CHAPTER 16 INTEGRATION IN VECTOR FIELDS

16.1 LINE INTEGRALS

1. $\mathbf{r} = t\mathbf{i} + (1-t)\mathbf{j} \Rightarrow x = t$ and $y = 1 - t \Rightarrow y = 1 - x \Rightarrow$ (c)

2. $\mathbf{r} = \mathbf{i} + \mathbf{j} + t\mathbf{k} \Rightarrow x = 1, y = 1$, and $z = t \Rightarrow$ (e)

3. $\mathbf{r} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} \Rightarrow x = 2\cos t$ and $y = 2\sin t \Rightarrow x^2 + y^2 = 4 \Rightarrow$ (g)

4. $\mathbf{r} = t\mathbf{i} \Rightarrow x = t, y = 0$, and $z = 0 \Rightarrow$ (a)

5. $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} \Rightarrow x = t, y = t$, and $z = t \Rightarrow$ (d)

6. $\mathbf{r} = t\mathbf{j} + (2 - 2t)\mathbf{k} \Rightarrow y = t$ and $z = 2 - 2t \Rightarrow z = 2 - 2y \Rightarrow$ (b)

7. $\mathbf{r} = (t^2 - 1)\,\mathbf{j} + 2t\mathbf{k} \Rightarrow y = t^2 - 1$ and $z = 2t \Rightarrow y = \frac{z^2}{4} - 1 \Rightarrow$ (f)

8. $\mathbf{r} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{k} \Rightarrow x = 2\cos t$ and $z = 2\sin t \Rightarrow x^2 + z^2 = 4 \Rightarrow$ (h)

9. $\mathbf{r}(t) = t\mathbf{i} + (1-t)\mathbf{j}, 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{2}\,\mathbf{j}$; $x = t$ and $y = 1 - t \Rightarrow x + y = t + (1 - t) = 1$
$\Rightarrow \int_C f(x,y,z)\,ds = \int_0^1 f(t, 1-t, 0)\left|\frac{d\mathbf{r}}{dt}\right| dt = \int_0^1 (1)\left(\sqrt{2}\right) dt = \left[\sqrt{2}\,t\right]_0^1 = \sqrt{2}$

10. $\mathbf{r}(t) = t\mathbf{i} + (1-t)\mathbf{j} + \mathbf{k}, 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{2}$; $x = t, y = 1 - t$, and $z = 1 \Rightarrow x - y + z - 2$
$= t - (1 - t) + 1 - 2 = 2t - 2 \Rightarrow \int_C f(x,y,z)\,ds = \int_0^1 (2t - 2)\sqrt{2}\,dt = \sqrt{2}\left[t^2 - 2t\right]_0^1 = -\sqrt{2}$

11. $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + (2 - 2t)\mathbf{k}, 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{4 + 1 + 4} = 3$; $xy + y + z$
$= (2t)t + t + (2 - 2t) \Rightarrow \int_C f(x,y,z)\,ds = \int_0^1 (2t^2 - t + 2)\,3\,dt = 3\left[\frac{2}{3}t^3 - \frac{1}{2}t^2 + 2t\right]_0^1 = 3\left(\frac{2}{3} - \frac{1}{2} + 2\right) = \frac{13}{2}$

12. $\mathbf{r}(t) = (4\cos t)\mathbf{i} + (4\sin t)\mathbf{j} + 3t\mathbf{k}, -2\pi \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-4\sin t)\mathbf{i} + (4\cos t)\mathbf{j} + 3\mathbf{k}$
$\Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{16\sin^2 t + 16\cos^2 t + 9} = 5$; $\sqrt{x^2 + y^2} = \sqrt{16\cos^2 t + 16\sin^2 t} = 4 \Rightarrow \int_C f(x,y,z)\,ds = \int_{-2\pi}^{2\pi} (4)(5)\,dt$
$= [20t]_{-2\pi}^{2\pi} = 80\pi$

13. $\mathbf{r}(t) = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + t(-\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = (1 - t)\mathbf{i} + (2 - 3t)\mathbf{j} + (3 - 2t)\mathbf{k}, 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$
$\Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1 + 9 + 4} = \sqrt{14}$; $x + y + z = (1 - t) + (2 - 3t) + (3 - 2t) = 6 - 6t \Rightarrow \int_C f(x,y,z)\,ds$
$= \int_0^1 (6 - 6t)\sqrt{14}\,dt = 6\sqrt{14}\left[t - \frac{t^2}{2}\right]_0^1 = \left(6\sqrt{14}\right)\left(\frac{1}{2}\right) = 3\sqrt{14}$

14. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, 1 \le t \le \infty \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{3}$; $\frac{\sqrt{3}}{x^2 + y^2 + z^2} = \frac{\sqrt{3}}{t^2 + t^2 + t^2} = \frac{\sqrt{3}}{3t^2}$
$\Rightarrow \int_C f(x,y,z)\,ds = \int_1^\infty \left(\frac{\sqrt{3}}{3t^2}\right)\sqrt{3}\,dt = \left[-\frac{1}{t}\right]_1^\infty = \lim_{b \to \infty}\left(-\frac{1}{b} + 1\right) = 1$

15. C_1: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}\,,\ 0 \le t \le 1 \ \Rightarrow\ \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \ \Rightarrow\ \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1+4t^2}\,;\ x + \sqrt{y} - z^2 = t + \sqrt{t^2} - 0 = t + |t| = 2t$ since $t \ge 0 \ \Rightarrow \int_{C_1} f(x,y,z)\,ds = \int_0^1 2t\sqrt{1+4t^2}\,dt = \left[\frac{1}{6}\left(1+4t^2\right)^{3/2}\right]_0^1 = \frac{1}{6}(5)^{3/2} - \frac{1}{6} = \frac{1}{6}\left(5\sqrt{5}-1\right)$;
C_2: $\mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k},\ 0 \le t \le 1 \ \Rightarrow\ \frac{d\mathbf{r}}{dt} = \mathbf{k} \ \Rightarrow\ \left|\frac{d\mathbf{r}}{dt}\right| = 1;\ x + \sqrt{y} - z^2 = 1 + \sqrt{1} - t^2 = 2 - t^2$
$\Rightarrow \int_{C_2} f(x,y,z)\,ds = \int_0^1 \left(2-t^2\right)(1)\,dt = \left[2t - \frac{1}{3}t^3\right]_0^1 = 2 - \frac{1}{3} = \frac{5}{3}$; therefore $\int_C f(x,y,z)\,ds$
$= \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds = \frac{5}{6}\sqrt{5} + \frac{3}{2}$

16. C_1: $\mathbf{r}(t) = t\mathbf{k}\,,\ 0 \le t \le 1 \ \Rightarrow\ \frac{d\mathbf{r}}{dt} = \mathbf{k} \ \Rightarrow\ \left|\frac{d\mathbf{r}}{dt}\right| = 1;\ x + \sqrt{y} - z^2 = 0 + \sqrt{0} - t^2 = -t^2$
$\Rightarrow \int_{C_1} f(x,y,z)\,ds = \int_0^1 \left(-t^2\right)(1)\,dt = \left[-\frac{t^3}{3}\right]_0^1 = -\frac{1}{3}$;
C_2: $\mathbf{r}(t) = t\mathbf{j} + \mathbf{k},\ 0 \le t \le 1 \ \Rightarrow\ \frac{d\mathbf{r}}{dt} = \mathbf{j} \ \Rightarrow\ \left|\frac{d\mathbf{r}}{dt}\right| = 1;\ x + \sqrt{y} - z^2 = 0 + \sqrt{t} - 1 = \sqrt{t} - 1$
$\Rightarrow \int_{C_2} f(x,y,z)\,ds = \int_0^1 \left(\sqrt{t}-1\right)(1)\,dt = \left[\frac{2}{3}t^{3/2} - t\right]_0^1 = \frac{2}{3} - 1 = -\frac{1}{3}$;
C_3: $\mathbf{r}(t) = t\mathbf{i} + \mathbf{j} + \mathbf{k}\,,\ 0 \le t \le 1 \ \Rightarrow\ \frac{d\mathbf{r}}{dt} = \mathbf{i} \ \Rightarrow\ \left|\frac{d\mathbf{r}}{dt}\right| = 1;\ x + \sqrt{y} - z^2 = t + \sqrt{1} - 1 = t$
$\Rightarrow \int_{C_3} f(x,y,z)\,ds = \int_0^1 (t)(1)\,dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2} \ \Rightarrow\ \int_C f(x,y,z)\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \int_{C_3} f\,ds = -\frac{1}{3} + \left(-\frac{1}{3}\right) + \frac{1}{2}$
$= -\frac{1}{6}$

17. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}\,,\ 0 < a \le t \le b \ \Rightarrow\ \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \ \Rightarrow\ \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{3}\,;\ \frac{x+y+z}{x^2+y^2+z^2} = \frac{t+t+t}{t^2+t^2+t^2} = \frac{1}{t}$
$\Rightarrow \int_C f(x,y,z)\,ds = \int_a^b \left(\frac{1}{t}\right)\sqrt{3}\,dt = \left[\sqrt{3}\ln|t|\right]_a^b = \sqrt{3}\ln\left(\frac{b}{a}\right)$, since $0 < a \le b$

18. $\mathbf{r}(t) = (a\cos t)\mathbf{j} + (a\sin t)\mathbf{k}\,,\ 0 \le t \le 2\pi \ \Rightarrow\ \frac{d\mathbf{r}}{dt} = (-a\sin t)\mathbf{j} + (a\cos t)\mathbf{k} \ \Rightarrow\ \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = |a|$;
$-\sqrt{x^2+z^2} = -\sqrt{0 + a^2\sin^2 t} = \begin{cases} -|a|\sin t, & 0 \le t \le \pi \\ |a|\sin t, & \pi \le t \le 2\pi \end{cases} \ \Rightarrow \int_C f(x,y,z)\,ds = \int_0^\pi -|a|^2\sin t\,dt + \int_\pi^{2\pi} |a|^2\sin t\,dt$
$= \left[a^2\cos t\right]_0^\pi - \left[a^2\cos t\right]_\pi^{2\pi} = \left[a^2(-1) - a^2\right] - \left[a^2 - a^2(-1)\right] = -4a^2$

19. $\mathbf{r}(x) = x\mathbf{i} + y\mathbf{j} = x\mathbf{i} + \frac{x^2}{2}\mathbf{j}\,,\ 0 \le x \le 2 \ \Rightarrow\ \frac{d\mathbf{r}}{dx} = \mathbf{i} + x\mathbf{j} \ \Rightarrow\ \left|\frac{d\mathbf{r}}{dx}\right| = \sqrt{1+x^2}\,;\ f(x,y) = f\left(x, \frac{x^2}{2}\right) = \frac{x^3}{\left(\frac{x^2}{2}\right)} = 2x \ \Rightarrow \int_C f\,ds$
$= \int_0^2 (2x)\sqrt{1+x^2}\,dx = \left[\frac{2}{3}\left(1+x^2\right)^{3/2}\right]_0^2 = \frac{2}{3}\left(5^{3/2} - 1\right) = \frac{10\sqrt{5}-2}{3}$

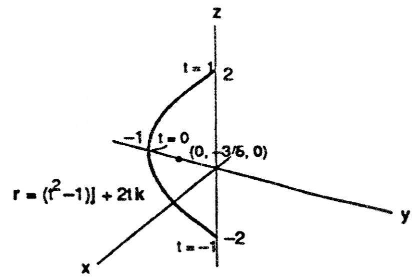
20. $\mathbf{r}(t) = (1-t)\mathbf{i} + \frac{1}{2}(1-t)^2\mathbf{j},\ 0 \le t \le 1 \ \Rightarrow\ \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1+(1-t)^2}\,;\ f(x,y) = f\left((1-t), \frac{1}{2}(1-t)^2\right) = \frac{(1-t)+\frac{1}{4}(1-t)^4}{\sqrt{1+(1-t)^2}}$
$\Rightarrow \int_C f\,ds = \int_0^1 \frac{(1-t)+\frac{1}{4}(1-t)^4}{\sqrt{1+(1-t)^2}}\sqrt{1+(1-t)^2}\,dt = \int_0^1 \left((1-t) + \frac{1}{4}(1-t)^4\right)dt = \left[-\frac{1}{2}(1-t)^2 - \frac{1}{20}(1-t)^5\right]_0^1$
$= 0 - \left(-\frac{1}{2} - \frac{1}{20}\right) = \frac{11}{20}$

21. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j}\,,\ 0 \le t \le \frac{\pi}{2} \ \Rightarrow\ \frac{d\mathbf{r}}{dt} = (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} \ \Rightarrow\ \left|\frac{d\mathbf{r}}{dt}\right| = 2;\ f(x,y) = f(2\cos t, 2\sin t)$
$= 2\cos t + 2\sin t \ \Rightarrow \int_C f\,ds = \int_0^{\pi/2} (2\cos t + 2\sin t)(2)\,dt = \left[4\sin t - 4\cos t\right]_0^{\pi/2} = 4 - (-4) = 8$

22. $\mathbf{r}(t) = (2\sin t)\mathbf{i} + (2\cos t)\mathbf{j}\,,\ 0 \le t \le \frac{\pi}{4} \ \Rightarrow\ \frac{d\mathbf{r}}{dt} = (2\cos t)\mathbf{i} + (-2\sin t)\mathbf{j} \ \Rightarrow\ \left|\frac{d\mathbf{r}}{dt}\right| = 2;\ f(x,y) = f(2\sin t, 2\cos t)$
$= 4\sin^2 t - 2\cos t \ \Rightarrow \int_C f\,ds = \int_0^{\pi/4} \left(4\sin^2 t - 2\cos t\right)(2)\,dt = \left[4t - 2\sin 2t - 4\sin t\right]_0^{\pi/4}$
$= \pi - 2\left(1+\sqrt{2}\right)$

23. $\mathbf{r}(t) = (t^2-1)\,\mathbf{j} + 2t\mathbf{k}\,,\ 0 \le t \le 1 \ \Rightarrow\ \frac{d\mathbf{r}}{dt} = 2t\mathbf{j} + 2\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 2\sqrt{t^2+1};\ M = \int_C \delta(x,y,z)\,ds = \int_0^1 \delta(t)\left(2\sqrt{t^2+1}\right)dt$
$= \int_0^1 \left(\frac{3}{2}\,t\right)\left(2\sqrt{t^2+1}\right)dt = \left[\left(t^2+1\right)^{3/2}\right]_0^1 = 2^{3/2} - 1 = 2\sqrt{2} - 1$

24. $\mathbf{r}(t) = (t^2-1)\,\mathbf{j} + 2t\mathbf{k}\,,\ -1 \le t \le 1 \ \Rightarrow\ \frac{d\mathbf{r}}{dt} = 2t\mathbf{j} + 2\mathbf{k}$
$\Rightarrow\ \left|\frac{d\mathbf{r}}{dt}\right| = 2\sqrt{t^2+1};\ M = \int\limits_C \delta(x,y,z)\,ds$
$= \int_{-1}^1 \left(15\sqrt{(t^2-1)+2}\right)\left(2\sqrt{t^2+1}\right)dt$
$= \int_{-1}^1 30\,(t^2+1)\,dt = \left[30\left(\frac{t^3}{3}+t\right)\right]_{-1}^1 = 60\left(\frac{1}{3}+1\right) = 80;$
$M_{xz} = \int_C y\delta(x,y,z)\,ds = \int_{-1}^1 (t^2-1)\,[30\,(t^2+1)]\,dt$
$= \int_{-1}^1 30\,(t^4-1)\,dt = \left[30\left(\frac{t^5}{5}-t\right)\right]_{-1}^1 = 60\left(\frac{1}{5}-1\right)$
$= -48 \ \Rightarrow\ \overline{y} = \frac{M_{xz}}{M} = -\frac{48}{80} = -\frac{3}{5}\,;\ M_{yz} = \int_C x\delta(x,y,z)\,ds = \int_C 0\,\delta\,ds = 0 \ \Rightarrow\ \overline{x} = 0;\ \overline{z} = 0$ by symmetry (since δ is independent of z) $\Rightarrow\ (\overline{x},\overline{y},\overline{z}) = \left(0, -\frac{3}{5}\,, 0\right)$



25. $\mathbf{r}(t) = \sqrt{2}t\,\mathbf{i} + \sqrt{2}t\,\mathbf{j} + (4-t^2)\,\mathbf{k}\,,\ 0 \le t \le 1 \ \Rightarrow\ \frac{d\mathbf{r}}{dt} = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} - 2t\mathbf{k} \ \Rightarrow\ \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{2+2+4t^2} = 2\sqrt{1+t^2}\,;$

(a) $M = \int_C \delta\,ds = \int_0^1 (3t)\left(2\sqrt{1+t^2}\right)dt = \left[2\left(1+t^2\right)^{3/2}\right]_0^1 = 2\left(2^{3/2}-1\right) = 4\sqrt{2} - 2$

(b) $M = \int_C \delta\,ds = \int_0^1 (1)\left(2\sqrt{1+t^2}\right)dt = \left[t\sqrt{1+t^2} + \ln\left(t+\sqrt{1+t^2}\right)\right]_0^1 = \left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right] - (0 + \ln 1)$
$= \sqrt{2} + \ln\left(1+\sqrt{2}\right)$

26. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + \frac{2}{3}\,t^{3/2}\mathbf{k}\,,\ 0 \le t \le 2 \ \Rightarrow\ \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2\mathbf{j} + t^{1/2}\mathbf{k} \ \Rightarrow\ \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1+4+t} = \sqrt{5+t};$
$M = \int_C \delta\,ds = \int_0^2 \left(3\sqrt{5+t}\right)\left(\sqrt{5+t}\right)dt = \int_0^2 3(5+t)\,dt = \left[\frac{3}{2}\,(5+t)^2\right]_0^2 = \frac{3}{2}\,(7^2-5^2) = \frac{3}{2}\,(24) = 36;$
$M_{yz} = \int_C x\delta\,ds = \int_0^2 t[3(5+t)]\,dt = \int_0^2 (15t+3t^2)\,dt = \left[\frac{15}{2}\,t^2 + t^3\right]_0^2 = 30 + 8 = 38;$
$M_{xz} = \int_C y\delta\,ds = \int_0^2 2t[3(5+t)]\,dt = 2\int_0^2 (15t+3t^2)\,dt = 76;\ M_{xy} = \int_C z\delta\,ds = \int_0^2 \frac{2}{3}\,t^{3/2}[3(5+t)]\,dt$
$= \int_0^2 \left(10t^{3/2} + 2t^{5/2}\right)dt = \left[4t^{5/2} + \frac{4}{7}\,t^{7/2}\right]_0^2 = 4(2)^{5/2} + \frac{4}{7}\,(2)^{7/2} = 16\sqrt{2} + \frac{32}{7}\,\sqrt{2} = \frac{144}{7}\,\sqrt{2} \ \Rightarrow\ \overline{x} = \frac{M_{yz}}{M}$
$= \frac{38}{36} = \frac{19}{18}\,,\ \overline{y} = \frac{M_{xz}}{M} = \frac{76}{36} = \frac{19}{9}\,,$ and $\overline{z} = \frac{M_{xy}}{M} = \frac{144\sqrt{2}}{7\cdot 36} = \frac{4}{7}\,\sqrt{2}$

27. Let $x = a\cos t$ and $y = a\sin t$, $0 \le t \le 2\pi$. Then $\frac{dx}{dt} = -a\sin t,\ \frac{dy}{dt} = a\cos t,\ \frac{dz}{dt} = 0$
$\Rightarrow\ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = a\,dt;\ I_z = \int_C (x^2+y^2)\,\delta\,ds = \int_0^{2\pi} (a^2\sin^2 t + a^2\cos^2 t)\,a\delta\,dt$
$= \int_0^{2\pi} a^3\delta\,dt = 2\pi\delta a^3;\ M = \int_C \delta(x,y,z)\,ds = \int_0^{2\pi} \delta a\,dt = 2\pi\delta a \ \Rightarrow\ R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{2\pi a^3\delta}{2\pi a\delta}} = a.$

28. $\mathbf{r}(t) = t\mathbf{j} + (2-2t)\mathbf{k}\,,\ 0 \le t \le 1 \ \Rightarrow\ \frac{d\mathbf{r}}{dt} = \mathbf{j} - 2\mathbf{k} \ \Rightarrow\ \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{5};\ M = \int_C \delta\,ds = \int_0^1 \delta\sqrt{5}\,dt = \delta\sqrt{5};$
$I_x = \int_C (y^2+z^2)\,\delta\,ds = \int_0^1 [t^2+(2-2t)^2]\,\delta\sqrt{5}\,dt = \int_0^1 (5t^2-8t+4)\,\delta\sqrt{5}\,dt = \delta\sqrt{5}\,\left[\frac{5}{3}\,t^3 - 4t^2 + 4t\right]_0^1 = \frac{5}{3}\,\delta\sqrt{5}\,;$
$I_y = \int_C (x^2+z^2)\,\delta\,ds = \int_0^1 [0^2+(2-2t)^2]\,\delta\sqrt{5}\,dt = \int_0^1 (4t^2-8t+4)\,\delta\sqrt{5}\,dt = \delta\sqrt{5}\,\left[\frac{4}{3}\,t^3 - 4t^2 + 4t\right]_0^1 = \frac{4}{3}\,\delta\sqrt{5}\,;$
$I_z = \int_C (x^2+y^2)\,\delta\,ds = \int_0^1 (0^2+t^2)\,\delta\sqrt{5}\,dt = \delta\sqrt{5}\,\left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3}\,\delta\sqrt{5} \ \Rightarrow\ R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{5}{3}}\,,\ R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}\,,$
and $R_z = \sqrt{\frac{I_z}{M}} = \frac{1}{\sqrt{3}}$

29. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, $0 \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$;

(a) $M = \int_C \delta\, ds = \int_0^{2\pi} \delta\sqrt{2}\, dt = 2\pi\delta\sqrt{2}$; $I_z = \int_C (x^2+y^2)\,\delta\, ds = \int_0^{2\pi} (\cos^2 t + \sin^2 t)\,\delta\sqrt{2}\, dt = 2\pi\delta\sqrt{2}$
$\Rightarrow R_z = \sqrt{\frac{I_z}{M}} = 1$

(b) $M = \int_C \delta(x,y,z)\, ds = \int_0^{4\pi} \delta\sqrt{2}\, dt = 4\pi\delta\sqrt{2}$ and $I_z = \int_C (x^2+y^2)\,\delta\, ds = \int_0^{4\pi} \delta\sqrt{2}\, dt = 4\pi\delta\sqrt{2}$
$\Rightarrow R_z = \sqrt{\frac{I_z}{M}} = 1$

30. $\mathbf{r}(t) = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + \frac{2\sqrt{2}}{3}t^{3/2}\mathbf{k}$, $0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} + \sqrt{2t}\,\mathbf{k}$
$\Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{(t+1)^2} = t+1$ for $0 \le t \le 1$; $M = \int_C \delta\, ds = \int_0^1 (t+1)\, dt = \left[\frac{1}{2}(t+1)^2\right]_0^1 = \frac{1}{2}(2^2 - 1^2) = \frac{3}{2}$;
$M_{xy} = \int_C z\delta\, ds = \int_0^1 \left(\frac{2\sqrt{2}}{3}t^{3/2}\right)(t+1)\, dt = \frac{2\sqrt{2}}{3}\int_0^1 \left(t^{5/2} + t^{3/2}\right) dt = \frac{2\sqrt{2}}{3}\left[\frac{2}{7}t^{7/2} + \frac{2}{5}t^{5/2}\right]_0^1$
$= \frac{2\sqrt{2}}{3}\left(\frac{2}{7} + \frac{2}{5}\right) = \frac{2\sqrt{2}}{3}\left(\frac{24}{35}\right) = \frac{16\sqrt{2}}{35} \Rightarrow \overline{z} = \frac{M_{xy}}{M} = \left(\frac{16\sqrt{2}}{35}\right)\left(\frac{2}{3}\right) = \frac{32\sqrt{2}}{105}$; $I_z = \int_C (x^2+y^2)\,\delta\, ds$
$= \int_0^1 (t^2\cos^2 t + t^2\sin^2 t)(t+1)\, dt = \int_0^1 (t^3 + t^2)\, dt = \left[\frac{t^4}{4} + \frac{t^3}{3}\right]_0^1 = \frac{1}{4} + \frac{1}{3} = \frac{7}{12} \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{7}{18}}$

31. $\delta(x,y,z) = 2 - z$ and $\mathbf{r}(t) = (\cos t)\mathbf{j} + (\sin t)\mathbf{k}$, $0 \le t \le \pi \Rightarrow M = 2\pi - 2$ as found in Example 4 of the text; also $\left|\frac{d\mathbf{r}}{dt}\right| = 1$; $I_x = \int_C (y^2+z^2)\,\delta\, ds = \int_0^\pi (\cos^2 t + \sin^2 t)(2 - \sin t)\, dt = \int_0^\pi (2 - \sin t)\, dt = 2\pi - 2 \Rightarrow R_x = \sqrt{\frac{I_x}{M}}$
$= 1$

32. $\mathbf{r}(t) = t\mathbf{i} + \frac{2\sqrt{2}}{3}t^{3/2}\mathbf{j} + \frac{t^2}{2}\mathbf{k}$, $0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + \sqrt{2}\,t^{1/2}\mathbf{j} + t\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1 + 2t + t^2} = \sqrt{(1+t)^2} = 1 + t$ for $0 \le t \le 2$; $M = \int_C \delta\, ds = \int_0^2 \left(\frac{1}{t+1}\right)(1+t)\, dt = \int_0^2 dt = 2$; $M_{yz} = \int_C x\delta\, ds = \int_0^2 t\left(\frac{1}{t+1}\right)(1+t)\, dt = \left[\frac{t^2}{2}\right]_0^2 = 2$;
$M_{xz} = \int_C y\delta\, ds = \int_0^2 \frac{2\sqrt{2}}{3}t^{3/2}\, dt = \left[\frac{4\sqrt{2}}{15}t^{5/2}\right]_0^2 = \frac{32}{15}$; $M_{xy} = \int_C z\delta\, ds = \int_0^2 \frac{t^2}{2}\, dt = \left[\frac{t^3}{6}\right]_0^2 = \frac{4}{3} \Rightarrow \overline{x} = \frac{M_{yz}}{M} = 1$,
$\overline{y} = \frac{M_{xz}}{M} = \frac{16}{15}$, and $\overline{z} = \frac{M_{xy}}{M} = \frac{2}{3}$; $I_x = \int_C (y^2+z^2)\,\delta\, ds = \int_0^2 \left(\frac{8}{9}t^3 + \frac{1}{4}t^4\right) dt = \left[\frac{2}{9}t^4 + \frac{t^5}{20}\right]_0^2 = \frac{32}{9} + \frac{32}{20} = \frac{232}{45}$;
$I_y = \int_C (x^2+z^2)\,\delta\, ds = \int_0^2 \left(t^2 + \frac{1}{4}t^4\right) dt = \left[\frac{t^3}{3} + \frac{t^5}{20}\right]_0^2 = \frac{8}{3} + \frac{32}{20} = \frac{64}{15}$; $I_z = \int_C (x^2+y^2)\,\delta\, ds$
$= \int_0^2 \left(t^2 + \frac{8}{9}t^3\right) dt = \left[\frac{t^3}{3} + \frac{2}{9}t^4\right]_0^2 = \frac{8}{3} + \frac{32}{9} = \frac{56}{9} \Rightarrow R_x = \sqrt{\frac{I_x}{M}} = \frac{2}{3}\sqrt{\frac{29}{5}}$, $R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{32}{15}}$, and
$R_z = \sqrt{\frac{I_z}{M}} = \frac{2}{3}\sqrt{7}$

33-36. Example CAS commands:

Maple:

```
f := (x,y,z) -> sqrt(1+30*x^2+10*y);
g := t -> t;
h := t -> t^2;
k := t -> 3*t^2;
a,b := 0,2;
ds := ( D(g)^2 + D(h)^2 + D(k)^2 )^(1/2):                    # (a)
'ds' = ds(t)*'dt';
F := f(g,h,k):                                                 # (b)
'F(t)' = F(t);
Int( f, s=C..NULL ) = Int( simplify(F(t)*ds(t)), t=a..b );    # (c)
`` = value(rhs(%));
```

Mathematica: (functions and domains may vary)

```
Clear[x, y, z, r, t, f]
f[x_,y_,z_]:= Sqrt[1 + 30x^2 + 10y]
```

```
{a,b}= {0, 2};
x[t_]:= t
y[t_]:= t^2
z[t_]:= 3t^2
r[t_]:= {x[t], y[t], z[t]}
v[t_]:= D[r[t], t]
mag[vector_]:=Sqrt[vector.vector]
Integrate[f[x[t],y[t],z[t]] mag[v[t]], {t, a, b}]
N[%]
```

16.2 VECTOR FIELDS, WORK, CIRCULATION, AND FLUX

1. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2} \Rightarrow \frac{\partial f}{\partial x} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2x) = -x(x^2 + y^2 + z^2)^{-3/2}$; similarly, $\frac{\partial f}{\partial y} = -y(x^2 + y^2 + z^2)^{-3/2}$ and $\frac{\partial f}{\partial z} = -z(x^2 + y^2 + z^2)^{-3/2} \Rightarrow \nabla f = \frac{-x\mathbf{i} - y\mathbf{j} - z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$

2. $f(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2} = \frac{1}{2}\ln(x^2 + y^2 + z^2) \Rightarrow \frac{\partial f}{\partial x} = \frac{1}{2}\left(\frac{1}{x^2+y^2+z^2}\right)(2x) = \frac{x}{x^2 + y^2 + z^2}$; similarly, $\frac{\partial f}{\partial y} = \frac{y}{x^2 + y^2 + z^2}$ and $\frac{\partial f}{\partial z} = \frac{z}{x^2 + y^2 + z^2} \Rightarrow \nabla f = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{x^2 + y^2 + z^2}$

3. $g(x, y, z) = e^z - \ln(x^2 + y^2) \Rightarrow \frac{\partial g}{\partial x} = -\frac{2x}{x^2 + y^2}$, $\frac{\partial g}{\partial y} = -\frac{2y}{x^2 + y^2}$ and $\frac{\partial g}{\partial z} = e^z$ $\Rightarrow \nabla g = \left(\frac{-2x}{x^2 + y^2}\right)\mathbf{i} - \left(\frac{2y}{x^2 + y^2}\right)\mathbf{j} + e^z\mathbf{k}$

4. $g(x, y, z) = xy + yz + xz \Rightarrow \frac{\partial g}{\partial x} = y + z$, $\frac{\partial g}{\partial y} = x + z$, and $\frac{\partial g}{\partial z} = y + x \Rightarrow \nabla g = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$

5. $|\mathbf{F}|$ inversely proportional to the square of the distance from (x, y) to the origin $\Rightarrow \sqrt{(M(x, y))^2 + (N(x, y))^2}$ $= \frac{k}{x^2 + y^2}$, $k > 0$; **F** points toward the origin $\Rightarrow$ **F** is in the direction of $\mathbf{n} = \frac{-x}{\sqrt{x^2 + y^2}}\mathbf{i} - \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$ $\Rightarrow \mathbf{F} = a\mathbf{n}$, for some constant $a > 0$. Then $M(x, y) = \frac{-ax}{\sqrt{x^2 + y^2}}$ and $N(x, y) = \frac{-ay}{\sqrt{x^2 + y^2}}$ $\Rightarrow \sqrt{(M(x, y))^2 + (N(x, y))^2} = a \Rightarrow a = \frac{k}{x^2 + y^2} \Rightarrow \mathbf{F} = \frac{-kx}{(x^2 + y^2)^{3/2}}\mathbf{i} - \frac{ky}{(x^2 + y^2)^{3/2}}\mathbf{j}$, for any constant $k > 0$

6. Given $x^2 + y^2 = a^2 + b^2$, let $x = \sqrt{a^2 + b^2}\cos t$ and $y = -\sqrt{a^2 + b^2}\sin t$. Then $\mathbf{r} = \left(\sqrt{a^2 + b^2}\cos t\right)\mathbf{i} - \left(\sqrt{a^2 + b^2}\sin t\right)\mathbf{j}$ traces the circle in a clockwise direction as t goes from 0 to 2π $\Rightarrow \mathbf{v} = \left(-\sqrt{a^2 + b^2}\sin t\right)\mathbf{i} - \left(\sqrt{a^2 + b^2}\cos t\right)\mathbf{j}$ is tangent to the circle in a clockwise direction. Thus, let $\mathbf{F} = \mathbf{v} \Rightarrow \mathbf{F} = y\mathbf{i} - x\mathbf{j}$ and $\mathbf{F}(0, 0) = \mathbf{0}$.

7. Substitute the parametric representations for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field **F**, and calculate the work $W = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

(a) $\mathbf{F} = 3t\mathbf{i} + 2t\mathbf{j} + 4t\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 9t \Rightarrow W = \int_0^1 9t\,dt = \frac{9}{2}$

(b) $\mathbf{F} = 3t^2\mathbf{i} + 2t\mathbf{j} + 4t^4\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 7t^2 + 16t^7 \Rightarrow W = \int_0^1 (7t^2 + 16t^7)\,dt = \left[\frac{7}{3}t^3 + 2t^8\right]_0^1$ $= \frac{7}{3} + 2 = \frac{13}{3}$

(c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = 3t\mathbf{i} + 2t\mathbf{j}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 5t \Rightarrow W_1 = \int_0^1 5t\,dt = \frac{5}{2}$; $\mathbf{F}_2 = 3\mathbf{i} + 2\mathbf{j} + 4t\mathbf{k}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 4t \Rightarrow W_2 = \int_0^1 4t\,dt = 2 \Rightarrow W = W_1 + W_2 = \frac{9}{2}$

8. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate the work $W = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

(a) $\mathbf{F} = \left(\frac{1}{t^2+1}\right)\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{1}{t^2+1} \Rightarrow W = \int_0^1 \frac{1}{t^2+1}\, dt = \left[\tan^{-1} t\right]_0^1 = \frac{\pi}{4}$

(b) $\mathbf{F} = \left(\frac{1}{t^2+1}\right)\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{2t}{t^2+1} \Rightarrow W = \int_0^1 \frac{2t}{t^2+1}\, dt = \left[\ln\left(t^2+1\right)\right]_0^1 = \ln 2$

(c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = \left(\frac{1}{t^2+1}\right)\mathbf{j}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = \frac{1}{t^2+1}$; $\mathbf{F}_2 = \frac{1}{2}\mathbf{j}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k}$
$\Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 0 \Rightarrow W = \int_0^1 \frac{1}{t^2+1}\, dt = \frac{\pi}{4}$

9. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate the work $W = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

(a) $\mathbf{F} = \sqrt{t}\mathbf{i} - 2t\mathbf{j} + \sqrt{t}\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2\sqrt{t} - 2t \Rightarrow W = \int_0^1 \left(2\sqrt{t} - 2t\right) dt = \left[\frac{4}{3}t^{3/2} - t^2\right]_0^1 = \frac{1}{3}$

(b) $\mathbf{F} = t^2\mathbf{i} - 2t\mathbf{j} + t\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 4t^4 - 3t^2 \Rightarrow W = \int_0^1 \left(4t^4 - 3t^2\right) dt = \left[\frac{4}{5}t^5 - t^3\right]_0^1 = -\frac{1}{5}$

(c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = -2t\mathbf{j} + \sqrt{t}\mathbf{k}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = -2t \Rightarrow W_1 = \int_0^1 -2t\, dt$
$= -1$; $\mathbf{F}_2 = \sqrt{t}\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 1 \Rightarrow W_2 = \int_0^1 dt = 1 \Rightarrow W = W_1 + W_2 = 0$

10. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate the work $W = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

(a) $\mathbf{F} = t^2\mathbf{i} + t^2\mathbf{j} + t^2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 3t^2 \Rightarrow W = \int_0^1 3t^2\, dt = 1$

(b) $\mathbf{F} = t^3\mathbf{i} - t^6\mathbf{j} + t^5\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t^3 + 2t^7 + 4t^8 \Rightarrow W = \int_0^1 \left(t^3 + 2t^7 + 4t^8\right) dt$
$= \left[\frac{t^4}{4} + \frac{t^8}{4} + \frac{4}{9}t^9\right]_0^1 = \frac{17}{18}$

(c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = t^2\mathbf{i}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = t^2 \Rightarrow W_1 = \int_0^1 t^2\, dt = \frac{1}{3}$;
$\mathbf{F}_2 = \mathbf{i} + t\mathbf{j} + t\mathbf{k}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = t \Rightarrow W_2 = \int_0^1 t\, dt = \frac{1}{2} \Rightarrow W = W_1 + W_2 = \frac{5}{6}$

11. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate the work $W = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

(a) $\mathbf{F} = \left(3t^2 - 3t\right)\mathbf{i} + 3t\mathbf{j} + \mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 3t^2 + 1 \Rightarrow W = \int_0^1 \left(3t^2 + 1\right) dt = \left[t^3 + t\right]_0^1 = 2$

(b) $\mathbf{F} = \left(3t^2 - 3t\right)\mathbf{i} + 3t^4\mathbf{j} + \mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 6t^5 + 4t^3 + 3t^2 - 3t$
$\Rightarrow W = \int_0^1 \left(6t^5 + 4t^3 + 3t^2 - 3t\right) dt = \left[t^6 + t^4 + t^3 - \frac{3}{2}t^2\right]_0^1 = \frac{3}{2}$

(c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = \left(3t^2 - 3t\right)\mathbf{i} + \mathbf{k}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 3t^2 - 3t$
$\Rightarrow W_1 = \int_0^1 \left(3t^2 - 3t\right) dt = \left[t^3 - \frac{3}{2}t^2\right]_0^1 = -\frac{1}{2}$; $\mathbf{F}_2 = 3t\mathbf{j} + \mathbf{k}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 1 \Rightarrow W_2 = \int_0^1 dt = 1$
$\Rightarrow W = W_1 + W_2 = \frac{1}{2}$

12. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate the work $W = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

(a) $\mathbf{F} = 2t\mathbf{i} + 2t\mathbf{j} + 2t\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 6t \Rightarrow W = \int_0^1 6t\, dt = \left[3t^2\right]_0^1 = 3$

(b) $\mathbf{F} = \left(t^2 + t^4\right)\mathbf{i} + \left(t^4 + t\right)\mathbf{j} + \left(t + t^2\right)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 6t^5 + 5t^4 + 3t^2$
$\Rightarrow W = \int_0^1 \left(6t^5 + 5t^4 + 3t^2\right) dt = \left[t^6 + t^5 + t^3\right]_0^1 = 3$

(c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 2t \Rightarrow W_1 = \int_0^1 2t\,dt = 1$;

$\mathbf{F}_2 = (1+t)\mathbf{i} + (t+1)\mathbf{j} + 2\mathbf{k}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 2 \Rightarrow W_2 = \int_0^1 2\,dt = 2 \Rightarrow W = W_1 + W_2 = 3$

13. $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$, and $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k} \Rightarrow \mathbf{F} = t^3\mathbf{i} + t^2\mathbf{j} - t^3\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}$
$\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2t^3 \Rightarrow$ work $= \int_0^1 2t^3\,dt = \frac{1}{2}$

14. $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + \frac{t}{6}\mathbf{k}$, $0 \le t \le 2\pi$, and $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} + (x+y)\mathbf{k}$
$\Rightarrow \mathbf{F} = (2\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + (\cos t + \sin t)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \frac{1}{6}\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$
$= 3\cos^2 t - 2\sin^2 t + \frac{1}{6}\cos t + \frac{1}{6}\sin t \Rightarrow$ work $= \int_0^{2\pi} \left(3\cos^2 t - 2\sin^2 t + \frac{1}{6}\cos t + \frac{1}{6}\sin t\right) dt$
$= \left[\frac{3}{2}t + \frac{3}{4}\sin 2t - t + \frac{\sin 2t}{2} + \frac{1}{6}\sin t - \frac{1}{6}\cos t\right]_0^{2\pi} = \pi$

15. $\mathbf{r} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + t\mathbf{k}$, $0 \le t \le 2\pi$, and $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k} \Rightarrow \mathbf{F} = t\mathbf{i} + (\sin t)\mathbf{j} + (\cos t)\mathbf{k}$ and
$\frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t\cos t - \sin^2 t + \cos t \Rightarrow$ work $= \int_0^{2\pi} (t\cos t - \sin^2 t + \cos t)\,dt$
$= \left[\cos t + t\sin t - \frac{t}{2} + \frac{\sin 2t}{4} + \sin t\right]_0^{2\pi} = -\pi$

16. $\mathbf{r} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \frac{t}{6}\mathbf{k}$, $0 \le t \le 2\pi$, and $\mathbf{F} = 6z\mathbf{i} + y^2\mathbf{j} + 12x\mathbf{k} \Rightarrow \mathbf{F} = t\mathbf{i} + (\cos^2 t)\mathbf{j} + (12\sin t)\mathbf{k}$ and
$\frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} + \frac{1}{6}\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t\cos t - \sin t\cos^2 t + 2\sin t$
$\Rightarrow$ work $= \int_0^{2\pi} (t\cos t - \sin t\cos^2 t + 2\sin t)\,dt = \left[\cos t + t\sin t + \frac{1}{3}\cos^3 t - 2\cos t\right]_0^{2\pi} = 0$

17. $x = t$ and $y = x^2 = t^2 \Rightarrow \mathbf{r} = t\mathbf{i} + t^2\mathbf{j}$, $-1 \le t \le 2$, and $\mathbf{F} = xy\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = t^3\mathbf{i} + (t + t^2)\mathbf{j}$ and
$\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t^3 + (2t^2 + 2t^3) = 3t^3 + 2t^2 \Rightarrow \int_C xy\,dx + (x+y)\,dy = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\,dt = \int_{-1}^2 (3t^3 + 2t^2)\,dt$
$= \left[\frac{3}{4}t^4 + \frac{2}{3}t^3\right]_{-1}^2 = \left(12 + \frac{16}{3}\right) - \left(\frac{3}{4} - \frac{2}{3}\right) = \frac{45}{4} + \frac{18}{3} = \frac{69}{4}$

18. Along $(0,0)$ to $(1,0)$: $\mathbf{r} = t\mathbf{i}$, $0 \le t \le 1$, and $\mathbf{F} = (x-y)\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = t\mathbf{i} + t\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t$;
Along $(1,0)$ to $(0,1)$: $\mathbf{r} = (1-t)\mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{F} = (x-y)\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = (1-2t)\mathbf{i} + \mathbf{j}$ and
$\frac{d\mathbf{r}}{dt} = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2t$;
Along $(0,1)$ to $(0,0)$: $\mathbf{r} = (1-t)\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{F} = (x-y)\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = (t-1)\mathbf{i} + (1-t)\mathbf{j}$ and
$\frac{d\mathbf{r}}{dt} = -\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t - 1 \Rightarrow \int_C (x-y)\,dx + (x+y)\,dy = \int_0^1 t\,dt + \int_0^1 2t\,dt + \int_0^1 (t-1)\,dt = \int_0^1 (4t-1)\,dt$
$= \left[2t^2 - t\right]_0^1 = 2 - 1 = 1$

19. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} = y^2\mathbf{i} + y\mathbf{j}$, $2 \ge y \ge -1$, and $\mathbf{F} = x^2\mathbf{i} - y\mathbf{j} = y^4\mathbf{i} - y\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dy} = 2y\mathbf{i} + \mathbf{j}$ and $\mathbf{F} \cdot \frac{d\mathbf{r}}{dy} = 2y^5 - y$
$\Rightarrow \int_C \mathbf{F} \cdot \mathbf{T}\,ds = \int_2^{-1} \mathbf{F} \cdot \frac{d\mathbf{r}}{dy}\,dy = \int_2^{-1} (2y^5 - y)\,dy = \left[\frac{1}{3}y^6 - \frac{1}{2}y^2\right]_2^{-1} = \left(\frac{1}{3} - \frac{1}{2}\right) - \left(\frac{64}{3} - \frac{4}{2}\right) = \frac{3}{2} - \frac{63}{3} = -\frac{39}{2}$

20. $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le \frac{\pi}{2}$, and $\mathbf{F} = y\mathbf{i} - x\mathbf{j} \Rightarrow \mathbf{F} = (\sin t)\mathbf{i} - (\cos t)\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$
$\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin^2 t - \cos^2 t = -1 \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (-1)\,dt = -\frac{\pi}{2}$

21. $\mathbf{r} = (\mathbf{i} + \mathbf{j}) + t(\mathbf{i} + 2\mathbf{j}) = (1+t)\mathbf{i} + (1+2t)\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{F} = xy\mathbf{i} + (y-x)\mathbf{j} \Rightarrow \mathbf{F} = (1 + 3t + 2t^2)\mathbf{i} + t\mathbf{j}$ and
$\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 1 + 5t + 2t^2 \Rightarrow$ work $= \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\,dt = \int_0^1 (1 + 5t + 2t^2)\,dt = \left[t + \frac{5}{2}t^2 + \frac{2}{3}t^3\right]_0^1 = \frac{25}{6}$

22. $\mathbf{r} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j}$, $0 \le t \le 2\pi$, and $\mathbf{F} = \nabla f = 2(x+y)\mathbf{i} + 2(x+y)\mathbf{j}$
$\Rightarrow \mathbf{F} = 4(\cos t + \sin t)\mathbf{i} + 4(\cos t + \sin t)\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$

$= -8(\sin t \cos t + \sin^2 t) + 8(\cos^2 t + \cos t \sin t) = 8(\cos^2 t - \sin^2 t) = 8 \cos 2t \Rightarrow \text{work} = \int_C \nabla f \cdot d\mathbf{r}$
$= \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^{2\pi} 8 \cos 2t\, dt = [4 \sin 2t]_0^{2\pi} = 0$

23. (a) $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j},\ 0 \le t \le 2\pi,\ \mathbf{F}_1 = x\mathbf{i} + y\mathbf{j}$, and $\mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$,
$\mathbf{F}_1 = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, and $\mathbf{F}_2 = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}}{dt} = 0$ and $\mathbf{F}_2 \cdot \frac{d\mathbf{r}}{dt} = \sin^2 t + \cos^2 t = 1$
$\Rightarrow \text{Circ}_1 = \int_0^{2\pi} 0\, dt = 0$ and $\text{Circ}_2 = \int_0^{2\pi} dt = 2\pi;\ \mathbf{n} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \mathbf{n} = \cos^2 t + \sin^2 t = 1$ and
$\mathbf{F}_2 \cdot \mathbf{n} = 0 \Rightarrow \text{Flux}_1 = \int_0^{2\pi} dt = 2\pi$ and $\text{Flux}_2 = \int_0^{2\pi} 0\, dt = 0$

(b) $\mathbf{r} = (\cos t)\mathbf{i} + (4 \sin t)\mathbf{j},\ 0 \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (4 \cos t)\mathbf{j},\ \mathbf{F}_1 = (\cos t)\mathbf{i} + (4 \sin t)\mathbf{j}$, and
$\mathbf{F}_2 = (-4 \sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}}{dt} = 15 \sin t \cos t$ and $\mathbf{F}_2 \cdot \frac{d\mathbf{r}}{dt} = 4 \Rightarrow \text{Circ}_1 = \int_0^{2\pi} 15 \sin t \cos t\, dt$
$= \left[\frac{15}{2} \sin^2 t\right]_0^{2\pi} = 0$ and $\text{Circ}_2 = \int_0^{2\pi} 4\, dt = 8\pi;\ \mathbf{n} = \left(\frac{4}{\sqrt{17}} \cos t\right)\mathbf{i} + \left(\frac{1}{\sqrt{17}} \sin t\right)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \mathbf{n}$
$= \frac{4}{\sqrt{17}} \cos^2 t + \frac{4}{\sqrt{17}} \sin^2 t$ and $\mathbf{F}_2 \cdot \mathbf{n} = -\frac{15}{\sqrt{17}} \sin t \cos t \Rightarrow \text{Flux}_1 = \int_0^{2\pi} (\mathbf{F}_1 \cdot \mathbf{n})\, |\mathbf{v}|\, dt = \int_0^{2\pi} \left(\frac{4}{\sqrt{17}}\right) \sqrt{17}\, dt$
$= 8\pi$ and $\text{Flux}_2 = \int_0^{2\pi} (\mathbf{F}_2 \cdot \mathbf{n})\, |\mathbf{v}|\, dt = \int_0^{2\pi} \left(-\frac{15}{\sqrt{17}} \sin t \cos t\right) \sqrt{17}\, dt = \left[-\frac{15}{2} \sin^2 t\right]_0^{2\pi} = 0$

