^n} = \sum_{n=0}^{\infty} 1$ which also diverges; the interval of convergence is $-4 < x < 2$; the series

$\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n} = \sum_{n=0}^{\infty} \left(\left(\frac{x+1}{3} \right)^2 \right)^n$ is a convergent geometric series when $-4 < x < 2$ and the sum is

$$\frac{1}{1 - \left(\frac{x+1}{3} \right)^2} = \frac{1}{\left[\frac{9 - (x+1)^2}{9} \right]} = \frac{9}{9 - x^2 - 2x - 1} = \frac{9}{8 - 2x - x^2}$$

$$35. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(\sqrt{x}-2)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(\sqrt{x}-2)^n} \right| < 1 \Rightarrow |\sqrt{x}-2| < 2 \Rightarrow -2 < \sqrt{x}-2 < 2 \Rightarrow 0 < \sqrt{x} < 4$$

$$\Rightarrow 0 < x < 16; \text{ when } x = 0 \text{ we have } \sum_{n=0}^{\infty} (-1)^n, \text{ a divergent series; when } x = 16 \text{ we have } \sum_{n=0}^{\infty} (1)^n, \text{ a divergent}$$

series; the interval of convergence is $0 < x < 16$; the series $\sum_{n=0}^{\infty} \left(\frac{\sqrt{x}-2}{2} \right)^n$ is a convergent geometric series when

$0 < x < 16$ and its sum is $\frac{1}{1 - \left(\frac{\sqrt{x}-2}{2} \right)} = \frac{1}{\left(\frac{2 - \sqrt{x} + 2}{2} \right)} = \frac{2}{4 - \sqrt{x}}$

$$36. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(\ln x)^{n+1}}{(\ln x)^n} \right| < 1 \Rightarrow |\ln x| < 1 \Rightarrow -1 < \ln x < 1 \Rightarrow e^{-1} < x < e; \text{ when } x = e^{-1} \text{ or } e \text{ we}$$

obtain the series $\sum_{n=0}^{\infty} 1^n$ and $\sum_{n=0}^{\infty} (-1)^n$ which both diverge; the interval of convergence is $e^{-1} < x < e$;

$$\sum_{n=0}^{\infty} (\ln x)^n = \frac{1}{1 - \ln x} \text{ when } e^{-1} < x < e$$

$$37. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \left(\frac{x^2+1}{3} \right)^{n+1} \cdot \left(\frac{3}{x^2+1} \right)^n \right| < 1 \Rightarrow \frac{x^2+1}{3} \lim_{n \rightarrow \infty} |1| < 1 \Rightarrow \frac{x^2+1}{3} < 1 \Rightarrow x^2 < 2$$

$$\Rightarrow |x| < \sqrt{2} \Rightarrow -\sqrt{2} < x < \sqrt{2}; \text{ at } x = \pm \sqrt{2} \text{ we have } \sum_{n=0}^{\infty} (1)^n \text{ which diverges; the interval of convergence is}$$

$-\sqrt{2} < x < \sqrt{2}$; the series $\sum_{n=0}^{\infty} \left(\frac{x^2+1}{3} \right)^n$ is a convergent geometric series when $-\sqrt{2} < x < \sqrt{2}$ and its sum is

$$\frac{1}{1 - \left(\frac{x^2+1}{3} \right)} = \frac{1}{\left(\frac{3 - x^2 - 1}{3} \right)} = \frac{3}{2 - x^2}$$

38. $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(x^2-1)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(x^2+1)^n} \right| < 1 \Rightarrow |x^2-1| < 2 \Rightarrow -\sqrt{3} < x < \sqrt{3}$; when $x = \pm \sqrt{3}$ we have $\sum_{n=0}^{\infty} 1^n$, a divergent series; the interval of convergence is $-\sqrt{3} < x < \sqrt{3}$; the series $\sum_{n=0}^{\infty} \left(\frac{x^2-1}{2} \right)^n$ is a convergent geometric series when $-\sqrt{3} < x < \sqrt{3}$ and its sum is $\frac{1}{1 - \left(\frac{x^2-1}{2} \right)} = \frac{1}{\left(\frac{2 - (x^2-1)}{2} \right)} = \frac{2}{3-x^2}$

39. $\lim_{n \rightarrow \infty} \left| \frac{(x-3)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(x-3)^n} \right| < 1 \Rightarrow |x-3| < 2 \Rightarrow 1 < x < 5$; when $x = 1$ we have $\sum_{n=1}^{\infty} (1)^n$ which diverges; when $x = 5$ we have $\sum_{n=1}^{\infty} (-1)^n$ which also diverges; the interval of convergence is $1 < x < 5$; the sum of this convergent geometric series is $\frac{1}{1 - \left(\frac{x-3}{2} \right)} = \frac{2}{x-1}$. If $f(x) = 1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \dots + \left(-\frac{1}{2}\right)^n (x-3)^n + \dots = \frac{2}{x-1}$ then $f'(x) = -\frac{1}{2} + \frac{1}{2}(x-3) + \dots + \left(-\frac{1}{2}\right)^n n(x-3)^{n-1} + \dots$ is convergent when $1 < x < 5$, and diverges when $x = 1$ or 5 . The sum for $f'(x)$ is $\frac{-2}{(x-1)^2}$, the derivative of $\frac{2}{x-1}$.

40. If $f(x) = 1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \dots + \left(-\frac{1}{2}\right)^n (x-3)^n + \dots = \frac{2}{x-1}$ then $\int f(x) dx = x - \frac{(x-3)^2}{4} + \frac{(x-3)^3}{12} + \dots + \left(-\frac{1}{2}\right)^n \frac{(x-3)^{n+1}}{n+1} + \dots$. At $x = 1$ the series $\sum_{n=1}^{\infty} \frac{-2}{n+1}$ diverges; at $x = 5$ the series $\sum_{n=1}^{\infty} \frac{(-1)^n 2}{n+1}$ converges. Therefore the interval of convergence is $1 < x \leq 5$ and the sum is $2 \ln |x-1| + (3 - \ln 4)$, since $\int \frac{2}{x-1} dx = 2 \ln |x-1| + C$, where $C = 3 - \ln 4$ when $x = 3$.

41. (a) Differentiate the series for $\sin x$ to get $\cos x = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \frac{9x^8}{9!} - \frac{11x^{10}}{11!} + \dots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots$. The series converges for all values of x since

$$\lim_{n \rightarrow \infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right| = x^2 \lim_{n \rightarrow \infty} \left(\frac{1}{(2n+1)(2n+2)} \right) = 0 < 1 \text{ for all } x.$$

(b) $\sin 2x = 2x - \frac{2^3 x^3}{3!} + \frac{2^5 x^5}{5!} - \frac{2^7 x^7}{7!} + \frac{2^9 x^9}{9!} - \frac{2^{11} x^{11}}{11!} + \dots = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \frac{512x^9}{9!} - \frac{2048x^{11}}{11!} + \dots$

(c) $2 \sin x \cos x = 2 \left[(0 \cdot 1) + (0 \cdot 0 + 1 \cdot 1)x + (0 \cdot \frac{-1}{2} + 1 \cdot 0 + 0 \cdot 1)x^2 + (0 \cdot 0 - 1 \cdot \frac{1}{2} + 0 \cdot 0 - 1 \cdot \frac{1}{3!})x^3 + (0 \cdot \frac{1}{4!} + 1 \cdot 0 - 0 \cdot \frac{1}{2} - 0 \cdot \frac{1}{3!} + 0 \cdot 1)x^4 + (0 \cdot 0 + 1 \cdot \frac{1}{4!} + 0 \cdot 0 + \frac{1}{2} \cdot \frac{1}{3!} + 0 \cdot 0 + 1 \cdot \frac{1}{5!})x^5 + (0 \cdot \frac{1}{6!} + 1 \cdot 0 + 0 \cdot \frac{1}{4!} + 0 \cdot \frac{1}{3!} + 0 \cdot \frac{1}{2} + 0 \cdot \frac{1}{5!} + 0 \cdot 1)x^6 + \dots \right] = 2 \left[x - \frac{4x^3}{3!} + \frac{16x^5}{5!} - \dots \right] = 2x - \frac{2^3 x^3}{3!} + \frac{2^5 x^5}{5!} - \frac{2^7 x^7}{7!} + \frac{2^9 x^9}{9!} - \frac{2^{11} x^{11}}{11!} + \dots$

42. (a) $\frac{d}{dx}(e^x) = 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \frac{5x^4}{5!} + \dots = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = e^x$; thus the derivative of e^x is e^x itself

(b) $\int e^x dx = e^x + C = x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots + C$, which is the general antiderivative of e^x

(c) $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \dots$; $e^{-x} \cdot e^x = 1 \cdot 1 + (1 \cdot 1 - 1 \cdot 1)x + (1 \cdot \frac{1}{2!} - 1 \cdot 1 + \frac{1}{2!} \cdot 1)x^2 + (1 \cdot \frac{1}{3!} - 1 \cdot \frac{1}{2!} + \frac{1}{2!} \cdot 1 - \frac{1}{3!} \cdot 1 + \frac{1}{4!} \cdot 1)x^3 + (1 \cdot \frac{1}{4!} - 1 \cdot \frac{1}{3!} + \frac{1}{2!} \cdot \frac{1}{2!} - \frac{1}{3!} \cdot 1 + \frac{1}{4!} \cdot 1)x^4 + (1 \cdot \frac{1}{5!} - 1 \cdot \frac{1}{4!} + \frac{1}{2!} \cdot \frac{1}{3!} - \frac{1}{3!} \cdot \frac{1}{2!} + \frac{1}{4!} \cdot 1 - \frac{1}{5!} \cdot 1)x^5 + \dots = 1 + 0 + 0 + 0 + 0 + 0 + \dots$

43. (a) $\ln |\sec x| + C = \int \tan x dx = \int \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \dots \right) dx = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \frac{31x^{10}}{14,175} + \dots + C$; $x = 0 \Rightarrow C = 0 \Rightarrow \ln |\sec x| = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \frac{31x^{10}}{14,175} + \dots$, converges when $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(b) $\sec^2 x = \frac{d(\tan x)}{dx} = \frac{d}{dx} \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \dots \right) = 1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \frac{62x^8}{315} + \dots$, converges when $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$$\begin{aligned}
 \text{(c) } \sec^2 x &= (\sec x)(\sec x) = \left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \dots\right) \left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \dots\right) \\
 &= 1 + \left(\frac{1}{2} + \frac{1}{2}\right)x^2 + \left(\frac{5}{24} + \frac{1}{4} + \frac{5}{24}\right)x^4 + \left(\frac{61}{720} + \frac{5}{48} + \frac{5}{48} + \frac{61}{720}\right)x^6 + \dots \\
 &= 1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \frac{62x^8}{315} + \dots, -\frac{\pi}{2} < x < \frac{\pi}{2}
 \end{aligned}$$

$$44. \text{ (a) } \ln |\sec x + \tan x| + C = \int \sec x \, dx = \int \left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \dots\right) dx$$

$$= x + \frac{x^3}{6} + \frac{x^5}{24} + \frac{61x^7}{5040} + \frac{277x^9}{72,576} + \dots + C; x = 0 \Rightarrow C = 0 \Rightarrow \ln |\sec x + \tan x|$$

$$= x + \frac{x^3}{6} + \frac{x^5}{24} + \frac{61x^7}{5040} + \frac{277x^9}{72,576} + \dots, \text{ converges when } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\text{(b) } \sec x \tan x = \frac{d(\sec x)}{dx} = \frac{d}{dx} \left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \dots\right) = x + \frac{5x^3}{6} + \frac{61x^5}{120} + \frac{277x^7}{1008} + \dots, \text{ converges}$$

$$\text{when } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\text{(c) } (\sec x)(\tan x) = \left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \dots\right) \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots\right)$$

$$= x + \left(\frac{1}{3} + \frac{1}{2}\right)x^3 + \left(\frac{2}{15} + \frac{1}{6} + \frac{5}{24}\right)x^5 + \left(\frac{17}{315} + \frac{1}{15} + \frac{5}{72} + \frac{61}{720}\right)x^7 + \dots = x + \frac{5x^3}{6} + \frac{61x^5}{120} + \frac{277x^7}{1008} + \dots, -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$45. \text{ (a) If } f(x) = \sum_{n=0}^{\infty} a_n x^n, \text{ then } f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-(k-1)) a_n x^{n-k} \text{ and } f^{(k)}(0) = k! a_k$$

$$\Rightarrow a_k = \frac{f^{(k)}(0)}{k!}; \text{ likewise if } f(x) = \sum_{n=0}^{\infty} b_n x^n, \text{ then } b_k = \frac{f^{(k)}(0)}{k!} \Rightarrow a_k = b_k \text{ for every nonnegative integer } k$$

$$\text{(b) If } f(x) = \sum_{n=0}^{\infty} a_n x^n = 0 \text{ for all } x, \text{ then } f^{(k)}(x) = 0 \text{ for all } x \Rightarrow \text{from part (a) that } a_k = 0 \text{ for every nonnegative integer } k$$

$$46. \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots \Rightarrow x \left[\frac{1}{(1-x)^2} \right] = x(1 + 2x + 3x^2 + 4x^3 + \dots) \Rightarrow \frac{x}{(1-x)^2}$$

$$= x + 2x^2 + 3x^3 + 4x^4 + \dots \Rightarrow x \left[\frac{1+x}{(1-x)^3} \right] = x(1 + 4x + 9x^2 + 16x^3 + \dots) \Rightarrow \frac{x+x^2}{(1-x)^3}$$

$$= x + 4x^2 + 9x^3 + 16x^4 + \dots \Rightarrow \frac{\left(\frac{1}{2} + \frac{1}{4}\right)}{\left(\frac{1}{8}\right)} = \frac{1}{2} + \frac{4}{4} + \frac{9}{8} + \frac{16}{16} + \dots \Rightarrow \sum_{n=1}^{\infty} \frac{n^2}{2^n} = 6$$

$$47. \text{ The series } \sum_{n=1}^{\infty} \frac{x^n}{n} \text{ converges conditionally at the left-hand endpoint of its interval of convergence } [-1, 1); \text{ the}$$

$$\text{series } \sum_{n=1}^{\infty} \frac{x^n}{(n^2)} \text{ converges absolutely at the left-hand endpoint of its interval of convergence } [-1, 1]$$

48. Answers will vary. For instance:

$$\text{(a) } \sum_{n=1}^{\infty} \left(\frac{x}{3}\right)^n$$

$$\text{(b) } \sum_{n=1}^{\infty} (x+1)^n$$

$$\text{(c) } \sum_{n=1}^{\infty} \left(\frac{x-3}{2}\right)^n$$

11.8 TAYLOR AND MACLAURIN SERIES

$$1. f(x) = \ln x, f'(x) = \frac{1}{x}, f''(x) = -\frac{1}{x^2}, f'''(x) = \frac{2}{x^3}; f(1) = \ln 1 = 0, f'(1) = 1, f''(1) = -1, f'''(1) = 2 \Rightarrow P_0(x) = 0, P_1(x) = (x-1), P_2(x) = (x-1) - \frac{1}{2}(x-1)^2, P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

$$2. f(x) = \ln(1+x), f'(x) = \frac{1}{1+x} = (1+x)^{-1}, f''(x) = -(1+x)^{-2}, f'''(x) = 2(1+x)^{-3}; f(0) = \ln 1 = 0, f'(0) = \frac{1}{1} = 1, f''(0) = -(1)^{-2} = -1, f'''(0) = 2(1)^{-3} = 2 \Rightarrow P_0(x) = 0, P_1(x) = x, P_2(x) = x - \frac{x^2}{2}, P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

3. $f(x) = \frac{1}{x} = x^{-1}$, $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -6x^{-4}$; $f(2) = \frac{1}{2}$, $f'(2) = -\frac{1}{4}$, $f''(2) = \frac{1}{4}$, $f'''(x) = -\frac{3}{8}$
 $\Rightarrow P_0(x) = \frac{1}{2}$, $P_1(x) = \frac{1}{2} - \frac{1}{4}(x-2)$, $P_2(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2$,
 $P_3(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3$
4. $f(x) = (x+2)^{-1}$, $f'(x) = -(x+2)^{-2}$, $f''(x) = 2(x+2)^{-3}$, $f'''(x) = -6(x+2)^{-4}$; $f(0) = (2)^{-1} = \frac{1}{2}$, $f'(0) = -(2)^{-2} = -\frac{1}{4}$, $f''(0) = 2(2)^{-3} = \frac{1}{4}$, $f'''(0) = -6(2)^{-4} = -\frac{3}{8} \Rightarrow P_0(x) = \frac{1}{2}$, $P_1(x) = \frac{1}{2} - \frac{x}{4}$, $P_2(x) = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8}$,
 $P_3(x) = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16}$
5. $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$; $f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $f'\left(\frac{\pi}{4}\right) = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$,
 $f''\left(\frac{\pi}{4}\right) = -\sin \frac{\pi}{4} = -\frac{\sqrt{2}}{2}$, $f'''\left(\frac{\pi}{4}\right) = -\cos \frac{\pi}{4} = -\frac{\sqrt{2}}{2} \Rightarrow P_0 = \frac{\sqrt{2}}{2}$, $P_1(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$,
 $P_2(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2$, $P_3(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 + \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3$
6. $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$; $f\left(\frac{\pi}{4}\right) = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$,
 $f'\left(\frac{\pi}{4}\right) = -\sin \frac{\pi}{4} = -\frac{1}{\sqrt{2}}$, $f''\left(\frac{\pi}{4}\right) = -\cos \frac{\pi}{4} = -\frac{1}{\sqrt{2}}$, $f'''\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \Rightarrow P_0(x) = \frac{1}{\sqrt{2}}$,
 $P_1(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$, $P_2(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2$,
 $P_3(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{6\sqrt{2}}\left(x - \frac{\pi}{4}\right)^3$
7. $f(x) = \sqrt{x} = x^{1/2}$, $f'(x) = \left(\frac{1}{2}\right)x^{-1/2}$, $f''(x) = \left(-\frac{1}{4}\right)x^{-3/2}$, $f'''(x) = \left(\frac{3}{8}\right)x^{-5/2}$; $f(4) = \sqrt{4} = 2$,
 $f'(4) = \left(\frac{1}{2}\right)4^{-1/2} = \frac{1}{4}$, $f''(4) = \left(-\frac{1}{4}\right)4^{-3/2} = -\frac{1}{32}$, $f'''(4) = \left(\frac{3}{8}\right)4^{-5/2} = \frac{3}{256} \Rightarrow P_0(x) = 2$, $P_1(x) = 2 + \frac{1}{4}(x-4)$,
 $P_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$, $P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$
8. $f(x) = (x+4)^{1/2}$, $f'(x) = \left(\frac{1}{2}\right)(x+4)^{-1/2}$, $f''(x) = \left(-\frac{1}{4}\right)(x+4)^{-3/2}$, $f'''(x) = \left(\frac{3}{8}\right)(x+4)^{-5/2}$; $f(0) = (4)^{1/2} = 2$,
 $f'(0) = \left(\frac{1}{2}\right)(4)^{-1/2} = \frac{1}{4}$, $f''(0) = \left(-\frac{1}{4}\right)(4)^{-3/2} = -\frac{1}{32}$, $f'''(0) = \left(\frac{3}{8}\right)(4)^{-5/2} = \frac{3}{256} \Rightarrow P_0(x) = 2$,
 $P_1(x) = 2 + \frac{1}{4}x$, $P_2(x) = 2 + \frac{1}{4}x - \frac{1}{64}x^2$, $P_3(x) = 2 + \frac{1}{4}x - \frac{1}{64}x^2 + \frac{1}{512}x^3$
9. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots$
10. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{x/2} = \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^n}{n!} = 1 + \frac{x}{2} + \frac{x^2}{4 \cdot 2!} + \frac{x^3}{2^3 \cdot 3!} + \frac{x^4}{2^4 \cdot 4!} + \dots$
11. $f(x) = (1+x)^{-1} \Rightarrow f'(x) = -(1+x)^{-2}$, $f''(x) = 2(1+x)^{-3}$, $f'''(x) = -3!(1+x)^{-4} \Rightarrow \dots f^{(k)}(x) = (-1)^k k! (1+x)^{-k-1}$; $f(0) = 1$, $f'(0) = -1$, $f''(0) = 2$, $f'''(0) = -3!$, \dots , $f^{(k)}(0) = (-1)^k k!$
 $\Rightarrow \frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$
12. $f(x) = (1-x)^{-1} \Rightarrow f'(x) = (1-x)^{-2}$, $f''(x) = 2(1-x)^{-3}$, $f'''(x) = 3!(1-x)^{-4} \Rightarrow \dots f^{(k)}(x) = k!(1-x)^{-k-1}$; $f(0) = 1$, $f'(0) = 1$, $f''(0) = 2$, $f'''(0) = 3!$, \dots , $f^{(k)}(0) = k!$
 $\Rightarrow \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$
13. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin 3x = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{(2n+1)!} = 3x - \frac{3^3 x^3}{3!} + \frac{3^5 x^5}{5!} - \dots$

$$14. \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin \frac{x}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}(2n+1)!} = \frac{x}{2} - \frac{x^3}{2^3 \cdot 3!} + \frac{x^5}{2^5 \cdot 5!} + \dots$$

$$15. 7 \cos(-x) = 7 \cos x = 7 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 7 - \frac{7x^2}{2!} + \frac{7x^4}{4!} - \frac{7x^6}{6!} + \dots, \text{ since the cosine is an even function}$$

$$16. \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow 5 \cos \pi x = 5 \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!} = 5 - \frac{5\pi^2 x^2}{2!} + \frac{5\pi^4 x^4}{4!} - \frac{5\pi^6 x^6}{6!} + \dots$$

$$17. \cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2} \left[\left(1 + x^2 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots \right) \right] = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

$$18. \sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2} \left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots \right) \right] = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$19. f(x) = x^4 - 2x^3 - 5x + 4 \Rightarrow f'(x) = 4x^3 - 6x^2 - 5, f''(x) = 12x^2 - 12x, f'''(x) = 24x - 12, f^{(4)}(x) = 24$$

$$\Rightarrow f^{(n)}(x) = 0 \text{ if } n \geq 5; f(0) = 4, f'(0) = -5, f''(0) = 0, f'''(0) = -12, f^{(4)}(0) = 24, f^{(n)}(0) = 0 \text{ if } n \geq 5$$

$$\Rightarrow x^4 - 2x^3 - 5x + 4 = 4 - 5x - \frac{12}{3!}x^3 + \frac{24}{4!}x^4 = x^4 - 2x^3 - 5x + 4 \text{ itself}$$

$$20. f(x) = (x+1)^2 \Rightarrow f'(x) = 2(x+1); f''(x) = 2 \Rightarrow f^{(n)}(x) = 0 \text{ if } n \geq 3; f(0) = 1, f'(0) = 2, f''(0) = 2, f^{(n)}(0) = 0 \text{ if } n \geq 3$$

$$\Rightarrow (x+1)^2 = 1 + 2x + \frac{2}{2!}x^2 = 1 + 2x + x^2$$

$$21. f(x) = x^3 - 2x + 4 \Rightarrow f'(x) = 3x^2 - 2, f''(x) = 6x, f'''(x) = 6 \Rightarrow f^{(n)}(x) = 0 \text{ if } n \geq 4; f(2) = 8, f'(2) = 10,$$

$$f''(2) = 12, f'''(2) = 6, f^{(n)}(2) = 0 \text{ if } n \geq 4 \Rightarrow x^3 - 2x + 4 = 8 + 10(x-2) + \frac{12}{2!}(x-2)^2 + \frac{6}{3!}(x-2)^3$$

$$= 8 + 10(x-2) + 6(x-2)^2 + (x-2)^3$$

$$22. f(x) = 2x^3 + x^2 + 3x - 8 \Rightarrow f'(x) = 6x^2 + 2x + 3, f''(x) = 12x + 2, f'''(x) = 12 \Rightarrow f^{(n)}(x) = 0 \text{ if } n \geq 4; f(1) = -2,$$

$$f'(1) = 11, f''(1) = 14, f'''(1) = 12, f^{(n)}(1) = 0 \text{ if } n \geq 4 \Rightarrow 2x^3 + x^2 + 3x - 8$$

$$= -2 + 11(x-1) + \frac{14}{2!}(x-1)^2 + \frac{12}{3!}(x-1)^3 = -2 + 11(x-1) + 7(x-1)^2 + 2(x-1)^3$$

$$23. f(x) = x^4 + x^2 + 1 \Rightarrow f'(x) = 4x^3 + 2x, f''(x) = 12x^2 + 2, f'''(x) = 24x, f^{(4)}(x) = 24, f^{(n)}(x) = 0 \text{ if } n \geq 5;$$

$$f(-2) = 21, f'(-2) = -36, f''(-2) = 50, f'''(-2) = -48, f^{(4)}(-2) = 24, f^{(n)}(-2) = 0 \text{ if } n \geq 5 \Rightarrow x^4 + x^2 + 1$$

$$= 21 - 36(x+2) + \frac{50}{2!}(x+2)^2 - \frac{48}{3!}(x+2)^3 + \frac{24}{4!}(x+2)^4 = 21 - 36(x+2) + 25(x+2)^2 - 8(x+2)^3 + (x+2)^4$$

$$24. f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2 \Rightarrow f'(x) = 15x^4 - 4x^3 + 6x^2 + 2x, f''(x) = 60x^3 - 12x^2 + 12x + 2,$$

$$f'''(x) = 180x^2 - 24x + 12, f^{(4)}(x) = 360x - 24, f^{(5)}(x) = 360, f^{(n)}(x) = 0 \text{ if } n \geq 6; f(-1) = -7,$$

$$f'(-1) = 23, f''(-1) = -82, f'''(-1) = 216, f^{(4)}(-1) = -384, f^{(5)}(-1) = 360, f^{(n)}(-1) = 0 \text{ if } n \geq 6$$

$$\Rightarrow 3x^5 - x^4 + 2x^3 + x^2 - 2 = -7 + 23(x+1) - \frac{82}{2!}(x+1)^2 + \frac{216}{3!}(x+1)^3 - \frac{384}{4!}(x+1)^4 + \frac{360}{5!}(x+1)^5$$

$$= -7 + 23(x+1) - 41(x+1)^2 + 36(x+1)^3 - 16(x+1)^4 + 3(x+1)^5$$

$$25. f(x) = x^{-2} \Rightarrow f'(x) = -2x^{-3}, f''(x) = 3!x^{-4}, f'''(x) = -4!x^{-5} \Rightarrow f^{(n)}(x) = (-1)^n(n+1)!x^{-n-2};$$

$$f(1) = 1, f'(1) = -2, f''(1) = 3!, f'''(1) = -4!, f^{(n)}(1) = (-1)^n(n+1)! \Rightarrow \frac{1}{x^2}$$

$$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \dots = \sum_{n=0}^{\infty} (-1)^n(n+1)(x-1)^n$$

$$26. f(x) = \frac{x}{1-x} \Rightarrow f'(x) = (1-x)^{-2}, f''(x) = 2(1-x)^{-3}, f'''(x) = 3!(1-x)^{-4} \Rightarrow f^{(n)}(x) = n!(1-x)^{-n-1};$$

$$f(0) = 0, f'(0) = 1, f''(0) = 2, f'''(0) = 3! \Rightarrow \frac{x}{1-x} = x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^{n+1}$$

$$27. f(x) = e^x \Rightarrow f'(x) = e^x, f''(x) = e^x \Rightarrow f^{(n)}(x) = e^x; f(2) = e^2, f'(2) = e^2, \dots, f^{(n)}(2) = e^2$$

$$\Rightarrow e^x = e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^2}{3!}(x-2)^3 + \dots = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$$

$$28. f(x) = 2^x \Rightarrow f'(x) = 2^x \ln 2, f''(x) = 2^x (\ln 2)^2, f'''(x) = 2^x (\ln 2)^3 \Rightarrow f^{(n)}(x) = 2^x (\ln 2)^n; f(1) = 2, f'(1) = 2 \ln 2,$$

$$f''(1) = 2(\ln 2)^2, f'''(1) = 2(\ln 2)^3, \dots, f^{(n)}(1) = 2(\ln 2)^n$$

$$\Rightarrow 2^x = 2 + (2 \ln 2)(x-1) + \frac{2(\ln 2)^2}{2}(x-1)^2 + \frac{2(\ln 2)^3}{3!}(x-1)^3 + \dots = \sum_{n=0}^{\infty} \frac{2(\ln 2)^n (x-1)^n}{n!}$$

$$29. \text{ If } e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \text{ and } f(x) = e^x, \text{ we have } f^{(n)}(a) = e^a \text{ for all } n = 0, 1, 2, 3, \dots$$

$$\Rightarrow e^x = e^a \left[\frac{(x-a)^0}{0!} + \frac{(x-a)^1}{1!} + \frac{(x-a)^2}{2!} + \dots \right] = e^a \left[1 + (x-a) + \frac{(x-a)^2}{2!} + \dots \right] \text{ at } x = a$$

$$30. f(x) = e^x \Rightarrow f^{(n)}(x) = e^x \text{ for all } n \Rightarrow f^{(n)}(1) = e \text{ for all } n = 0, 1, 2, \dots$$

$$\Rightarrow e^x = e + e(x-1) + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \dots = e \left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \dots \right]$$

$$31. f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots \Rightarrow f'(x)$$

$$= f'(a) + f''(a)(x-a) + \frac{f'''(a)}{3!}3(x-a)^2 + \dots \Rightarrow f''(x) = f''(a) + f'''(a)(x-a) + \frac{f^{(4)}(a)}{4!}4 \cdot 3(x-a)^2 + \dots$$

$$\Rightarrow f^{(n)}(x) = f^{(n)}(a) + f^{(n+1)}(a)(x-a) + \frac{f^{(n+2)}(a)}{2}(x-a)^2 + \dots$$

$$\Rightarrow f(a) = f(a) + 0, f'(a) = f'(a) + 0, \dots, f^{(n)}(a) = f^{(n)}(a) + 0$$

$$32. E(x) = f(x) - b_0 - b_1(x-a) - b_2(x-a)^2 - b_3(x-a)^3 - \dots - b_n(x-a)^n$$

$$\Rightarrow 0 = E(a) = f(a) - b_0 \Rightarrow b_0 = f(a); \text{ from condition (b),}$$

$$\lim_{x \rightarrow a} \frac{f(x) - f(a) - b_1(x-a) - b_2(x-a)^2 - b_3(x-a)^3 - \dots - b_n(x-a)^n}{(x-a)^n} = 0$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{f'(x) - b_1 - 2b_2(x-a) - 3b_3(x-a)^2 - \dots - nb_n(x-a)^{n-1}}{n(x-a)^{n-1}} = 0$$

$$\Rightarrow b_1 = f'(a) \Rightarrow \lim_{x \rightarrow a} \frac{f''(x) - 2b_2 - 3!b_3(x-a) - \dots - n(n-1)b_n(x-a)^{n-2}}{n(n-1)(x-a)^{n-2}} = 0$$

$$\Rightarrow b_2 = \frac{1}{2}f''(a) \Rightarrow \lim_{x \rightarrow a} \frac{f'''(x) - 3!b_3 - \dots - n(n-1)(n-2)b_n(x-a)^{n-3}}{n(n-1)(n-2)(x-a)^{n-3}} = 0$$

$$= b_3 = \frac{1}{3!}f'''(a) \Rightarrow \lim_{x \rightarrow a} \frac{f^{(n)}(x) - n!b_n}{n!} = 0 \Rightarrow b_n = \frac{1}{n!}f^{(n)}(a); \text{ therefore,}$$

$$g(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n = P_n(x)$$

$$33. f(x) = \ln(\cos x) \Rightarrow f'(x) = -\tan x \text{ and } f''(x) = -\sec^2 x; f(0) = 0, f'(0) = 0, f''(0) = -1$$

$$\Rightarrow L(x) = 0 \text{ and } Q(x) = -\frac{x^2}{2}$$

$$34. f(x) = e^{\sin x} \Rightarrow f'(x) = (\cos x)e^{\sin x} \text{ and } f''(x) = (-\sin x)e^{\sin x} + (\cos x)^2 e^{\sin x}; f(0) = 1, f'(0) = 1,$$

$$f''(0) = 1 \Rightarrow L(x) = 1 + x \text{ and } Q(x) = 1 + x + \frac{x^2}{2}$$

$$35. f(x) = (1-x^2)^{-1/2} \Rightarrow f'(x) = x(1-x^2)^{-3/2} \text{ and } f''(x) = (1-x^2)^{-3/2} + 3x^2(1-x^2)^{-5/2}; f(0) = 1,$$

$$f'(0) = 0, f''(0) = 1 \Rightarrow L(x) = 1 \text{ and } Q(x) = 1 + \frac{x^2}{2}$$

$$36. f(x) = \cosh x \Rightarrow f'(x) = \sinh x \text{ and } f''(x) = \cosh x; f(0) = 1, f'(0) = 0, f''(0) = 1 \Rightarrow L(x) = 1 \text{ and } Q(x) = 1 + \frac{x^2}{2}$$

$$37. f(x) = \sin x \Rightarrow f'(x) = \cos x \text{ and } f''(x) = -\sin x; f(0) = 0, f'(0) = 1, f''(0) = 0 \Rightarrow L(x) = x \text{ and } Q(x) = x$$

$$38. f(x) = \tan x \Rightarrow f'(x) = \sec^2 x \text{ and } f''(x) = 2 \sec^2 x \tan x; f(0) = 0, f'(0) = 1, f'' = 0 \Rightarrow L(x) = x \text{ and } Q(x) = x$$

11.9 CONVERGENCE OF TAYLOR SERIES; ERROR ESTIMATES

$$1. e^x = 1 + x + \frac{x^2}{2!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-5x} = 1 + (-5x) + \frac{(-5x)^2}{2!} + \dots = 1 - 5x + \frac{5^2 x^2}{2!} - \frac{5^3 x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n 5^n x^n}{n!}$$

$$2. e^x = 1 + x + \frac{x^2}{2!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x/2} = 1 + \left(\frac{-x}{2}\right) + \frac{\left(\frac{-x}{2}\right)^2}{2!} + \dots = 1 - \frac{x}{2} + \frac{x^2}{2^2 2!} - \frac{x^3}{2^3 3!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n n!}$$

$$3. \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow 5 \sin(-x) = 5 \left[(-x) - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} - \dots \right]$$

$$= \sum_{n=0}^{\infty} \frac{5(-1)^{n+1} x^{2n+1}}{(2n+1)!}$$

$$4. \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin \frac{\pi x}{2} = \frac{\pi x}{2} - \frac{\left(\frac{\pi x}{2}\right)^3}{3!} + \frac{\left(\frac{\pi x}{2}\right)^5}{5!} - \frac{\left(\frac{\pi x}{2}\right)^7}{7!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1} x^{2n+1}}{2^{2n+1} (2n+1)!}$$

$$5. \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos \sqrt{x+1} = \sum_{n=0}^{\infty} \frac{(-1)^n \left[(x+1)^{1/2}\right]^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (x+1)^n}{(2n)!} = 1 - \frac{x+1}{2!} + \frac{(x+1)^2}{4!} - \frac{(x+1)^3}{6!} + \dots$$

$$6. \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos\left(\frac{x^{3/2}}{\sqrt{2}}\right) = \cos\left(\left(\frac{x^3}{2}\right)^{1/2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\left(\frac{x^3}{2}\right)^{1/2}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{2^n (2n)!}$$

$$= 1 - \frac{x^3}{2 \cdot 2!} + \frac{x^6}{2^2 \cdot 4!} - \frac{x^9}{2^3 \cdot 6!} + \dots$$

$$7. e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow x e^x = x \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \dots$$

$$8. \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow x^2 \sin x = x^2 \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!} = x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \frac{x^9}{7!} + \dots$$

$$9. \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \frac{x^2}{2} - 1 + \cos x = \frac{x^2}{2} - 1 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \frac{x^2}{2} - 1 + 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots$$

$$= \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots = \sum_{n=2}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$10. \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin x - x + \frac{x^3}{3!} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right) - x + \frac{x^3}{3!}$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots \right) - x + \frac{x^3}{3!} = \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots = \sum_{n=2}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$11. \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow x \cos \pi x = x \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n} x^{2n+1}}{(2n)!} = x - \frac{\pi^2 x^3}{2!} + \frac{\pi^4 x^5}{4!} - \frac{\pi^6 x^7}{6!} + \dots$$

$$12. \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow x^2 \cos(x^2) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n)!} = x^2 - \frac{x^6}{2!} + \frac{x^{10}}{4!} - \frac{x^{14}}{6!} + \dots$$

$$13. \cos^2 x = \frac{1}{2} + \frac{\cos 2x}{2} = \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \frac{1}{2} + \frac{1}{2} \left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} - \dots \right]$$

$$= 1 - \frac{(2x)^2}{2 \cdot 2!} + \frac{(2x)^4}{2 \cdot 4!} - \frac{(2x)^6}{2 \cdot 6!} + \frac{(2x)^8}{2 \cdot 8!} - \dots = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n}}{2 \cdot (2n)!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!}$$

$$14. \sin^2 x = \left(\frac{1 - \cos 2x}{2} \right) = \frac{1}{2} - \frac{1}{2} \cos 2x = \frac{1}{2} - \frac{1}{2} \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots \right) = \frac{(2x)^2}{2 \cdot 2!} - \frac{(2x)^4}{2 \cdot 4!} + \frac{(2x)^6}{2 \cdot 6!} - \dots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2x)^{2n}}{2 \cdot (2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!}$$

$$15. \frac{x^2}{1-2x} = x^2 \left(\frac{1}{1-2x} \right) = x^2 \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^{n+2} = x^2 + 2x^3 + 2^2 x^4 + 2^3 x^5 + \dots$$

$$16. x \ln(1+2x) = x \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (2x)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^n x^{n+1}}{n} = 2x^2 - \frac{2^2 x^3}{2} + \frac{2^3 x^4}{4} - \frac{2^4 x^5}{5} + \dots$$

$$17. \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \Rightarrow \frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots = \sum_{n=1}^{\infty} n x^{n-1}$$

$$= \sum_{n=0}^{\infty} (n+1) x^n$$

$$18. \frac{2}{(1-x)^3} = \frac{d^2}{dx^2} \left(\frac{1}{1-x} \right) = \frac{d}{dx} \left(\frac{1}{(1-x)^2} \right) = \frac{d}{dx} (1 + 2x + 3x^2 + \dots) = 2 + 6x + 12x^2 + \dots = \sum_{n=2}^{\infty} n(n-1) x^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) x^n$$

$$19. \text{By the Alternating Series Estimation Theorem, the error is less than } \frac{|x|^5}{5!} \Rightarrow |x|^5 < (5!) (5 \times 10^{-4})$$

$$\Rightarrow |x|^5 < 600 \times 10^{-4} \Rightarrow |x| < \sqrt[5]{6 \times 10^{-2}} \approx 0.56968$$

$$20. \text{If } \cos x = 1 - \frac{x^2}{2} \text{ and } |x| < 0.5, \text{ then the error is less than } \left| \frac{(-5)^4}{24} \right| = 0.0026, \text{ by Alternating Series Estimation Theorem;}$$

since the next term in the series is positive, the approximation $1 - \frac{x^2}{2}$ is too small, by the Alternating Series Estimation Theorem

$$21. \text{If } \sin x = x \text{ and } |x| < 10^{-3}, \text{ then the error is less than } \frac{(10^{-3})^3}{3!} \approx 1.67 \times 10^{-10}, \text{ by Alternating Series Estimation Theorem;}$$

The Alternating Series Estimation Theorem says $R_2(x)$ has the same sign as $-\frac{x^3}{3!}$. Moreover, $x < \sin x$

$$\Rightarrow 0 < \sin x - x = R_2(x) \Rightarrow x < 0 \Rightarrow -10^{-3} < x < 0.$$

$$22. \sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \dots. \text{ By the Alternating Series Estimation Theorem the } |\text{error}| < \left| \frac{-x^2}{8} \right| < \frac{(0.01)^2}{8}$$

$$= 1.25 \times 10^{-5}$$

$$23. |R_2(x)| = \left| \frac{e^c x^3}{3!} \right| < \frac{3^{(0.1)} (0.1)^3}{3!} < 1.87 \times 10^{-4}, \text{ where } c \text{ is between } 0 \text{ and } x$$

$$24. |R_2(x)| = \left| \frac{e^c x^3}{3!} \right| < \frac{(0.1)^3}{3!} = 1.67 \times 10^{-4}, \text{ where } c \text{ is between } 0 \text{ and } x$$

$$25. |R_4(x)| < \left| \frac{\cosh c}{5!} x^5 \right| = \left| \frac{e^c + e^{-c}}{2} \frac{x^5}{5!} \right| < \frac{1.65 + \frac{1}{1.65}}{2} \cdot \frac{(0.5)^5}{5!} = (1.13) \frac{(0.5)^5}{5!} \approx 0.000294$$

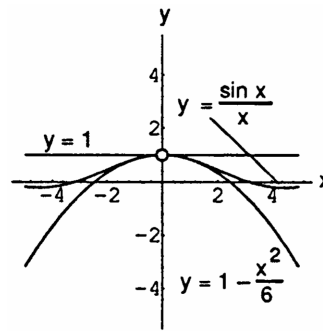
$$26. \text{ If we approximate } e^h \text{ with } 1 + h \text{ and } 0 \leq h \leq 0.01, \text{ then } |\text{error}| < \left| \frac{e^c h^2}{2} \right| \leq \frac{e^{0.01} h \cdot h}{2} \leq \left(\frac{e^{0.01}(0.01)}{2} \right) h \\ = 0.00505h < 0.006h = (0.6\%)h, \text{ where } c \text{ is between } 0 \text{ and } h.$$

$$27. |R_1| = \left| \frac{1}{(1+c)^2} \frac{x^2}{2!} \right| < \frac{x^2}{2} = \left| \frac{x}{2} \right| |x| < .01 |x| = (1\%) |x| \Rightarrow \left| \frac{x}{2} \right| < .01 \Rightarrow 0 < |x| < .02$$

$$28. \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \Rightarrow \frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots; |\text{error}| < \frac{1}{2n+1} < .01 \\ \Rightarrow 2n+1 > 100 \Rightarrow n > 49$$

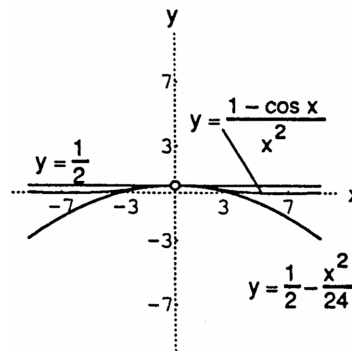
$$29. (a) \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \Rightarrow \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots, s_1 = 1 \text{ and } s_2 = 1 - \frac{x^2}{6}; \text{ if } L \text{ is the sum of the} \\ \text{series representing } \frac{\sin x}{x}, \text{ then by the Alternating Series Estimation Theorem, } L - s_1 = \frac{\sin x}{x} - 1 < 0 \text{ and} \\ L - s_2 = \frac{\sin x}{x} - \left(1 - \frac{x^2}{6} \right) > 0. \text{ Therefore } 1 - \frac{x^2}{6} < \frac{\sin x}{x} < 1$$

- (b) The graph of $y = \frac{\sin x}{x}$, $x \neq 0$, is bounded below by the graph of $y = 1 - \frac{x^2}{6}$ and above by the graph of $y = 1$ as derived in part (a).



$$30. (a) \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \Rightarrow 1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \dots \Rightarrow \frac{1 - \cos x}{x^2} = \frac{1}{2} - \frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^6}{8!} + \dots; \\ \text{if } L \text{ is the sum of the series representing } \frac{1 - \cos x}{x^2}, \text{ then by the Alternating Series Estimation Theorem} \\ L - s_1 = \frac{1 - \cos x}{x^2} - \frac{1}{2} < 0 \text{ and } \frac{1 - \cos x}{x^2} - \left(\frac{1}{2} - \frac{x^2}{24} \right) > 0. \text{ Therefore } \frac{1}{2} - \frac{x^2}{24} < \frac{1 - \cos x}{x^2} < \frac{1}{2}.$$

- (b) The graph of $y = \frac{1 - \cos x}{x^2}$ is bounded below by the graph of $y = \frac{1}{2} - \frac{x^2}{24}$ and above by the graph of $y = \frac{1}{2}$ as indicated in part (a).



$$31. \sin x \text{ when } x = 0.1; \text{ the sum is } \sin(0.1) \approx 0.099833417$$

$$32. \cos x \text{ when } x = \frac{\pi}{4}; \text{ the sum is } \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \approx 0.707106781$$

$$33. \tan^{-1} x \text{ when } x = \frac{\pi}{3}; \text{ the sum is } \tan^{-1}\left(\frac{\pi}{3}\right) \approx 0.808448$$

$$34. \ln(1+x) \text{ when } x = \pi; \text{ the sum is } \ln(1+\pi) \approx 1.421080$$

$$35. e^x \sin x = 0 + x + x^2 + x^3 \left(-\frac{1}{3!} + \frac{1}{2!}\right) + x^4 \left(-\frac{1}{3!} + \frac{1}{3!}\right) + x^5 \left(\frac{1}{5!} - \frac{1}{2!} \frac{1}{3!} + \frac{1}{4!}\right) + x^6 \left(\frac{1}{5!} - \frac{1}{3!} \frac{1}{3!} + \frac{1}{5!}\right) + \dots$$

$$= x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 - \frac{1}{90}x^6 + \dots$$

$$36. e^x \cos x = 1 + x + x^2 \left(-\frac{1}{2!} + \frac{1}{2!}\right) + x^3 \left(-\frac{1}{2!} + \frac{1}{3!}\right) + x^4 \left(\frac{1}{4!} - \frac{1}{2!} \frac{1}{2!} + \frac{1}{4!}\right) + x^5 \left(\frac{1}{4!} - \frac{1}{2!} \frac{1}{3!} + \frac{1}{5!}\right) + \dots$$

$$= 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{30}x^5 + \dots$$

$$37. \sin^2 x = \left(\frac{1 - \cos 2x}{2}\right) = \frac{1}{2} - \frac{1}{2} \cos 2x = \frac{1}{2} - \frac{1}{2} \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots\right) = \frac{2x^2}{2!} - \frac{2^3 x^4}{4!} + \frac{2^5 x^6}{6!} - \dots$$

$$\Rightarrow \frac{d}{dx}(\sin^2 x) = \frac{d}{dx} \left(\frac{2x^2}{2!} - \frac{2^3 x^4}{4!} + \frac{2^5 x^6}{6!} - \dots\right) = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \dots \Rightarrow 2 \sin x \cos x$$

$$= 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \dots = \sin 2x, \text{ which checks}$$

$$38. \cos^2 x = \cos 2x + \sin^2 x = \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} + \dots\right) + \left(\frac{2x^2}{2!} - \frac{2^3 x^4}{4!} + \frac{2^5 x^6}{6!} - \frac{2^7 x^8}{8!} + \dots\right)$$

$$= 1 - \frac{2x^2}{2!} + \frac{2^3 x^4}{4!} - \frac{2^5 x^6}{6!} + \dots = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \frac{1}{315}x^8 - \dots$$

39. A special case of Taylor's Theorem is $f(b) = f(a) + f'(c)(b - a)$, where c is between a and $b \Rightarrow f(b) - f(a) = f'(c)(b - a)$, the Mean Value Theorem.

40. If $f(x)$ is twice differentiable and at $x = a$ there is a point of inflection, then $f''(a) = 0$. Therefore, $L(x) = Q(x) = f(a) + f'(a)(x - a)$.

41. (a) $f'' \leq 0$, $f'(a) = 0$ and $x = a$ interior to the interval $I \Rightarrow f(x) - f(a) = \frac{f''(c_2)}{2}(x - a)^2 \leq 0$ throughout I
 $\Rightarrow f(x) \leq f(a)$ throughout $I \Rightarrow f$ has a local maximum at $x = a$
 (b) similar reasoning gives $f(x) - f(a) = \frac{f''(c_2)}{2}(x - a)^2 \geq 0$ throughout $I \Rightarrow f(x) \geq f(a)$ throughout $I \Rightarrow f$ has a local minimum at $x = a$

$$42. (a) f(x) = (1 - x)^{-1} \Rightarrow f'(x) = (1 - x)^{-2} \Rightarrow f''(x) = 2(1 - x)^{-3} \Rightarrow f^{(3)}(x) = 6(1 - x)^{-4}$$

$$\Rightarrow f^{(4)}(x) = 24(1 - x)^{-5}; \text{ therefore } \frac{1}{1-x} \approx 1 + x + x^2 + x^3$$

$$(b) |x| < 0.1 \Rightarrow \frac{10}{11} < \frac{1}{1-x} < \frac{10}{9} \Rightarrow \left|\frac{1}{(1-x)^5}\right| < \left(\frac{10}{9}\right)^5 \Rightarrow \left|\frac{x^4}{(1-x)^5}\right| < x^4 \left(\frac{10}{9}\right)^5 \Rightarrow \text{the error}$$

$$e_3 \leq \left|\frac{\max f^{(4)}(x) x^4}{4!}\right| < (0.1)^4 \left(\frac{10}{9}\right)^5 = 0.00016935 < 0.00017, \text{ since } \left|\frac{f^{(4)}(x)}{4!}\right| = \left|\frac{1}{(1-x)^5}\right|.$$

$$43. (a) f(x) = (1 + x)^k \Rightarrow f'(x) = k(1 + x)^{k-1} \Rightarrow f''(x) = k(k-1)(1 + x)^{k-2}; f(0) = 1, f'(0) = k, \text{ and } f''(0) = k(k-1)$$

$$\Rightarrow Q(x) = 1 + kx + \frac{k(k-1)}{2}x^2$$

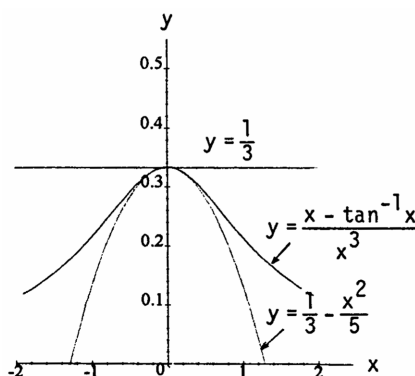
$$(b) |R_2(x)| = \left|\frac{3 \cdot 2 \cdot 1}{3!} x^3\right| < \frac{1}{100} \Rightarrow |x^3| < \frac{1}{100} \Rightarrow 0 < x < \frac{1}{100^{1/3}} \text{ or } 0 < x < .21544$$

44. Let $P = x + \pi \Rightarrow |x| = |P - \pi| < .5 \times 10^{-n}$ since P approximates π accurate to n decimals. Then,
 $P + \sin P = (\pi + x) + \sin(\pi + x) = (\pi + x) - \sin x = \pi + (x - \sin x) \Rightarrow |(P + \sin P) - \pi|$
 $= |\sin x - x| \leq \frac{|x|^3}{3!} < \frac{0.125}{3!} \times 10^{-3n} < .5 \times 10^{-3n} \Rightarrow P + \sin P$ gives an approximation to π correct to $3n$ decimals.

45. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)(n-2) \cdots (n-k+1) a_n x^{n-k}$ and $f^{(k)}(0) = k! a_k$
 $\Rightarrow a_k = \frac{f^{(k)}(0)}{k!}$ for k a nonnegative integer. Therefore, the coefficients of $f(x)$ are identical with the corresponding coefficients in the Maclaurin series of $f(x)$ and the statement follows.

46. Note: f even $\Rightarrow f(-x) = f(x) \Rightarrow -f'(-x) = f'(x) \Rightarrow f'(-x) = -f'(x) \Rightarrow f'$ odd;
 f odd $\Rightarrow f(-x) = -f(x) \Rightarrow -f'(-x) = -f'(x) \Rightarrow f'(-x) = f'(x) \Rightarrow f'$ even;
 also, f odd $\Rightarrow f(0) = f(0) \Rightarrow 2f(0) = 0 \Rightarrow f(0) = 0$
- (a) If $f(x)$ is even, then any odd-order derivative is odd and equal to 0 at $x = 0$. Therefore,
 $a_1 = a_3 = a_5 = \dots = 0$; that is, the Maclaurin series for f contains only even powers.
- (b) If $f(x)$ is odd, then any even-order derivative is odd and equal to 0 at $x = 0$. Therefore,
 $a_0 = a_2 = a_4 = \dots = 0$; that is, the Maclaurin series for f contains only odd powers.
47. (a) Suppose $f(x)$ is a continuous periodic function with period p . Let x_0 be an arbitrary real number. Then f assumes a minimum m_1 and a maximum m_2 in the interval $[x_0, x_0 + p]$; i.e., $m_1 \leq f(x) \leq m_2$ for all x in $[x_0, x_0 + p]$. Since f is periodic it has exactly the same values on all other intervals $[x_0 + p, x_0 + 2p]$, $[x_0 + 2p, x_0 + 3p]$, \dots , and $[x_0 - p, x_0]$, $[x_0 - 2p, x_0 - p]$, \dots , and so forth. That is, for all real numbers $-\infty < x < \infty$ we have $m_1 \leq f(x) \leq m_2$. Now choose $M = \max\{|m_1|, |m_2|\}$. Then $-M \leq -|m_1| \leq m_1 \leq f(x) \leq m_2 \leq |m_2| \leq M \Rightarrow |f(x)| \leq M$ for all x .
- (b) The dominate term in the n th order Taylor polynomial generated by $\cos x$ about $x = a$ is $\frac{\sin(a)}{n!}(x - a)^n$ or $\frac{\cos(a)}{n!}(x - a)^n$. In both cases, as $|x|$ increases the absolute value of these dominate terms tends to ∞ , causing the graph of $P_n(x)$ to move away from $\cos x$.

48. (b) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \Rightarrow \frac{x - \tan^{-1} x}{x^3}$
 $= \frac{1}{3} - \frac{x^2}{5} + \dots$; from the Alternating Series
 Estimation Theorem, $\frac{x - \tan^{-1} x}{x^3} - \frac{1}{3} < 0$
 $\Rightarrow \frac{x - \tan^{-1} x}{x^3} - \left(\frac{1}{3} - \frac{x^2}{5}\right) > 0 \Rightarrow \frac{1}{3} < \frac{x - \tan^{-1} x}{x^3}$
 $< \frac{1}{3} - \frac{x^2}{5}$; therefore, the $\lim_{x \rightarrow 0} \frac{x - \tan^{-1} x}{x^3} = \frac{1}{3}$



49. (a) $e^{-i\pi} = \cos(-\pi) + i \sin(-\pi) = -1 + i(0) = -1$
 (b) $e^{i\pi/4} = \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}\right)(1 + i)$
 (c) $e^{-i\pi/2} = \cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right) = 0 + i(-1) = -i$
50. $e^{i\theta} = \cos \theta + i \sin \theta \Rightarrow e^{-i\theta} = e^{i(-\theta)} = \cos(-\theta) + i \sin(-\theta) = \cos \theta - i \sin \theta$;
 $e^{i\theta} + e^{-i\theta} = \cos \theta + i \sin \theta + \cos \theta - i \sin \theta = 2 \cos \theta \Rightarrow \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$;
 $e^{i\theta} - e^{-i\theta} = \cos \theta + i \sin \theta - (\cos \theta - i \sin \theta) = 2i \sin \theta \Rightarrow \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$
51. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \Rightarrow e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots$ and
 $e^{-i\theta} = 1 - i\theta + \frac{(-i\theta)^2}{2!} + \frac{(-i\theta)^3}{3!} + \frac{(-i\theta)^4}{4!} + \dots = 1 - i\theta + \frac{(i\theta)^2}{2!} - \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} - \dots$
 $\Rightarrow \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{\left(1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots\right) + \left(1 - i\theta + \frac{(i\theta)^2}{2!} - \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} - \dots\right)}{2}$
 $= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots = \cos \theta$;
 $\frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{\left(1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots\right) - \left(1 - i\theta + \frac{(i\theta)^2}{2!} - \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} - \dots\right)}{2i}$
 $= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots = \sin \theta$

$$52. e^{i\theta} = \cos \theta + i \sin \theta \Rightarrow e^{-i\theta} = e^{i(-\theta)} = \cos(-\theta) + i \sin(-\theta) = \cos \theta - i \sin \theta$$

$$(a) e^{i\theta} + e^{-i\theta} = (\cos \theta + i \sin \theta) + (\cos \theta - i \sin \theta) = 2 \cos \theta \Rightarrow \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cosh i\theta$$

$$(b) e^{i\theta} - e^{-i\theta} = (\cos \theta + i \sin \theta) - (\cos \theta - i \sin \theta) = 2i \sin \theta \Rightarrow i \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2} = \sinh i\theta$$

$$53. e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)$$

$$= (1)x + (1)x^2 + \left(-\frac{1}{6} + \frac{1}{2}\right)x^3 + \left(-\frac{1}{6} + \frac{1}{6}\right)x^4 + \left(\frac{1}{120} - \frac{1}{12} + \frac{1}{24}\right)x^5 + \dots = x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \dots;$$

$$e^x \cdot e^{ix} = e^{(1+i)x} = e^x (\cos x + i \sin x) = e^x \cos x + i(e^x \sin x) \Rightarrow e^x \sin x \text{ is the series of the imaginary part}$$

$$\text{of } e^{(1+i)x} \text{ which we calculate next; } e^{(1+i)x} = \sum_{n=0}^{\infty} \frac{(x+ix)^n}{n!} = 1 + (x+ix) + \frac{(x+ix)^2}{2!} + \frac{(x+ix)^3}{3!} + \frac{(x+ix)^4}{4!} + \dots$$

$$= 1 + x + ix + \frac{1}{2!}(2ix^2) + \frac{1}{3!}(2ix^3 - 2x^3) + \frac{1}{4!}(-4x^4) + \frac{1}{5!}(-4x^5 - 4ix^5) + \frac{1}{6!}(-8ix^6) + \dots \Rightarrow \text{the imaginary part}$$

$$\text{of } e^{(1+i)x} \text{ is } x + \frac{2}{2!}x^2 + \frac{2}{3!}x^3 - \frac{4}{5!}x^5 - \frac{8}{6!}x^6 + \dots = x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 - \frac{1}{90}x^6 + \dots \text{ in agreement with our}$$

$$\text{product calculation. The series for } e^x \sin x \text{ converges for all values of } x.$$

$$54. \frac{d}{dx} (e^{(a+ib)x}) = \frac{d}{dx} [e^{ax}(\cos bx + i \sin bx)] = ae^{ax}(\cos bx + i \sin bx) + e^{ax}(-b \sin bx + bi \cos bx)$$

$$= ae^{ax}(\cos bx + i \sin bx) + bie^{ax}(\cos bx + i \sin bx) = ae^{(a+ib)x} + ibe^{(a+ib)x} = (a+ib)e^{(a+ib)x}$$

$$55. (a) e^{i\theta_1} e^{i\theta_2} = (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) = (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \sin \theta_2 \cos \theta_1)$$

$$= \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) = e^{i(\theta_1 + \theta_2)}$$

$$(b) e^{-i\theta} = \cos(-\theta) + i \sin(-\theta) = \cos \theta - i \sin \theta = (\cos \theta - i \sin \theta) \left(\frac{\cos \theta + i \sin \theta}{\cos \theta + i \sin \theta} \right) = \frac{1}{\cos \theta + i \sin \theta} = \frac{1}{e^{i\theta}}$$

$$56. \frac{a-bi}{a^2+b^2} e^{(a+bi)x} + C_1 + iC_2 = \left(\frac{a-bi}{a^2+b^2} \right) e^{ax}(\cos bx + i \sin bx) + C_1 + iC_2$$

$$= \frac{e^{ax}}{a^2+b^2} (a \cos bx + ia \sin bx - ib \cos bx + b \sin bx) + C_1 + iC_2$$

$$= \frac{e^{ax}}{a^2+b^2} [(a \cos bx + b \sin bx) + (a \sin bx - b \cos bx)i] + C_1 + iC_2$$

$$= \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2+b^2} + C_1 + \frac{ie^{ax}(a \sin bx - b \cos bx)}{a^2+b^2} + iC_2;$$

$$e^{(a+bi)x} = e^{ax} e^{ibx} = e^{ax}(\cos bx + i \sin bx) = e^{ax} \cos bx + ie^{ax} \sin bx, \text{ so that given}$$

$$\int e^{(a+bi)x} dx = \frac{a-bi}{a^2+b^2} e^{(a+bi)x} + C_1 + iC_2 \text{ we conclude that } \int e^{ax} \cos bx dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2+b^2} + C_1$$

$$\text{and } \int e^{ax} \sin bx dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2+b^2} + C_2$$