24. $\mathbf{r} = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j},\ 0 \le t \le 2\pi,\ \mathbf{F}_1 = 2x\mathbf{i} - 3y\mathbf{j}$, and $\mathbf{F}_2 = 2x\mathbf{i} + (x - y)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j}$,
$\mathbf{F}_1 = (2a \cos t)\mathbf{i} - (3a \sin t)\mathbf{j}$, and $\mathbf{F}_2 = (2a \cos t)\mathbf{i} + (a \cos t - a \sin t)\mathbf{j} \Rightarrow \mathbf{n}\, |\mathbf{v}| = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$,
$\mathbf{F}_1 \cdot \mathbf{n}\, |\mathbf{v}| = 2a^2 \cos^2 t - 3a^2 \sin^2 t$, and $\mathbf{F}_2 \cdot \mathbf{n}\, |\mathbf{v}| = 2a^2 \cos^2 t + a^2 \sin t \cos t - a^2 \sin^2 t$
$\Rightarrow \text{Flux}_1 = \int_0^{2\pi} (2a^2 \cos^2 t - 3a^2 \sin^2 t)\, dt = 2a^2 \left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} - 3a^2 \left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = -\pi a^2$, and
$\text{Flux}_2 = \int_0^{2\pi} (2a^2 \cos^2 t - a^2 \sin t \cos t - a^2 \sin^2 t)\, dt = 2a^2 \left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} + \frac{a^2}{2} [\sin^2 t]_0^{2\pi} - a^2 \left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = \pi a^2$

25. $\mathbf{F}_1 = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j},\ \frac{d\mathbf{r}_1}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 0 \Rightarrow \text{Circ}_1 = 0;\ M_1 = a \cos t$,
$N_1 = a \sin t,\ dx = -a \sin t\, dt,\ dy = a \cos t\, dt \Rightarrow \text{Flux}_1 = \int_C M_1\, dy - N_1\, dx = \int_0^{\pi} (a^2 \cos^2 t + a^2 \sin^2 t)\, dt$
$= \int_0^{\pi} a^2\, dt = a^2\pi;$
$\mathbf{F}_2 = t\mathbf{i},\ \frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = t \Rightarrow \text{Circ}_2 = \int_{-a}^{a} t\, dt = 0;\ M_2 = t,\ N_2 = 0,\ dx = dt,\ dy = 0 \Rightarrow \text{Flux}_2$
$= \int_C M_2\, dy - N_2\, dx = \int_{-a}^{a} 0\, dt = 0$; therefore, $\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = 0$ and $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = a^2\pi$

26. $\mathbf{F}_1 = (a^2 \cos^2 t)\,\mathbf{i} + (a^2 \sin^2 t)\,\mathbf{j},\ \frac{d\mathbf{r}_1}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = -a^3 \sin t \cos^2 t + a^3 \cos t \sin^2 t$
$\Rightarrow \text{Circ}_1 = \int_0^{\pi} (-a^3 \sin t \cos^2 t + a^3 \cos t \sin^2 t)\, dt = -\frac{2a^3}{3};\ M_1 = a^2 \cos^2 t,\ N_1 = a^2 \sin^2 t,\ dy = a \cos t\, dt$,
$dx = -a \sin t\, dt \Rightarrow \text{Flux}_1 = \int_C M_1\, dy - N_1\, dx = \int_0^{\pi} (a^3 \cos^3 t + a^3 \sin^3 t)\, dt = \frac{4}{3} a^3;$
$\mathbf{F}_2 = t^2\mathbf{i},\ \frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = t^2 \Rightarrow \text{Circ}_2 = \int_{-a}^{a} t^2\, dt = \frac{2a^3}{3};\ M_2 = t^2,\ N_2 = 0,\ dy = 0,\ dx = dt$
$\Rightarrow \text{Flux}_2 = \int_C M_2\, dy - N_2\, dx = 0$; therefore, $\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = 0$ and $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = \frac{4}{3} a^3$

27. $\mathbf{F}_1 = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j},\ \frac{d\mathbf{r}_1}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = a^2 \sin^2 t + a^2 \cos^2 t = a^2$
$\Rightarrow \text{Circ}_1 = \int_0^{\pi} a^2\, dt = a^2\pi;\ M_1 = -a \sin t,\ N_1 = a \cos t,\ dx = -a \sin t\, dt,\ dy = a \cos t\, dt$
$\Rightarrow \text{Flux}_1 = \int_C M_1\, dy - N_1\, dx = \int_0^{\pi} (-a^2 \sin t \cos t + a^2 \sin t \cos t)\, dt = 0;\ \mathbf{F}_2 = t\mathbf{j},\ \frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 0$
$\Rightarrow \text{Circ}_2 = 0;\ M_2 = 0,\ N_2 = t,\ dx = dt,\ dy = 0 \Rightarrow \text{Flux}_2 = \int_C M_2\, dy - N_2\, dx = \int_{-a}^{a} -t\, dt = 0$; therefore,
$\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = a^2\pi$ and $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = 0$

28. $\mathbf{F}_1 = (-a^2 \sin^2 t)\,\mathbf{i} + (a^2 \cos^2 t)\,\mathbf{j}$, $\frac{d\mathbf{r}_1}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = a^3 \sin^3 t + a^3 \cos^3 t$
$\Rightarrow \text{Circ}_1 = \int_0^{\pi} (a^3 \sin^3 t + a^3 \cos^3 t)\, dt = \frac{4}{3} a^3$; $M_1 = -a^2 \sin^2 t$, $N_1 = a^2 \cos^2 t$, $dy = a \cos t\, dt$, $dx = -a \sin t\, dt$
$\Rightarrow \text{Flux}_1 = \int_C M_1\, dy - N_1\, dx = \int_0^{\pi} (-a^3 \cos t \sin^2 t + a^3 \sin t \cos^2 t)\, dt = \frac{2}{3} a^3$; $\mathbf{F}_2 = t^2\mathbf{j}$, $\frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 0$
$\Rightarrow \text{Circ}_2 = 0$; $M_2 = 0$, $N_2 = t^2$, $dy = 0$, $dx = dt \Rightarrow \text{Flux}_2 = \int_C M_2\, dy - N_2\, dx = \int_{-a}^{a} -t^2\, dt = -\frac{2}{3} a^3$; therefore,
$\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = \frac{4}{3} a^3$ and $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = 0$

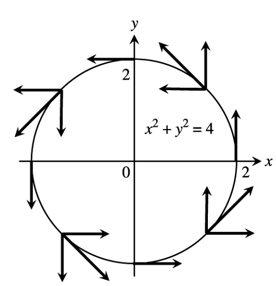
29. (a) $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le \pi$, and $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\,\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$ and
$\mathbf{F} = (\cos t + \sin t)\mathbf{i} - (\cos^2 t + \sin^2 t)\,\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t \cos t - \sin^2 t - \cos t \Rightarrow \int_C \mathbf{F} \cdot \mathbf{T}\, ds$
$= \int_0^{\pi} (-\sin t \cos t - \sin^2 t - \cos t)\, dt = \left[-\frac{1}{2} \sin^2 t - \frac{t}{2} + \frac{\sin 2t}{4} - \sin t\right]_0^{\pi} = -\frac{\pi}{2}$

(b) $\mathbf{r} = (1 - 2t)\mathbf{i}$, $0 \le t \le 1$, and $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\,\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = -2\mathbf{i}$ and $\mathbf{F} = (1 - 2t)\mathbf{i} - (1 - 2t)^2\,\mathbf{j} \Rightarrow$
$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 4t - 2 \Rightarrow \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_0^1 (4t - 2)\, dt = [2t^2 - 2t]_0^1 = 0$

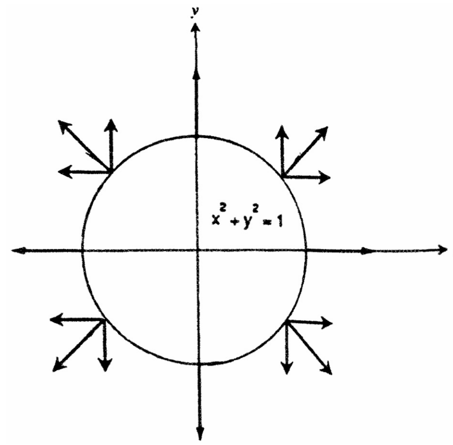
(c) $\mathbf{r}_1 = (1 - t)\mathbf{i} - t\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\,\mathbf{j} \Rightarrow \frac{d\mathbf{r}_1}{dt} = -\mathbf{i} - \mathbf{j}$ and $\mathbf{F} = (1 - 2t)\mathbf{i} - (1 - 2t + 2t^2)\,\mathbf{j}$
$\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}_1}{dt} = (2t - 1) + (1 - 2t + 2t^2) = 2t^2 \Rightarrow \text{Flow}_1 = \int_{C_1} \mathbf{F} \cdot \frac{d\mathbf{r}_1}{dt} = \int_0^1 2t^2\, dt = \frac{2}{3}$; $\mathbf{r}_2 = -t\mathbf{i} + (t - 1)\mathbf{j}$,
$0 \le t \le 1$, and $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\,\mathbf{j} \Rightarrow \frac{d\mathbf{r}_2}{dt} = -\mathbf{i} + \mathbf{j}$ and $\mathbf{F} = -\mathbf{i} - (t^2 + t^2 - 2t + 1)\,\mathbf{j}$
$= -\mathbf{i} - (2t^2 - 2t + 1)\,\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}_2}{dt} = 1 - (2t^2 - 2t + 1) = 2t - 2t^2 \Rightarrow \text{Flow}_2 = \int_{C_2} \mathbf{F} \cdot \frac{d\mathbf{r}_2}{dt} = \int_0^1 (2t - 2t^2)\, dt$
$= \left[t^2 - \frac{2}{3} t^3\right]_0^1 = \frac{1}{3} \Rightarrow \text{Flow} = \text{Flow}_1 + \text{Flow}_2 = \frac{2}{3} + \frac{1}{3} = 1$

30. From $(1, 0)$ to $(0, 1)$: $\mathbf{r}_1 = (1 - t)\mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\,\mathbf{j} \Rightarrow \frac{d\mathbf{r}_1}{dt} = -\mathbf{i} + \mathbf{j}$,
$\mathbf{F} = \mathbf{i} - (1 - 2t + 2t^2)\,\mathbf{j}$, and $\mathbf{n}_1\,|\mathbf{v}_1| = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_1\,|\mathbf{v}_1| = 2t - 2t^2 \Rightarrow \text{Flux}_1 = \int_0^1 (2t - 2t^2)\, dt$
$= \left[t^2 - \frac{2}{3} t^3\right]_0^1 = \frac{1}{3}$;
From $(0, 1)$ to $(-1, 0)$: $\mathbf{r}_2 = -t\mathbf{i} + (1 - t)\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\,\mathbf{j} \Rightarrow \frac{d\mathbf{r}_2}{dt} = -\mathbf{i} - \mathbf{j}$,
$\mathbf{F} = (1 - 2t)\mathbf{i} - (1 - 2t + 2t^2)\,\mathbf{j}$, and $\mathbf{n}_2\,|\mathbf{v}_2| = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_2\,|\mathbf{v}_2| = (2t - 1) + (-1 + 2t - 2t^2) = -2 + 4t - 2t^2$
$\Rightarrow \text{Flux}_2 = \int_0^1 (-2 + 4t - 2t^2)\, dt = \left[-2t + 2t^2 - \frac{2}{3} t^3\right]_0^1 = -\frac{2}{3}$;
From $(-1, 0)$ to $(1, 0)$: $\mathbf{r}_3 = (-1 + 2t)\mathbf{i}$, $0 \le t \le 1$, and $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\,\mathbf{j} \Rightarrow \frac{d\mathbf{r}_3}{dt} = 2\mathbf{i}$,
$\mathbf{F} = (-1 + 2t)\mathbf{i} - (1 - 4t + 4t^2)\,\mathbf{j}$, and $\mathbf{n}_3\,|\mathbf{v}_3| = -2\mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_3\,|\mathbf{v}_3| = 2\,(1 - 4t + 4t^2)$
$\Rightarrow \text{Flux}_3 = 2 \int_0^1 (1 - 4t + 4t^2)\, dt = 2\left[t - 2t^2 + \frac{4}{3} t^3\right]_0^1 = \frac{2}{3} \Rightarrow \text{Flux} = \text{Flux}_1 + \text{Flux}_2 + \text{Flux}_3 = \frac{1}{3} - \frac{2}{3} + \frac{2}{3} = \frac{1}{3}$

31. $\mathbf{F} = -\frac{y}{\sqrt{x^2 + y^2}}\,\mathbf{i} + \frac{x}{\sqrt{x^2 + y^2}}\,\mathbf{j}$ on $x^2 + y^2 = 4$;
at $(2, 0)$, $\mathbf{F} = \mathbf{j}$; at $(0, 2)$, $\mathbf{F} = -\mathbf{i}$; at $(-2, 0)$,
$\mathbf{F} = -\mathbf{j}$; at $(0, -2)$, $\mathbf{F} = \mathbf{i}$; at $\left(\sqrt{2}, \sqrt{2}\right)$, $\mathbf{F} = -\frac{\sqrt{3}}{2}\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}$;
at $\left(\sqrt{2}, -\sqrt{2}\right)$, $\mathbf{F} = \frac{\sqrt{3}}{2}\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}$; at $\left(-\sqrt{2}, \sqrt{2}\right)$,
$\mathbf{F} = -\frac{\sqrt{3}}{2}\,\mathbf{i} - \frac{1}{2}\,\mathbf{j}$; at $\left(-\sqrt{2}, -\sqrt{2}\right)$, $\mathbf{F} = \frac{\sqrt{3}}{2}\,\mathbf{i} - \frac{1}{2}\,\mathbf{j}$



32. $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ on $x^2 + y^2 = 1$; at $(1, 0)$, $\mathbf{F} = \mathbf{i}$; at $(-1, 0)$, $\mathbf{F} = -\mathbf{i}$; at $(0, 1)$, $\mathbf{F} = \mathbf{j}$; at $(0, -1)$, $\mathbf{F} = -\mathbf{j}$; at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\mathbf{F} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$;
at $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\mathbf{F} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$;
at $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, $\mathbf{F} = \frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$;
at $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, $\mathbf{F} = -\frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$.



33. (a) $\mathbf{G} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is to have a magnitude $\sqrt{a^2 + b^2}$ and to be tangent to $x^2 + y^2 = a^2 + b^2$ in a counterclockwise direction. Thus $x^2 + y^2 = a^2 + b^2 \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}$ is the slope of the tangent line at any point on the circle $\Rightarrow y' = -\frac{a}{b}$ at (a, b). Let $\mathbf{v} = -b\mathbf{i} + a\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{a^2 + b^2}$, with $\mathbf{v}$ in a counterclockwise direction and tangent to the circle. Then let $P(x, y) = -y$ and $Q(x, y) = x$ $\Rightarrow \mathbf{G} = -y\mathbf{i} + x\mathbf{j} \Rightarrow$ for (a, b) on $x^2 + y^2 = a^2 + b^2$ we have $\mathbf{G} = -b\mathbf{i} + a\mathbf{j}$ and $|\mathbf{G}| = \sqrt{a^2 + b^2}$.

(b) $\mathbf{G} = \left(\sqrt{x^2 + y^2}\right)\mathbf{F} = \left(\sqrt{a^2 + b^2}\right)\mathbf{F}$.

34. (a) From Exercise 33, part a, $-y\mathbf{i} + x\mathbf{j}$ is a vector tangent to the circle and pointing in a counterclockwise direction $\Rightarrow y\mathbf{i} - x\mathbf{j}$ is a vector tangent to the circle pointing in a clockwise direction $\Rightarrow \mathbf{G} = \frac{y\mathbf{i} - x\mathbf{j}}{\sqrt{x^2 + y^2}}$ is a unit vector tangent to the circle and pointing in a clockwise direction.

(b) $\mathbf{G} = -\mathbf{F}$

35. The slope of the line through (x, y) and the origin is $\frac{y}{x} \Rightarrow \mathbf{v} = x\mathbf{i} + y\mathbf{j}$ is a vector parallel to that line and pointing away from the origin $\Rightarrow \mathbf{F} = -\frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}$ is the unit vector pointing toward the origin.

36. (a) From Exercise 35, $-\frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}$ is a unit vector through (x, y) pointing toward the origin and we want $|\mathbf{F}|$ to have magnitude $\sqrt{x^2 + y^2} \Rightarrow \mathbf{F} = \sqrt{x^2 + y^2}\left(-\frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}\right) = -x\mathbf{i} - y\mathbf{j}$.

(b) We want $|\mathbf{F}| = \frac{C}{\sqrt{x^2 + y^2}}$ where $C \neq 0$ is a constant $\Rightarrow \mathbf{F} = \frac{C}{\sqrt{x^2 + y^2}}\left(-\frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}\right) = -C\left(\frac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2}\right)$.

37. $\mathbf{F} = -4t^3\mathbf{i} + 8t^2\mathbf{j} + 2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 12t^3 \Rightarrow \text{Flow} = \int_0^2 12t^3\, dt = \left[3t^4\right]_0^2 = 48$

38. $\mathbf{F} = 12t^2\mathbf{j} + 9t^2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = 3\mathbf{j} + 4\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 72t^2 \Rightarrow \text{Flow} = \int_0^1 72t^2\, dt = \left[24t^3\right]_0^1 = 24$

39. $\mathbf{F} = (\cos t - \sin t)\mathbf{i} + (\cos t)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t \cos t + 1$
$\Rightarrow \text{Flow} = \int_0^{\pi} (-\sin t \cos t + 1)\, dt = \left[\frac{1}{2}\cos^2 t + t\right]_0^{\pi} = \left(\frac{1}{2} + \pi\right) - \left(\frac{1}{2} + 0\right) = \pi$

40. $\mathbf{F} = (-2 \sin t)\mathbf{i} - (2 \cos t)\mathbf{j} + 2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -4 \sin^2 t - 4 \cos^2 t + 4 = 0$
$\Rightarrow \text{Flow} = 0$

41. C_1: $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, $0 \le t \le \frac{\pi}{2} \Rightarrow \mathbf{F} = (2 \cos t)\mathbf{i} + 2t\mathbf{j} + (2 \sin t)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}$
$\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -2 \cos t \sin t + 2t \cos t + 2 \sin t = -\sin 2t + 2t \cos t + 2 \sin t$

$\Rightarrow \text{Flow}_1 = \int_0^{\pi/2} (-\sin 2t + 2t\cos t + 2\sin t)\,dt = \left[\frac{1}{2}\cos 2t + 2t\sin t + 2\cos t - 2\cos t\right]_0^{\pi/2} = -1 + \pi;$

C_2: $\mathbf{r} = \mathbf{j} + \frac{\pi}{2}(1-t)\mathbf{k},\ 0 \le t \le 1 \Rightarrow \mathbf{F} = \pi(1-t)\mathbf{j} + 2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = -\frac{\pi}{2}\mathbf{k} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = -\pi$

$\Rightarrow \text{Flow}_2 = \int_0^1 -\pi\,dt = [-\pi t]_0^1 = -\pi;$

C_3: $\mathbf{r} = t\mathbf{i} + (1-t)\mathbf{j},\ 0 \le t \le 1 \Rightarrow \mathbf{F} = 2t\mathbf{i} + 2(1-t)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = 2t$

$\Rightarrow \text{Flow}_3 = \int_0^1 2t\,dt = [t^2]_0^1 = 1 \Rightarrow \text{Circulation} = (-1+\pi) - \pi + 1 = 0$

42. $\mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$, where $f(x,y,z) = \frac{1}{2}(x^2+y^2+x^2) \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(f(\mathbf{r}(t)))$ by the chain rule $\Rightarrow$ Circulation $= \int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt = \int_a^b \frac{d}{dt}(f(\mathbf{r}(t)))\,dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. Since C is an entire ellipse, $\mathbf{r}(b) = \mathbf{r}(a)$, thus the Circulation $= 0$.

43. Let $x = t$ be the parameter $\Rightarrow y = x^2 = t^2$ and $z = x = t \Rightarrow \mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k},\ 0 \le t \le 1$ from $(0,0,0)$ to $(1,1,1)$ $\Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}$ and $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k} = t^3\mathbf{i} + t^2\mathbf{j} - t^3\mathbf{k} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = t^3 + 2t^3 - t^3 = 2t^3 \Rightarrow \text{Flow} = \int_0^1 2t^3\,dt = \frac{1}{2}$

44. (a) $\mathbf{F} = \nabla(xy^2z^3) \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial z}{\partial z}\frac{dz}{dt} = \frac{df}{dt}$, where $f(x,y,z) = xy^2z^3 \Rightarrow \oint_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt = \int_a^b \frac{d}{dt}(f(\mathbf{r}(t)))\,dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = 0$ since C is an entire ellipse.

(b) $\int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = \int_{(1,1,1)}^{(2,1,-1)} \frac{d}{dt}(xy^2z^3)\,dt = [xy^2z^3]_{(1,1,1)}^{(2,1,-1)} = (2)(1)^2(-1)^3 - (1)(1)^2(1)^3 = -2 - 1 = -3$

45. Yes. The work and area have the same numerical value because work $= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C y\mathbf{i}\cdot d\mathbf{r}$

$= \int_b^a [f(t)\mathbf{i}]\cdot\left[\mathbf{i} + \frac{df}{dt}\mathbf{j}\right]dt$ [On the path, y equals f(t)]

$= \int_a^b f(t)\,dt$ = Area under the curve [because $f(t) > 0$]

46. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} = x\mathbf{i} + f(x)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dx} = \mathbf{i} + f'(x)\mathbf{j}$; $\mathbf{F} = \frac{k}{\sqrt{x^2+y^2}}(x\mathbf{i} + y\mathbf{j})$ has constant magnitude k and points away from the origin $\Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dx} = \frac{kx}{\sqrt{x^2+y^2}} + \frac{k\cdot y\cdot f'(x)}{\sqrt{x^2+y^2}} = \frac{kx + k\cdot f(x)\cdot f'(x)}{\sqrt{x^2+[f(x)]^2}} = k\frac{d}{dx}\sqrt{x^2+[f(x)]^2}$, by the chain rule

$\Rightarrow \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dx}\,dx = \int_a^b k\frac{d}{dx}\sqrt{x^2+[f(x)]^2}\,dx = k\left[\sqrt{x^2+[f(x)]^2}\right]_a^b$

$= k\left(\sqrt{b^2+[f(b)]^2} - \sqrt{a^2+[f(a)]^2}\right)$, as claimed.

47-52. Example CAS commands:

Maple:

```
with( LinearAlgebra );#47
F := r -> < r[1]*r[2]^6 | 3*r[1]*(r[1]*r[2]^5+2) >;
r := t -> < 2*cos(t) | sin(t) >;
a,b := 0,2*Pi;
dr := map(diff,r(t),t);                          # (a)
F(r(t));                                          # (b)
q1 := simplify( F(r(t)) . dr ) assuming t::real;  # (c)
q2 := Int( q1, t=a..b );
value( q2 );
```

Mathematica: (functions and bounds will vary):

Exercises 47 and 48 use vectors in 2 dimensions

```
Clear[x, y, t, f, r, v]
f[x_, y_]:= {x y^6, 3x (x y^5 + 2)}
```

```
{a, b}={0, 2π};
x[t_]:= 2 Cos[t]
y[t_]:= Sin[t]
r[t_]:={x[t], y[t]}
v[t_]:= r'[t]
integrand= f[x[t], y[t]] . v[t] //Simplify
Integrate[integrand,{t, a, b}]
N[%]
```

If the integration takes too long or cannot be done, use NIntegrate to integrate numerically. This is suggested for exercises 49 - 52 that use vectors in 3 dimensions. Be certain to leave spaces between variables to be multiplied.

```
Clear[x, y, z, t, f, r, v]
f[x_, y_, z_]:= {y + y z Cos[x y z], x^2 + x z Cos[x y z], z + x y Cos[x y z]}
{a, b}={0, 2π};
x[t_]:= 2 Cos[t]
y[t_]:= 3 Sin[t]
z[t_]:= 1
r[t_]:={x[t], y[t], z[t]}
v[t_]:= r'[t]
integrand= f[x[t], y[t],z[t]] . v[t] //Simplify
NIntegrate[integrand,{t, a, b}]
```

16.3 PATH INDEPENDENCE, POTENTIAL FUNCTIONS, AND CONSERVATIVE FIELDS

1. $\frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = z = \frac{\partial M}{\partial y} \Rightarrow$ Conservative

2. $\frac{\partial P}{\partial y} = x \cos z = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = y \cos z = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = \sin z = \frac{\partial M}{\partial y} \Rightarrow$ Conservative

3. $\frac{\partial P}{\partial y} = -1 \neq 1 = \frac{\partial N}{\partial z} \Rightarrow$ Not Conservative

4. $\frac{\partial N}{\partial x} = 1 \neq -1 = \frac{\partial M}{\partial y} \Rightarrow$ Not Conservative

5. $\frac{\partial N}{\partial x} = 0 \neq 1 = \frac{\partial M}{\partial y} \Rightarrow$ Not Conservative

6. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = -e^x \sin y = \frac{\partial M}{\partial y} \Rightarrow$ Conservative

7. $\frac{\partial f}{\partial x} = 2x \Rightarrow f(x,y,z) = x^2 + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 3y \Rightarrow g(y,z) = \frac{3y^2}{2} + h(z) \Rightarrow f(x,y,z) = x^2 + \frac{3y^2}{2} + h(z)$
$\Rightarrow \frac{\partial f}{\partial z} = h'(z) = 4z \Rightarrow h(z) = 2z^2 + C \Rightarrow f(x,y,z) = x^2 + \frac{3y^2}{2} + 2z^2 + C$

8. $\frac{\partial f}{\partial x} = y + z \Rightarrow f(x,y,z) = (y+z)x + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = x + z \Rightarrow \frac{\partial g}{\partial y} = z \Rightarrow g(y,z) = zy + h(z)$
$\Rightarrow f(x,y,z) = (y+z)x + zy + h(z) \Rightarrow \frac{\partial f}{\partial z} = x + y + h'(z) = x + y \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z)$
$= (y+z)x + zy + C$

9. $\frac{\partial f}{\partial x} = e^{y+2z} \Rightarrow f(x,y,z) = xe^{y+2z} + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = xe^{y+2z} + \frac{\partial g}{\partial y} = xe^{y+2z} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow f(x,y,z)$
$= xe^{y+2z} + h(z) \Rightarrow \frac{\partial f}{\partial z} = 2xe^{y+2z} + h'(z) = 2xe^{y+2z} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z) = xe^{y+2z} + C$

10. $\frac{\partial f}{\partial x} = y \sin z \Rightarrow f(x,y,z) = xy \sin z + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = x \sin z + \frac{\partial g}{\partial y} = x \sin z \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z)$
$\Rightarrow f(x,y,z) = xy \sin z + h(z) \Rightarrow \frac{\partial f}{\partial z} = xy \cos z + h'(z) = xy \cos z \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z)$
$= xy \sin z + C$

11. $\frac{\partial f}{\partial z} = \frac{z}{y^2+z^2} \Rightarrow f(x, y, z) = \frac{1}{2} \ln(y^2 + z^2) + g(x, y) \Rightarrow \frac{\partial f}{\partial x} = \frac{\partial g}{\partial x} = \ln x + \sec^2(x + y) \Rightarrow g(x, y)$
$= (x \ln x - x) + \tan(x + y) + h(y) \Rightarrow f(x, y, z) = \frac{1}{2} \ln(y^2 + z^2) + (x \ln x - x) + \tan(x + y) + h(y)$
$\Rightarrow \frac{\partial f}{\partial y} = \frac{y}{y^2+z^2} + \sec^2(x + y) + h'(y) = \sec^2(x + y) + \frac{y}{y^2+z^2} \Rightarrow h'(y) = 0 \Rightarrow h(y) = C \Rightarrow f(x, y, z)$
$= \frac{1}{2} \ln(y^2 + z^2) + (x \ln x - x) + \tan(x + y) + C$

12. $\frac{\partial f}{\partial x} = \frac{y}{1+x^2y^2} \Rightarrow f(x, y, z) = \tan^{-1}(xy) + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{x}{1+x^2y^2} + \frac{\partial g}{\partial y} = \frac{x}{1+x^2y^2} + \frac{z}{\sqrt{1-y^2z^2}}$
$\Rightarrow \frac{\partial g}{\partial y} = \frac{z}{\sqrt{1-y^2z^2}} \Rightarrow g(y, z) = \sin^{-1}(yz) + h(z) \Rightarrow f(x, y, z) = \tan^{-1}(xy) + \sin^{-1}(yz) + h(z)$
$\Rightarrow \frac{\partial f}{\partial z} = \frac{y}{\sqrt{1-y^2z^2}} + h'(z) = \frac{y}{\sqrt{1-y^2z^2}} + \frac{1}{z} \Rightarrow h'(z) = \frac{1}{z} \Rightarrow h(z) = \ln|z| + C$
$\Rightarrow f(x, y, z) = \tan^{-1}(xy) + \sin^{-1}(yz) + \ln|z| + C$

13. Let $\mathbf{F}(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = 2x \Rightarrow f(x, y, z) = x^2 + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 2y \Rightarrow g(y, z) = y^2 + h(z) \Rightarrow f(x, y, z) = x^2 + y^2 = h(z)$
$\Rightarrow \frac{\partial f}{\partial z} = h'(z) = 2z \Rightarrow h(z) = z^2 + C \Rightarrow f(x, y, z) = x^2 + y^2 + z^2 + C \Rightarrow \int_{(0,0,0)}^{(2,3,-6)} 2x\,dx + 2y\,dy + 2z\,dz$
$= f(2, 3, -6) - f(0, 0, 0) = 2^2 + 3^2 + (-6)^2 = 49$

14. Let $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = z = \frac{\partial M}{\partial y} \Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = yz \Rightarrow f(x, y, z) = xyz + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = xz + \frac{\partial g}{\partial y} = xz \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow f(x, y, z)$
$= xyz + h(z) \Rightarrow \frac{\partial f}{\partial z} = xy + h'(z) = xy \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = xyz + C$
$\Rightarrow \int_{(1,1,2)}^{(3,5,0)} yz\,dx + xz\,dy + xy\,dz = f(3, 5, 0) - f(1, 1, 2) = 0 - 2 = -2$

15. Let $\mathbf{F}(x, y, z) = 2xy\mathbf{i} + (x^2 - z^2)\mathbf{j} - 2yz\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = -2z = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 2x = \frac{\partial M}{\partial y}$
$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = 2xy \Rightarrow f(x, y, z) = x^2y + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = x^2 - z^2 \Rightarrow \frac{\partial g}{\partial y} = -z^2$
$\Rightarrow g(y, z) = -yz^2 + h(z) \Rightarrow f(x, y, z) = x^2y - yz^2 + h(z) \Rightarrow \frac{\partial f}{\partial z} = -2yz + h'(z) = -2yz \Rightarrow h'(z) = 0 \Rightarrow h(z) = C$
$\Rightarrow f(x, y, z) = x^2y - yz^2 + C \Rightarrow \int_{(0,0,0)}^{(1,2,3)} 2xy\,dx + (x^2 - z^2)\,dy - 2yz\,dz = f(1, 2, 3) - f(0, 0, 0) = 2 - 2(3)^2 = -16$

16. Let $\mathbf{F}(x, y, z) = 2x\mathbf{i} - y^2\mathbf{j} - \left(\frac{4}{1+z^2}\right)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y}$
$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = 2x \Rightarrow f(x, y, z) = x^2 + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = -y^2 \Rightarrow g(y, z) = -\frac{y^3}{3} + h(z)$
$\Rightarrow f(x, y, z) = x^2 - \frac{y^3}{3} + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = -\frac{4}{1+z^2} \Rightarrow h(z) = -4\tan^{-1} z + C \Rightarrow f(x, y, z)$
$= x^2 - \frac{y^3}{3} - 4\tan^{-1} z + C \Rightarrow \int_{(0,0,0)}^{(3,3,1)} 2x\,dx - y^2\,dy - \frac{4}{1-z^2}\,dz = f(3, 3, 1) - f(0, 0, 0)$
$= \left(9 - \frac{27}{3} - 4 \cdot \frac{\pi}{4}\right) - (0 - 0 - 0) = -\pi$

17. Let $\mathbf{F}(x, y, z) = (\sin y \cos x)\mathbf{i} + (\cos y \sin x)\mathbf{j} + \mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \cos y \cos x = \frac{\partial M}{\partial y}$
$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = \sin y \cos x \Rightarrow f(x, y, z) = \sin y \sin x + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \cos y \sin x + \frac{\partial g}{\partial y}$
$= \cos y \sin x \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow f(x, y, z) = \sin y \sin x + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = 1 \Rightarrow h(z) = z + C$
$\Rightarrow f(x, y, z) = \sin y \sin x + z + C \Rightarrow \int_{(1,0,0)}^{(0,1,1)} \sin y \cos x\,dx + \cos y \sin x\,dy + dz = f(0, 1, 1) - f(1, 0, 0)$
$= (0 + 1) - (0 + 0) = 1$

18. Let $\mathbf{F}(x, y, z) = (2\cos y)\mathbf{i} + \left(\frac{1}{y} - 2x \sin y\right)\mathbf{j} + \left(\frac{1}{z}\right)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -2\sin y = \frac{\partial M}{\partial y}$
$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = 2\cos y \Rightarrow f(x, y, z) = 2x \cos y + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = -2x \sin y + \frac{\partial g}{\partial y}$
$= \frac{1}{y} - 2x \sin y \Rightarrow \frac{\partial g}{\partial y} = \frac{1}{y} \Rightarrow g(y, z) = \ln|y| + h(z) \Rightarrow f(x, y, z) = 2x \cos y + \ln|y| + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = \frac{1}{z}$

$\Rightarrow h(z) = \ln|z| + C \Rightarrow f(x,y,z) = 2x\cos y + \ln|y| + \ln|z| + C$

$\Rightarrow \int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y\,dx + \left(\frac{1}{y} - 2x\sin y\right)dy + \frac{1}{z}\,dz = f\left(1, \frac{\pi}{2}, 2\right) - f(0,2,1)$

$= \left(2\cdot 0 + \ln\frac{\pi}{2} + \ln 2\right) - (0\cdot\cos 2 + \ln 2 + \ln 1) = \ln\frac{\pi}{2}$

19. Let $\mathbf{F}(x,y,z) = 3x^2\mathbf{i} + \left(\frac{z^2}{y}\right)\mathbf{j} + (2z\ln y)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = \frac{2z}{y} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y}$

$\Rightarrow$ M dx + N dy + P dz is exact; $\frac{\partial f}{\partial x} = 3x^2 \Rightarrow f(x,y,z) = x^3 + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = \frac{z^2}{y} \Rightarrow g(y,z) = z^2\ln y + h(z)$

$\Rightarrow f(x,y,z) = x^3 + z^2\ln y + h(z) \Rightarrow \frac{\partial f}{\partial z} = 2z\ln y + h'(z) = 2z\ln y \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z)$

$= x^3 + z^2\ln y + C \Rightarrow \int_{(1,1,1)}^{(1,2,3)} 3x^2\,dx + \frac{z^2}{y}\,dy + 2z\ln y\,dz = f(1,2,3) - f(1,1,1)$

$= (1 + 9\ln 2 + C) - (1 + 0 + C) = 9\ln 2$

20. Let $\mathbf{F}(x,y,z) = (2x\ln y - yz)\mathbf{i} + \left(\frac{x^2}{y} - xz\right)\mathbf{j} - (xy)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = -x = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -y = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{2x}{y} - z = \frac{\partial M}{\partial y}$

$\Rightarrow$ M dx + N dy + P dz is exact; $\frac{\partial f}{\partial x} = 2x\ln y - yz \Rightarrow f(x,y,z) = x^2\ln y - xyz + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{x^2}{y} - xz + \frac{\partial g}{\partial y}$

$= \frac{x^2}{y} - xz \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z) \Rightarrow f(x,y,z) = x^2\ln y - xyz + h(z) \Rightarrow \frac{\partial f}{\partial z} = -xy + h'(z) = -xy \Rightarrow h'(z) = 0$

$\Rightarrow h(z) = C \Rightarrow f(x,y,z) = x^2\ln y - xyz + C \Rightarrow \int_{(1,2,1)}^{(2,1,1)} (2x\ln y - yz)\,dx + \left(\frac{x^2}{y} - xz\right)dy - xy\,dz$

$= f(2,1,1) - f(1,2,1) = (4\ln 1 - 2 + C) - (\ln 2 - 2 + C) = -\ln 2$

21. Let $\mathbf{F}(x,y,z) = \left(\frac{1}{y}\right)\mathbf{i} + \left(\frac{1}{z} - \frac{x}{y^2}\right)\mathbf{j} - \left(\frac{y}{z^2}\right)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = -\frac{1}{z^2} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{1}{y^2} = \frac{\partial M}{\partial y}$

$\Rightarrow$ M dx + N dy + P dz is exact; $\frac{\partial f}{\partial x} = \frac{1}{y} \Rightarrow f(x,y,z) = \frac{x}{y} + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = -\frac{x}{y^2} + \frac{\partial g}{\partial y} = \frac{1}{z} - \frac{x}{y^2}$

$\Rightarrow \frac{\partial g}{\partial y} = \frac{1}{z} \Rightarrow g(y,z) = \frac{y}{z} + h(z) \Rightarrow f(x,y,z) = \frac{x}{y} + \frac{y}{z} + h(z) \Rightarrow \frac{\partial f}{\partial z} = -\frac{y}{z^2} + h'(z) = -\frac{y}{z^2} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C$

$\Rightarrow f(x,y,z) = \frac{x}{y} + \frac{y}{z} + C \Rightarrow \int_{(1,1,1)}^{(2,2,2)} \frac{1}{y}\,dx + \left(\frac{1}{z} - \frac{x}{y^2}\right)dy - \frac{y}{z^2}\,dz = f(2,2,2) - f(1,1,1) = \left(\frac{2}{2} + \frac{2}{2} + C\right) - \left(\frac{1}{1} + \frac{1}{1} + C\right)$

$= 0$

22. Let $\mathbf{F}(x,y,z) = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{x^2 + y^2 + z^2}$ $\left(\text{and let } \rho^2 = x^2 + y^2 + z^2 \Rightarrow \frac{\partial\rho}{\partial x} = \frac{x}{\rho}, \frac{\partial\rho}{\partial y} = \frac{y}{\rho}, \frac{\partial\rho}{\partial z} = \frac{z}{\rho}\right)$

$\Rightarrow \frac{\partial P}{\partial y} = -\frac{4yz}{\rho^4} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -\frac{4xz}{\rho^4} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{4xy}{\rho^4} = \frac{\partial M}{\partial y} \Rightarrow$ M dx + N dy + P dz is exact;

$\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2 + z^2} \Rightarrow f(x,y,z) = \ln(x^2 + y^2 + z^2) + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2 + z^2} + \frac{\partial g}{\partial y} = \frac{2y}{x^2 + y^2 + z^2}$

$\Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z) \Rightarrow f(x,y,z) = \ln(x^2 + y^2 + z^2) + h(z) \Rightarrow \frac{\partial f}{\partial z} = \frac{2z}{x^2 + y^2 + z^2} + h'(z)$

$= \frac{2z}{x^2 + y^2 + z^2} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z) = \ln(x^2 + y^2 + z^2) + C$

$\Rightarrow \int_{(-1,-1,-1)}^{(2,2,2)} \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2 + y^2 + z^2} = f(2,2,2) - f(-1,-1,-1) = \ln 12 - \ln 3 = \ln 4$

23. $\mathbf{r} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) + t(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = (1+t)\mathbf{i} + (1+2t)\mathbf{j} + (1-2t)\mathbf{k}, 0 \le t \le 1 \Rightarrow dx = dt, dy = 2\,dt, dz = -2\,dt$

$\Rightarrow \int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz = \int_0^1 (2t+1)\,dt + (t+1)(2\,dt) + 4(-2)\,dt = \int_0^1 (4t-5)\,dt = \left[2t^2 - 5t\right]_0^1 = -3$

24. $\mathbf{r} = t(3\mathbf{j} + 4\mathbf{k}), 0 \le t \le 1 \Rightarrow dx = 0, dy = 3\,dt, dz = 4\,dt \Rightarrow \int_{(0,0,0)}^{(0,3,4)} x^2\,dx + yz\,dy + \left(\frac{y^2}{2}\right)dz$

$= \int_0^1 (12t^2)(3\,dt) + \left(\frac{9t^2}{2}\right)(4\,dt) = \int_0^1 54t^2\,dt = \left[18t^2\right]_0^1 = 18$

25. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 2z = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow$ M dx + N dy + P dz is exact $\Rightarrow$ **F** is conservative

$\Rightarrow$ path independence

26. $\frac{\partial P}{\partial y} = -\frac{yz}{\left(\sqrt{x^2 + y^2 + z^2}\right)^3} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -\frac{xz}{\left(\sqrt{x^2 + y^2 + z^2}\right)^3} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{xy}{\left(\sqrt{x^2 + y^2 + z^2}\right)^3} = \frac{\partial M}{\partial y}$

$\Rightarrow$ M dx + N dy + P dz is exact $\Rightarrow$ **F** is conservative $\Rightarrow$ path independence

27. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{2x}{y^2} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative $\Rightarrow$ there exists an f so that $\mathbf{F} = \nabla f$;
$\frac{\partial f}{\partial x} = \frac{2x}{y} \Rightarrow f(x, y) = \frac{x^2}{y} + g(y) \Rightarrow \frac{\partial f}{\partial y} = -\frac{x^2}{y^2} + g'(y) = \frac{1 - x^2}{y^2} \Rightarrow g'(y) = \frac{1}{y^2} \Rightarrow g(y) = -\frac{1}{y} + C$
$\Rightarrow f(x, y) = \frac{x^2}{y} - \frac{1}{y} + C \Rightarrow \mathbf{F} = \nabla\left(\frac{x^2 - 1}{y}\right)$

28. $\frac{\partial P}{\partial y} = \cos z = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{e^x}{y} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative $\Rightarrow$ there exists an f so that $\mathbf{F} = \nabla f$;
$\frac{\partial f}{\partial x} = e^x \ln y \Rightarrow f(x, y, z) = e^x \ln y + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{e^x}{y} + \frac{\partial g}{\partial y} = \frac{e^x}{y} + \sin z \Rightarrow \frac{\partial g}{\partial y} = \sin z \Rightarrow g(y, z)$
$= y \sin z + h(z) \Rightarrow f(x, y, z) = e^x \ln y + y \sin z + h(z) \Rightarrow \frac{\partial f}{\partial z} = y \cos z + h'(z) = y \cos z \Rightarrow h'(z) = 0$
$\Rightarrow h(z) = C \Rightarrow f(x, y, z) = e^x \ln y + y \sin z + C \Rightarrow \mathbf{F} = \nabla(e^x \ln y + y \sin z)$

29. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative $\Rightarrow$ there exists an f so that $\mathbf{F} = \nabla f$;
$\frac{\partial f}{\partial x} = x^2 + y \Rightarrow f(x, y, z) = \frac{1}{3}x^3 + xy + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = y^2 + x \Rightarrow \frac{\partial g}{\partial y} = y^2 \Rightarrow g(y, z) = \frac{1}{3}y^3 + h(z)$
$\Rightarrow f(x, y, z) = \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = ze^z \Rightarrow h(z) = ze^z - e^z + C \Rightarrow f(x, y, z)$
$= \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z + C \Rightarrow \mathbf{F} = \nabla\left(\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right)$

(a) work $= \int_A^B \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} = \left(\frac{1}{3} + 0 + 0 + e - e\right) - \left(\frac{1}{3} + 0 + 0 - 1\right)$
$= 1$

(b) work $= \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} = 1$

(c) work $= \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} = 1$

Note: Since **F** is conservative, $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path from $(1, 0, 0)$ to $(1, 0, 1)$.

30. $\frac{\partial P}{\partial y} = xe^{yz} + xyze^{yz} + \cos y = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = ye^{yz} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = ze^{yz} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative $\Rightarrow$ there exists an f so that $\mathbf{F} = \nabla f$; $\frac{\partial f}{\partial x} = e^{yz} \Rightarrow f(x, y, z) = xe^{yz} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = xze^{yz} + \frac{\partial g}{\partial y} = xze^{yz} + z \cos y \Rightarrow \frac{\partial g}{\partial y} = z \cos y$
$\Rightarrow g(y, z) = z \sin y + h(z) \Rightarrow f(x, y, z) = xe^{yz} + z \sin y + h(z) \Rightarrow \frac{\partial f}{\partial z} = xye^{yz} + \sin y + h'(z) = xye^{yz} + \sin y$
$\Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = xe^{yz} + z \sin y + C \Rightarrow \mathbf{F} = \nabla(xe^{yz} + z \sin y)$

(a) work $= \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[xe^{yz} + z \sin y\right]_{(1,0,1)}^{(1,\pi/2,0)} = (1 + 0) - (1 + 0) = 0$

(b) work $= \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[xe^{yz} + z \sin y\right]_{(1,0,1)}^{(1,\pi/2,0)} = 0$

(c) work $= \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[xe^{yz} + z \sin y\right]_{(1,0,1)}^{(1,\pi/2,0)} = 0$

Note: Since **F** is conservative, $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path from $(1, 0, 1)$ to $\left(1, \frac{\pi}{2}, 0\right)$.