57-62. Example CAS commands:

Maple:

```
f := x -> 1/sqrt(1+x);
```

```
x0 := -3/4;
```

```
x1 := 3/4;
```

```
# Step 1:
```

```
plot( f(x), x=x0..x1, title="Step 1: #57 (Section 11.9)" );
```

```
# Step 2:
```

```
P1 := unapply( TaylorApproximation(f(x), x = 0, order=1), x );
```

```
P2 := unapply( TaylorApproximation(f(x), x = 0, order=2), x );
```

```
P3 := unapply( TaylorApproximation(f(x), x = 0, order=3), x );
```

```
# Step 3:
```

```
D2f := D(D(f));
```

```
D3f := D(D(D(f)));
```

```
D4f := D(D(D(D(f))));
```

```
plot( [D2f(x), D3f(x), D4f(x)], x=x0..x1, thickness=[0,2,4], color=[red,blue,green], title="Step 3: #57 (Section 11.9)" );
```

```
c1 := x0;
```

```
M1 := abs( D2f(c1) );
```

```

c2 := x0;
M2 := abs( D3f(c2) );
c3 := x0;
M3 := abs( D4f(c3) );
# Step 4:
R1 := unapply( abs(M1/2!*(x-0)^2), x );
R2 := unapply( abs(M2/3!*(x-0)^3), x );
R3 := unapply( abs(M3/4!*(x-0)^4), x );
plot( [R1(x),R2(x),R3(x)], x=x0..x1, thickness=[0,2,4], color=[red,blue,green], title="Step 4: #57 (Section 11.9)" );
# Step 5:
E1 := unapply( abs(f(x)-P1(x)), x );
E2 := unapply( abs(f(x)-P2(x)), x );
E3 := unapply( abs(f(x)-P3(x)), x );
plot( [E1(x),E2(x),E3(x),R1(x),R2(x),R3(x)], x=x0..x1, thickness=[0,2,4], color=[red,blue,green],
linestyle=[1,1,1,3,3,3], title="Step 5: #57 (Section 11.9)" );
# Step 6:
TaylorApproximation( f(x), view=[x0..x1,DEFAULT], x=0, output=animation, order=1..3 );
L1 := fsolve( abs(f(x)-P1(x))=0.01, x=x0/2 );          # (a)
R1 := fsolve( abs(f(x)-P1(x))=0.01, x=x1/2 );
L2 := fsolve( abs(f(x)-P2(x))=0.01, x=x0/2 );
R2 := fsolve( abs(f(x)-P2(x))=0.01, x=x1/2 );
L3 := fsolve( abs(f(x)-P3(x))=0.01, x=x0/2 );
R3 := fsolve( abs(f(x)-P3(x))=0.01, x=x1/2 );
plot( [E1(x),E2(x),E3(x),0.01], x=min(L1,L2,L3)..max(R1,R2,R3), thickness=[0,2,4,0], linestyle=[0,0,0,2],
color=[red,blue,green,black], view=[DEFAULT,0..0.01], title="#57(a) (Section 11.9)" );
abs( f(x) - P[1](x) ) <= evalf( E1(x0) );              # (b)
abs( f(x) - P[2](x) ) <= evalf( E2(x0) );
abs( f(x) - P[3](x) ) <= evalf( E3(x0) );

```

Mathematica: (assigned function and values for a, b, c, and n may vary)

```

Clear[x, f, c]
f[x_] = (1 + x)^(3/2)
{a, b} = {-1/2, 2};
pf = Plot[ f[x], {x, a, b}];
poly1[x_] = Series[f[x], {x, 0, 1}]/Normal
poly2[x_] = Series[f[x], {x, 0, 2}]/Normal
poly3[x_] = Series[f[x], {x, 0, 3}]/Normal
Plot[{f[x], poly1[x], poly2[x], poly3[x]}, {x, a, b},
PlotStyle -> {RGBColor[1,0,0], RGBColor[0,1,0], RGBColor[0,0,1], RGBColor[0,.5,.5]}];

```

The above defines the approximations. The following analyzes the derivatives to determine their maximum values.

```

f''[c]
Plot[f''[x], {x, a, b}];
f'''[c]
Plot[f'''[x], {x, a, b}];
f''''[c]
Plot[f''''[x], {x, a, b}];

```

Noting the upper bound for each of the above derivatives occurs at $x = a$, the upper bounds m_1 , m_2 , and m_3 can be defined and bounds for remainders viewed as functions of x .

```

m1 = f''[a]
m2 = -f'''[a]

```

```

m3=f'''[a]
r1[x_]=m1 x^2 /2!
Plot[r1[x], {x, a, b}];
r2[x_]=m2 x^3 /3!
Plot[r2[x], {x, a, b}];
r3[x_]=m3 x^4 /4!
Plot[r3[x], {x, a, b}];

```

A three dimensional look at the error functions, allowing both c and x to vary can also be viewed. Recall that c must be a value between 0 and x , so some points on the surfaces where c is not in that interval are meaningless.

```

Plot3D[f''[c] x^2 /2!, {x, a, b}, {c, a, b}, PlotRange -> All]
Plot3D[f'''[c] x^3 /3!, {x, a, b}, {c, a, b}, PlotRange -> All]
Plot3D[f''''[c] x^4 /4!, {x, a, b}, {c, a, b}, PlotRange -> All]

```

11.10 APPLICATIONS OF POWER SERIES

- $(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{(\frac{1}{2})(-\frac{1}{2})x^2}{2!} + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})x^3}{3!} + \dots = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots$
- $(1+x)^{1/3} = 1 + \frac{1}{3}x + \frac{(\frac{1}{3})(-\frac{2}{3})x^2}{2!} + \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})x^3}{3!} + \dots = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \dots$
- $(1-x)^{-1/2} = 1 - \frac{1}{2}(-x) + \frac{(-\frac{1}{2})(-\frac{3}{2})(-x)^2}{2!} + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-x)^3}{3!} + \dots = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots$
- $(1-2x)^{1/2} = 1 + \frac{1}{2}(-2x) + \frac{(\frac{1}{2})(-\frac{1}{2})(-2x)^2}{2!} + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-2x)^3}{3!} + \dots = 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 - \dots$
- $(1+\frac{x}{2})^{-2} = 1 - 2(\frac{x}{2}) + \frac{(-2)(-3)(\frac{x}{2})^2}{2!} + \frac{(-2)(-3)(-4)(\frac{x}{2})^3}{3!} + \dots = 1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3$
- $(1-\frac{x}{2})^{-2} = 1 - 2(-\frac{x}{2}) + \frac{(-2)(-3)(-\frac{x}{2})^2}{2!} + \frac{(-2)(-3)(-4)(-\frac{x}{2})^3}{3!} + \dots = 1 + x + \frac{3}{4}x^2 + \frac{1}{2}x^3 + \dots$
- $(1+x^3)^{-1/2} = 1 - \frac{1}{2}x^3 + \frac{(-\frac{1}{2})(-\frac{3}{2})(x^3)^2}{2!} + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(x^3)^3}{3!} + \dots = 1 - \frac{1}{2}x^3 + \frac{3}{8}x^6 - \frac{5}{16}x^9 + \dots$
- $(1+x^2)^{-1/3} = 1 - \frac{1}{3}x^2 + \frac{(-\frac{1}{3})(-\frac{4}{3})(x^2)^2}{2!} + \frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})(x^2)^3}{3!} + \dots = 1 - \frac{1}{3}x^2 + \frac{2}{9}x^4 - \frac{14}{81}x^6 + \dots$
- $(1+\frac{1}{x})^{1/2} = 1 + \frac{1}{2}(\frac{1}{x}) + \frac{(\frac{1}{2})(-\frac{1}{2})(\frac{1}{x})^2}{2!} + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(\frac{1}{x})^3}{3!} + \dots = 1 + \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{16x^3}$
- $(1-\frac{2}{x})^{1/3} = 1 + \frac{1}{3}(-\frac{2}{x}) + \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{2}{x})^2}{2!} + \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})(-\frac{2}{x})^3}{3!} + \dots = 1 - \frac{2}{3x} - \frac{4}{9x^2} - \frac{40}{81x^3} - \dots$
- $(1+x)^4 = 1 + 4x + \frac{(4)(3)x^2}{2!} + \frac{(4)(3)(2)x^3}{3!} + \frac{(4)(3)(2)x^4}{4!} = 1 + 4x + 6x^2 + 4x^3 + x^4$
- $(1+x^2)^3 = 1 + 3x^2 + \frac{(3)(2)(x^2)^2}{2!} + \frac{(3)(2)(1)(x^2)^3}{3!} = 1 + 3x^2 + 3x^4 + x^6$
- $(1-2x)^3 = 1 + 3(-2x) + \frac{(3)(2)(-2x)^2}{2!} + \frac{(3)(2)(1)(-2x)^3}{3!} = 1 - 6x + 12x^2 - 8x^3$
- $(1-\frac{x}{2})^4 = 1 + 4(-\frac{x}{2}) + \frac{(4)(3)(-\frac{x}{2})^2}{2!} + \frac{(4)(3)(2)(-\frac{x}{2})^3}{3!} + \frac{(4)(3)(2)(1)(-\frac{x}{2})^4}{4!} = 1 - 2x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{16}x^4$

15. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n + \dots$

$$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \dots + na_nx^{n-1} + \dots$$

$$\Rightarrow \frac{dy}{dx} + y = (a_1 + a_0) + (2a_2 + a_1)x + (3a_3 + a_2)x^2 + \dots + (na_n + a_{n-1})x^{n-1} + \dots = 0$$

$\Rightarrow a_1 + a_0 = 0, 2a_2 + a_1 = 0, 3a_3 + a_2 = 0$ and in general $na_n + a_{n-1} = 0$. Since $y = 1$ when $x = 0$ we have

$$a_0 = 1. \text{ Therefore } a_1 = -1, a_2 = \frac{-a_1}{2 \cdot 1} = \frac{1}{2}, a_3 = \frac{-a_2}{3} = -\frac{1}{3 \cdot 2}, \dots, a_n = \frac{-a_{n-1}}{n} = \frac{(-1)^n}{n!}$$

$$\Rightarrow y = 1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \dots + \frac{(-1)^n}{n!}x^n + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = e^{-x}$$

16. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n + \dots$

$$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \dots + na_nx^{n-1} + \dots$$

$$\Rightarrow \frac{dy}{dx} - 2y = (a_1 - 2a_0) + (2a_2 - 2a_1)x + (3a_3 - 2a_2)x^2 + \dots + (na_n - 2a_{n-1})x^{n-1} + \dots = 0$$

$\Rightarrow a_1 - 2a_0 = 0, 2a_2 - 2a_1 = 0, 3a_3 - 2a_2 = 0$ and in general $na_n - 2a_{n-1} = 0$. Since $y = 1$ when $x = 0$ we have

$$a_0 = 1. \text{ Therefore } a_1 = 2a_0 = 2(1) = 2, a_2 = \frac{2}{2}a_1 = \frac{2}{2}(2) = \frac{2^2}{2}, a_3 = \frac{2}{3}a_2 = \frac{2}{3}\left(\frac{2^2}{2}\right) = \frac{2^3}{3 \cdot 2}, \dots,$$

$$a_n = \left(\frac{2}{n}\right)a_{n-1} = \left(\frac{2}{n}\right)\left(\frac{2^{n-1}}{n-1}\right)a_{n-2} = \frac{2^n}{n!} \Rightarrow y = 1 + 2x + \frac{2^2}{2}x^2 + \frac{2^3}{3!}x^3 + \dots + \frac{2^n}{n!}x^n + \dots$$

$$= 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \dots + \frac{(2x)^n}{n!} + \dots = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = e^{2x}$$

17. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n + \dots$

$$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \dots + na_nx^{n-1} + \dots$$

$$\Rightarrow \frac{dy}{dx} - y = (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \dots + (na_n - a_{n-1})x^{n-1} + \dots = 1$$

$\Rightarrow a_1 - a_0 = 1, 2a_2 - a_1 = 0, 3a_3 - a_2 = 0$ and in general $na_n - a_{n-1} = 0$. Since $y = 0$ when $x = 0$ we have

$$a_0 = 0. \text{ Therefore } a_1 = 1, a_2 = \frac{a_1}{2} = \frac{1}{2}, a_3 = \frac{a_2}{3} = \frac{1}{3 \cdot 2}, a_4 = \frac{a_3}{4} = \frac{1}{4 \cdot 3 \cdot 2}, \dots, a_n = \frac{a_{n-1}}{n} = \frac{1}{n!}$$

$$\Rightarrow y = 0 + 1x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + \dots + \frac{1}{n!}x^n + \dots$$

$$= \left(1 + 1x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + \dots + \frac{1}{n!}x^n + \dots\right) - 1 = \sum_{n=0}^{\infty} \frac{x^n}{n!} - 1 = e^x - 1$$

18. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n + \dots$

$$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \dots + na_nx^{n-1} + \dots$$

$$\Rightarrow \frac{dy}{dx} + y = (a_1 + a_0) + (2a_2 + a_1)x + (3a_3 + a_2)x^2 + \dots + (na_n + a_{n-1})x^{n-1} + \dots = 1$$

$\Rightarrow a_1 + a_0 = 1, 2a_2 + a_1 = 0, 3a_3 + a_2 = 0$ and in general $na_n + a_{n-1} = 0$. Since $y = 2$ when $x = 0$ we have

$$a_0 = 2. \text{ Therefore } a_1 = 1 - a_0 = -1, a_2 = \frac{-a_1}{2 \cdot 1} = \frac{1}{2}, a_3 = \frac{-a_2}{3} = -\frac{1}{3 \cdot 2}, \dots, a_n = \frac{-a_{n-1}}{n} = \frac{(-1)^n}{n!}$$

$$\Rightarrow y = 2 - x + \frac{1}{2}x^2 - \frac{1}{3 \cdot 2}x^3 + \dots + \frac{(-1)^n}{n!}x^n + \dots = 1 + \left(1 - x + \frac{1}{2}x^2 - \frac{1}{3 \cdot 2}x^3 + \dots + \frac{(-1)^n}{n!}x^n + \dots\right)$$

$$= 1 + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = 1 + e^{-x}$$

19. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n + \dots$

$$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \dots + na_nx^{n-1} + \dots$$

$$\Rightarrow \frac{dy}{dx} - y = (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \dots + (na_n - a_{n-1})x^{n-1} + \dots = x$$

$\Rightarrow a_1 - a_0 = 0, 2a_2 - a_1 = 1, 3a_3 - a_2 = 0$ and in general $na_n - a_{n-1} = 0$. Since $y = 0$ when $x = 0$ we have

$$a_0 = 0. \text{ Therefore } a_1 = 0, a_2 = \frac{1+a_1}{2} = \frac{1}{2}, a_3 = \frac{a_2}{3} = \frac{1}{3 \cdot 2}, a_4 = \frac{a_3}{4} = \frac{1}{4 \cdot 3 \cdot 2}, \dots, a_n = \frac{a_{n-1}}{n} = \frac{1}{n!}$$

$$\Rightarrow y = 0 + 0x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + \dots + \frac{1}{n!}x^n + \dots$$

$$= \left(1 + 1x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + \dots + \frac{1}{n!}x^n + \dots\right) - 1 - x = \sum_{n=0}^{\infty} \frac{x^n}{n!} - 1 - x = e^x - x - 1$$

20. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n + \dots$

$$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \dots + na_nx^{n-1} + \dots$$

$$\begin{aligned}
&\Rightarrow \frac{dy}{dx} + y = (a_1 + a_0) + (2a_2 + a_1)x + (3a_3 + a_2)x^2 + \dots + (na_n + a_{n-1})x^{n-1} + \dots = 2x \\
&\Rightarrow a_1 + a_0 = 0, 2a_2 + a_1 = 2, 3a_3 + a_2 = 0 \text{ and in general } na_n + a_{n-1} = 0. \text{ Since } y = -1 \text{ when } x = 0 \text{ we have} \\
&a_0 = -1. \text{ Therefore } a_1 = 1, a_2 = \frac{2-a_1}{2} = \frac{1}{2}, a_3 = \frac{-a_2}{3} = -\frac{1}{3 \cdot 2}, \dots, a_n = \frac{-a_{n-1}}{n} = \frac{(-1)^n}{n!} \\
&\Rightarrow y = -1 + 1x + \frac{1}{2}x^2 - \frac{1}{3 \cdot 2}x^3 + \dots + \frac{(-1)^n}{n!}x^n + \dots \\
&= \left(1 - 1x + \frac{1}{2}x^2 - \frac{1}{3 \cdot 2}x^3 + \dots + \frac{(-1)^n}{n!}x^n + \dots\right) - 2 + 2x = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} - 2 + 2x = e^{-x} + 2x - 2
\end{aligned}$$

21. $y' - xy = a_1 + (2a_2 - a_0)x + (3a_3 - a_1)x + \dots + (na_n - a_{n-2})x^{n-1} + \dots = 0 \Rightarrow a_1 = 0, 2a_2 - a_0 = 0, 3a_3 - a_1 = 0,$
 $4a_4 - a_2 = 0$ and in general $na_n - a_{n-2} = 0$. Since $y = 1$ when $x = 0$, we have $a_0 = 1$. Therefore $a_2 = \frac{a_0}{2} = \frac{1}{2},$
 $a_3 = \frac{a_1}{3} = 0, a_4 = \frac{a_2}{4} = \frac{1}{2 \cdot 4}, a_5 = \frac{a_3}{5} = 0, \dots, a_{2n} = \frac{1}{2 \cdot 4 \cdot 6 \dots 2n}$ and $a_{2n+1} = 0$

$$\Rightarrow y = 1 + \frac{1}{2}x^2 + \frac{1}{2 \cdot 4}x^4 + \frac{1}{2 \cdot 4 \cdot 6}x^6 + \dots + \frac{1}{2 \cdot 4 \cdot 6 \dots 2n}x^{2n} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = \sum_{n=0}^{\infty} \frac{\left(\frac{x^2}{2}\right)^n}{n!} = e^{x^2/2}$$

22. $y' - x^2y = a_1 + 2a_2x + (3a_3 - a_0)x^2 + (4a_4 - a_1)x^3 + \dots + (na_n - a_{n-3})x^{n-1} + \dots = 0 \Rightarrow a_1 = 0, a_2 = 0,$
 $3a_3 - a_0 = 0, 4a_4 - a_1 = 0$ and in general $na_n - a_{n-3} = 0$. Since $y = 1$ when $x = 0$, we have $a_0 = 1$. Therefore
 $a_3 = \frac{a_0}{3} = \frac{1}{3}, a_4 = \frac{a_1}{4} = 0, a_5 = \frac{a_2}{5} = 0, a_6 = \frac{a_3}{6} = \frac{1}{3 \cdot 6}, \dots, a_{3n} = \frac{1}{3 \cdot 6 \cdot 9 \dots 3n}, a_{3n+1} = 0$ and $a_{3n+2} = 0$

$$\Rightarrow y = 1 + \frac{1}{3}x^3 + \frac{1}{3 \cdot 6}x^6 + \frac{1}{3 \cdot 6 \cdot 9}x^9 + \dots + \frac{1}{3 \cdot 6 \cdot 9 \dots 3n}x^{3n} + \dots = \sum_{n=0}^{\infty} \frac{x^{3n}}{3^n n!} = \sum_{n=0}^{\infty} \frac{\left(\frac{x^3}{3}\right)^n}{n!} = e^{x^3/3}$$

23. $(1-x)y' - y = (a_1 - a_0) + (2a_2 - a_1 - a_1)x + (3a_3 - 2a_2 - a_2)x^2 + (4a_4 - 3a_3 - a_3)x^3 + \dots$
 $+ (na_n - (n-1)a_{n-1} - a_{n-1})x^{n-1} + \dots = 0 \Rightarrow a_1 - a_0 = 0, 2a_2 - 2a_1 = 0, 3a_3 - 3a_2 = 0$ and in
 general $(na_n - na_{n-1}) = 0$. Since $y = 2$ when $x = 0$, we have $a_0 = 2$. Therefore

$$a_1 = 2, a_2 = 2, \dots, a_n = 2 \Rightarrow y = 2 + 2x + 2x^2 + \dots = \sum_{n=0}^{\infty} 2x^n = \frac{2}{1-x}$$

24. $(1+x^2)y' + 2xy = a_1 + (2a_2 + 2a_0)x + (3a_3 + 2a_1 + a_1)x^2 + (4a_4 + 2a_2 + 2a_2)x^3 + \dots + (na_n + na_{n-2})x^{n-1} + \dots$
 $= 0 \Rightarrow a_1 = 0, 2a_2 + 2a_0 = 0, 3a_3 + 3a_1 = 0, 4a_4 + 4a_2 = 0$ and in general $na_n + na_{n-2} = 0$. Since $y = 3$ when
 $x = 0$, we have $a_0 = 3$. Therefore $a_2 = -3, a_3 = 0, a_4 = 3, \dots, a_{2n+1} = 0, a_{2n} = (-1)^n 3$

$$\Rightarrow y = 3 - 3x^2 + 3x^4 - \dots = \sum_{n=0}^{\infty} 3(-1)^n x^{2n} = \sum_{n=0}^{\infty} 3(-x^2)^n = \frac{3}{1+x^2}$$

25. $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots \Rightarrow y'' = 2a_2 + 3 \cdot 2a_3x + \dots + n(n-1)a_nx^{n-2} + \dots \Rightarrow y'' - y$
 $= (2a_2 - a_0) + (3 \cdot 2a_3 - a_1)x + (4 \cdot 3a_4 - a_2)x^2 + \dots + (n(n-1)a_n - a_{n-2})x^{n-2} + \dots = 0 \Rightarrow 2a_2 - a_0 = 0,$
 $3 \cdot 2a_3 - a_1 = 0, 4 \cdot 3a_4 - a_2 = 0$ and in general $n(n-1)a_n - a_{n-2} = 0$. Since $y' = 1$ and $y = 0$ when $x = 0$,
 we have $a_0 = 0$ and $a_1 = 1$. Therefore $a_2 = 0, a_3 = \frac{1}{3 \cdot 2}, a_4 = 0, a_5 = \frac{1}{5 \cdot 4 \cdot 3 \cdot 2}, \dots, a_{2n+1} = \frac{1}{(2n+1)!}$ and

$$a_{2n} = 0 \Rightarrow y = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sinh x$$

26. $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots \Rightarrow y'' = 2a_2 + 3 \cdot 2a_3x + \dots + n(n-1)a_nx^{n-2} + \dots \Rightarrow y'' + y$
 $= (2a_2 + a_0) + (3 \cdot 2a_3 + a_1)x + (4 \cdot 3a_4 + a_2)x^2 + \dots + (n(n-1)a_n + a_{n-2})x^{n-2} + \dots = 0 \Rightarrow 2a_2 + a_0 = 0,$
 $3 \cdot 2a_3 + a_1 = 0, 4 \cdot 3a_4 + a_2 = 0$ and in general $n(n-1)a_n + a_{n-2} = 0$. Since $y' = 0$ and $y = 1$ when $x = 0$,
 we have $a_0 = 1$ and $a_1 = 0$. Therefore $a_2 = -\frac{1}{2}, a_3 = 0, a_4 = \frac{1}{4 \cdot 3 \cdot 2}, a_5 = 0, \dots, a_{2n+1} = 0$ and $a_{2n} = \frac{(-1)^n}{(2n)!}$

$$\Rightarrow y = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos x$$

27. $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots \Rightarrow y'' = 2a_2 + 3 \cdot 2a_3x + \dots + n(n-1)a_nx^{n-2} + \dots \Rightarrow y'' + y$
 $= (2a_2 + a_0) + (3 \cdot 2a_3 + a_1)x + (4 \cdot 3a_4 + a_2)x^2 + \dots + (n(n-1)a_n + a_{n-2})x^{n-2} + \dots = x \Rightarrow 2a_2 + a_0 = 0,$
 $3 \cdot 2a_3 + a_1 = 1, 4 \cdot 3a_4 + a_2 = 0$ and in general $n(n-1)a_n + a_{n-2} = 0$. Since $y' = 1$ and $y = 2$ when $x = 0$,

we have $a_0 = 2$ and $a_1 = 1$. Therefore $a_2 = -1$, $a_3 = 0$, $a_4 = \frac{1}{4 \cdot 3}$, $a_5 = 0$, \dots , $a_{2n} = -2 \cdot \frac{(-1)^{n+1}}{(2n)!}$ and

$$a_{2n+1} = 0 \Rightarrow y = 2 + x - x^2 + 2 \cdot \frac{x^4}{4!} + \dots = 2 + x - 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!} = x + \cos 2x$$

28. $y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots \Rightarrow y'' = 2a_2 + 3 \cdot 2a_3 x + \dots + n(n-1)a_n x^{n-2} + \dots \Rightarrow y'' - y$
 $= (2a_2 - a_0) + (3 \cdot 2a_3 - a_1)x + (4 \cdot 3a_4 - a_2)x^2 + \dots + (n(n-1)a_n - a_{n-2})x^{n-2} + \dots = x \Rightarrow 2a_2 - a_0 = 0$,
 $3 \cdot 2a_3 - a_1 = 1$, $4 \cdot 3a_4 - a_2 = 0$ and in general $n(n-1)a_n - a_{n-2} = 0$. Since $y' = 2$ and $y = -1$ when $x = 0$,
 we have $a_0 = -1$ and $a_1 = 2$. Therefore $a_2 = \frac{-1}{2}$, $a_3 = \frac{1}{2}$, $a_4 = \frac{-1}{2 \cdot 3 \cdot 4}$, $a_5 = \frac{1}{5 \cdot 4 \cdot 2} = \frac{3}{5!}$, \dots , $a_{2n} = \frac{-1}{(2n)!}$

$$\text{and } a_{2n+1} = \frac{3}{(2n+1)!} \Rightarrow y = -1 + 2x - \frac{1}{2}x^2 + \frac{3}{3!}x^3 - \dots = -1 + 2x - \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} + \sum_{n=1}^{\infty} \frac{3x^{2n+1}}{(2n+1)!}$$

29. $y = a_0 + a_1(x-2) + a_2(x-2)^2 + \dots + a_n(x-2)^n + \dots$
 $\Rightarrow y'' = 2a_2 + 3 \cdot 2a_3(x-2) + \dots + n(n-1)a_n(x-2)^{n-2} + \dots \Rightarrow y'' - y$
 $= (2a_2 - a_0) + (3 \cdot 2a_3 - a_1)(x-2) + (4 \cdot 3a_4 - a_2)(x-2)^2 + \dots + (n(n-1)a_n - a_{n-2})(x-2)^{n-2} + \dots = -x$
 $= -(x-2) - 2 \Rightarrow 2a_2 - a_0 = -2$, $3 \cdot 2a_3 - a_1 = -1$, and $n(n-1)a_n - a_{n-2} = 0$ for $n > 3$. Since $y = 0$ when $x = 2$,
 we have $a_0 = 0$, and since $y' = -2$ when $x = 2$, we have $a_1 = -2$. Therefore $a_2 = -1$, $a_3 = -\frac{1}{2}$, $a_4 = \frac{1}{4 \cdot 3}(-1) = \frac{-2}{4 \cdot 3 \cdot 2 \cdot 1}$,
 $a_5 = \frac{1}{5 \cdot 4}(-\frac{1}{2}) = \frac{-3}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$, \dots , $a_{2n} = \frac{-2}{(2n)!}$, and $a_{2n+1} = \frac{-3}{(2n+1)!}$. Since $a_1 = -2$, we have $a_1(x-2) = (-2)(x-2)$ and
 $(-2)(x-2) = (-3+1)(x-2) = (-3)(x-2) + (1)(x-2) = x-2-3(x-2)$.
 $\Rightarrow y = x-2-3(x-2) - \frac{2}{2!}(x-2)^2 - \frac{3}{3!}(x-2)^3 - \frac{2}{4!}(x-2)^4 - \frac{3}{5!}(x-2)^5 - \dots$
 $\Rightarrow y = x-2 - \frac{2}{2!}(x-2)^2 - \frac{2}{4!}(x-2)^4 - \dots - 3(x-2)x - \frac{3}{3!}(x-2)^3 - \frac{3}{5!}(x-2)^5 - \dots$
 $\Rightarrow y = x - 2 \sum_{n=0}^{\infty} \frac{(x-2)^{2n}}{(2n)!} - 3 \sum_{n=0}^{\infty} \frac{(x-2)^{2n+1}}{(2n+1)!}$

30. $y'' - x^2 y = 2a_2 + 6a_3 x + (4 \cdot 3a_4 - a_0)x^2 + \dots + (n(n-1)a_n - a_{n-4})x^{n-2} + \dots = 0 \Rightarrow 2a_2 = 0$, $6a_3 = 0$,
 $4 \cdot 3a_4 - a_0 = 0$, $5 \cdot 4a_5 - a_1 = 0$, and in general $n(n-1)a_n - a_{n-4} = 0$. Since $y' = b$ and $y = a$ when $x = 0$,
 we have $a_0 = a$, $a_1 = b$, $a_2 = 0$, $a_3 = 0$, $a_4 = \frac{a}{3 \cdot 4}$, $a_5 = \frac{b}{4 \cdot 5}$, $a_6 = 0$, $a_7 = 0$, $a_8 = \frac{a}{3 \cdot 4 \cdot 7 \cdot 8}$, $a_9 = \frac{b}{4 \cdot 5 \cdot 8 \cdot 9}$
 $\Rightarrow y = a + bx + \frac{a}{3 \cdot 4}x^4 + \frac{b}{4 \cdot 5}x^5 + \frac{a}{3 \cdot 4 \cdot 7 \cdot 8}x^8 + \frac{b}{4 \cdot 5 \cdot 8 \cdot 9}x^9 + \dots$

31. $y'' + x^2 y = 2a_2 + 6a_3 x + (4 \cdot 3a_4 + a_0)x^2 + \dots + (n(n-1)a_n + a_{n-4})x^{n-2} + \dots = x \Rightarrow 2a_2 = 0$, $6a_3 = 1$,
 $4 \cdot 3a_4 + a_0 = 0$, $5 \cdot 4a_5 + a_1 = 0$, and in general $n(n-1)a_n + a_{n-4} = 0$. Since $y' = b$ and $y = a$ when $x = 0$,
 we have $a_0 = a$ and $a_1 = b$. Therefore $a_2 = 0$, $a_3 = \frac{1}{2 \cdot 3}$, $a_4 = -\frac{a}{3 \cdot 4}$, $a_5 = -\frac{b}{4 \cdot 5}$, $a_6 = 0$, $a_7 = \frac{-1}{2 \cdot 3 \cdot 6 \cdot 7}$
 $\Rightarrow y = a + bx + \frac{1}{2 \cdot 3}x^3 - \frac{a}{3 \cdot 4}x^4 - \frac{b}{4 \cdot 5}x^5 - \frac{1}{2 \cdot 3 \cdot 6 \cdot 7}x^7 + \frac{ax^8}{3 \cdot 4 \cdot 7 \cdot 8} + \frac{bx^9}{4 \cdot 5 \cdot 8 \cdot 9} + \dots$

32. $y'' - 2y' + y = (2a_2 - 2a_1 + a_0) + (2 \cdot 3a_3 - 4a_2 + a_1)x + (3 \cdot 4a_4 - 2 \cdot 3a_3 + a_2)x^2 + \dots$
 $+ ((n-1)a_n - 2(n-1)a_{n-1} + a_{n-2})x^{n-2} + \dots = 0 \Rightarrow 2a_2 - 2a_1 + a_0 = 0$, $2 \cdot 3a_3 - 4a_2 + a_1 = 0$,
 $3 \cdot 4a_4 - 2 \cdot 3a_3 + a_2 = 0$ and in general $(n-1)a_n - 2(n-1)a_{n-1} + a_{n-2} = 0$. Since $y' = 1$ and $y = 0$ when
 when $x = 0$, we have $a_0 = 0$ and $a_1 = 1$. Therefore $a_2 = 1$, $a_3 = \frac{1}{2}$, $a_4 = \frac{1}{6}$, $a_5 = \frac{1}{24}$ and $a_n = \frac{1}{(n-1)!}$
 $\Rightarrow y = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \dots = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x \sum_{n=0}^{\infty} \frac{x^n}{n!} = xe^x$

33. $\int_0^{0.2} \sin x^2 dx = \int_0^{0.2} \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \dots \right) dx = \left[\frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \dots \right]_0^{0.2} \approx \left[\frac{x^3}{3} \right]_0^{0.2} \approx 0.00267$ with error
 $|E| \leq \frac{(0.2)^7}{7 \cdot 3!} \approx 0.0000003$

34. $\int_0^{0.2} \frac{e^{-x}-1}{x} dx = \int_0^{0.2} \frac{1}{x} \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots - 1 \right) dx = \int_0^{0.2} \left(-1 + \frac{x}{2} - \frac{x^2}{6} + \frac{x^3}{24} - \dots \right) dx$
 $= \left[-x + \frac{x^2}{4} - \frac{x^3}{18} + \dots \right]_0^{0.2} \approx -0.19044$ with error $|E| \leq \frac{(0.2)^4}{96} \approx 0.00002$

$$35. \int_0^{0.1} \frac{1}{\sqrt{1+x^4}} dx = \int_0^{0.1} \left(1 - \frac{x^4}{2} + \frac{3x^8}{8} - \dots\right) dx = \left[x - \frac{x^5}{10} + \dots\right]_0^{0.1} \approx [x]_0^{0.1} \approx 0.1 \text{ with error}$$

$$|E| \leq \frac{(0.1)^5}{10} = 0.000001$$

$$36. \int_0^{0.25} \sqrt[3]{1+x^2} dx = \int_0^{0.25} \left(1 + \frac{x^2}{3} - \frac{x^4}{9} + \dots\right) dx = \left[x + \frac{x^3}{9} - \frac{x^5}{45} + \dots\right]_0^{0.25} \approx \left[x + \frac{x^3}{9}\right]_0^{0.25} \approx 0.25174 \text{ with error}$$

$$|E| \leq \frac{(0.25)^5}{45} \approx 0.0000217$$

$$37. \int_0^{0.1} \frac{\sin x}{x} dx = \int_0^{0.1} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots\right) dx = \left[x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \dots\right]_0^{0.1} \approx \left[x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!}\right]_0^{0.1}$$

$$\approx 0.0999444611, |E| \leq \frac{(0.1)^7}{7 \cdot 7!} \approx 2.8 \times 10^{-12}$$

$$38. \int_0^{0.1} \exp(-x^2) dx = \int_0^{0.1} \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \dots\right) dx = \left[x - \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42} + \dots\right]_0^{0.1} \approx \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42}\right]_0^{0.1}$$

$$\approx 0.0996676643, |E| \leq \frac{(0.1)^9}{216} \approx 4.6 \times 10^{-12}$$

$$39. (1+x^4)^{1/2} = (1)^{1/2} + \frac{(\frac{1}{2})}{1} (1)^{-1/2} (x^4) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!} (1)^{-3/2} (x^4)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!} (1)^{-5/2} (x^4)^3$$

$$+ \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!} (1)^{-7/2} (x^4)^4 + \dots = 1 + \frac{x^4}{2} - \frac{x^8}{8} + \frac{x^{12}}{16} - \frac{5x^{16}}{128} + \dots$$

$$\Rightarrow \int_0^{0.1} \left(1 + \frac{x^4}{2} - \frac{x^8}{8} + \frac{x^{12}}{16} - \frac{5x^{16}}{128} + \dots\right) dx \approx \left[x + \frac{x^5}{10}\right]_0^{0.1} \approx 0.100001, |E| \leq \frac{(0.1)^9}{72} \approx 1.39 \times 10^{-11}$$

$$40. \int_0^1 \left(\frac{1-\cos x}{x^2}\right) dx = \int_0^1 \left(\frac{1}{2} - \frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^6}{8!} + \frac{x^8}{10!} - \dots\right) dx \approx \left[\frac{x}{2} - \frac{x^3}{3 \cdot 4!} + \frac{x^5}{5 \cdot 6!} - \frac{x^7}{7 \cdot 8!} + \frac{x^9}{9 \cdot 10!}\right]_0^1$$

$$\approx 0.4863853764, |E| \leq \frac{1}{11 \cdot 12!} \approx 1.9 \times 10^{-10}$$

$$41. \int_0^1 \cos t^2 dt = \int_0^1 \left(1 - \frac{t^4}{2} + \frac{t^8}{4!} - \frac{t^{12}}{6!} + \dots\right) dt = \left[t - \frac{t^5}{10} + \frac{t^9}{9 \cdot 4!} - \frac{t^{13}}{13 \cdot 6!} + \dots\right]_0^1 \Rightarrow |\text{error}| < \frac{1}{13 \cdot 6!} \approx .00011$$

$$42. \int_0^1 \cos \sqrt{t} dt = \int_0^1 \left(1 - \frac{t}{2} + \frac{t^2}{4!} - \frac{t^3}{6!} + \frac{t^4}{8!} - \dots\right) dt = \left[t - \frac{t^2}{4} + \frac{t^3}{3 \cdot 4!} - \frac{t^4}{4 \cdot 6!} + \frac{t^5}{5 \cdot 8!} - \dots\right]_0^1$$

$$\Rightarrow |\text{error}| < \frac{1}{5 \cdot 8!} \approx 0.000004960$$

$$43. F(x) = \int_0^x \left(t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \frac{t^{14}}{7!} + \dots\right) dt = \left[\frac{t^3}{3} - \frac{t^7}{7 \cdot 3!} + \frac{t^{11}}{11 \cdot 5!} - \frac{t^{15}}{15 \cdot 7!} + \dots\right]_0^x \approx \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!}$$

$$\Rightarrow |\text{error}| < \frac{1}{15 \cdot 7!} \approx 0.000013$$

$$44. F(x) = \int_0^x \left(t^2 - t^4 + \frac{t^6}{2!} - \frac{t^8}{3!} + \frac{t^{10}}{4!} - \frac{t^{12}}{5!} + \dots\right) dt = \left[\frac{t^3}{3} - \frac{t^5}{5} + \frac{t^7}{7 \cdot 2!} - \frac{t^9}{9 \cdot 3!} + \frac{t^{11}}{11 \cdot 4!} - \frac{t^{13}}{13 \cdot 5!} + \dots\right]_0^x$$

$$\approx \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7 \cdot 2!} - \frac{x^9}{9 \cdot 3!} + \frac{x^{11}}{11 \cdot 4!} \Rightarrow |\text{error}| < \frac{1}{13 \cdot 5!} \approx 0.00064$$

$$45. (a) F(x) = \int_0^x \left(t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \dots\right) dt = \left[\frac{t^2}{2} - \frac{t^4}{12} + \frac{t^6}{30} - \dots\right]_0^x \approx \frac{x^2}{2} - \frac{x^4}{12} \Rightarrow |\text{error}| < \frac{(0.5)^6}{30} \approx .00052$$

$$(b) |\text{error}| < \frac{1}{33 \cdot 34} \approx .00089 \text{ when } F(x) \approx \frac{x^2}{2} - \frac{x^4}{3 \cdot 4} + \frac{x^6}{5 \cdot 6} - \frac{x^8}{7 \cdot 8} + \dots + (-1)^{15} \frac{x^{32}}{31 \cdot 32}$$

$$46. (a) F(x) = \int_0^x \left(1 - \frac{t}{2} + \frac{t^2}{3} - \frac{t^3}{4} + \dots\right) dt = \left[t - \frac{t^2}{2 \cdot 2} + \frac{t^3}{3 \cdot 3} - \frac{t^4}{4 \cdot 4} + \frac{t^5}{5 \cdot 5} - \dots\right]_0^x \approx x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \frac{x^5}{5^2}$$

$$\Rightarrow |\text{error}| < \frac{(0.5)^6}{6^2} \approx .00043$$

$$(b) |\text{error}| < \frac{1}{32^2} \approx .00097 \text{ when } F(x) \approx x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \dots + (-1)^{31} \frac{x^{31}}{31^2}$$

$$47. \frac{1}{x^2} (e^x - (1+x)) = \frac{1}{x^2} \left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots \right) - 1 - x \right) = \frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \dots \Rightarrow \lim_{x \rightarrow 0} \frac{e^x - (1+x)}{x^2} \\ = \lim_{x \rightarrow 0} \left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \dots \right) = \frac{1}{2}$$

$$48. \frac{1}{x} (e^x - e^{-x}) = \frac{1}{x} \left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots \right) \right] = \frac{1}{x} \left(2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \frac{2x^7}{7!} + \dots \right) \\ = 2 + \frac{2x^2}{3!} + \frac{2x^4}{5!} + \frac{2x^6}{7!} + \dots \Rightarrow \lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{x} = \lim_{x \rightarrow \infty} \left(2 + \frac{2x^2}{3!} + \frac{2x^4}{5!} + \frac{2x^6}{7!} + \dots \right) = 2$$

$$49. \frac{1}{t^4} \left(1 - \cos t - \frac{t^2}{2} \right) = \frac{1}{t^4} \left[1 - \frac{t^2}{2} - \left(1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots \right) \right] = -\frac{1}{4!} + \frac{t^2}{6!} - \frac{t^4}{8!} + \dots \Rightarrow \lim_{t \rightarrow 0} \frac{1 - \cos t - \left(\frac{t^2}{2} \right)}{t^4} \\ = \lim_{t \rightarrow 0} \left(-\frac{1}{4!} + \frac{t^2}{6!} - \frac{t^4}{8!} + \dots \right) = -\frac{1}{24}$$

$$50. \frac{1}{\theta^5} \left(-\theta + \frac{\theta^3}{6} + \sin \theta \right) = \frac{1}{\theta^5} \left(-\theta + \frac{\theta^3}{6} + \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \right) = \frac{1}{5!} - \frac{\theta^2}{7!} + \frac{\theta^4}{9!} - \dots \Rightarrow \lim_{\theta \rightarrow 0} \frac{\sin \theta - \theta + \left(\frac{\theta^3}{6} \right)}{\theta^5} \\ = \lim_{\theta \rightarrow 0} \left(\frac{1}{5!} - \frac{\theta^2}{7!} + \frac{\theta^4}{9!} - \dots \right) = \frac{1}{120}$$

$$51. \frac{1}{y^3} (y - \tan^{-1} y) = \frac{1}{y^3} \left[y - \left(y - \frac{y^3}{3} + \frac{y^5}{5} - \dots \right) \right] = \frac{1}{3} - \frac{y^2}{5} + \frac{y^4}{7} - \dots \Rightarrow \lim_{y \rightarrow 0} \frac{y - \tan^{-1} y}{y^3} = \lim_{y \rightarrow 0} \left(\frac{1}{3} - \frac{y^2}{5} + \frac{y^4}{7} - \dots \right) \\ = \frac{1}{3}$$

$$52. \frac{\tan^{-1} y - \sin y}{y^3 \cos y} = \frac{\left(y - \frac{y^3}{3} + \frac{y^5}{5} - \dots \right) - \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots \right)}{y^3 \cos y} = \frac{\left(-\frac{y^3}{6} + \frac{23y^5}{5!} - \dots \right)}{y^3 \cos y} = \frac{\left(-\frac{1}{6} + \frac{23y^2}{5!} - \dots \right)}{\cos y} \\ \Rightarrow \lim_{y \rightarrow 0} \frac{\tan^{-1} y - \sin y}{y^3 \cos y} = \lim_{y \rightarrow 0} \frac{\left(-\frac{1}{6} + \frac{23y^2}{5!} - \dots \right)}{\cos y} = -\frac{1}{6}$$

$$53. x^2 \left(-1 + e^{-1/x^2} \right) = x^2 \left(-1 + 1 - \frac{1}{x^2} + \frac{1}{2x^4} - \frac{1}{6x^6} + \dots \right) = -1 + \frac{1}{2x^2} - \frac{1}{6x^4} + \dots \Rightarrow \lim_{x \rightarrow \infty} x^2 \left(e^{-1/x^2} - 1 \right) \\ = \lim_{x \rightarrow \infty} \left(-1 + \frac{1}{2x^2} - \frac{1}{6x^4} + \dots \right) = -1$$

$$54. (x+1) \sin \left(\frac{1}{x+1} \right) = (x+1) \left(\frac{1}{x+1} - \frac{1}{3!(x+1)^3} + \frac{1}{5!(x+1)^5} - \dots \right) = 1 - \frac{1}{3!(x+1)^2} + \frac{1}{5!(x+1)^4} - \dots \\ \Rightarrow \lim_{x \rightarrow \infty} (x+1) \sin \left(\frac{1}{x+1} \right) = \lim_{x \rightarrow \infty} \left(1 - \frac{1}{3!(x+1)^2} + \frac{1}{5!(x+1)^4} - \dots \right) = 1$$

$$55. \frac{\ln(1+x^2)}{1-\cos x} = \frac{\left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \dots \right)}{1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right)} = \frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{3} - \dots \right)}{\left(\frac{1}{2!} - \frac{x^2}{4!} + \dots \right)} \Rightarrow \lim_{x \rightarrow 0} \frac{\ln(1+x^2)}{1-\cos x} = \lim_{x \rightarrow 0} \frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{3} - \dots \right)}{\left(\frac{1}{2!} - \frac{x^2}{4!} + \dots \right)} = 2! = 2$$

$$56. \frac{x^2-4}{\ln(x-1)} = \frac{(x-2)(x+2)}{\left[(x-2) - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3} - \dots \right]} = \frac{x+2}{\left[1 - \frac{x-2}{2} + \frac{(x-2)^2}{3} - \dots \right]} \Rightarrow \lim_{x \rightarrow 2} \frac{x^2-4}{\ln(x-1)} \\ = \lim_{x \rightarrow 2} \frac{x+2}{\left[1 - \frac{x-2}{2} + \frac{(x-2)^2}{3} - \dots \right]} = 4$$

$$57. \ln \left(\frac{1+x}{1-x} \right) = \ln(1+x) - \ln(1-x) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \right) = 2 \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots \right)$$

$$58. \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^{n-1}x^n}{n} + \dots \Rightarrow |\text{error}| = \left| \frac{(-1)^{n-1}x^n}{n} \right| = \frac{1}{n10^n} \text{ when } x = 0.1; \\ \frac{1}{n10^n} < \frac{1}{10^8} \Rightarrow n10^n > 10^8 \text{ when } n \geq 8 \Rightarrow 7 \text{ terms}$$

59. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots + \frac{(-1)^{n-1}x^{2n-1}}{2n-1} + \dots \Rightarrow |\text{error}| = \left| \frac{(-1)^{n-1}x^{2n-1}}{2n-1} \right| = \frac{1}{2n-1}$ when $x = 1$;
 $\frac{1}{2n-1} < \frac{1}{10^3} \Rightarrow n > \frac{1001}{2} = 500.5 \Rightarrow$ the first term not used is the 501st \Rightarrow we must use 500 terms

60. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots + \frac{(-1)^{n-1}x^{2n-1}}{2n-1} + \dots$ and $\lim_{n \rightarrow \infty} \left| \frac{x^{2n+1}}{2n+1} \cdot \frac{2n-1}{x^{2n-1}} \right| = x^2 \lim_{n \rightarrow \infty} \left| \frac{2n-1}{2n+1} \right| = x^2$
 $\Rightarrow \tan^{-1} x$ converges for $|x| < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ which is a convergent series; when $x = 1$
 we have $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ which is a convergent series \Rightarrow the series representing $\tan^{-1} x$ diverges for $|x| > 1$

61. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots + \frac{(-1)^{n-1}x^{2n-1}}{2n-1} + \dots$ and when the series representing $48 \tan^{-1} \left(\frac{1}{18} \right)$ has an error less than $\frac{1}{3} \cdot 10^{-6}$, then the series representing the sum
 $48 \tan^{-1} \left(\frac{1}{18} \right) + 32 \tan^{-1} \left(\frac{1}{57} \right) - 20 \tan^{-1} \left(\frac{1}{239} \right)$ also has an error of magnitude less than 10^{-6} ; thus
 $|\text{error}| = 48 \frac{\left(\frac{1}{18} \right)^{2n-1}}{2n-1} < \frac{1}{3 \cdot 10^6} \Rightarrow n \geq 4$ using a calculator $\Rightarrow 4$ terms

62. $\ln(\sec x) = \int_0^x \tan t \, dt = \int_0^x \left(t + \frac{t^3}{3} + \frac{2t^5}{15} + \dots \right) dt \approx \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \dots$

63. (a) $(1 - x^2)^{-1/2} \approx 1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16} \Rightarrow \sin^{-1} x \approx x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112}$; Using the Ratio Test:

$$\lim_{n \rightarrow \infty} \left| \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)x^{2n+3}}{2 \cdot 4 \cdot 6 \cdots (2n)(2n+2)(2n+3)} \cdot \frac{2 \cdot 4 \cdot 6 \cdots (2n)(2n+1)}{1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n+1}} \right| < 1 \Rightarrow x^2 \lim_{n \rightarrow \infty} \left| \frac{(2n+1)(2n+1)}{(2n+2)(2n+3)} \right| < 1$$

$\Rightarrow |x| < 1 \Rightarrow$ the radius of convergence is 1. See Exercise 69.

(b) $\frac{d}{dx} (\cos^{-1} x) = -(1 - x^2)^{-1/2} \Rightarrow \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x \approx \frac{\pi}{2} - \left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} \right) \approx \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} - \frac{5x^7}{112}$

64. (a) $(1 + t^2)^{-1/2} \approx (1)^{-1/2} + \left(-\frac{1}{2} \right) (1)^{-3/2} (t^2) + \frac{\left(-\frac{1}{2} \right) \left(-\frac{3}{2} \right) (1)^{-5/2} (t^2)^2}{2!} + \frac{\left(-\frac{1}{2} \right) \left(-\frac{3}{2} \right) \left(-\frac{5}{2} \right) (1)^{-7/2} (t^2)^3}{3!}$
 $= 1 - \frac{t^2}{2} + \frac{3t^4}{2^2 \cdot 2!} - \frac{3 \cdot 5t^6}{2^3 \cdot 3!} \Rightarrow \sinh^{-1} x \approx \int_0^x \left(1 - \frac{t^2}{2} + \frac{3t^4}{8} - \frac{5t^6}{16} \right) dt = x - \frac{x^3}{6} + \frac{3x^5}{40} - \frac{5x^7}{112}$

(b) $\sinh^{-1} \left(\frac{1}{4} \right) \approx \frac{1}{4} - \frac{1}{384} + \frac{3}{40,960} = 0.24746908$; the error is less than the absolute value of the first unused term, $\frac{5x^7}{112}$, evaluated at $t = \frac{1}{4}$ since the series is alternating $\Rightarrow |\text{error}| < \frac{5 \left(\frac{1}{4} \right)^7}{112} \approx 2.725 \times 10^{-6}$

65. $\frac{-1}{1+x} = -\frac{1}{1-(-x)} = -1 + x - x^2 + x^3 - \dots \Rightarrow \frac{d}{dx} \left(\frac{-1}{1+x} \right) = \frac{1}{1+x^2} = \frac{d}{dx} (-1 + x - x^2 + x^3 - \dots)$
 $= 1 - 2x + 3x^2 - 4x^3 + \dots$

66. $\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \dots \Rightarrow \frac{d}{dx} \left(\frac{1}{1-x^2} \right) = \frac{2x}{(1-x^2)^2} = \frac{d}{dx} (1 + x^2 + x^4 + x^6 + \dots) = 2x + 4x^3 + 6x^5 + \dots$

67. Wallis' formula gives the approximation $\pi \approx 4 \left[\frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots (2n-2) \cdot (2n)}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdots (2n-1) \cdot (2n-1)} \right]$ to produce the table

n	$\sim \pi$
10	3.221088998
20	3.181104886
30	3.167880758
80	3.151425420
90	3.150331383
93	3.150049112
94	3.149959030
95	3.149870848
100	3.149456425

At $n = 1929$ we obtain the first approximation accurate to 3 decimals: 3.141999845. At $n = 30,000$ we still do not obtain accuracy to 4 decimals: 3.141617732, so the convergence to π is very slow. Here is a Maple CAS procedure to produce these approximations:

```
pie :=
proc(n)
local i,j;
a(2) := evalf(8/9);
for i from 3 to n do a(i) := evalf(2*(2*i-2)*i/(2*i-1)^2*a(i-1)) od;
[[j,4*a(j)] $ (j = n-5 .. n)]
end
```

$$\begin{aligned}
68. \quad \ln 1 &= 0; \ln 2 = \ln \frac{1 + \left(\frac{1}{3}\right)}{1 - \left(\frac{1}{3}\right)} \approx 2 \left(\frac{1}{3} + \frac{\left(\frac{1}{3}\right)^3}{3} + \frac{\left(\frac{1}{3}\right)^5}{5} + \frac{\left(\frac{1}{3}\right)^7}{7} \right) \approx 0.69314; \ln 3 = \ln 2 + \ln \left(\frac{3}{2}\right) = \ln 2 + \ln \frac{1 + \left(\frac{1}{5}\right)}{1 - \left(\frac{1}{5}\right)} \\
&\approx \ln 2 + 2 \left(\frac{1}{5} + \frac{\left(\frac{1}{5}\right)^3}{3} + \frac{\left(\frac{1}{5}\right)^5}{5} + \frac{\left(\frac{1}{5}\right)^7}{7} \right) \approx 1.09861; \ln 4 = 2 \ln 2 \approx 1.38628; \ln 5 = \ln 4 + \ln \left(\frac{5}{4}\right) = \ln 4 + \ln \frac{1 + \left(\frac{1}{9}\right)}{1 - \left(\frac{1}{9}\right)} \\
&\approx 1.60943; \ln 6 = \ln 2 + \ln 3 \approx 1.79175; \ln 7 = \ln 6 + \ln \left(\frac{7}{6}\right) = \ln 6 + \ln \frac{1 + \left(\frac{1}{13}\right)}{1 - \left(\frac{1}{13}\right)} \approx 1.94591; \ln 8 = 3 \ln 2 \\
&\approx 2.07944; \ln 9 = 2 \ln 3 \approx 2.19722; \ln 10 = \ln 2 + \ln 5 \approx 2.30258
\end{aligned}$$

$$\begin{aligned}
69. \quad (1 - x^2)^{-1/2} &= (1 + (-x^2))^{-1/2} = (1)^{-1/2} + \left(-\frac{1}{2}\right)(1)^{-3/2}(-x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1)^{-5/2}(-x^2)^2}{2!} \\
&+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(1)^{-7/2}(-x^2)^3}{3!} + \dots = 1 + \frac{x^2}{2} + \frac{1 \cdot 3x^4}{2^2 \cdot 2!} + \frac{1 \cdot 3 \cdot 5x^6}{2^3 \cdot 3!} + \dots = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n}}{2^n \cdot n!} \\
\Rightarrow \sin^{-1} x &= \int_0^x (1 - t^2)^{-1/2} dt = \int_0^x \left(1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)t^{2n}}{2^n \cdot n!} \right) dt = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n+1}}{2 \cdot 4 \cdots (2n)(2n+1)},
\end{aligned}$$

where $|x| < 1$

$$\begin{aligned}
70. \quad [\tan^{-1} t]_x^{\infty} &= \frac{\pi}{2} - \tan^{-1} x = \int_x^{\infty} \frac{dt}{1+t^2} = \int_x^{\infty} \left[\frac{\left(\frac{1}{t^2}\right)}{1 + \left(\frac{1}{t^2}\right)} \right] dt = \int_x^{\infty} \frac{1}{t^2} \left(1 - \frac{1}{t^2} + \frac{1}{t^4} - \frac{1}{t^6} + \dots \right) dt \\
&= \int_x^{\infty} \left(\frac{1}{t^2} - \frac{1}{t^4} + \frac{1}{t^6} - \frac{1}{t^8} + \dots \right) dt = \lim_{b \rightarrow \infty} \left[-\frac{1}{t} + \frac{1}{3t^3} - \frac{1}{5t^5} + \frac{1}{7t^7} - \dots \right]_x^b = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + \dots \\
\Rightarrow \tan^{-1} x &= \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \dots, x > 1; [\tan^{-1} t]_{-\infty}^x = \tan^{-1} x + \frac{\pi}{2} = \int_{-\infty}^x \frac{dt}{1+t^2} \\
&= \lim_{b \rightarrow -\infty} \left[-\frac{1}{t} + \frac{1}{3t^3} - \frac{1}{5t^5} + \frac{1}{7t^7} - \dots \right]_b^x = -\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \dots \Rightarrow \tan^{-1} x = -\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \dots, \\
&x < -1
\end{aligned}$$

$$\begin{aligned}
71. \quad (a) \quad \tan(\tan^{-1}(n+1) - \tan^{-1}(n-1)) &= \frac{\tan(\tan^{-1}(n+1)) - \tan(\tan^{-1}(n-1))}{1 + \tan(\tan^{-1}(n+1)) \tan(\tan^{-1}(n-1))} = \frac{(n+1) - (n-1)}{1 + (n+1)(n-1)} = \frac{2}{n^2} \\
(b) \quad \sum_{n=1}^N \tan^{-1} \left(\frac{2}{n^2} \right) &= \sum_{n=1}^N [\tan^{-1}(n+1) - \tan^{-1}(n-1)] = (\tan^{-1} 2 - \tan^{-1} 0) + (\tan^{-1} 3 - \tan^{-1} 1) \\
&+ (\tan^{-1} 4 - \tan^{-1} 2) + \dots + (\tan^{-1}(N+1) - \tan^{-1}(N-1)) = \tan^{-1}(N+1) + \tan^{-1} N - \frac{\pi}{4} \\
(c) \quad \sum_{n=1}^{\infty} \tan^{-1} \left(\frac{2}{n^2} \right) &= \lim_{n \rightarrow \infty} \left[\tan^{-1}(N+1) + \tan^{-1} N - \frac{\pi}{4} \right] = \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{4} = \frac{3\pi}{4}
\end{aligned}$$

11.11 FOURIER SERIES

$$\begin{aligned}
1. \quad a_0 &= \frac{1}{2\pi} \int_0^{2\pi} 1 \, dx = 1, a_k = \frac{1}{\pi} \int_0^{2\pi} \cos kx \, dx = \frac{1}{\pi} \left[\frac{\sin kx}{k} \right]_0^{2\pi} = 0, b_k = \frac{1}{\pi} \int_0^{2\pi} \sin kx \, dx = \frac{1}{\pi} \left[-\frac{\cos kx}{k} \right]_0^{2\pi} = 0. \\
&\text{Thus, the Fourier series for } f(x) \text{ is 1.}
\end{aligned}$$

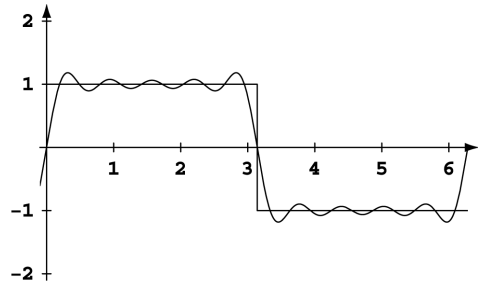


$$2. \quad a_0 = \frac{1}{2\pi} \left[\int_0^\pi 1 \, dx + \int_\pi^{2\pi} -1 \, dx \right] = 0, \quad a_k = \frac{1}{\pi} \left[\int_0^\pi \cos kx \, dx - \int_\pi^{2\pi} \cos kx \, dx \right] = \frac{1}{\pi} \left[\frac{\sin kx}{k} \Big|_0^\pi - \frac{\sin kx}{k} \Big|_\pi^{2\pi} \right] = 0,$$

$$b_k = \frac{1}{\pi} \left[\int_0^\pi \sin kx \, dx - \int_\pi^{2\pi} \sin kx \, dx \right] = \frac{1}{\pi} \left[-\frac{\cos kx}{k} \Big|_0^\pi + \frac{\cos kx}{k} \Big|_\pi^{2\pi} \right] = \frac{1}{k\pi} [(-\cos k\pi + 1) + (\cos 2\pi k - \cos \pi k)]$$

$$= \frac{1}{k\pi} (2 - 2 \cos k\pi) = \begin{cases} \frac{4}{k\pi}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}.$$

Thus, the Fourier series for $f(x)$ is $\frac{4}{\pi} \left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right]$.