31. (a) $\mathbf{F} = \nabla(x^3y^2) \Rightarrow \mathbf{F} = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j}$; let C_1 be the path from $(-1, 1)$ to $(0, 0) \Rightarrow x = t - 1$ and $y = -t + 1, 0 \le t \le 1 \Rightarrow \mathbf{F} = 3(t-1)^2(-t+1)^2\mathbf{i} + 2(t-1)^3(-t+1)\mathbf{j} = 3(t-1)^4\mathbf{i} - 2(t-1)^4\mathbf{j}$ and $\mathbf{r}_1 = (t-1)\mathbf{i} + (-t+1)\mathbf{j} \Rightarrow d\mathbf{r}_1 = dt\,\mathbf{i} - dt\,\mathbf{j} \Rightarrow \int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 = \int_0^1 \left[3(t-1)^4 + 2(t-1)^4\right] dt$
$= \int_0^1 5(t-1)^4\, dt = \left[(t-1)^5\right]_0^1 = 1$; let C_2 be the path from $(0, 0)$ to $(1, 1) \Rightarrow x = t$ and $y = t$,
$0 \le t \le 1 \Rightarrow \mathbf{F} = 3t^4\mathbf{i} + 2t^4\mathbf{j}$ and $\mathbf{r}_2 = t\mathbf{i} + t\mathbf{j} \Rightarrow d\mathbf{r}_2 = dt\,\mathbf{i} + dt\,\mathbf{j} \Rightarrow \int_{C_2} \mathbf{F} \cdot d\mathbf{r}_2 = \int_0^1 (3t^4 + 2t^4)\, dt$
$= \int_0^1 5t^4\, dt = 1 \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}_2 = 2$

(b) Since $f(x, y) = x^3y^2$ is a potential function for **F**, $\int_{(-1,1)}^{(1,1)} \mathbf{F} \cdot d\mathbf{r} = f(1, 1) - f(-1, 1) = 2$

32. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = -2x \sin y = \frac{\partial M}{\partial y} \Rightarrow$ **F** is conservative $\Rightarrow$ there exists an f so that $\mathbf{F} = \nabla f$; $\frac{\partial f}{\partial x} = 2x \cos y \Rightarrow f(x, y, z) = x^2 \cos y + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = -x^2 \sin y + \frac{\partial g}{\partial y} = -x^2 \sin y \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z)$ $\Rightarrow f(x, y, z) = x^2 \cos y + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = x^2 \cos y + C \Rightarrow \mathbf{F} = \nabla (x^2 \cos y)$

(a) $\int_C 2x \cos y \, dx - x^2 \sin y \, dy = \left[x^2 \cos y\right]_{(1,0)}^{(0,1)} = 0 - 1 = -1$

(b) $\int_C 2x \cos y \, dx - x^2 \sin y \, dy = \left[x^2 \cos y\right]_{(-1,\pi)}^{(1,0)} = 1 - (-1) = 2$

(c) $\int_C 2x \cos y \, dx - x^2 \sin y \, dy = \left[x^2 \cos y\right]_{(-1,0)}^{(1,0)} = 1 - 1 = 0$

(d) $\int_C 2x \cos y \, dx - x^2 \sin y \, dy = \left[x^2 \cos y\right]_{(1,0)}^{(1,0)} = 1 - 1 = 0$

33. (a) If the differential form is exact, then $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \Rightarrow 2ay = cy$ for all y $\Rightarrow 2a = c$, $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \Rightarrow 2cx = 2cx$ for all x, and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \Rightarrow by = 2ay$ for all y $\Rightarrow b = 2a$ and $c = 2a$

(b) $\mathbf{F} = \nabla f \Rightarrow$ the differential form with $a = 1$ in part (a) is exact $\Rightarrow b = 2$ and $c = 2$

34. $\mathbf{F} = \nabla f \Rightarrow g(x, y, z) = \int_{(0,0,0)}^{(x,y,z)} \mathbf{F} \cdot d\mathbf{r} = \int_{(0,0,0)}^{(x,y,z)} \nabla f \cdot d\mathbf{r} = f(x, y, z) - f(0, 0, 0) \Rightarrow \frac{\partial g}{\partial x} = \frac{\partial f}{\partial x} - 0$, $\frac{\partial g}{\partial y} = \frac{\partial f}{\partial y} - 0$, and $\frac{\partial g}{\partial z} = \frac{\partial f}{\partial z} - 0 \Rightarrow \nabla g = \nabla f = \mathbf{F}$, as claimed

35. The path will not matter; the work along any path will be the same because the field is conservative.

36. The field is not conservative, for otherwise the work would be the same along C_1 and C_2.

37. Let the coordinates of points A and B be (x_A, y_A, z_A) and (x_B, y_B, z_B), respectively. The force $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is conservative because all the partial derivatives of M, N, and P are zero. Therefore, the potential function is $f(x, y, z) = ax + by + cz + C$, and the work done by the force in moving a particle along any path from A to B is $f(B) - f(A) = f(x_B, y_B, z_B) - f(x_A, y_A, z_A) = (ax_B + by_B + cz_B + C) - (ax_A + by_A + cz_A + C)$ $= a(x_B - x_A) + b(y_B - y_A) + c(z_B - z_A) = \mathbf{F} \cdot \overrightarrow{BA}$

38. (a) Let $-GmM = C \Rightarrow \mathbf{F} = C\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\,\mathbf{i} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\,\mathbf{j} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\,\mathbf{k}\right]$ $\Rightarrow \frac{\partial P}{\partial y} = \frac{-3yzC}{(x^2+y^2+z^2)^{5/2}} = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = \frac{-3xzC}{(x^2+y^2+z^2)^{5/2}} = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = \frac{-3xyC}{(x^2+y^2+z^2)^{5/2}} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F} = \nabla f$ for some f; $\frac{\partial f}{\partial x} = \frac{xC}{(x^2+y^2+z^2)^{3/2}} \Rightarrow f(x, y, z) = -\frac{C}{(x^2+y^2+z^2)^{1/2}} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{yC}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial g}{\partial y}$ $= \frac{yC}{(x^2+y^2+z^2)^{3/2}} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow \frac{\partial f}{\partial z} = \frac{zC}{(x^2+y^2+z^2)^{3/2}} + h'(z) = \frac{zC}{(x^2+y^2+z^2)^{3/2}}$ $\Rightarrow h(z) = C_1 \Rightarrow f(x, y, z) = -\frac{C}{(x^2+y^2+z^2)^{1/2}} + C_1$. Let $C_1 = 0 \Rightarrow f(x, y, z) = \frac{GmM}{(x^2+y^2+z^2)^{1/2}}$ is a potential function for **F**.

(b) If s is the distance of (x, y, z) from the origin, then $s = \sqrt{x^2 + y^2 + z^2}$. The work done by the gravitational field **F** is work $= \int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r} = \left[\frac{GmM}{\sqrt{x^2+y^2+z^2}}\right]_{P_1}^{P_2} = \frac{GmM}{s_2} - \frac{GmM}{s_1} = GmM\left(\frac{1}{s_2} - \frac{1}{s_1}\right)$, as claimed.

16.4 GREEN'S THEOREM IN THE PLANE

1. $M = -y = -a \sin t$, $N = x = a \cos t$, $dx = -a \sin t \, dt$, $dy = a \cos t \, dt \Rightarrow \frac{\partial M}{\partial x} = 0$, $\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = 1$, and $\frac{\partial N}{\partial y} = 0$;

Equation (11): $\oint_C M\,dy - N\,dx = \int_0^{2\pi} [(-a \sin t)(a \cos t) - (a \cos t)(-a \sin t)]\,dt = \int_0^{2\pi} 0\,dt = 0$; $\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R 0\,dx\,dy = 0$, Flux

Equation (12): $\oint_C M\,dx + N\,dy = \int_0^{2\pi} [(-a \sin t)(-a \sin t) - (a \cos t)(a \cos t)]\,dt = \int_0^{2\pi} a^2\,dt = 2\pi a^2$;
$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_{-a}^{a}\int_{-c}^{\sqrt{a^2-x^2}} 2\,dy\,dx = \int_{-a}^{a} 4\sqrt{a^2-x^2}\,dx = 4\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{-a}^{a}$
$= 2a^2\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 2a^2\pi$, Circulation

2. $M = y = a \sin t$, $N = 0$, $dx = -a \sin t\,dt$, $dy = a \cos t\,dt \Rightarrow \frac{\partial M}{\partial x} = 0$, $\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = 0$, and $\frac{\partial N}{\partial y} = 0$;
Equation (11): $\oint_C M\,dy - N\,dx = \int_0^{2\pi} a^2 \sin t \cos t\,dt = a^2\left[\frac{1}{2}\sin^2 t\right]_0^{2\pi} = 0$; $\iint_R 0\,dx\,dy = 0$, Flux
Equation (12): $\oint_C M\,dx + N\,dy = \int_0^{2\pi}(-a^2 \sin^2 t)\,dt = -a^2\left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = -\pi a^2$; $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$
$= \iint_R -1\,dx\,dy = \int_0^{2\pi}\int_0^{a} -r\,dr\,d\theta = \int_0^{2\pi} -\frac{a^2}{2}\,d\theta = -\pi a^2$, Circulation

3. $M = 2x = 2a \cos t$, $N = -3y = -3a \sin t$, $dx = -a \sin t\,dt$, $dy = a \cos t\,dt \Rightarrow \frac{\partial M}{\partial x} = 2$, $\frac{\partial M}{\partial y} = 0$, $\frac{\partial N}{\partial x} = 0$, and $\frac{\partial N}{\partial y} = -3$;
Equation (11): $\oint_C M\,dy - N\,dx = \int_0^{2\pi} [(2a \cos t)(a \cos t) + (3a \sin t)(-a \sin t)]\,dt$
$= \int_0^{2\pi}(2a^2 \cos^2 t - 3a^2 \sin^2 t)\,dt = 2a^2\left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} - 3a^2\left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = 2\pi a^2 - 3\pi a^2 = -\pi a^2$;
$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) = \iint_R -1\,dx\,dy = \int_0^{2\pi}\int_0^{a} -r\,dr\,d\theta = \int_0^{2\pi} -\frac{a^2}{2}\,d\theta = -\pi a^2$, Flux
Equation (12): $\oint_C M\,dx + N\,dy = \int_0^{2\pi} [(2a \cos t)(-a \sin t) + (-3a \sin t)(a \cos t)]\,dt$
$= \int_0^{2\pi}(-2a^2 \sin t \cos t - 3a^2 \sin t \cos t)\,dt = -5a^2\left[\frac{1}{2}\sin^2 t\right]_0^{2\pi} = 0$; $\iint_R 0\,dx\,dy = 0$, Circulation

4. $M = -x^2 y = -a^3 \cos^2 t$, $N = xy^2 = a^3 \cos t \sin^2 t$, $dx = -a \sin t\,dt$, $dy = a \cos t\,dt$
$\Rightarrow \frac{\partial M}{\partial x} = -2xy$, $\frac{\partial M}{\partial y} = -x^2$, $\frac{\partial N}{\partial x} = y^2$, and $\frac{\partial N}{\partial y} = 2xy$;
Equation (11): $\oint_C M\,dy - N\,dx = \int_0^{2\pi}(-a^4 \cos^3 t \sin t + a^4 \cos t \sin^3 t) = \left[\frac{a^4}{4}\cos^4 t + \frac{a^4}{4}\sin^4 t\right]_0^{2\pi} = 0$;
$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R (-2xy + 2xy)\,dx\,dy = 0$, Flux
Equation (12): $\oint_C M\,dx + N\,dy = \int_0^{2\pi}(a^4 \cos^2 t \sin^2 t + a^4 \cos^2 t \sin^2 t)\,dt = \int_0^{2\pi}(2a^4 \cos^2 t \sin^2 t)\,dt$
$= \int_0^{2\pi} \frac{1}{2} a^4 \sin^2 2t\,dt = \frac{a^4}{4}\int_0^{4\pi} \sin^2 u\,du = \frac{a^4}{4}\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{4\pi} = \frac{\pi a^4}{2}$; $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (y^2 + x^2)\,dx\,dy$
$= \int_0^{2\pi}\int_0^{a} r^2 \cdot r\,dr\,d\theta = \int_0^{2\pi} \frac{a^4}{4}\,d\theta = \frac{\pi a^4}{2}$, Circulation

5. $M = x - y$, $N = y - x \Rightarrow \frac{\partial M}{\partial x} = 1$, $\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = -1$, $\frac{\partial N}{\partial y} = 1 \Rightarrow$ Flux $= \iint_R 2\,dx\,dy = \int_0^1\int_0^1 2\,dx\,dy = 2$;
Circ $= \iint_R [-1 - (-1)]\,dx\,dy = 0$

6. $M = x^2 + 4y$, $N = x + y^2 \Rightarrow \frac{\partial M}{\partial x} = 2x$, $\frac{\partial M}{\partial y} = 4$, $\frac{\partial N}{\partial x} = 1$, $\frac{\partial N}{\partial y} = 2y \Rightarrow$ Flux $= \iint_R (2x + 2y)\,dx\,dy$
$= \int_0^1\int_0^1 (2x + 2y)\,dx\,dy = \int_0^1 [x^2 + 2xy]_0^1\,dy = \int_0^1 (1 + 2y)\,dy = [y + y^2]_0^1 = 2$; Circ $= \iint_R (1 - 4)\,dx\,dy$
$= \int_0^1\int_0^1 -3\,dx\,dy = -3$

7. $M = y^2 - x^2, N = x^2 + y^2 \Rightarrow \frac{\partial M}{\partial x} = -2x, \frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = 2x, \frac{\partial N}{\partial y} = 2y \Rightarrow \text{Flux} = \iint_R (-2x + 2y)\,dx\,dy$
$= \int_0^3 \int_0^x (-2x + 2y)\,dy\,dx = \int_0^3 (-2x^2 + x^2)\,dx = \left[-\frac{1}{3}x^3\right]_0^3 = -9$; $\text{Circ} = \iint_R (2x - 2y)\,dx\,dy$
$= \int_0^3 \int_0^x (2x - 2y)\,dy\,dx = \int_0^3 x^2\,dx = 9$

8. $M = x + y, N = -(x^2 + y^2) \Rightarrow \frac{\partial M}{\partial x} = 1, \frac{\partial M}{\partial y} = 1, \frac{\partial N}{\partial x} = -2x, \frac{\partial N}{\partial y} = -2y \Rightarrow \text{Flux} = \iint_R (1 - 2y)\,dx\,dy$
$= \int_0^1 \int_0^x (1 - 2y)\,dy\,dx = \int_0^1 (x - x^2)\,dx = \frac{1}{6}$; $\text{Circ} = \iint_R (-2x - 1)\,dx\,dy = \int_0^1 \int_0^x (-2x - 1)\,dy\,dx$
$= \int_0^1 (-2x^2 - x)\,dx = -\frac{7}{6}$

9. $M = x + e^x \sin y, N = x + e^x \cos y \Rightarrow \frac{\partial M}{\partial x} = 1 + e^x \sin y, \frac{\partial M}{\partial y} = e^x \cos y, \frac{\partial N}{\partial x} = 1 + e^x \cos y, \frac{\partial N}{\partial y} = -e^x \sin y$
$\Rightarrow \text{Flux} = \iint_R dx\,dy = \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} r\,dr\,d\theta = \int_{-\pi/4}^{\pi/4} \left(\frac{1}{2}\cos 2\theta\right) d\theta = \left[\frac{1}{4}\sin 2\theta\right]_{-\pi/4}^{\pi/4} = \frac{1}{2}$;
$\text{Circ} = \iint_R (1 + e^x \cos y - e^x \cos y)\,dx\,dy = \iint_R dx\,dy = \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} r\,dr\,d\theta = \int_{-\pi/4}^{\pi/4} \left(\frac{1}{2}\cos 2\theta\right) d\theta = \frac{1}{2}$

10. $M = \tan^{-1}\frac{y}{x}, N = \ln(x^2 + y^2) \Rightarrow \frac{\partial M}{\partial x} = \frac{-y}{x^2 + y^2}, \frac{\partial M}{\partial y} = \frac{x}{x^2 + y^2}, \frac{\partial N}{\partial x} = \frac{2x}{x^2 + y^2}, \frac{\partial N}{\partial y} = \frac{2y}{x^2 + y^2}$
$\Rightarrow \text{Flux} = \iint_R \left(\frac{-y}{x^2 + y^2} + \frac{2y}{x^2 + y^2}\right) dx\,dy = \int_0^\pi \int_1^2 \left(\frac{r \sin\theta}{r^2}\right) r\,dr\,d\theta = \int_0^\pi \sin\theta\,d\theta = 2$;
$\text{Circ} = \iint_R \left(\frac{2x}{x^2 + y^2} - \frac{x}{x^2 + y^2}\right) dx\,dy = \int_0^\pi \int_1^2 \left(\frac{r\cos\theta}{r^2}\right) r\,dr\,d\theta = \int_0^\pi \cos\theta\,d\theta = 0$

11. $M = xy, N = y^2 \Rightarrow \frac{\partial M}{\partial x} = y, \frac{\partial M}{\partial y} = x, \frac{\partial N}{\partial x} = 0, \frac{\partial N}{\partial y} = 2y \Rightarrow \text{Flux} = \iint_R (y + 2y)\,dy\,dx = \int_0^1 \int_{x^2}^x 3y\,dy\,dx$
$= \int_0^1 \left(\frac{3x^2}{2} - \frac{3x^4}{2}\right) dx = \frac{1}{5}$; $\text{Circ} = \iint_R -x\,dy\,dx = \int_0^1 \int_{x^2}^x -x\,dy\,dx = \int_0^1 (-x^2 + x^3)\,dx = -\frac{1}{12}$

12. $M = -\sin y, N = x\cos y \Rightarrow \frac{\partial M}{\partial x} = 0, \frac{\partial M}{\partial y} = -\cos y, \frac{\partial N}{\partial x} = \cos y, \frac{\partial N}{\partial y} = -x\sin y$
$\Rightarrow \text{Flux} = \iint_R (-x\sin y)\,dx\,dy = \int_0^{\pi/2} \int_0^{\pi/2} (-x\sin y)\,dx\,dy = \int_0^{\pi/2} \left(-\frac{\pi^2}{8}\sin y\right) dy = -\frac{\pi^2}{8}$;
$\text{Circ} = \iint_R [\cos y - (-\cos y)]\,dx\,dy = \int_0^{\pi/2} \int_0^{\pi/2} 2\cos y\,dx\,dy = \int_0^{\pi/2} \pi\cos y\,dy = [\pi \sin y]_0^{\pi/2} = \pi$

13. $M = 3xy - \frac{x}{1 + y^2}, N = e^x + \tan^{-1} y \Rightarrow \frac{\partial M}{\partial x} = 3y - \frac{1}{1 + y^2}, \frac{\partial N}{\partial y} = \frac{1}{1 + y^2}$
$\Rightarrow \text{Flux} = \iint_R \left(3y - \frac{1}{1 + y^2} + \frac{1}{1 + y^2}\right) dx\,dy = \iint_R 3y\,dx\,dy = \int_0^{2\pi} \int_0^{a(1 + \cos\theta)} (3r\sin\theta)\,r\,dr\,d\theta$
$= \int_0^{2\pi} a^3(1 + \cos\theta)^3(\sin\theta)\,d\theta = \left[-\frac{a^3}{4}(1 + \cos\theta)^4\right]_0^{2\pi} = -4a^3 - (-4a^3) = 0$

14. $M = y + e^x \ln y, N = \frac{e^x}{y} \Rightarrow \frac{\partial M}{\partial y} = 1 + \frac{e^x}{y}, \frac{\partial N}{\partial x} = \frac{e^x}{y} \Rightarrow \text{Circ} = \iint_R \left[\frac{e^x}{y} - \left(1 + \frac{e^x}{y}\right)\right] dx\,dy = \iint_R (-1)\,dx\,dy$
$= \int_{-1}^1 \int_{x^4 + 1}^{3 - x^2} -dy\,dx = -\int_{-1}^1 [(3 - x^2) - (x^4 + 1)]\,dx = \int_{-1}^1 (x^4 + x^2 - 2)\,dx = -\frac{44}{15}$

15. $M = 2xy^3, N = 4x^2y^2 \Rightarrow \frac{\partial M}{\partial y} = 6xy^2, \frac{\partial N}{\partial x} = 8xy^2 \Rightarrow \text{work} = \oint_C 2xy^3\,dx + 4x^2y^2\,dy = \iint_R (8xy^2 - 6xy^2)\,dx\,dy$
$= \int_0^1 \int_0^{x^3} 2xy^2\,dy\,dx = \int_0^1 \frac{2}{3}x^{10}\,dx = \frac{2}{33}$

16. $M = 4x - 2y,\ N = 2x - 4y \Rightarrow \frac{\partial M}{\partial y} = -2,\ \frac{\partial N}{\partial x} = 2 \Rightarrow$ work $= \oint_C (4x - 2y)\,dx + (2x - 4y)\,dy$
$= \iint_R [2 - (-2)]\,dx\,dy = 4 \iint_R dx\,dy = 4(\text{Area of the circle}) = 4(\pi \cdot 4) = 16\pi$

17. $M = y^2,\ N = x^2 \Rightarrow \frac{\partial M}{\partial y} = 2y,\ \frac{\partial N}{\partial x} = 2x \Rightarrow \oint_C y^2\,dx + x^2\,dy = \iint_R (2x - 2y)\,dy\,dx$
$= \int_0^1 \int_0^{1-x} (2x - 2y)\,dy\,dx = \int_0^1 (-3x^2 + 4x - 1)\,dx = \left[-x^3 + 2x^2 - x\right]_0^1 = -1 + 2 - 1 = 0$

18. $M = 3y,\ N = 2x \Rightarrow \frac{\partial M}{\partial y} = 3,\ \frac{\partial N}{\partial x} = 2 \Rightarrow \oint_C 3y\,dx + 2x\,dy = \iint_R (2 - 3)\,dx\,dy = \int_0^{\pi} \int_0^{\sin x} -1\,dy\,dx$
$= -\int_0^{\pi} \sin x\,dx = -2$

19. $M = 6y + x,\ N = y + 2x \Rightarrow \frac{\partial M}{\partial y} = 6,\ \frac{\partial N}{\partial x} = 2 \Rightarrow \oint_C (6y + x)\,dx + (y + 2x)\,dy = \iint_R (2 - 6)\,dy\,dx$
$= -4(\text{Area of the circle}) = -16\pi$

20. $M = 2x + y^2,\ N = 2xy + 3y \Rightarrow \frac{\partial M}{\partial y} = 2y,\ \frac{\partial N}{\partial x} = 2y \Rightarrow \oint_C (2x + y^2)\,dx + (2xy + 3y)\,dy = \iint_R (2y - 2y)\,dx\,dy = 0$

21. $M = x = a\cos t,\ N = y = a\sin t \Rightarrow dx = -a\sin t\,dt,\ dy = a\cos t\,dt \Rightarrow \text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx$
$= \frac{1}{2}\int_0^{2\pi} (a^2\cos^2 t + a^2\sin^2 t)\,dt = \frac{1}{2}\int_0^{2\pi} a^2\,dt = \pi a^2$

22. $M = x = a\cos t,\ N = y = b\sin t \Rightarrow dx = -a\sin t\,dt,\ dy = b\cos t\,dt \Rightarrow \text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx$
$= \frac{1}{2}\int_0^{2\pi} (ab\cos^2 t + ab\sin^2 t)\,dt = \frac{1}{2}\int_0^{2\pi} ab\,dt = \pi ab$

23. $M = x = a\cos^3 t,\ N = y = \sin^3 t \Rightarrow dx = -3\cos^2 t\sin t\,dt,\ dy = 3\sin^2 t\cos t\,dt \Rightarrow \text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx$
$= \frac{1}{2}\int_0^{2\pi} (3\sin^2 t\cos^2 t)(\cos^2 t + \sin^2 t)\,dt = \frac{1}{2}\int_0^{2\pi} (3\sin^2 t\cos^2 t)\,dt = \frac{3}{8}\int_0^{2\pi} \sin^2 2t\,dt = \frac{3}{16}\int_0^{4\pi} \sin^2 u\,du$
$= \frac{3}{16}\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{4\pi} = \frac{3}{8}\pi$

24. $M = x = t^2,\ N = y = \frac{t^3}{3} - t \Rightarrow dx = 2t\,dt,\ dy = (t^2 - 1)\,dt \Rightarrow \text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx$
$= \frac{1}{2}\int_{-\sqrt{3}}^{\sqrt{3}} \left[t^2(t^2 - 1) - \left(\frac{t^3}{3} - t\right)(2t)\right] dt = \frac{1}{2}\int_{-\sqrt{3}}^{\sqrt{3}} \left(\frac{1}{3}t^4 + t^2\right) dt = \frac{1}{2}\left[\frac{1}{15}t^5 + -\frac{1}{3}t^3\right]_{-\sqrt{3}}^{\sqrt{3}} = \frac{1}{15}\left(9\sqrt{3} + 15\sqrt{3}\right)$
$= \frac{8}{5}\sqrt{3}$

25. (a) $M = f(x),\ N = g(y) \Rightarrow \frac{\partial M}{\partial y} = 0,\ \frac{\partial N}{\partial x} = 0 \Rightarrow \oint_C f(x)\,dx + g(y)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$
$= \iint_R 0\,dx\,dy = 0$

(b) $M = ky,\ N = hx \Rightarrow \frac{\partial M}{\partial y} = k,\ \frac{\partial N}{\partial x} = h \Rightarrow \oint_C ky\,dx + hx\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$
$= \iint_R (h - k)\,dx\,dy = (h - k)(\text{Area of the region})$

26. $M = xy^2,\ N = x^2y + 2x \Rightarrow \frac{\partial M}{\partial y} = 2xy,\ \frac{\partial N}{\partial x} = 2xy + 2 \Rightarrow \oint_C xy^2\,dx + (x^2y + 2x)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$
$= \iint_R (2xy + 2 - 2xy)\,dx\,dy = 2\iint_R dx\,dy = 2$ times the area of the square

27. The integral is 0 for any simple closed plane curve C. The reasoning: By the tangential form of Green's Theorem, with $M = 4x^3y$ and $N = x^4$, $\oint_C 4x^3y\,dx + x^4\,dy = \iint_R \left[\frac{\partial}{\partial x}(x^4) - \frac{\partial}{\partial y}(4x^3y)\right] dx\,dy$
$= \iint_R \underbrace{(4x^3 - 4x^3)}_{0}\,dx\,dy = 0.$

28. The integral is 0 for any simple closed curve C. The reasoning: By the normal form of Green's theorem, with $M = x^3$ and $N = -y^3$, $\oint_C -y^3\,dy + x^3\,dx = \iint_R \left[\underbrace{\frac{\partial}{\partial x}(-y^3)}_{0} - \underbrace{\frac{\partial}{\partial y}(x^3)}_{0}\right] dx\,dy = 0.$

29. Let $M = x$ and $N = 0 \Rightarrow \frac{\partial M}{\partial x} = 1$ and $\frac{\partial N}{\partial y} = 0 \Rightarrow \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy \Rightarrow \oint_C x\,dy$
$= \iint_R (1+0)\,dx\,dy \Rightarrow$ Area of $R = \iint_R dx\,dy = \oint_C x\,dy$; similarly, $M = y$ and $N = 0 \Rightarrow \frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 0 \Rightarrow \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y}\right) dy\,dx \Rightarrow \oint_C y\,dx = \iint_R (0-1)\,dy\,dx \Rightarrow -\oint_C y\,dx$
$= \iint_R dx\,dy =$ Area of R

30. $\int_a^b f(x)\,dx =$ Area of $R = -\oint_C y\,dx$, from Exercise 29

31. Let $\delta(x,y) = 1 \Rightarrow \overline{x} = \frac{M_y}{M} = \frac{\iint_R x\,\delta(x,y)\,dA}{\iint_R \delta(x,y)\,dA} = \frac{\iint_R x\,dA}{\iint_R dA} = \frac{\iint_R x\,dA}{A} \Rightarrow A\overline{x} = \iint_R x\,dA = \iint_R (x+0)\,dx\,dy$
$= \oint_C \frac{x^2}{2}\,dy$, $A\overline{x} = \iint_R x\,dA = \iint_R (0+x)\,dx\,dy = -\oint_C xy\,dx$, and $A\overline{x} = \iint_R x\,dA = \iint_R \left(\frac{2}{3}x + \frac{1}{3}x\right) dx\,dy$
$= \oint_C \frac{1}{3}x^2\,dy - \frac{1}{3}xy\,dx \Rightarrow \frac{1}{2}\oint_C x^2\,dy = -\oint_C xy\,dx = \frac{1}{3}\oint_C x^2\,dy - xy\,dx = A\overline{x}$

32. If $\delta(x,y) = 1$, then $I_y = \iint_R x^2\,\delta(x,y)\,dA = \iint_R x^2\,dA = \iint_R (x^2+0)\,dy\,dx = \frac{1}{3}\oint_C x^3\,dy$,
$\iint_R x^2\,dA = \iint_R (0+x^2)\,dy\,dx = -\oint_C x^2y\,dx$, and $\iint_R x^2\,dA = \iint_R \left(\frac{3}{4}x^2 + \frac{1}{4}x^2\right) dy\,dx$
$= \oint_C \frac{1}{4}x^3\,dy - \frac{1}{4}x^2y\,dx = \frac{1}{4}\oint_C x^3\,dy - x^2y\,dx \Rightarrow \frac{1}{3}\oint_C x^3\,dy = -\oint_C x^2y\,dx = \frac{1}{4}\oint_C x^3\,dy - x^2y\,dx = I_y$

33. $M = \frac{\partial f}{\partial y}$, $N = -\frac{\partial f}{\partial x} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y^2}$, $\frac{\partial N}{\partial x} = -\frac{\partial^2 f}{\partial x^2} \Rightarrow \oint_C \frac{\partial f}{\partial y}\,dx - \frac{\partial f}{\partial x}\,dy = \iint_R \left(-\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right) dx\,dy = 0$ for such curves C

34. $M = \frac{1}{4}x^2y + \frac{1}{3}y^3$, $N = x \Rightarrow \frac{\partial M}{\partial y} = \frac{1}{4}x^2 + y^2$, $\frac{\partial N}{\partial x} = 1 \Rightarrow \text{Curl} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - \left(\frac{1}{4}x^2 + y^2\right) > 0$ in the interior of the ellipse $\frac{1}{4}x^2 + y^2 = 1 \Rightarrow$ work $= \int_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \left(1 - \frac{1}{4}x^2 - y^2\right) dx\,dy$ will be maximized on the region $R = \{(x,y) \mid \text{curl}\,\mathbf{F}\} \ge 0$ or over the region enclosed by $1 = \frac{1}{4}x^2 + y^2$

35. (a) $\nabla f = \left(\frac{2x}{x^2+y^2}\right)\mathbf{i} + \left(\frac{2y}{x^2+y^2}\right)\mathbf{j} \Rightarrow M = \frac{2x}{x^2+y^2}$, $N = \frac{2y}{x^2+y^2}$; since M, N are discontinuous at $(0,0)$, we compute $\int_C \nabla f \cdot \mathbf{n}\,ds$ directly since Green's Theorem does not apply. Let $x = a\cos t$, $y = a\sin t \Rightarrow dx = -a\sin t\,dt$, $dy = a\cos t\,dt$, $M = \frac{2}{a}\cos t$, $N = \frac{2}{a}\sin t$, $0 \le t \le 2\pi$, so $\int_C \nabla f \cdot \mathbf{n}\,ds = \int_C M\,dy - N\,dx$
$= \int_0^{2\pi} \left[\left(\frac{2}{a}\cos t\right)(a\cos t) - \left(\frac{2}{a}\sin t\right)(-a\sin t)\right] dt = \int_0^{2\pi} 2(\cos^2 t + \sin^2 t)\,dt = 4\pi$. Note that this holds for any

$a > 0$, so $\int_C \nabla f \cdot \mathbf{n}\, ds = 4\pi$ for any circle C centered at $(0,0)$ traversed counterclockwise and $\int_C \nabla f \cdot \mathbf{n}\, ds = -4\pi$ if C is traversed clockwise.

(b) If K does not enclose the point $(0,0)$ we may apply Green's Theorem: $\int_C \nabla f \cdot \mathbf{n}\, ds = \int_C M\,dy - N\,dx$ $= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R \left(\frac{2(y^2-x^2)}{(x^2+y^2)^2} + \frac{2(x^2-y^2)}{(x^2+y^2)^2}\right) dx\,dy = \iint_R 0\,dx\,dy = 0$. If K does enclose the point $(0,0)$ we proceed as in Example 6:

Choose a small enough so that the circle C centered at $(0,0)$ of radius a lies entirely within K. Green's Theorem applies to the region R that lies between K and C. Thus, as before, $0 = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$

$= \int_K M\,dy - N\,dx + \int_C M\,dy - N\,dx$ where K is traversed counterclockwise and C is traversed clockwise.

Hence by part (a) $0 = \left[\int_K M\,dy - N\,dx\right] - 4\pi \Rightarrow 4\pi = \int_K M\,dy - N\,dx = \int_K \nabla f \cdot \mathbf{n}\, ds$. We have shown:

$$\int_K \nabla f \cdot \mathbf{n}\, ds = \begin{cases} 0 & \text{if } (0,0) \text{ lies inside K} \\ 4\pi & \text{if } (0,0) \text{ lies outside K} \end{cases}$$

36. Assume a particle has a closed trajectory in R and let C_1 be the path $\Rightarrow$ C_1 encloses a simply connected region R_1 $\Rightarrow$ C_1 is a simple closed curve. Then the flux over R_1 is $\oint_{C_1} \mathbf{F} \cdot \mathbf{n}\, ds = 0$, since the velocity vectors $\mathbf{F}$ are tangent to C_1. But $0 = \oint_{C_1} \mathbf{F} \cdot \mathbf{n}\, ds = \oint_{C_1} M\,dy - N\,dx = \iint_{R_1} \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy \Rightarrow M_x + N_y = 0$, which is a contradiction. Therefore, C_1 cannot be a closed trajectory.

37. $\int_{g_1(y)}^{g_2(y)} \frac{\partial N}{\partial x}\, dx\,dy = N(g_2(y), y) - N(g_1(y), y) \Rightarrow \int_c^d \int_{g_1(y)}^{g_2(y)} \left(\frac{\partial N}{\partial x}\, dx\right) dy = \int_c^d [N(g_2(y), y) - N(g_1(y), y)]\, dy$
$= \int_c^d N(g_2(y), y)\, dy - \int_c^d N(g_1(y), y)\, dy = \int_c^d N(g_2(y), y)\, dy + \int_d^c N(g_1(y), y)\, dy = \int_{C_2} N\,dy + \int_{C_1} N\,dy$
$= \oint_C dy \Rightarrow \oint_C N\,dy = \iint_R \frac{\partial N}{\partial x}\, dx\,dy$

38. $\int_a^b \int_c^d \frac{\partial M}{\partial y}\, dy\,dx = \int_a^b [M(x,d) - M(x,c)]\, dx = \int_a^b M(x,d)\, dx + \int_a^b M(x,c)\, dx = -\int_{C_3} M\,dx - \int_{C_1} M\,dx$.
Because x is constant along C_2 and C_4, $\int_{C_2} M\,dx = \int_{C_4} M\,dx = 0$
$\Rightarrow -\left(\int_{C_1} M\,dx + \int_{C_2} M\,dx + \int_{C_3} M\,dx + \int_{C_4} M\,dx\right) = -\oint_C M\,dx \Rightarrow \int_a^b \int_c^d \frac{\partial M}{\partial y}\, dy\,dx = -\oint_C M\,dx$.

39. The curl of a conservative two-dimensional field is zero. The reasoning: A two-dimensional field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ can be considered to be the restriction to the xy-plane of a three-dimensional field whose k component is zero, and whose $\mathbf{i}$ and $\mathbf{j}$ components are independent of z. For such a field to be conservative, we must have $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ by the component test in Section 16.3 $\Rightarrow$ curl $\mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$.

40. Green's theorem tells us that the circulation of a conservative two-dimensional field around any simple closed curve in the xy-plane is zero. The reasoning: For a conservative field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$, we have $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ (component test for conservative fields, Section 16.3, Eq. (2)), so curl $\mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$. By Green's theorem, the counterclockwise circulation around a simple closed plane curve C must equal the integral of curl $\mathbf{F}$ over the region R enclosed by C. Since curl $\mathbf{F} = 0$, the latter integral is zero and, therefore, so is the circulation.
The circulation $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$ is the same as the work $\oint_C \mathbf{F} \cdot d\mathbf{r}$ done by $\mathbf{F}$ around C, so our observation that circulation of a conservative two-dimensional field is zero agrees with the fact that the work done by a conservative field around a closed curve is always 0.

41-44. Example CAS commands:

Maple:

```
with( plots );#41
M := (x,y) -> 2*x-y;
N := (x,y) -> x+3*y;
C := x^2 + 4*y^2 = 4;
implicitplot( C, x=-2..2, y=-2..2, scaling=constrained, title="#41(a) (Section 16.4)" );
curlF_k := D[1](N) - D[2](M):                    # (b)
'curlF_k' = curlF_k(x,y);
top,bot := solve( C, y );                        # (c)
left,right := -2, 2;
q1 := Int( Int( curlF_k(x,y), y=bot..top ), x=left..right );
value( q1 );
```

Mathematica: (functions and bounds will vary)

The **ImplicitPlot** command will be useful for 41 and 42, but is not needed for 43 and 44. In 44, the equation of the line from (0, 4) to (2, 0) must be determined first.

```
Clear[x, y, f]
<<Graphics`ImplicitPlot`
f[x_, y_]:= {2x − y, x + 3y}
curve= x^2 + 4y^2 ==4
ImplicitPlot[curve, {x, −3, 3},{y, −2, 2}, AspectRatio → Automatic, AxesLabel → {x, y}];
ybounds= Solve[curve, y]
{y1, y2}=y/.ybounds;
integrand:=D[f[x,y][[2]], x] − D[f[x,y][[1]], y]//Simplify
Integrate[integrand, {x, −2, 2}, {y, y1, y2}]
N[%]
```

Bounds for y are determined differently in 43 and 44. In 44, note equation of the line from (0, 4) to (2, 0).

```
Clear[x, y, f]
f[x_, y_]:= {x Exp[y], 4x^2 Log[y]}
ybound = 4 − 2x
Plot[{0, ybound}, {x, 0,2. 1}, AspectRatio → Automatic, AxesLabel → {x, y}];
integrand:=D[f[x, y][[2]], x] − D[f[x, y][[1]], y]//Simplify
Integrate[integrand, {x, 0, 2}, {y, 0, ybound}]
N[%]
```

16.5 SURFACE AREA AND SURFACE INTEGRALS

1. $\mathbf{p}=\mathbf{k}$, $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{(2x)^2+(2y)^2+(-1)^2} = \sqrt{4x^2+4y^2+1}$ and $|\nabla f \cdot \mathbf{p}| = 1$; $z=2 \Rightarrow x^2+y^2=2$; thus $S = \iint\limits_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint\limits_R \sqrt{4x^2+4y^2+1}\, dx\, dy$
$= \iint\limits_R \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + 1}\; r\, dr\, d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{4r^2+1}\; r\, dr\, d\theta = \int_0^{2\pi} \left[\frac{1}{12}\left(4r^2+1\right)^{3/2}\right]_0^{\sqrt{2}} d\theta$
$= \int_0^{2\pi} \frac{13}{6}\, d\theta = \frac{13}{3}\pi$

2. $\mathbf{p}=\mathbf{k}$, $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2+4y^2+1}$ and $|\nabla f \cdot \mathbf{p}| = 1$; $2 \le x^2+y^2 \le 6$
$\Rightarrow S = \iint\limits_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint\limits_R \sqrt{4x^2+4y^2+1}\, dx\, dy = \iint\limits_R \sqrt{4r^2+1}\; r\, dr\, d\theta = \int_0^{2\pi}\int_{\sqrt{2}}^{\sqrt{6}} \sqrt{4r^2+1}\; r\, dr\, d\theta$
$= \int_0^{2\pi} \left[\frac{1}{12}\left(4r^2+1\right)^{3/2}\right]_{\sqrt{2}}^{\sqrt{6}} d\theta = \int_0^{2\pi} \frac{49}{6}\, d\theta = \frac{49}{3}\pi$

3. $\mathbf{p} = \mathbf{k}$, $\nabla f = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow |\nabla f| = 3$ and $|\nabla f \cdot \mathbf{p}| = 2$; $x = y^2$ and $x = 2 - y^2$ intersect at $(1, 1)$ and $(1, -1)$
$\Rightarrow S = \iint\limits_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint\limits_R \frac{3}{2}\, dx\, dy = \int_{-1}^{1}\int_{y^2}^{2-y^2} \frac{3}{2}\, dx\, dy = \int_{-1}^{1} (3 - 3y^2)\, dy = 4$

4. $\mathbf{p} = \mathbf{k}$, $\nabla f = 2x\mathbf{i} - 2\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4} = 2\sqrt{x^2 + 1}$ and $|\nabla f \cdot \mathbf{p}| = 2 \Rightarrow S = \iint\limits_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA$
$= \iint\limits_R \frac{2\sqrt{x^2+1}}{2}\, dx\, dy = \int_0^{\sqrt{3}}\int_0^{x} \sqrt{x^2 + 1}\, dy\, dx = \int_0^{\sqrt{3}} x\sqrt{x^2 + 1}\, dx = \left[\frac{1}{3}\left(x^2 + 1\right)^{3/2}\right]_0^{\sqrt{3}} = \frac{1}{3}(4)^{3/2} - \frac{1}{3} = \frac{7}{3}$

5. $\mathbf{p} = \mathbf{k}$, $\nabla f = 2x\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} \Rightarrow |\nabla f| = \sqrt{(2x)^2 + (-2)^2 + (-2)^2} = \sqrt{4x^2 + 8} = 2\sqrt{x^2 + 2}$ and $|\nabla f \cdot \mathbf{p}| = 2$
$\Rightarrow S = \iint\limits_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint\limits_R \frac{2\sqrt{x^2+2}}{2}\, dx\, dy = \int_0^{2}\int_0^{3x} \sqrt{x^2 + 2}\, dy\, dx = \int_0^{2} 3x\sqrt{x^2 + 2}\, dx = \left[\left(x^2 + 2\right)^{3/2}\right]_0^2$
$= 6\sqrt{6} - 2\sqrt{2}$

6. $\mathbf{p} = \mathbf{k}$, $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = \sqrt{8} = 2\sqrt{2}$ and $|\nabla f \cdot \mathbf{p}| = 2z$; $x^2 + y^2 + z^2 = 2$ and $z = \sqrt{x^2 + y^2} \Rightarrow x^2 + y^2 = 1$; thus, $S = \iint\limits_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint\limits_R \frac{2\sqrt{2}}{2z}\, dA = \sqrt{2}\iint\limits_R \frac{1}{z}\, dA$
$= \sqrt{2}\iint\limits_R \frac{1}{\sqrt{2 - (x^2 + y^2)}}\, dA = \sqrt{2}\int_0^{2\pi}\int_0^{1} \frac{r\, dr\, d\theta}{\sqrt{2 - r^2}} = \sqrt{2}\int_0^{2\pi}\left(-1 + \sqrt{2}\right) d\theta = 2\pi\left(2 - \sqrt{2}\right)$

7. $\mathbf{p} = \mathbf{k}$, $\nabla f = c\mathbf{i} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{c^2 + 1}$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow S = \iint\limits_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint\limits_R \sqrt{c^2 + 1}\, dx\, dy$
$= \int_0^{2\pi}\int_0^{1} \sqrt{c^2 + 1}\, r\, dr\, d\theta = \int_0^{2\pi} \frac{\sqrt{c^2+1}}{2}\, d\theta = \pi\sqrt{c^2 + 1}$

8. $\mathbf{p} = \mathbf{k}$, $\nabla f = 2x\mathbf{i} + 2z\mathbf{j} \Rightarrow |\nabla f| = \sqrt{(2x)^2 + (2z)^2} = 2$ and $|\nabla f \cdot \mathbf{p}| = 2z$ for the upper surface, $z \ge 0$
$\Rightarrow S = \iint\limits_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint\limits_R \frac{2}{2z}\, dA = \iint\limits_R \frac{1}{\sqrt{1 - x^2}}\, dy\, dx = 2\int_{-1/2}^{1/2}\int_0^{1/2} \frac{1}{\sqrt{1 - x^2}}\, dy\, dx = \int_{-1/2}^{1/2} \frac{1}{\sqrt{1 - x^2}}\, dx$
$= \left[\sin^{-1} x\right]_{-1/2}^{1/2} = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$

9. $\mathbf{p} = \mathbf{i}$, $\nabla f = \mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{1^2 + (2y)^2 + (2z)^2} = \sqrt{1 + 4y^2 + 4z^2}$ and $|\nabla f \cdot \mathbf{p}| = 1$; $1 \le y^2 + z^2 \le 4$
$\Rightarrow S = \iint\limits_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint\limits_R \sqrt{1 + 4y^2 + 4z^2}\, dy\, dz = \int_0^{2\pi}\int_1^{2} \sqrt{1 + 4r^2\cos^2\theta + 4r^2\sin^2\theta}\, r\, dr\, d\theta$
$= \int_0^{2\pi}\int_1^{2} \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi}\left[\frac{1}{12}\left(1 + 4r^2\right)^{3/2}\right]_1^2 d\theta = \int_0^{2\pi} \frac{1}{12}\left(17\sqrt{17} - 5\sqrt{5}\right) d\theta = \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$

10. $\mathbf{p} = \mathbf{j}$, $\nabla f = 2x\mathbf{i} + \mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4z^2 + 1}$ and $|\nabla f \cdot \mathbf{p}| = 1$; $y = 0$ and $x^2 + y + z^2 = 2 \Rightarrow x^2 + z^2 = 2$; thus, $S = \iint\limits_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint\limits_R \sqrt{4x^2 + 4z^2 + 1}\, dx\, dz = \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{4r^2 + 1}\, r\, dr\, d\theta = \int_0^{2\pi} \frac{13}{6}\, d\theta = \frac{13}{3}\pi$

11. $\mathbf{p} = \mathbf{k}$, $\nabla f = \left(2x - \frac{2}{x}\right)\mathbf{i} + \sqrt{15}\,\mathbf{j} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{\left(2x - \frac{2}{x}\right)^2 + \left(\sqrt{15}\right)^2 + (-1)^2} = \sqrt{4x^2 + 8 + \frac{4}{x^2}} = \sqrt{\left(2x + \frac{2}{x}\right)^2}$
$= 2x + \frac{2}{x}$, on $1 \le x \le 2$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow S = \iint\limits_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint\limits_R \left(2x + 2x^{-1}\right) dx\, dy$
$= \int_0^{1}\int_1^{2} \left(2x + 2x^{-1}\right) dx\, dy = \int_0^{1} \left[x^2 + 2\ln x\right]_1^2 dy = \int_0^{1} (3 + 2\ln 2)\, dy = 3 + 2\ln 2$

12. $\mathbf{p} = \mathbf{k}$, $\nabla f = 3\sqrt{x}\,\mathbf{i} + 3\sqrt{y}\,\mathbf{j} - 3\mathbf{k} \Rightarrow |\nabla f| = \sqrt{9x + 9y + 9} = 3\sqrt{x + y + 1}$ and $|\nabla f \cdot \mathbf{p}| = 3$
$\Rightarrow S = \iint\limits_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint\limits_R \sqrt{x + y + 1}\, dx\, dy = \int_0^{1}\int_0^{1} \sqrt{x + y + 1}\, dx\, dy = \int_0^{1} \left[\frac{2}{3}(x + y + 1)^{3/2}\right]_0^1 dy$
$= \int_0^{1} \left[\frac{2}{3}(y + 2)^{3/2} - \frac{2}{3}(y + 1)^{3/2}\right] dy = \left[\frac{4}{15}(y + 2)^{5/2} - \frac{4}{15}(y + 1)^{5/2}\right]_0^1 = \frac{4}{15}\left[(3)^{5/2} - (2)^{5/2} - (2)^{5/2} + 1\right]$

$= \frac{4}{15}\left(9\sqrt{3} - 8\sqrt{2} + 1\right)$

13. The bottom face S of the cube is in the xy-plane $\Rightarrow$ $z = 0$ $\Rightarrow$ $g(x, y, 0) = x + y$ and $f(x, y, z) = z = 0$ $\Rightarrow$ $\mathbf{p} = \mathbf{k}$ and $\nabla f = \mathbf{k}$ $\Rightarrow$ $|\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1$ $\Rightarrow$ $d\sigma = dx\,dy$ $\Rightarrow$ $\iint_S g\,d\sigma = \iint_R (x + y)\,dx\,dy$ $= \int_0^a \int_0^a (x + y)\,dx\,dy = \int_0^a \left(\frac{a^2}{2} + ay\right) dy = a^3$. Because of symmetry, we also get a^3 over the face of the cube in the xz-plane and a^3 over the face of the cube in the yz-plane. Next, on the top of the cube, $g(x, y, z)$ $= g(x, y, a) = x + y + a$ and $f(x, y, z) = z = a$ $\Rightarrow$ $\mathbf{p} = \mathbf{k}$ and $\nabla f = \mathbf{k}$ $\Rightarrow$ $|\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx\,dy$
$\iint_S g\,d\sigma = \iint_R (x + y + a)\,dx\,dy = \int_0^a \int_0^a (x + y + a)\,dx\,dy = \int_0^a \int_0^a (x + y)\,dx\,dy + \int_0^a \int_0^a a\,dx\,dy = 2a^3$.
Because of symmetry, the integral is also $2a^3$ over each of the other two faces. Therefore,
$\iint_{\text{cube}} (x + y + z)\,d\sigma = 3\left(a^3 + 2a^3\right) = 9a^3$.