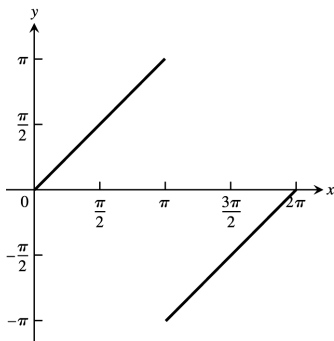
$$3. \quad a_0 = \frac{1}{2\pi} \left[\int_0^\pi x \, dx + \int_\pi^{2\pi} (x - 2\pi) \, dx \right] = \frac{1}{2\pi} \left[\frac{1}{2}\pi^2 + \frac{1}{2}(4\pi^2 - \pi^2) - 2\pi^2 \right] = 0. \quad \text{Note,}$$

$$\int_\pi^{2\pi} (x - 2\pi) \cos kx \, dx = -\int_0^\pi u \cos ku \, du \quad (\text{Let } u = 2\pi - x). \quad \text{So } a_k = \frac{1}{\pi} \left[\int_0^\pi x \cos kx \, dx + \int_\pi^{2\pi} (x - 2\pi) \cos kx \, dx \right] = 0.$$

Note, $\int_\pi^{2\pi} (x - 2\pi) \sin kx \, dx = \int_0^\pi u \sin ku \, du$ (Let $u = 2\pi - x$). So $b_k = \frac{1}{\pi} \left[\int_0^\pi x \sin kx \, dx + \int_\pi^{2\pi} (x - 2\pi) \sin kx \, dx \right]$

$$= \frac{2}{\pi} \int_0^\pi x \sin kx \, dx = \frac{2}{\pi} \left[-\frac{x}{k} \cos kx + \frac{1}{k^2} \sin kx \right]_0^\pi = -\frac{2}{k} \cos k\pi = \frac{2}{k} (-1)^{k+1}.$$

Thus, the Fourier series for $f(x)$ is $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2 \sin kx}{k}$.



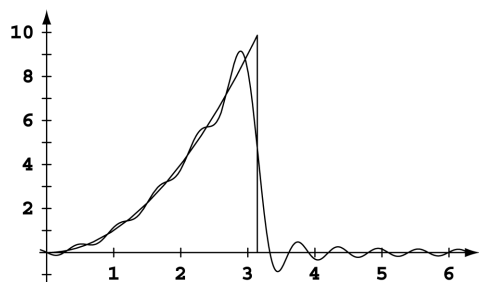
$$4. \quad a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) \, dx = \frac{1}{2\pi} \int_0^\pi x^2 \, dx = \frac{1}{6}\pi^2, \quad a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \, dx = \frac{1}{\pi} \int_0^\pi x^2 \cos kx \, dx$$

$$= \frac{1}{\pi} \left[\left(\frac{x^2}{k} - \frac{2}{k^3} \right) \sin kx + \frac{2}{k^2} x \cos kx \right]_0^\pi = \frac{2}{k^2} \cos k\pi = (-1)^k \left(\frac{2}{k^2} \right), \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx \, dx = \frac{1}{\pi} \int_0^\pi x^2 \sin kx \, dx =$$

$$= \frac{1}{\pi} \left[\left(\frac{2}{k^3} - \frac{x^2}{k} \right) \cos kx + \frac{2}{k^2} x \sin kx \right]_0^\pi = \frac{1}{\pi} \left[\left(\frac{2}{k^3} - \frac{\pi^2}{k} \right) (-1)^k - \frac{2}{k^3} \right] = \frac{1}{\pi} \left[\left((-1)^k - 1 \right) \frac{2}{k^3} \right] - \frac{\pi}{k} (-1)^k$$

$$= \begin{cases} -\frac{4}{\pi k^3} + \frac{\pi}{k}, & k \text{ odd} \\ -\frac{\pi}{k}, & k \text{ even} \end{cases}.$$

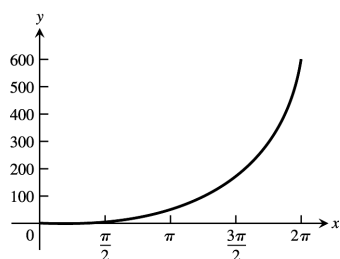
Thus, the Fourier series for $f(x)$ is $\frac{1}{6}\pi^2 - 2\cos x + \left(\frac{\pi^2-4}{\pi}\right)\sin x + \frac{1}{2}\cos 2x - \frac{\pi}{2}\sin 2x - \frac{2}{9}\cos 3x + \left(\frac{9\pi^2-4}{27\pi}\right)\sin 3x + \dots$



$$5. \quad a_0 = \frac{1}{2\pi} \int_0^{2\pi} e^x dx = \frac{1}{2\pi} (e^{2\pi} - 1), \quad a_k = \frac{1}{\pi} \int_0^{2\pi} e^x \cos kx dx = \frac{1}{\pi} \left[\frac{e^x}{1+k^2} (\cos kx + k \sin kx) \right]_0^{2\pi} = \frac{e^{2\pi}-1}{\pi(1+k^2)},$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} e^x \sin kx dx = \frac{1}{\pi} \left[\frac{e^x}{1+k^2} (\sin kx - k \cos kx) \right]_0^{2\pi} = \frac{k(1-e^{2\pi})}{\pi(1+k^2)}.$$

Thus, the Fourier series for $f(x)$ is $\frac{1}{2\pi}(e^{2\pi}-1) + \frac{e^{2\pi}-1}{\pi} \sum_{k=1}^{\infty} \left(\frac{\cos kx}{1+k^2} - \frac{k \sin kx}{1+k^2} \right).$



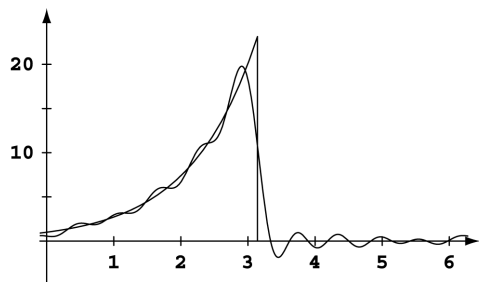
$$6. \quad a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx = \frac{1}{2\pi} \int_0^{\pi} e^x dx = \frac{e^{\pi}-1}{2\pi}, \quad a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx dx = \frac{1}{\pi} \int_0^{\pi} e^x \cos kx dx = \frac{1}{\pi} \left[\frac{e^x}{1+k^2} (\cos kx + k \sin kx) \right]_0^{\pi}$$

$$= \frac{1}{\pi(1+k^2)} [e^{\pi}(-1)^k - 1] = \begin{cases} \frac{-(1+e^{\pi})}{\pi(1+k^2)}, & k \text{ odd} \\ \frac{e^{\pi}-1}{\pi(1+k^2)}, & k \text{ even} \end{cases}. \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx dx = \frac{1}{\pi} \int_0^{\pi} e^x \sin kx dx$$

$$= \frac{1}{\pi} \left[\frac{e^x}{1+k^2} (\sin kx - k \cos kx) \right]_0^{\pi} = \frac{-k}{\pi(1+k^2)} [e^{\pi}(-1)^k - 1] = \begin{cases} \frac{k(1+e^{\pi})}{\pi(1+k^2)}, & k \text{ odd} \\ \frac{1-e^{\pi}}{\pi(1+k^2)}, & k \text{ even} \end{cases}.$$

Thus, the Fourier series for $f(x)$ is

$$\frac{e^{\pi}-1}{2\pi} - \frac{(1+e^{\pi})}{2\pi} \cos x + \frac{(1+e^{\pi})}{2\pi} \sin x + \frac{e^{\pi}-1}{5\pi} \cos 2x + \frac{2(1-e^{\pi})}{5\pi} \sin 2x - \frac{(1+e^{\pi})}{10\pi} \cos 3x + \frac{3(1+e^{\pi})}{10\pi} \sin 3x + \dots$$

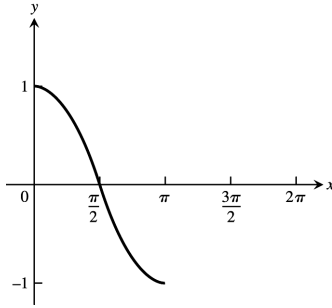


$$7. \quad a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx = \frac{1}{2\pi} \int_0^{2\pi} \cos x dx = 0, \quad a_k = \frac{1}{\pi} \int_0^{2\pi} \cos x \cos kx dx = \begin{cases} \frac{1}{\pi} \left[\frac{\sin(k-1)x}{2(k-1)} + \frac{\sin(k+1)x}{2(k+1)} \right]_0^{2\pi}, & k \neq 1 \\ \frac{1}{\pi} \left[\frac{1}{2}x + \frac{1}{4}\sin 2x \right]_0^{2\pi}, & k = 1 \end{cases}$$

$$= \begin{cases} 0, & k \neq 1 \\ \frac{1}{2}, & k = 1 \end{cases}.$$

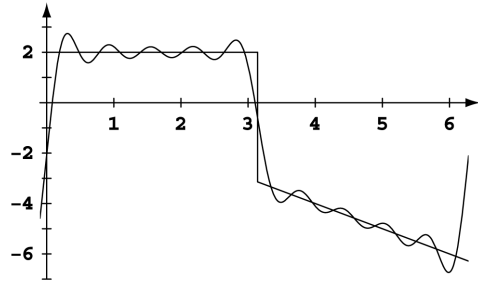
$$b_k = \frac{1}{\pi} \int_0^{2\pi} \cos x \sin kx \, dx = \begin{cases} -\frac{1}{\pi} \left[\frac{\cos(k-1)x}{2(k-1)} + \frac{\cos(k+1)x}{2(k+1)} \right]_0^\pi, & k \neq 1 \\ -\frac{1}{4\pi} \cos 2x \Big|_0^\pi, & k = 1 \end{cases} = \begin{cases} 0, & k \text{ odd} \\ \frac{2k}{\pi(k^2-1)}, & k \text{ even} \end{cases}.$$

Thus, the Fourier series for $f(x)$ is $\frac{1}{2}\cos x + \sum_{k \text{ even}} \frac{2k}{\pi(k^2-1)} \sin kx$.



$$\begin{aligned} 8. \quad a_0 &= \frac{1}{2\pi} \int_0^{2\pi} f(x) \, dx = \frac{1}{2\pi} \left[\int_0^\pi 2 \, dx + \int_\pi^{2\pi} -x \, dx \right] = 1 - \frac{3}{4}\pi, \quad a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \, dx \\ &= \frac{1}{\pi} \left[\int_0^\pi 2 \cos kx \, dx + \int_\pi^{2\pi} -x \cos kx \, dx \right] = -\frac{1}{\pi} \left[\frac{\cos kx}{k^2} + \frac{x \sin kx}{k} \right]_\pi^{2\pi} = \frac{-1 + (-1)^k}{\pi k^2} = \begin{cases} -\frac{2}{\pi k^2}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases} \\ b_k &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx \, dx = \frac{1}{\pi} \left[\int_0^\pi 2 \sin kx \, dx + \int_\pi^{2\pi} -x \sin kx \, dx \right] = \frac{1}{\pi} \left[-\frac{2}{k} \cos kx \Big|_0^\pi + \left(\frac{x \cos kx}{k} - \frac{\sin kx}{k^2} \right) \Big|_\pi^{2\pi} \right] \\ &= \begin{cases} \frac{1}{k} \left(\frac{4}{\pi} + 3 \right), & k \text{ odd} \\ \frac{1}{k}, & k \text{ even} \end{cases}. \end{aligned}$$

Thus, the Fourier series for $f(x)$ is $1 - \frac{3}{4}\pi - \frac{2}{\pi} \cos x + \left(\frac{4}{\pi} + 3 \right) \sin x + \frac{1}{2} \sin 2x - \frac{2}{9\pi} \cos 3x + \frac{1}{3} \left(\frac{4}{\pi} + 3 \right) \sin 3x + \dots$



$$9. \quad \int_0^{2\pi} \cos px \, dx = \frac{1}{p} \sin px \Big|_0^{2\pi} = 0 \text{ if } p \neq 0.$$

$$10. \quad \int_0^{2\pi} \sin px \, dx = -\frac{1}{p} \cos px \Big|_0^{2\pi} = -\frac{1}{p} [1 - 1] = 0 \text{ if } p \neq 0.$$

$$11. \quad \int_0^{2\pi} \cos px \cos qx \, dx = \int_0^{2\pi} \frac{1}{2} [\cos(p+q)x + \cos(p-q)x] \, dx = \frac{1}{2} \left[\frac{1}{p+q} \sin(p+q)x + \frac{1}{p-q} \sin(p-q)x \right]_0^{2\pi} = 0 \text{ if } p \neq q.$$

$$\text{If } p = q \text{ then } \int_0^{2\pi} \cos px \cos qx \, dx = \int_0^{2\pi} \cos^2 px \, dx = \int_0^{2\pi} \frac{1}{2} (1 + \cos 2px) \, dx = \frac{1}{2} \left(x + \frac{1}{2p} \sin 2px \right) \Big|_0^{2\pi} = \pi.$$

$$12. \quad \int_0^{2\pi} \sin px \sin qx \, dx = \int_0^{2\pi} \frac{1}{2} [\cos(p-q)x - \cos(p+q)x] \, dx = \frac{1}{2} \left[\frac{1}{p-q} \sin(p-q)x - \frac{1}{p+q} \sin(p+q)x \right]_0^{2\pi} = 0 \text{ if } p \neq q.$$

$$\text{If } p = q \text{ then } \int_0^{2\pi} \sin px \sin qx \, dx = \int_0^{2\pi} \sin^2 px \, dx = \int_0^{2\pi} \frac{1}{2} (1 - \cos 2px) \, dx = \frac{1}{2} \left(x - \frac{1}{2p} \sin 2px \right) \Big|_0^{2\pi} = \pi.$$

13. $\int_0^{2\pi} \sin px \cos qx \, dx = \int_0^{2\pi} \frac{1}{2} [\sin(p+q)x + \sin(p-q)x] \, dx = -\frac{1}{2} \left[\frac{1}{p+q} \cos(p+q)x + \frac{1}{p-q} \cos(p-q)x \right]_0^{2\pi}$
 $= -\frac{1}{2} \left[(1-1) \frac{1}{p+q} + (1-1) \frac{1}{p-q} \right] = 0$. If $p = q$ then $\int_0^{2\pi} \sin px \cos qx \, dx = \int_0^{2\pi} \sin px \cos px \, dx = \int_0^{2\pi} \frac{1}{2} \sin 2px \, dx$
 $= -\frac{1}{4\pi} \cos 2px \Big|_0^{2\pi} = -\frac{1}{4\pi} (1-1) = 0$.
14. Yes. Note that if f is continuous at c , then the expression $\frac{f(c^+) + f(c^-)}{2} = f(c)$ since $f(c^+) = \lim_{x \rightarrow c^+} f(x) = f(c)$ and $f(c^-) = \lim_{x \rightarrow c^-} f(x) = f(c)$. Now since the sum of two piecewise continuous functions on $[0, 2\pi]$ is also continuous on $[0, 2\pi]$, the function $f + g$ satisfies the hypothesis of Theorem 24, and so its Fourier series converges to $\frac{(f+g)(c^+) + (f+g)(c^-)}{2}$ for $0 < c < 2\pi$. Let $s_f(x)$ denote the Fourier series for $f(x)$. Then for any c in the interval $(0, 2\pi)$
 $s_{f+g}(c) = \frac{(f+g)(c^+) + (f+g)(c^-)}{2} = \frac{1}{2} \left[\lim_{x \rightarrow c^+} (f+g)(x) + \lim_{x \rightarrow c^-} (f+g)(x) \right] = \frac{1}{2} \left[\lim_{x \rightarrow c^+} f(x) + \lim_{x \rightarrow c^+} g(x) + \lim_{x \rightarrow c^-} f(x) + \lim_{x \rightarrow c^-} g(x) \right]$
 $= \frac{1}{2} [(f(c^+) + g(c^+)) + (f(c^-) + g(c^-))] = s_f(c) + s_g(c)$, since f and g satisfy the hypothesis of Theorem 24.
15. (a) $f(x)$ is piecewise continuous on $[0, 2\pi]$ and $f'(x) = 1$ for all $x \neq \pi \Rightarrow f'(x)$ is piecewise continuous on $[0, 2\pi]$. Then by Theorem 24, the Fourier series for $f(x)$ converges to $f(x)$ for all $x \neq \pi$ and converges to $\frac{1}{2}(f(\pi^+) + f(\pi^-)) = \frac{1}{2}(-\pi + \pi) = 0$ at $x = \pi$.
- (b) The Fourier series for $f(x)$ is $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2 \sin kx}{k}$. If we differentiate this series term by term we get the series $\sum_{k=1}^{\infty} (-1)^{k+1} 2 \cos kx$, which diverges by the n^{th} term test for divergence for any x since $\lim_{k \rightarrow \infty} (-1)^{k+1} 2 \cos kx \neq 0$.
16. Since the Fourier series is discontinuous at $x = \pi$, by Theorem 24, the Fourier series will converge to $\frac{f(c^+) + f(c^-)}{2}$. Thus, at $x = \pi$ we have $\frac{f(\pi^+) + f(\pi^-)}{2} = \frac{1}{6}\pi^2 - 2 \cos x + \left(\frac{\pi^2-4}{\pi}\right) \sin x + \frac{1}{2} \cos 2x - \frac{\pi}{2} \sin 2x - \frac{2}{9} \cos 3x + \left(\frac{9\pi^2-4}{27\pi}\right) \sin 3x + \dots$
 $\Rightarrow \frac{0+\pi^2}{2} = \frac{1}{6}\pi^2 - 2 \cos \pi + \left(\frac{\pi^2-4}{\pi}\right) \sin \pi + \frac{1}{2} \cos 2\pi - \frac{\pi}{2} \sin 2\pi - \frac{2}{9} \cos 3\pi + \left(\frac{9\pi^2-4}{27\pi}\right) \sin 3\pi + \dots$
 $\Rightarrow \frac{0+\pi^2}{2} = \frac{1}{6}\pi^2 + 2 + \frac{1}{2} + \frac{2}{9} + \dots = \frac{1}{6}\pi^2 + 2\left(1 + \frac{1}{4} + \frac{1}{9} + \dots\right) = \frac{1}{6}\pi^2 + 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \frac{\pi^2}{2} = \frac{\pi^2}{6} + 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$
 $\frac{\pi^2}{2} - \frac{\pi^2}{6} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \frac{\pi^2}{3} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$.

CHAPTER 11 PRACTICE EXERCISES

- converges to 1, since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(1 + \frac{(-1)^n}{n}\right) = 1$
- converges to 0, since $0 \leq a_n \leq \frac{2}{\sqrt{n}}$, $\lim_{n \rightarrow \infty} 0 = 0$, $\lim_{n \rightarrow \infty} \frac{2}{\sqrt{n}} = 0$ using the Sandwich Theorem for Sequences
- converges to -1 , since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{1-2^n}{2^n}\right) = \lim_{n \rightarrow \infty} \left(\frac{1}{2^n} - 1\right) = -1$
- converges to 1, since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} [1 + (0.9)^n] = 1 + 0 = 1$
- diverges, since $\left\{\sin \frac{n\pi}{2}\right\} = \{0, 1, 0, -1, 0, 1, \dots\}$
- converges to 0, since $\{\sin n\pi\} = \{0, 0, 0, \dots\}$
- converges to 0, since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{\ln n^2}{n} = 2 \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n}\right)}{1} = 0$

8. converges to 0, since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{\ln(2n+1)}{n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{2}{2n+1}\right)}{1} = 0$
9. converges to 1, since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{n+\ln n}{n}\right) = \lim_{n \rightarrow \infty} \frac{1+\left(\frac{1}{n}\right)}{1} = 1$
10. converges to 0, since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{\ln(2n^3+1)}{n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{6n^2}{2n^3+1}\right)}{1} = \lim_{n \rightarrow \infty} \frac{12n}{6n^2} = \lim_{n \rightarrow \infty} \frac{2}{n} = 0$
11. converges to e^{-5} , since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{n-5}{n}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{(-5)}{n}\right)^n = e^{-5}$ by Theorem 5
12. converges to $\frac{1}{e}$, since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{-n} = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e}$ by Theorem 5
13. converges to 3, since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{3n}{n}\right)^{1/n} = \lim_{n \rightarrow \infty} \frac{3}{n^{1/n}} = \frac{3}{1} = 3$ by Theorem 5
14. converges to 1, since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{3}{n}\right)^{1/n} = \lim_{n \rightarrow \infty} \frac{3^{1/n}}{n^{1/n}} = \frac{1}{1} = 1$ by Theorem 5
15. converges to $\ln 2$, since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} n(2^{1/n} - 1) = \lim_{n \rightarrow \infty} \frac{2^{1/n} - 1}{\left(\frac{1}{n}\right)} = \lim_{n \rightarrow \infty} \frac{\left[\frac{(-2^{1/n} \ln 2)}{n^2}\right]}{\left(\frac{-1}{n^2}\right)} = \lim_{n \rightarrow \infty} 2^{1/n} \ln 2$
 $= 2^0 \cdot \ln 2 = \ln 2$
16. converges to 1, since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sqrt[n]{2n+1} = \lim_{n \rightarrow \infty} \exp\left(\frac{\ln(2n+1)}{n}\right) = \lim_{n \rightarrow \infty} \exp\left(\frac{\frac{2}{2n+1}}{1}\right) = e^0 = 1$
17. diverges, since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} = \lim_{n \rightarrow \infty} (n+1) = \infty$
18. converges to 0, since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{(-4)^n}{n!} = 0$ by Theorem 5
19. $\frac{1}{(2n-3)(2n-1)} = \frac{\left(\frac{1}{2}\right)}{2n-3} - \frac{\left(\frac{1}{2}\right)}{2n-1} \Rightarrow s_n = \left[\frac{\left(\frac{1}{2}\right)}{3} - \frac{\left(\frac{1}{2}\right)}{5}\right] + \left[\frac{\left(\frac{1}{2}\right)}{5} - \frac{\left(\frac{1}{2}\right)}{7}\right] + \dots + \left[\frac{\left(\frac{1}{2}\right)}{2n-3} - \frac{\left(\frac{1}{2}\right)}{2n-1}\right] = \frac{\left(\frac{1}{2}\right)}{3} - \frac{\left(\frac{1}{2}\right)}{2n-1}$
 $\Rightarrow \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left[\frac{1}{6} - \frac{\left(\frac{1}{2}\right)}{2n-1}\right] = \frac{1}{6}$
20. $\frac{-2}{n(n+1)} = \frac{-2}{n} + \frac{2}{n+1} \Rightarrow s_n = \left(\frac{-2}{2} + \frac{2}{3}\right) + \left(\frac{-2}{3} + \frac{2}{4}\right) + \dots + \left(\frac{-2}{n} + \frac{2}{n+1}\right) = -\frac{2}{2} + \frac{2}{n+1} \Rightarrow \lim_{n \rightarrow \infty} s_n$
 $= \lim_{n \rightarrow \infty} \left(-1 + \frac{2}{n+1}\right) = -1$
21. $\frac{9}{(3n-1)(3n+2)} = \frac{3}{3n-1} - \frac{3}{3n+2} \Rightarrow s_n = \left(\frac{3}{2} - \frac{3}{5}\right) + \left(\frac{3}{5} - \frac{3}{8}\right) + \left(\frac{3}{8} - \frac{3}{11}\right) + \dots + \left(\frac{3}{3n-1} - \frac{3}{3n+2}\right)$
 $= \frac{3}{2} - \frac{3}{3n+2} \Rightarrow \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(\frac{3}{2} - \frac{3}{3n+2}\right) = \frac{3}{2}$
22. $\frac{-8}{(4n-3)(4n+1)} = \frac{-2}{4n-3} + \frac{2}{4n+1} \Rightarrow s_n = \left(\frac{-2}{9} + \frac{2}{13}\right) + \left(\frac{-2}{13} + \frac{2}{17}\right) + \left(\frac{-2}{17} + \frac{2}{21}\right) + \dots + \left(\frac{-2}{4n-3} + \frac{2}{4n+1}\right)$
 $= -\frac{2}{9} + \frac{2}{4n+1} \Rightarrow \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(-\frac{2}{9} + \frac{2}{4n+1}\right) = -\frac{2}{9}$
23. $\sum_{n=0}^{\infty} e^{-n} = \sum_{n=0}^{\infty} \frac{1}{e^n}$, a convergent geometric series with $r = \frac{1}{e}$ and $a = 1 \Rightarrow$ the sum is $\frac{1}{1 - \left(\frac{1}{e}\right)} = \frac{e}{e-1}$

24. $\sum_{n=1}^{\infty} (-1)^n \frac{3}{4^n} = \sum_{n=0}^{\infty} \left(-\frac{3}{4}\right) \left(\frac{-1}{4}\right)^n$ a convergent geometric series with $r = -\frac{1}{4}$ and $a = \frac{-3}{4} \Rightarrow$ the sum is $\frac{(-\frac{3}{4})}{1 - (-\frac{1}{4})} = -\frac{3}{5}$
25. diverges, a p-series with $p = \frac{1}{2}$
26. $\sum_{n=1}^{\infty} \frac{-5}{n} = -5 \sum_{n=1}^{\infty} \frac{1}{n}$, diverges since it is a nonzero multiple of the divergent harmonic series
27. Since $f(x) = \frac{1}{x^{1/2}} \Rightarrow f'(x) = -\frac{1}{2x^{3/2}} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow a_{n+1} < a_n$, and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test. Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, the given series converges conditionally.
28. converges absolutely by the Direct Comparison Test since $\frac{1}{2n^3} < \frac{1}{n^3}$ for $n \geq 1$, which is the n th term of a convergent p-series
29. The given series does not converge absolutely by the Direct Comparison Test since $\frac{1}{\ln(n+1)} > \frac{1}{n+1}$, which is the n th term of a divergent series. Since $f(x) = \frac{1}{\ln(x+1)} \Rightarrow f'(x) = -\frac{1}{(\ln(x+1))^2(x+1)} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow a_{n+1} < a_n$, and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{\ln(n+1)} = 0$, the given series converges conditionally by the Alternating Series Test.
30. $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x(\ln x)^2} dx = \lim_{b \rightarrow \infty} [-(\ln x)^{-1}]_2^b = -\lim_{b \rightarrow \infty} \left(\frac{1}{\ln b} - \frac{1}{\ln 2}\right) = \frac{1}{\ln 2} \Rightarrow$ the series converges absolutely by the Integral Test
31. converges absolutely by the Direct Comparison Test since $\frac{\ln n}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}$, the n th term of a convergent p-series
32. diverges by the Direct Comparison Test for $e^n > n \Rightarrow \ln(e^n) > \ln n \Rightarrow n^n > \ln n \Rightarrow \ln n^n > \ln(\ln n) \Rightarrow n \ln n > \ln(\ln n) \Rightarrow \frac{\ln n}{\ln(\ln n)} > \frac{1}{n}$, the n th term of the divergent harmonic series
33. $\lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n\sqrt{n^2+1}}\right)}{\left(\frac{1}{n^2}\right)} = \sqrt{\lim_{n \rightarrow \infty} \frac{n^2}{n^2+1}} = \sqrt{1} = 1 \Rightarrow$ converges absolutely by the Limit Comparison Test
34. Since $f(x) = \frac{3x^2}{x^3+1} \Rightarrow f'(x) = \frac{3x(2-x^3)}{(x^3+1)^2} < 0$ when $x \geq 2 \Rightarrow a_{n+1} < a_n$ for $n \geq 2$ and $\lim_{n \rightarrow \infty} \frac{3n^2}{n^3+1} = 0$, the series converges by the Alternating Series Test. The series does not converge absolutely: By the Limit Comparison Test, $\lim_{n \rightarrow \infty} \frac{\left(\frac{3n^2}{n^3+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \rightarrow \infty} \frac{3n^3}{n^3+1} = 3$. Therefore the convergence is conditional.
35. converges absolutely by the Ratio Test since $\lim_{n \rightarrow \infty} \left[\frac{n+2}{(n+1)!} \cdot \frac{n!}{n+1} \right] = \lim_{n \rightarrow \infty} \frac{n+2}{(n+1)^2} = 0 < 1$
36. diverges since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{(-1)^n(n^2+1)}{2n^2+n-1}$ does not exist
37. converges absolutely by the Ratio Test since $\lim_{n \rightarrow \infty} \left[\frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} \right] = \lim_{n \rightarrow \infty} \frac{3}{n+1} = 0 < 1$

38. converges absolutely by the Root Test since $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{2^n 3^n}{n^n}} = \lim_{n \rightarrow \infty} \frac{6}{n} = 0 < 1$

39. converges absolutely by the Limit Comparison Test since $\lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n^{3/2}}\right)}{\left(\frac{1}{\sqrt{n(n+1)(n+2)}}\right)} = \sqrt{\lim_{n \rightarrow \infty} \frac{n(n+1)(n+2)}{n^3}} = 1$

40. converges absolutely by the Limit Comparison Test since $\lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n\sqrt{n^2-1}}\right)} = \sqrt{\lim_{n \rightarrow \infty} \frac{n^2(n^2-1)}{n^4}} = 1$

41. $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(x+4)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{(x+4)^n} \right| < 1 \Rightarrow \frac{|x+4|}{3} \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right) < 1 \Rightarrow \frac{|x+4|}{3} < 1$
 $\Rightarrow |x+4| < 3 \Rightarrow -3 < x+4 < 3 \Rightarrow -7 < x < -1$; at $x = -7$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, the alternating harmonic series, which converges conditionally; at $x = -1$ we have $\sum_{n=1}^{\infty} \frac{3^n}{n 3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$, the divergent harmonic series

(a) the radius is 3; the interval of convergence is $-7 \leq x < -1$

(b) the interval of absolute convergence is $-7 < x < -1$

(c) the series converges conditionally at $x = -7$

42. $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(x-1)^{2n}}{(2n+1)!} \cdot \frac{(2n-1)!}{(x-1)^{2n-2}} \right| < 1 \Rightarrow (x-1)^2 \lim_{n \rightarrow \infty} \frac{1}{(2n)(2n+1)} = 0 < 1$, which holds for all x

(a) the radius is ∞ ; the series converges for all x

(b) the series converges absolutely for all x

(c) there are no values for which the series converges conditionally

43. $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(3x-1)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(3x-1)^n} \right| < 1 \Rightarrow |3x-1| \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} < 1 \Rightarrow |3x-1| < 1$
 $\Rightarrow -1 < 3x-1 < 1 \Rightarrow 0 < 3x < 2 \Rightarrow 0 < x < \frac{2}{3}$; at $x = 0$ we have $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n^2}$
 $= -\sum_{n=1}^{\infty} \frac{1}{n^2}$, a nonzero constant multiple of a convergent p -series, which is absolutely convergent; at $x = \frac{2}{3}$ we

have $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$, which converges absolutely

(a) the radius is $\frac{1}{3}$; the interval of convergence is $0 \leq x \leq \frac{2}{3}$

(b) the interval of absolute convergence is $0 \leq x \leq \frac{2}{3}$

(c) there are no values for which the series converges conditionally

44. $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{n+2}{2n+3} \cdot \frac{(2x+1)^{n+1}}{2^{n+1}} \cdot \frac{2n+1}{n+1} \cdot \frac{2^n}{(2x+1)^n} \right| < 1 \Rightarrow \frac{|2x+1|}{2} \lim_{n \rightarrow \infty} \left| \frac{n+2}{2n+3} \cdot \frac{2n+1}{n+1} \right| < 1$
 $\Rightarrow \frac{|2x+1|}{2} (1) < 1 \Rightarrow |2x+1| < 2 \Rightarrow -2 < 2x+1 < 2 \Rightarrow -3 < 2x < 1 \Rightarrow -\frac{3}{2} < x < \frac{1}{2}$; at $x = -\frac{3}{2}$ we have
 $\sum_{n=1}^{\infty} \frac{n+1}{2n+1} \cdot \frac{(-2)^n}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{2n+1}$ which diverges by the n th-Term Test for Divergence since
 $\lim_{n \rightarrow \infty} \left(\frac{n+1}{2n+1} \right) = \frac{1}{2} \neq 0$; at $x = \frac{1}{2}$ we have $\sum_{n=1}^{\infty} \frac{n+1}{2n+1} \cdot \frac{2^n}{2^n} = \sum_{n=1}^{\infty} \frac{n+1}{2n+1}$, which diverges by the n th-

Term Test

(a) the radius is 1; the interval of convergence is $-\frac{3}{2} < x < \frac{1}{2}$

(b) the interval of absolute convergence is $-\frac{3}{2} < x < \frac{1}{2}$

(c) there are no values for which the series converges conditionally

$$45. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{x^n} \right| < 1 \Rightarrow |x| \lim_{n \rightarrow \infty} \left| \left(\frac{n}{n+1} \right)^n \left(\frac{1}{n+1} \right) \right| < 1 \Rightarrow \frac{|x|}{e} \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} \right) < 1$$

$$\Rightarrow \frac{|x|}{e} \cdot 0 < 1, \text{ which holds for all } x$$

- (a) the radius is ∞ ; the series converges for all x
 (b) the series converges absolutely for all x
 (c) there are no values for which the series converges conditionally

$$46. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n} \right| < 1 \Rightarrow |x| \lim_{n \rightarrow \infty} \sqrt{\frac{n}{n+1}} < 1 \Rightarrow |x| < 1; \text{ when } x = -1 \text{ we have}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}, \text{ which converges by the Alternating Series Test; when } x = 1 \text{ we have } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}, \text{ a divergent}$$

p-series

- (a) the radius is 1; the interval of convergence is $-1 \leq x < 1$
 (b) the interval of absolute convergence is $-1 < x < 1$
 (c) the series converges conditionally at $x = -1$

$$47. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(n+2)x^{2n+1}}{3^{n+1}} \cdot \frac{3^n}{(n+1)x^{2n-1}} \right| < 1 \Rightarrow \frac{x^2}{3} \lim_{n \rightarrow \infty} \left(\frac{n+2}{n+1} \right) < 1 \Rightarrow -\sqrt{3} < x < \sqrt{3};$$

the series $\sum_{n=1}^{\infty} -\frac{n+1}{\sqrt{3}}$ and $\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{3}}$, obtained with $x = \pm \sqrt{3}$, both diverge

- (a) the radius is $\sqrt{3}$; the interval of convergence is $-\sqrt{3} < x < \sqrt{3}$
 (b) the interval of absolute convergence is $-\sqrt{3} < x < \sqrt{3}$
 (c) there are no values for which the series converges conditionally

$$48. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(x-1)x^{2n+3}}{2n+3} \cdot \frac{2n+1}{(x-1)^{2n+1}} \right| < 1 \Rightarrow (x-1)^2 \lim_{n \rightarrow \infty} \left(\frac{2n+1}{2n+3} \right) < 1 \Rightarrow (x-1)^2(1) < 1$$

$$\Rightarrow (x-1)^2 < 1 \Rightarrow |x-1| < 1 \Rightarrow -1 < x-1 < 1 \Rightarrow 0 < x < 2; \text{ at } x = 0 \text{ we have } \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^{2n+1}}{2n+1}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{3n+1}}{2n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1} \text{ which converges conditionally by the Alternating Series Test and the fact}$$

$$\text{that } \sum_{n=1}^{\infty} \frac{1}{2n+1} \text{ diverges; at } x = 2 \text{ we have } \sum_{n=1}^{\infty} \frac{(-1)^n(1)^{2n+1}}{2n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}, \text{ which also converges}$$

conditionally

- (a) the radius is 1; the interval of convergence is $0 \leq x \leq 2$
 (b) the interval of absolute convergence is $0 < x < 2$
 (c) the series converges conditionally at $x = 0$ and $x = 2$

$$49. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{\operatorname{csch}(n+1)x^{n+1}}{\operatorname{csch}(n)x^n} \right| < 1 \Rightarrow |x| \lim_{n \rightarrow \infty} \left| \frac{\left(\frac{2}{e^{n+1} - e^{-n-1}} \right)}{\left(\frac{2}{e^n - e^{-n}} \right)} \right| < 1$$

$$\Rightarrow |x| \lim_{n \rightarrow \infty} \left| \frac{e^{-1} - e^{-2n-1}}{1 - e^{-2n-2}} \right| < 1 \Rightarrow \frac{|x|}{e} < 1 \Rightarrow -e < x < e; \text{ the series } \sum_{n=1}^{\infty} (\pm e)^n \operatorname{csch} n, \text{ obtained with } x = \pm e,$$

both diverge since $\lim_{n \rightarrow \infty} (\pm e)^n \operatorname{csch} n \neq 0$

- (a) the radius is e ; the interval of convergence is $-e < x < e$
 (b) the interval of absolute convergence is $-e < x < e$
 (c) there are no values for which the series converges conditionally

$$50. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{x^{n+1} \coth(n+1)}{x^n \coth(n)} \right| < 1 \Rightarrow |x| \lim_{n \rightarrow \infty} \left| \frac{1 + e^{-2n-2}}{1 - e^{-2n-2}} \cdot \frac{1 - e^{-2n}}{1 + e^{-2n}} \right| < 1 \Rightarrow |x| < 1$$

$$\Rightarrow -1 < x < 1; \text{ the series } \sum_{n=1}^{\infty} (\pm 1)^n \coth n, \text{ obtained with } x = \pm 1, \text{ both diverge since } \lim_{n \rightarrow \infty} (\pm 1)^n \coth n \neq 0$$

- (a) the radius is 1; the interval of convergence is $-1 < x < 1$
 (b) the interval of absolute convergence is $-1 < x < 1$

(c) there are no values for which the series converges conditionally

51. The given series has the form $1 - x + x^2 - x^3 + \dots + (-x)^n + \dots = \frac{1}{1+x}$, where $x = \frac{1}{4}$; the sum is $\frac{1}{1+(\frac{1}{4})} = \frac{4}{5}$ 52. The given series has the form $x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} + \dots = \ln(1+x)$, where $x = \frac{2}{3}$; the sum is $\ln\left(\frac{5}{3}\right) \approx 0.510825624$ 53. The given series has the form $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots = \sin x$, where $x = \pi$; the sum is $\sin \pi = 0$ 54. The given series has the form $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots = \cos x$, where $x = \frac{\pi}{3}$; the sum is $\cos \frac{\pi}{3} = \frac{1}{2}$ 55. The given series has the form $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots = e^x$, where $x = \ln 2$; the sum is $e^{\ln(2)} = 2$ 56. The given series has the form $x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)} + \dots = \tan^{-1} x$, where $x = \frac{1}{\sqrt{3}}$; the sum is $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ 57. Consider $\frac{1}{1-2x}$ as the sum of a convergent geometric series with $a = 1$ and $r = 2x \Rightarrow \frac{1}{1-2x} = 1 + (2x) + (2x)^2 + (2x)^3 + \dots = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n$ where $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$ 58. Consider $\frac{1}{1+x^3}$ as the sum of a convergent geometric series with $a = 1$ and $r = -x^3 \Rightarrow \frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = 1 + (-x^3) + (-x^3)^2 + (-x^3)^3 + \dots = \sum_{n=0}^{\infty} (-1)^n x^{3n}$ where $|-x^3| < 1 \Rightarrow |x^3| < 1 \Rightarrow |x| < 1$ 59. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin \pi x = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1} x^{2n+1}}{(2n+1)!}$ 60. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin \frac{2x}{3} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{2x}{3}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{3^{2n+1} (2n+1)!}$ 61. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos(x^{5/2}) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{5/2})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n}}{(2n)!}$ 62. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos \sqrt{5x} = \cos((5x)^{1/2}) = \sum_{n=0}^{\infty} \frac{(-1)^n ((5x)^{1/2})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 5^n x^n}{(2n)!}$ 63. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{(\pi x/2)} = \sum_{n=0}^{\infty} \frac{(\frac{\pi x}{2})^n}{n!} = \sum_{n=0}^{\infty} \frac{\pi^n x^n}{2^n n!}$ 64. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$ 65. $f(x) = \sqrt{3+x^2} = (3+x^2)^{1/2} \Rightarrow f'(x) = x(3+x^2)^{-1/2} \Rightarrow f''(x) = -x^2(3+x^2)^{-3/2} + (3+x^2)^{-1/2}$
 $\Rightarrow f'''(x) = 3x^3(3+x^2)^{-5/2} - 3x(3+x^2)^{-3/2}$; $f(-1) = 2$, $f'(-1) = -\frac{1}{2}$, $f''(-1) = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}$,
 $f'''(-1) = -\frac{3}{32} + \frac{3}{8} = \frac{9}{32} \Rightarrow \sqrt{3+x^2} = 2 - \frac{(x+1)}{2 \cdot 1!} + \frac{3(x+1)^2}{2^3 \cdot 2!} + \frac{9(x+1)^3}{2^5 \cdot 3!} + \dots$

$$66. f(x) = \frac{1}{1-x} = (1-x)^{-1} \Rightarrow f'(x) = (1-x)^{-2} \Rightarrow f''(x) = 2(1-x)^{-3} \Rightarrow f'''(x) = 6(1-x)^{-4}; f(2) = -1, f'(2) = 1, \\ f''(2) = -2, f'''(2) = 6 \Rightarrow \frac{1}{1-x} = -1 + (x-2) - (x-2)^2 + (x-2)^3 - \dots$$

$$67. f(x) = \frac{1}{x+1} = (x+1)^{-1} \Rightarrow f'(x) = -(x+1)^{-2} \Rightarrow f''(x) = 2(x+1)^{-3} \Rightarrow f'''(x) = -6(x+1)^{-4}; f(3) = \frac{1}{4}, \\ f'(3) = -\frac{1}{4^2}, f''(3) = \frac{2}{4^3}, f'''(3) = -\frac{6}{4^4} \Rightarrow \frac{1}{x+1} = \frac{1}{4} - \frac{1}{4^2}(x-3) + \frac{1}{4^3}(x-3)^2 - \frac{1}{4^4}(x-3)^3 + \dots$$

$$68. f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = -x^{-2} \Rightarrow f''(x) = 2x^{-3} \Rightarrow f'''(x) = -6x^{-4}; f(a) = \frac{1}{a}, f'(a) = -\frac{1}{a^2}, f''(a) = \frac{2}{a^3}, \\ f'''(a) = -\frac{6}{a^4} \Rightarrow \frac{1}{x} = \frac{1}{a} - \frac{1}{a^2}(x-a) + \frac{1}{a^3}(x-a)^2 - \frac{1}{a^4}(x-a)^3 + \dots$$

$$69. \text{ Assume the solution has the form } y = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n + \dots \\ \Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \dots + na_nx^{n-1} + \dots \Rightarrow \frac{dy}{dx} + y \\ = (a_1 + a_0) + (2a_2 + a_1)x + (3a_3 + a_2)x^2 + \dots + (na_n + a_{n-1})x^{n-1} + \dots = 0 \Rightarrow a_1 + a_0 = 0, 2a_2 + a_1 = 0, \\ 3a_3 + a_2 = 0 \text{ and in general } na_n + a_{n-1} = 0. \text{ Since } y = -1 \text{ when } x = 0 \text{ we have } a_0 = -1. \text{ Therefore } a_1 = 1, \\ a_2 = \frac{-a_1}{2 \cdot 1} = -\frac{1}{2}, a_3 = \frac{-a_2}{3} = \frac{1}{3 \cdot 2}, a_4 = \frac{-a_3}{4} = -\frac{1}{4 \cdot 3 \cdot 2}, \dots, a_n = \frac{-a_{n-1}}{n} = \frac{-1}{n} \frac{(-1)^{n-1}}{(n-1)!} = \frac{(-1)^{n+1}}{n!} \\ \Rightarrow y = -1 + x - \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 - \dots + \frac{(-1)^{n+1}}{n!}x^n + \dots = -\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = -e^{-x}$$

$$70. \text{ Assume the solution has the form } y = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n + \dots \\ \Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \dots + na_nx^{n-1} + \dots \Rightarrow \frac{dy}{dx} - y \\ = (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \dots + (na_n - a_{n-1})x^{n-1} + \dots = 0 \Rightarrow a_1 - a_0 = 0, 2a_2 - a_1 = 0, \\ 3a_3 - a_2 = 0 \text{ and in general } na_n - a_{n-1} = 0. \text{ Since } y = -3 \text{ when } x = 0 \text{ we have } a_0 = -3. \text{ Therefore } a_1 = -3, \\ a_2 = \frac{a_1}{2} = \frac{-3}{2}, a_3 = \frac{a_2}{3} = \frac{-3}{3 \cdot 2}, a_n = \frac{a_{n-1}}{n} = \frac{-3}{n!} \Rightarrow y = -3 - 3x - \frac{3}{2 \cdot 1}x^2 - \frac{3}{3 \cdot 2}x^3 - \dots - \frac{3}{n!}x^n + \dots \\ = -3 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots \right) = -3 \sum_{n=0}^{\infty} \frac{x^n}{n!} = -3e^x$$

$$71. \text{ Assume the solution has the form } y = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n + \dots \\ \Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \dots + na_nx^{n-1} + \dots \Rightarrow \frac{dy}{dx} + 2y \\ = (a_1 + 2a_0) + (2a_2 + 2a_1)x + (3a_3 + 2a_2)x^2 + \dots + (na_n + 2a_{n-1})x^{n-1} + \dots = 0. \text{ Since } y = 3 \text{ when } x = 0 \text{ we} \\ \text{have } a_0 = 3. \text{ Therefore } a_1 = -2a_0 = -2(3) = -3(2), a_2 = -\frac{2}{2}a_1 = -\frac{2}{2}(-2 \cdot 3) = 3 \left(\frac{2^2}{2} \right), a_3 = -\frac{2}{3}a_2 \\ = -\frac{2}{3} \left[3 \left(\frac{2^2}{2} \right) \right] = -3 \left(\frac{2^3}{3 \cdot 2} \right), \dots, a_n = \left(-\frac{2}{n} \right) a_{n-1} = \left(-\frac{2}{n} \right) \left(3 \left(\frac{(-1)^{n-1} 2^{n-1}}{(n-1)!} \right) \right) = 3 \left(\frac{(-1)^n 2^n}{n!} \right) \\ \Rightarrow y = 3 - 3(2x) + 3 \frac{(2)^2}{2} x^2 - 3 \frac{(2)^3}{3 \cdot 2} x^3 + \dots + 3 \frac{(-1)^n 2^n}{n!} x^n + \dots \\ = 3 \left[1 - (2x) + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + \dots + \frac{(-1)^n (2x)^n}{n!} + \dots \right] = 3 \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^n}{n!} = 3e^{-2x}$$

$$72. \text{ Assume the solution has the form } y = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n + \dots \\ \Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \dots + na_nx^{n-1} + \dots \Rightarrow \frac{dy}{dx} + y \\ = (a_1 + a_0) + (2a_2 + a_1)x + (3a_3 + a_2)x^2 + \dots + (na_n + a_{n-1})x^{n-1} + \dots = 1 \Rightarrow a_1 + a_0 = 1, 2a_2 + a_1 = 0, \\ 3a_3 + a_2 = 0 \text{ and in general } na_n + a_{n-1} = 0 \text{ for } n > 1. \text{ Since } y = 0 \text{ when } x = 0 \text{ we have } a_0 = 0. \text{ Therefore} \\ a_1 = 1 - a_0 = 1, a_2 = \frac{-a_1}{2 \cdot 1} = -\frac{1}{2}, a_3 = \frac{-a_2}{3} = \frac{1}{3 \cdot 2}, a_4 = \frac{-a_3}{4} = -\frac{1}{4 \cdot 3 \cdot 2}, \dots, a_n \\ = \frac{-a_{n-1}}{n} = \left(\frac{-1}{n} \right) \frac{(-1)^{n-1}}{(n-1)!} = \frac{(-1)^{n+1}}{n!} \Rightarrow y = 0 + x - \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 - \dots + \frac{(-1)^{n+1}}{n!}x^n + \dots \\ = -1 \left[1 - x + \frac{1}{2}x^2 - \frac{1}{3 \cdot 2}x^3 + \dots + \frac{(-1)^n}{n!}x^n + \dots \right] + 1 = -\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} + 1 = 1 - e^{-x}$$

$$73. \text{ Assume the solution has the form } y = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n + \dots \\ \Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \dots + na_nx^{n-1} + \dots \Rightarrow \frac{dy}{dx} - y \\ = (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \dots + (na_n - a_{n-1})x^{n-1} + \dots = 3x \Rightarrow a_1 - a_0 = 0, 2a_2 - a_1 = 3,$$

$$\begin{aligned}
3a_3 - a_2 &= 0 \text{ and in general } na_n - a_{n-1} = 0 \text{ for } n > 2. \text{ Since } y = -1 \text{ when } x = 0 \text{ we have } a_0 = -1. \text{ Therefore} \\
a_1 &= -1, a_2 = \frac{3+a_1}{2} = \frac{2}{2}, a_3 = \frac{a_2}{3} = \frac{2}{3 \cdot 2}, a_4 = \frac{a_3}{4} = \frac{2}{4 \cdot 3 \cdot 2}, \dots, a_n = \frac{a_{n-1}}{n} = \frac{2}{n!} \\
&\Rightarrow y = -1 - x + \left(\frac{2}{2}\right)x^2 + \frac{2}{3 \cdot 2}x^3 + \frac{2}{4 \cdot 3 \cdot 2}x^4 + \dots + \frac{2}{n!}x^n + \dots \\
&= 2\left(1 + x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + \dots + \frac{1}{n!}x^n + \dots\right) - 3 - 3x = 2\sum_{n=0}^{\infty} \frac{x^n}{n!} - 3 - 3x = 2e^x - 3x - 3
\end{aligned}$$

74. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n + \dots$

$$\begin{aligned}
&\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \dots + na_nx^{n-1} + \dots \Rightarrow \frac{dy}{dx} + y \\
&= (a_1 + a_0) + (2a_2 + a_1)x + (3a_3 + a_2)x^2 + \dots + (na_n + a_{n-1})x^{n-1} + \dots = x \Rightarrow a_1 + a_0 = 0, 2a_2 + a_1 = 1, \\
&3a_3 + a_2 = 0 \text{ and in general } na_n + a_{n-1} = 0 \text{ for } n > 2. \text{ Since } y = 0 \text{ when } x = 0 \text{ we have } a_0 = 0. \text{ Therefore} \\
&a_1 = 0, a_2 = \frac{1-a_1}{2} = \frac{1}{2}, a_3 = \frac{-a_2}{3} = -\frac{1}{3 \cdot 2}, \dots, a_n = \frac{-a_{n-1}}{n} = \frac{(-1)^n}{n!} \\
&\Rightarrow y = 0 - 0x + \frac{1}{2}x^2 - \frac{1}{3 \cdot 2}x^3 + \dots + \frac{(-1)^n}{n!}x^n + \dots = \left(1 - x + \frac{1}{2}x^2 - \frac{1}{3 \cdot 2}x^3 + \dots + \frac{(-1)^n}{n!}x^n + \dots\right) - 1 + x \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} - 1 + x = e^{-x} + x - 1
\end{aligned}$$

75. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n + \dots$

$$\begin{aligned}
&\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \dots + na_nx^{n-1} + \dots \Rightarrow \frac{dy}{dx} - y \\
&= (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \dots + (na_n - a_{n-1})x^{n-1} + \dots = x \Rightarrow a_1 - a_0 = 0, 2a_2 - a_1 = 1, \\
&3a_3 - a_2 = 0 \text{ and in general } na_n - a_{n-1} = 0 \text{ for } n > 2. \text{ Since } y = 1 \text{ when } x = 0 \text{ we have } a_0 = 1. \text{ Therefore} \\
&a_1 = 1, a_2 = \frac{1+a_1}{2} = \frac{2}{2}, a_3 = \frac{a_2}{3} = \frac{2}{3 \cdot 2}, a_4 = \frac{a_3}{4} = \frac{2}{4 \cdot 3 \cdot 2}, \dots, a_n = \frac{a_{n-1}}{n} = \frac{2}{n!} \\
&\Rightarrow y = 1 + x + \left(\frac{2}{2}\right)x^2 + \frac{2}{3 \cdot 2}x^3 + \frac{2}{4 \cdot 3 \cdot 2}x^4 + \dots + \frac{2}{n!}x^n + \dots \\
&= 2\left(1 + x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + \dots + \frac{1}{n!}x^n + \dots\right) - 1 - x = 2\sum_{n=0}^{\infty} \frac{x^n}{n!} - 1 - x = 2e^x - x - 1
\end{aligned}$$

76. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n + \dots$

$$\begin{aligned}
&\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \dots + na_nx^{n-1} + \dots \Rightarrow \frac{dy}{dx} - y \\
&= (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \dots + (na_n - a_{n-1})x^{n-1} + \dots = -x \Rightarrow a_1 - a_0 = 0, 2a_2 - a_1 = -1, \\
&3a_3 - a_2 = 0 \text{ and in general } na_n - a_{n-1} = 0 \text{ for } n > 2. \text{ Since } y = 2 \text{ when } x = 0 \text{ we have } a_0 = 2. \text{ Therefore} \\
&a_1 = 2, a_2 = \frac{-1+a_1}{2} = \frac{1}{2}, a_3 = \frac{a_2}{3} = \frac{1}{3 \cdot 2}, a_4 = \frac{a_3}{4} = \frac{1}{4 \cdot 3 \cdot 2}, \dots, a_n = \frac{a_{n-1}}{n} = \frac{1}{n!} \\
&\Rightarrow y = 2 + 2x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + \dots + \frac{1}{n!}x^n + \dots \\
&= \left(1 + x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + \dots + \frac{1}{n!}x^n + \dots\right) + 1 + x = \sum_{n=0}^{\infty} \frac{x^n}{n!} + 1 + x = e^x + x + 1
\end{aligned}$$

77. $\int_0^{1/2} \exp(-x^3) dx = \int_0^{1/2} \left(1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \frac{x^{12}}{4!} + \dots\right) dx = \left[x - \frac{x^4}{4} + \frac{x^7}{7 \cdot 2!} - \frac{x^{10}}{10 \cdot 3!} + \frac{x^{13}}{13 \cdot 4!} - \dots\right]_0^{1/2}$

$$\approx \frac{1}{2} - \frac{1}{2^4 \cdot 4} + \frac{1}{2^7 \cdot 7 \cdot 2!} - \frac{1}{2^{10} \cdot 10 \cdot 3!} + \frac{1}{2^{13} \cdot 13 \cdot 4!} - \frac{1}{2^{16} \cdot 16 \cdot 5!} \approx 0.484917143$$

78. $\int_0^1 x \sin(x^3) dx = \int_0^1 x \left(x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \frac{x^{27}}{9!} + \dots\right) dx = \int_0^1 \left(x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \frac{x^{22}}{7!} + \frac{x^{28}}{9!} - \dots\right) dx$

$$= \left[\frac{x^5}{5} - \frac{x^{11}}{11 \cdot 3!} + \frac{x^{17}}{17 \cdot 5!} - \frac{x^{23}}{23 \cdot 7!} + \frac{x^{29}}{29 \cdot 9!} - \dots\right]_0^1 \approx 0.185330149$$