14. On the face S in the xz-plane, we have $y = 0$ $\Rightarrow$ $f(x, y, z) = y = 0$ and $g(x, y, z) = g(x, 0, z) = z$ $\Rightarrow$ $\mathbf{p} = \mathbf{j}$ and $\nabla f = \mathbf{j}$ $\Rightarrow$ $|\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1$ $\Rightarrow$ $d\sigma = dx\,dz$ $\Rightarrow$ $\iint_S g\,d\sigma = \iint_S (y + z)\,d\sigma = \int_0^1 \int_0^2 z\,dx\,dz = \int_0^1 2z\,dz$ $= 1$.
On the face in the xy-plane, we have $z = 0$ $\Rightarrow$ $f(x, y, z) = z = 0$ and $g(x, y, z) = g(x, y, 0) = y$ $\Rightarrow$ $\mathbf{p} = \mathbf{k}$ and $\nabla f = \mathbf{k}$ $\Rightarrow$ $|\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1$ $\Rightarrow$ $d\sigma = dx\,dy$ $\Rightarrow \iint_S g\,d\sigma = \iint_S y\,d\sigma = \int_0^1 \int_0^2 y\,dx\,dy = 1$.
On the triangular face in the plane $x = 2$ we have $f(x, y, z) = x = 2$ and $g(x, y, z) = g(2, y, z) = y + z$ $\Rightarrow$ $\mathbf{p} = \mathbf{i}$ and $\nabla f = \mathbf{i}$ $\Rightarrow$ $|\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1$ $\Rightarrow$ $d\sigma = dz\,dy$ $\Rightarrow$ $\iint_S g\,d\sigma = \iint_S (y + z)\,d\sigma = \int_0^1 \int_0^{1-y} (y + z)\,dz\,dy$ $= \int_0^1 \frac{1}{2}\left(1 - y^2\right) dy = \frac{1}{3}$.
On the triangular face in the yz-plane, we have $x = 0$ $\Rightarrow$ $f(x, y, z) = x = 0$ and $g(x, y, z) = g(0, y, z) = y + z$ $\Rightarrow$ $\mathbf{p} = \mathbf{i}$ and $\nabla f = \mathbf{i}$ $\Rightarrow$ $|\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1$ $\Rightarrow$ $d\sigma = dz\,dy$ $\Rightarrow$ $\iint_S g\,d\sigma = \iint_S (y + z)\,d\sigma$ $= \int_0^1 \int_0^{1-y} (y + z)\,dz\,dy = \frac{1}{3}$.
Finally, on the sloped face, we have $y + z = 1$ $\Rightarrow$ $f(x, y, z) = y + z = 1$ and $g(x, y, z) = y + z = 1$ $\Rightarrow$ $\mathbf{p} = \mathbf{k}$ and $\nabla f = \mathbf{j} + \mathbf{k}$ $\Rightarrow$ $|\nabla f| = \sqrt{2}$ and $|\nabla f \cdot \mathbf{p}| = 1$ $\Rightarrow$ $d\sigma = \sqrt{2}\,dx\,dy$ $\Rightarrow$ $\iint_S g\,d\sigma = \iint_S (y + z)\,d\sigma$ $= \int_0^1 \int_0^2 \sqrt{2}\,dx\,dy = 2\sqrt{2}$. Therefore, $\iint_{\text{wedge}} g(x, y, z)\,d\sigma = 1 + 1 + \frac{1}{3} + \frac{1}{3} + 2\sqrt{2} = \frac{8}{3} + 2\sqrt{2}$

15. On the faces in the coordinate planes, $g(x, y, z) = 0$ $\Rightarrow$ the integral over these faces is 0.
On the face $x = a$, we have $f(x, y, z) = x = a$ and $g(x, y, z) = g(a, y, z) = ayz$ $\Rightarrow$ $\mathbf{p} = \mathbf{i}$ and $\nabla f = \mathbf{i}$ $\Rightarrow$ $|\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1$ $\Rightarrow$ $d\sigma = dy\,dz$ $\Rightarrow$ $\iint_S g\,d\sigma = \iint_S ayz\,d\sigma = \int_0^c \int_0^b ayz\,dy\,dz = \frac{ab^2c^2}{4}$.
On the face $y = b$, we have $f(x, y, z) = y = b$ and $g(x, y, z) = g(x, b, z) = bxz$ $\Rightarrow$ $\mathbf{p} = \mathbf{j}$ and $\nabla f = \mathbf{j}$ $\Rightarrow$ $|\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1$ $\Rightarrow$ $d\sigma = dx\,dz$ $\Rightarrow$ $\iint_S g\,d\sigma = \iint_S bxz\,d\sigma = \int_0^c \int_0^a bxz\,dx\,dz = \frac{a^2bc^2}{4}$.
On the face $z = c$, we have $f(x, y, z) = z = c$ and $g(x, y, z) = g(x, y, c) = cxy$ $\Rightarrow$ $\mathbf{p} = \mathbf{k}$ and $\nabla f = \mathbf{k}$ $\Rightarrow$ $|\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1$ $\Rightarrow$ $d\sigma = dy\,dx$ $\Rightarrow$ $\iint_S g\,d\sigma = \iint_S cxy\,d\sigma = \int_0^b \int_0^a cxy\,dx\,dy = \frac{a^2b^2c}{4}$. Therefore,
$\iint_S g(x, y, z)\,d\sigma = \frac{abc(ab + ac + bc)}{4}$.

16. On the face $x = a$, we have $f(x,y,z) = x = a$ and $g(x,y,z) = g(a,y,z) = ayz \Rightarrow \mathbf{p} = \mathbf{i}$ and $\nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dz\,dy \Rightarrow \iint_S g\,d\sigma = \iint_S ayz\,d\sigma = \int_{-b}^{b}\int_{-c}^{c} ayz\,dz\,dy = 0$. Because of the symmetry of g on all the other faces, all the integrals are 0, and $\iint_S g(x,y,z)\,d\sigma = 0$.

17. $f(x,y,z) = 2x + 2y + z = 2 \Rightarrow \nabla f = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $g(x,y,z) = x + y + (2 - 2x - 2y) = 2 - x - y \Rightarrow \mathbf{p} = \mathbf{k}$, $|\nabla f| = 3$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = 3\,dy\,dx$; $z = 0 \Rightarrow 2x + 2y = 2 \Rightarrow y = 1 - x \Rightarrow \iint_S g\,d\sigma = \iint_S (2 - x - y)\,d\sigma$ $= 3\int_0^1\int_0^{1-x}(2 - x - y)\,dy\,dx = 3\int_0^1 \left[(2-x)(1-x) - \frac{1}{2}(1-x)^2\right] dx = 3\int_0^1 \left(\frac{3}{2} - 2x + \frac{x^2}{2}\right) dx = 2$

18. $f(x,y,z) = y^2 + 4z = 16 \Rightarrow \nabla f = 2y\mathbf{j} + 4\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 16} = 2\sqrt{y^2+4}$ and $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 4$ $\Rightarrow d\sigma = \frac{2\sqrt{y^2+4}}{4}\,dx\,dy \Rightarrow \iint_S g\,d\sigma = \int_{-4}^{4}\int_0^1 \left(x\sqrt{y^2+4}\right)\left(\frac{\sqrt{y^2+4}}{2}\right) dx\,dy = \int_{-4}^{4}\int_0^1 \frac{x\,(y^2+4)}{2}\,dx\,dy$ $= \int_{-4}^{4} \frac{1}{4}(y^2+4)\,dy = \frac{1}{2}\left[\frac{y^3}{3} + 4y\right]_0^4 = \frac{1}{2}\left(\frac{64}{3} + 16\right) = \frac{56}{3}$

19. $g(x,y,z) = z$, $\mathbf{p} = \mathbf{k} \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1$ and $|\nabla g \cdot \mathbf{p}| = 1 \Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\,d\sigma = \iint_R (\mathbf{F} \cdot \mathbf{k})\,dA$ $= \int_0^2\int_0^3 3\,dy\,dx = 18$

20. $g(x,y,z) = y$, $\mathbf{p} = -\mathbf{j} \Rightarrow \nabla g = \mathbf{j} \Rightarrow |\nabla g| = 1$ and $|\nabla g \cdot \mathbf{p}| = 1 \Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\,d\sigma = \iint_R (\mathbf{F} \cdot -\mathbf{j})\,dA$ $= \int_{-1}^{2}\int_2^7 2\,dz\,dx = \int_{-1}^{2} 2(7-2)\,dx = 10(2+1) = 30$

21. $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a$; $\mathbf{n} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2+y^2+z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a} \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{z^2}{a}$; $|\nabla g \cdot \mathbf{k}| = 2z \Rightarrow d\sigma = \frac{2a}{2z}\,dA \Rightarrow \text{Flux} = \iint_R \left(\frac{z^2}{a}\right)\left(\frac{a}{z}\right) dA = \iint_R z\,dA = \iint_R \sqrt{a^2 - (x^2+y^2)}\,dx\,dy$ $= \int_0^{\pi/2}\int_0^a \sqrt{a^2 - r^2}\,r\,dr\,d\theta = \frac{\pi a^3}{6}$

22. $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a$; $\mathbf{n} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2+y^2+z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a} \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{-xy}{a} + \frac{xy}{a}$ $= 0$; $|\nabla g \cdot \mathbf{k}| = 2z \Rightarrow d\sigma = \frac{2a}{2z}\,dA \Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\,d\sigma = \iint_S 0\,d\sigma = 0$

23. From Exercise 21, $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}$ and $d\sigma = \frac{a}{z}\,dA \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{xy}{a} - \frac{xy}{a} + \frac{z}{a} = \frac{z}{a} \Rightarrow \text{Flux} = \iint_R \left(\frac{z}{a}\right)\left(\frac{a}{z}\right) dA$ $= \iint_R 1\,dA = \frac{\pi a^2}{4}$

24. From Exercise 21, $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}$ and $d\sigma = \frac{a}{z}\,dA \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{zx^2}{a} + \frac{zy^2}{a} + \frac{z^3}{a} = z\left(\frac{x^2+y^2+z^2}{a}\right) = az$ $\Rightarrow \text{Flux} = \iint_R (za)\left(\frac{a}{z}\right) dx\,dy = \iint_R a^2\,dx\,dy = a^2(\text{Area of R}) = \frac{1}{4}\pi a^4$

25. From Exercise 21, $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}$ and $d\sigma = \frac{a}{z}\,dA \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{x^2}{a} + \frac{y^2}{a} + \frac{z^2}{a} = a \Rightarrow$ Flux $= \iint_R a\left(\frac{a}{z}\right) dA = \iint_R \frac{a^2}{z}\,dA = \iint_R \frac{a^2}{\sqrt{a^2 - (x^2+y^2)}}\,dA = \int_0^{\pi/2}\int_0^a \frac{a^2}{\sqrt{a^2 - r^2}}\,r\,dr\,d\theta$ $= \int_0^{\pi/2} a^2\left[-\sqrt{a^2 - r^2}\right]_0^a d\theta = \frac{\pi a^3}{2}$

26. From Exercise 21, $\mathbf{n} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a}$ and $d\sigma = \frac{a}{z}\,dA \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{\left(\frac{x^2}{a}\right)+\left(\frac{y^2}{a}\right)+\left(\frac{z^2}{a}\right)}{\sqrt{x^2+y^2+z^2}} = \frac{\left(\frac{a^2}{a}\right)}{a} = 1$

$\Rightarrow$ Flux $= \iint\limits_R \frac{a}{z}\,dx\,dy = \iint\limits_R \frac{a}{\sqrt{a^2-(x^2+y^2)}}\,dx\,dy = \int_0^{\pi/2}\int_0^a \frac{a}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta = \frac{\pi a^2}{2}$

27. $g(x,y,z) = y^2+z = 4 \Rightarrow \nabla g = 2y\mathbf{j}+\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4y^2+1} \Rightarrow \mathbf{n} = \frac{2y\mathbf{j}+\mathbf{k}}{\sqrt{4y^2+1}}$

$\Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{2xy-3z}{\sqrt{4y^2+1}}$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla g\cdot\mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{4y^2+1}\,dA \Rightarrow$ Flux

$= \iint\limits_R \left(\frac{2xy-3z}{\sqrt{4y^2+1}}\right)\sqrt{4y^2+1}\,dA = \iint\limits_R (2xy-3z)\,dA$; $z = 0$ and $z = 4-y^2 \Rightarrow y^2 = 4$

$\Rightarrow$ Flux $= \iint\limits_R [2xy-3(4-y^2)]\,dA = \int_0^1\int_{-2}^2 (2xy-12+3y^2)\,dy\,dx = \int_0^1 \left[xy^2-12y+y^3\right]_{-2}^2\,dx$

$= \int_0^1 -32\,dx = -32$

28. $g(x,y,z) = x^2+y^2-z = 0 \Rightarrow \nabla g = 2x\mathbf{i}+2y\mathbf{j}-\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2+4y^2+1} = \sqrt{4(x^2+y^2)+1}$

$\Rightarrow \mathbf{n} = \frac{2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}}{\sqrt{4(x^2+y^2)+1}} \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{8x^2+8y^2-2}{\sqrt{4(x^2+y^2)+1}}$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla g\cdot\mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{4(x^2+y^2)+1}\,dA$

$\Rightarrow$ Flux $= \iint\limits_R \left(\frac{8x^2+8y^2-2}{\sqrt{4(x^2+y^2)+1}}\right)\sqrt{4(x^2+y^2)+1}\,dA = \iint\limits_R (8x^2+8y^2-2)\,dA$; $z = 1$ and $x^2+y^2 = z$

$\Rightarrow x^2+y^2 = 1 \Rightarrow$ Flux $= \int_0^{2\pi}\int_0^1 (8r^2-2)\,r\,dr\,d\theta = 2\pi$

29. $g(x,y,z) = y-e^x = 0 \Rightarrow \nabla g = -e^x\mathbf{i}+\mathbf{j} \Rightarrow |\nabla g| = \sqrt{e^{2x}+1} \Rightarrow \mathbf{n} = \frac{e^x\mathbf{i}-\mathbf{j}}{\sqrt{e^{2x}+1}} \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{-2e^x-2y}{\sqrt{e^{2x}+1}}$; $\mathbf{p} = \mathbf{i}$

$\Rightarrow |\nabla g\cdot\mathbf{p}| = e^x \Rightarrow d\sigma = \frac{\sqrt{e^{2x}+1}}{e^x}\,dA \Rightarrow$ Flux $= \iint\limits_R \left(\frac{-2e^x-2y}{\sqrt{e^{2x}+1}}\right)\left(\frac{\sqrt{e^{2x}+1}}{e^x}\right)dA = \iint\limits_R \frac{-2e^x-2e^x}{e^x}\,dA$

$= \iint\limits_R -4\,dA = \int_0^1\int_1^2 -4\,dy\,dz = -4$

30. $g(x,y,z) = y-\ln x = 0 \Rightarrow \nabla g = -\frac{1}{x}\mathbf{i}+\mathbf{j} \Rightarrow |\nabla g| = \sqrt{\frac{1}{x^2}+1} = \frac{\sqrt{1+x^2}}{x}$ since $1 \le x \le e$

$\Rightarrow \mathbf{n} = \frac{\left(-\frac{1}{x}\mathbf{i}+\mathbf{j}\right)}{\left(\frac{\sqrt{1+x^2}}{x}\right)} = \frac{-\mathbf{i}+x\mathbf{j}}{\sqrt{1+x^2}} \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{2xy}{\sqrt{1+x^2}}$; $\mathbf{p} = \mathbf{j} \Rightarrow |\nabla g\cdot\mathbf{p}| = 1 \Rightarrow d\sigma = \frac{\sqrt{1+x^2}}{x}\,dA$

$\Rightarrow$ Flux $= \iint\limits_R \left(\frac{2xy}{\sqrt{1+x^2}}\right)\left(\frac{\sqrt{1+x^2}}{x}\right)dA = \int_0^1\int_1^e 2y\,dx\,dz = \int_1^e\int_0^1 2\ln x\,dz\,dx = \int_1^e 2\ln x\,dx$

$= 2\left[x\ln x - x\right]_1^e = 2(e-e)-2(0-1) = 2$

31. On the face $z = a$: $g(x,y,z) = z \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1$; $\mathbf{n} = \mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n} = 2xz = 2ax$ since $z = a$;

$d\sigma = dx\,dy \Rightarrow$ Flux $= \iint\limits_R 2ax\,dx\,dy = \int_0^a\int_0^a 2ax\,dx\,dy = a^4$.

On the face $z = 0$: $g(x,y,z) = z \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1$; $\mathbf{n} = -\mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n} = -2xz = 0$ since $z = 0$;

$d\sigma = dx\,dy \Rightarrow$ Flux $= \iint\limits_R 0\,dx\,dy = 0$.

On the face $x = a$: $g(x,y,z) = x \Rightarrow \nabla g = \mathbf{i} \Rightarrow |\nabla g| = 1$; $\mathbf{n} = \mathbf{i} \Rightarrow \mathbf{F}\cdot\mathbf{n} = 2xy = 2ay$ since $x = a$;

$d\sigma = dy\,dz \Rightarrow$ Flux $= \int_0^a\int_0^a 2ay\,dy\,dz = a^4$.

On the face $x = 0$: $g(x,y,z) = x \Rightarrow \nabla g = \mathbf{i} \Rightarrow |\nabla g| = 1$; $\mathbf{n} = -\mathbf{i} \Rightarrow \mathbf{F}\cdot\mathbf{n} = -2xy = 0$ since $x = 0$

$\Rightarrow$ Flux $= 0$.

On the face $y = a$: $g(x,y,z) = y \Rightarrow \nabla g = \mathbf{j} \Rightarrow |\nabla g| = 1$; $\mathbf{n} = \mathbf{j} \Rightarrow \mathbf{F}\cdot\mathbf{n} = 2yz = 2az$ since $y = a$;

$d\sigma = dz\,dx \Rightarrow$ Flux $= \int_0^a\int_0^a 2az\,dz\,dx = a^4$.

On the face $y = 0$: $g(x,y,z) = y \Rightarrow \nabla g = \mathbf{j} \Rightarrow |\nabla g| = 1$; $\mathbf{n} = -\mathbf{j} \Rightarrow \mathbf{F}\cdot\mathbf{n} = -2yz = 0$ since $y = 0$

$\Rightarrow$ Flux $= 0$. Therefore, Total Flux $= 3a^4$.

32. Across the cap: $g(x, y, z) = x^2 + y^2 + z^2 = 25 \Rightarrow \nabla g = 2x\,\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2 + 4y^2 + 4z^2} = 10$
$\Rightarrow \mathbf{n} = \frac{\nabla g}{|\nabla g|} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{5} \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{x^2 z}{5} + \frac{y^2 z}{5} + \frac{z}{5}$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla g \cdot \mathbf{p}| = 2z$ since $z \ge 0 \Rightarrow d\sigma = \frac{10}{2z}\,dA$
$\Rightarrow \text{Flux}_{\text{cap}} = \iint\limits_{\text{cap}} \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint\limits_{R} \left(\frac{x^2 z}{5} + \frac{y^2 z}{5} + \frac{z}{5}\right)\left(\frac{5}{z}\right) dA = \iint\limits_{R} (x^2 + y^2 + 1)\,dx\,dy = \int_0^{2\pi}\int_0^4 (r^2 + 1)\,r\,dr\,d\theta$
$= \int_0^{2\pi} 72\,d\theta = 144\pi.$
Across the bottom: $g(x, y, z) = z = 3 \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1 \Rightarrow \mathbf{n} = -\mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n} = -1$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla g \cdot \mathbf{p}| = 1$
$\Rightarrow d\sigma = dA \Rightarrow \text{Flux}_{\text{bottom}} = \iint\limits_{\text{bottom}} \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint\limits_{R} -1\,dA = -1(\text{Area of the circular region}) = -16\pi$. Therefore,
$\text{Flux} = \text{Flux}_{\text{cap}} + \text{Flux}_{\text{bottom}} = 128\pi$

33. $\nabla f = 2x\,\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z$ since $z \ge 0 \Rightarrow d\sigma = \frac{2a}{2z}\,dA$
$= \frac{a}{z}\,dA$; $M = \iint\limits_{S} \delta\,d\sigma = \frac{\delta}{8}$ (surface area of sphere) $= \frac{\delta\pi a^2}{2}$; $M_{xy} = \iint\limits_{S} z\delta\,d\sigma = \delta\iint\limits_{R} z\left(\frac{a}{z}\right) dA$
$= a\delta\iint\limits_{R} dA = a\delta\int_0^{\pi/2}\int_0^a r\,dr\,d\theta = \frac{\delta\pi a^3}{4} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{\delta\pi a^3}{4}\right)\left(\frac{2}{\delta\pi a^2}\right) = \frac{a}{2}$. Because of symmetry, $\bar{x} = \bar{y}$
$= \frac{a}{2} \Rightarrow$ the centroid is $\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$.

34. $\nabla f = 2y\,\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 4z^2} = \sqrt{4(y^2 + z^2)} = 6$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{k}| = 2z$ since $z \ge 0 \Rightarrow d\sigma = \frac{6}{2z}\,dA$
$= \frac{3}{z}\,dA$; $M = \iint\limits_{S} 1\,d\sigma = \int_{-3}^{3}\int_0^3 \frac{3}{z}\,dx\,dy = \int_{-3}^{3}\int_0^3 \frac{3}{\sqrt{9 - y^2}}\,dx\,dy = 9\pi$; $M_{xy} = \iint\limits_{S} z\,d\sigma$
$= \int_{-3}^{3}\int_0^3 z\left(\frac{3}{z}\right) dx\,dy = 54$; $M_{xz} = \iint\limits_{S} y\,d\sigma = \int_{-3}^{3}\int_0^3 y\left(\frac{3}{z}\right) dx\,dy = \int_{-3}^{3}\int_0^3 \frac{3y}{\sqrt{9 - y^2}}\,dx\,dy = 0$;
$M_{yz} = \iint\limits_{S} x\,d\sigma = \int_{-3}^{3}\int_0^3 \frac{3x}{\sqrt{9 - y^2}}\,dx\,dy = \frac{27}{2}\pi$. Therefore, $\bar{x} = \frac{\left(\frac{27}{2}\pi\right)}{9\pi} = \frac{3}{2}$, $\bar{y} = 0$, and $\bar{z} = \frac{54}{9\pi} = \frac{6}{\pi}$

35. Because of symmetry, $\bar{x} = \bar{y} = 0$; $M = \iint\limits_{S} \delta\,d\sigma = \delta\iint\limits_{S} d\sigma = (\text{Area of S})\delta = 3\pi\sqrt{2}\,\delta$; $\nabla f = 2x\,\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}$
$\Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2}$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \Rightarrow d\sigma = \frac{2\sqrt{x^2 + y^2 + z^2}}{2z}\,dA$
$= \frac{\sqrt{x^2 + y^2 + (x^2 + y^2)}}{z}\,dA = \frac{\sqrt{2}\sqrt{x^2 + y^2}}{z}\,dA \Rightarrow M_{xy} = \delta\iint\limits_{R} z\left(\frac{\sqrt{2}\sqrt{x^2 + y^2}}{z}\right) dA$
$= \delta\iint\limits_{R} \sqrt{2}\sqrt{x^2 + y^2}\,dA = \delta\int_0^{2\pi}\int_1^2 \sqrt{2}\,r^2\,dr\,d\theta = \frac{14\pi\sqrt{2}}{3}\delta \Rightarrow \bar{z} = \frac{\left(\frac{14\pi\sqrt{2}}{3}\delta\right)}{3\pi\sqrt{2}\,\delta} = \frac{14}{9}$
$\Rightarrow (\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{14}{9}\right)$. Next, $I_z = \iint\limits_{S} (x^2 + y^2)\,\delta\,d\sigma = \iint\limits_{R} (x^2 + y^2)\left(\frac{\sqrt{2}\sqrt{x^2 + y^2}}{z}\right)\delta\,dA$
$= \delta\sqrt{2}\iint\limits_{R} (x^2 + y^2)\,dA = \delta\sqrt{2}\int_0^{2\pi}\int_1^2 r^3\,dr\,d\theta = \frac{15\pi\sqrt{2}}{2}\delta \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \frac{\sqrt{10}}{2}$

36. $f(x, y, z) = 4x^2 + 4y^2 - z^2 = 0 \Rightarrow \nabla f = 8x\mathbf{i} + 8y\mathbf{j} - 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{64x^2 + 64y^2 + 4z^2}$
$= 2\sqrt{16x^2 + 16y^2 + z^2} = 2\sqrt{4z^2 + z^2} = 2\sqrt{5}\,z$ since $z \ge 0$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \Rightarrow d\sigma = \frac{2\sqrt{5}\,z}{2z}\,dA = \sqrt{5}\,dA$
$\Rightarrow I_z = \iint\limits_{S} (x^2 + y^2)\,\delta\,d\sigma = \delta\sqrt{5}\iint\limits_{R} (x^2 + y^2)\,dx\,dy = \delta\sqrt{5}\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3\,dr\,d\theta = \frac{3\sqrt{5}\pi\delta}{2}$

37. (a) Let the diameter lie on the z-axis and let $f(x, y, z) = x^2 + y^2 + z^2 = a^2$, $z \ge 0$ be the upper hemisphere
$\Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a$, $a > 0$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z$ since $z \ge 0$
$\Rightarrow d\sigma = \frac{a}{z}\,dA \Rightarrow I_z = \iint\limits_{S} \delta(x^2 + y^2)\left(\frac{a}{z}\right) d\sigma = a\delta\iint\limits_{R} \frac{x^2 + y^2}{\sqrt{a^2 - (x^2 + y^2)}}\,dA = a\delta\int_0^{2\pi}\int_0^a \frac{r^2}{\sqrt{a^2 - r^2}}\,r\,dr\,d\theta$
$= a\delta\int_0^{2\pi}\left[-r^2\sqrt{a^2 - r^2} - \frac{2}{3}(a^2 - r^2)^{3/2}\right]_0^a d\theta = a\delta\int_0^{2\pi} \frac{2}{3}a^3\,d\theta = \frac{4\pi}{3}a^4\delta \Rightarrow$ the moment of inertia is $\frac{8\pi}{3}a^4\delta$ for

the whole sphere

(b) $I_L = I_{c.m.} + mh^2$, where m is the mass of the body and h is the distance between the parallel lines; now, $I_{c.m.} = \frac{8\pi}{3} a^4\delta$ (from part a) and $\frac{m}{2} = \iint\limits_S \delta\, d\sigma = \delta \iint\limits_R \left(\frac{a}{z}\right) dA = a\delta \iint\limits_R \frac{1}{\sqrt{a^2-(x^2+y^2)}}\, dy\, dx$
$= a\delta \int_0^{2\pi}\int_0^a \frac{1}{\sqrt{a^2-r^2}}\, r\, dr\, d\theta = a\delta \int_0^{2\pi} \left[-\sqrt{a^2-r^2}\right]_0^a d\theta = a\delta \int_0^{2\pi} a\, d\theta = 2\pi a^2\delta$ and $h = a$
$\Rightarrow\ I_L = \frac{8\pi}{3} a^4\delta + 4\pi a^2\delta a^2 = \frac{20\pi}{3} a^4\delta$

38. (a) Let $z = \frac{h}{a}\sqrt{x^2+y^2}$ be the cone from $z = 0$ to $z = h$, $h > 0$. Because of symmetry, $\overline{x} = 0$ and $\overline{y} = 0$; $z = \frac{h}{a}\sqrt{x^2+y^2} \Rightarrow f(x,y,z) = \frac{h^2}{a^2}(x^2+y^2) - z^2 = 0 \Rightarrow \nabla f = \frac{2xh^2}{a^2}\mathbf{i} + \frac{2yh^2}{a^2}\mathbf{j} - 2z\mathbf{k}$
$\Rightarrow |\nabla f| = \sqrt{\frac{4x^2h^4}{a^4} + \frac{4y^2h^4}{a^4} + 4z^2} = 2\sqrt{\frac{h^4}{a^4}(x^2+y^2) + \frac{h^2}{a^2}(x^2+y^2)} = 2\sqrt{\left(\frac{h^2}{a^2}\right)(x^2+y^2)\left(\frac{h^2}{a^2}+1\right)}$
$= 2\sqrt{z^2\left(\frac{h^2+a^2}{a^2}\right)} = \left(\frac{2z}{a}\right)\sqrt{h^2+a^2}$ since $z \ge 0$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \Rightarrow d\sigma = \frac{\left(\frac{2z}{a}\right)\sqrt{h^2+a^2}}{2z}\, dA$
$= \frac{\sqrt{h^2+a^2}}{a}\, dA$; $M = \iint\limits_S d\sigma = \iint\limits_R \frac{\sqrt{h^2+a^2}}{a}\, dA = \frac{\sqrt{h^2+a^2}}{a}(\pi a^2) = \pi a\sqrt{h^2+a^2}$;
$M_{xy} = \iint\limits_S z\, d\sigma = \iint\limits_R z\left(\frac{\sqrt{h^2+a^2}}{a}\right) dA = \frac{\sqrt{h^2+a^2}}{a}\iint\limits_R \frac{h}{a}\sqrt{x^2+y^2}\, dx\, dy = \frac{h\sqrt{h^2+a^2}}{a^2}\int_0^{2\pi}\int_0^a r^2\, dr\, d\theta$
$= \frac{2\pi ah\sqrt{h^2+a^2}}{3} \Rightarrow \overline{z} = \frac{M_{xy}}{M} = \frac{2h}{3} \Rightarrow$ the centroid is $\left(0, 0, \frac{2h}{3}\right)$

(b) The base is a circle of radius a and center at $(0,0,h) \Rightarrow (0,0,h)$ is the centroid of the base and the mass is $M = \iint\limits_S d\sigma = \pi a^2$. In Pappus' formula, let $\mathbf{c}_1 = \frac{2h}{3}\mathbf{k}$, $\mathbf{c}_2 = h\mathbf{k}$, $m_1 = \pi a\sqrt{h^2+a^2}$, and $m_2 = \pi a^2$
$\Rightarrow \mathbf{c} = \frac{\pi a\sqrt{h^2+a^2}\left(\frac{2h}{3}\right)\mathbf{k} + \pi a^2 h\mathbf{k}}{\pi a\sqrt{h^2+a^2} + \pi a^2} = \frac{2h\sqrt{h^2+a^2}+3ah}{3\left(\sqrt{h^2+a^2}+a\right)}\mathbf{k} \Rightarrow$ the centroid is $\left(0, 0, \frac{2h\sqrt{h^2+a^2}+3ah}{3\left(\sqrt{h^2+a^2}+a\right)}\right)$

(c) If the hemisphere is sitting so its base is in the plane $z = h$, then its centroid is $\left(0, 0, h + \frac{a}{2}\right)$ and its mass is $2\pi a^2$. In Pappus' formula, let $\mathbf{c}_1 = \frac{2h}{3}\mathbf{k}$, $\mathbf{c}_2 = \left(h + \frac{a}{2}\right)\mathbf{k}$, $m_1 = \pi a\sqrt{h^2+a^2}$, and $m_2 = 2\pi a^2$
$\Rightarrow \mathbf{c} = \frac{\pi a\sqrt{h^2+a^2}\left(\frac{2h}{3}\right)\mathbf{k} + 2\pi a^2\left(h+\frac{a}{2}\right)\mathbf{k}}{\pi a\sqrt{h^2+a^2}+2\pi a^2} = \frac{2h\sqrt{h^2+a^2}+6ah+3a^2}{3\left(\sqrt{h^2+a^2}+2a\right)}\mathbf{k} \Rightarrow$ the centroid is
$\left(0, 0, \frac{2h\sqrt{h^2+a^2}+6ah+3a^2}{3\left(\sqrt{h^2+a^2}+2a\right)}\right)$. Thus, for the centroid to be in the plane of the bases we must have $z = h$
$\Rightarrow \frac{2h\sqrt{h^2+a^2}+6ah+3a^2}{3\left(\sqrt{h^2+a^2}+2a\right)} = h \Rightarrow 2h\sqrt{h^2+a^2} + 6ah + 3a^2 = 3h\sqrt{h^2+a^2} + 6ah \Rightarrow 3a^2 = h\sqrt{h^2+a^2}$
$\Rightarrow 9a^4 = h^2(h^2+a^2) \Rightarrow h^4 + a^2h^2 - 9a^4 = 0 \Rightarrow h^2 = \frac{\left(\sqrt{37}-1\right)a^2}{2}$ (the positive root) $\Rightarrow h = \frac{\sqrt{2\sqrt{37}-2}}{2}\, a$

39. $f_x(x,y) = 2x$, $f_y(x,y) = 2y \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{4x^2+4y^2+1} \Rightarrow \text{Area} = \iint\limits_R \sqrt{4x^2+4y^2+1}\, dx\, dy$
$= \int_0^{2\pi}\int_0^{\sqrt{3}} \sqrt{4r^2+1}\, r\, dr\, d\theta = \frac{\pi}{6}\left(13\sqrt{13}-1\right)$

40. $f_y(y,z) = -2y$, $f_z(y,z) = -2z \Rightarrow \sqrt{f_y^2 + f_z^2 + 1} = \sqrt{4y^2+4z^2+1} \Rightarrow \text{Area} = \iint\limits_R \sqrt{4y^2+4z^2+1}\, dy\, dz$
$= \int_0^{2\pi}\int_0^1 \sqrt{4r^2+1}\, r\, dr\, d\theta = \frac{\pi}{6}\left(5\sqrt{5}-1\right)$

41. $f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}}$, $f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}} \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1} = \sqrt{2}$
$\Rightarrow \text{Area} = \iint\limits_{R_{xy}} \sqrt{2}\, dx\, dy = \sqrt{2}$(Area between the ellipse and the circle) $= \sqrt{2}(6\pi - \pi) = 5\pi\sqrt{2}$

42. Over R_{xy}: $z = 2 - \frac{2}{3}x - 2y \Rightarrow f_x(x,y) = -\frac{2}{3}$, $f_y(x,y) = -2 \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{\frac{4}{9} + 4 + 1} = \frac{7}{3}$
$\Rightarrow$ Area $= \iint\limits_{R_{xy}} \frac{7}{3}\,dA = \frac{7}{3}$ (Area of the shadow triangle in the xy-plane) $= \left(\frac{7}{3}\right)\left(\frac{3}{2}\right) = \frac{7}{2}$.
Over R_{xz}: $y = 1 - \frac{1}{3}x - \frac{1}{2}z \Rightarrow f_x(x,z) = -\frac{1}{3}$, $f_z(x,z) = -\frac{1}{2} \Rightarrow \sqrt{f_x^2 + f_z^2 + 1} = \sqrt{\frac{1}{9} + \frac{1}{4} + 1} = \frac{7}{6}$
$\Rightarrow$ Area $= \iint\limits_{R_{xz}} \frac{7}{6}\,dA = \frac{7}{6}$ (Area of the shadow triangle in the xz-plane) $= \left(\frac{7}{6}\right)(3) = \frac{7}{2}$.
Over R_{yz}: $x = 3 - 3y - \frac{3}{2}z \Rightarrow f_y(y,z) = -3$, $f_z(y,z) = -\frac{3}{2} \Rightarrow \sqrt{f_y^2 + f_z^2 + 1} = \sqrt{9 + \frac{9}{4} + 1} = \frac{7}{2}$
$\Rightarrow$ Area $= \iint\limits_{R_{yz}} \frac{7}{2}\,dA = \frac{7}{2}$ (Area of the shadow triangle in the yz-plane) $= \left(\frac{7}{2}\right)(1) = \frac{7}{2}$.

43. $y = \frac{2}{3}z^{3/2} \Rightarrow f_x(x,z) = 0$, $f_z(x,z) = z^{1/2} \Rightarrow \sqrt{f_x^2 + f_z^2 + 1} = \sqrt{z+1}$; $y = \frac{16}{3} \Rightarrow \frac{16}{3} = \frac{2}{3}z^{3/2} \Rightarrow z = 4$
$\Rightarrow$ Area $= \int_0^4 \int_0^1 \sqrt{z+1}\,dx\,dz = \int_0^4 \sqrt{z+1}\,dz = \frac{2}{3}\left(5\sqrt{5} - 1\right)$

44. $y = 4 - z \Rightarrow f_x(x,z) = 0$, $f_z(x,z) = -1 \Rightarrow \sqrt{f_x^2 + f_z^2 + 1} = \sqrt{2} \Rightarrow$ Area $= \iint\limits_{R_{xz}} \sqrt{2}\,dA = \int_0^2 \int_0^{4-z^2} \sqrt{2}\,dx\,dz$
$= \sqrt{2}\int_0^2 (4 - z^2)\,dz = \frac{16\sqrt{2}}{3}$

16.6 PARAMETRIZED SURFACES

1. In cylindrical coordinates, let $x = r\cos\theta$, $y = r\sin\theta$, $z = \left(\sqrt{x^2+y^2}\right)^2 = r^2$. Then
$\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r^2\mathbf{k}$, $0 \le r \le 2$, $0 \le \theta \le 2\pi$.

2. In cylindrical coordinates, let $x = r\cos\theta$, $y = r\sin\theta$, $z = 9 - x^2 - y^2 = 9 - r^2$. Then
$\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (9 - r^2)\,\mathbf{k}$; $z \ge 0 \Rightarrow 9 - r^2 \ge 0 \Rightarrow r^2 \le 9 \Rightarrow -3 \le r \le 3$, $0 \le \theta \le 2\pi$. But
$-3 \le r \le 0$ gives the same points as $0 \le r \le 3$, so let $0 \le r \le 3$.

3. In cylindrical coordinates, let $x = r\cos\theta$, $y = r\sin\theta$, $z = \frac{\sqrt{x^2+y^2}}{2} \Rightarrow z = \frac{r}{2}$. Then
$\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(\frac{r}{2}\right)\mathbf{k}$. For $0 \le z \le 3$, $0 \le \frac{r}{2} \le 3 \Rightarrow 0 \le r \le 6$; to get only the first octant, let
$0 \le \theta \le \frac{\pi}{2}$.

4. In cylindrical coordinates, let $x = r\cos\theta$, $y = r\sin\theta$, $z = 2\sqrt{x^2+y^2} \Rightarrow z = 2r$. Then
$\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + 2r\mathbf{k}$. For $2 \le z \le 4$, $2 \le 2r \le 4 \Rightarrow 1 \le r \le 2$, and let $0 \le \theta \le 2\pi$.

5. In cylindrical coordinates, let $x = r\cos\theta$, $y = r\sin\theta$ since $x^2 + y^2 = r^2 \Rightarrow z^2 = 9 - (x^2+y^2) = 9 - r^2$
$\Rightarrow z = \sqrt{9 - r^2}$, $z \ge 0$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \sqrt{9 - r^2}\mathbf{k}$. Let $0 \le \theta \le 2\pi$. For the domain
of r: $z = \sqrt{x^2+y^2}$ and $x^2 + y^2 + z^2 = 9 \Rightarrow x^2 + y^2 + \left(\sqrt{x^2+y^2}\right)^2 = 9 \Rightarrow 2(x^2+y^2) = 9 \Rightarrow 2r^2 = 9$
$\Rightarrow r = \frac{3}{\sqrt{2}} \Rightarrow 0 \le r \le \frac{3}{\sqrt{2}}$.

6. In cylindrical coordinates, $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \sqrt{4 - r^2}\,\mathbf{k}$ (see Exercise 5 above with $x^2 + y^2 + z^2 = 4$,
instead of $x^2 + y^2 + z^2 = 9$). For the first octant, let $0 \le \theta \le \frac{\pi}{2}$. For the domain of r: $z = \sqrt{x^2+y^2}$ and
$x^2 + y^2 + z^2 = 4 \Rightarrow x^2 + y^2 + \left(\sqrt{x^2+y^2}\right)^2 = 4 \Rightarrow 2(x^2+y^2) = 4 \Rightarrow 2r^2 = 4 \Rightarrow r = \sqrt{2}$. Thus, let $\sqrt{2} \le r \le 2$
(to get the portion of the sphere between the cone and the xy-plane).

7. In spherical coordinates, $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $\rho = \sqrt{x^2+y^2+z^2} \Rightarrow \rho^2 = 3 \Rightarrow \rho = \sqrt{3}$ $\Rightarrow z = \sqrt{3}\cos\phi$ for the sphere; $z = \frac{\sqrt{3}}{2} = \sqrt{3}\cos\phi \Rightarrow \cos\phi = \frac{1}{2} \Rightarrow \phi = \frac{\pi}{3}$; $z = -\frac{\sqrt{3}}{2} \Rightarrow -\frac{\sqrt{3}}{2} = \sqrt{3}\cos\phi$ $\Rightarrow \cos\phi = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$. Then $\mathbf{r}(\phi,\theta) = \left(\sqrt{3}\sin\phi\cos\theta\right)\mathbf{i} + \left(\sqrt{3}\sin\phi\sin\theta\right)\mathbf{j} + \left(\sqrt{3}\cos\phi\right)\mathbf{k}$, $\frac{\pi}{3} \le \phi \le \frac{2\pi}{3}$ and $0 \le \theta \le 2\pi$.

8. In spherical coordinates, $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $\rho = \sqrt{x^2+y^2+z^2} \Rightarrow \rho^2 = 8 \Rightarrow \rho = \sqrt{8} = 2\sqrt{2}$ $\Rightarrow x = 2\sqrt{2}\sin\phi\cos\theta$, $y = 2\sqrt{2}\sin\phi\sin\theta$, and $z = 2\sqrt{2}\cos\phi$. Thus let $\mathbf{r}(\phi,\theta) = \left(2\sqrt{2}\sin\phi\cos\theta\right)\mathbf{i} + \left(2\sqrt{2}\sin\phi\sin\theta\right)\mathbf{j} + \left(2\sqrt{2}\cos\phi\right)\mathbf{k}$; $z = -2 \Rightarrow -2 = 2\sqrt{2}\cos\phi$ $\Rightarrow \cos\phi = -\frac{1}{\sqrt{2}} \Rightarrow \phi = \frac{3\pi}{4}$; $z = 2\sqrt{2} \Rightarrow 2\sqrt{2} = 2\sqrt{2}\cos\phi \Rightarrow \cos\phi = 1 \Rightarrow \phi = 0$. Thus $0 \le \phi \le \frac{3\pi}{4}$ and $0 \le \theta \le 2\pi$.

9. Since $z = 4 - y^2$, we can let $\mathbf{r}$ be a function of x and y $\Rightarrow \mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + (4-y^2)\mathbf{k}$. Then $z = 0$ $\Rightarrow 0 = 4 - y^2 \Rightarrow y = \pm 2$. Thus, let $-2 \le y \le 2$ and $0 \le x \le 2$.

10. Since $y = x^2$, we can let $\mathbf{r}$ be a function of x and z $\Rightarrow \mathbf{r}(x,z) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$. Then $y = 2$ $\Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$. Thus, let $-\sqrt{2} \le x \le \sqrt{2}$ and $0 \le z \le 3$.

11. When $x = 0$, let $y^2 + z^2 = 9$ be the circular section in the yz-plane. Use polar coordinates in the yz-plane $\Rightarrow y = 3\cos\theta$ and $z = 3\sin\theta$. Thus let $x = u$ and $\theta = v \Rightarrow \mathbf{r}(u,v) = u\mathbf{i} + (3\cos v)\mathbf{j} + (3\sin v)\mathbf{k}$ where $0 \le u \le 3$, and $0 \le v \le 2\pi$.

12. When $y = 0$, let $x^2 + z^2 = 4$ be the circular section in the xz-plane. Use polar coordinates in the xz-plane $\Rightarrow x = 2\cos\theta$ and $z = 2\sin\theta$. Thus let $y = u$ and $\theta = v \Rightarrow \mathbf{r}(u,v) = (2\cos v)\mathbf{i} + u\mathbf{j} + (3\sin v)\mathbf{k}$ where $-2 \le u \le 2$, and $0 \le v \le \pi$ (since we want the portion above the xy-plane).

13. (a) $x + y + z = 1 \Rightarrow z = 1 - x - y$. In cylindrical coordinates, let $x = r\cos\theta$ and $y = r\sin\theta$ $\Rightarrow z = 1 - r\cos\theta - r\sin\theta \Rightarrow \mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (1 - r\cos\theta - r\sin\theta)\mathbf{k}$, $0 \le \theta \le 2\pi$ and $0 \le r \le 3$.

(b) In a fashion similar to cylindrical coordinates, but working in the yz-plane instead of the xy-plane, let $y = u\cos v$, $z = u\sin v$ where $u = \sqrt{y^2+z^2}$ and v is the angle formed by (x,y,z), $(x,0,0)$, and $(x,y,0)$ with $(x,0,0)$ as vertex. Since $x + y + z = 1 \Rightarrow x = 1 - y - z \Rightarrow x = 1 - u\cos v - u\sin v$, then $\mathbf{r}$ is a function of u and v $\Rightarrow \mathbf{r}(u,v) = (1 - u\cos v - u\sin v)\mathbf{i} + (u\cos v)\mathbf{j} + (u\sin v)\mathbf{k}$, $0 \le u \le 3$ and $0 \le v \le 2\pi$.

14. (a) In a fashion similar to cylindrical coordinates, but working in the xz-plane instead of the xy-plane, let $x = u\cos v$, $z = u\sin v$ where $u = \sqrt{x^2+z^2}$ and v is the angle formed by (x,y,z), $(y,0,0)$, and $(x,y,0)$ with vertex $(y,0,0)$. Since $x - y + 2z = 2 \Rightarrow y = x + 2z - 2$, then $\mathbf{r}(u,v)$ $= (u\cos v)\mathbf{i} + (u\cos v + 2u\sin v - 2)\mathbf{j} + (u\sin v)\mathbf{k}$, $0 \le u \le \sqrt{3}$ and $0 \le v \le 2\pi$.

(b) In a fashion similar to cylindrical coordinates, but working in the yz-plane instead of the xy-plane, let $y = u\cos v$, $z = u\sin v$ where $u = \sqrt{y^2+z^2}$ and v is the angle formed by (x,y,z), $(x,0,0)$, and $(x,y,0)$ with vertex $(x,0,0)$. Since $x - y + 2z = 2 \Rightarrow x = y - 2z + 2$, then $\mathbf{r}(u,v)$ $= (u\cos v - 2u\sin v + 2)\mathbf{i} + (u\cos v)\mathbf{j} + (u\sin v)\mathbf{k}$, $0 \le u \le \sqrt{2}$ and $0 \le v \le 2\pi$.

15. Let $x = w\cos v$ and $z = w\sin v$. Then $(x-2)^2 + z^2 = 4 \Rightarrow x^2 - 4x + z^2 = 0 \Rightarrow w^2\cos^2 v - 4w\cos v + w^2\sin^2 v$ $= 0 \Rightarrow w^2 - 4w\cos v = 0 \Rightarrow w = 0$ or $w - 4\cos v = 0 \Rightarrow w = 0$ or $w = 4\cos v$. Now $w = 0 \Rightarrow x = 0$ and $y = 0$, which is a line not a cylinder. Therefore, let $w = 4\cos v \Rightarrow x = (4\cos v)(\cos v) = 4\cos^2 v$ and $z = 4\cos v\sin v$. Finally, let $y = u$. Then $\mathbf{r}(u,v) = (4\cos^2 v)\,\mathbf{i} + u\mathbf{j} + (4\cos v\sin v)\mathbf{k}$, $-\frac{\pi}{2} \le v \le \frac{\pi}{2}$ and $0 \le u \le 3$.

16. Let $y = w\cos v$ and $z = w\sin v$. Then $y^2 + (z-5)^2 = 25 \Rightarrow y^2 + z^2 - 10z = 0$
$\Rightarrow w^2\cos^2 v + w^2\sin^2 v - 10w\sin v = 0 \Rightarrow w^2 - 10w\sin v = 0 \Rightarrow w(w - 10\sin v) = 0 \Rightarrow w = 0$ or $w = 10\sin v$. Now $w = 0 \Rightarrow y = 0$ and $z = 0$, which is a line not a cylinder. Therefore, let $w = 10\sin v$
$\Rightarrow y = 10\sin v\cos v$ and $z = 10\sin^2 v$. Finally, let $x = u$. Then $\mathbf{r}(u,v) = u\mathbf{i} + (10\sin v\cos v)\mathbf{j} + (10\sin^2 v)\,\mathbf{k}$, $0 \le u \le 10$ and $0 \le v \le \pi$.

17. Let $x = r\cos\theta$ and $y = r\sin\theta$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(\frac{2 - r\sin\theta}{2}\right)\mathbf{k}$, $0 \le r \le 1$ and $0 \le \theta \le 2\pi$
$\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - \left(\frac{\sin\theta}{2}\right)\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} - \left(\frac{r\cos\theta}{2}\right)\mathbf{k}$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -\frac{\sin\theta}{2} \\ -r\sin\theta & r\cos\theta & -\frac{r\cos\theta}{2} \end{vmatrix}$$

$$= \left(\frac{-r\sin\theta\cos\theta}{2} + \frac{(\sin\theta)(r\cos\theta)}{2}\right)\mathbf{i} + \left(\frac{r\sin^2\theta}{2} + \frac{r\cos^2\theta}{2}\right)\mathbf{j} + (r\cos^2\theta + r\sin^2\theta)\,\mathbf{k} = \frac{r}{2}\mathbf{j} + r\mathbf{k}$$

$$\Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{\frac{r^2}{4} + r^2} = \frac{\sqrt{5}\,r}{2} \Rightarrow A = \int_0^{2\pi}\int_0^1 \frac{\sqrt{5}\,r}{2}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{\sqrt{5}\,r^2}{4}\right]_0^1 d\theta = \int_0^{2\pi} d\theta = \frac{\pi\sqrt{5}}{2}$$

18. Let $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow z = -x = -r\cos\theta$, $0 \le r \le 2$ and $0 \le \theta \le 2\pi$. Then
$\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} - (r\cos\theta)\mathbf{k} \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - (\cos\theta)\mathbf{k}$ and
$\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} + (r\sin\theta)\mathbf{k}$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -\cos\theta \\ -r\sin\theta & r\cos\theta & r\sin\theta \end{vmatrix}$$

$$= (r\sin^2\theta + r\cos^2\theta)\,\mathbf{i} + (r\sin\theta\cos\theta - r\sin\theta\cos\theta)\mathbf{j} + (r\cos^2\theta + r\sin^2\theta)\,\mathbf{k} = r\mathbf{i} + r\mathbf{k}$$

$$\Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{r^2 + r^2} = r\sqrt{2} \Rightarrow A = \int_0^{2\pi}\int_0^2 r\sqrt{2}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2\sqrt{2}}{2}\right]_0^2 d\theta = \int_0^{2\pi} 2\sqrt{2}\,d\theta = 4\pi\sqrt{2}$$

19. Let $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow z = 2\sqrt{x^2 + y^2} = 2r$, $1 \le r \le 3$ and $0 \le \theta \le 2\pi$. Then
$\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + 2r\mathbf{k} \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 2 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-2r\cos\theta)\mathbf{i} - (2r\sin\theta)\mathbf{j} + (r\cos^2\theta + r\sin^2\theta)\,\mathbf{k}$$

$$= (-2r\cos\theta)\mathbf{i} - (2r\sin\theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + r^2} = \sqrt{5r^2} = r\sqrt{5}$$

$$\Rightarrow A = \int_0^{2\pi}\int_1^3 r\sqrt{5}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2\sqrt{5}}{2}\right]_1^3 d\theta = \int_0^{2\pi} 4\sqrt{5}\,d\theta = 8\pi\sqrt{5}$$

20. Let $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow z = \frac{\sqrt{x^2+y^2}}{3} = \frac{r}{3}$, $3 \le r \le 4$ and $0 \le \theta \le 2\pi$. Then
$\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(\frac{r}{3}\right)\mathbf{k} \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \left(\frac{1}{3}\right)\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & \frac{1}{3} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \left(-\frac{1}{3}r\cos\theta\right)\mathbf{i} - \left(\frac{1}{3}r\sin\theta\right)\mathbf{j} + (r\cos^2\theta + r\sin^2\theta)\,\mathbf{k}$$

$$= \left(-\frac{1}{3}r\cos\theta\right)\mathbf{i} - \left(\frac{1}{3}r\sin\theta\right)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{\frac{1}{9}r^2\cos^2\theta + \frac{1}{9}r^2\sin^2\theta + r^2} = \sqrt{\frac{10r^2}{9}} = \frac{r\sqrt{10}}{3}$$

$$\Rightarrow A = \int_0^{2\pi}\int_3^4 \frac{r\sqrt{10}}{3}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2\sqrt{10}}{6}\right]_3^4 d\theta = \int_0^{2\pi}\frac{7\sqrt{10}}{6}\,d\theta = \frac{7\pi\sqrt{10}}{3}$$

21. Let $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow r^2 = x^2 + y^2 = 1$, $1 \le z \le 4$ and $0 \le \theta \le 2\pi$. Then
$\mathbf{r}(z,\theta) = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + z\mathbf{k} \Rightarrow \mathbf{r}_z = \mathbf{k}$ and $\mathbf{r}_\theta = (-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta)\mathbf{i} + (\sin\theta)\,\mathbf{j} \Rightarrow |\mathbf{r}_\theta \times \mathbf{r}_z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$\Rightarrow\ A = \int_0^{2\pi}\int_1^4 1\, dr\, d\theta = \int_0^{2\pi} 3\, d\theta = 6\pi$

22. Let $x = u\cos v$ and $z = u\sin v \Rightarrow u^2 = x^2 + z^2 = 10$, $-1 \le y \le 1$, $0 \le v \le 2\pi$. Then
$\mathbf{r}(y, v) = (u\cos v)\mathbf{i} + y\mathbf{j} + (u\sin v)\mathbf{k} = \left(\sqrt{10}\cos v\right)\mathbf{i} + y\mathbf{j} + \left(\sqrt{10}\sin v\right)\mathbf{k}$
$\Rightarrow \mathbf{r}_v = \left(-\sqrt{10}\sin v\right)\mathbf{i} + \left(\sqrt{10}\cos v\right)\mathbf{k}$ and $\mathbf{r}_y = \mathbf{j} \Rightarrow \mathbf{r}_v \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sqrt{10}\sin v & 0 & \sqrt{10}\cos v \\ 0 & 1 & 0 \end{vmatrix}$
$= \left(-\sqrt{10}\cos v\right)\mathbf{i} - \left(\sqrt{10}\sin v\right)\mathbf{k} \Rightarrow |\mathbf{r}_v \times \mathbf{r}_y| = \sqrt{10} \Rightarrow A = \int_0^{2\pi}\int_{-1}^{1}\sqrt{10}\, du\, dv = \int_0^{2\pi}\left[\sqrt{10}u\right]_{-1}^{1} dv$
$= \int_0^{2\pi} 2\sqrt{10}\, dv = 4\pi\sqrt{10}$

23. $z = 2 - x^2 - y^2$ and $z = \sqrt{x^2 + y^2} \Rightarrow z = 2 - z^2 \Rightarrow z^2 + z - 2 = 0 \Rightarrow z = -2$ or $z = 1$. Since $z = \sqrt{x^2 + y^2} \ge 0$, we get $z = 1$ where the cone intersects the paraboloid. When $x = 0$ and $y = 0$, $z = 2 \Rightarrow$ the vertex of the paraboloid is $(0, 0, 2)$. Therefore, z ranges from 1 to 2 on the "cap" $\Rightarrow$ r ranges from 1 (when $x^2 + y^2 = 1$) to 0 (when $x = 0$ and $y = 0$ at the vertex). Let $x = r\cos\theta$, $y = r\sin\theta$, and $z = 2 - r^2$. Then
$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (2 - r^2)\mathbf{k}$, $0 \le r \le 1$, $0 \le \theta \le 2\pi \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - 2r\mathbf{k}$ and
$\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$
$= (2r^2\cos\theta)\mathbf{i} + (2r^2\sin\theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2} = r\sqrt{4r^2 + 1}$
$\Rightarrow A = \int_0^{2\pi}\int_0^1 r\sqrt{4r^2 + 1}\, dr\, d\theta = \int_0^{2\pi}\left[\frac{1}{12}(4r^2 + 1)^{3/2}\right]_0^1 d\theta = \int_0^{2\pi}\left(\frac{5\sqrt{5} - 1}{12}\right) d\theta = \frac{\pi}{6}\left(5\sqrt{5} - 1\right)$

24. Let $x = r\cos\theta$, $y = r\sin\theta$ and $z = x^2 + y^2 = r^2$. Then $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r^2\mathbf{k}$, $1 \le r \le 2$, $0 \le \theta \le 2\pi \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2r\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}$
$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-2r^2\cos\theta)\mathbf{i} - (2r^2\sin\theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta|$
$= \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2} = r\sqrt{4r^2 + 1} \Rightarrow A = \int_0^{2\pi}\int_1^2 r\sqrt{4r^2 + 1}\, dr\, d\theta = \int_0^{2\pi}\left[\frac{1}{12}(4r^2 + 1)^{3/2}\right]_1^2 d\theta$
$= \int_0^{2\pi}\left(\frac{17\sqrt{17} - 5\sqrt{5}}{12}\right) d\theta = \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$

25. Let $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi \Rightarrow \rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{2}$ on the sphere. Next, $x^2 + y^2 + z^2 = 2$ and $z = \sqrt{x^2 + y^2} \Rightarrow z^2 + z^2 = 2 \Rightarrow z^2 = 1 \Rightarrow z = 1$ since $z \ge 0 \Rightarrow \phi = \frac{\pi}{4}$. For the lower portion of the sphere cut by the cone, we get $\phi = \pi$. Then
$\mathbf{r}(\phi, \theta) = \left(\sqrt{2}\sin\phi\cos\theta\right)\mathbf{i} + \left(\sqrt{2}\sin\phi\sin\theta\right)\mathbf{j} + \left(\sqrt{2}\cos\phi\right)\mathbf{k}$, $\frac{\pi}{4} \le \phi \le \pi$, $0 \le \theta \le 2\pi$
$\Rightarrow \mathbf{r}_\phi = \left(\sqrt{2}\cos\phi\cos\theta\right)\mathbf{i} + \left(\sqrt{2}\cos\phi\sin\theta\right)\mathbf{j} - \left(\sqrt{2}\sin\phi\right)\mathbf{k}$ and $\mathbf{r}_\theta = \left(-\sqrt{2}\sin\phi\sin\theta\right)\mathbf{i} + \left(\sqrt{2}\sin\phi\cos\theta\right)\mathbf{j}$
$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{2}\cos\phi\cos\theta & \sqrt{2}\cos\phi\sin\theta & -\sqrt{2}\sin\phi \\ -\sqrt{2}\sin\phi\sin\theta & \sqrt{2}\sin\phi\cos\theta & 0 \end{vmatrix}$
$= (2\sin^2\phi\cos\theta)\mathbf{i} + (2\sin^2\phi\sin\theta)\mathbf{j} + (2\sin\phi\cos\phi)\mathbf{k}$
$\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{4\sin^4\phi\cos^2\theta + 4\sin^4\phi\sin^2\theta + 4\sin^2\phi\cos^2\phi} = \sqrt{4\sin^2\phi} = 2|\sin\phi| = 2\sin\phi$
$\Rightarrow A = \int_0^{2\pi}\int_{\pi/4}^{\pi} 2\sin\phi\, d\phi\, d\theta = \int_0^{2\pi}\left(2 + \sqrt{2}\right) d\theta = \left(4 + 2\sqrt{2}\right)\pi$

26. Let $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi \Rightarrow \rho = \sqrt{x^2 + y^2 + z^2} = 2$ on the sphere. Next, $z = -1 \Rightarrow -1 = 2\cos\phi \Rightarrow \cos\phi = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$; $z = \sqrt{3} \Rightarrow \sqrt{3} = 2\cos\phi \Rightarrow \cos\phi = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6}$. Then

$\mathbf{r}(\phi,\theta) = (2\sin\phi\cos\theta)\mathbf{i} + (2\sin\phi\sin\theta)\mathbf{j} + (2\cos\phi)\mathbf{k}$, $\frac{\pi}{6} \le \phi \le \frac{2\pi}{3}$, $0 \le \theta \le 2\pi$

$\Rightarrow\ \mathbf{r}_\phi = (2\cos\phi\cos\theta)\mathbf{i} + (2\cos\phi\sin\theta)\mathbf{j} - (2\sin\phi)\mathbf{k}$ and

$\mathbf{r}_\theta = (-2\sin\phi\sin\theta)\mathbf{i} + (2\sin\phi\cos\theta)\,\mathbf{j}$

$$\Rightarrow\ \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix}$$

$= (4\sin^2\phi\cos\theta)\,\mathbf{i} + (4\sin^2\phi\sin\theta)\,\mathbf{j} + (4\sin\phi\cos\phi)\mathbf{k}$

$\Rightarrow\ |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{16\sin^4\phi\cos^2\theta + 16\sin^4\phi\sin^2\theta + 16\sin^2\phi\cos^2\phi} = \sqrt{16\sin^2\phi} = 4\,|\sin\phi| = 4\sin\phi$

$\Rightarrow\ A = \int_0^{2\pi}\int_{\pi/6}^{2\pi/3} 4\sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\left(2+2\sqrt{3}\right)d\theta = \left(4+4\sqrt{3}\right)\pi$

27. Let the parametrization be $\mathbf{r}(x,z) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}\ \Rightarrow\ \mathbf{r}_x = \mathbf{i} + 2x\mathbf{j}$ and $\mathbf{r}_z = \mathbf{k}\ \Rightarrow\ \mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= 2x\mathbf{i} + \mathbf{j}\ \Rightarrow\ |\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{4x^2+1}\ \Rightarrow\ \iint_S G(x,y,z)\,d\sigma = \int_0^3\int_0^2 x\sqrt{4x^2+1}\,dx\,dz = \int_0^3\left[\frac{1}{12}\left(4x^2+1\right)^{3/2}\right]_0^2 dz$

$= \int_0^3 \frac{1}{12}\left(17\sqrt{17}-1\right)dz = \frac{17\sqrt{17}-1}{4}$

28. Let the parametrization be $\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + \sqrt{4-y^2}\mathbf{k}$, $-2 \le y \le 2\ \Rightarrow\ \mathbf{r}_x = \mathbf{i}$ and $\mathbf{r}_y = \mathbf{j} - \frac{y}{\sqrt{4-y^2}}\,\mathbf{k}$

$$\Rightarrow\ \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & -\frac{y}{\sqrt{4-y^2}} \end{vmatrix} = \frac{y}{\sqrt{4-y^2}}\,\mathbf{j} + \mathbf{k}\ \Rightarrow\ |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\frac{y^2}{4-y^2}+1} = \frac{2}{\sqrt{4-y^2}}$$

$\Rightarrow\ \iint_S G(x,y,z)\,d\sigma = \int_1^4\int_{-2}^2 \sqrt{4-y^2}\left(\frac{2}{\sqrt{4-y^2}}\right)dy\,dx = 24$

29. Let the parametrization be $\mathbf{r}(\phi,\theta) = (\sin\phi\cos\theta)\mathbf{i} + (\sin\phi\sin\theta)\mathbf{j} + (\cos\phi)\mathbf{k}$ (spherical coordinates with $\rho = 1$ on the sphere), $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi\ \Rightarrow\ \mathbf{r}_\phi = (\cos\phi\cos\theta)\mathbf{i} + (\cos\phi\sin\theta)\mathbf{j} - (\sin\phi)\mathbf{k}$ and

$$\mathbf{r}_\theta = (-\sin\phi\sin\theta)\mathbf{i} + (\sin\phi\cos\theta)\mathbf{j}\ \Rightarrow\ \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix}$$

$= (\sin^2\phi\cos\theta)\,\mathbf{i} + (\sin^2\phi\sin\theta)\,\mathbf{j} + (\sin\phi\cos\phi)\mathbf{k}\ \Rightarrow\ |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi}$

$= \sin\phi$; $x = \sin\phi\cos\theta\ \Rightarrow\ G(x,y,z) = \cos^2\theta\sin^2\phi\ \Rightarrow\ \iint_S G(x,y,z)\,d\sigma = \int_0^{2\pi}\int_0^{\pi}(\cos^2\theta\sin^2\phi)(\sin\phi)\,d\phi\,d\theta$

$= \int_0^{2\pi}\int_0^{\pi}(\cos^2\theta)\left(1-\cos^2\phi\right)(\sin\phi)\,d\phi\,d\theta;\ \begin{bmatrix} u = \cos\phi \\ du = -\sin\phi\,d\phi \end{bmatrix}\ \to\ \int_0^{2\pi}\int_1^{-1}(\cos^2\theta)\left(u^2-1\right)du\,d\theta$

$= \int_0^{2\pi}(\cos^2\theta)\left[\frac{u^3}{3}-u\right]_1^{-1}d\theta = \frac{4}{3}\int_0^{2\pi}\cos^2\theta\,d\theta = \frac{4}{3}\left[\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_0^{2\pi} = \frac{4\pi}{3}$

30. Let the parametrization be $\mathbf{r}(\phi,\theta) = (a\sin\phi\cos\theta)\mathbf{i} + (a\sin\phi\sin\theta)\mathbf{j} + (a\cos\phi)\mathbf{k}$ (spherical coordinates with $\rho = a$, $a \ge 0$, on the sphere), $0 \le \phi \le \frac{\pi}{2}$ (since $z \ge 0$), $0 \le \theta \le 2\pi$

$\Rightarrow\ \mathbf{r}_\phi = (a\cos\phi\cos\theta)\mathbf{i} + (a\cos\phi\sin\theta)\mathbf{j} - (a\sin\phi)\mathbf{k}$ and

$$\mathbf{r}_\theta = (-a\sin\phi\sin\theta)\mathbf{i} + (a\sin\phi\cos\theta)\mathbf{j}\ \Rightarrow\ \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$= (a^2\sin^2\phi\cos\theta)\,\mathbf{i} + (a^2\sin^2\phi\sin\theta)\,\mathbf{j} + (a^2\sin\phi\cos\phi)\mathbf{k}$

$\Rightarrow\ |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\sin^2\phi\cos^2\phi} = a^2\sin\phi$; $z = a\cos\phi$

$\Rightarrow\ G(x,y,z) = a^2\cos^2\phi\ \Rightarrow\ \iint_S G(x,y,z)\,d\sigma = \int_0^{2\pi}\int_0^{\pi/2}(a^2\cos^2\phi)(a^2\sin\phi)\,d\phi\,d\theta = \frac{2}{3}\pi a^4$

31. Let the parametrization be $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (4 - x - y)\mathbf{k} \Rightarrow \mathbf{r}_x = \mathbf{i} - \mathbf{k}$ and $\mathbf{r}_y = \mathbf{j} - \mathbf{k}$

$$\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{3} \Rightarrow \iint_S F(x, y, z)\, d\sigma = \int_0^1 \int_0^1 (4 - x - y)\sqrt{3}\, dy\, dx$$

$$= \int_0^1 \sqrt{3}\left[4y - xy - \tfrac{y^2}{2}\right]_0^1 dx = \int_0^1 \sqrt{3}\left(\tfrac{7}{2} - x\right) dx = \sqrt{3}\left[\tfrac{7}{2}x - \tfrac{x^2}{2}\right]_0^1 = 3\sqrt{3}$$

32. Let the parametrization be $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $0 \le r \le 1$ (since $0 \le z \le 1$) and $0 \le \theta \le 2\pi$

$$\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k} \text{ and } \mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= (-r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{(-r\cos\theta)^2 + (-r\sin\theta)^2 + r^2} = r\sqrt{2};\ z = r \text{ and } x = r\cos\theta$$

$$\Rightarrow F(x, y, z) = r - r\cos\theta \Rightarrow \iint_S F(x, y, z)\, d\sigma = \int_0^{2\pi}\int_0^1 (r - r\cos\theta)\left(r\sqrt{2}\right) dr\, d\theta = \sqrt{2}\int_0^{2\pi}\int_0^1 (1 - \cos\theta)\, r^2\, dr\, d\theta$$

$$= \tfrac{2\pi\sqrt{2}}{3}$$

33. Let the parametrization be $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(1 - r^2\right)\mathbf{k}$, $0 \le r \le 1$ (since $0 \le z \le 1$) and $0 \le \theta \le 2\pi$

$$\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - 2r\mathbf{k} \text{ and } \mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \left(2r^2\cos\theta\right)\mathbf{i} + \left(2r^2\sin\theta\right)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{\left(2r^2\cos\theta\right)^2 + \left(2r^2\sin\theta\right) + r^2} = r\sqrt{1 + 4r^2};\ z = 1 - r^2 \text{ and}$$

$$x = r\cos\theta \Rightarrow H(x, y, z) = \left(r^2\cos^2\theta\right)\sqrt{1 + 4r^2} \Rightarrow \iint_S H(x, y, z)\, d\sigma$$

$$= \int_0^{2\pi}\int_0^1 \left(r^2\cos^2\theta\right)\left(\sqrt{1 + 4r^2}\right)\left(r\sqrt{1 + 4r^2}\right) dr\, d\theta = \int_0^{2\pi}\int_0^1 r^3\left(1 + 4r^2\right)\cos^2\theta\, dr\, d\theta = \tfrac{11\pi}{12}$$

34. Let the parametrization be $\mathbf{r}(\phi, \theta) = (2\sin\phi\cos\theta)\mathbf{i} + (2\sin\phi\sin\theta)\mathbf{j} + (2\cos\phi)\mathbf{k}$ (spherical coordinates with $\rho = 2$ on the sphere), $0 \le \phi \le \frac{\pi}{4}$; $x^2 + y^2 + z^2 = 4$ and $z = \sqrt{x^2 + y^2} \Rightarrow z^2 + z^2 = 4 \Rightarrow z^2 = 2 \Rightarrow z = \sqrt{2}$ (since $z \ge 0$) $\Rightarrow 2\cos\phi = \sqrt{2} \Rightarrow \cos\phi = \frac{\sqrt{2}}{2} \Rightarrow \phi = \frac{\pi}{4}$, $0 \le \theta \le 2\pi$; $\mathbf{r}_\phi = (2\cos\phi\cos\theta)\mathbf{i} + (2\cos\phi\sin\theta)\mathbf{j} - (2\sin\phi)\mathbf{k}$

$$\text{and } r_\theta = (-2\sin\phi\sin\theta)\mathbf{i} + (2\sin\phi\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \left(4\sin^2\phi\cos\theta\right)\mathbf{i} + \left(4\sin^2\phi\sin\theta\right)\mathbf{j} + (4\sin\phi\cos\phi)\mathbf{k}$$

$$\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{16\sin^4\phi\cos^2\theta + 16\sin^4\phi\sin^2\theta + 16\sin^2\phi\cos^2\phi} = 4\sin\phi;\ y = 2\sin\phi\sin\theta \text{ and}$$

$$z = 2\cos\phi \Rightarrow H(x, y, z) = 4\cos\phi\sin\phi\sin\theta \Rightarrow \iint_S H(x, y, z)\, d\sigma = \int_0^{2\pi}\int_0^{\pi/4} (4\cos\phi\sin\phi\sin\theta)(4\sin\phi)\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} 16\sin^2\phi\cos\phi\sin\theta\, d\phi\, d\theta = 0$$

35. Let the parametrization be $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + \left(4 - y^2\right)\mathbf{k}$, $0 \le x \le 1$, $-2 \le y \le 2$; $z = 0 \Rightarrow 0 = 4 - y^2$

$$\Rightarrow y = \pm 2;\ \mathbf{r}_x = \mathbf{i} \text{ and } \mathbf{r}_y = \mathbf{j} - 2y\mathbf{k} \Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & -2y \end{vmatrix} = 2y\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

$$= \mathbf{F} \cdot \tfrac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|}\, |\mathbf{r}_x \times \mathbf{r}_y|\, dy\, dx = (2xy - 3z)\, dy\, dx = \left[2xy - 3\left(4 - y^2\right)\right] dy\, dx \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

$$= \int_0^1 \int_{-2}^2 \left(2xy + 3y^2 - 12\right) dy\, dx = \int_0^1 \left[xy^2 + y^3 - 12y\right]_{-2}^2 dx = \int_0^1 -32\, dx = -32$$

36. Let the parametrization be $\mathbf{r}(x,y) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$, $-1 \le x \le 1$, $0 \le z \le 2 \Rightarrow \mathbf{r}_x = \mathbf{i} + 2x\mathbf{j}$ and $\mathbf{r}_z = \mathbf{k}$

$$\Rightarrow \mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\mathbf{i} - \mathbf{j} \Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\tfrac{\mathbf{r}_x\times\mathbf{r}_z}{|\mathbf{r}_x\times\mathbf{r}_z|}\,|\mathbf{r}_x\times\mathbf{r}_z|\,dz\,dx = -x^2\,dz\,dx$$

$$\Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_{-1}^{1}\int_0^2 -x^2\,dz\,dx = -\tfrac{4}{3}$$

37. Let the parametrization be $\mathbf{r}(\phi,\theta) = (a\ \sin\phi\cos\theta)\mathbf{i} + (a\sin\phi\sin\theta)\mathbf{j} + (a\cos\phi)\mathbf{k}$ (spherical coordinates with $\rho = a$, $a \ge 0$, on the sphere), $0 \le \phi \le \frac{\pi}{2}$ (for the first octant) , $0 \le \theta \le \frac{\pi}{2}$ (for the first octant)

$\Rightarrow \mathbf{r}_\phi = (a\cos\phi\cos\theta)\mathbf{i} + (a\cos\phi\sin\theta)\mathbf{j} - (a\sin\phi)\mathbf{k}$ and $\mathbf{r}_\theta = (-a\sin\phi\sin\theta)\mathbf{i} + (a\sin\phi\cos\theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= (a^2\sin^2\phi\cos\theta)\,\mathbf{i} + (a^2\sin^2\phi\sin\theta)\,\mathbf{j} + (a^2\sin\phi\cos\phi)\,\mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\tfrac{\mathbf{r}_\phi\times\mathbf{r}_\theta}{|\mathbf{r}_\phi\times\mathbf{r}_\theta|}\,|\mathbf{r}_\phi\times\mathbf{r}_\theta|\,d\theta\,d\phi$$

$$= a^3\cos^2\phi\sin\phi\,d\theta\,d\phi \text{ since } \mathbf{F} = z\mathbf{k} = (a\cos\phi)\mathbf{k} \Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{\pi/2}\int_0^{\pi/2} a^3\cos^2\phi\sin\phi\,d\phi\,d\theta = \tfrac{\pi a^3}{6}$$

38. Let the parametrization be $\mathbf{r}(\phi,\theta) = (a\sin\phi\cos\theta)\mathbf{i} + (a\sin\phi\sin\theta)\mathbf{j} + (a\cos\phi)\mathbf{k}$ (spherical coordinates with $\rho = a$, $a \ge 0$, on the sphere), $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$

$\Rightarrow \mathbf{r}_\phi = (a\cos\phi\cos\theta)\mathbf{i} + (a\cos\phi\sin\theta)\mathbf{j} - (a\sin\phi)\mathbf{k}$ and $\mathbf{r}_\theta = (-a\sin\phi\sin\theta)\mathbf{i} + (a\sin\phi\cos\theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= (a^2\sin^2\phi\cos\theta)\,\mathbf{i} + (a^2\sin^2\phi\sin\theta)\,\mathbf{j} + (a^2\sin\phi\cos\phi)\,\mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\tfrac{\mathbf{r}_\phi\times\mathbf{r}_\theta}{|\mathbf{r}_\phi\times\mathbf{r}_\theta|}\,|\mathbf{r}_\phi\times\mathbf{r}_\theta|\,d\theta\,d\phi$$

$$= (a^3\sin^3\phi\cos^2\phi + a^3\sin^3\phi\sin^2\theta + a^3\sin\phi\cos^2\phi)\,d\theta\,d\phi = a^3\sin\phi\,d\theta\,d\phi \text{ since } \mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$= (a\sin\phi\cos\theta)\mathbf{i} + (a\sin\phi\sin\theta)\mathbf{j} + (a\cos\phi)\mathbf{k} \Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^{\pi} a^3\sin\phi\,d\phi\,d\theta = 4\pi a^3$$

39. Let the parametrization be $\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + (2a - x - y)\mathbf{k}$, $0 \le x \le a$, $0 \le y \le a \Rightarrow \mathbf{r}_x = \mathbf{i} - \mathbf{k}$ and $\mathbf{r}_y = \mathbf{j} - \mathbf{k}$

$$\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\tfrac{\mathbf{r}_x\times\mathbf{r}_y}{|\mathbf{r}_x\times\mathbf{r}_y|}\,|\mathbf{r}_x\times\mathbf{r}_y|\,dy\,dx$$

$$= [2xy + 2y(2a - x - y) + 2x(2a - x - y)]\,dy\,dx \text{ since } \mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$$

$$= 2xy\mathbf{i} + 2y(2a - x - y)\mathbf{j} + 2x(2a - x - y)\mathbf{k} \Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$$

$$= \int_0^a\int_0^a [2xy + 2y(2a - x - y) + 2x(2a - x - y)]\,dy\,dx = \int_0^a\int_0^a (4ay - 2y^2 + 4ax - 2x^2 - 2xy)\,dy\,dx$$

$$= \int_0^a \left(\tfrac{4}{3}a^3 + 3a^2x - 2ax^2\right)dx = \left(\tfrac{4}{3} + \tfrac{3}{2} - \tfrac{2}{3}\right)a^4 = \tfrac{13a^4}{6}$$

40. Let the parametrization be $\mathbf{r}(\theta,z) = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + z\mathbf{k}$, $0 \le z \le a$, $0 \le \theta \le 2\pi$ (where $r = \sqrt{x^2 + y^2} = 1$ on the cylinder) $\Rightarrow \mathbf{r}_\theta = (-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$ and $\mathbf{r}_z = \mathbf{k} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$

$$\Rightarrow \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\tfrac{\mathbf{r}_\theta\times\mathbf{r}_z}{|\mathbf{r}_\theta\times\mathbf{r}_z|}\,|\mathbf{r}_\theta\times\mathbf{r}_z|\,dz\,d\theta = (\cos^2\theta + \sin^2\theta)\,dz\,d\theta = dz\,d\theta, \text{ since } \mathbf{F} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + z\mathbf{k}$$

$$\Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^a 1\,dz\,d\theta = 2\pi a$$

41. Let the parametrization be $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $0 \le r \le 1$ (since $0 \le z \le 1$) and $0 \le \theta \le 2\pi$

$\Rightarrow\ \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}\ \Rightarrow\ \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix}$

$= (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} - r\mathbf{k}\ \Rightarrow\ \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\frac{\mathbf{r}_\theta\times\mathbf{r}_r}{|\mathbf{r}_\theta\times\mathbf{r}_r|}\,|\mathbf{r}_\theta\times\mathbf{r}_r|\,d\theta\,dr = (r^3\sin\theta\cos^2\theta + r^2)\,d\theta\,dr$ since

$\mathbf{F} = (r^2\sin\theta\cos\theta)\,\mathbf{i} - r\mathbf{k}\ \Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^1 (r^3\sin\theta\cos^2\theta + r^2)\,dr\,d\theta = \int_0^{2\pi}\left(\frac{1}{4}\sin\theta\cos^2\theta + \frac{1}{3}\right)d\theta$

$= \left[-\frac{1}{12}\cos^3\theta + \frac{\theta}{3}\right]_0^{2\pi} = \frac{2\pi}{3}$

42. Let the parametrization be $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + 2r\mathbf{k}$, $0 \le r \le 1$ (since $0 \le z \le 2$) and $0 \le \theta \le 2\pi$

$\Rightarrow\ \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}\ \Rightarrow\ \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 2 \end{vmatrix}$

$= (2r\cos\theta)\mathbf{i} + (2r\sin\theta)\mathbf{j} - r\mathbf{k}\ \Rightarrow\ \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\frac{\mathbf{r}_\theta\times\mathbf{r}_r}{|\mathbf{r}_\theta\times\mathbf{r}_r|}\,|\mathbf{r}_\theta\times\mathbf{r}_r|\,d\theta\,dr$

$= (2r^3\sin^2\theta\cos\theta + 4r^3\cos\theta\sin\theta + r)\,d\theta\,dr$ since

$\mathbf{F} = (r^2\sin^2\theta)\,\mathbf{i} + (2r^2\cos\theta)\,\mathbf{j} - \mathbf{k}\ \Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^1 (2r^3\sin^2\theta\cos\theta + 4r^3\cos\theta\sin\theta + r)\,dr\,d\theta$

$= \int_0^{2\pi}\left(\frac{1}{2}\sin^2\theta\cos\theta + \cos\theta\sin\theta + \frac{1}{2}\right)d\theta = \left[\frac{1}{6}\sin^3\theta + \frac{1}{2}\sin^2\theta + \frac{1}{2}\theta\right]_0^{2\pi} = \pi$

43. Let the parametrization be $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $1 \le r \le 2$ (since $1 \le z \le 2$) and $0 \le \theta \le 2\pi$

$\Rightarrow\ \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}\ \Rightarrow\ \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix}$

$= (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} - r\mathbf{k}\ \Rightarrow\ \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\frac{\mathbf{r}_\theta\times\mathbf{r}_r}{|\mathbf{r}_\theta\times\mathbf{r}_r|}\,|\mathbf{r}_\theta\times\mathbf{r}_r|\,d\theta\,dr = (-r^2\cos^2\theta - r^2\sin^2\theta - r^3)\,d\theta\,dr$

$= (-r^2 - r^3)\,d\theta\,dr$ since $\mathbf{F} = (-r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + r^2\mathbf{k}\ \Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_1^2 (-r^2 - r^3)\,dr\,d\theta = -\frac{73\pi}{6}$

44. Let the parametrization be $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r^2\mathbf{k}$, $0 \le r \le 1$ (since $0 \le z \le 1$) and $0 \le \theta \le 2\pi$

$\Rightarrow\ \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2r\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}\ \Rightarrow\ \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 2r \end{vmatrix}$

$= (2r^2\cos\theta)\,\mathbf{i} + (2r^2\sin\theta)\,\mathbf{j} - r\mathbf{k}\ \Rightarrow\ \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\frac{\mathbf{r}_\theta\times\mathbf{r}_r}{|\mathbf{r}_\theta\times\mathbf{r}_r|}\,|\mathbf{r}_\theta\times\mathbf{r}_r|\,d\theta\,dr = (8r^3\cos^2\theta + 8r^3\sin^2\theta - 2r)\,d\theta\,dr$

$= (8r^3 - 2r)\,d\theta\,dr$ since $\mathbf{F} = (4r\cos\theta)\mathbf{i} + (4r\sin\theta)\mathbf{j} + 2\mathbf{k}\ \Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^1 (8r^3 - 2r)\,dr\,d\theta = 2\pi$

45. Let the parametrization be $\mathbf{r}(\phi,\theta) = (a\sin\phi\cos\theta)\mathbf{i} + (a\sin\phi\sin\theta)\mathbf{j} + (a\cos\phi)\mathbf{k}$, $0 \le \phi \le \frac{\pi}{2}$, $0 \le \theta \le \frac{\pi}{2}$

$\Rightarrow\ \mathbf{r}_\phi = (a\cos\phi\cos\theta)\mathbf{i} + (a\cos\phi\sin\theta)\mathbf{j} - (a\sin\phi)\mathbf{k}$ and $\mathbf{r}_\theta = (-a\sin\phi\sin\theta)\mathbf{i} + (a\sin\phi\cos\theta)\mathbf{j}$

$\Rightarrow\ \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$

$= (a^2\sin^2\phi\cos\theta)\,\mathbf{i} + (a^2\sin^2\phi\sin\theta)\,\mathbf{j} + (a^2\sin\phi\cos\phi)\,\mathbf{k}$

$\Rightarrow\ |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\sin^2\phi\cos^2\phi} = \sqrt{a^4\sin^2\phi} = a^2\sin\phi$. The mass is

$M = \iint_S d\sigma = \int_0^{\pi/2}\int_0^{\pi/2} (a^2\sin\phi)\,d\phi\,d\theta = \frac{a^2\pi}{2}$; the first moment is $M_{yz} = \iint_S x\,d\sigma$

$= \int_0^{\pi/2}\int_0^{\pi/2} (a\sin\phi\cos\theta)(a^2\sin\phi)\,d\phi\,d\theta = \frac{a^3\pi}{4}\ \Rightarrow\ \overline{x} = \frac{\left(\frac{a^3\pi}{4}\right)}{\left(\frac{a^2\pi}{2}\right)} = \frac{a}{2}\ \Rightarrow$ the centroid is located at $\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$ by symmetry

46. Let the parametrization be $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $1 \le r \le 2$ (since $1 \le z \le 2$) and $0 \le \theta \le 2\pi$

$\Rightarrow\ \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}\ \Rightarrow\ \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix}$

$= (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} - r\mathbf{k}\ \Rightarrow\ |\mathbf{r}_\theta \times \mathbf{r}_r| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = r\sqrt{2}$. The mass is

$M = \iint\limits_S \delta\, d\sigma = \int_0^{2\pi}\int_1^2 \delta\, r\sqrt{2}\, dr\, d\theta = \left(3\sqrt{2}\right)\pi\delta$; the first moment is $M_{xy} = \iint\limits_S \delta z\, d\sigma = \int_0^{2\pi}\int_1^2 \delta r\left(r\sqrt{2}\right) dr\, d\theta$

$= \frac{\left(14\sqrt{2}\right)\pi\delta}{3}\ \Rightarrow\ \bar{z} = \frac{\left(\frac{\left(14\sqrt{2}\right)\pi\delta}{3}\right)}{\left(3\sqrt{2}\right)\pi\delta} = \frac{14}{9}\ \Rightarrow$ the center of mass is located at $\left(0, 0, \frac{14}{9}\right)$ by symmetry. The

moment of inertia is $I_z = \iint\limits_S \delta\left(x^2+y^2\right) d\sigma = \int_0^{2\pi}\int_1^2 \delta r^2\left(r\sqrt{2}\right) dr\, d\theta = \frac{\left(15\sqrt{2}\right)\pi\delta}{2}\ \Rightarrow$ the radius of gyration is

$R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{5}{2}}$

47. Let the parametrization be $\mathbf{r}(\phi,\theta) = (a\sin\phi\cos\theta)\mathbf{i} + (a\sin\phi\sin\theta)\mathbf{j} + (a\cos\phi)\mathbf{k}$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$

$\Rightarrow\ \mathbf{r}_\phi = (a\cos\phi\cos\theta)\mathbf{i} + (a\cos\phi\sin\theta)\mathbf{j} - (a\sin\phi)\mathbf{k}$ and $\mathbf{r}_\theta = (-a\sin\phi\sin\theta)\mathbf{i} + (a\sin\phi\cos\theta)\mathbf{j}$

$\Rightarrow\ \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$

$= \left(a^2\sin^2\phi\cos\theta\right)\mathbf{i} + \left(a^2\sin^2\phi\sin\theta\right)\mathbf{j} + \left(a^2\sin\phi\cos\phi\right)\mathbf{k}$

$\Rightarrow\ |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\sin^2\phi\cos^2\phi} = \sqrt{a^4\sin^2\phi} = a^2\sin\phi$. The moment of

inertia is $I_z = \iint\limits_S \delta\left(x^2+y^2\right) d\sigma = \int_0^{2\pi}\int_0^{\pi} \delta\left[(a\sin\phi\cos\theta)^2 + (a\sin\phi\sin\theta)^2\right]\left(a^2\sin\phi\right) d\phi\, d\theta$

$= \int_0^{2\pi}\int_0^{\pi} \delta\left(a^2\sin^2\phi\right)\left(a^2\sin\phi\right) d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} \delta a^4\sin^3\phi\, d\phi\, d\theta = \int_0^{2\pi} \delta a^4\left[\left(-\frac{1}{3}\cos\phi\right)\left(\sin^2\phi + 2\right)\right]_0^{\pi} d\theta = \frac{8\delta\pi a^4}{3}$

48. Let the parametrization be $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $0 \le r \le 1$ (since $0 \le z \le 1$) and $0 \le \theta \le 2\pi$

$\Rightarrow\ \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}\ \Rightarrow\ \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix}$

$= (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} - r\mathbf{k}\ \Rightarrow\ |\mathbf{r}_\theta \times \mathbf{r}_r| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = r\sqrt{2}$. The moment of inertia is

$I_z = \iint\limits_S \delta\left(x^2+y^2\right) d\sigma = \int_0^{2\pi}\int_0^1 \delta r^2\left(r\sqrt{2}\right) dr\, d\theta = \frac{\pi\delta\sqrt{2}}{2}$

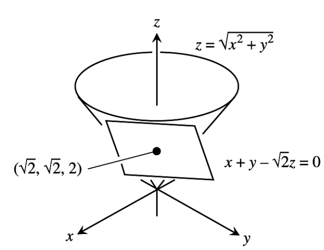
49. The parametrization $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$

at $P_0 = \left(\sqrt{2}, \sqrt{2}, 2\right)\ \Rightarrow\ \theta = \frac{\pi}{4}$, $r = 2$,

$\mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} + \mathbf{k}$ and

$\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$

$\Rightarrow\ \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{2}/2 & \sqrt{2}/2 & 1 \\ -\sqrt{2} & \sqrt{2} & 0 \end{vmatrix}$

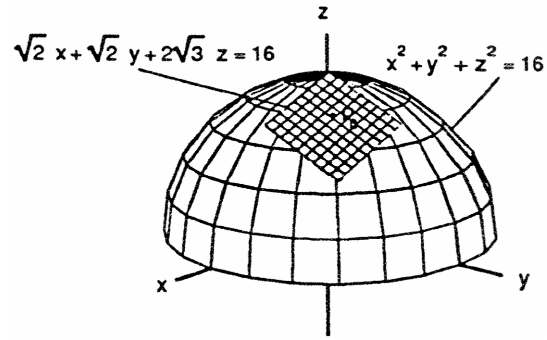


$= -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j} + 2\mathbf{k}\ \Rightarrow$ the tangent plane is

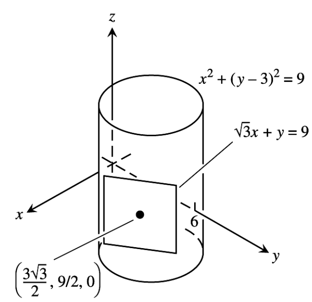
$0 = \left(-\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j} + 2\mathbf{k}\right) \cdot \left[\left(x - \sqrt{2}\right)\mathbf{i} + \left(y - \sqrt{2}\right)\mathbf{j} + (z-2)\mathbf{k}\right] \Rightarrow \sqrt{2}x + \sqrt{2}y - 2z = 0$, or $x + y - \sqrt{2}z = 0$.

The parametrization $\mathbf{r}(r,\theta)\ \Rightarrow\ x = r\cos\theta$, $y = r\sin\theta$ and $z = r\ \Rightarrow\ x^2 + y^2 = r^2 = z^2\ \Rightarrow$ the surface is $z = \sqrt{x^2+y^2}$.

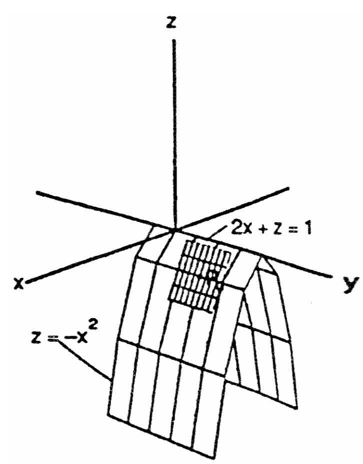
50. The parametrization $\mathbf{r}(\phi, \theta)$
$= (4 \sin \phi \cos \theta)\mathbf{i} + (4 \sin \phi \sin \theta)\mathbf{j} + (4 \cos \phi)\mathbf{k}$
at $P_0 = \left(\sqrt{2}, \sqrt{2}, 2\sqrt{3}\right) \Rightarrow \rho = 4$ and $z = 2\sqrt{3}$
$= 4 \cos \phi \Rightarrow \phi = \frac{\pi}{6}$; also $x = \sqrt{2}$ and $y = \sqrt{2}$
$\Rightarrow \theta = \frac{\pi}{4}$. Then $\mathbf{r}_\phi$
$= (4 \cos \phi \cos \theta)\mathbf{i} + (4 \cos \phi \sin \theta)\mathbf{j} - (4 \sin \phi)\mathbf{k}$
$= \sqrt{6}\mathbf{i} + \sqrt{6}\mathbf{j} - 2\mathbf{k}$ and
$\mathbf{r}_\theta = (-4 \sin \phi \sin \theta)\mathbf{i} + (4 \sin \phi \cos \theta)\mathbf{j}$
$= -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$ at $P_0 \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{6} & \sqrt{6} & -2 \\ -\sqrt{2} & \sqrt{2} & 0 \end{vmatrix}$
$= 2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} + 4\sqrt{3}\mathbf{k} \Rightarrow$ the tangent plane is
$\left(2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} + 4\sqrt{3}\mathbf{k}\right) \cdot \left[\left(x - \sqrt{2}\right)\mathbf{i} + \left(y - \sqrt{2}\right)\mathbf{j} + \left(z - 2\sqrt{3}\right)\mathbf{k}\right] = 0 \Rightarrow \sqrt{2}x + \sqrt{2}y + 2\sqrt{3}z = 16$,
or $x + y + \sqrt{6}z = 8\sqrt{2}$. The parametrization $\Rightarrow x = 4 \sin \phi \cos \theta$, $y = 4 \sin \phi \sin \theta$, $z = 4 \cos \phi$
$\Rightarrow$ the surface is $x^2 + y^2 + z^2 = 16$, $z \geq 0$.