79. $\int_1^{1/2} \frac{\tan^{-1} x}{x} dx = \int_1^{1/2} \left(1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \frac{x^8}{9} - \frac{x^{10}}{11} + \dots\right) dx = \left[x - \frac{x^3}{9} + \frac{x^5}{25} - \frac{x^7}{49} + \frac{x^9}{81} - \frac{x^{11}}{121} + \dots\right]_0^{1/2}$

$$\approx \frac{1}{2} - \frac{1}{9 \cdot 2^3} + \frac{1}{5^2 \cdot 2^5} - \frac{1}{7^2 \cdot 2^7} + \frac{1}{9^2 \cdot 2^9} - \frac{1}{11^2 \cdot 2^{11}} + \frac{1}{13^2 \cdot 2^{13}} - \frac{1}{15^2 \cdot 2^{15}} + \frac{1}{17^2 \cdot 2^{17}} - \frac{1}{19^2 \cdot 2^{19}} + \frac{1}{21^2 \cdot 2^{21}}$$

$$\approx 0.4872223583$$

$$80. \int_0^{1/64} \frac{\tan^{-1} x}{\sqrt{x}} dx = \int_0^{1/64} \frac{1}{\sqrt{x}} \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \right) dx = \int_0^{1/64} \left(x^{1/2} - \frac{1}{3} x^{5/2} + \frac{1}{5} x^{9/2} - \frac{1}{7} x^{13/2} + \dots \right) dx$$

$$= \left[\frac{2}{3} x^{3/2} - \frac{2}{21} x^{7/2} + \frac{2}{55} x^{11/2} - \frac{2}{105} x^{15/2} + \dots \right]_0^{1/64} = \left(\frac{2}{3 \cdot 8^3} - \frac{2}{21 \cdot 8^7} + \frac{2}{55 \cdot 8^{11}} - \frac{2}{105 \cdot 8^{15}} + \dots \right) \approx 0.0013020379$$

$$81. \lim_{x \rightarrow 0} \frac{7 \sin x}{e^{2x} - 1} = \lim_{x \rightarrow 0} \frac{7 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right)}{\left(2x + \frac{2^2 x^2}{2!} + \frac{2^3 x^3}{3!} + \dots \right)} = \lim_{x \rightarrow 0} \frac{7 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots \right)}{\left(2 + \frac{2^2 x}{2!} + \frac{2^3 x^2}{3!} + \dots \right)} = \frac{7}{2}$$

$$82. \lim_{\theta \rightarrow 0} \frac{e^\theta - e^{-\theta} - 2\theta}{\theta - \sin \theta} = \lim_{\theta \rightarrow 0} \frac{\left(1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \dots \right) - \left(1 - \theta + \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \dots \right) - 2\theta}{\theta - \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \right)} = \lim_{\theta \rightarrow 0} \frac{2 \left(\frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots \right)}{\left(\frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \dots \right)}$$

$$= \lim_{\theta \rightarrow 0} \frac{2 \left(\frac{1}{3!} + \frac{\theta^2}{5!} + \dots \right)}{\left(\frac{1}{3!} - \frac{\theta^2}{5!} + \dots \right)} = 2$$

$$83. \lim_{t \rightarrow 0} \left(\frac{1}{2 - 2 \cos t} - \frac{1}{t^2} \right) = \lim_{t \rightarrow 0} \frac{t^2 - 2 + 2 \cos t}{2t^2(1 - \cos t)} = \lim_{t \rightarrow 0} \frac{t^2 - 2 + 2 \left(1 - \frac{t^2}{2} + \frac{t^4}{4!} - \dots \right)}{2t^2 \left(1 - 1 + \frac{t^2}{2} - \frac{t^4}{4!} + \dots \right)} = \lim_{t \rightarrow 0} \frac{2 \left(\frac{t^4}{4!} - \frac{t^6}{6!} + \dots \right)}{\left(t^4 - \frac{2t^6}{4!} + \dots \right)}$$

$$= \lim_{t \rightarrow 0} \frac{2 \left(\frac{1}{4!} - \frac{t^2}{6!} + \dots \right)}{\left(1 - \frac{2t^2}{4!} + \dots \right)} = \frac{1}{12}$$

$$84. \lim_{h \rightarrow 0} \frac{\left(\frac{\sin h}{h} \right) - \cos h}{h^2} = \lim_{h \rightarrow 0} \frac{\left(1 - \frac{h^2}{3!} + \frac{h^4}{5!} - \dots \right) - \left(1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \dots \right)}{h^2}$$

$$= \lim_{h \rightarrow 0} \frac{\left(\frac{h^2}{2!} - \frac{h^2}{3!} + \frac{h^4}{5!} - \frac{h^4}{4!} + \frac{h^6}{6!} - \frac{h^6}{7!} + \dots \right)}{h^2} = \lim_{h \rightarrow 0} \left(\frac{1}{2!} - \frac{1}{3!} + \frac{h^2}{5!} - \frac{h^2}{4!} + \frac{h^4}{6!} - \frac{h^4}{7!} + \dots \right) = \frac{1}{3}$$

$$85. \lim_{z \rightarrow 0} \frac{1 - \cos^2 z}{\ln(1 - z) + \sin z} = \lim_{z \rightarrow 0} \frac{1 - \left(1 - z^2 + \frac{z^4}{3} - \dots \right)}{\left(-z - \frac{z^2}{2} - \frac{z^3}{3} - \dots \right) + \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right)} = \lim_{z \rightarrow 0} \frac{\left(z^2 - \frac{z^4}{3} + \dots \right)}{\left(-\frac{z^2}{2} - \frac{2z^3}{3} - \frac{z^4}{4} - \dots \right)}$$

$$= \lim_{z \rightarrow 0} \frac{\left(1 - \frac{z^2}{3} + \dots \right)}{\left(-\frac{1}{2} - \frac{2z}{3} - \frac{z^2}{4} - \dots \right)} = -2$$

$$86. \lim_{y \rightarrow 0} \frac{y^2}{\cos y - \cosh y} = \lim_{y \rightarrow 0} \frac{y^2}{\left(1 - \frac{y^2}{2} + \frac{y^4}{4!} - \frac{y^6}{6!} + \dots \right) - \left(1 + \frac{y^2}{2!} + \frac{y^4}{4!} + \frac{y^6}{6!} + \dots \right)} = \lim_{y \rightarrow 0} \frac{y^2}{\left(-\frac{2y^2}{2} - \frac{2y^6}{6!} - \dots \right)}$$

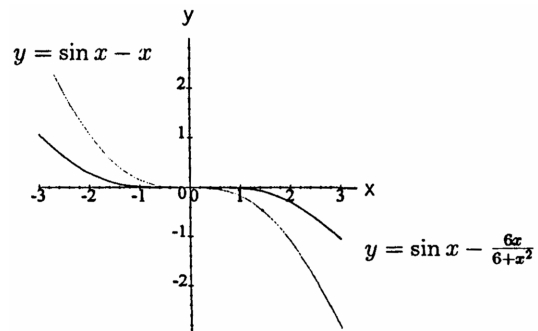
$$= \lim_{y \rightarrow 0} \frac{1}{\left(-1 - \frac{2y^4}{6!} - \dots \right)} = -1$$

$$87. \lim_{x \rightarrow 0} \left(\frac{\sin 3x}{x^3} + \frac{r}{x^2} + s \right) = \lim_{x \rightarrow 0} \left[\frac{\left(3x - \frac{(3x)^3}{6} + \frac{(3x)^5}{120} - \dots \right)}{x^3} + \frac{r}{x^2} + s \right] = \lim_{x \rightarrow 0} \left(\frac{3}{x^2} - \frac{9}{2} + \frac{81x^2}{40} + \dots + \frac{r}{x^2} + s \right) = 0$$

$$\Rightarrow \frac{r}{x^2} + \frac{3}{x^2} = 0 \text{ and } s - \frac{9}{2} = 0 \Rightarrow r = -3 \text{ and } s = \frac{9}{2}$$

$$88. (a) \csc x \approx \frac{1}{x} + \frac{x}{6} \Rightarrow \csc x \approx \frac{6+x^2}{6x} \Rightarrow \sin x \approx \frac{6x}{6+x^2}$$

(b) The approximation $\sin x \approx \frac{6x}{6+x^2}$ is better than $\sin x \approx x$.



$$89. (a) \sum_{n=1}^{\infty} \left(\sin \frac{1}{2n} - \sin \frac{1}{2n+1} \right) = \left(\sin \frac{1}{2} - \sin \frac{1}{3} \right) + \left(\sin \frac{1}{4} - \sin \frac{1}{5} \right) + \left(\sin \frac{1}{6} - \sin \frac{1}{7} \right) + \dots + \left(\sin \frac{1}{2n} - \sin \frac{1}{2n+1} \right) \\ + \dots = \sum_{n=2}^{\infty} (-1)^n \sin \frac{1}{n}; f(x) = \sin \frac{1}{x} \Rightarrow f'(x) = \frac{-\cos\left(\frac{1}{x}\right)}{x^2} < 0 \text{ if } x \geq 2 \Rightarrow \sin \frac{1}{n+1} < \sin \frac{1}{n}, \text{ and}$$

$$\lim_{n \rightarrow \infty} \sin \frac{1}{n} = 0 \Rightarrow \sum_{n=2}^{\infty} (-1)^n \sin \frac{1}{n} \text{ converges by the Alternating Series Test}$$

$$(b) |\text{error}| < \left| \sin \frac{1}{42} \right| \approx 0.02381 \text{ and the sum is an underestimate because the remainder is positive}$$

$$90. (a) \sum_{n=1}^{\infty} \left(\tan \frac{1}{2n} - \tan \frac{1}{2n+1} \right) = \sum_{n=2}^{\infty} (-1)^n \tan \frac{1}{n} \text{ (see Exercise 89); } f(x) = \tan \frac{1}{x} \Rightarrow f'(x) = \frac{-\sec^2\left(\frac{1}{x}\right)}{x^2} < 0 \\ \Rightarrow \tan \frac{1}{n+1} < \tan \frac{1}{n}, \text{ and } \lim_{n \rightarrow \infty} \tan \frac{1}{n} = 0 \Rightarrow \sum_{n=2}^{\infty} (-1)^n \tan \frac{1}{n} \text{ converges by the Alternating Series}$$

Test

$$(b) |\text{error}| < \left| \tan \frac{1}{42} \right| \approx 0.02382 \text{ and the sum is an underestimate because the remainder is positive}$$

$$91. \lim_{n \rightarrow \infty} \left| \frac{2 \cdot 5 \cdot 8 \cdots (3n-1)(3n+2)x^{n+1}}{2 \cdot 4 \cdot 6 \cdots (2n)(2n+2)} \cdot \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{2 \cdot 5 \cdot 8 \cdots (3n-1)x^n} \right| < 1 \Rightarrow |x| \lim_{n \rightarrow \infty} \left| \frac{3n+2}{2n+2} \right| < 1 \Rightarrow |x| < \frac{2}{3} \\ \Rightarrow \text{the radius of convergence is } \frac{2}{3}$$

$$92. \lim_{n \rightarrow \infty} \left| \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)(x-1)^{n+1}}{4 \cdot 9 \cdot 14 \cdots (5n-1)(5n+4)} \cdot \frac{4 \cdot 9 \cdot 14 \cdots (5n-1)}{3 \cdot 5 \cdot 7 \cdots (2n+1)x^n} \right| < 1 \Rightarrow |x| \lim_{n \rightarrow \infty} \left| \frac{2n+3}{5n+4} \right| < 1 \Rightarrow |x| < \frac{5}{2} \\ \Rightarrow \text{the radius of convergence is } \frac{5}{2}$$

$$93. \sum_{k=2}^n \ln \left(1 - \frac{1}{k^2} \right) = \sum_{k=2}^n \left[\ln \left(1 + \frac{1}{k} \right) + \ln \left(1 - \frac{1}{k} \right) \right] = \sum_{k=2}^n [\ln(k+1) - \ln k + \ln(k-1) - \ln k] \\ = [\ln 3 - \ln 2 + \ln 1 - \ln 2] + [\ln 4 - \ln 3 + \ln 2 - \ln 3] + [\ln 5 - \ln 4 + \ln 3 - \ln 4] + [\ln 6 - \ln 5 + \ln 4 - \ln 5] \\ + \dots + [\ln(n+1) - \ln n + \ln(n-1) - \ln n] = [\ln 1 - \ln 2] + [\ln(n+1) - \ln n] \quad \text{after cancellation} \\ \Rightarrow \sum_{k=2}^n \ln \left(1 - \frac{1}{k^2} \right) = \ln \left(\frac{n+1}{2n} \right) \Rightarrow \sum_{k=2}^{\infty} \ln \left(1 - \frac{1}{k^2} \right) = \lim_{n \rightarrow \infty} \ln \left(\frac{n+1}{2n} \right) = \ln \frac{1}{2} \text{ is the sum}$$

$$94. \sum_{k=2}^n \frac{1}{k^2-1} = \frac{1}{2} \sum_{k=2}^n \left(\frac{1}{k-1} - \frac{1}{k+1} \right) = \frac{1}{2} \left[\left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \dots + \left(\frac{1}{n-2} - \frac{1}{n} \right) \right. \\ \left. + \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \right] = \frac{1}{2} \left(\frac{1}{1} + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1} \right) = \frac{1}{2} \left(\frac{3}{2} - \frac{1}{n} - \frac{1}{n+1} \right) = \frac{1}{2} \left[\frac{3n(n+1) - 2(n+1) - 2n}{2n(n+1)} \right] = \frac{3n^2 - n - 2}{4n(n+1)} \\ \Rightarrow \sum_{k=2}^{\infty} \frac{1}{k^2-1} = \lim_{n \rightarrow \infty} \frac{1}{2} \left(\frac{3}{2} - \frac{1}{n} - \frac{1}{n+1} \right) = \frac{3}{4}$$

$$95. (a) \lim_{n \rightarrow \infty} \left| \frac{1 \cdot 4 \cdot 7 \cdots (3n-2)(3n+1)x^{3n+3}}{(3n+3)!} \cdot \frac{(3n)!}{1 \cdot 4 \cdot 7 \cdots (3n-2)x^{3n}} \right| < 1 \Rightarrow |x^3| \lim_{n \rightarrow \infty} \frac{(3n+1)}{(3n+1)(3n+2)(3n+3)} \\ = |x^3| \cdot 0 < 1 \Rightarrow \text{the radius of convergence is } \infty$$

$$(b) y = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 4 \cdot 7 \cdots (3n-2)}{(3n)!} x^{3n} \Rightarrow \frac{dy}{dx} = \sum_{n=1}^{\infty} \frac{1 \cdot 4 \cdot 7 \cdots (3n-2)}{(3n-1)!} x^{3n-1} \\ \Rightarrow \frac{d^2y}{dx^2} = \sum_{n=1}^{\infty} \frac{1 \cdot 4 \cdot 7 \cdots (3n-2)}{(3n-2)!} x^{3n-2} = x + \sum_{n=2}^{\infty} \frac{1 \cdot 4 \cdot 7 \cdots (3n-5)}{(3n-3)!} x^{3n-2} \\ = x \left(1 + \sum_{n=1}^{\infty} \frac{1 \cdot 4 \cdot 7 \cdots (3n-2)}{(3n)!} x^{3n} \right) = xy + 0 \Rightarrow a = 1 \text{ and } b = 0$$

$$96. (a) \frac{x^2}{1+x} = \frac{x^2}{1-(-x)} = x^2 + x^2(-x) + x^2(-x)^2 + x^2(-x)^3 + \dots = x^2 - x^3 + x^4 - x^5 + \dots = \sum_{n=2}^{\infty} (-1)^n x^n \text{ which} \\ \text{converges absolutely for } |x| < 1$$

$$(b) x = 1 \Rightarrow \sum_{n=2}^{\infty} (-1)^n x^n = \sum_{n=2}^{\infty} (-1)^n \text{ which diverges}$$

97. Yes, the series $\sum_{n=1}^{\infty} a_n b_n$ converges as we now show. Since $\sum_{n=1}^{\infty} a_n$ converges it follows that $a_n \rightarrow 0 \Rightarrow a_n < 1$
for $n > \text{some index } N \Rightarrow a_n b_n < b_n$ for $n > N \Rightarrow \sum_{n=1}^{\infty} a_n b_n$ converges by the Direct Comparison Test with $\sum_{n=1}^{\infty} b_n$

98. No, the series $\sum_{n=1}^{\infty} a_n b_n$ might diverge (as it would if a_n and b_n both equaled n) or it might converge (as it would if a_n and b_n both equaled $\frac{1}{n}$).

99. $\sum_{n=1}^{\infty} (x_{n+1} - x_n) = \lim_{n \rightarrow \infty} \sum_{k=1}^n (x_{k+1} - x_k) = \lim_{n \rightarrow \infty} (x_{n+1} - x_1) = \lim_{n \rightarrow \infty} (x_{n+1}) - x_1 \Rightarrow$ both the series and sequence must either converge or diverge.

100. It converges by the Limit Comparison Test since $\lim_{n \rightarrow \infty} \frac{\left(\frac{a_n}{1+a_n}\right)}{a_n} = \lim_{n \rightarrow \infty} \frac{1}{1+a_n} = 1$ because $\sum_{n=1}^{\infty} a_n$ converges and so $a_n \rightarrow 0$.

101. Newton's method gives $x_{n+1} = x_n - \frac{(x_n-1)^{40}}{40(x_n-1)^{39}} = \frac{39}{40}x_n + \frac{1}{40}$, and if the sequence $\{x_n\}$ has the limit L , then $L = \frac{39}{40}L + \frac{1}{40} \Rightarrow L = 1$ and $\{x_n\}$ converges since $\left| \frac{f(x)f''(x)}{[f'(x)]^2} \right| = \frac{39}{40} < 1$

102. $\sum_{n=1}^{\infty} \frac{a_n}{n} = a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \frac{a_4}{4} + \dots \geq a_1 + \left(\frac{1}{2}\right)a_2 + \left(\frac{1}{3} + \frac{1}{4}\right)a_4 + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)a_8$
 $+ \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \dots + \frac{1}{16}\right)a_{16} + \dots \geq \frac{1}{2}(a_2 + a_4 + a_8 + a_{16} + \dots)$ which is a divergent series

103. $a_n = \frac{1}{\ln n}$ for $n \geq 2 \Rightarrow a_2 \geq a_3 \geq a_4 \geq \dots$, and $\frac{1}{\ln 2} + \frac{1}{\ln 4} + \frac{1}{\ln 8} + \dots = \frac{1}{\ln 2} + \frac{1}{2 \ln 2} + \frac{1}{3 \ln 2} + \dots$
 $= \frac{1}{\ln 2} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots\right)$ which diverges so that $1 + \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by the Integral Test.

104. (a) $T = \frac{(\frac{1}{2})}{2} \left(0 + 2 \left(\frac{1}{2}\right)^2 e^{1/2} + e\right) = \frac{1}{8} e^{1/2} + \frac{1}{4} e \approx 0.885660616$

(b) $x^2 e^x = x^2 \left(1 + x + \frac{x^2}{2} + \dots\right) = x^2 + x^3 + \frac{x^4}{2} + \dots \Rightarrow \int_0^1 \left(x^2 + x^3 + \frac{x^4}{2}\right) dx = \left[\frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{10}\right]_0^1 = \frac{41}{60} = 0.683\bar{3}$

(c) If the second derivative is positive, the curve is concave upward and the polygonal line segments used in the trapezoidal rule lie above the curve. The trapezoidal approximation is therefore greater than the actual area under the graph.

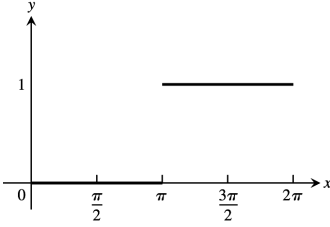
(d) All terms in the Maclaurin series are positive. If we truncate the series, we are omitting positive terms and hence the estimate is too small.

(e) $\int_0^1 x^2 e^x dx = [x^2 e^x - 2x e^x + 2e^x]_0^1 = e - 2e + 2e - 2 = e - 2 \approx 0.7182818285$

105. $a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx = \frac{1}{2\pi} \int_{\pi}^{2\pi} 1 dx = \frac{1}{2}$, $a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx dx = \frac{1}{\pi} \int_{\pi}^{2\pi} \cos kx dx = 0$.

$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx dx = \frac{1}{\pi} \int_{\pi}^{2\pi} \sin kx dx = -\frac{\cos kx}{\pi k} \Big|_{\pi}^{2\pi} = -\frac{1}{\pi k} (1 - (-1)^k) = \begin{cases} -\frac{2}{\pi k}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$.

Thus, the Fourier series of $f(x)$ is $\frac{1}{2} - \sum_{k \text{ odd}} \frac{2}{\pi k} \sin kx$



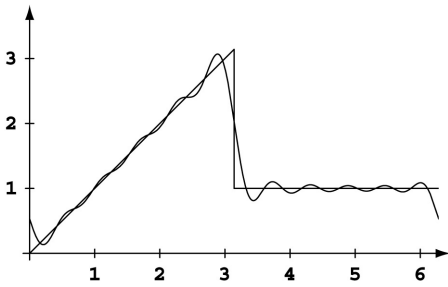
$$106. \quad a_0 = \frac{1}{2\pi} \left[\int_0^\pi x \, dx + \int_\pi^{2\pi} 1 \, dx \right] = \frac{1}{2} + \frac{1}{4}\pi, \quad a_k = \frac{1}{\pi} \left[\int_0^\pi x \cos kx \, dx + \int_\pi^{2\pi} \cos kx \, dx \right] = \frac{1}{\pi} \left[\frac{\cos kx}{k^2} + \frac{x \sin kx}{k} \right]_0^\pi$$

$$= \frac{1}{\pi k^2} \left((-1)^k - 1 \right) = \begin{cases} -\frac{2}{\pi k^2}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}.$$

$$b_k = \frac{1}{\pi} \left[\int_0^\pi x \sin kx \, dx + \int_\pi^{2\pi} \sin kx \, dx \right] = \frac{1}{\pi} \left[\frac{\sin kx}{k^2} - \frac{x \cos kx}{k} \right]_0^\pi - \frac{\cos kx}{\pi k} \Big|_\pi^{2\pi} = \frac{(-1)^{k+1}}{k} - \frac{1}{\pi k} \left(1 - (-1)^k \right)$$

$$= \begin{cases} \frac{1}{k} \left(1 - \frac{2}{\pi} \right), & k \text{ odd} \\ -\frac{1}{k}, & k \text{ even} \end{cases}.$$

Thus, the Fourier series of $f(x)$ is $\frac{1}{2} + \frac{1}{4}\pi - \frac{2}{\pi} \cos x + \left(1 - \frac{2}{\pi} \right) \sin x - \frac{1}{2} \sin 2x - \frac{2}{9\pi} \cos 3x + \frac{1}{3} \left(1 - \frac{2}{\pi} \right) \sin 3x + \dots$



$$107. \quad a_0 = \frac{1}{2\pi} \left[\int_0^\pi (\pi - x) \, dx + \int_\pi^{2\pi} (x - 2\pi) \, dx \right] = \frac{1}{2\pi} \left[\int_0^\pi (\pi - x) \, dx - \int_0^\pi (\pi - u) \, du \right] = 0 \text{ where we used the}$$

substitution $u = x - \pi$ in the second integral. We have $a_k = \frac{1}{\pi} \left[\int_0^\pi (\pi - x) \cos kx \, dx + \int_\pi^{2\pi} (x - 2\pi) \cos kx \, dx \right]$. Using

the substitution $u = x - \pi$ in the second integral gives $\int_\pi^{2\pi} (x - 2\pi) \cos kx \, dx = \int_0^\pi -(\pi - u) \cos(ku + k\pi) \, du$

$$= \begin{cases} \int_0^\pi (\pi - u) \cos ku \, du, & k \text{ odd} \\ \int_0^\pi -(\pi - u) \cos ku \, du, & k \text{ even} \end{cases}.$$

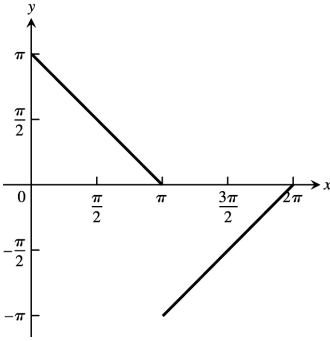
$$\text{Thus, } a_k = \begin{cases} \frac{2}{\pi} \int_0^\pi (\pi - x) \cos kx \, dx, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}.$$

Now, since k is odd, letting $v = \pi - x \Rightarrow \frac{2}{\pi} \int_0^\pi (\pi - x) \cos kx \, dx = -\frac{2}{\pi} \int_0^\pi v \cos kv \, dv = -\frac{2}{\pi} \left(-\frac{2}{k^2} \right) = \frac{4}{\pi k^2}$, k odd. (See

Exercise 106). So, $a_k = \begin{cases} \frac{4}{\pi k^2}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}.$

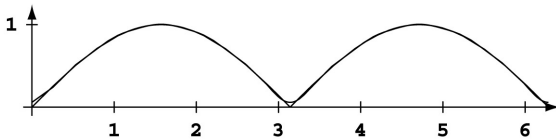
$$\text{Using similar techniques we see that } b_k = \begin{cases} \frac{2}{\pi} \int_0^\pi (\pi - u) \sin ku \, du, & k \text{ odd} \\ 0, & k \text{ even} \end{cases} = \begin{cases} \frac{2}{k}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}.$$

Thus, the Fourier series of $f(x)$ is $\sum_{k \text{ odd}} \left(\frac{4}{\pi k^2} \cos kx + \frac{2}{k} \sin kx \right)$.



108. $a_0 = \frac{1}{2\pi} \int_0^{2\pi} |\sin x| dx = \frac{1}{\pi} \int_0^{\pi} \sin x dx = \frac{2}{\pi}$. We have $a_k = \frac{1}{\pi} \int_0^{2\pi} |\sin x| \cos kx dx$
 $= \frac{1}{\pi} \left[\int_0^{\pi} \sin x \cos kx dx - \int_{\pi}^{2\pi} \sin x \cos kx dx \right]$. Using techniques similar to those used in Exercise 107, we find
 $a_k = \begin{cases} 0, & k \text{ odd} \\ \frac{2}{\pi} \int_0^{\pi} \sin x \cos kx dx, & k \text{ even} \end{cases} = \begin{cases} 0, & k \text{ odd} \\ \frac{-4}{(k^2-1)\pi}, & k \text{ even} \end{cases}$
 $b_k = \frac{1}{\pi} \int_0^{2\pi} |\sin x| \sin kx dx = \frac{1}{\pi} \left[\int_0^{\pi} \sin x \sin kx dx - \int_{\pi}^{2\pi} \sin x \sin kx dx \right] = \begin{cases} 0, & k \text{ odd} \\ \frac{2}{\pi} \int_0^{\pi} \sin x \sin kx dx, & k \text{ even} \end{cases} = 0$
for all k .

Thus, the Fourier series of $f(x)$ is $\frac{2}{\pi} + \sum_{k \text{ even}} \left(\frac{-4}{(k^2-1)\pi} \cos kx \right)$.



CHAPTER 11 ADDITIONAL AND ADVANCED EXERCISES

- converges since $\frac{1}{(3n-2)^{(2n+1)/2}} < \frac{1}{(3n-2)^{3/2}}$ and $\sum_{n=1}^{\infty} \frac{1}{(3n-2)^{3/2}}$ converges by the Limit Comparison Test:

$$\lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n^{3/2}} \right)}{\left(\frac{1}{(3n-2)^{3/2}} \right)} = \lim_{n \rightarrow \infty} \left(\frac{3n-2}{n} \right)^{3/2} = 3^{3/2}$$
- converges by the Integral Test: $\int_1^{\infty} (\tan^{-1} x)^2 \frac{dx}{x^2+1} = \lim_{b \rightarrow \infty} \left[\frac{(\tan^{-1} x)^3}{3} \right]_1^b = \lim_{b \rightarrow \infty} \left[\frac{(\tan^{-1} b)^3}{3} - \frac{\pi^3}{192} \right]$
 $= \left(\frac{\pi^3}{24} - \frac{\pi^3}{192} \right) = \frac{7\pi^3}{192}$
- diverges by the n th-Term Test since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (-1)^n \tanh n = \lim_{b \rightarrow \infty} (-1)^n \left(\frac{1-e^{-2n}}{1+e^{-2n}} \right) = \lim_{n \rightarrow \infty} (-1)^n$
does not exist
- converges by the Direct Comparison Test: $n! < n^n \Rightarrow \ln(n!) < n \ln(n) \Rightarrow \frac{\ln(n!)}{\ln(n)} < n$
 $\Rightarrow \log_n(n!) < n \Rightarrow \frac{\log_n(n!)}{n^3} < \frac{1}{n^2}$, which is the n th-term of a convergent p -series
- converges by the Direct Comparison Test: $a_1 = 1 = \frac{12}{(1)(3)(2)^2}$, $a_2 = \frac{1 \cdot 2}{3 \cdot 4} = \frac{12}{(2)(4)(3)^2}$, $a_3 = \left(\frac{2 \cdot 3}{4 \cdot 5} \right) \left(\frac{1 \cdot 2}{3 \cdot 4} \right)$
 $= \frac{12}{(3)(5)(4)^2}$, $a_4 = \left(\frac{3 \cdot 4}{5 \cdot 6} \right) \left(\frac{2 \cdot 3}{4 \cdot 5} \right) \left(\frac{1 \cdot 2}{3 \cdot 4} \right) = \frac{12}{(4)(6)(5)^2}$, $\dots \Rightarrow 1 + \sum_{n=1}^{\infty} \frac{12}{(n+1)(n+3)(n+2)^2}$ represents the

given series and $\frac{12}{(n+1)(n+3)(n+2)^2} < \frac{12}{n^4}$, which is the n th-term of a convergent p -series

6. converges by the Ratio Test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{n}{(n-1)(n+1)} = 0 < 1$
7. diverges by the n th-Term Test since if $a_n \rightarrow L$ as $n \rightarrow \infty$, then $L = \frac{1}{1+L} \Rightarrow L^2 + L - 1 = 0 \Rightarrow L = \frac{-1 \pm \sqrt{5}}{2} \neq 0$
8. Split the given series into $\sum_{n=1}^{\infty} \frac{1}{3^{2n+1}}$ and $\sum_{n=1}^{\infty} \frac{2n}{3^{2n}}$; the first subseries is a convergent geometric series and the second converges by the Root Test: $\lim_{n \rightarrow \infty} \sqrt[n]{\frac{2n}{3^{2n}}} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{2} \cdot \sqrt[n]{n}}{9} = \frac{1 \cdot 1}{9} = \frac{1}{9} < 1$
9. $f(x) = \cos x$ with $a = \frac{\pi}{3} \Rightarrow f\left(\frac{\pi}{3}\right) = 0.5, f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}, f''\left(\frac{\pi}{3}\right) = -0.5, f'''\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}, f^{(4)}\left(\frac{\pi}{3}\right) = 0.5$;
 $\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3 + \dots$
10. $f(x) = \sin x$ with $a = 2\pi \Rightarrow f(2\pi) = 0, f'(2\pi) = 1, f''(2\pi) = 0, f'''(2\pi) = -1, f^{(4)}(2\pi) = 0, f^{(5)}(2\pi) = 1, f^{(6)}(2\pi) = 0, f^{(7)}(2\pi) = -1$; $\sin x = (x - 2\pi) - \frac{(x - 2\pi)^3}{3!} + \frac{(x - 2\pi)^5}{5!} - \frac{(x - 2\pi)^7}{7!} + \dots$
11. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ with $a = 0$
12. $f(x) = \ln x$ with $a = 1 \Rightarrow f(1) = 0, f'(1) = 1, f''(1) = -1, f'''(1) = 2, f^{(4)}(1) = -6$;
 $\ln x = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \dots$
13. $f(x) = \cos x$ with $a = 22\pi \Rightarrow f(22\pi) = 1, f'(22\pi) = 0, f''(22\pi) = -1, f'''(22\pi) = 0, f^{(4)}(22\pi) = 1, f^{(5)}(22\pi) = 0, f^{(6)}(22\pi) = -1$; $\cos x = 1 - \frac{1}{2}(x - 22\pi)^2 + \frac{1}{4!}(x - 22\pi)^4 - \frac{1}{6!}(x - 22\pi)^6 + \dots$
14. $f(x) = \tan^{-1} x$ with $a = 1 \Rightarrow f(1) = \frac{\pi}{4}, f'(1) = \frac{1}{2}, f''(1) = -\frac{1}{2}, f'''(1) = \frac{1}{2}$;
 $\tan^{-1} x = \frac{\pi}{4} + \frac{(x - 1)}{2} - \frac{(x - 1)^2}{4} + \frac{(x - 1)^3}{12} + \dots$
15. Yes, the sequence converges: $c_n = (a^n + b^n)^{1/n} \Rightarrow c_n = b \left(\left(\frac{a}{b}\right)^n + 1 \right)^{1/n} \Rightarrow \lim_{n \rightarrow \infty} c_n = \ln b + \lim_{n \rightarrow \infty} \frac{\ln \left(\left(\frac{a}{b}\right)^n + 1 \right)}{n}$
 $= \ln b + \lim_{n \rightarrow \infty} \frac{\left(\frac{a}{b}\right)^n \ln \left(\frac{a}{b}\right)}{\left(\frac{a}{b}\right)^n + 1} = \ln b + \frac{0 \cdot \ln \left(\frac{a}{b}\right)}{0 + 1} = \ln b$ since $0 < a < b$. Thus, $\lim_{n \rightarrow \infty} c_n = e^{\ln b} = b$.
16. $1 + \frac{2}{10} + \frac{3}{10^2} + \frac{7}{10^3} + \frac{2}{10^4} + \frac{3}{10^5} + \frac{7}{10^6} + \dots = 1 + \sum_{n=1}^{\infty} \frac{2}{10^{3n-2}} + \sum_{n=1}^{\infty} \frac{3}{10^{3n-1}} + \sum_{n=1}^{\infty} \frac{7}{10^{3n}}$
 $= 1 + \sum_{n=0}^{\infty} \frac{2}{10^{3n+1}} + \sum_{n=0}^{\infty} \frac{3}{10^{3n+2}} + \sum_{n=0}^{\infty} \frac{7}{10^{3n+3}} = 1 + \frac{\left(\frac{2}{10}\right)}{1 - \left(\frac{1}{10}\right)^3} + \frac{\left(\frac{3}{10^2}\right)}{1 - \left(\frac{1}{10}\right)^3} + \frac{\left(\frac{7}{10^3}\right)}{1 - \left(\frac{1}{10}\right)^3}$
 $= 1 + \frac{200}{999} + \frac{30}{999} + \frac{7}{999} = \frac{999 + 237}{999} = \frac{412}{333}$
17. $s_n = \sum_{k=0}^{n-1} \int_k^{k+1} \frac{dx}{1+x^2} \Rightarrow s_n = \int_0^1 \frac{dx}{1+x^2} + \int_1^2 \frac{dx}{1+x^2} + \dots + \int_{n-1}^n \frac{dx}{1+x^2} \Rightarrow s_n = \int_0^n \frac{dx}{1+x^2}$
 $\Rightarrow \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} (\tan^{-1} n - \tan^{-1} 0) = \frac{\pi}{2}$
18. $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)x^{n+1}}{(n+2)(2x+1)^{n+1}} \cdot \frac{(n+1)(2x+1)^n}{nx^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{2x+1} \cdot \frac{(n+1)^2}{n(n+2)} \right| = \left| \frac{x}{2x+1} \right| < 1$
 $\Rightarrow |x| < |2x+1|$; if $x > 0, |x| < |2x+1| \Rightarrow x < 2x+1 \Rightarrow x > -1$; if $-\frac{1}{2} < x < 0, |x| < |2x+1|$
 $\Rightarrow -x < 2x+1 \Rightarrow 3x > -1 \Rightarrow x > -\frac{1}{3}$; if $x < -\frac{1}{2}, |x| < |2x+1| \Rightarrow -x < -2x-1 \Rightarrow x < -1$. Therefore,

the series converges absolutely for $x < -1$ and $x > -\frac{1}{3}$.

19. (a) Each A_{n+1} fits into the corresponding upper triangular region, whose vertices are:

$(n, f(n) - f(n+1))$, $(n+1, f(n+1))$ and $(n, f(n))$ along the line whose slope is $f(n+1) - f(n)$.

All the A_n 's fit into the first upper triangular region whose area is $\frac{f(1)-f(2)}{2} \Rightarrow \sum_{n=1}^{\infty} A_n < \frac{f(1)-f(2)}{2}$

- (b) If $A_k = \frac{f(k+1)+f(k)}{2} - \int_k^{k+1} f(x) dx$, then

$$\begin{aligned} \sum_{k=1}^{n-1} A_k &= \frac{f(1)+f(2)+f(2)+f(3)+f(3)+\dots+f(n-1)+f(n)}{2} - \int_1^2 f(x) dx - \int_2^3 f(x) dx - \dots - \int_{n-1}^n f(x) dx \\ &= \frac{f(1)+f(n)}{2} + \sum_{k=2}^{n-1} f(k) - \int_1^n f(x) dx \Rightarrow \sum_{k=1}^{n-1} A_k = \sum_{k=1}^n f(k) - \frac{f(1)+f(n)}{2} - \int_1^n f(x) dx < \frac{f(1)-f(2)}{2}, \text{ from} \end{aligned}$$

part (a). The sequence $\left\{ \sum_{k=1}^{n-1} A_k \right\}$ is bounded above and increasing, so it converges and the limit in question must exist.

- (c) Let $L = \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n f(k) - \int_1^n f(x) dx - \frac{1}{2}(f(1) + f(n)) \right]$, which exists by part (b). Since f is positive and

decreasing $\lim_{n \rightarrow \infty} f(n) = M \geq 0$ exists. Thus $\lim_{n \rightarrow \infty} \left[\sum_{k=1}^n f(k) - \int_1^n f(x) dx \right] = L + \frac{1}{2}(f(1) + M)$.

20. The number of triangles removed at stage n is 3^{n-1} ; the side length at stage n is $\frac{b}{2^{n-1}}$; the area of a triangle at stage n is $\frac{\sqrt{3}}{4} \left(\frac{b}{2^{n-1}} \right)^2$.

(a) $\frac{\sqrt{3}}{4} b^2 + 3 \frac{\sqrt{3}}{4} \left(\frac{b^2}{2^2} \right) + 3^2 \frac{\sqrt{3}}{4} \left(\frac{b^2}{2^4} \right) + 3^3 \frac{\sqrt{3}}{4} \left(\frac{b^2}{2^6} \right) + \dots = \frac{\sqrt{3}}{4} b^2 \sum_{n=0}^{\infty} \frac{3^n}{2^{2n}} = \frac{\sqrt{3}}{4} b^2 \sum_{n=0}^{\infty} \left(\frac{3}{4} \right)^n$

(b) a geometric series with sum $\frac{\left(\frac{\sqrt{3}}{4} b^2 \right)}{1 - \left(\frac{3}{4} \right)} = \sqrt{3} b^2$

- (c) No; for instance, the three vertices of the original triangle are not removed. However the total area removed is $\sqrt{3} b^2$ which equals the area of the original triangle. Thus the set of points not removed has area 0.

21. (a) No, the limit does not appear to depend on the value of the constant a
 (b) Yes, the limit depends on the value of b

(c) $s = \left(1 - \frac{\cos\left(\frac{a}{n}\right)}{n} \right)^n \Rightarrow \ln s = \frac{\ln\left(1 - \frac{\cos\left(\frac{a}{n}\right)}{n}\right)}{\left(\frac{1}{n}\right)} \Rightarrow \lim_{n \rightarrow \infty} \ln s = \frac{\left(\frac{1}{1 - \cos\left(\frac{a}{n}\right)} \right) \left(\frac{-\frac{a}{n} \sin\left(\frac{a}{n}\right) + \cos\left(\frac{a}{n}\right)}{n^2} \right)}{\left(-\frac{1}{n^2} \right)}$
 $= \lim_{n \rightarrow \infty} \frac{\frac{a}{n} \sin\left(\frac{a}{n}\right) - \cos\left(\frac{a}{n}\right)}{1 - \cos\left(\frac{a}{n}\right)} = \frac{0-1}{1-0} = -1 \Rightarrow \lim_{n \rightarrow \infty} s = e^{-1} \approx 0.3678794412$; similarly,
 $\lim_{n \rightarrow \infty} \left(1 - \frac{\cos\left(\frac{a}{bn}\right)}{bn} \right)^n = e^{-1/b}$

22. $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \lim_{n \rightarrow \infty} a_n = 0$; $\lim_{n \rightarrow \infty} \left[\left(\frac{1+\sin a_n}{2} \right)^n \right]^{1/n} = \lim_{n \rightarrow \infty} \left(\frac{1+\sin a_n}{2} \right) = \frac{1+\sin\left(\lim_{n \rightarrow \infty} a_n\right)}{2} = \frac{1+\sin 0}{2} = \frac{1}{2} \Rightarrow$ the series converges by the n th-Root Test

23. $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{b^{n+1} x^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{b^n x^n} \right| < 1 \Rightarrow |bx| < 1 \Rightarrow -\frac{1}{b} < x < \frac{1}{b} = 5 \Rightarrow b = \pm \frac{1}{5}$

24. A polynomial has only a finite number of nonzero terms in its Taylor series, but the functions $\sin x$, $\ln x$ and e^x have infinitely many nonzero terms in their Taylor expansions.

$$\begin{aligned}
25. \lim_{x \rightarrow 0} \frac{\sin(ax) - \sin x - x}{x^3} &= \lim_{x \rightarrow 0} \frac{\left(ax - \frac{a^3 x^3}{3!} + \dots\right) - \left(x - \frac{x^3}{3!} + \dots\right) - x}{x^3} \\
&= \lim_{x \rightarrow 0} \left[\frac{a-2}{x^2} - \frac{a^3}{3!} + \frac{1}{3!} - \left(\frac{a^5}{5!} - \frac{1}{5!}\right)x^2 + \dots \right] \text{ is finite if } a-2=0 \Rightarrow a=2; \\
\lim_{x \rightarrow 0} \frac{\sin 2x - \sin x - x}{x^3} &= -\frac{2^3}{3!} + \frac{1}{3!} = -\frac{7}{6}
\end{aligned}$$

$$\begin{aligned}
26. \lim_{x \rightarrow 0} \frac{\cos ax - b}{2x^2} = -1 &\Rightarrow \lim_{x \rightarrow 0} \frac{\left(1 - \frac{a^2 x^2}{2} + \frac{a^4 x^4}{4!} - \dots\right) - b}{2x^2} = -1 \Rightarrow \lim_{x \rightarrow 0} \left(\frac{1-b}{2x^2} - \frac{a^2}{4} + \frac{a^2 x^2}{48} - \dots\right) = -1 \\
&\Rightarrow b = 1 \text{ and } a = \pm 2
\end{aligned}$$

$$27. (a) \frac{u_n}{u_{n+1}} = \frac{(n+1)^2}{n^2} = 1 + \frac{2}{n} + \frac{1}{n^2} \Rightarrow C = 2 > 1 \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges}$$

$$(b) \frac{u_n}{u_{n+1}} = \frac{n+1}{n} = 1 + \frac{1}{n} + \frac{0}{n^2} \Rightarrow C = 1 \leq 1 \text{ and } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

$$\begin{aligned}
28. \frac{u_n}{u_{n+1}} &= \frac{2n(2n+1)}{(2n-1)^2} = \frac{4n^2+2n}{4n^2-4n+1} = 1 + \frac{\left(\frac{6}{4}\right)}{4n^2-4n+1} = 1 + \frac{\left(\frac{3}{2}\right)}{n} + \frac{\left[\frac{5n^2}{(4n^2-4n+1)}\right]}{n^2} \text{ after long division} \\
&\Rightarrow C = \frac{3}{2} > 1 \text{ and } |f(n)| = \frac{5n^2}{4n^2-4n+1} = \frac{5}{\left(4-\frac{4}{n}+\frac{1}{n^2}\right)} \leq 5 \Rightarrow \sum_{n=1}^{\infty} u_n \text{ converges by Raabe's Test}
\end{aligned}$$

$$29. (a) \sum_{n=1}^{\infty} a_n = L \Rightarrow a_n^2 \leq a_n \sum_{n=1}^{\infty} a_n = a_n L \Rightarrow \sum_{n=1}^{\infty} a_n^2 \text{ converges by the Direct Comparison Test}$$

$$\begin{aligned}
(b) \text{ converges by the Limit Comparison Test: } \lim_{n \rightarrow \infty} \frac{\left(\frac{a_n}{1-a_n}\right)}{a_n} &= \lim_{n \rightarrow \infty} \frac{1}{1-a_n} = 1 \text{ since } \sum_{n=1}^{\infty} a_n \text{ converges and} \\
\text{therefore } \lim_{n \rightarrow \infty} a_n &= 0
\end{aligned}$$

$$\begin{aligned}
30. \text{ If } 0 < a_n < 1 \text{ then } |\ln(1-a_n)| &= -\ln(1-a_n) = a_n + \frac{a_n^2}{2} + \frac{a_n^3}{3} + \dots < a_n + a_n^2 + a_n^3 + \dots = \frac{a_n}{1-a_n}, \\
&\text{a positive term of a convergent series, by the Limit Comparison Test and Exercise 29b}
\end{aligned}$$

$$\begin{aligned}
31. (1-x)^{-1} &= 1 + \sum_{n=1}^{\infty} x^n \text{ where } |x| < 1 \Rightarrow \frac{1}{(1-x)^2} = \frac{d}{dx} (1-x)^{-1} = \sum_{n=1}^{\infty} n x^{n-1} \text{ and when } x = \frac{1}{2} \text{ we have} \\
4 &= 1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \dots + n\left(\frac{1}{2}\right)^{n-1} + \dots
\end{aligned}$$

$$\begin{aligned}
32. (a) \sum_{n=1}^{\infty} x^{n+1} &= \frac{x^2}{1-x} \Rightarrow \sum_{n=1}^{\infty} (n+1)x^n = \frac{2x-x^2}{(1-x)^2} \Rightarrow \sum_{n=1}^{\infty} n(n+1)x^{n-1} = \frac{2}{(1-x)^3} \Rightarrow \sum_{n=1}^{\infty} n(n+1)x^n = \frac{2x}{(1-x)^3} \\
&\Rightarrow \sum_{n=1}^{\infty} \frac{n(n+1)}{x^n} = \frac{\frac{2}{x}}{\left(1-\frac{1}{x}\right)^3} = \frac{2x^2}{(x-1)^3}, |x| > 1 \\
(b) x &= \sum_{n=1}^{\infty} \frac{n(n+1)}{x^n} \Rightarrow x = \frac{2x^2}{(x-1)^3} \Rightarrow x^3 - 3x^2 + x - 1 = 0 \Rightarrow x = 1 + \left(1 + \frac{\sqrt{57}}{9}\right)^{1/3} + \left(1 - \frac{\sqrt{57}}{9}\right)^{1/3} \\
&\approx 2.769292, \text{ using a CAS or calculator}
\end{aligned}$$

$$\begin{aligned}
33. \text{ The sequence } \{x_n\} \text{ converges to } \frac{\pi}{2} \text{ from below so } \epsilon_n &= \frac{\pi}{2} - x_n > 0 \text{ for each } n. \text{ By the Alternating Series} \\
&\text{Estimation Theorem } \epsilon_{n+1} \approx \frac{1}{3!} (\epsilon_n)^3 \text{ with } |\text{error}| < \frac{1}{5!} (\epsilon_n)^5, \text{ and since the remainder is negative this is an} \\
&\text{overestimate } \Rightarrow 0 < \epsilon_{n+1} < \frac{1}{6} (\epsilon_n)^3.
\end{aligned}$$

$$\begin{aligned}
34. \text{ Yes, the series } \sum_{n=1}^{\infty} \ln(1+a_n) \text{ converges by the Direct Comparison Test: } 1+a_n &< 1+a_n + \frac{a_n^2}{2!} + \frac{a_n^3}{3!} + \dots \\
&\Rightarrow 1+a_n < e^{a_n} \Rightarrow \ln(1+a_n) < a_n
\end{aligned}$$

35. (a) $\frac{1}{(1-x)^2} = \frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{d}{dx} (1 + x + x^2 + x^3 + \dots) = 1 + 2x + 3x^2 + 4x^3 + \dots = \sum_{n=1}^{\infty} nx^{n-1}$
- (b) from part (a) we have $\sum_{n=1}^{\infty} n \left(\frac{5}{6} \right)^{n-1} \left(\frac{1}{6} \right) = \left(\frac{1}{6} \right) \left[\frac{1}{1 - (\frac{5}{6})} \right]^2 = 6$
- (c) from part (a) we have $\sum_{n=1}^{\infty} np^{n-1}q = \frac{q}{(1-p)^2} = \frac{q}{q^2} = \frac{1}{q}$
36. (a) $\sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} 2^{-k} = \frac{(\frac{1}{2})}{1 - (\frac{1}{2})} = 1$ and $E(x) = \sum_{k=1}^{\infty} kp_k = \sum_{k=1}^{\infty} k2^{-k} = \frac{1}{2} \sum_{k=1}^{\infty} k2^{1-k} = \left(\frac{1}{2} \right) \frac{1}{[1 - (\frac{1}{2})]^2} = 2$
by Exercise 35(a)
- (b) $\sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} \frac{5^{k-1}}{6^k} = \frac{1}{5} \sum_{k=1}^{\infty} \left(\frac{5}{6} \right)^k = \left(\frac{1}{5} \right) \left[\frac{(\frac{5}{6})}{1 - (\frac{5}{6})} \right] = 1$ and $E(x) = \sum_{k=1}^{\infty} kp_k = \sum_{k=1}^{\infty} k \frac{5^{k-1}}{6^k} = \frac{1}{6} \sum_{k=1}^{\infty} k \left(\frac{5}{6} \right)^{k-1}$
 $= \left(\frac{1}{6} \right) \frac{1}{[1 - (\frac{5}{6})]^2} = 6$
- (c) $\sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \lim_{k \rightarrow \infty} \left(1 - \frac{1}{k+1} \right) = 1$ and $E(x) = \sum_{k=1}^{\infty} kp_k = \sum_{k=1}^{\infty} k \left(\frac{1}{k(k+1)} \right)$
 $= \sum_{k=1}^{\infty} \frac{1}{k+1}$, a divergent series so that $E(x)$ does not exist
37. (a) $R_n = C_0 e^{-kt_0} + C_0 e^{-2kt_0} + \dots + C_0 e^{-nkt_0} = \frac{C_0 e^{-kt_0} (1 - e^{-nkt_0})}{1 - e^{-kt_0}} \Rightarrow R = \lim_{n \rightarrow \infty} R_n = \frac{C_0 e^{-kt_0}}{1 - e^{-kt_0}} = \frac{C_0}{e^{kt_0} - 1}$
- (b) $R_n = \frac{e^{-1} (1 - e^{-n})}{1 - e^{-1}} \Rightarrow R_1 = e^{-1} \approx 0.36787944$ and $R_{10} = \frac{e^{-1} (1 - e^{-10})}{1 - e^{-1}} \approx 0.58195028$;
 $R = \frac{1}{e-1} \approx 0.58197671$; $R - R_{10} \approx 0.00002643 \Rightarrow \frac{R - R_{10}}{R} < 0.0001$
- (c) $R_n = \frac{e^{-1} (1 - e^{-1/n})}{1 - e^{-1}} = \frac{R}{2} = \frac{1}{2} \left(\frac{1}{e-1} \right) \approx 0.29093835$; $R_n > \frac{R}{2} \Rightarrow \frac{1 - e^{-1/n}}{1 - e^{-1}} > \left(\frac{1}{2} \right) \left(\frac{1}{e-1} \right)$
 $\Rightarrow 1 - e^{-1/n} > \frac{1}{2} \Rightarrow e^{-1/n} < \frac{1}{2} \Rightarrow -\frac{1}{n} < \ln \left(\frac{1}{2} \right) \Rightarrow \frac{1}{n} > -\ln \left(\frac{1}{2} \right) \Rightarrow n > 6.93 \Rightarrow n = 7$
38. (a) $R = \frac{C_0}{e^{kt_0} - 1} \Rightarrow R e^{kt_0} = R + C_0 = C_H \Rightarrow e^{kt_0} = \frac{C_H}{C_0} \Rightarrow t_0 = \frac{1}{k} \ln \left(\frac{C_H}{C_0} \right)$
- (b) $t_0 = \frac{1}{0.05} \ln e = 20$ hrs
- (c) Give an initial dose that produces a concentration of 2 mg/ml followed every $t_0 = \frac{1}{0.02} \ln \left(\frac{2}{0.5} \right) \approx 69.31$ hrs
by a dose that raises the concentration by 1.5 mg/ml
- (d) $t_0 = \frac{1}{0.2} \ln \left(\frac{0.1}{0.03} \right) = 5 \ln \left(\frac{10}{3} \right) \approx 6$ hrs
39. The convergence of $\sum_{n=1}^{\infty} |a_n|$ implies that $\lim_{n \rightarrow \infty} |a_n| = 0$. Let $N > 0$ be such that $|a_n| < \frac{1}{2} \Rightarrow 1 - |a_n| > \frac{1}{2}$
 $\Rightarrow \frac{|a_n|}{1 - |a_n|} < 2 |a_n|$ for all $n > N$. Now $|\ln(1 + a_n)| = \left| a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} - \frac{a_n^4}{4} + \dots \right| \leq |a_n| + \left| \frac{a_n^2}{2} \right| + \left| \frac{a_n^3}{3} \right| + \left| \frac{a_n^4}{4} \right| + \dots$
 $< |a_n| + |a_n|^2 + |a_n|^3 + |a_n|^4 + \dots = \frac{|a_n|}{1 - |a_n|} < 2 |a_n|$. Therefore $\sum_{n=1}^{\infty} \ln(1 + a_n)$ converges by the Direct
Comparison Test since $\sum_{n=1}^{\infty} |a_n|$ converges.
40. $\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln(\ln n))^p}$ converges if $p > 1$ and diverges otherwise by the Integral Test: when $p = 1$ we have
 $\lim_{b \rightarrow \infty} \int_3^b \frac{dx}{x \ln x (\ln(\ln x))} = \lim_{b \rightarrow \infty} [\ln(\ln(\ln x))]_3^b = \infty$; when $p \neq 1$ we have $\lim_{b \rightarrow \infty} \int_3^b \frac{dx}{x \ln x (\ln(\ln x))^p}$
 $= \lim_{b \rightarrow \infty} \left[\frac{(\ln(\ln x))^{-p+1}}{1-p} \right]_3^b = \begin{cases} \frac{(\ln(\ln 3))^{-p+1}}{1-p}, & \text{if } p > 1 \\ \infty, & \text{if } p < 1 \end{cases}$

41. (a) $s_{2n+1} = \frac{c_1}{1} + \frac{c_2}{2} + \frac{c_3}{3} + \dots + \frac{c_{2n+1}}{2n+1} = \frac{t_1}{1} + \frac{t_2-t_1}{2} + \frac{t_3-t_2}{3} + \dots + \frac{t_{2n+1}-t_{2n}}{2n+1}$
 $= t_1 \left(1 - \frac{1}{2}\right) + t_2 \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + t_{2n} \left(\frac{1}{2n} - \frac{1}{2n+1}\right) + \frac{t_{2n+1}}{2n+1} = \sum_{k=1}^{2n} \frac{t_k}{k(k+1)} + \frac{t_{2n+1}}{2n+1}.$
- (b) $\{c_n\} = \{(-1)^n\} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges
- (c) $\{c_n\} = \{1, -1, -1, 1, 1, -1, -1, 1, 1, \dots\} \Rightarrow$ the series $1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \dots$ converges
42. (a) $(1 - t + t^2 - t^3 + \dots + (-1)^n t^n)(1 + t) = 1 - t + t^2 - t^3 + \dots + (-1)^n t^n + t - t^2 + t^3 - t^4 + \dots + (-1)^n t^{n+1}$
 $= 1 + (-1)^n t^{n+1} \Rightarrow 1 - t + t^2 - t^3 + \dots + (-1)^n t^n - \frac{(-1)^n t^{n+1}}{1+t} = \frac{1}{1+t}$
- (b) $\int_0^x \frac{1}{1+t} dt = \int_0^x \left[1 - t + t^2 + \dots + (-1)^n t^n + \frac{(-1)^{n+1} t^{n+1}}{1+t}\right] dt \Rightarrow [\ln |1+t|]_0^x$
 $= \left[t - \frac{t^2}{2} + \frac{t^3}{3} + \dots + \frac{(-1)^n t^{n+1}}{n+1}\right]_0^x + \int_0^x \frac{(-1)^{n+1} t^{n+1}}{n+1} dt \Rightarrow \ln |1+x|$
 $= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^n x^{n+1}}{n+1} + R_{n+1}, \text{ where } R_{n+1} = \int_0^x \frac{(-1)^{n+1} t^{n+1}}{n+1} dt$
- (c) $x > 0$ and $R_{n+1} = (-1)^{n+1} \int_0^x \frac{t^{n+1}}{1+t} dt \Rightarrow |R_{n+1}| = \int_0^x \frac{t^{n+1}}{1+t} dt \leq \int_0^x t^{n+1} dt = \frac{x^{n+2}}{n+2}$
- (d) $-1 < x < 0$ and $R_{n+1} = (-1)^{n+1} \int_0^x \frac{t^{n+1}}{1+t} dt \Rightarrow |R_{n+1}| = \left| \int_0^x \frac{t^{n+1}}{1+t} dt \right| \leq \int_0^x \left| \frac{t^{n+1}}{1+t} \right| dt$
 $\leq \int_0^x \frac{|t|^{n+1}}{1-|x|} dx = \frac{|x|^{n+2}}{(1-|x|)(n+2)}$ since $|1+t| \geq 1-|x|$
- (e) From part (d) we have $|R_{n+1}| \leq \frac{|x|^{n+2}}{(1-|x|)(n+2)} \Rightarrow$ the given series converges since
 $\lim_{n \rightarrow \infty} \frac{|x|^{n+2}}{(1-|x|)(n+2)} = 0 \Rightarrow |R_{n+1}| \rightarrow 0$ when $|x| < 1$. If $x = 1$, by part (c) $|R_{n+1}| \leq \frac{|x|^{n+2}}{n+2} = \frac{1}{n+2} \rightarrow 0$.
 Thus the given series converges to $\ln(1+x)$ for $-1 < x \leq 1$.