51. The parametrization $\mathbf{r}(\theta, z) = (3 \sin 2\theta)\mathbf{i} + (6 \sin^2 \theta)\,\mathbf{j} + z\mathbf{k}$
at $P_0 = \left(\frac{3\sqrt{3}}{2}, \frac{9}{2}, 0\right) \Rightarrow \theta = \frac{\pi}{3}$ and $z = 0$. Then
$\mathbf{r}_\theta = (6 \cos 2\theta)\mathbf{i} + (12 \sin \theta \cos \theta)\mathbf{j}$
$= -3\mathbf{i} + 3\sqrt{3}\mathbf{j}$ and $\mathbf{r}_z = \mathbf{k}$ at P_0
$\Rightarrow \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 3\sqrt{3} & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3\sqrt{3}\mathbf{i} + 3\mathbf{j}$
$\Rightarrow$ the tangent plane is
$\left(3\sqrt{3}\mathbf{i} + 3\mathbf{j}\right) \cdot \left[\left(x - \frac{3\sqrt{3}}{2}\right)\mathbf{i} + \left(y - \frac{9}{2}\right)\mathbf{j} + (z - 0)\mathbf{k}\right] = 0$
$\Rightarrow \sqrt{3}x + y = 9$. The parametrization $\Rightarrow x = 3 \sin 2\theta$
and $y = 6 \sin^2 \theta \Rightarrow x^2 + y^2 = 9 \sin^2 2\theta + \left(6 \sin^2 \theta\right)^2$
$= 9\left(4 \sin^2 \theta \cos^2 \theta\right) + 36 \sin^4 \theta = 6\left(6 \sin^2 \theta\right) = 6y \Rightarrow x^2 + y^2 - 6y + 9 = 9 \Rightarrow x^2 + (y - 3)^2 = 9$



52. The parametrization $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} - x^2\mathbf{k}$ at
$P_0 = (1, 2, -1) \Rightarrow \mathbf{r}_x = \mathbf{i} - 2x\mathbf{k} = \mathbf{i} - 2\mathbf{k}$ and $\mathbf{r}_y = \mathbf{j}$ at P_0
$\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{vmatrix} = 2\mathbf{i} + \mathbf{k} \Rightarrow$ the tangent plane
is $(2\mathbf{i} + \mathbf{k}) \cdot [(x - 1)\mathbf{i} + (y - 2)\mathbf{j} + (z + 1)\mathbf{k}] = 0$
$\Rightarrow 2x + z = 1$. The parametrization $\Rightarrow x = x$, $y = y$ and
$z = -x^2 \Rightarrow$ the surface is $z = -x^2$



53. (a) An arbitrary point on the circle C is $(x, z) = (R + r \cos u, r \sin u) \Rightarrow (x, y, z)$ is on the torus with
$x = (R + r \cos u) \cos v$, $y = (R + r \cos u) \sin v$, and $z = r \sin u$, $0 \leq u \leq 2\pi$, $0 \leq v \leq 2\pi$

(b) $\mathbf{r}_u = (-r\sin u\cos v)\mathbf{i} - (r\sin u\sin v)\mathbf{j} + (r\cos u)\mathbf{k}$ and $\mathbf{r}_v = (-(R + r\cos u)\sin v)\mathbf{i} + ((R + r\cos u)\cos v)\mathbf{j}$

$$\Rightarrow \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin u\cos v & -r\sin u\sin v & r\cos u \\ -(R + r\cos u)\sin v & (R + r\cos u)\cos v & 0 \end{vmatrix}$$

$= -(R + r\cos u)(r\cos v\cos u)\mathbf{i} - (R + r\cos u)(r\sin v\cos u)\mathbf{j} + (-r\sin u)(R + r\cos u)\mathbf{k}$

$\Rightarrow |\mathbf{r}_u \times \mathbf{r}_v|^2 = (R + r\cos u)^2\left(r^2\cos^2 v\cos^2 u + r^2\sin^2 v\cos^2 u + r^2\sin^2 u\right) \Rightarrow |\mathbf{r}_u \times \mathbf{r}_v| = r(R + r\cos u)$

$\Rightarrow A = \int_0^{2\pi}\int_0^{2\pi} (rR + r^2\cos u)\,du\,dv = \int_0^{2\pi} 2\pi rR\,dv = 4\pi^2 rR$

54. (a) The point (x, y, z) is on the surface for fixed $x = f(u)$ when $y = g(u)\sin\left(\frac{\pi}{2} - v\right)$ and $z = g(u)\cos\left(\frac{\pi}{2} - v\right)$ $\Rightarrow x = f(u)$, $y = g(u)\cos v$, and $z = g(u)\sin v \Rightarrow \mathbf{r}(u, v) = f(u)\mathbf{i} + (g(u)\cos v)\mathbf{j} + (g(u)\sin v)\mathbf{k}$, $0 \le v \le 2\pi$, $a \le u \le b$

(b) Let $u = y$ and $x = u^2 \Rightarrow f(u) = u^2$ and $g(u) = u \Rightarrow \mathbf{r}(u, v) = u^2\mathbf{i} + (u\cos v)\mathbf{j} + (u\sin v)\mathbf{k}$, $0 \le v \le 2\pi$, $0 \le u$

55. (a) Let $w^2 + \frac{z^2}{c^2} = 1$ where $w = \cos\phi$ and $\frac{z}{c} = \sin\phi \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\phi \Rightarrow \frac{x}{a} = \cos\phi\cos\theta$ and $\frac{y}{b} = \cos\phi\sin\theta$
$\Rightarrow x = a\cos\theta\cos\phi$, $y = b\sin\theta\cos\phi$, and $z = c\sin\phi$
$\Rightarrow \mathbf{r}(\theta, \phi) = (a\cos\theta\cos\phi)\mathbf{i} + (b\sin\theta\cos\phi)\mathbf{j} + (c\sin\phi)\mathbf{k}$

(b) $\mathbf{r}_\theta = (-a\sin\theta\cos\phi)\mathbf{i} + (b\cos\theta\cos\phi)\mathbf{j}$ and $\mathbf{r}_\phi = (-a\cos\theta\sin\phi)\mathbf{i} - (b\sin\theta\sin\phi)\mathbf{j} + (c\cos\phi)\mathbf{k}$

$$\Rightarrow \mathbf{r}_\theta \times \mathbf{r}_\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin\theta\cos\phi & b\cos\theta\cos\phi & 0 \\ -a\cos\theta\sin\phi & -b\sin\theta\sin\phi & c\cos\phi \end{vmatrix}$$

$= (bc\cos\theta\cos^2\phi)\,\mathbf{i} + (ac\sin\theta\cos^2\phi)\,\mathbf{j} + (ab\sin\phi\cos\phi)\mathbf{k}$

$\Rightarrow |\mathbf{r}_\theta \times \mathbf{r}_\phi|^2 = b^2c^2\cos^2\theta\cos^4\phi + a^2c^2\sin^2\theta\cos^4\phi + a^2b^2\sin^2\phi\cos^2\phi$, and the result follows.

$A \Rightarrow \int_0^{2\pi}\int_0^{\pi} |\mathbf{r}_\theta \times \mathbf{r}_\phi|\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi} \left[a^2b^2\sin^2\phi\cos^2\phi + b^2c^2\cos^2\theta\cos^4\phi + a^2c^2\sin^2\theta\cos^4\phi\right]^{1/2} d\phi\,d\theta$

56. (a) $\mathbf{r}(\theta, u) = (\cosh u\cos\theta)\mathbf{i} + (\cosh u\sin\theta)\mathbf{j} + (\sinh u)\mathbf{k}$

(b) $\mathbf{r}(\theta, u) = (a\cosh u\cos\theta)\mathbf{i} + (b\cosh u\sin\theta)\mathbf{j} + (c\sinh u)\mathbf{k}$

57. $\mathbf{r}(\theta, u) = (5\cosh u\cos\theta)\mathbf{i} + (5\cosh u\sin\theta)\mathbf{j} + (5\sinh u)\mathbf{k} \Rightarrow \mathbf{r}_\theta = (-5\cosh u\sin\theta)\mathbf{i} + (5\cosh u\cos\theta)\mathbf{j}$ and $\mathbf{r}_u = (5\sinh u\cos\theta)\mathbf{i} + (5\sinh u\sin\theta)\mathbf{j} + (5\cosh u)\mathbf{k}$

$$\Rightarrow r_\theta \times \mathbf{r}_u = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5\cosh u\sin\theta & 5\cosh u\cos\theta & 0 \\ 5\sinh u\cos\theta & 5\sinh u\sin\theta & 5\cosh u \end{vmatrix}$$

$= (25\cosh^2 u\cos\theta)\,\mathbf{i} + (25\cosh^2 u\sin\theta)\,\mathbf{j} - (25\cosh u\sinh u)\mathbf{k}$. At the point $(x_0, y_0, 0)$, where $x_0^2 + y_0^2 = 25$ we have $5\sinh u = 0 \Rightarrow u = 0$ and $x_0 = 25\cos\theta$, $y_0 = 25\sin\theta \Rightarrow$ the tangent plane is $5(x_0\mathbf{i} + y_0\mathbf{j}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + z\mathbf{k}] = 0 \Rightarrow x_0x - x_0^2 + y_0y - y_0^2 = 0 \Rightarrow x_0x + y_0y = 25$

58. Let $\frac{z^2}{c^2} - w^2 = 1$ where $\frac{z}{c} = \cosh u$ and $w = \sinh u \Rightarrow w^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} \Rightarrow \frac{x}{a} = w\cos\theta$ and $\frac{y}{b} = w\sin\theta$
$\Rightarrow x = a\sinh u\cos\theta$, $y = b\sinh u\sin\theta$, and $z = c\cosh u$
$\Rightarrow \mathbf{r}(\theta, u) = (a\sinh u\cos\theta)\mathbf{i} + (b\sinh u\sin\theta)\mathbf{j} + (c\cosh u)\mathbf{k}$, $0 \le \theta \le 2\pi$, $-\infty < u < \infty$

16.7 STOKES' THEOREM

1. $$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2x & z^2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + (2 - 0)\mathbf{k} = 2\mathbf{k} \text{ and } \mathbf{n} = \mathbf{k} \Rightarrow \text{curl}\,\mathbf{F} \cdot \mathbf{n} = 2 \Rightarrow d\sigma = dx\,dy$$

$\Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R 2\,dA = 2(\text{Area of the ellipse}) = 4\pi$

2. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3x & -z^2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + (3-2)\mathbf{k} = \mathbf{k}$ and $\mathbf{n} = \mathbf{k} \Rightarrow$ curl $\mathbf{F} \cdot \mathbf{n} = 1 \Rightarrow d\sigma = dx\,dy$

$\Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R dx\,dy = \text{Area of circle} = 9\pi$

3. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & x^2 \end{vmatrix} = -x\mathbf{i} - 2x\mathbf{j} + (z-1)\mathbf{k}$ and $\mathbf{n} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}} \Rightarrow$ curl $\mathbf{F} \cdot \mathbf{n}$

$= \frac{1}{\sqrt{3}}(-x - 2x + z - 1) \Rightarrow d\sigma = \frac{\sqrt{3}}{1}\,dA \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \frac{1}{\sqrt{3}}(-3x + z - 1)\sqrt{3}\,dA$

$= \int_0^1 \int_0^{1-x} [-3x + (1 - x - y) - 1]\,dy\,dx = \int_0^1 \int_0^{1-x} (-4x - y)\,dy\,dx = \int_0^1 -\left[4x(1-x) + \frac{1}{2}(1-x)^2\right] dx$

$= -\int_0^1 \left(\frac{1}{2} + 3x - \frac{7}{2}x^2\right) dx = -\frac{5}{6}$

4. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & x^2+z^2 & x^2+y^2 \end{vmatrix} = (2y-2z)\mathbf{i} + (2z-2x)\mathbf{j} + (2x-2y)\mathbf{k}$ and $\mathbf{n} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$

$\Rightarrow$ curl $\mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{3}}(2y - 2z + 2z - 2x + 2x - 2y) = 0 \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S 0\,d\sigma = 0$

5. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & x^2+y^2 & x^2+y^2 \end{vmatrix} = 2y\mathbf{i} + (2z-2x)\mathbf{j} + (2x-2y)\mathbf{k}$ and $\mathbf{n} = \mathbf{k}$

$\Rightarrow$ curl $\mathbf{F} \cdot \mathbf{n} = 2x - 2y \Rightarrow d\sigma = dx\,dy \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^1 \int_{-1}^1 (2x - 2y)\,dx\,dy = \int_{-1}^1 \left[x^2 - 2xy\right]_{-1}^1 dy$

$= \int_{-1}^1 -4y\,dy = 0$

6. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & 1 & z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} - 3x^2y^2\mathbf{k}$ and $\mathbf{n} = \frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{2\sqrt{x^2+y^2+z^2}} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{4}$

$\Rightarrow$ curl $\mathbf{F} \cdot \mathbf{n} = -\frac{3}{4}x^2y^2z$; $d\sigma = \frac{4}{z}\,dA$ (Section 16.5, Example 5, with a = 4) $\Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r}$

$= \iint_R \left(-\frac{3}{4}x^2y^2z\right)\left(\frac{4}{z}\right) dA = -3\int_0^{2\pi}\int_0^2 (r^2\cos^2\theta)(r^2\sin^2\theta)\,r\,dr\,d\theta = -3\int_0^{2\pi}\left[\frac{r^6}{6}\right]_0^2 (\cos\theta\sin\theta)^2\,d\theta$

$= -32\int_0^{2\pi} \frac{1}{4}\sin^2 2\theta\,d\theta = -4\int_0^{4\pi} \sin^2 u\,du = -4\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{4\pi} = -8\pi$

7. $x = 3\cos t$ and $y = 2\sin t \Rightarrow \mathbf{F} = (2\sin t)\mathbf{i} + (9\cos^2 t)\,\mathbf{j} + (9\cos^2 t + 16\sin^4 t)\sin e^{\sqrt{(6\sin t\cos t)(0)}}\mathbf{k}$ at the base of the shell; $\mathbf{r} = (3\cos t)\mathbf{i} + (2\sin t)\mathbf{j} \Rightarrow d\mathbf{r} = (-3\sin t)\mathbf{i} + (2\cos t)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -6\sin^2 t + 18\cos^3 t$

$\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\,d\sigma = \int_0^{2\pi} (-6\sin^2 t + 18\cos^3 t)\,dt = \left[-3t + \frac{3}{2}\sin 2t + 6(\sin t)(\cos^2 t + 2)\right]_0^{2\pi} = -6\pi$

8. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z + \frac{1}{2+x} & \tan^{-1} y & x + \frac{1}{4+z} \end{vmatrix} = -2\mathbf{j}$; $f(x,y,z) = 4x^2 + y + z^2 \Rightarrow \nabla f = 8x\mathbf{i} + \mathbf{j} + 2z\mathbf{k}$

$\Rightarrow \mathbf{n} = \frac{\nabla f}{|\nabla f|}$ and $\mathbf{p} = \mathbf{j} \Rightarrow |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\,dA = |\nabla f|\,dA$; $\nabla \times \mathbf{F} \cdot \mathbf{n} = \frac{1}{|\nabla f|}(-2\mathbf{j} \cdot \nabla f) = \frac{-2}{|\nabla f|}$

$\Rightarrow \nabla \times \mathbf{F} \cdot \mathbf{n}\,d\sigma = -2\,dA \Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\,d\sigma = \iint_R -2\,dA = -2(\text{Area of R}) = -2(\pi \cdot 1 \cdot 2) = -4\pi$, where R is the elliptic region in the xz-plane enclosed by $4x^2 + z^2 = 4$.

9. Flux of $\nabla \times \mathbf{F} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r}$, so let C be parametrized by $\mathbf{r} = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$, $0 \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ay \sin t + ax \cos t = a^2 \sin^2 t + a^2 \cos^2 t = a^2$
$\Rightarrow$ Flux of $\nabla \times \mathbf{F} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} a^2\, dt = 2\pi a^2$

10. $\nabla \times (y\mathbf{i}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & 0 \end{vmatrix} = -\mathbf{k}\,;\ \mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2+y^2+z^2}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
$\Rightarrow \nabla \times (y\mathbf{i}) \cdot \mathbf{n} = -z;\ d\sigma = \frac{1}{z}\, dA$ (Section 16.5, Example 5, with $a = 1$) $\Rightarrow \iint_S \nabla \times (y\mathbf{i}) \cdot \mathbf{n}\, d\sigma$
$= \iint_R (-z)\left(\frac{1}{z}\, dA\right) = -\iint_R dA = -\pi$, where R is the disk $x^2 + y^2 \le 1$ in the xy-plane.

11. Let S_1 and S_2 be oriented surfaces that span C and that induce the same positive direction on C. Then
$\iint_{S_1} \nabla \times \mathbf{F} \cdot \mathbf{n}_1\, d\sigma_1 = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \nabla \times \mathbf{F} \cdot \mathbf{n}_2\, d\sigma_2$

12. $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{S_1} \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma + \iint_{S_2} \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$, and since S_1 and S_2 are joined by the simple closed curve C, each of the above integrals will be equal to a circulation integral on C. But for one surface the circulation will be counterclockwise, and for the other surface the circulation will be clockwise. Since the integrands are the same, the sum will be 0 $\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0$.

13. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & 3x & 5y \end{vmatrix} = 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\,;\ \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - 2r\mathbf{k}$ and $\mathbf{r}_\theta = (-r \sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}$
$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (2r^2 \cos\theta)\,\mathbf{i} + (2r^2 \sin\theta)\,\mathbf{j} + r\mathbf{k}\,;\ \mathbf{n} = \frac{\mathbf{r}_r \times \mathbf{r}_\theta}{|\mathbf{r}_r \times \mathbf{r}_\theta|}$ and $d\sigma = |\mathbf{r}_r \times \mathbf{r}_\theta|\, dr\, d\theta$
$\Rightarrow \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta)\, dr\, d\theta = (10r^2 \cos\theta + 4r^2 \sin\theta + 3r)\, dr\, d\theta \Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$
$= \int_0^{2\pi}\int_0^2 (10r^2\cos\theta + 4r^2\sin\theta + 3r)\, dr\, d\theta = \int_0^{2\pi} \left[\frac{10}{3} r^3 \cos\theta + \frac{4}{3} r^3 \sin\theta + \frac{3}{2} r^2\right]_0^2 d\theta$
$= \int_0^{2\pi} \left(\frac{80}{3}\cos\theta + \frac{32}{3}\sin\theta + 6\right) d\theta = 6(2\pi) = 12\pi$

14. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z & z - x & x + z \end{vmatrix} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}\,;\ \mathbf{r}_r \times \mathbf{r}_\theta = (2r^2\cos\theta)\,\mathbf{i} + (2r^2\sin\theta)\,\mathbf{j} + r\mathbf{k}$ and
$\nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta)\, dr\, d\theta$ (see Exercise 13 above) $\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$
$= \int_0^{2\pi}\int_0^3 (-2r^2\cos\theta - 4r^2\sin\theta - 2r)\, dr\, d\theta = \int_0^{2\pi} \left[-\frac{2}{3} r^3\cos\theta - \frac{4}{3} r^3 \sin\theta - r^2\right]_0^3 d\theta$
$= \int_0^{2\pi} (-18\cos\theta - 36\sin\theta - 9)\, d\theta = -9(2\pi) = -18\pi$

15. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & 2y^3 z & 3z \end{vmatrix} = -2y^3\mathbf{i} + 0\mathbf{j} - x^2\mathbf{k}\,;\ \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$
$= (-r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + r\mathbf{k}$ and $\nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta)\, dr\, d\theta$ (see Exercise 13 above)
$\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_R (2ry^3\cos\theta - rx^2)\, dr\, d\theta = \int_0^{2\pi}\int_0^1 (2r^4 \sin^3\theta\cos\theta - r^3\cos^2\theta)\, dr\, d\theta$

$= \int_0^{2\pi} \left(\frac{2}{5}\sin^3\theta\cos\theta - \frac{1}{4}\cos^2\theta\right) d\theta = \left[\frac{1}{10}\sin^4\theta - \frac{1}{4}\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\right]_0^{2\pi} = -\frac{\pi}{4}$

16. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-y & y-z & z-x \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}\,;\ \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$

$= (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$ and $\nabla \times \mathbf{F}\cdot\mathbf{n}\,d\sigma = (\nabla \times \mathbf{F})\cdot(\mathbf{r}_r \times \mathbf{r}_\theta)\,dr\,d\theta$ (see Exercise 13 above)

$\Rightarrow \iint_S \nabla \times \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^5 (r\cos\theta + r\sin\theta + r)\,dr\,d\theta = \int_0^{2\pi}\left[(\cos\theta + \sin\theta + 1)\frac{r^2}{2}\right]_0^5 d\theta = \left(\frac{25}{2}\right)(2\pi) = 25\pi$

17. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & 5-2x & z^2-2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} - 5\mathbf{k}\,;$

$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{3}\cos\phi\cos\theta & \sqrt{3}\cos\phi\sin\theta & -\sqrt{3}\sin\phi \\ -\sqrt{3}\sin\phi\sin\theta & \sqrt{3}\sin\phi\cos\theta & 0 \end{vmatrix}$

$= (3\sin^2\phi\cos\theta)\,\mathbf{i} + (3\sin^2\phi\sin\theta)\,\mathbf{j} + (3\sin\phi\cos\phi)\mathbf{k}\,;\ \nabla \times \mathbf{F}\cdot\mathbf{n}\,d\sigma = (\nabla \times \mathbf{F})\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta)\,d\phi\,d\theta$ (see Exercise 13 above) $\Rightarrow \iint_S \nabla \times \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^{\pi/2} -15\cos\phi\sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\left[\frac{15}{2}\cos^2\phi\right]_0^{\pi/2} d\theta = \int_0^{2\pi} -\frac{15}{2}\,d\theta = -15\pi$

18. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x \end{vmatrix} = -2z\mathbf{i} - \mathbf{j} - 2y\mathbf{k}\,;$

$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix}$

$= (4\sin^2\phi\cos\theta)\,\mathbf{i} + (4\sin^2\phi\sin\theta)\,\mathbf{j} + (4\sin\phi\cos\phi)\mathbf{k}\,;\ \nabla \times \mathbf{F}\cdot\mathbf{n}\,d\sigma = (\nabla \times \mathbf{F})\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta)\,d\phi\,d\theta$ (see Exercise 13 above) $\Rightarrow \iint_S \nabla \times \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_R (-8z\sin^2\phi\cos\theta - 4\sin^2\phi\sin\theta - 8y\sin\phi\cos\theta)\,d\phi\,d\theta$

$= \int_0^{2\pi}\int_0^{\pi/2} (-16\sin^2\phi\cos\phi\cos\theta - 4\sin^2\phi\sin\theta - 16\sin^2\phi\sin\theta\cos\theta)\,d\phi\,d\theta$

$= \int_0^{2\pi}\left[-\frac{16}{3}\sin^3\phi\cos\theta - 4\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right)(\sin\theta) - 16\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right)(\sin\theta\cos\theta)\right]_0^{\pi/2} d\theta$

$= \int_0^{2\pi}\left(-\frac{16}{3}\cos\theta - \pi\sin\theta - 4\pi\sin\theta\cos\theta\right) d\theta = \left[-\frac{16}{3}\sin\theta + \pi\cos\theta - 2\pi\sin^2\theta\right]_0^{2\pi} = 0$

19. (a) $\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow \text{curl}\,\mathbf{F} = \mathbf{0} \Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S 0\,d\sigma = 0$

(b) Let $f(x, y, z) = x^2y^2z^3 \Rightarrow \nabla \times \mathbf{F} = \nabla \times \nabla f = \mathbf{0} \Rightarrow \text{curl}\,\mathbf{F} = \mathbf{0} \Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F}\cdot\mathbf{n}\,d\sigma$

$= \iint_S 0\,d\sigma = 0$

(c) $\mathbf{F} = \nabla \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \mathbf{0} \Rightarrow \nabla \times \mathbf{F} = \mathbf{0} \Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S 0\,d\sigma = 0$

(d) $\mathbf{F} = \nabla f \Rightarrow \nabla \times \mathbf{F} = \nabla \times \nabla f = \mathbf{0} \Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S 0\,d\sigma = 0$

20. $\mathbf{F} = \nabla f = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2x)\mathbf{i} - \frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2y)\mathbf{j} - \frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2z)\mathbf{k}$

$= -x(x^2 + y^2 + z^2)^{-3/2}\mathbf{i} - y(x^2 + y^2 + z^2)^{-3/2}\mathbf{j} - z(x^2 + y^2 + z^2)^{-3/2}\mathbf{k}$

(a) $\mathbf{r} = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j},\ 0 \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j}$

$\Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = -x(x^2 + y^2 + z^2)^{-3/2}(-a\sin t) - y(x^2 + y^2 + z^2)^{-3/2}(a\cos t)$

$= \left(-\frac{a\cos t}{a^3}\right)(-a\sin t) - \left(\frac{a\sin t}{a^3}\right)(a\cos t) = 0 \Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = 0$

(b) $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S \nabla \times \nabla f \cdot \mathbf{n}\, d\sigma = \iint_S \mathbf{0} \cdot \mathbf{n}\, d\sigma = \iint_S 0\, d\sigma = 0$

21. Let $\mathbf{F} = 2y\mathbf{i} + 3z\mathbf{j} - x\mathbf{k} \Rightarrow \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3z & -x \end{vmatrix} = -3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$; $\mathbf{n} = \frac{2\mathbf{i}+2\mathbf{j}+\mathbf{k}}{3}$

$\Rightarrow \nabla \times \mathbf{F} \cdot \mathbf{n} = -2 \Rightarrow \oint_C 2y\, dx + 3z\, dy - x\, dz = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S -2\, d\sigma$

$= -2 \iint_S d\sigma$, where $\iint_S d\sigma$ is the area of the region enclosed by C on the plane S: $2x + 2y + z = 2$

22. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$

23. Suppose $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ exists such that $\nabla \times \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$

$= x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then $\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) = \frac{\partial}{\partial x}(x) \Rightarrow \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} = 1$. Likewise, $\frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) = \frac{\partial}{\partial y}(y)$

$\Rightarrow \frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 P}{\partial y \partial x} = 1$ and $\frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{\partial}{\partial z}(z) \Rightarrow \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} = 1$. Summing the calculated equations

$\Rightarrow \left(\frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 P}{\partial y \partial x}\right) + \left(\frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 N}{\partial x \partial z}\right) + \left(\frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 M}{\partial z \partial y}\right) = 3$ or $0 = 3$ (assuming the second mixed partials are equal). This result is a contradiction, so there is no field $\mathbf{F}$ such that curl $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

24. Yes: If $\nabla \times \mathbf{F} = \mathbf{0}$, then the circulation of $\mathbf{F}$ around the boundary C of any oriented surface S in the domain of $\mathbf{F}$ is zero. The reason is this: By Stokes's theorem, circulation $= \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S \mathbf{0} \cdot \mathbf{n}\, d\sigma$

$= 0$.

25. $r = \sqrt{x^2 + y^2} \Rightarrow r^4 = (x^2 + y^2)^2 \Rightarrow \mathbf{F} = \nabla(r^4) = 4x(x^2 + y^2)\mathbf{i} + 4y(x^2 + y^2)\mathbf{j} = M\mathbf{i} + N\mathbf{j}$

$\Rightarrow \oint_C \nabla(r^4) \cdot \mathbf{n}\, ds = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \oint_C M\, dy - N\, dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy$

$= \iint_R \left[4(x^2 + y^2) + 8x^2 + 4(x^2 + y^2) + 8y^2\right] dA = \iint_R 16(x^2 + y^2)\, dA = 16 \iint_R x^2\, dA + 16 \iint_R y^2\, dA$

$= 16I_y + 16I_x$.

26. $\frac{\partial P}{\partial y} = 0$, $\frac{\partial N}{\partial z} = 0$, $\frac{\partial M}{\partial z} = 0$, $\frac{\partial P}{\partial x} = 0$, $\frac{\partial N}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$, $\frac{\partial M}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \Rightarrow$ curl $\mathbf{F} = \left[\frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2}\right]\mathbf{k} = \mathbf{0}$.

However, $x^2 + y^2 = 1 \Rightarrow \mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$

$\Rightarrow \mathbf{F} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \sin^2 t + \cos^2 t = 1 \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_0^{2\pi} 1\, dt = 2\pi$ which is not zero.

16.8 THE DIVERGENCE THEOREM AND A UNIFIED THEORY

1. $\mathbf{F} = \frac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2 + y^2}} \Rightarrow$ div $\mathbf{F} = \frac{xy - xy}{(x^2 + y^2)^{3/2}} = 0$

2. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} \Rightarrow$ div $\mathbf{F} = 1 + 1 = 2$

3. $\mathbf{F} = -\frac{GM(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{(x^2 + y^2 + z^2)^{3/2}} \Rightarrow$ div $\mathbf{F} = -GM\left[\frac{(x^2 + y^2 + z^2)^{3/2} - 3x^2(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3}\right]$

$- GM\left[\frac{(x^2 + y^2 + z^2)^{3/2} - 3y^2(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3}\right] - GM\left[\frac{(x^2 + y^2 + z^2)^{3/2} - 3z^2(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3}\right]$

$= -GM\left[\frac{3(x^2+y^2+z^2)^2 - 3(x^2+y^2+z^2)(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{7/2}}\right] = 0$

4. $z = a^2 - r^2$ in cylindrical coordinates $\Rightarrow z = a^2 - (x^2+y^2) \Rightarrow \mathbf{v} = (a^2 - x^2 - y^2)\,\mathbf{k} \Rightarrow \text{div}\,\mathbf{v} = 0$

5. $\frac{\partial}{\partial x}(y - x) = -1,\ \frac{\partial}{\partial y}(z - y) = -1,\ \frac{\partial}{\partial z}(y - x) = 0 \Rightarrow \nabla \cdot \mathbf{F} = -2 \Rightarrow \text{Flux} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} -2\,dx\,dy\,dz = -2\left(2^3\right)$
$= -16$

6. $\frac{\partial}{\partial x}(x^2) = 2x,\ \frac{\partial}{\partial y}(y^2) = 2y,\ \frac{\partial}{\partial x}(z^2) = 2z \Rightarrow \nabla \cdot \mathbf{F} = 2x + 2y + 2z$
   (a) Flux $= \int_0^1\int_0^1\int_0^1 (2x + 2y + 2z)\,dx\,dy\,dz = \int_0^1\int_0^1 \left[x^2 + 2x(y+z)\right]_0^1 dy\,dz = \int_0^1\int_0^1 (1 + 2y + 2z)\,dy\,dz$
   $= \int_0^1 \left[y(1+2z) + y^2\right]_0^1 dz = \int_0^1 (2 + 2z)\,dz = \left[2z + z^2\right]_0^1 = 3$
   (b) Flux $= \int_{-1}^1\int_{-1}^1\int_{-1}^1 (2x + 2y + 2z)\,dx\,dy\,dz = \int_{-1}^1\int_{-1}^1 \left[x^2 + 2x(y+z)\right]_{-1}^1 dy\,dz = \int_{-1}^1\int_{-1}^1 (4y + 4z)\,dy\,dz$
   $= \int_{-1}^1 \left[2y^2 + 4yz\right]_{-1}^1 dz = \int_{-1}^1 8z\,dz = \left[4z^2\right]_{-1}^1 = 0$
   (c) In cylindrical coordinates, Flux $= \iiint_D (2x + 2y + 2z)\,dx\,dy\,dz$
   $= \int_0^1\int_0^{2\pi}\int_0^2 (2r\cos\theta + 2r\sin\theta + 2z)\,r\,dr\,d\theta\,dz = \int_0^1\int_0^{2\pi}\left[\frac{2}{3}r^3\cos\theta + \frac{2}{3}r^3\sin\theta + zr^2\right]_0^2 d\theta\,dz$
   $= \int_0^1\int_0^{2\pi}\left(\frac{16}{3}\cos\theta + \frac{16}{3}\sin\theta + 4z\right)d\theta\,dz = \int_0^1\left[\frac{16}{3}\sin\theta - \frac{16}{3}\cos\theta + 4z\theta\right]_0^{2\pi} dz = \int_0^1 8\pi z\,dz = \left[4\pi z^2\right]_0^1 = 4\pi$

7. $\frac{\partial}{\partial x}(y) = 0,\ \frac{\partial}{\partial y}(xy) = x,\ \frac{\partial}{\partial z}(-z) = -1 \Rightarrow \nabla \cdot \mathbf{F} = x - 1;\ z = x^2 + y^2 \Rightarrow z = r^2$ in cylindrical coordinates
$\Rightarrow \text{Flux} = \iiint_D (x - 1)\,dz\,dy\,dx = \int_0^{2\pi}\int_0^2\int_0^{r^2} (r\cos\theta - 1)\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (r^3\cos\theta - r^2)\,r\,dr\,d\theta$
$= \int_0^{2\pi}\left[\frac{r^5}{5}\cos\theta - \frac{r^4}{4}\right]_0^2 d\theta = \int_0^{2\pi}\left(\frac{32}{5}\cos\theta - 4\right)d\theta = \left[\frac{32}{5}\sin\theta - 4\theta\right]_0^{2\pi} = -8\pi$

8. $\frac{\partial}{\partial x}(x^2) = 2x,\ \frac{\partial}{\partial y}(xz) = 0,\ \frac{\partial}{\partial z}(3z) = 3 \Rightarrow \nabla \cdot \mathbf{F} = 2x + 3 \Rightarrow \text{Flux} = \iiint_D (2x + 3)\,dV$
$= \int_0^{2\pi}\int_0^{\pi}\int_0^2 (2\rho\sin\phi\cos\theta + 3)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\left[\frac{\rho^4}{2}\sin\phi\cos\theta + \rho^3\right]_0^2 \sin\phi\,d\phi\,d\theta$
$= \int_0^{2\pi}\int_0^{\pi}(8\sin\phi\cos\theta + 8)\sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\left[8\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right)\cos\theta - 8\cos\phi\right]_0^{\pi} d\theta = \int_0^{2\pi}(4\pi\cos\theta + 16)\,d\theta$
$= 32\pi$

9. $\frac{\partial}{\partial x}(x^2) = 2x,\ \frac{\partial}{\partial y}(-2xy) = -2x,\ \frac{\partial}{\partial z}(3xz) = 3x \Rightarrow \text{Flux} = \iiint_D 3x\,dx\,dy\,dz$
$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^2 (3\rho\sin\phi\cos\theta)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta = \int_0^{\pi/2}\int_0^{\pi/2} 12\sin^2\phi\cos\theta\,d\phi\,d\theta = \int_0^{\pi/2} 3\pi\cos\theta\,d\theta = 3\pi$

10. $\frac{\partial}{\partial x}(6x^2 + 2xy) = 12x + 2y,\ \frac{\partial}{\partial y}(2y + x^2z) = 2,\ \frac{\partial}{\partial z}(4x^2y^3) = 0 \Rightarrow \nabla \cdot \mathbf{F} = 12x + 2y + 2$
$\Rightarrow \text{Flux} = \iiint_D (12x + 2y + 2)\,dV = \int_0^3\int_0^{\pi/2}\int_0^2 (12r\cos\theta + 2r\sin\theta + 2)\,r\,dr\,d\theta\,dz$
$= \int_0^3\int_0^{\pi/2}\left(32\cos\theta + \frac{16}{3}\sin\theta + 4\right)d\theta\,dz = \int_0^3\left(32 + 2\pi + \frac{16}{3}\right)dz = 112 + 6\pi$

11. $\frac{\partial}{\partial x}(2xz) = 2z,\ \frac{\partial}{\partial y}(-xy) = -x,\ \frac{\partial}{\partial z}(-z^2) = -2z \Rightarrow \nabla \cdot \mathbf{F} = -x \Rightarrow \text{Flux} = \iiint_D -x\,dV$
$= \int_0^2\int_0^{\sqrt{16-4x^2}}\int_0^{4-y} -x\,dz\,dy\,dx = \int_0^2\int_0^{\sqrt{16-4x^2}} (xy - 4x)\,dy\,dx = \int_0^2\left[\frac{1}{2}x(16 - 4x^2) - 4x\sqrt{16 - 4x^2}\right]dx$
$= \left[4x^2 - \frac{1}{2}x^4 + \frac{1}{3}(16 - 4x^2)^{3/2}\right]_0^2 = -\frac{40}{3}$

12. $\frac{\partial}{\partial x}(x^3) = 3x^2$, $\frac{\partial}{\partial y}(y^3) = 3y^2$, $\frac{\partial}{\partial z}(z^3) = 3z^2 \Rightarrow \nabla \cdot \mathbf{F} = 3x^2 + 3y^2 + 3z^2 \Rightarrow \text{Flux} = \iiint_D 3(x^2+y^2+z^2)\,dV$
$= 3\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2(\rho^2 \sin\phi)\,d\rho\,d\phi\,d\theta = 3\int_0^{2\pi}\int_0^{\pi} \frac{a^5}{5}\sin\phi\,d\phi\,d\theta = 3\int_0^{2\pi} \frac{2a^5}{5}\,d\theta = \frac{12\pi a^5}{5}$

13. Let $\rho = \sqrt{x^2+y^2+z^2}$. Then $\frac{\partial \rho}{\partial x} = \frac{x}{\rho}$, $\frac{\partial \rho}{\partial y} = \frac{y}{\rho}$, $\frac{\partial \rho}{\partial z} = \frac{z}{\rho} \Rightarrow \frac{\partial}{\partial x}(\rho x) = \left(\frac{\partial \rho}{\partial x}\right)x + \rho = \frac{x^2}{\rho} + \rho$, $\frac{\partial}{\partial y}(\rho y) = \left(\frac{\partial \rho}{\partial y}\right)y + \rho$
$= \frac{y^2}{\rho} + \rho$, $\frac{\partial}{\partial z}(\rho z) = \left(\frac{\partial \rho}{\partial z}\right)z + \rho = \frac{z^2}{\rho} + \rho \Rightarrow \nabla \cdot \mathbf{F} = \frac{x^2+y^2+z^2}{\rho} + 3\rho = 4\rho$, since $\rho = \sqrt{x^2+y^2+z^2}$
$\Rightarrow \text{Flux} = \iiint_D 4\rho\,dV = \int_0^{2\pi}\int_0^{\pi}\int_1^{\sqrt{2}} (4\rho)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi} 3\sin\phi\,d\phi\,d\theta = \int_0^{2\pi} 6\,d\theta = 12\pi$

14. Let $\rho = \sqrt{x^2+y^2+z^2}$. Then $\frac{\partial \rho}{\partial x} = \frac{x}{\rho}$, $\frac{\partial \rho}{\partial y} = \frac{y}{\rho}$, $\frac{\partial \rho}{\partial z} = \frac{z}{\rho} \Rightarrow \frac{\partial}{\partial x}\left(\frac{x}{\rho}\right) = \frac{1}{\rho} - \left(\frac{x}{\rho^2}\right)\frac{\partial \rho}{\partial x} = \frac{1}{\rho} - \frac{x^2}{\rho^3}$. Similarly,
$\frac{\partial}{\partial y}\left(\frac{y}{\rho}\right) = \frac{1}{\rho} - \frac{y^2}{\rho^3}$ and $\frac{\partial}{\partial z}\left(\frac{z}{\rho}\right) = \frac{1}{\rho} - \frac{z^2}{\rho^3} \Rightarrow \nabla \cdot \mathbf{F} = \frac{3}{\rho} - \frac{x^2+y^2+z^2}{\rho^3} = \frac{2}{\rho}$
$\Rightarrow \text{Flux} = \iiint_D \frac{2}{\rho}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_1^{2} \left(\frac{2}{\rho}\right)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi} 3\sin\phi\,d\phi\,d\theta = \int_0^{2\pi} 6\,d\theta = 12\pi$

15. $\frac{\partial}{\partial x}(5x^3 + 12xy^2) = 15x^2 + 12y^2$, $\frac{\partial}{\partial y}(y^3 + e^y \sin z) = 3y^2 + e^y \sin z$, $\frac{\partial}{\partial z}(5z^3 + e^y \cos z) = 15z^2 - e^y \sin z$
$\Rightarrow \nabla \cdot \mathbf{F} = 15x^2 + 15y^2 + 15z^2 = 15\rho^2 \Rightarrow \text{Flux} = \iiint_D 15\rho^2\,dV = \int_0^{2\pi}\int_0^{\pi}\int_1^{\sqrt{2}} (15\rho^2)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$
$= \int_0^{2\pi}\int_0^{\pi}\left(12\sqrt{2} - 3\right)\sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\left(24\sqrt{2} - 6\right)d\theta = \left(48\sqrt{2} - 12\right)\pi$

16. $\frac{\partial}{\partial x}\left[\ln(x^2+y^2)\right] = \frac{2x}{x^2+y^2}$, $\frac{\partial}{\partial y}\left(-\frac{2z}{x}\tan^{-1}\frac{y}{x}\right) = \left(-\frac{2z}{x}\right)\left[\frac{\left(\frac{1}{x}\right)}{1+\left(\frac{y}{x}\right)^2}\right] = -\frac{2z}{x^2+y^2}$, $\frac{\partial}{\partial z}\left(z\sqrt{x^2+y^2}\right) = \sqrt{x^2+y^2}$
$\Rightarrow \nabla \cdot \mathbf{F} = \frac{2x}{x^2+y^2} - \frac{2z}{x^2+y^2} + \sqrt{x^2+y^2} \Rightarrow \text{Flux} = \iiint_D \left(\frac{2x}{x^2+y^2} - \frac{2z}{x^2+y^2} + \sqrt{x^2+y^2}\right)dz\,dy\,dx$
$= \int_0^{2\pi}\int_1^{\sqrt{2}}\int_{-1}^{2} \left(\frac{2r\cos\theta}{r^2} - \frac{2z}{r^2} + r\right)dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_1^{\sqrt{2}} \left(6\cos\theta - \frac{3}{r} + 3r^2\right)dr\,d\theta$
$= \int_0^{2\pi}\left[6\left(\sqrt{2}-1\right)\cos\theta - 3\ln\sqrt{2} + 2\sqrt{2} - 1\right]d\theta = 2\pi\left(-\frac{3}{2}\ln 2 + 2\sqrt{2} - 1\right)$

17. (a) $\mathbf{G} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} \Rightarrow \nabla \times \mathbf{G} = \text{curl}\,\mathbf{G} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{k} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} \Rightarrow \nabla \cdot \nabla \times \mathbf{G}$
$= \text{div}(\text{curl}\,\mathbf{G}) = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$
$= \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} + \frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} = 0$ if all first and second partial derivatives are continuous

(b) By the Divergence Theorem, the outward flux of $\nabla \times \mathbf{G}$ across a closed surface is zero because
outward flux of $\nabla \times \mathbf{G} = \iint_S (\nabla \times \mathbf{G}) \cdot \mathbf{n}\,d\sigma$
$= \iiint_D \nabla \cdot \nabla \times \mathbf{G}\,dV$ [Divergence Theorem with $\mathbf{F} = \nabla \times \mathbf{G}$]
$= \iiint_D (0)\,dV = 0$ [by part (a)]

18. (a) Let $\mathbf{F}_1 = M_1\mathbf{i} + N_1\mathbf{j} + P_1\mathbf{k}$ and $\mathbf{F}_2 = M_2\mathbf{i} + N_2\mathbf{j} + P_2\mathbf{k} \Rightarrow a\mathbf{F}_1 + b\mathbf{F}_2$
$= (aM_1 + bM_2)\mathbf{i} + (aN_1 + bN_2)\mathbf{j} + (aP_1 + bP_2)\mathbf{k} \Rightarrow \nabla \cdot (a\mathbf{F}_1 + b\mathbf{F}_2)$
$= \left(a\frac{\partial M_1}{\partial x} + b\frac{\partial M_2}{\partial x}\right) + \left(a\frac{\partial N_1}{\partial y} + b\frac{\partial N_2}{\partial y}\right) + \left(a\frac{\partial P_1}{\partial z} + b\frac{\partial P_2}{\partial z}\right)$
$= a\left(\frac{\partial M_1}{\partial x} + \frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z}\right) + b\left(\frac{\partial M_2}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z}\right) = a(\nabla \cdot \mathbf{F}_1) + b(\nabla \cdot \mathbf{F}_2)$

(b) Define $\mathbf{F}_1$ and $\mathbf{F}_2$ as in part a $\Rightarrow \nabla \times (a\mathbf{F}_1 + b\mathbf{F}_2)$
$= \left[\left(a\frac{\partial P_1}{\partial y} + b\frac{\partial P_2}{\partial y}\right) - \left(a\frac{\partial N_1}{\partial z} + b\frac{\partial N_2}{\partial z}\right)\right]\mathbf{i} + \left[\left(a\frac{\partial M_1}{\partial z} + b\frac{\partial M_2}{\partial z}\right) - \left(a\frac{\partial P_1}{\partial x} + b\frac{\partial P_2}{\partial x}\right)\right]\mathbf{j}$

$+\left[\left(a\,\frac{\partial N_1}{\partial x}+b\,\frac{\partial N_2}{\partial x}\right)-\left(a\,\frac{\partial M_1}{\partial y}+b\,\frac{\partial M_2}{\partial y}\right)\right]\mathbf{k}=a\left[\left(\frac{\partial P_1}{\partial y}-\frac{\partial N_1}{\partial z}\right)\mathbf{i}+\left(\frac{\partial M_1}{\partial z}-\frac{\partial P_1}{\partial x}\right)\mathbf{j}+\left(\frac{\partial N_1}{\partial x}-\frac{\partial M_1}{\partial y}\right)\mathbf{k}\right]$

$+b\left[\left(\frac{\partial P_2}{\partial y}-\frac{\partial N_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial M_2}{\partial z}-\frac{\partial P_2}{\partial x}\right)\mathbf{j}+\left(\frac{\partial N_2}{\partial x}-\frac{\partial M_2}{\partial y}\right)\mathbf{k}\right]=a\nabla\times\mathbf{F}_1+b\nabla\times\mathbf{F}_2$

(c) $\mathbf{F}_1\times\mathbf{F}_2=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ M_1&N_1&P_1\\ M_2&N_2&P_2\end{vmatrix}=(N_1P_2-P_1N_2)\mathbf{i}-(M_1P_2-P_1M_2)\mathbf{j}+(M_1N_2-N_1M_2)\mathbf{k}\;\Rightarrow\;\nabla\cdot(\mathbf{F}_1\times\mathbf{F}_2)$

$=\nabla\cdot[(N_1P_2-P_1N_2)\mathbf{i}-(M_1P_2-P_1M_2)\mathbf{j}+(M_1N_2-N_1M_2)\mathbf{k}]$

$=\frac{\partial}{\partial x}(N_1P_2-P_1N_2)-\frac{\partial}{\partial y}(M_1P_2-P_1M_2)+\frac{\partial}{\partial z}(M_1N_2-N_1M_2)=\left(P_2\,\frac{\partial N_1}{\partial x}+N_1\,\frac{\partial P_2}{\partial x}-N_2\,\frac{\partial P_1}{\partial x}-P_1\,\frac{\partial N_2}{\partial x}\right)$

$-\left(M_1\,\frac{\partial P_2}{\partial y}+P_2\,\frac{\partial M_1}{\partial y}-P_1\,\frac{\partial M_2}{\partial y}-M_2\,\frac{\partial P_1}{\partial y}\right)+\left(M_1\,\frac{\partial N_2}{\partial z}+N_2\,\frac{\partial M_1}{\partial z}-N_1\,\frac{\partial M_2}{\partial z}-M_2\,\frac{\partial N_1}{\partial z}\right)$

$=M_2\left(\frac{\partial P_1}{\partial y}-\frac{\partial N_1}{\partial z}\right)+N_2\left(\frac{\partial M_1}{\partial z}-\frac{\partial P_1}{\partial x}\right)+P_2\left(\frac{\partial N_1}{\partial x}-\frac{\partial M_1}{\partial y}\right)+M_1\left(\frac{\partial N_2}{\partial z}-\frac{\partial P_2}{\partial y}\right)+N_1\left(\frac{\partial P_2}{\partial x}-\frac{\partial M_2}{\partial z}\right)$

$+P_1\left(\frac{\partial M_2}{\partial y}-\frac{\partial N_2}{\partial x}\right)=\mathbf{F}_2\cdot\nabla\times\mathbf{F}_1-\mathbf{F}_1\cdot\nabla\times\mathbf{F}_2$

19. (a) $\operatorname{div}(g\mathbf{F})=\nabla\cdot g\mathbf{F}=\frac{\partial}{\partial x}(gM)+\frac{\partial}{\partial y}(gN)+\frac{\partial}{\partial z}(gP)=\left(g\,\frac{\partial M}{\partial x}+M\,\frac{\partial g}{\partial x}\right)+\left(g\,\frac{\partial N}{\partial y}+N\,\frac{\partial g}{\partial y}\right)+\left(g\,\frac{\partial P}{\partial z}+P\,\frac{\partial g}{\partial z}\right)$

$=\left(M\,\frac{\partial g}{\partial x}+N\,\frac{\partial g}{\partial y}+P\,\frac{\partial g}{\partial z}\right)+g\left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}\right)=g\nabla\cdot\mathbf{F}+\nabla g\cdot\mathbf{F}$

(b) $\nabla\times(g\mathbf{F})=\left[\frac{\partial}{\partial y}(gP)-\frac{\partial}{\partial z}(gN)\right]\mathbf{i}+\left[\frac{\partial}{\partial z}(gM)-\frac{\partial}{\partial x}(gP)\right]\mathbf{j}+\left[\frac{\partial}{\partial x}(gN)-\frac{\partial}{\partial y}(gM)\right]\mathbf{k}$

$=\left(P\,\frac{\partial g}{\partial y}+g\,\frac{\partial P}{\partial y}-N\,\frac{\partial g}{\partial z}-g\,\frac{\partial N}{\partial z}\right)\mathbf{i}+\left(M\,\frac{\partial g}{\partial z}+g\,\frac{\partial M}{\partial z}-P\,\frac{\partial g}{\partial x}-g\,\frac{\partial P}{\partial x}\right)\mathbf{j}+\left(N\,\frac{\partial g}{\partial x}+g\,\frac{\partial N}{\partial x}-M\,\frac{\partial g}{\partial y}-g\,\frac{\partial M}{\partial y}\right)\mathbf{k}$

$=\left(P\,\frac{\partial g}{\partial y}-N\,\frac{\partial g}{\partial z}\right)\mathbf{i}+\left(g\,\frac{\partial P}{\partial y}-g\,\frac{\partial N}{\partial z}\right)\mathbf{i}+\left(M\,\frac{\partial g}{\partial z}-P\,\frac{\partial g}{\partial x}\right)\mathbf{j}+\left(g\,\frac{\partial M}{\partial z}-g\,\frac{\partial P}{\partial x}\right)\mathbf{j}+\left(N\,\frac{\partial g}{\partial x}-M\,\frac{\partial g}{\partial y}\right)\mathbf{k}$

$+\left(g\,\frac{\partial N}{\partial x}-g\,\frac{\partial M}{\partial y}\right)\mathbf{k}=g\nabla\times\mathbf{F}+\nabla g\times\mathbf{F}$

20. Let $\mathbf{F}_1=M_1\mathbf{i}+N_1\mathbf{j}+P_1\mathbf{k}$ and $\mathbf{F}_2=M_2\mathbf{i}+N_2\mathbf{j}+P_2\mathbf{k}$.

(a) $\mathbf{F}_1\times\mathbf{F}_2=(N_1P_2-P_1N_2)\mathbf{i}+(P_1M_2-M_1P_2)\mathbf{j}+(M_1N_2-N_1M_2)\mathbf{k}\;\Rightarrow\;\nabla\times(\mathbf{F}_1\times\mathbf{F}_2)$

$=\left[\frac{\partial}{\partial y}(M_1N_2-N_1M_2)-\frac{\partial}{\partial z}(P_1M_2-M_1P_2)\right]\mathbf{i}+\left[\frac{\partial}{\partial z}(N_1P_2-P_1N_2)-\frac{\partial}{\partial x}(M_1N_2-N_1M_2)\right]\mathbf{j}$

$+\left[\frac{\partial}{\partial x}(P_1M_2-M_1P_2)-\frac{\partial}{\partial y}(N_1P_2-P_1N_2)\right]\mathbf{k}$

and consider the **i**-component only: $\frac{\partial}{\partial y}(M_1N_2-N_1M_2)-\frac{\partial}{\partial z}(P_1M_2-M_1P_2)$

$=N_2\,\frac{\partial M_1}{\partial y}+M_1\,\frac{\partial N_2}{\partial y}-M_2\,\frac{\partial N_1}{\partial y}-N_1\,\frac{\partial M_2}{\partial y}-M_2\,\frac{\partial P_1}{\partial z}-P_1\,\frac{\partial M_2}{\partial z}+P_2\,\frac{\partial M_1}{\partial z}+M_1\,\frac{\partial P_2}{\partial z}$

$=\left(N_2\,\frac{\partial M_1}{\partial y}+P_2\,\frac{\partial M_1}{\partial z}\right)-\left(N_1\,\frac{\partial M_2}{\partial y}+P_1\,\frac{\partial M_2}{\partial z}\right)+\left(\frac{\partial N_2}{\partial y}+\frac{\partial P_2}{\partial z}\right)M_1-\left(\frac{\partial N_1}{\partial y}+\frac{\partial P_1}{\partial z}\right)M_2$

$=\left(M_2\,\frac{\partial M_1}{\partial x}+N_2\,\frac{\partial M_1}{\partial y}+P_2\,\frac{\partial M_1}{\partial z}\right)-\left(M_1\,\frac{\partial M_2}{\partial x}+N_1\,\frac{\partial M_2}{\partial y}+P_1\,\frac{\partial M_2}{\partial z}\right)+\left(\frac{\partial M_2}{\partial x}+\frac{\partial N_2}{\partial y}+\frac{\partial P_2}{\partial z}\right)M_1$

$-\left(\frac{\partial M_1}{\partial x}+\frac{\partial N_1}{\partial y}+\frac{\partial P_1}{\partial z}\right)M_2$. Now, **i**-comp of $(\mathbf{F}_2\cdot\nabla)\mathbf{F}_1=\left(M_2\,\frac{\partial}{\partial x}+N_2\,\frac{\partial}{\partial y}+P_2\,\frac{\partial}{\partial z}\right)M_1$

$=\left(M_2\,\frac{\partial M_1}{\partial x}+N_2\,\frac{\partial M_1}{\partial y}+P_2\,\frac{\partial M_1}{\partial z}\right)$; likewise, **i**-comp of $(\mathbf{F}_1\cdot\nabla)\mathbf{F}_2=\left(M_1\,\frac{\partial M_2}{\partial x}+N_1\,\frac{\partial M_2}{\partial y}+P_1\,\frac{\partial M_2}{\partial z}\right)$;

i-comp of $(\nabla\cdot\mathbf{F}_2)\mathbf{F}_1=\left(\frac{\partial M_2}{\partial x}+\frac{\partial N_2}{\partial y}+\frac{\partial P_2}{\partial z}\right)M_1$ and **i**-comp of $(\nabla\cdot\mathbf{F}_1)\mathbf{F}_2=\left(\frac{\partial M_1}{\partial x}+\frac{\partial N_1}{\partial y}+\frac{\partial P_1}{\partial z}\right)M_2$.

Similar results hold for the **j** and **k** components of $\nabla\times(\mathbf{F}_1\times\mathbf{F}_2)$. In summary, since the corresponding components are equal, we have the result

$\nabla\times(\mathbf{F}_1\times\mathbf{F}_2)=(\mathbf{F}_2\cdot\nabla)\mathbf{F}_1-(\mathbf{F}_1\cdot\nabla)\mathbf{F}_2+(\nabla\cdot\mathbf{F}_2)\mathbf{F}_1-(\nabla\cdot\mathbf{F}_1)\mathbf{F}_2$

(b) Here again we consider only the **i**-component of each expression. Thus, the **i**-comp of $\nabla(\mathbf{F}_1\cdot\mathbf{F}_2)$

$=\frac{\partial}{\partial x}(M_1M_2+N_1N_2+P_1P_2)=\left(M_1\,\frac{\partial M_2}{\partial x}+M_2\,\frac{\partial M_1}{\partial x}+N_1\,\frac{\partial N_2}{\partial x}+N_2\,\frac{\partial N_1}{\partial x}+P_1\,\frac{\partial P_2}{\partial x}+P_2\,\frac{\partial P_1}{\partial x}\right)$

i-comp of $(\mathbf{F}_1\cdot\nabla)\mathbf{F}_2=\left(M_1\,\frac{\partial M_2}{\partial x}+N_1\,\frac{\partial M_2}{\partial y}+P_1\,\frac{\partial M_2}{\partial z}\right)$,

i-comp of $(\mathbf{F}_2\cdot\nabla)\mathbf{F}_1=\left(M_2\,\frac{\partial M_1}{\partial x}+N_2\,\frac{\partial M_1}{\partial y}+P_2\,\frac{\partial M_1}{\partial z}\right)$,

i-comp of $\mathbf{F}_1\times(\nabla\times\mathbf{F}_2)=N_1\left(\frac{\partial N_2}{\partial x}-\frac{\partial M_2}{\partial y}\right)-P_1\left(\frac{\partial M_2}{\partial z}-\frac{\partial P_2}{\partial x}\right)$, and

i-comp of $\mathbf{F}_2 \times (\nabla \times \mathbf{F}_1) = N_2\left(\frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y}\right) - P_2\left(\frac{\partial M_1}{\partial z} - \frac{\partial P_1}{\partial x}\right)$.

Since corresponding components are equal, we see that

$\nabla(\mathbf{F}_1 \cdot \mathbf{F}_2) = (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 + \mathbf{F}_1 \times (\nabla \times \mathbf{F}_2) + \mathbf{F}_2 \times (\nabla \times \mathbf{F}_1)$, as claimed.

21. The integral's value never exceeds the surface area of S. Since $|\mathbf{F}| \le 1$, we have $|\mathbf{F} \cdot \mathbf{n}| = |\mathbf{F}|\,|\mathbf{n}| \le (1)(1) = 1$ and

$$\begin{aligned} \iiint_D \nabla \cdot \mathbf{F}\, d\sigma &= \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma && \text{[Divergence Theorem]} \\ &\le \iint_S |\mathbf{F} \cdot \mathbf{n}|\, d\sigma && \text{[A property of integrals]} \\ &\le \iint_S (1)\, d\sigma && [|\mathbf{F} \cdot \mathbf{n}| \le 1] \\ &= \text{Area of S.} \end{aligned}$$

22. Yes, the outward flux through the top is 5. The reason is this: Since $\nabla \cdot \mathbf{F} = \nabla \cdot (x\mathbf{i} - 2y\mathbf{j} + (z+3)\mathbf{k}) = 1 - 2 + 1 = 0$, the outward flux across the closed cubelike surface is 0 by the Divergence Theorem. The flux across the top is therefore the negative of the flux across the sides and base. Routine calculations show that the sum of these latter fluxes is -5. (The flux across the sides that lie in the xz-plane and the yz-plane are 0, while the flux across the xy-plane is -3.) Therefore the flux across the top is 5.

23. (a) $\frac{\partial}{\partial x}(x) = 1,\ \frac{\partial}{\partial y}(y) = 1,\ \frac{\partial}{\partial z}(z) = 1 \Rightarrow \nabla \cdot \mathbf{F} = 3 \Rightarrow \text{Flux} = \iiint_D 3\, dV = 3\iiint_D dV$

$= 3(\text{Volume of the solid})$

(b) If **F** is orthogonal to **n** at every point of S, then $\mathbf{F} \cdot \mathbf{n} = 0$ everywhere $\Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0$.

But the flux is 3(Volume of the solid) $\neq 0$, so **F** is not orthogonal to **n** at every point.

24. $\nabla \cdot \mathbf{F} = -2x - 4y - 6z + 12 \Rightarrow \text{Flux} = \int_0^a \int_0^b \int_0^1 (-2x - 4y - 6z + 12)\, dz\, dy\, dx$
$= \int_0^a \int_0^b (-2x - 4y + 9)\, dy\, dx = \int_0^a (-2xb - 2b^2 + 9b)\, dx = -a^2b - 2ab^2 + 9ab = ab(-a - 2b + 9) = f(a, b)$;
$\frac{\partial f}{\partial a} = -2ab - 2b^2 + 9b$ and $\frac{\partial f}{\partial b} = -a^2 - 4ab + 9a$ so that $\frac{\partial f}{\partial a} = 0$ and $\frac{\partial f}{\partial b} = 0 \Rightarrow b(-2a - 2b + 9) = 0$ and $a(-a - 4b + 9) = 0 \Rightarrow b = 0$ or $-2a - 2b + 9 = 0$, and $a = 0$ or $-a - 4b + 9 = 0$. Now $b = 0$ or $a = 0$ $\Rightarrow \text{Flux} = 0$; $-2a - 2b + 9 = 0$ and $-a - 4b + 9 = 0 \Rightarrow 3a - 9 = 0 \Rightarrow a = 3 \Rightarrow b = \frac{3}{2}$ so that $f\left(3, \frac{3}{2}\right) = \frac{27}{2}$ is the maximum flux.

25. $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV = \iiint_D 3\, dV \Rightarrow \frac{1}{3}\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D dV = \text{Volume of D}$

26. $\mathbf{F} = \mathbf{C} \Rightarrow \nabla \cdot \mathbf{F} = 0 \Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV = \iiint_D 0\, dV = 0$

27. (a) From the Divergence Theorem, $\iint_S \nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \nabla f\, dV = \iiint_D \nabla^2 f\, dV = \iiint_D 0\, dV = 0$

(b) From the Divergence Theorem, $\iint_S f\nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot f\nabla f\, dV$. Now,

$f\nabla f = \left(f\frac{\partial f}{\partial x}\right)\mathbf{i} + \left(f\frac{\partial f}{\partial y}\right)\mathbf{j} + \left(f\frac{\partial f}{\partial z}\right)\mathbf{k} \Rightarrow \nabla \cdot f\nabla f = \left[f\frac{\partial^2 f}{\partial x^2} + \left(\frac{\partial f}{\partial x}\right)^2\right] + \left[f\frac{\partial^2 f}{\partial y^2} + \left(\frac{\partial f}{\partial y}\right)^2\right] + \left[f\frac{\partial^2 f}{\partial z^2} + \left(\frac{\partial f}{\partial z}\right)^2\right]$

$= f\nabla^2 f + |\nabla f|^2 = 0 + |\nabla f|^2$ since f is harmonic $\Rightarrow \iint_S f\nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D |\nabla f|^2\, dV$, as claimed.

28. From the Divergence Theorem, $\iint_S \nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \nabla f\, dV = \iiint_D \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) dV$. Now,

$f(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2} = \frac{1}{2}\ln(x^2 + y^2 + z^2) \Rightarrow \frac{\partial f}{\partial x} = \frac{x}{x^2 + y^2 + z^2},\ \frac{\partial f}{\partial y} = \frac{y}{x^2 + y^2 + z^2},\ \frac{\partial f}{\partial z} = \frac{z}{x^2 + y^2 + z^2}$

$\Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2}, \frac{\partial^2 f}{\partial y^2} = \frac{x^2-y^2+z^2}{(x^2+y^2+z^2)^2}, \frac{\partial^2 f}{\partial z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}, \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
$= \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = \frac{1}{x^2+y^2+z^2} \Rightarrow \iint_S \nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D \frac{dV}{x^2+y^2+z^2} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \frac{\rho^2 \sin\phi}{\rho^2}\, d\rho\, d\phi\, d\theta$
$= \int_0^{\pi/2}\int_0^{\pi/2} a \sin\phi\, d\phi\, d\theta = \int_0^{\pi/2} [-a\cos\phi]_0^{\pi/2}\, d\theta = \int_0^{\pi/2} a\, d\theta = \frac{\pi a}{2}$

29. $\iint_S f\nabla g \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot f\nabla g\, dV = \iiint_D \nabla \cdot \left(f\frac{\partial g}{\partial x}\mathbf{i} + f\frac{\partial g}{\partial y}\mathbf{j} + f\frac{\partial g}{\partial z}\mathbf{k}\right) dV$
$= \iiint_D \left(f\frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial y^2} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + f\frac{\partial^2 g}{\partial z^2} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right) dV$
$= \iiint_D \left[f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right)\right] dV = \iiint_D (f\nabla^2 g + \nabla f \cdot \nabla g)\, dV$

30. By Exercise 29, $\iint_S f\nabla g \cdot \mathbf{n}\, d\sigma = \iiint_D (f\nabla^2 g + \nabla f \cdot \nabla g)\, dV$ and by interchanging the roles of f and g, $\iint_S g\nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D (g\nabla^2 f + \nabla g \cdot \nabla f)\, dV$. Subtracting the second equation from the first yields: $\iint_S (f\nabla g - g\nabla f) \cdot \mathbf{n}\, d\sigma = \iiint_D (f\nabla^2 g - g\nabla^2 f)\, dV$ since $\nabla f \cdot \nabla g = \nabla g \cdot \nabla f$.

31. (a) The integral $\iiint_D p(t,x,y,z)\, dV$ represents the mass of the fluid at any time t. The equation says that the instantaneous rate of change of mass is flux of the fluid through the surface S enclosing the region D: the mass decreases if the flux is outward (so the fluid flows out of D), and increases if the flow is inward (interpreting **n** as the outward pointing unit normal to the surface).

(b) $\iiint_D \frac{\partial p}{\partial t}\, dV = \frac{d}{dt}\iiint_D p\, dV = -\iint_S p\mathbf{v} \cdot \mathbf{n}\, d\sigma = -\iiint_D \nabla \cdot p\mathbf{v}\, dV \Rightarrow \frac{\partial \rho}{\partial t} = -\nabla \cdot p\mathbf{v}$

Since the law is to hold for all regions D, $\nabla \cdot p\mathbf{v} + \frac{\partial p}{\partial t} = 0$, as claimed

32. (a) ∇T points in the direction of maximum change of the temperature, so if the solid is heating up at the point the temperature is greater in a region surrounding the point $\Rightarrow \nabla T$ points away from the point $\Rightarrow -\nabla T$ points toward the point $\Rightarrow -\nabla T$ points in the direction the heat flows.

(b) Assuming the Law of Conservation of Mass (Exercise 31) with $-k\nabla T = p\mathbf{v}$ and $c\rho T = p$, we have $\frac{d}{dt}\iiint_D c\rho T\, dV = -\iint_S -k\nabla T \cdot \mathbf{n}\, d\sigma \Rightarrow$ the continuity equation, $\nabla \cdot (-k\nabla T) + \frac{\partial}{\partial t}(c\rho T) = 0$
$\Rightarrow c\rho\frac{\partial T}{\partial t} = -\nabla \cdot (-k\nabla T) = k\nabla^2 T \Rightarrow \frac{\partial T}{\partial t} = \frac{k}{c\rho}\nabla^2 T = K\nabla^2 T$, as claimed

CHAPTER 16 PRACTICE EXERCISES

1. Path 1: $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} \Rightarrow x = t, y = t, z = t, 0 \le t \le 1 \Rightarrow f(g(t), h(t), k(t)) = 3 - 3t^2$ and $\frac{dx}{dt} = 1, \frac{dy}{dt} = 1,$
$\frac{dz}{dt} = 1 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \sqrt{3}\, dt \Rightarrow \int_C f(x,y,z)\, ds = \int_0^1 \sqrt{3}(3 - 3t^2)\, dt = 2\sqrt{3}$

Path 2: $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}, 0 \le t \le 1 \Rightarrow x = t, y = t, z = 0 \Rightarrow f(g(t), h(t), k(t)) = 2t - 3t^2 + 3$ and $\frac{dx}{dt} = 1, \frac{dy}{dt} = 1,$
$\frac{dz}{dt} = 0 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \sqrt{2}\, dt \Rightarrow \int_{C_1} f(x,y,z)\, ds = \int_0^1 \sqrt{2}(2t - 3t^2 + 3)\, dt = 3\sqrt{2};$

$\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k} \Rightarrow x = 1, y = 1, z = t \Rightarrow f(g(t), h(t), k(t)) = 2 - 2t$ and $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0, \frac{dz}{dt} = 1$
$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = dt \Rightarrow \int_{C_2} f(x,y,z)\, ds = \int_0^1 (2 - 2t)\, dt = 1$
$\Rightarrow \int_C f(x,y,z)\, ds = \int_{C_1} f(x,y,z)\, ds + \int_{C_2} f(x,y,z) = 3\sqrt{2} + 1$

2. Path 1: $\mathbf{r}_1 = t\mathbf{i} \Rightarrow x = t, y = 0, z = 0 \Rightarrow f(g(t), h(t), k(t)) = t^2$ and $\frac{dx}{dt} = 1, \frac{dy}{dt} = 0, \frac{dz}{dt} = 0$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = dt \Rightarrow \int_{C_1} f(x, y, z)\,ds = \int_0^1 t^2\,dt = \frac{1}{3}$;

$\mathbf{r}_2 = \mathbf{i} + t\mathbf{j} \Rightarrow x = 1, y = t, z = 0 \Rightarrow f(g(t), h(t), k(t)) = 1 + t$ and $\frac{dx}{dt} = 0, \frac{dy}{dt} = 1, \frac{dz}{dt} = 0$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = dt \Rightarrow \int_{C_2} f(x, y, z)\,ds = \int_0^1 (1 + t)\,dt = \frac{3}{2}$;

$\mathbf{r}_3 = \mathbf{i} + \mathbf{j} + t\mathbf{k} \Rightarrow x = 1, y = 1, z = t \Rightarrow f(g(t), h(t), k(t)) = 2 - t$ and $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0, \frac{dz}{dt} = 1$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = dt \Rightarrow \int_{C_3} f(x, y, z)\,ds = \int_0^1 (2 - t)\,dt = \frac{3}{2}$

$\Rightarrow \int_{\text{Path 1}} f(x, y, z)\,ds = \int_{C_1} f(x, y, z)\,ds + \int_{C_2} f(x, y, z)\,ds + \int_{C_3} f(x, y, z)\,ds = \frac{10}{3}$

Path 2: $\mathbf{r}_4 = t\mathbf{i} + t\mathbf{j} \Rightarrow x = t, y = t, z = 0 \Rightarrow f(g(t), h(t), k(t)) = t^2 + t$ and $\frac{dx}{dt} = 1, \frac{dy}{dt} = 1, \frac{dz}{dt} = 0$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \sqrt{2}\,dt \Rightarrow \int_{C_4} f(x, y, z)\,ds = \int_0^1 \sqrt{2}\,(t^2 + t)\,dt = \frac{5}{6}\sqrt{2}$;

$\mathbf{r}_3 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$ (see above) $\Rightarrow \int_{C_3} f(x, y, z)\,ds = \frac{3}{2}$

$\Rightarrow \int_{\text{Path 2}} f(x, y, z)\,ds = \int_{C_3} f(x, y, z)\,ds + \int_{C_4} f(x, y, z)\,ds = \frac{5}{6}\sqrt{2} + \frac{3}{2} = \frac{5\sqrt{2}+9}{6}$

Path 3: $\mathbf{r}_5 = t\mathbf{k} \Rightarrow x = 0, y = 0, z = t, 0 \le t \le 1 \Rightarrow f(g(t), h(t), k(t)) = -t$ and $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0, \frac{dz}{dt} = 1$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = dt \Rightarrow \int_{C_5} f(x, y, z)\,ds = \int_0^1 -t\,dt = -\frac{1}{2}$;

$\mathbf{r}_6 = t\mathbf{j} + \mathbf{k} \Rightarrow x = 0, y = t, z = 1, 0 \le t \le 1 \Rightarrow f(g(t), h(t), k(t)) = t - 1$ and $\frac{dx}{dt} = 0, \frac{dy}{dt} = 1, \frac{dz}{dt} = 0$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = dt \Rightarrow \int_{C_6} f(x, y, z)\,ds = \int_0^1 (t - 1)\,dt = -\frac{1}{2}$;

$\mathbf{r}_7 = t\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow x = t, y = 1, z = 1, 0 \le t \le 1 \Rightarrow f(g(t), h(t), k(t)) = t^2$ and $\frac{dx}{dt} = 1, \frac{dy}{dt} = 0, \frac{dz}{dt} = 0$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = dt \Rightarrow \int_{C_7} f(x, y, z)\,ds = \int_0^1 t^2\,dt = \frac{1}{3}$

$\Rightarrow \int_{\text{Path 3}} f(x, y, z)\,ds = \int_{C_5} f(x, y, z)\,ds + \int_{C_6} f(x, y, z)\,ds + \int_{C_7} f(x, y, z)\,ds = -\frac{1}{2} - \frac{1}{2} + \frac{1}{3} = -\frac{2}{3}$

3. $\mathbf{r} = (a\cos t)\mathbf{j} + (a\sin t)\mathbf{k} \Rightarrow x = 0, y = a\cos t, z = a\sin t \Rightarrow f(g(t), h(t), k(t)) = \sqrt{a^2\sin^2 t} = a\,|\sin t|$ and

$\frac{dx}{dt} = 0, \frac{dy}{dt} = -a\sin t, \frac{dz}{dt} = a\cos t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = a\,dt$

$\Rightarrow \int_C f(x, y, z)\,ds = \int_0^{2\pi} a^2\,|\sin t|\,dt = \int_0^{\pi} a^2\sin t\,dt + \int_{\pi}^{2\pi} -a^2\sin t\,dt = 4a^2$

4. $\mathbf{r} = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} \Rightarrow x = \cos t + t\sin t, y = \sin t - t\cos t, z = 0$

$\Rightarrow f(g(t), h(t), k(t)) = \sqrt{(\cos t + t\sin t)^2 + (\sin t - t\cos t)^2} = \sqrt{1 + t^2}$ and $\frac{dx}{dt} = -\sin t + \sin t + t\cos t$

$= t\cos t, \frac{dy}{dt} = \cos t - \cos t + t\sin t = t\sin t, \frac{dz}{dt} = 0 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$

$= \sqrt{t^2\cos^2 t + t^2\sin^2 t}\,dt = |t|\,dt = t\,dt$ since $0 \le t \le \sqrt{3} \Rightarrow \int_C f(x, y, z)\,ds = \int_0^{\sqrt{3}} t\sqrt{1 + t^2}\,dt = \frac{7}{3}$

5. $\frac{\partial P}{\partial y} = -\frac{1}{2}(x + y + z)^{-3/2} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -\frac{1}{2}(x + y + z)^{-3/2} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{1}{2}(x + y + z)^{-3/2} = \frac{\partial M}{\partial y}$

$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = \frac{1}{\sqrt{x+y+z}} \Rightarrow f(x, y, z) = 2\sqrt{x + y + z} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{1}{\sqrt{x+y+z}} + \frac{\partial g}{\partial y}$

$= \frac{1}{\sqrt{x+y+z}} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow f(x, y, z) = 2\sqrt{x + y + z} + h(z) \Rightarrow \frac{\partial f}{\partial z} = \frac{1}{\sqrt{x+y+z}} + h'(z)$

$= \frac{1}{\sqrt{x+y+z}} \Rightarrow h'(x) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = 2\sqrt{x + y + z} + C \Rightarrow \int_{(-1,1,1)}^{(4,-3,0)} \frac{dx + dy + dz}{\sqrt{x+y+z}}$

$= f(4, -3, 0) - f(-1, 1, 1) = 2\sqrt{1} - 2\sqrt{1} = 0$

6. $\frac{\partial P}{\partial y} = -\frac{1}{2\sqrt{yz}} = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow$ M dx + N dy + P dz is exact; $\frac{\partial f}{\partial x} = 1 \Rightarrow f(x, y, z)$
$= x + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = -\sqrt{\frac{z}{y}} \Rightarrow g(y, z) = -2\sqrt{yz} + h(z) \Rightarrow f(x, y, z) = x - 2\sqrt{yz} + h(z)$
$\Rightarrow \frac{\partial f}{\partial z} = -\sqrt{\frac{y}{z}} + h'(z) = -\sqrt{\frac{y}{z}} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = x - 2\sqrt{yz} + C$
$\Rightarrow \int_{(1,1,1)}^{(10,3,3)} dx - \sqrt{\frac{z}{y}}\, dy - \sqrt{\frac{y}{z}}\, dz = f(10, 3, 3) - f(1, 1, 1) = (10 - 2 \cdot 3) - (1 - 2 \cdot 1) = 4 + 1 = 5$

7. $\frac{\partial M}{\partial z} = -y \cos z \neq y \cos z = \frac{\partial P}{\partial x} \Rightarrow \mathbf{F}$ is not conservative; $\mathbf{r} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} - \mathbf{k}, 0 \le t \le 2\pi$
$\Rightarrow d\mathbf{r} = (-2 \sin t)\mathbf{i} - (2 \cos t)\mathbf{j} \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} [-(-2 \sin t)(\sin(-1))(-2 \sin t) + (2 \cos t)(\sin(-1))(-2 \cos t)]\, dt$
$= 4 \sin(1) \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = 8\pi \sin(1)$

8. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = 3x^2 = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative $\Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = 0$

9. Let $M = 8x \sin y$ and $N = -8y \cos x \Rightarrow \frac{\partial M}{\partial y} = 8x \cos y$ and $\frac{\partial N}{\partial x} = 8y \sin x \Rightarrow \int_C 8x \sin y\, dx - 8y \cos x\, dy$
$= \iint_R (8y \sin x - 8x \cos y)\, dy\, dx = \int_0^{\pi/2} \int_0^{\pi/2} (8y \sin x - 8x \cos y)\, dy\, dx = \int_0^{\pi/2} (\pi^2 \sin x - 8x)\, dx$
$= -\pi^2 + \pi^2 = 0$

10. Let $M = y^2$ and $N = x^2 \Rightarrow \frac{\partial M}{\partial y} = 2y$ and $\frac{\partial N}{\partial x} = 2x \Rightarrow \int_C y^2\, dx + x^2\, dy = \iint_R (2x - 2y)\, dx\, dy$
$= \int_0^{2\pi} \int_0^2 (2r \cos\theta - 2r \sin\theta)\, r\, dr\, d\theta = \int_0^{2\pi} \frac{16}{3} (\cos\theta - \sin\theta)\, d\theta = 0$

11. Let $z = 1 - x - y \Rightarrow f_x(x, y) = -1$ and $f_y(x, y) = -1 \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{3} \Rightarrow$ Surface Area $= \iint_R \sqrt{3}\, dx\, dy$
$= \sqrt{3}$(Area of the circular region in the xy-plane) $= \pi\sqrt{3}$

12. $\nabla f = -3\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\mathbf{p} = \mathbf{i} \Rightarrow |\nabla f| = \sqrt{9 + 4y^2 + 4z^2}$ and $|\nabla f \cdot \mathbf{p}| = 3$
$\Rightarrow$ Surface Area $= \iint_R \frac{\sqrt{9 + 4y^2 + 4z^2}}{3}\, dy\, dz = \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{\sqrt{9 + 4r^2}}{3}\, r\, dr\, d\theta = \frac{1}{3} \int_0^{2\pi} \left(\frac{7}{4}\sqrt{21} - \frac{9}{4}\right) d\theta = \frac{\pi}{6}\left(7\sqrt{21} - 9\right)$

13. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 2$ and $|\nabla f \cdot \mathbf{p}| = |2z| = 2z$ since
$z \ge 0 \Rightarrow$ Surface Area $= \iint_R \frac{2}{2z}\, dA = \iint_R \frac{1}{z}\, dA = \iint_R \frac{1}{\sqrt{1 - x^2 - y^2}}\, dx\, dy = \int_0^{2\pi} \int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1 - r^2}}\, r\, dr\, d\theta$
$\int_0^{2\pi} \left[-\sqrt{1 - r^2}\right]_0^{1/\sqrt{2}} d\theta = \int_0^{2\pi} \left(1 - \frac{1}{\sqrt{2}}\right) d\theta = 2\pi\left(1 - \frac{1}{\sqrt{2}}\right)$

14. (a) $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 4$ and $|\nabla f \cdot \mathbf{p}| = 2z$ since
$z \ge 0 \Rightarrow$ Surface Area $= \iint_R \frac{4}{2z}\, dA = \iint_R \frac{2}{z}\, dA = 2 \int_0^{\pi/2} \int_0^{2\cos\theta} \frac{2}{\sqrt{4 - r^2}}\, r\, dr\, d\theta = 4\pi - 8$

(b) $\mathbf{r} = 2 \cos\theta \Rightarrow d\mathbf{r} = -2 \sin\theta\, d\theta$; $ds^2 = r^2\, d\theta^2 + dr^2$ (Arc length in polar coordinates)
$\Rightarrow ds^2 = (2 \cos\theta)^2\, d\theta^2 + dr^2 = 4 \cos^2\theta\, d\theta^2 + 4 \sin^2\theta\, d\theta^2 = 4\, d\theta^2 \Rightarrow ds = 2\, d\theta$; the height of the cylinder is $z = \sqrt{4 - r^2} = \sqrt{4 - 4\cos^2\theta} = 2\,|\sin\theta| = 2 \sin\theta$ if $0 \le \theta \le \frac{\pi}{2}$
$\Rightarrow$ Surface Area $= \int_{-\pi/2}^{\pi/2} h\, ds = 2 \int_0^{\pi/2} (2 \sin\theta)(2\, d\theta) = 8$

15. $f(x, y, z) = \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \Rightarrow \nabla f = \left(\frac{1}{a}\right)\mathbf{i} + \left(\frac{1}{b}\right)\mathbf{j} + \left(\frac{1}{c}\right)\mathbf{k} \Rightarrow |\nabla f| = \sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$ and $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = \frac{1}{c}$ since $c > 0 \Rightarrow$ Surface Area $= \iint_R \frac{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}{\left(\frac{1}{c}\right)}\, dA = c\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} \iint_R dA = \frac{1}{2}\, abc\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$, since the area of the triangular region R is $\frac{1}{2}$ ab. To check this result, let $\mathbf{v} = a\mathbf{i} + c\mathbf{k}$ and $\mathbf{w} = -a\mathbf{i} + b\mathbf{j}$; the area can be found by computing $\frac{1}{2}|\mathbf{v} \times \mathbf{w}|$.

16. (a) $\nabla f = 2y\mathbf{j} - \mathbf{k}$, $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 1}$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{4y^2 + 1}\, dx\, dy$
$\Rightarrow \iint_S g(x, y, z)\, d\sigma = \iint_R \frac{yz}{\sqrt{4y^2+1}} \sqrt{4y^2 + 1}\, dx\, dy = \iint_R y\,(y^2 - 1)\, dx\, dy = \int_{-1}^{1}\int_0^3 (y^3 - y)\, dx\, dy$
$= \int_{-1}^{1} 3\,(y^3 - y)\, dy = 3\left[\frac{y^4}{4} - \frac{y^2}{2}\right]_{-1}^{1} = 0$

(b) $\iint_S g(x, y, z)\, d\sigma = \iint_R \frac{z}{\sqrt{4y^2+1}} \sqrt{4y^2 + 1}\, dx\, dy = \int_{-1}^{1}\int_0^3 (y^2 - 1)\, dx\, dy = \int_{-1}^{1} 3\,(y^2 - 1)\, dy$
$= 3\left[\frac{y^3}{3} - y\right]_{-1}^{1} = -4$

17. $\nabla f = 2y\mathbf{j} + 2z\mathbf{k}$, $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 4z^2} = 2\sqrt{y^2 + z^2} = 10$ and $|\nabla f \cdot \mathbf{p}| = 2z$ since $z \ge 0$
$\Rightarrow d\sigma = \frac{10}{2z}\, dx\, dy = \frac{5}{z}\, dx\, dy = \iint_S g(x, y, z)\, d\sigma = \iint_R (x^4 y)\,(y^2 + z^2)\left(\frac{5}{z}\right) dx\, dy$
$= \iint_R (x^4 y)\,(25)\left(\frac{5}{\sqrt{25 - y^2}}\right) dx\, dy = \int_0^4\int_0^1 \frac{125y}{\sqrt{25 - y^2}}\, x^4\, dx\, dy = \int_0^4 \frac{25y}{\sqrt{25 - y^2}}\, dy = 50$

18. Define the coordinate system so that the origin is at the center of the earth, the z-axis is the earth's axis (north is the positive z direction), and the xz-plane contains the earth's prime meridian. Let S denote the surface which is Wyoming so then S is part of the surface $z = (R^2 - x^2 - y^2)^{1/2}$. Let R_{xy} be the projection of S onto the xy-plane. The surface area of Wyoming is $\iint_S 1\, d\sigma = \iint_{R_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$
$\iint_{R_{xy}} \sqrt{\frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2} + 1}\, dA = \iint_{R_{xy}} \frac{R}{(R^2 - x^2 - y^2)^{1/2}}\, dA = \int_{\theta_1}^{\theta_2}\int_{R\sin 45°}^{R\sin 49°} R\,(R^2 - r^2)^{-1/2}\, r\, dr\, d\theta$
(where θ_1 and θ_2 are the radian equivalent to $104°3'$ and $111°3'$, respectively)
$= \int_{\theta_1}^{\theta_2} -R\,(R^2 - r^2)^{1/2}\Big|_{R\sin 45°}^{R\sin 49°} = \int_{\theta_1}^{\theta_2} R\,(R^2 - R^2\sin^2 45°)^{1/2} - R\,(R^2 - R^2\sin^2 49°)^{1/2}\, d\theta$
$= (\theta_2 - \theta_1)R^2(\cos 45° - \cos 49°) = \frac{7\pi}{180}\, R^2(\cos 45° - \cos 49°) = \frac{7\pi}{180}\,(3959)^2(\cos 45° - \cos 49°)$
$\approx 97{,}751$ sq. mi.

19. A possible parametrization is $\mathbf{r}(\phi, \theta) = (6\sin\phi\cos\theta)\mathbf{i} + (6\sin\phi\sin\theta)\mathbf{j} + (6\cos\phi)\mathbf{k}$ (spherical coordinates); now $\rho = 6$ and $z = -3 \Rightarrow -3 = 6\cos\phi \Rightarrow \cos\phi = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$ and $z = 3\sqrt{3} \Rightarrow 3\sqrt{3} = 6\cos\phi$
$\Rightarrow \cos\phi = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6} \Rightarrow \frac{\pi}{6} \le \phi \le \frac{2\pi}{3}$; also $0 \le \theta \le 2\pi$

20. A possible parametrization is $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} - \left(\frac{r^2}{2}\right)\mathbf{k}$ (cylindrical coordinates); now $r = \sqrt{x^2 + y^2} \Rightarrow z = -\frac{r^2}{2}$ and $-2 \le z \le 0 \Rightarrow -2 \le -\frac{r^2}{2} \le 0 \Rightarrow 4 \ge r^2 \ge 0 \Rightarrow 0 \le r \le 2$ since $r \ge 0$; also $0 \le \theta \le 2\pi$

21. A possible parametrization is $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (1 + r)\mathbf{k}$ (cylindrical coordinates); now $r = \sqrt{x^2 + y^2} \Rightarrow z = 1 + r$ and $1 \le z \le 3 \Rightarrow 1 \le 1 + r \le 3 \Rightarrow 0 \le r \le 2$; also $0 \le \theta \le 2\pi$

22. A possible parametrization is $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + \left(3 - x - \frac{y}{2}\right)\mathbf{k}$ for $0 \le x \le 2$ and $0 \le y \le 2$

23. Let $x = u\cos v$ and $z = u\sin v$, where $u = \sqrt{x^2+z^2}$ and v is the angle in the xz-plane with the x-axis $\Rightarrow\ \mathbf{r}(u,v) = (u\cos v)\mathbf{i} + 2u^2\mathbf{j} + (u\sin v)\mathbf{k}$ is a possible parametrization; $0 \le y \le 2\ \Rightarrow\ 2u^2 \le 2\ \Rightarrow\ u^2 \le 1$ $\Rightarrow\ 0 \le u \le 1$ since $u \ge 0$; also, for just the upper half of the paraboloid, $0 \le v \le \pi$

24. A possible parametrization is $\left(\sqrt{10}\sin\phi\cos\theta\right)\mathbf{i} + \left(\sqrt{10}\sin\phi\sin\theta\right)\mathbf{j} + \left(\sqrt{10}\cos\phi\right)\mathbf{k}$, $0 \le \phi \le \frac{\pi}{2}$ and $0 \le \theta \le \frac{\pi}{2}$

25. $\mathbf{r}_u = \mathbf{i}+\mathbf{j}$, $\mathbf{r}_v = \mathbf{i}-\mathbf{j}+\mathbf{k}\ \Rightarrow\ \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \mathbf{i}-\mathbf{j}-2\mathbf{k}\ \Rightarrow\ |\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{6}$

$\Rightarrow$ Surface Area $= \iint\limits_{R_{uv}} |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv = \int_0^1\int_0^1 \sqrt{6}\,du\,dv = \sqrt{6}$

26. $\iint\limits_S (xy - z^2)\,d\sigma = \int_0^1\int_0^1 [(u+v)(u-v) - v^2]\sqrt{6}\,du\,dv = \sqrt{6}\int_0^1\int_0^1 (u^2 - 2v^2)\,du\,dv$

$= \sqrt{6}\int_0^1 \left[\frac{u^3}{3} - 2uv^2\right]_0^1 dv = \sqrt{6}\int_0^1 \left(\frac{1}{3} - 2v^2\right) dv = \sqrt{6}\left[\frac{1}{3}v - \frac{2}{3}v^3\right]_0^1 = -\frac{\sqrt{6}}{3} = -\sqrt{\frac{2}{3}}$

27. $\mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$, $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} + \mathbf{k}\ \Rightarrow\ \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 1 \end{vmatrix}$

$= (\sin\theta)\mathbf{i} - (\cos\theta)\mathbf{j} + r\mathbf{k}\ \Rightarrow\ |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{\sin^2\theta + \cos^2\theta + r^2} = \sqrt{1+r^2}\ \Rightarrow$ Surface Area $= \iint\limits_{R_{r\theta}} |\mathbf{r}_r \times \mathbf{r}_\theta|\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r}{2}\sqrt{1+r^2} + \frac{1}{2}\ln\left(r + \sqrt{1+r^2}\right)\right]_0^1 d\theta = \int_0^{2\pi}\left[\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln\left(1+\sqrt{2}\right)\right] d\theta$

$= \pi\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$

28. $\iint\limits_S \sqrt{x^2+y^2+1}\,d\sigma = \int_0^{2\pi}\int_0^1 \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + 1}\,\sqrt{1+r^2}\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (1+r^2)\,dr\,d\theta$

$= \int_0^{2\pi}\left[r + \frac{r^3}{3}\right]_0^1 d\theta = \int_0^{2\pi} \frac{4}{3}\,d\theta = \frac{8}{3}\pi$

29. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y}\ \Rightarrow$ Conservative

30. $\frac{\partial P}{\partial y} = \frac{-3zy}{(x^2+y^2+z^2)^{-5/2}} = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = \frac{-3xz}{(x^2+y^2+z^2)^{-5/2}} = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = \frac{-3xy}{(x^2+y^2+z^2)^{-5/2}} = \frac{\partial M}{\partial y}\ \Rightarrow$ Conservative

31. $\frac{\partial P}{\partial y} = 0 \ne ye^z = \frac{\partial N}{\partial z}\ \Rightarrow$ Not Conservative

32. $\frac{\partial P}{\partial y} = \frac{x}{(x+yz)^2} = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = \frac{-y}{(x+yz)^2} = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = \frac{-z}{(x+yz)^2} = \frac{\partial M}{\partial y}\ \Rightarrow$ Conservative

33. $\frac{\partial f}{\partial x} = 2\ \Rightarrow\ f(x,y,z) = 2x + g(y,z)\ \Rightarrow\ \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 2y + z\ \Rightarrow\ g(y,z) = y^2 + zy + h(z)$

$\Rightarrow\ f(x,y,z) = 2x + y^2 + zy + h(z)\ \Rightarrow\ \frac{\partial f}{\partial z} = y + h'(z) = y + 1\ \Rightarrow\ h'(z) = 1\ \Rightarrow\ h(z) = z + C$

$\Rightarrow\ f(x,y,z) = 2x + y^2 + zy + z + C$

34. $\frac{\partial f}{\partial x} = z\cos xz\ \Rightarrow\ f(x,y,z) = \sin xz + g(y,z)\ \Rightarrow\ \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = e^y\ \Rightarrow\ g(y,z) = e^y + h(z)$

$\Rightarrow\ f(x,y,z) = \sin xz + e^y + h(z)\ \Rightarrow\ \frac{\partial f}{\partial z} = x\cos xz + h'(z) = x\cos xz\ \Rightarrow\ h'(z) = 0\ \Rightarrow\ h(z) = C$

$\Rightarrow\ f(x,y,z) = \sin xz + e^y + C$

35. Over Path 1: $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1 \Rightarrow x = t, y = t, z = t$ and $d\mathbf{r} = (\mathbf{i} + \mathbf{j} + \mathbf{k})\,dt \Rightarrow \mathbf{F} = 2t^2\,\mathbf{i} + \mathbf{j} + t^2\mathbf{k}$
$\Rightarrow \mathbf{F} \cdot d\mathbf{r} = (3t^2 + 1)\,dt \Rightarrow \text{Work} = \int_0^1 (3t^2 + 1)\,dt = 2;$
Over Path 2: $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$, $0 \le t \le 1 \Rightarrow x = t, y = t, z = 0$ and $d\mathbf{r}_1 = (\mathbf{i} + \mathbf{j})\,dt \Rightarrow \mathbf{F}_1 = 2t^2\,\mathbf{i} + \mathbf{j} + t^2\mathbf{k}$
$\Rightarrow \mathbf{F}_1 \cdot d\mathbf{r}_1 = (2t^2 + 1)\,dt \Rightarrow \text{Work}_1 = \int_0^1 (2t^2 + 1)\,dt = \frac{5}{3}$; $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$, $0 \le t \le 1 \Rightarrow x = 1, y = 1, z = t$ and $d\mathbf{r}_2 = \mathbf{k}\,dt \Rightarrow \mathbf{F}_2 = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot d\mathbf{r}_2 = dt \Rightarrow \text{Work}_2 = \int_0^1 dt = 1 \Rightarrow \text{Work} = \text{Work}_1 + \text{Work}_2 = \frac{5}{3} + 1 = \frac{8}{3}$

36. Over Path 1: $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1 \Rightarrow x = t, y = t, z = t$ and $d\mathbf{r} = (\mathbf{i} + \mathbf{j} + \mathbf{k})\,dt \Rightarrow \mathbf{F} = 2t^2\,\mathbf{i} + t^2\mathbf{j} + \mathbf{k}$
$\Rightarrow \mathbf{F} \cdot d\mathbf{r} = (3t^2 + 1)\,dt \Rightarrow \text{Work} = \int_0^1 (3t^2 + 1)\,dt = 2;$
Over Path 2: Since f is conservative, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ around any simple closed curve C. Thus consider $\int_{\text{curve}} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where C_1 is the path from $(0,0,0)$ to $(1,1,0)$ to $(1,1,1)$ and C_2 is the path from $(1,1,1)$ to $(0,0,0)$. Now, from Path 1 above, $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -2 \Rightarrow 0 = \int_{\text{curve}} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + (-2)$
$\Rightarrow \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 2$

37. (a) $\mathbf{r} = (e^t \cos t)\,\mathbf{i} + (e^t \sin t)\,\mathbf{j} \Rightarrow x = e^t \cos t, y = e^t \sin t$ from $(1,0)$ to $(e^{2\pi}, 0) \Rightarrow 0 \le t \le 2\pi$
$\Rightarrow \frac{d\mathbf{r}}{dt} = (e^t \cos t - e^t \sin t)\,\mathbf{i} + (e^t \sin t + e^t \cos t)\,\mathbf{j}$ and $\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j}}{(x^2+y^2)^{3/2}} = \frac{(e^t \cos t)\,\mathbf{i} + (e^t \sin t)\,\mathbf{j}}{(e^{2t}\cos^2 t + e^{2t}\sin^2 t)^{3/2}}$
$= \left(\frac{\cos t}{e^{2t}}\right)\mathbf{i} + \left(\frac{\sin t}{e^{2t}}\right)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \left(\frac{\cos^2 t}{e^t} - \frac{\sin t \cos t}{e^t} + \frac{\sin^2 t}{e^t} + \frac{\sin t \cos t}{e^t}\right) = e^{-t}$
$\Rightarrow \text{Work} = \int_0^{2\pi} e^{-t}\,dt = 1 - e^{-2\pi}$
(b) $\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j}}{(x^2+y^2)^{3/2}} \Rightarrow \frac{\partial f}{\partial x} = \frac{x}{(x^2+y^2)^{3/2}} \Rightarrow f(x,y,z) = -(x^2+y^2)^{-1/2} + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{y}{(x^2+y^2)^{3/2}} + \frac{\partial g}{\partial y}$
$= \frac{y}{(x^2+y^2)^{3/2}} \Rightarrow g(y,z) = C \Rightarrow f(x,y,z) = -(x^2+y^2)^{-1/2}$ is a potential function for $\mathbf{F} \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r}$
$= f(e^{2\pi}, 0) - f(1,0) = 1 - e^{-2\pi}$

38. (a) $\mathbf{F} = \nabla(x^2 z e^y) \Rightarrow \mathbf{F}$ is conservative $\Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for <u>any</u> closed path C
(b) $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{(1,0,0)}^{(1,0,2\pi)} \nabla(x^2 z e^y) \cdot d\mathbf{r} = (x^2 z e^y)\big|_{(1,0,2\pi)} - (x^2 z e^y)\big|_{(1,0,0)} = 2\pi - 0 = 2\pi$

39. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & -y & 3z^2 \end{vmatrix} = -2y\mathbf{k}$; unit normal to the plane is $\mathbf{n} = \frac{2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}}{\sqrt{4+36+9}} = \frac{2}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{3}{7}\mathbf{k}$
$\Rightarrow \nabla \times \mathbf{F} \cdot \mathbf{n} = \frac{6}{7}y$; $\mathbf{p} = \mathbf{k}$ and $f(x,y,z) = 2x + 6y - 3z \Rightarrow |\nabla f \cdot \mathbf{p}| = 3 \Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\,dA = \frac{7}{3}\,dA$
$\Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \frac{6}{7}y\,d\sigma = \iint_R \left(\frac{6}{7}y\right)\left(\frac{7}{3}\,dA\right) = \iint_R 2y\,dA = \int_0^{2\pi}\int_0^1 2r\sin\theta\, r\,dr\,d\theta = \int_0^{2\pi} \frac{2}{3}\sin\theta\,d\theta = 0$

40. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y & x + y & 4y^2 - z \end{vmatrix} = 8y\mathbf{i}$; the circle lies in the plane $f(x,y,z) = y + z = 0$ with unit normal $\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k} \Rightarrow \nabla \times \mathbf{F} \cdot \mathbf{n} = 0 \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{n}\,d\sigma = \iint_R 0\,d\sigma = 0$

41. (a) $\mathbf{r} = \sqrt{2}t\mathbf{i} + \sqrt{2}t\mathbf{j} + (4 - t^2)\,\mathbf{k}$, $0 \le t \le 1 \Rightarrow x = \sqrt{2}t, y = \sqrt{2}t, z = 4 - t^2 \Rightarrow \frac{dx}{dt} = \sqrt{2}, \frac{dy}{dt} = \sqrt{2}, \frac{dz}{dt} = -2t$
$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \sqrt{4 + 4t^2}\,dt \Rightarrow M = \int_C \delta(x,y,z)\,ds = \int_0^1 3t\sqrt{4+4t^2}\,dt = \left[\frac{1}{4}(4+4t)^{3/2}\right]_0^1$
$= 4\sqrt{2} - 2$

(b) $M = \int_C \delta(x,y,z)\,ds = \int_0^1 \sqrt{4+4t^2}\,dt = \left[t\sqrt{1+t^2} + \ln\left(t+\sqrt{1+t^2}\right)\right]_0^1 = \sqrt{2} + \ln\left(1+\sqrt{2}\right)$

42. $\mathbf{r} = t\mathbf{i} + 2t\mathbf{j} + \frac{2}{3}t^{3/2}\mathbf{k},\ 0 \le t \le 2 \Rightarrow x = t,\ y = 2t,\ z = \frac{2}{3}t^{3/2} \Rightarrow \frac{dx}{dt} = 1,\ \frac{dy}{dt} = 2,\ \frac{dz}{dt} = t^{1/2}$
$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \sqrt{t+5}\,dt \Rightarrow M = \int_C \delta(x,y,z)\,ds = \int_0^2 3\sqrt{5+t}\,\sqrt{t+5}\,dt$
$= \int_0^2 3(t+5)\,dt = 36;\ M_{yz} = \int_C x\delta\,ds = \int_0^2 3t(t+5)\,dt = 38;\ M_{xz} = \int_C y\delta\,ds = \int_0^2 6t(t+5)\,dt = 76;$
$M_{xy} = \int_C z\delta\,ds = \int_0^2 2t^{3/2}(t+5)\,dt = \frac{144}{7}\sqrt{2} \Rightarrow \overline{x} = \frac{M_{yz}}{M} = \frac{38}{36} = \frac{19}{18},\ \overline{y} = \frac{M_{xz}}{M} = \frac{76}{36} = \frac{19}{9},\ \overline{z} = \frac{M_{xy}}{M} = \frac{\left(\frac{144}{7}\sqrt{2}\right)}{36}$
$= \frac{4}{7}\sqrt{2}$

43. $\mathbf{r} = t\mathbf{i} + \left(\frac{2\sqrt{2}}{3}t^{3/2}\right)\mathbf{j} + \left(\frac{t^2}{2}\right)\mathbf{k},\ 0 \le t \le 2 \Rightarrow x = t,\ y = \frac{2\sqrt{2}}{3}t^{3/2},\ z = \frac{t^2}{2} \Rightarrow \frac{dx}{dt} = 1,\ \frac{dy}{dt} = \sqrt{2}\,t^{1/2},\ \frac{dz}{dt} = t$
$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \sqrt{1+2t+t^2}\,dt = \sqrt{(t+1)^2}\,dt = |t+1|\,dt = (t+1)\,dt$ on the domain given.
Then $M = \int_C \delta\,ds = \int_0^2 \left(\frac{1}{t+1}\right)(t+1)\,dt = \int_0^2 dt = 2;\ M_{yz} = \int_C x\delta\,ds = \int_0^2 t\left(\frac{1}{t+1}\right)(t+1)\,dt = \int_0^2 t\,dt = 2;$
$M_{xz} = \int_C y\delta\,ds = \int_0^2 \left(\frac{2\sqrt{2}}{3}t^{3/2}\right)\left(\frac{1}{t+1}\right)(t+1)\,dt = \int_0^2 \frac{2\sqrt{2}}{3}t^{3/2}\,dt = \frac{32}{15};\ M_{xy} = \int_C z\delta\,ds$
$= \int_0^2 \left(\frac{t^2}{2}\right)\left(\frac{1}{t+1}\right)(t+1)\,dt = \int_0^2 \frac{t^2}{2}\,dt = \frac{4}{3} \Rightarrow \overline{x} = \frac{M_{yz}}{M} = \frac{2}{2} = 1;\ \overline{y} = \frac{M_{xz}}{M} = \frac{\left(\frac{32}{15}\right)}{2} = \frac{16}{15};\ \overline{z} = \frac{M_{xy}}{M}$
$= \frac{\left(\frac{4}{3}\right)}{2} = \frac{2}{3};\ I_x = \int_C (y^2+z^2)\,\delta\,ds = \int_0^2 \left(\frac{8}{9}t^3 + \frac{t^4}{4}\right)dt = \frac{232}{45};\ I_y = \int_C (x^2+z^2)\,\delta\,ds = \int_0^2 \left(t^2 + \frac{t^4}{4}\right)dt = \frac{64}{15};$
$I_z = \int_C (y^2+x^2)\,\delta\,ds = \int_0^2 \left(t^2 + \frac{8}{9}t^3\right)dt = \frac{56}{9};\ R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{\left(\frac{232}{45}\right)}{2}} = \frac{2\sqrt{29}}{3\sqrt{5}};\ R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{\left(\frac{64}{15}\right)}{2}} = \frac{4\sqrt{2}}{\sqrt{15}};$
$R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{\left(\frac{56}{9}\right)}{2}} = \frac{2\sqrt{7}}{3}$

44. $\overline{z} = 0$ because the arch is in the xy-plane, and $\overline{x} = 0$ because the mass is distributed symmetrically with respect to the y-axis; $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j},\ 0 \le t \le \pi \Rightarrow ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$
$= \sqrt{(-a\sin t)^2 + (a\cos t)^2}\,dt = a\,dt$, since $a \ge 0$; $M = \int_C \delta\,ds = \int_C (2a-y)\,ds = \int_0^\pi (2a - a\sin t)\,a\,dt$
$= 2a^2\pi - 2a^2;\ M_{xz} = \int_C y\delta\,dt = \int_C y(2a-y)\,ds = \int_0^\pi (a\sin t)(2a - a\sin t)\,dt = \int_0^\pi (2a^2\sin t - a^2\sin^2 t)\,dt$
$= \left[-2a^2\cos t - a^2\left(\frac{t}{2} - \frac{\sin 2t}{4}\right)\right]_0^\pi = 4a^2 - \frac{a^2\pi}{2} \Rightarrow \overline{y} = \frac{\left(4a^2 - \frac{a^2\pi}{2}\right)}{2a^2\pi - 2a^2} = \frac{8-\pi}{4\pi-4} \Rightarrow (\overline{x},\overline{y},\overline{z}) = \left(0, \frac{8-\pi}{4\pi-4}, 0\right)$

45. $\mathbf{r}(t) = (e^t\cos t)\,\mathbf{i} + (e^t\sin t)\,\mathbf{j} + e^t\mathbf{k},\ 0 \le t \le \ln 2 \Rightarrow x = e^t\cos t,\ y = e^t\sin t,\ z = e^t \Rightarrow \frac{dx}{dt} = (e^t\cos t - e^t\sin t),$
$\frac{dy}{dt} = (e^t\sin t + e^t\cos t),\ \frac{dz}{dt} = e^t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$
$= \sqrt{(e^t\cos t - e^t\sin t)^2 + (e^t\sin t + e^t\cos t)^2 + (e^t)^2}\,dt = \sqrt{3e^{2t}}\,dt = \sqrt{3}\,e^t\,dt;\ M = \int_C \delta\,ds = \int_0^{\ln 2} \sqrt{3}\,e^t\,dt$
$= \sqrt{3};\ M_{xy} = \int_C z\delta\,ds = \int_0^{\ln 2} \left(\sqrt{3}\,e^t\right)(e^t)\,dt = \int_0^{\ln 2} \sqrt{3}\,e^{2t}\,dt = \frac{3\sqrt{3}}{2} \Rightarrow \overline{z} = \frac{M_{xy}}{M} = \frac{\left(\frac{3\sqrt{3}}{2}\right)}{\sqrt{3}} = \frac{3}{2};$
$I_z = \int_C (x^2+y^2)\,\delta\,ds = \int_0^{\ln 2} (e^{2t}\cos^2 t + e^{2t}\sin^2 t)\left(\sqrt{3}\,e^t\right)dt = \int_0^{\ln 2} \sqrt{3}\,e^{3t}\,dt = \frac{7\sqrt{3}}{3} \Rightarrow R_z = \sqrt{\frac{I_z}{M}}$
$= \sqrt{\frac{7\sqrt{3}}{3\sqrt{3}}} = \sqrt{\frac{7}{3}}$

46. $\mathbf{r}(t) = (2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + 3t\mathbf{k},\ 0 \le t \le 2\pi \Rightarrow x = 2\sin t,\ y = 2\cos t,\ z = 3t \Rightarrow \frac{dx}{dt} = 2\cos t,\ \frac{dy}{dt} = -2\sin t,$
$\frac{dz}{dt} = 3 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \sqrt{4+9}\,dt = \sqrt{13}\,dt;\ M = \int_C \delta\,ds = \int_0^{2\pi} \delta\sqrt{13}\,dt = 2\pi\delta\sqrt{13};$

$M_{xy} = \int_C z\delta\, ds = \int_0^{2\pi} (3t)\left(\delta\sqrt{13}\right) dt = 6\delta\pi^2\sqrt{13}$; $M_{yz} = \int_C x\delta\, ds = \int_0^{2\pi} (2\sin t)\left(\delta\sqrt{13}\right) dt = 0$;
$M_{xz} = \int_C y\delta\, ds = \int_0^{2\pi} (2\cos t)\left(\delta\sqrt{13}\right) dt = 0 \Rightarrow \overline{x} = \overline{y} = 0$ and $\overline{z} = \frac{M_{xy}}{M} = \frac{6\delta\pi^2\sqrt{13}}{2\delta\pi\sqrt{13}} = 3\pi \Rightarrow (0, 0, 3\pi)$ is the center of mass

47. Because of symmetry $\overline{x} = \overline{y} = 0$. Let $f(x, y, z) = x^2 + y^2 + z^2 = 25 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$
$\Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 10$ and $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z$, since $z \geq 0 \Rightarrow M = \iint_R \delta(x, y, z)\, d\sigma$
$= \iint_R z\left(\frac{10}{2z}\right) dA = \iint_R 5\, dA = 5(\text{Area of the circular region}) = 80\pi$; $M_{xy} = \iint_R z\delta\, d\sigma = \iint_R 5z\, dA$
$= \iint_R 5\sqrt{25 - x^2 - y^2}\, dx\, dy = \int_0^{2\pi}\int_0^4 \left(5\sqrt{25 - r^2}\right) r\, dr\, d\theta = \int_0^{2\pi} \frac{490}{3}\, d\theta = \frac{980}{3}\pi \Rightarrow \overline{z} = \frac{\left(\frac{980}{3}\pi\right)}{80\pi} = \frac{49}{12}$
$\Rightarrow (\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{49}{12}\right)$; $I_z = \iint_R (x^2 + y^2)\,\delta\, d\sigma = \iint_R 5\,(x^2 + y^2)\, dx\, dy = \int_0^{2\pi}\int_0^4 5r^3\, dr\, d\theta = \int_0^{2\pi} 320\, d\theta = 640\pi$;
$R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{640\pi}{80\pi}} = 2\sqrt{2}$

48. On the face $z = 1$: $g(x, y, z) = z = 1$ and $\mathbf{p} = \mathbf{k} \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1$ and $|\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA$
$\Rightarrow I = \iint_R (x^2 + y^2)\, dA = 2\int_0^{\pi/4}\int_0^{\sec\theta} r^3\, dr\, d\theta = \frac{2}{3}$; On the face $z = 0$: $g(x, y, z) = z = 0 \Rightarrow \nabla g = \mathbf{k}$ and $\mathbf{p} = \mathbf{k}$
$\Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA \Rightarrow I = \iint_R (x^2 + y^2)\, dA = \frac{2}{3}$; On the face $y = 0$: $g(x, y, z) = y = 0$
$\Rightarrow \nabla g = \mathbf{j}$ and $\mathbf{p} = \mathbf{j} \Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA \Rightarrow I = \iint_R (x^2 + 0)\, dA = \int_0^1\int_0^1 x^2\, dx\, dz = \frac{1}{3}$;
On the face $y = 1$: $g(x, y, z) = y = 1 \Rightarrow \nabla g = \mathbf{j}$ and $\mathbf{p} = \mathbf{j} \Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA$
$\Rightarrow I = \iint_R (x^2 + 1^2)\, dA = \int_0^1\int_0^1 (x^2 + 1)\, dx\, dz = \frac{4}{3}$; On the face $x = 1$: $g(x, y, z) = x = 1 \Rightarrow \nabla g = \mathbf{i}$ and $\mathbf{p} = \mathbf{i}$
$\Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA \Rightarrow I = \iint_R (1^2 + y^2)\, dA = \int_0^1\int_0^1 (1 + y^2)\, dy\, dz = \frac{4}{3}$; On the face
$x = 0$: $g(x, y, z) = x = 0 \Rightarrow \nabla g = \mathbf{i}$ and $\mathbf{p} = \mathbf{i} \Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA$
$\Rightarrow I = \iint_R (0^2 + y^2)\, dA = \int_0^1\int_0^1 y^2\, dy\, dz = \frac{1}{3} \Rightarrow I_z = \frac{2}{3} + \frac{2}{3} + \frac{1}{3} + \frac{4}{3} + \frac{4}{3} + \frac{1}{3} = \frac{14}{3}$

49. $M = 2xy + x$ and $N = xy - y \Rightarrow \frac{\partial M}{\partial x} = 2y + 1,\ \frac{\partial M}{\partial y} = 2x,\ \frac{\partial N}{\partial x} = y,\ \frac{\partial N}{\partial y} = x - 1 \Rightarrow \text{Flux} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy$
$= \iint_R (2y + 1 + x - 1)\, dy\, dx = \int_0^1\int_0^1 (2y + x)\, dy\, dx = \frac{3}{2}$; $\text{Circ} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$
$= \iint_R (y - 2x)\, dy\, dx = \int_0^1\int_0^1 (y - 2x)\, dy\, dx = -\frac{1}{2}$

50. $M = y - 6x^2$ and $N = x + y^2 \Rightarrow \frac{\partial M}{\partial x} = -12x,\ \frac{\partial M}{\partial y} = 1,\ \frac{\partial N}{\partial x} = 1,\ \frac{\partial N}{\partial y} = 2y \Rightarrow \text{Flux} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy$
$= \iint_R (-12x + 2y)\, dx\, dy = \int_0^1\int_y^1 (-12x + 2y)\, dx\, dy = \int_0^1 (4y^2 + 2y - 6)\, dy = -\frac{11}{3}$;
$\text{Circ} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \iint_R (1 - 1)\, dx\, dy = 0$

51. $M = -\frac{\cos y}{x}$ and $N = \ln x \sin y \Rightarrow \frac{\partial M}{\partial y} = \frac{\sin y}{x}$ and $\frac{\partial N}{\partial x} = \frac{\sin y}{x} \Rightarrow \oint_C \ln x \sin y\, dy - \frac{\cos y}{x}\, dx$
$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \iint_R \left(\frac{\sin y}{x} - \frac{\sin y}{x}\right) dx\, dy = 0$

52. (a) Let $M = x$ and $N = y \Rightarrow \frac{\partial M}{\partial x} = 1,\ \frac{\partial M}{\partial y} = 0,\ \frac{\partial N}{\partial x} = 0,\ \frac{\partial N}{\partial y} = 1 \Rightarrow$ Flux $= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$

$= \iint_R (1+1)\,dx\,dy = 2\iint_R dx\,dy = 2(\text{Area of the region})$

(b) Let C be a closed curve to which Green's Theorem applies and let **n** be the unit normal vector to C. Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ and assume **F** is orthogonal to **n** at every point of C. Then the flux density of **F** at every point of C is 0 since $\mathbf{F} \cdot \mathbf{n} = 0$ at every point of C $\Rightarrow \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0$ at every point of C

$\Rightarrow$ Flux $= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R 0\,dx\,dy = 0$. But part (a) above states that the flux is 2(Area of the region) $\Rightarrow$ the area of the region would be 0 $\Rightarrow$ contradiction. Therefore, **F** cannot be orthogonal to **n** at every point of C.

53. $\frac{\partial}{\partial x}(2xy) = 2y,\ \frac{\partial}{\partial y}(2yz) = 2z,\ \frac{\partial}{\partial z}(2xz) = 2x \Rightarrow \nabla \cdot \mathbf{F} = 2y + 2z + 2x \Rightarrow$ Flux $= \iiint_D (2x + 2y + 2z)\,dV$

$= \int_0^1 \int_0^1 \int_0^1 (2x + 2y + 2z)\,dx\,dy\,dz = \int_0^1 \int_0^1 (1 + 2y + 2z)\,dy\,dz = \int_0^1 (2 + 2z)\,dz = 3$

54. $\frac{\partial}{\partial x}(xz) = z,\ \frac{\partial}{\partial y}(yz) = z,\ \frac{\partial}{\partial z}(1) = 0 \Rightarrow \nabla \cdot \mathbf{F} = 2z \Rightarrow$ Flux $= \iiint_D 2z\,r\,dr\,d\theta\,dz$

$= \int_0^{2\pi} \int_0^4 \int_3^{\sqrt{25-r^2}} 2z\,dz\,r\,dr\,d\theta = \int_0^{2\pi} \int_0^4 r(16 - r^2)\,dr\,d\theta = \int_0^{2\pi} 64\,d\theta = 128\pi$

55. $\frac{\partial}{\partial x}(-2x) = -2,\ \frac{\partial}{\partial y}(-3y) = -3,\ \frac{\partial}{\partial z}(z) = 1 \Rightarrow \nabla \cdot \mathbf{F} = -4;\ x^2 + y^2 + z^2 = 2$ and $x^2 + y^2 = z \Rightarrow z = 1$

$\Rightarrow x^2 + y^2 = 1 \Rightarrow$ Flux $= \iiint_D -4\,dV = -4\int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} dz\,r\,dr\,d\theta = -4\int_0^{2\pi} \int_0^1 \left(r\sqrt{2 - r^2} - r^3\right) dr\,d\theta$

$= -4\int_0^{2\pi} \left(-\frac{7}{12} + \frac{2}{3}\sqrt{2}\right) d\theta = \frac{2}{3}\pi\left(7 - 8\sqrt{2}\right)$

56. $\frac{\partial}{\partial x}(6x + y) = 6,\ \frac{\partial}{\partial y}(-x - z) = 0,\ \frac{\partial}{\partial z}(4yz) = 4y \Rightarrow \nabla \cdot \mathbf{F} = 6 + 4y;\ z = \sqrt{x^2 + y^2} = r$

$\Rightarrow$ Flux $= \iiint_D (6 + 4y)\,dV = \int_0^{\pi/2} \int_0^1 \int_0^r (6 + 4r\sin\theta)\,dz\,r\,dr\,d\theta = \int_0^{\pi/2} \int_0^1 (6r^2 + 4r^3 \sin\theta)\,dr\,d\theta$

$= \int_0^{\pi/2} (2 + \sin\theta)\,d\theta = \pi + 1$

57. $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k} \Rightarrow \nabla \cdot \mathbf{F} = 0 \Rightarrow$ Flux $= \iint_S \mathbf{F} \cdot \mathbf{n}\,d\sigma = \iiint_D \nabla \cdot \mathbf{F}\,dV = 0$

58. $\mathbf{F} = 3xz^2\mathbf{i} + y\mathbf{j} - z^3\mathbf{k} \Rightarrow \nabla \cdot \mathbf{F} = 3z^2 + 1 - 3z^2 = 1 \Rightarrow$ Flux $= \iint_S \mathbf{F} \cdot \mathbf{n}\,d\sigma = \iiint_D \nabla \cdot \mathbf{F}\,dV$

$= \int_0^4 \int_0^{\sqrt{16-x^2}/2} \int_0^{y/2} 1\,dz\,dy\,dx = \int_0^4 \left(\frac{16-x^2}{16}\right) dx = \left[x - \frac{x^3}{48}\right]_0^4 = \frac{8}{3}$

59. $\mathbf{F} = xy^2\mathbf{i} + x^2y\mathbf{j} + y\mathbf{k} \Rightarrow \nabla \cdot \mathbf{F} = y^2 + x^2 + 0 \Rightarrow$ Flux $= \iint_S \mathbf{F} \cdot \mathbf{n}\,d\sigma = \iiint_D \nabla \cdot \mathbf{F}\,dV$

$= \iiint_D (x^2 + y^2)\,dV = \int_0^{2\pi} \int_0^1 \int_{-1}^1 r^2\,dz\,r\,dr\,d\theta = \int_0^{2\pi} \int_0^1 2r^3\,dr\,d\theta = \int_0^{2\pi} \frac{1}{2}\,d\theta = \pi$

60. (a) $\mathbf{F} = (3z + 1)\mathbf{k} \Rightarrow \nabla \cdot \mathbf{F} = 3 \Rightarrow$ Flux across the hemisphere $= \iint_S \mathbf{F} \cdot \mathbf{n}\,d\sigma = \iiint_D \nabla \cdot \mathbf{F}\,dV$

$= \iiint_D 3\,dV = 3\left(\frac{1}{2}\right)\left(\frac{4}{3}\pi a^3\right) = 2\pi a^3$

(b) $f(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = \sqrt{4a^2} = 2a$ since $a \ge 0 \Rightarrow \mathbf{n} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2a} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a} \Rightarrow \mathbf{F} \cdot \mathbf{n} = (3z + 1)\left(\frac{z}{a}\right);\ \mathbf{p} = \mathbf{k} \Rightarrow \nabla f \cdot \mathbf{p} = \nabla f \cdot \mathbf{k} = 2z$

$\Rightarrow |\nabla f \cdot \mathbf{p}| = 2z$ since $z \ge 0 \Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} = \frac{2a}{2z}\, dA = \frac{a}{z}\, dA \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{R_{xy}} (3z+1)\left(\frac{z}{a}\right)\left(\frac{a}{z}\right) dA$

$= \iint_{R_{xy}} (3z+1)\, dx\, dy = \iint_{R_{xy}} \left(3\sqrt{a^2 - x^2 - y^2} + 1\right) dx\, dy = \int_0^{2\pi}\int_0^a \left(3\sqrt{a^2 - r^2} + 1\right) r\, dr\, d\theta$

$= \int_0^{2\pi} \left(\frac{a^2}{2} + a^3\right) d\theta = \pi a^2 + 2\pi a^3$, which is the flux across the hemisphere. Across the base we find $\mathbf{F} = [3(0)+1]\mathbf{k} = \mathbf{k}$ since $z = 0$ in the xy-plane $\Rightarrow \mathbf{n} = -\mathbf{k}$ (outward normal) $\Rightarrow \mathbf{F} \cdot \mathbf{n} = -1 \Rightarrow$ Flux across the base $= \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{R_{xy}} -1\, dx\, dy = -\pi a^2$. Therefore, the total flux across the closed surface is $(\pi a^2 + 2\pi a^3) - \pi a^2 = 2\pi a^3$.

CHAPTER 16 ADDITIONAL AND ADVANCED EXERCISES

1. $dx = (-2\sin t + 2\sin 2t)\, dt$ and $dy = (2\cos t - 2\cos 2t)\, dt$; Area $= \frac{1}{2}\oint_C x\, dy - y\, dx$

$= \frac{1}{2}\int_0^{2\pi} [(2\cos t - \cos 2t)(2\cos t - 2\cos 2t) - (2\sin t - \sin 2t)(-2\sin t + 2\sin 2t)]\, dt$

$= \frac{1}{2}\int_0^{2\pi} [6 - (6\cos t\cos 2t + 6\sin t\sin 2t)]\, dt = \frac{1}{2}\int_0^{2\pi} (6 - 6\cos t)\, dt = 6\pi$

2. $dx = (-2\sin t - 2\sin 2t)\, dt$ and $dy = (2\cos t - 2\cos 2t)\, dt$; Area $= \frac{1}{2}\oint_C x\, dy - y\, dx$

$= \frac{1}{2}\int_0^{2\pi} [(2\cos t + \cos 2t)(2\cos t - 2\cos 2t) - (2\sin t - \sin 2t)(-2\sin t - 2\sin 2t)]\, dt$

$= \frac{1}{2}\int_0^{2\pi} [2 - 2(\cos t\cos 2t - \sin t\sin 2t)]\, dt = \frac{1}{2}\int_0^{2\pi} (2 - 2\cos 3t)\, dt = \frac{1}{2}\left[2t - \frac{2}{3}\sin 3t\right]_0^{2\pi} = 2\pi$

3. $dx = \cos 2t\, dt$ and $dy = \cos t\, dt$; Area $= \frac{1}{2}\oint_C x\, dy - y\, dx = \frac{1}{2}\int_0^{\pi} \left(\frac{1}{2}\sin 2t\cos t - \sin t\cos 2t\right) dt$

$= \frac{1}{2}\int_0^{\pi} [\sin t\cos^2 t - (\sin t)(2\cos^2 t - 1)]\, dt = \frac{1}{2}\int_0^{\pi} (-\sin t\cos^2 t + \sin t)\, dt = \frac{1}{2}\left[\frac{1}{3}\cos^3 t - \cos t\right]_0^{\pi} = -\frac{1}{3} + 1 = \frac{2}{3}$

4. $dx = (-2a\sin t - 2a\cos 2t)\, dt$ and $dy = (b\cos t)\, dt$; Area $= \frac{1}{2}\oint_C x\, dy - y\, dx$

$= \frac{1}{2}\int_0^{2\pi} [(2ab\cos^2 t - ab\cos t\sin 2t) - (-2ab\sin^2 t - 2ab\sin t\cos 2t)]\, dt$

$= \frac{1}{2}\int_0^{2\pi} [2ab - 2ab\cos^2 t\sin t + 2ab(\sin t)(2\cos^2 t - 1)]\, dt = \frac{1}{2}\int_0^{2\pi} (2ab + 2ab\cos^2 t\sin t - 2ab\sin t)\, dt$

$= \frac{1}{2}\left[2abt - \frac{2}{3}ab\cos^3 t + 2ab\cos t\right]_0^{2\pi} = 2\pi ab$

5. (a) $\mathbf{F}(x,y,z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ is $\mathbf{0}$ only at the point $(0,0,0)$, and curl $\mathbf{F}(x,y,z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$ is never $\mathbf{0}$.
(b) $\mathbf{F}(x,y,z) = z\mathbf{i} + y\mathbf{k}$ is $\mathbf{0}$ only on the line $x = t$, $y = 0$, $z = 0$ and curl $\mathbf{F}(x,y,z) = \mathbf{i} + \mathbf{j}$ is never $\mathbf{0}$.
(c) $\mathbf{F}(x,y,z) = z\mathbf{i}$ is $\mathbf{0}$ only when $z = 0$ (the xy-plane) and curl $\mathbf{F}(x,y,z) = \mathbf{j}$ is never $\mathbf{0}$.

6. $\mathbf{F} = yz^2\mathbf{i} + xz^2\mathbf{j} + 2xyz\mathbf{k}$ and $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{R}$, so $\mathbf{F}$ is parallel to $\mathbf{n}$ when $yz^2 = \frac{cx}{R}$, $xz^2 = \frac{cy}{R}$, and $2xyz = \frac{cz}{R} \Rightarrow \frac{yz^2}{x} = \frac{xz^2}{y} = 2xy \Rightarrow y^2 = x^2 \Rightarrow y = \pm x$ and $z^2 = \pm\frac{c}{R} = 2x^2 \Rightarrow z = \pm\sqrt{2}x$. Also, $x^2 + y^2 + z^2 = R^2 \Rightarrow x^2 + x^2 + 2x^2 = R^2 \Rightarrow 4x^2 = R^2 \Rightarrow x = \pm\frac{R}{2}$. Thus the points are: $\left(\frac{R}{2}, \frac{R}{2}, \frac{\sqrt{2}R}{2}\right)$, $\left(\frac{R}{2}, \frac{R}{2}, -\frac{\sqrt{2}R}{2}\right)$, $\left(-\frac{R}{2}, -\frac{R}{2}, \frac{\sqrt{2}R}{2}\right)$, $\left(-\frac{R}{2}, -\frac{R}{2}, -\frac{\sqrt{2}R}{2}\right)$, $\left(\frac{R}{2}, -\frac{R}{2}, \frac{\sqrt{2}R}{2}\right)$, $\left(\frac{R}{2}, -\frac{R}{2}, -\frac{\sqrt{2}R}{2}\right)$, $\left(-\frac{R}{2}, \frac{R}{2}, \frac{\sqrt{2}R}{2}\right)$, $\left(-\frac{R}{2}, \frac{R}{2}, -\frac{\sqrt{2}R}{2}\right)$

7. Set up the coordinate system so that $(a,b,c) = (0,R,0) \Rightarrow \delta(x,y,z) = \sqrt{x^2 + (y-R)^2 + z^2}$

$= \sqrt{x^2 + y^2 + z^2 - 2Ry + R^2} = \sqrt{2R^2 - 2Ry}$; let $f(x,y,z) = x^2 + y^2 + z^2 - R^2$ and $\mathbf{p} = \mathbf{i}$

$\Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = 2\sqrt{x^2+y^2+z^2} = 2R \Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{i}|}\, dz\, dy = \frac{2R}{2x}\, dz\, dy$

$\Rightarrow$ Mass $= \iint\limits_{S} \delta(x,y,z)\,d\sigma = \iint\limits_{R_{yz}} \sqrt{2R^2-2Ry}\left(\frac{R}{x}\right) dz\,dy = R\iint\limits_{R_{yz}} \frac{\sqrt{2R^2-2Ry}}{\sqrt{R^2-y^2-z^2}}\,dz\,dy$

$= 4R\int_{-R}^{R}\int_{0}^{\sqrt{R^2-y^2}} \frac{\sqrt{2R^2-2Ry}}{\sqrt{R^2-y^2-z^2}}\,dz\,dy = 4R\int_{-R}^{R}\sqrt{2R^2-2Ry}\,\sin^{-1}\left(\frac{z}{\sqrt{R^2-y^2}}\right)\Big|_0^{\sqrt{R^2-y^2}} dy$

$= 2\pi R\int_{-R}^{R}\sqrt{2R^2-2Ry}\,dy = 2\pi R\left(\frac{-1}{3R}\right)(2R^2-2Ry)^{3/2}\Big|_{-R}^{R} = \frac{16\pi R^3}{3}$

8. $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \theta\mathbf{k}$, $0 \le r \le 1$, $0 \le \theta \le 2\pi \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 1 \end{vmatrix}$

$= (\sin\theta)\mathbf{i} - (\cos\theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{1+r^2}$; $\delta = 2\sqrt{x^2+y^2} = 2\sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = 2r$

$\Rightarrow$ Mass $= \iint\limits_{S} \delta(x,y,z)\,d\sigma = \int_0^{2\pi}\int_0^1 2r\sqrt{1+r^2}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{2}{3}(1+r^2)^{3/2}\right]_0^1 d\theta = \int_0^{2\pi}\frac{2}{3}\left(2\sqrt{2}-1\right)d\theta$

$= \frac{4\pi}{3}\left(2\sqrt{2}-1\right)$

9. $M = x^2+4xy$ and $N = -6y \Rightarrow \frac{\partial M}{\partial x} = 2x+4y$ and $\frac{\partial N}{\partial x} = -6 \Rightarrow$ Flux $= \int_0^b\int_0^a (2x+4y-6)\,dx\,dy$

$= \int_0^b (a^2+4ay-6a)\,dy = a^2b+2ab^2-6ab$. We want to minimize $f(a,b) = a^2b+2ab^2-6ab = ab(a+2b-6)$. Thus, $f_a(a,b) = 2ab+2b^2-6b = 0$ and $f_b(a,b) = a^2+4ab-6a = 0 \Rightarrow b(2a+2b-6) = 0 \Rightarrow b = 0$ or $b = -a+3$. Now $b = 0 \Rightarrow a^2-6a = 0 \Rightarrow a = 0$ or $a = 6 \Rightarrow (0,0)$ and $(6,0)$ are critical points. On the other hand, $b = -a+3 \Rightarrow a^2+4a(-a+3)-6a = 0 \Rightarrow -3a^2+6a = 0 \Rightarrow a = 0$ or $a = 2 \Rightarrow (0,3)$ and $(2,1)$ are also critical points. The flux at $(0,0) = 0$, the flux at $(6,0) = 0$, the flux at $(0,3) = 0$ and the flux at $(2,1) = -4$. Therefore, the flux is minimized at $(2,1)$ with value -4.

10. A plane through the origin has equation $ax+by+cz = 0$. Consider first the case when $c \ne 0$. Assume the plane is given by $z = ax+by$ and let $f(x,y,z) = x^2+y^2+z^2 = 4$. Let C denote the circle of intersection of the plane with the sphere. By Stokes's Theorem, $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint\limits_{S} \nabla \times \mathbf{F}\cdot\mathbf{n}\,d\sigma$, where $\mathbf{n}$ is a unit normal to the plane. Let

$\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + (ax+by)\mathbf{k}$ be a parametrization of the surface. Then $\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & a \\ 0 & 1 & b \end{vmatrix} = -a\mathbf{i} - b\mathbf{j} + \mathbf{k}$

$\Rightarrow d\sigma = |\mathbf{r}_x \times \mathbf{r}_y|\,dx\,dy = \sqrt{a^2+b^2+1}\,dx\,dy$. Also, $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{n} = \frac{a\mathbf{i}+b\mathbf{j}-\mathbf{k}}{\sqrt{a^2+b^2+1}}$

$\Rightarrow \iint\limits_{S} \nabla \times \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint\limits_{R_{xy}} \frac{a+b-1}{\sqrt{a^2+b^2+1}}\sqrt{a^2+b^2+1}\,dx\,dy = \iint\limits_{R_{xy}} (a+b-1)\,dx\,dy = (a+b-1)\iint\limits_{R_{xy}} dx\,dy$. Now $x^2+y^2+(ax+by)^2 = 4 \Rightarrow \left(\frac{a^2+1}{4}\right)x^2 + \left(\frac{b^2+1}{4}\right)y^2 + \left(\frac{ab}{2}\right)xy = 1 \Rightarrow$ the region R_{xy} is the interior of the ellipse $Ax^2+Bxy+Cy^2 = 1$ in the xy-plane, where $A = \frac{a^2+1}{4}$, $B = \frac{ab}{2}$, and $C = \frac{b^2+1}{4}$. By Exercise 47 in Section 10.3, the area of the ellipse is $\frac{2\pi}{\sqrt{4AC-B^2}} = \frac{4\pi}{\sqrt{a^2+b^2+1}} \Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = h(a,b) = \frac{4\pi(a+b-1)}{\sqrt{a^2+b^2+1}}$.

Thus we optimize $H(a,b) = \frac{(a+b-1)^2}{a^2+b^2+1}$: $\frac{\partial H}{\partial a} = \frac{2(a+b-1)(b^2+1+a-ab)}{(a^2+b^2+1)^2} = 0$ and

$\frac{\partial H}{\partial b} = \frac{2(a+b-1)(a^2+1+b-ab)}{(a^2+b^2+1)^2} = 0 \Rightarrow a+b-1 = 0$, or $b^2+1+a-ab = 0$ and $a^2+1+b-ab = 0$

$\Rightarrow a+b-1 = 0$, or $a^2-b^2+(b-a) = 0 \Rightarrow a+b-1 = 0$, or $(a-b)(a+b-1) = 0 \Rightarrow a+b-1 = 0$ or $a = b$. The critical values $a+b-1 = 0$ give a saddle. If $a = b$, then $0 = b^2+1+a-ab \Rightarrow a^2+1+a-a^2 = 0$

$\Rightarrow a = -1 \Rightarrow b = -1$. Thus, the point $(a,b) = (-1,-1)$ gives a local extremum for $\oint_C \mathbf{F}\cdot d\mathbf{r} \Rightarrow z = -x-y$

$\Rightarrow x+y+z = 0$ is the desired plane, if $c \ne 0$.

Note: Since $h(-1,-1)$ is negative, the circulation about $\mathbf{n}$ is clockwise, so $-\mathbf{n}$ is the correct pointing normal for the counterclockwise circulation. Thus $\iint_S \nabla \times \mathbf{F} \cdot (-\mathbf{n})\, d\sigma$ actually gives the <u>maximum</u> circulation.

If $c = 0$, one can see that the corresponding problem is equivalent to the calculation above when $b = 0$, which does not lead to a local extreme.

11. (a) Partition the string into small pieces. Let $\Delta_i s$ be the length of the i^{th} piece. Let (x_i, y_i) be a point in the i^{th} piece. The work done by gravity in moving the i^{th} piece to the x-axis is approximately $W_i = (g x_i y_i \Delta_i s) y_i$ where $x_i y_i \Delta_i s$ is approximately the mass of the i^{th} piece. The total work done by gravity in moving the string to the x-axis is $\sum_i W_i = \sum_i g x_i y_i^2 \Delta_i s \Rightarrow \text{Work} = \int_C gxy^2\, ds$

(b) $\text{Work} = \int_C gxy^2\, ds = \int_0^{\pi/2} g(2\cos t)(4\sin^2 t)\sqrt{4\sin^2 t + 4\cos^2 t}\, dt = 16g \int_0^{\pi/2} \cos t \sin^2 t\, dt$
$= \left[16g\left(\frac{\sin^3 t}{3}\right)\right]_0^{\pi/2} = \frac{16}{3} g$

(c) $\overline{x} = \frac{\int_C x(xy)\, ds}{\int_C xy\, ds}$ and $\overline{y} = \frac{\int_C y(xy)\, ds}{\int_C xy\, ds}$; the mass of the string is $\int_C xy\, ds$ and the weight of the string is $g\int_C xy\, ds$. Therefore, the work done in moving the point mass at $(\overline{x}, \overline{y})$ to the x-axis is $W = \left(g\int_C xy\, ds\right)\overline{y} = g\int_C xy^2\, ds = \frac{16}{3} g.$

12. (a) Partition the sheet into small pieces. Let $\Delta_i \sigma$ be the area of the i^{th} piece and select a point (x_i, y_i, z_i) in the i^{th} piece. The mass of the i^{th} piece is approximately $x_i y_i \Delta_i \sigma$. The work done by gravity in moving the i^{th} piece to the xy-plane is approximately $(g x_i y_i \Delta_i \sigma) z_i = g x_i y_i z_i \Delta_i \sigma \Rightarrow \text{Work} = \iint_S gxyz\, d\sigma$.

(b) $\iint_S gxyz\, d\sigma = g \iint_{R_{xy}} xy(1 - x - y)\sqrt{1 + (-1)^2 + (-1)^2}\, dA = \sqrt{3}g \int_0^1 \int_0^{1-x} (xy - x^2y - xy^2)\, dy\, dx$
$= \sqrt{3}g \int_0^1 \left[\frac{1}{2}xy^2 - \frac{1}{2}x^2y^2 - \frac{1}{3}xy^3\right]_0^{1-x} dx = \sqrt{3}g \int_0^1 \left[\frac{1}{6}x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^4\right] dx$
$= \sqrt{3}g \left[\frac{1}{12}x^2 - \frac{1}{6}x^3 + \frac{1}{6}x^4 - \frac{1}{30}x^5\right]_0^1 = \sqrt{3}g\left(\frac{1}{12} - \frac{1}{30}\right) = \frac{\sqrt{3}g}{20}$

(c) The center of mass of the sheet is the point $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{z} = \frac{M_{xy}}{M}$ with $M_{xy} = \iint_S xyz\, d\sigma$ and $M = \iint_S xy\, d\sigma$. The work done by gravity in moving the point mass at $(\overline{x}, \overline{y}, \overline{z})$ to the xy-plane is $gM\overline{z} = gM\left(\frac{M_{xy}}{M}\right) = gM_{xy} = \iint_S gxyz\, d\sigma = \frac{\sqrt{3}g}{20}$.

13. (a) Partition the sphere $x^2 + y^2 + (z - 2)^2 = 1$ into small pieces. Let $\Delta_i \sigma$ be the surface area of the i^{th} piece and let (x_i, y_i, z_i) be a point on the i^{th} piece. The force due to pressure on the i^{th} piece is approximately $w(4 - z_i)\Delta_i \sigma$. The total force on S is approximately $\sum_i w(4 - z_i)\Delta_i \sigma$. This gives the actual force to be $\iint_S w(4 - z)\, d\sigma$.

(b) The upward buoyant force is a result of the **k**-component of the force on the ball due to liquid pressure. The force on the ball at (x, y, z) is $w(4 - z)(-\mathbf{n}) = w(z - 4)\mathbf{n}$, where $\mathbf{n}$ is the outer unit normal at (x, y, z). Hence the **k**-component of this force is $w(z - 4)\mathbf{n} \cdot \mathbf{k} = w(z - 4)\mathbf{k} \cdot \mathbf{n}$. The (magnitude of the) buoyant force on the ball is obtained by adding up all these **k**-components to obtain $\iint_S w(z - 4)\mathbf{k} \cdot \mathbf{n}\, d\sigma$.

(c) The Divergence Theorem says $\iint_S w(z - 4)\mathbf{k} \cdot \mathbf{n}\, d\sigma = \iiint_D \operatorname{div}(w(z - 4)\mathbf{k})\, dV = \iiint_D w\, dV$, where D is $x^2 + y^2 + (z - 2)^2 \le 1 \Rightarrow \iint_S w(z - 4)\mathbf{k} \cdot \mathbf{n}\, d\sigma = w\iiint_D 1\, dV = \frac{4}{3}\pi w$, the weight of the fluid if it were to occupy the region D.

14. The surface S is $z = \sqrt{x^2 + y^2}$ from $z = 1$ to $z = 2$. Partition S into small pieces and let $\Delta_i\sigma$ be the area of the i^{th} piece. Let (x_i, y_i, z_i) be a point on the i^{th} piece. Then the magnitude of the force on the i^{th} piece due to liquid pressure is approximately $F_i = w(2 - z_i)\Delta_i\sigma \Rightarrow$ the total force on S is approximately $\sum_i F_i = \Sigma\, w(2 - z_i)\Delta_i\sigma \Rightarrow$ the actual force is $\iint_S w(2 - z)\, d\sigma = \iint_{R_{xy}} w\left(2 - \sqrt{x^2 + y^2}\right)\sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}\, dA$
$= \iint_{R_{xy}} \sqrt{2}\, w\left(2 - \sqrt{x^2 + y^2}\right) dA = \int_0^{2\pi}\int_1^2 \sqrt{2}w(2 - r)\, r\, dr\, d\theta = \int_0^{2\pi} \sqrt{2}w\left[r^2 - \frac{1}{3}r^3\right]_1^2 d\theta = \int_0^{2\pi} \frac{2\sqrt{2}w}{3}\, d\theta$
$= \frac{4\sqrt{2}\pi w}{3}$

15. Assume that S is a surface to which Stokes's Theorem applies. Then $\oint_C \mathbf{E} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{E}) \cdot \mathbf{n}\, d\sigma$
$= \iint_S -\frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, d\sigma = -\frac{\partial}{\partial t}\iint_S \mathbf{B} \cdot \mathbf{n}\, d\sigma$. Thus the voltage around a loop equals the negative of the rate of change of magnetic flux through the loop.

16. According to Gauss's Law, $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 4\pi GmM$ for any surface enclosing the origin. But if $\mathbf{F} = \nabla \times \mathbf{H}$ then the integral over such a closed surface would have to be 0 by the Divergence Theorem since div $\mathbf{F} = 0$.

17. $\oint_C f\nabla g \cdot d\mathbf{r} = \iint_S \nabla \times (f\nabla g) \cdot \mathbf{n}\, d\sigma$ (Stokes's Theorem)
$= \iint_S (f\nabla \times \nabla g + \nabla f \times \nabla g) \cdot \mathbf{n}\, d\sigma$ (Section 16.8, Exercise 19b)
$= \iint_S [(f)(\mathbf{0}) + \nabla f \times \nabla g] \cdot \mathbf{n}\, d\sigma$ (Section 16.7, Equation 8)
$= \iint_S (\nabla f \times \nabla g) \cdot \mathbf{n}\, d\sigma$

18. $\nabla \times \mathbf{F}_1 = \nabla \times \mathbf{F}_2 \Rightarrow \nabla \times (\mathbf{F}_2 - \mathbf{F}_1) = \mathbf{0} \Rightarrow \mathbf{F}_2 - \mathbf{F}_1$ is conservative $\Rightarrow \mathbf{F}_2 - \mathbf{F}_1 = \nabla f$; also, $\nabla \cdot \mathbf{F}_1 = \nabla \cdot \mathbf{F}_2$ $\Rightarrow \nabla \cdot (\mathbf{F}_2 - \mathbf{F}_1) = 0 \Rightarrow \nabla^2 f = 0$ (so f is harmonic). Finally, on the surface S, $\nabla f \cdot \mathbf{n} = (\mathbf{F}_2 - \mathbf{F}_1) \cdot \mathbf{n}$ $= \mathbf{F}_2 \cdot \mathbf{n} - \mathbf{F}_1 \cdot \mathbf{n} = 0$. Now, $\nabla \cdot (f\nabla f) = \nabla f \cdot \nabla f + f\nabla^2 f$ so the Divergence Theorem gives $\iiint_D |\nabla f|^2\, dV + \iiint_D f\nabla^2 f\, dV = \iiint_D \nabla \cdot (f\nabla f)\, dV = \iint_S f\nabla f \cdot \mathbf{n}\, d\sigma = 0$, and since $\nabla^2 f = 0$ we have $\iiint_D |\nabla f|^2\, dV + 0 = 0 \Rightarrow \iiint_D |\mathbf{F}_2 - \mathbf{F}_1|^2\, dV = 0 \Rightarrow \mathbf{F}_2 - \mathbf{F}_1 = \mathbf{0} \Rightarrow \mathbf{F}_2 = \mathbf{F}_1$, as claimed.

19. False; let $\mathbf{F} = y\mathbf{i} + x\mathbf{j} \neq \mathbf{0} \Rightarrow \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) = 0$ and $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

20. $|\mathbf{r}_u \times \mathbf{r}_v|^2 = |\mathbf{r}_u|^2 |\mathbf{r}_v|^2 \sin^2\theta = |\mathbf{r}_u|^2 |\mathbf{r}_v|^2 (1 - \cos^2\theta) = |\mathbf{r}_u|^2 |\mathbf{r}_v|^2 - |\mathbf{r}_u|^2 |\mathbf{r}_v|^2 \cos^2\theta = |\mathbf{r}_u|^2 |\mathbf{r}_v|^2 - (\mathbf{r}_u \cdot \mathbf{r}_v)^2$ $\Rightarrow |\mathbf{r}_u \times \mathbf{r}_v|^2 = \sqrt{EG - F^2} \Rightarrow d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv = \sqrt{EG - F^2}\, du\, dv$

21. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \Rightarrow \nabla \cdot \mathbf{r} = 1 + 1 + 1 = 3 \Rightarrow \iiint_D \nabla \cdot \mathbf{r}\, dV = 3\iiint_D dV = 3V \Rightarrow V = \frac{1}{3}\iiint_D \nabla \cdot \mathbf{r}\, dV$ $= \frac{1}{3}\iint_S \mathbf{r} \cdot \mathbf{n}\, d\sigma$, by the Divergence Theorem

NOTES: