6. $\lim\limits_{(x,y)\to(0,0)} \cos\left(\frac{x^2+y^3}{x+y+1}\right) = \cos\left(\frac{0^2+0^3}{0+0+1}\right) = \cos 0 = 1$

7. $\lim\limits_{(x,y)\to(0,\ln 2)} e^{x-y} = e^{0-\ln 2} = e^{\ln\left(\frac{1}{2}\right)} = \frac{1}{2}$

8. $\lim\limits_{(x,y)\to(1,1)} \ln|1+x^2y^2| = \ln|1+(1)^2(1)^2| = \ln 2$

9. $\lim\limits_{(x,y)\to(0,0)} \frac{e^y \sin x}{x} = \lim\limits_{(x,y)\to(0,0)} (e^y)\left(\frac{\sin x}{x}\right) = e^0 \cdot \lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right) = 1\cdot 1 = 1$

10. $\lim\limits_{(x,y)\to(1,1)} \cos\left(\sqrt[3]{|xy|-1}\right) = \cos\left(\sqrt[3]{(1)(1)-1}\right) = \cos 0 = 1$

11. $\lim\limits_{(x,y)\to(1,0)} \frac{x\sin y}{x^2+1} = \frac{1\cdot\sin 0}{1^2+1} = \frac{0}{2} = 0$

12. $\lim\limits_{(x,y)\to\left(\frac{\pi}{2},0\right)} \frac{\cos y+1}{y-\sin x} = \frac{(\cos 0)+1}{0-\sin\left(\frac{\pi}{2}\right)} = \frac{1+1}{-1} = -2$

13. $\lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-2xy+y^2}{x-y} = \lim\limits_{(x,y)\to(1,1)} \frac{(x-y)^2}{x-y} = \lim\limits_{(x,y)\to(1,1)} (x-y) = (1-1) = 0$

14. $\lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-y^2}{x-y} = \lim\limits_{(x,y)\to(1,1)} \frac{(x+y)(x-y)}{x-y} = \lim\limits_{(x,y)\to(1,1)} (x+y) = (1+1) = 2$

15. $\lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \frac{xy-y-2x+2}{x-1} = \lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \frac{(x-1)(y-2)}{x-1} = \lim\limits_{(x,y)\to(1,1)} (y-2) = (1-2) = -1$

16. $\lim\limits_{\substack{(x,y)\to(2,-4)\\ y\neq -4,\ x\neq x^2}} \frac{y+4}{x^2y-xy+4x^2-4x} = \lim\limits_{\substack{(x,y)\to(2,-4)\\ y\neq -4,\ x\neq x^2}} \frac{y+4}{x(x-1)(y+4)} = \lim\limits_{\substack{(x,y)\to(2,-4)\\ x\neq x^2}} \frac{1}{x(x-1)} = \frac{1}{2(2-1)} = \frac{1}{2}$

17. $\lim\limits_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}} = \lim\limits_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{\left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}+2\right)}{\sqrt{x}-\sqrt{y}} = \lim\limits_{(x,y)\to(0,0)} \left(\sqrt{x}+\sqrt{y}+2\right)$

$= \left(\sqrt{0}+\sqrt{0}+2\right) = 2$

Note: (x, y) must approach $(0, 0)$ through the first quadrant only with $x \neq y$.

18. $\lim\limits_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \frac{x+y-4}{\sqrt{x+y}-2} = \lim\limits_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \frac{\left(\sqrt{x+y}+2\right)\left(\sqrt{x+y}-2\right)}{\sqrt{x+y}-2} = \lim\limits_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \left(\sqrt{x+y}+2\right)$

$= \left(\sqrt{2+2}+2\right) = 2+2 = 4$

19. $\lim\limits_{\substack{(x,y)\to(2,0)\\ 2x-y\neq 4}} \frac{\sqrt{2x-y}-2}{2x-y-4} = \lim\limits_{\substack{(x,y)\to(2,0)\\ 2x-y\neq 4}} \frac{\sqrt{2x-y}-2}{\left(\sqrt{2x-y}+2\right)\left(\sqrt{2x-y}-2\right)} = \lim\limits_{(x,y)\to(2,0)} \frac{1}{\sqrt{2x-y}+2}$

$= \frac{1}{\sqrt{(2)(2)-0}+2} = \frac{1}{2+2} = \frac{1}{4}$

20. $\lim\limits_{\substack{(x,y)\to(4,3)\\ x-y\neq 1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1} = \lim\limits_{\substack{(x,y)\to(4,3)\\ x-y\neq 1}} \frac{\sqrt{x}-\sqrt{y+1}}{\left(\sqrt{x}+\sqrt{y+1}\right)\left(\sqrt{x}-\sqrt{y+1}\right)} = \lim\limits_{(x,y)\to(4,3)} \frac{1}{\sqrt{x}+\sqrt{y+1}}$

$= \frac{1}{\sqrt{4+\sqrt{3+1}}} = \frac{1}{2+2} = \frac{1}{4}$

21. $\lim\limits_{P \to (1,3,4)} \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = \frac{1}{1} + \frac{1}{3} + \frac{1}{4} = \frac{12+4+3}{12} = \frac{19}{12}$

22. $\lim\limits_{P \to (1,-1,-1)} \frac{2xy+yz}{x^2+z^2} = \frac{2(1)(-1)+(-1)(-1)}{1^2+(-1)^2} = \frac{-2+1}{1+1} = -\frac{1}{2}$

23. $\lim\limits_{P \to (3,3,0)} (\sin^2 x + \cos^2 y + \sec^2 z) = (\sin^2 3 + \cos^2 3) + \sec^2 0 = 1 + 1^2 = 2$

24. $\lim\limits_{P \to \left(-\frac{1}{4}, \frac{\pi}{2}, 2\right)} \tan^{-1}(xyz) = \tan^{-1}\left(-\frac{1}{4} \cdot \frac{\pi}{2} \cdot 2\right) = \tan^{-1}\left(-\frac{\pi}{4}\right)$

25. $\lim\limits_{P \to (\pi,0,3)} ze^{-2y} \cos 2x = 3e^{-2(0)} \cos 2\pi = (3)(1)(1) = 3$

26. $\lim\limits_{P \to (0,-2,0)} \ln \sqrt{x^2+y^2+z^2} = \ln \sqrt{0^2+(-2)^2+0^2} = \ln \sqrt{4} = \ln 2$

27. (a) All (x, y)
(b) All (x, y) except $(0, 0)$

28. (a) All (x, y) so that $x \neq y$
(b) All (x, y)

29. (a) All (x, y) except where $x = 0$ or $y = 0$
(b) All (x, y)

30. (a) All (x, y) so that $x^2 - 3x + 2 \neq 0 \Rightarrow (x-2)(x-1) \neq 0 \Rightarrow x \neq 2$ and $x \neq 1$
(b) All (x, y) so that $y \neq x^2$

31. (a) All (x, y, z)
(b) All (x, y, z) except the interior of the cylinder $x^2 + y^2 = 1$

32. (a) All (x, y, z) so that $xyz > 0$
(b) All (x, y, z)

33. (a) All (x, y, z) with $z \neq 0$
(b) All (x, y, z) with $x^2 + z^2 \neq 1$

34. (a) All (x, y, z) except $(x, 0, 0)$
(b) All (x, y, z) except $(0, y, 0)$ or $(x, 0, 0)$

35. $\lim\limits_{\substack{(x,y) \to (0,0) \\ \text{along } y=x \\ x>0}} -\frac{x}{\sqrt{x^2+y^2}} = \lim\limits_{x \to 0^+} -\frac{x}{\sqrt{x^2+x^2}} = \lim\limits_{x \to 0^+} -\frac{x}{\sqrt{2}\,|x|} = \lim\limits_{x \to 0^+} -\frac{x}{\sqrt{2}\,x} = \lim\limits_{x \to 0^+} -\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$;

$\lim\limits_{\substack{(x,y) \to (0,0) \\ \text{along } y=x \\ x<0}} -\frac{x}{\sqrt{x^2+y^2}} = \lim\limits_{x \to 0^-} -\frac{x}{\sqrt{2}\,|x|} = \lim\limits_{x \to 0^-} -\frac{x}{\sqrt{2}(-x)} = \lim\limits_{x \to 0^-} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

36. $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } y=0}} \frac{x^4}{x^4+y^2} = \lim\limits_{x\to 0} \frac{x^4}{x^4+0^2} = 1;\ \lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } y=x^2}} \frac{x^4}{x^4+y^2} = \lim\limits_{x\to 0} \frac{x^4}{x^4+(x^2)^2} = \lim\limits_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}$

37. $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } y=kx^2}} \frac{x^4-y^2}{x^4+y^2} = \lim\limits_{x\to 0} \frac{x^4-(kx^2)^2}{x^4+(kx^2)^2} = \lim\limits_{x\to 0} \frac{x^4-k^2x^4}{x^4+k^2x^4} = \frac{1-k^2}{1+k^2} \Rightarrow$ different limits for different values of k

38. $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } y=kx\\ k\neq 0}} \frac{xy}{|xy|} = \lim\limits_{x\to 0} \frac{x(kx)}{|x(kx)|} = \lim\limits_{x\to 0} \frac{kx^2}{|kx^2|} = \lim\limits_{x\to 0} \frac{k}{|k|}$; if $k > 0$, the limit is 1; but if $k < 0$, the limit is -1

39. $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } y=kx\\ k\neq -1}} \frac{x-y}{x+y} = \lim\limits_{x\to 0} \frac{x-kx}{x+kx} = \frac{1-k}{1+k} \Rightarrow$ different limits for different values of k, $k \neq -1$

40. $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } y=kx\\ k\neq 1}} \frac{x+y}{x-y} = \lim\limits_{x\to 0} \frac{x+kx}{x-kx} = \frac{1+k}{1-k} \Rightarrow$ different limits for different values of k, $k \neq 1$

41. $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } y=kx^2\\ k\neq 0}} \frac{x^2+y}{y} = \lim\limits_{x\to 0} \frac{x^2+kx^2}{kx^2} = \frac{1+k}{k} \Rightarrow$ different limits for different values of k, $k \neq 0$

42. $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } y=kx^2\\ k\neq 1}} \frac{x^2}{x^2-y} = \lim\limits_{x\to 0} \frac{x^2}{x^2-kx^2} = \frac{1}{1-k} \Rightarrow$ different limits for different values of k, $k \neq 1$

43. No, the limit depends only on the values $f(x,y)$ has when $(x,y) \neq (x_0, y_0)$

44. If f is continuous at (x_0, y_0), then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ must equal $f(x_0, y_0) = 3$. If f is not continuous at (x_0, y_0), the limit could have any value different from 3, and need not even exist.

45. $\lim\limits_{(x,y)\to(0,0)} \left(1 - \frac{x^2y^2}{3}\right) = 1$ and $\lim\limits_{(x,y)\to(0,0)} 1 = 1 \Rightarrow \lim\limits_{(x,y)\to(0,0)} \frac{\tan^{-1} xy}{xy} = 1$, by the Sandwich Theorem

46. If $xy > 0$, $\lim\limits_{(x,y)\to(0,0)} \frac{2|xy| - \left(\frac{x^2y^2}{6}\right)}{|xy|} = \lim\limits_{(x,y)\to(0,0)} \frac{2xy - \left(\frac{x^2y^2}{6}\right)}{xy} = \lim\limits_{(x,y)\to(0,0)} \left(2 - \frac{xy}{6}\right) = 2$ and $\lim\limits_{(x,y)\to(0,0)} \frac{2|xy|}{|xy|} = \lim\limits_{(x,y)\to(0,0)} 2 = 2$; if $xy < 0$, $\lim\limits_{(x,y)\to(0,0)} \frac{2|xy| - \left(\frac{x^2y^2}{6}\right)}{|xy|} = \lim\limits_{(x,y)\to(0,0)} \frac{-2xy - \left(\frac{x^2y^2}{6}\right)}{-xy}$ $= \lim\limits_{(x,y)\to(0,0)} \left(2 + \frac{xy}{6}\right) = 2$ and $\lim\limits_{(x,y)\to(0,0)} \frac{2|xy|}{|xy|} = 2 \Rightarrow \lim\limits_{(x,y)\to(0,0)} \frac{4 - 4\cos\sqrt{|xy|}}{|xy|} = 2$, by the Sandwich Theorem

47. The limit is 0 since $\left|\sin\left(\frac{1}{x}\right)\right| \le 1 \Rightarrow -1 \le \sin\left(\frac{1}{x}\right) \le 1 \Rightarrow -y \le y\sin\left(\frac{1}{x}\right) \le y$ for $y \ge 0$, and $-y \ge y\sin\left(\frac{1}{x}\right) \ge y$ for $y \le 0$. Thus as $(x,y) \to (0,0)$, both $-y$ and y approach $0 \Rightarrow y\sin\left(\frac{1}{x}\right) \to 0$, by the Sandwich Theorem.

48. The limit is 0 since $\left|\cos\left(\frac{1}{y}\right)\right| \le 1 \Rightarrow -1 \le \cos\left(\frac{1}{y}\right) \le 1 \Rightarrow -x \le x\cos\left(\frac{1}{y}\right) \le x$ for $x \ge 0$, and $-x \ge x\cos\left(\frac{1}{y}\right) \ge x$ for $x \le 0$. Thus as $(x,y) \to (0,0)$, both $-x$ and x approach $0 \Rightarrow x\cos\left(\frac{1}{y}\right) \to 0$, by the Sandwich Theorem.

49. (a) $f(x, y)\big|_{y=mx} = \frac{2m}{1+m^2} = \frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$. The value of $f(x, y) = \sin 2\theta$ varies with θ, which is the line's angle of inclination.

(b) Since $f(x, y)\big|_{y=mx} = \sin 2\theta$ and since $-1 \le \sin 2\theta \le 1$ for every θ, $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ varies from -1 to 1 along $y = mx$.

50. $|xy\,(x^2 - y^2)| = |xy|\,|x^2 - y^2| \le |x|\,|y|\,|x^2 + y^2| = \sqrt{x^2}\,\sqrt{y^2}\,|x^2 + y^2| \le \sqrt{x^2 + y^2}\,\sqrt{x^2 + y^2}\,|x^2 + y^2|$ $= (x^2 + y^2)^2 \Rightarrow \left|\frac{xy\,(x^2 - y^2)}{x^2 + y^2}\right| \le \frac{(x^2 + y^2)^2}{x^2 + y^2} = x^2 + y^2 \Rightarrow -(x^2 + y^2) \le \frac{xy\,(x^2 - y^2)}{x^2 + y^2} \le (x^2 + y^2)$ $\Rightarrow \lim\limits_{(x,y)\to(0,0)} xy\,\frac{x^2 - y^2}{x^2 + y^2} = 0$ by the Sandwich Theorem, since $\lim\limits_{(x,y)\to(0,0)} \pm(x^2 + y^2) = 0$; thus, define $f(0, 0) = 0$

51. $\lim\limits_{(x,y)\to(0,0)} \frac{x^3 - xy^2}{x^2 + y^2} = \lim\limits_{r\to 0} \frac{r^3\cos^3\theta - (r\cos\theta)(r^2\sin^2\theta)}{r^2\cos^2\theta + r^2\sin^2\theta} = \lim\limits_{r\to 0} \frac{r\,(\cos^3\theta - \cos\theta\sin^2\theta)}{1} = 0$

52. $\lim\limits_{(x,y)\to(0,0)} \cos\left(\frac{x^3 - y^3}{x^2 + y^2}\right) = \lim\limits_{r\to 0} \cos\left(\frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\right) = \lim\limits_{r\to 0} \cos\left[\frac{r\,(\cos^3\theta - \sin^3\theta)}{1}\right] = \cos 0 = 1$

53. $\lim\limits_{(x,y)\to(0,0)} \frac{y^2}{x^2 + y^2} = \lim\limits_{r\to 0} \frac{r^2\sin^2\theta}{r^2} = \lim\limits_{r\to 0} (\sin^2\theta) = \sin^2\theta$; the limit does not exist since $\sin^2\theta$ is between 0 and 1 depending on θ

54. $\lim\limits_{(x,y)\to(0,0)} \frac{2x}{x^2 + x + y^2} = \lim\limits_{r\to 0} \frac{2r\cos\theta}{r^2 + r\cos\theta} = \lim\limits_{r\to 0} \frac{2\cos\theta}{r + \cos\theta} = \frac{2\cos\theta}{\cos\theta}$; the limit does not exist for $\cos\theta = 0$

55. $\lim\limits_{(x,y)\to(0,0)} \tan^{-1}\left[\frac{|x| + |y|}{x^2 + y^2}\right] = \lim\limits_{r\to 0} \tan^{-1}\left[\frac{|r\cos\theta| + |r\sin\theta|}{r^2}\right] = \lim\limits_{r\to 0} \tan^{-1}\left[\frac{|r|\,(|\cos\theta| + |\sin\theta|)}{r^2}\right]$; if $r \to 0^+$, then $\lim\limits_{r\to 0^+} \tan^{-1}\left[\frac{|r|\,(|\cos\theta| + |\sin\theta|)}{r^2}\right] = \lim\limits_{r\to 0^+} \tan^{-1}\left[\frac{|\cos\theta| + |\sin\theta|}{r}\right] = \frac{\pi}{2}$; if $r \to 0^-$, then $\lim\limits_{r\to 0^-} \tan^{-1}\left[\frac{|r|\,(|\cos\theta| + |\sin\theta|)}{r^2}\right] = \lim\limits_{r\to 0^-} \tan^{-1}\left(\frac{|\cos\theta| + |\sin\theta|}{-r}\right) = \frac{\pi}{2} \Rightarrow$ the limit is $\frac{\pi}{2}$

56. $\lim\limits_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim\limits_{r\to 0} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2} = \lim\limits_{r\to 0} (\cos^2\theta - \sin^2\theta) = \lim\limits_{r\to 0} (\cos 2\theta)$ which ranges between -1 and 1 depending on $\theta \Rightarrow$ the limit does not exist

57. $\lim\limits_{(x,y)\to(0,0)} \ln\left(\frac{3x^2 - x^2y^2 + 3y^2}{x^2 + y^2}\right) = \lim\limits_{r\to 0} \ln\left(\frac{3r^2\cos^2\theta - r^4\cos^2\theta\sin^2\theta + 3r^2\sin^2\theta}{r^2}\right)$ $= \lim\limits_{r\to 0} \ln(3 - r^2\cos^2\theta\sin^2\theta) = \ln 3 \Rightarrow$ define $f(0, 0) = \ln 3$

58. $\lim\limits_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2} = \lim\limits_{r\to 0} \frac{(2r\cos\theta)(r^2\sin^2\theta)}{r^2} = \lim\limits_{r\to 0} 2r\cos\theta\sin^2\theta = 0 \Rightarrow$ define $f(0, 0) = 0$

59. Let $\delta = 0.1$. Then $\sqrt{x^2 + y^2} < \delta \Rightarrow \sqrt{x^2 + y^2} < 0.1 \Rightarrow x^2 + y^2 < 0.01 \Rightarrow |x^2 + y^2 - 0| < 0.01 \Rightarrow |f(x, y) - f(0, 0)|$ $< 0.01 = \epsilon$.

60. Let $\delta = 0.05$. Then $|x| < \delta$ and $|y| < \delta \Rightarrow |f(x, y) - f(0, 0)| = \left|\frac{y}{x^2 + 1} - 0\right| = \left|\frac{y}{x^2 + 1}\right| \le |y| < 0.05 = \epsilon$.

61. Let $\delta = 0.005$. Then $|x| < \delta$ and $|y| < \delta \Rightarrow |f(x, y) - f(0, 0)| = \left|\frac{x + y}{x^2 + 1} - 0\right| = \left|\frac{x + y}{x^2 + 1}\right| \le |x + y| < |x| + |y|$ $< 0.005 + 0.005 = 0.01 = \epsilon$.

62. Let $\delta = 0.01$. Since $-1 \le \cos x \le 1 \Rightarrow 1 \le 2 + \cos x \le 3 \Rightarrow \frac{1}{3} \le \frac{1}{2+\cos x} \le 1 \Rightarrow \frac{|x+y|}{3} \le \left|\frac{x+y}{2+\cos x}\right| \le |x+y|$ $\le |x| + |y|$. Then $|x| < \delta$ and $|y| < \delta \Rightarrow |f(x,y) - f(0,0)| = \left|\frac{x+y}{2+\cos x} - 0\right| = \left|\frac{x+y}{2+\cos x}\right| \le |x| + |y| < 0.01 + 0.01$ $= 0.02 = \epsilon$.

63. Let $\delta = \sqrt{0.015}$. Then $\sqrt{x^2+y^2+z^2} < \delta \Rightarrow |f(x,y,z) - f(0,0,0)| = |x^2+y^2+z^2 - 0| = |x^2+y^2+z^2|$ $= \left(\sqrt{x^2+t^2+x^2}\right)^2 < \left(\sqrt{0.015}\right)^2 = 0.015 = \epsilon$.

64. Let $\delta = 0.2$. Then $|x| < \delta$, $|y| < \delta$, and $|z| < \delta \Rightarrow |f(x,y,z) - f(0,0,0)| = |xyz - 0| = |xyz| = |x|\,|y|\,|z| < (0.2)^3$ $= 0.008 = \epsilon$.

65. Let $\delta = 0.005$. Then $|x| < \delta$, $|y| < \delta$, and $|z| < \delta \Rightarrow |f(x,y,z) - f(0,0,0)| = \left|\frac{x+y+z}{x^2+y^2+z^2+1} - 0\right|$ $= \left|\frac{x+y+z}{x^2+y^2+z^2+1}\right| \le |x+y+z| \le |x| + |y| + |z| < 0.005 + 0.005 + 0.005 = 0.015 = \epsilon$.

66. Let $\delta = \tan^{-1}(0.1)$. Then $|x| < \delta$, $|y| < \delta$, and $|z| < \delta \Rightarrow |f(x,y,z) - f(0,0,0)| = |\tan^2 x + \tan^2 y + \tan^2 z|$ $\le |\tan^2 x| + |\tan^2 y| + |\tan^2 z| = \tan^2 x + \tan^2 y + \tan^2 z < \tan^2\delta + \tan^2\delta + \tan^2\delta = 0.01 + 0.01 + 0.01 = 0.03$ $= \epsilon$.

67. $\lim\limits_{(x,y,z) \to (x_0,y_0,z_0)} f(x,y,z) = \lim\limits_{(x,y,z) \to (x_0,y_0,z_0)} (x+y+z) = x_0 + y_0 + z_0 = f(x_0,y_0,z_0) \Rightarrow$ f is continuous at every (x_0,y_0,z_0)

68. $\lim\limits_{(x,y,z) \to (x_0,y_0,z_0)} f(x,y,z) = \lim\limits_{(x,y,z) \to (x_0,y_0,z_0)} (x^2+y^2+z^2) = x_0^2 + y_0^2 + z_0^2 = f(x_0,y_0,z_0) \Rightarrow$ f is continuous at every point (x_0,y_0,z_0)

14.3 PARTIAL DERIVATIVES

1. $\frac{\partial f}{\partial x} = 4x$, $\frac{\partial f}{\partial y} = -3$

2. $\frac{\partial f}{\partial x} = 2x - y$, $\frac{\partial f}{\partial y} = -x + 2y$

3. $\frac{\partial f}{\partial x} = 2x(y+2)$, $\frac{\partial f}{\partial y} = x^2 - 1$

4. $\frac{\partial f}{\partial x} = 5y - 14x + 3$, $\frac{\partial f}{\partial y} = 5x - 2y - 6$

5. $\frac{\partial f}{\partial x} = 2y(xy-1)$, $\frac{\partial f}{\partial y} = 2x(xy-1)$

6. $\frac{\partial f}{\partial x} = 6(2x-3y)^2$, $\frac{\partial f}{\partial y} = -9(2x-3y)^2$

7. $\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$, $\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$

8. $\frac{\partial f}{\partial x} = \frac{2x^2}{\sqrt[3]{x^3+\left(\frac{y}{2}\right)}}$, $\frac{\partial f}{\partial y} = \frac{1}{3\sqrt[3]{x^3+\left(\frac{y}{2}\right)}}$

9. $\frac{\partial f}{\partial x} = -\frac{1}{(x+y)^2} \cdot \frac{\partial}{\partial x}(x+y) = -\frac{1}{(x+y)^2}$, $\frac{\partial f}{\partial y} = -\frac{1}{(x+y)^2} \cdot \frac{\partial}{\partial y}(x+y) = -\frac{1}{(x+y)^2}$

10. $\frac{\partial f}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$, $\frac{\partial f}{\partial y} = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}$

11. $\frac{\partial f}{\partial x} = = \frac{(xy-1)(1) - (x+y)(y)}{(xy-1)^2} = \frac{-y^2-1}{(xy-1)^2}$, $\frac{\partial f}{\partial y} = \frac{(xy-1)(1) - (x+y)(x)}{(xy-1)^2} = \frac{-x^2-1}{(xy-1)^2}$

12. $\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = -\frac{y}{x^2\left[1+\left(\frac{y}{x}\right)^2\right]} = -\frac{y}{x^2+y^2}$, $\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{x\left[1+\left(\frac{y}{x}\right)^2\right]} = \frac{x}{x^2+y^2}$

13. $\frac{\partial f}{\partial x} = e^{(x+y+1)} \cdot \frac{\partial}{\partial x}(x+y+1) = e^{(x+y+1)}$, $\frac{\partial f}{\partial y} = e^{(x+y+1)} \cdot \frac{\partial}{\partial y}(x+y+1) = e^{(x+y+1)}$

14. $\frac{\partial f}{\partial x} = -e^{-x} \sin(x+y) + e^{-x} \cos(x+y)$, $\frac{\partial f}{\partial y} = e^{-x} \cos(x+y)$

15. $\frac{\partial f}{\partial x} = \frac{1}{x+y} \cdot \frac{\partial}{\partial x}(x+y) = \frac{1}{x+y}$, $\frac{\partial f}{\partial y} = \frac{1}{x+y} \cdot \frac{\partial}{\partial y}(x+y) = \frac{1}{x+y}$

16. $\frac{\partial f}{\partial x} = e^{xy} \cdot \frac{\partial}{\partial x}(xy) \cdot \ln y = ye^{xy} \ln y$, $\frac{\partial f}{\partial y} = e^{xy} \cdot \frac{\partial}{\partial y}(xy) \cdot \ln y + e^{xy} \cdot \frac{1}{y} = xe^{xy} \ln y + \frac{e^{xy}}{y}$

17. $\frac{\partial f}{\partial x} = 2\sin(x-3y) \cdot \frac{\partial}{\partial x}\sin(x-3y) = 2\sin(x-3y)\cos(x-3y) \cdot \frac{\partial}{\partial x}(x-3y) = 2\sin(x-3y)\cos(x-3y)$,
$\frac{\partial f}{\partial y} = 2\sin(x-3y) \cdot \frac{\partial}{\partial y}\sin(x-3y) = 2\sin(x-3y)\cos(x-3y) \cdot \frac{\partial}{\partial y}(x-3y) = -6\sin(x-3y)\cos(x-3y)$

18. $\frac{\partial f}{\partial x} = 2\cos(3x-y^2) \cdot \frac{\partial}{\partial x}\cos(3x-y^2) = -2\cos(3x-y^2)\sin(3x-y^2) \cdot \frac{\partial}{\partial x}(3x-y^2)$
$= -6\cos(3x-y^2)\sin(3x-y^2)$,
$\frac{\partial f}{\partial y} = 2\cos(3x-y^2) \cdot \frac{\partial}{\partial y}\cos(3x-y^2) = -2\cos(3x-y^2)\sin(3x-y^2) \cdot \frac{\partial}{\partial y}(3x-y^2)$
$= 4y\cos(3x-y^2)\sin(3x-y^2)$

19. $\frac{\partial f}{\partial x} = yx^{y-1}$, $\frac{\partial f}{\partial y} = x^y \ln x$

20. $f(x,y) = \frac{\ln x}{\ln y} \Rightarrow \frac{\partial f}{\partial x} = \frac{1}{x \ln y}$ and $\frac{\partial f}{\partial y} = \frac{-\ln x}{y(\ln y)^2}$

21. $\frac{\partial f}{\partial x} = -g(x)$, $\frac{\partial f}{\partial y} = g(y)$

22. $f(x,y) = \sum_{n=0}^{\infty}(xy)^n$, $|xy| < 1 \Rightarrow f(x,y) = \frac{1}{1-xy} \Rightarrow \frac{\partial f}{\partial x} = -\frac{1}{(1-xy)^2} \cdot \frac{\partial}{\partial x}(1-xy) = \frac{y}{(1-xy)^2}$ and
$\frac{\partial f}{\partial y} = -\frac{1}{(1-xy)^2} \cdot \frac{\partial}{\partial y}(1-xy) = \frac{x}{(1-xy)^2}$

23. $f_x = 1 + y^2$, $f_y = 2xy$, $f_z = -4z$

24. $f_x = y + z$, $f_y = x + z$, $f_z = y + x$

25. $f_x = 1$, $f_y = -\frac{y}{\sqrt{y^2+z^2}}$, $f_z = -\frac{z}{\sqrt{y^2+z^2}}$

26. $f_x = -x(x^2+y^2+z^2)^{-3/2}$, $f_y = -y(x^2+y^2+z^2)^{-3/2}$, $f_z = -z(x^2+y^2+z^2)^{-3/2}$

27. $f_x = \frac{yz}{\sqrt{1-x^2y^2z^2}}$, $f_y = \frac{xz}{\sqrt{1-x^2y^2z^2}}$, $f_z = \frac{xy}{\sqrt{1-x^2y^2z^2}}$

28. $f_x = \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}$, $f_y = \frac{z}{|x+yz|\sqrt{(x+yz)^2-1}}$, $f_z = \frac{y}{|x+yz|\sqrt{(x+yz)^2-1}}$

29. $f_x = \frac{1}{x+2y+3z}$, $f_y = \frac{2}{x+2y+3z}$, $f_z = \frac{3}{x+2y+3z}$

30. $f_x = yz \cdot \frac{1}{xy} \cdot \frac{\partial}{\partial x}(xy) = \frac{(yz)(y)}{xy} = \frac{yz}{x}$, $f_y = z\ln(xy) + yz \cdot \frac{\partial}{\partial y}\ln(xy) = z\ln(xy) + \frac{yz}{xy} \cdot \frac{\partial}{\partial y}(xy) = z\ln(xy) + z$,
$f_z = y\ln(xy) + yz \cdot \frac{\partial}{\partial z}\ln(xy) = y\ln(xy)$

31. $f_x = -2xe^{-(x^2+y^2+z^2)}$, $f_y = -2ye^{-(x^2+y^2+z^2)}$, $f_z = -2ze^{-(x^2+y^2+z^2)}$

32. $f_x = -yze^{-xyz}$, $f_y = -xze^{-xyz}$, $f_z = -xye^{-xyz}$

33. $f_x = \operatorname{sech}^2(x+2y+3z)$, $f_y = 2\operatorname{sech}^2(x+2y+3z)$, $f_z = 3\operatorname{sech}^2(x+2y+3z)$

34. $f_x = y\cosh(xy-z^2)$, $f_y = x\cosh(xy-z^2)$, $f_z = -2z\cosh(xy-z^2)$

35. $\frac{\partial f}{\partial t} = -2\pi \sin(2\pi t - \alpha)$, $\frac{\partial f}{\partial \alpha} = \sin(2\pi t - \alpha)$

36. $\frac{\partial g}{\partial u} = v^2 e^{(2u/v)} \cdot \frac{\partial}{\partial u}\left(\frac{2u}{v}\right) = 2ve^{(2u/v)}$, $\frac{\partial g}{\partial v} = 2ve^{(2u/v)} + v^2 e^{(2u/v)} \cdot \frac{\partial}{\partial v}\left(\frac{2u}{v}\right) = 2ve^{(2u/v)} - 2ue^{(2u/v)}$

37. $\frac{\partial h}{\partial \rho} = \sin\phi \cos\theta$, $\frac{\partial h}{\partial \phi} = \rho \cos\phi \cos\theta$, $\frac{\partial h}{\partial \theta} = -\rho \sin\phi \sin\theta$

38. $\frac{\partial g}{\partial r} = 1 - \cos\theta$, $\frac{\partial g}{\partial \theta} = r \sin\theta$, $\frac{\partial g}{\partial z} = -1$

39. $W_p = V$, $W_v = P + \frac{\delta v^2}{2g}$, $W_\delta = \frac{Vv^2}{2g}$, $W_v = \frac{2V\delta v}{2g} = \frac{V\delta v}{g}$, $W_g = -\frac{V\delta v^2}{2g^2}$

40. $\frac{\partial A}{\partial c} = m$, $\frac{\partial A}{\partial h} = \frac{q}{2}$, $\frac{\partial A}{\partial k} = \frac{m}{q}$, $\frac{\partial A}{\partial m} = \frac{k}{q} + c$, $\frac{\partial A}{\partial q} = -\frac{km}{q^2} + \frac{h}{2}$

41. $\frac{\partial f}{\partial x} = 1 + y$, $\frac{\partial f}{\partial y} = 1 + x$, $\frac{\partial^2 f}{\partial x^2} = 0$, $\frac{\partial^2 f}{\partial y^2} = 0$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 1$

42. $\frac{\partial f}{\partial x} = y \cos xy$, $\frac{\partial f}{\partial y} = x \cos xy$, $\frac{\partial^2 f}{\partial x^2} = -y^2 \sin xy$, $\frac{\partial^2 f}{\partial y^2} = -x^2 \sin xy$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \cos xy - xy \sin xy$

43. $\frac{\partial g}{\partial x} = 2xy + y \cos x$, $\frac{\partial g}{\partial y} = x^2 - \sin y + \sin x$, $\frac{\partial^2 g}{\partial x^2} = 2y - y \sin x$, $\frac{\partial^2 g}{\partial y^2} = -\cos y$, $\frac{\partial^2 g}{\partial y \partial x} = \frac{\partial^2 g}{\partial x \partial y} = 2x + \cos x$

44. $\frac{\partial h}{\partial x} = e^y$, $\frac{\partial h}{\partial y} = xe^y + 1$, $\frac{\partial^2 h}{\partial x^2} = 0$, $\frac{\partial^2 h}{\partial y^2} = xe^y$, $\frac{\partial^2 h}{\partial y \partial x} = \frac{\partial^2 h}{\partial x \partial y} = e^y$

45. $\frac{\partial r}{\partial x} = \frac{1}{x+y}$, $\frac{\partial r}{\partial y} = \frac{1}{x+y}$, $\frac{\partial^2 r}{\partial x^2} = \frac{-1}{(x+y)^2}$, $\frac{\partial^2 r}{\partial y^2} = \frac{-1}{(x+y)^2}$, $\frac{\partial^2 r}{\partial y \partial x} = \frac{\partial^2 r}{\partial x \partial y} = \frac{-1}{(x+y)^2}$

46. $\frac{\partial s}{\partial x} = \left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \left(-\frac{y}{x^2}\right)\left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] = \frac{-y}{x^2+y^2}$, $\frac{\partial s}{\partial y} = \left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \left(\frac{1}{x}\right)\left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] = \frac{x}{x^2+y^2}$,
$\frac{\partial^2 s}{\partial x^2} = \frac{y(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$, $\frac{\partial^2 s}{\partial y^2} = \frac{-x(2y)}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}$,
$\frac{\partial^2 s}{\partial y \partial x} = \frac{\partial^2 s}{\partial x \partial y} = \frac{(x^2+y^2)(-1)+y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

47. $\frac{\partial w}{\partial x} = \frac{2}{2x+3y}$, $\frac{\partial w}{\partial y} = \frac{3}{2x+3y}$, $\frac{\partial^2 w}{\partial y \partial x} = \frac{-6}{(2x+3y)^2}$, and $\frac{\partial^2 w}{\partial x \partial y} = \frac{-6}{(2x+3y)^2}$

48. $\frac{\partial w}{\partial x} = e^x + \ln y + \frac{y}{x}$, $\frac{\partial w}{\partial y} = \frac{x}{y} + \ln x$, $\frac{\partial^2 w}{\partial y \partial x} = \ = \frac{1}{y} + \frac{1}{x}$, and $\frac{\partial^2 w}{\partial x \partial y} = \frac{1}{y} + \frac{1}{x}$

49. $\frac{\partial w}{\partial x} = y^2 + 2xy^3 + 3x^2y^4$, $\frac{\partial w}{\partial y} = 2xy + 3x^2y^2 + 4x^3y^3$, $\frac{\partial^2 w}{\partial y \partial x} = 2y + 6xy^2 + 12x^2y^3$, and
$\frac{\partial^2 w}{\partial x \partial y} = 2y + 6xy^2 + 12x^2y^3$

50. $\frac{\partial w}{\partial x} = \sin y + y \cos x + y$, $\frac{\partial w}{\partial y} = x \cos y + \sin x + x$, $\frac{\partial^2 w}{\partial y \partial x} = \cos y + \cos x + 1$, and
$\frac{\partial^2 w}{\partial x \partial y} = \cos y + \cos x + 1$

51. (a) x first (b) y first (c) x first (d) x first (e) y first (f) y first

52. (a) y first three times (b) y first three times (c) y first twice (d) x first twice

53. $f_x(1,2) = \lim\limits_{h \to 0} \frac{f(1+h,2) - f(1,2)}{h} = \lim\limits_{h \to 0} \frac{[1-(1+h)+2-6(1+h)^2] - (2-6)}{h} = \lim\limits_{h \to 0} \frac{-h-6(1+2h+h^2)+6}{h}$
$= \lim\limits_{h \to 0} \frac{-13h-6h^2}{h} = \lim\limits_{h \to 0} (-13-6h) = -13,$
$f_y(1,2) = \lim\limits_{h \to 0} \frac{f(1,2+h) - f(1,2)}{h} = \lim\limits_{h \to 0} \frac{[1-1+(2+h)-3(2+h)] - (2-6)}{h} = \lim\limits_{h \to 0} \frac{(2-6-2h)-(2-6)}{h}$
$= \lim\limits_{h \to 0} (-2) = -2$

54. $f_x(-2,1) = \lim\limits_{h \to 0} \frac{f(-2+h,1) - f(-2,1)}{h} = \lim\limits_{h \to 0} \frac{[4+2(-2+h)-3-(-2+h)] - (-3+2)}{h}$
$= \lim\limits_{h \to 0} \frac{(2h-1-h)+1}{h} = \lim\limits_{h \to 0} 1 = 1,$
$f_y(-2,1) = \lim\limits_{h \to 0} \frac{f(-2,1+h) - f(-2,1)}{h} = \lim\limits_{h \to 0} \frac{[4-4-3(1+h)+2(1+h^2)] - (-3+2)}{h}$
$= \lim\limits_{h \to 0} \frac{(-3-3h+2+4h+2h^2)+1}{h} = \lim\limits_{h \to 0} \frac{h+2h^2}{h} = \lim\limits_{h \to 0} (1+2h) = 1$

55. $f_z(x_0, y_0, z_0) = \lim\limits_{h \to 0} \frac{f(x_0, y_0, z_0+h) - f(x_0, y_0, z_0)}{h}$;
$f_z(1,2,3) = \lim\limits_{h \to 0} \frac{f(1,2,3+h) - f(1,2,3)}{h} = \lim\limits_{h \to 0} \frac{2(3+h)^2 - 2(9)}{h} = \lim\limits_{h \to 0} \frac{12h+2h^2}{h} = \lim\limits_{h \to 0} (12+2h) = 12$

56. $f_y(x_0, y_0, z_0) = \lim\limits_{h \to 0} \frac{f(x_0, y_0+h, z_0) - f(x_0, y_0, z_0)}{h}$;
$f_y(-1,0,3) = \lim\limits_{h \to 0} \frac{f(-1,h,3) - f(-1,0,3)}{h} = \lim\limits_{h \to 0} \frac{(2h^2+9h)-0}{h} = \lim\limits_{h \to 0} (2h+9) = 9$

57. $y + \left(3z^2 \frac{\partial z}{\partial x}\right) x + z^3 - 2y \frac{\partial z}{\partial x} = 0 \Rightarrow (3xz^2 - 2y) \frac{\partial z}{\partial x} = -y - z^3 \Rightarrow$ at $(1,1,1)$ we have $(3-2) \frac{\partial z}{\partial x} = -1-1$ or $\frac{\partial z}{\partial x} = -2$

58. $\left(\frac{\partial x}{\partial z}\right) z + x + \left(\frac{y}{x}\right) \frac{\partial x}{\partial z} - 2x \frac{\partial x}{\partial z} = 0 \Rightarrow \left(z + \frac{y}{x} - 2x\right) \frac{\partial x}{\partial z} = -x \Rightarrow$ at $(1,-1,-3)$ we have $(-3-1-2) \frac{\partial x}{\partial z} = -1$ or $\frac{\partial x}{\partial z} = \frac{1}{6}$

59. $a^2 = b^2 + c^2 - 2bc \cos A \Rightarrow 2a = (2bc \sin A) \frac{\partial A}{\partial a} \Rightarrow \frac{\partial A}{\partial a} = \frac{a}{bc \sin A}$; also $0 = 2b - 2c \cos A + (2bc \sin A) \frac{\partial A}{\partial b}$
$\Rightarrow 2c \cos A - 2b = (2bc \sin A) \frac{\partial A}{\partial b} \Rightarrow \frac{\partial A}{\partial b} = \frac{c \cos A - b}{bc \sin A}$

60. $\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{(\sin A) \frac{\partial a}{\partial A} - a \cos A}{\sin^2 A} = 0 \Rightarrow (\sin A) \frac{\partial a}{\partial x} - a \cos A = 0 \Rightarrow \frac{\partial a}{\partial A} = \frac{a \cos A}{\sin A}$; also
$\left(\frac{1}{\sin A}\right) \frac{\partial a}{\partial B} = b(-\csc B \cot B) \Rightarrow \frac{\partial a}{\partial B} = -b \csc B \cot B \sin A$

61. Differentiating each equation implicitly gives $1 = v_x \ln u + \left(\frac{v}{u}\right) u_x$ and $0 = u_x \ln v + \left(\frac{u}{v}\right) v_x$ or

$$\left.\begin{aligned} (\ln u)\, v_x \quad + \left(\tfrac{v}{u}\right) u_x &= 1 \\ \left(\tfrac{u}{v}\right) v_x + (\ln v)\, u_x &= 0 \end{aligned}\right\} \Rightarrow v_x = \frac{\begin{vmatrix} 1 & \frac{v}{u} \\ 0 & \ln v \end{vmatrix}}{\begin{vmatrix} \ln u & \frac{v}{u} \\ \frac{u}{v} & \ln v \end{vmatrix}} = \frac{\ln v}{(\ln u)(\ln v) - 1}$$

62. Differentiating each equation implicitly gives $1 = (2x)x_u - (2y)y_u$ and $0 = (2x)x_u - y_u$ or

$$\left.\begin{aligned} (2x)x_u - (2y)y_u &= 1 \\ (2x)x_u - \quad\;\; y_u &= 0 \end{aligned}\right\} \Rightarrow x_u = \frac{\begin{vmatrix} 1 & -2y \\ 0 & -1 \end{vmatrix}}{\begin{vmatrix} 2x & -2y \\ 2x & -1 \end{vmatrix}} = \frac{-1}{-2x+4xy} = \frac{1}{2x-4xy} \text{ and}$$

$y_u = \frac{\begin{vmatrix} 2x & 1 \\ 2x & 0 \end{vmatrix}}{-2x+4xy} = \frac{-2x}{-2x+4xy} = \frac{2x}{2x-4xy} = \frac{1}{1-2y}$; next $s = x^2 + y^2 \Rightarrow \frac{\partial s}{\partial u} = 2x \frac{\partial x}{\partial u} + 2y \frac{\partial y}{\partial u}$
$= 2x \left(\frac{1}{2x-4xy}\right) + 2y \left(\frac{1}{1-2y}\right) = \frac{1}{1-2y} + \frac{2y}{1-2y} = \frac{1+2y}{1-2y}$

63. $\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z} = -4z \Rightarrow \frac{\partial^2 f}{\partial x^2} = 2, \frac{\partial^2 f}{\partial y^2} = 2, \frac{\partial^2 f}{\partial z^2} = -4 \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 2 + 2 + (-4) = 0$

64. $\frac{\partial f}{\partial x} = -6xz$, $\frac{\partial f}{\partial y} = -6yz$, $\frac{\partial f}{\partial z} = 6z^2 - 3\left(x^2 + y^2\right)$, $\frac{\partial^2 f}{\partial x^2} = -6z$, $\frac{\partial^2 f}{\partial y^2} = -6z$, $\frac{\partial^2 f}{\partial z^2} = 12z \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
$= -6z - 6z + 12z = 0$

65. $\frac{\partial f}{\partial x} = -2e^{-2y} \sin 2x$, $\frac{\partial f}{\partial y} = -2e^{-2y} \cos 2x$, $\frac{\partial^2 f}{\partial x^2} = -4e^{-2y} \cos 2x$, $\frac{\partial^2 f}{\partial y^2} = 4e^{-2y} \cos 2x \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$
$= -4e^{-2y} \cos 2x + 4e^{-2y} \cos 2x = 0$

66. $\frac{\partial f}{\partial x} = \frac{x}{x^2 + y^2}$, $\frac{\partial f}{\partial y} = \frac{y}{x^2 + y^2}$, $\frac{\partial^2 f}{\partial x^2} = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}$, $\frac{\partial^2 f}{\partial y^2} = \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2} \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2} + \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2} = 0$

67. $\frac{\partial f}{\partial x} = -\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2}(2x) = -x\left(x^2 + y^2 + z^2\right)^{-3/2}$, $\frac{\partial f}{\partial y} = -\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2}(2y)$
$= -y\left(x^2 + y^2 + z^2\right)^{-3/2}$, $\frac{\partial f}{\partial z} = -\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2}(2z) = -z\left(x^2 + y^2 + z^2\right)^{-3/2}$;
$\frac{\partial^2 f}{\partial x^2} = -\left(x^2 + y^2 + z^2\right)^{-3/2} + 3x^2\left(x^2 + y^2 + z^2\right)^{-5/2}$, $\frac{\partial^2 f}{\partial y^2} = -\left(x^2 + y^2 + z^2\right)^{-3/2} + 3y^2\left(x^2 + y^2 + z^2\right)^{-5/2}$,
$\frac{\partial^2 f}{\partial z^2} = -\left(x^2 + y^2 + z^2\right)^{-3/2} + 3z^2\left(x^2 + y^2 + z^2\right)^{-5/2} \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
$= \left[-\left(x^2 + y^2 + z^2\right)^{-3/2} + 3x^2\left(x^2 + y^2 + z^2\right)^{-5/2}\right] + \left[-\left(x^2 + y^2 + z^2\right)^{-3/2} + 3y^2\left(x^2 + y^2 + z^2\right)^{-5/2}\right]$
$+ \left[-\left(x^2 + y^2 + z^2\right)^{-3/2} + 3z^2\left(x^2 + y^2 + z^2\right)^{-5/2}\right] = -3\left(x^2 + y^2 + z^2\right)^{-3/2} + \left(3x^2 + 3y^2 + 3z^2\right)\left(x^2 + y^2 + z^2\right)^{-5/2}$
$= 0$

68. $\frac{\partial f}{\partial x} = 3e^{3x+4y} \cos 5z$, $\frac{\partial f}{\partial y} = 4e^{3x+4y} \cos 5z$, $\frac{\partial f}{\partial z} = -5e^{3x+4y} \sin 5z$; $\frac{\partial^2 f}{\partial x^2} = 9e^{3x+4y} \cos 5z$, $\frac{\partial^2 f}{\partial y^2} = 16e^{3x+4y} \cos 5z$,
$\frac{\partial^2 f}{\partial z^2} = -25e^{3x+4y} \cos 5z \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 9e^{3x+4y} \cos 5z + 16e^{3x+4y} \cos 5z - 25e^{3x+4y} \cos 5z = 0$

69. $\frac{\partial w}{\partial x} = \cos(x + ct)$, $\frac{\partial w}{\partial t} = c \cos(x + ct)$; $\frac{\partial^2 w}{\partial x^2} = -\sin(x + ct)$, $\frac{\partial^2 w}{\partial t^2} = -c^2 \sin(x + ct) \Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2\left[-\sin(x + ct)\right]$
$= c^2 \frac{\partial^2 w}{\partial x^2}$

70. $\frac{\partial w}{\partial x} = -2 \sin(2x + 2ct)$, $\frac{\partial w}{\partial t} = -2c \sin(2x + 2ct)$; $\frac{\partial^2 w}{\partial x^2} = -4 \cos(2x + 2ct)$, $\frac{\partial^2 w}{\partial t^2} = -4c^2 \cos(2x + 2ct)$
$\Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2[-4 \cos(2x + 2ct)] = c^2 \frac{\partial^2 w}{\partial x^2}$

71. $\frac{\partial w}{\partial x} = \cos(x + ct) - 2 \sin(2x + 2ct)$, $\frac{\partial w}{\partial t} = c \cos(x + ct) - 2c \sin(2x + 2ct)$;
$\frac{\partial^2 w}{\partial x^2} = -\sin(x + ct) - 4 \cos(2x + 2ct)$, $\frac{\partial^2 w}{\partial t^2} = -c^2 \sin(x + ct) - 4c^2 \cos(2x + 2ct)$
$\Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2[-\sin(x + ct) - 4 \cos(2x + 2ct)] = c^2 \frac{\partial^2 w}{\partial x^2}$

72. $\frac{\partial w}{\partial x} = \frac{1}{x + ct}$, $\frac{\partial w}{\partial t} = \frac{c}{x + ct}$; $\frac{\partial^2 w}{\partial x^2} = \frac{-1}{(x + ct)^2}$, $\frac{\partial^2 w}{\partial t^2} = \frac{-c^2}{(x + ct)^2} \Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2\left[\frac{-1}{(x + ct)^2}\right] = c^2 \frac{\partial^2 w}{\partial x^2}$

73. $\frac{\partial w}{\partial x} = 2 \sec^2(2x - 2ct)$, $\frac{\partial w}{\partial t} = -2c \sec^2(2x - 2ct)$; $\frac{\partial^2 w}{\partial x^2} = 8 \sec^2(2x - 2ct) \tan(2x - 2ct)$,
$\frac{\partial^2 w}{\partial t^2} = 8c^2 \sec^2(2x - 2ct) \tan(2x - 2ct) \Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2[8 \sec^2(2x - 2ct) \tan(2x - 2ct)] = c^2 \frac{\partial^2 w}{\partial x^2}$

74. $\frac{\partial w}{\partial x} = -15 \sin(3x + 3ct) + e^{x+ct}$, $\frac{\partial w}{\partial t} = -15c \sin(3x + 3ct) + ce^{x+ct}$; $\frac{\partial^2 w}{\partial x^2} = -45 \cos(3x + 3ct) + e^{x+ct}$,
$\frac{\partial^2 w}{\partial t^2} = -45c^2 \cos(3x + 3ct) + c^2 e^{x+ct} \Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2\left[-45 \cos(3x + 3ct) + e^{x+ct}\right] = c^2 \frac{\partial^2 w}{\partial x^2}$

75. $\frac{\partial w}{\partial t} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial t} = \frac{\partial f}{\partial u}(ac) \Rightarrow \frac{\partial^2 w}{\partial t^2} = (ac)\left(\frac{\partial^2 f}{\partial u^2}\right)(ac) = a^2c^2 \frac{\partial^2 f}{\partial u^2}$; $\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} = \frac{\partial f}{\partial u} \cdot a \Rightarrow \frac{\partial^2 w}{\partial x^2} = \left(a \frac{\partial^2 f}{\partial u^2}\right) \cdot a$
$= a^2 \frac{\partial^2 f}{\partial u^2} \Rightarrow \frac{\partial^2 w}{\partial t^2} = a^2c^2 \frac{\partial^2 f}{\partial u^2} = c^2\left(a^2 \frac{\partial^2 f}{\partial u^2}\right) = c^2 \frac{\partial^2 w}{\partial x^2}$

76. If the first partial derivatives are continuous throughout an open region R, then by Theorem 3 in this section of the text, $f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$, where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$. Then as $(x, y) \to (x_0, y_0)$, $\Delta x \to 0$ and $\Delta y \to 0 \Rightarrow \lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0) \Rightarrow$ f is continuous at every point (x_0, y_0) in R.

77. Yes, since f_{xx}, f_{yy}, f_{xy}, and f_{yx} are all continuous on R, use the same reasoning as in Exercise 76 with $f_x(x, y) = f_x(x_0, y_0) + f_{xx}(x_0, y_0)\,\Delta x + f_{xy}(x_0, y_0)\,\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$ and $f_y(x, y) = f_y(x_0, y_0) + f_{yx}(x_0, y_0)\,\Delta x + f_{yy}(x_0, y_0)\,\Delta y + \widehat{\epsilon}_1\Delta x + \widehat{\epsilon}_2\Delta y$. Then $\lim\limits_{(x,y)\to(x_0,y_0)} f_x(x, y) = f_x(x_0, y_0)$ and $\lim\limits_{(x,y)\to(x_0,y_0)} f_y(x, y) = f_y(x_0, y_0)$.

14.4 THE CHAIN RULE

1. (a) $\frac{\partial w}{\partial x} = 2x,\ \frac{\partial w}{\partial y} = 2y,\ \frac{dx}{dt} = -\sin t,\ \frac{dy}{dt} = \cos t \Rightarrow \frac{dw}{dt} = -2x \sin t + 2y \cos t = -2 \cos t \sin t + 2 \sin t \cos t$ $= 0$; $w = x^2 + y^2 = \cos^2 t + \sin^2 t = 1 \Rightarrow \frac{dw}{dt} = 0$
   (b) $\frac{dw}{dt}(\pi) = 0$

2. (a) $\frac{\partial w}{\partial x} = 2x,\ \frac{\partial w}{\partial y} = 2y,\ \frac{dx}{dt} = -\sin t + \cos t,\ \frac{dy}{dt} = -\sin t - \cos t \Rightarrow \frac{dw}{dt}$
   $= (2x)(-\sin t + \cos t) + (2y)(-\sin t - \cos t)$
   $= 2(\cos t + \sin t)(\cos t - \sin t) - 2(\cos t - \sin t)(\sin t + \cos t) = (2 \cos^2 t - 2 \sin^2 t) - (2 \cos^2 t - 2 \sin^2 t)$
   $= 0$; $w = x^2 + y^2 = (\cos t + \sin t)^2 + (\cos t - \sin t)^2 = 2 \cos^2 t + 2 \sin^2 t = 2 \Rightarrow \frac{dw}{dt} = 0$
   (b) $\frac{dw}{dt}(0) = 0$

3. (a) $\frac{\partial w}{\partial x} = \frac{1}{z},\ \frac{\partial w}{\partial y} = \frac{1}{z},\ \frac{\partial w}{\partial z} = \frac{-(x+y)}{z^2},\ \frac{dx}{dt} = -2 \cos t \sin t,\ \frac{dy}{dt} = 2 \sin t \cos t,\ \frac{dz}{dt} = -\frac{1}{t^2}$
   $\Rightarrow \frac{dw}{dt} = -\frac{2}{z} \cos t \sin t + \frac{2}{z} \sin t \cos t + \frac{x+y}{z^2t^2} = \frac{\cos^2 t + \sin^2 t}{\left(\frac{1}{t^2}\right)(t^2)} = 1$; $w = \frac{x}{z} + \frac{y}{z} = \frac{\cos^2 t}{\left(\frac{1}{t}\right)} + \frac{\sin^2 t}{\left(\frac{1}{t}\right)} = t \Rightarrow \frac{dw}{dt} = 1$
   (b) $\frac{dw}{dt}(3) = 1$

4. (a) $\frac{\partial w}{\partial x} = \frac{2x}{x^2+y^2+z^2},\ \frac{\partial w}{\partial y} = \frac{2y}{x^2+y^2+z^2},\ \frac{\partial w}{\partial z} = \frac{2z}{x^2+y^2+z^2},\ \frac{dx}{dt} = -\sin t,\ \frac{dy}{dt} = \cos t,\ \frac{dz}{dt} = 2t^{-1/2}$
   $\Rightarrow \frac{dw}{dt} = \frac{-2x \sin t}{x^2+y^2+z^2} + \frac{2y \cos t}{x^2+y^2+z^2} + \frac{4zt^{-1/2}}{x^2+y^2+z^2} = \frac{-2 \cos t \sin t + 2 \sin t \cos t + 4\left(4t^{1/2}\right)t^{-1/2}}{\cos^2 t + \sin^2 t + 16t}$
   $= \frac{16}{1+16t}$; $w = \ln(x^2 + y^2 + z^2) = \ln(\cos^2 t + \sin^2 t + 16t) = \ln(1 + 16t) \Rightarrow \frac{dw}{dt} = \frac{16}{1+16t}$
   (b) $\frac{dw}{dt}(3) = \frac{16}{49}$

5. (a) $\frac{\partial w}{\partial x} = 2ye^x,\ \frac{\partial w}{\partial y} = 2e^x,\ \frac{\partial w}{\partial z} = -\frac{1}{z},\ \frac{dx}{dt} = \frac{2t}{t^2+1},\ \frac{dy}{dt} = \frac{1}{t^2+1},\ \frac{dz}{dt} = e^t \Rightarrow \frac{dw}{dt} = \frac{4yte^x}{t^2+1} + \frac{2e^x}{t^2+1} - \frac{e^t}{z}$
   $= \frac{(4t)(\tan^{-1} t)(t^2+1)}{t^2+1} + \frac{2(t^2+1)}{t^2+1} - \frac{e^t}{e^t} = 4t \tan^{-1} t + 1$; $w = 2ye^x - \ln z = (2 \tan^{-1} t)(t^2 + 1) - t$
   $\Rightarrow \frac{dw}{dt} = \left(\frac{2}{t^2+1}\right)(t^2 + 1) + (2 \tan^{-1} t)(2t) - 1 = 4t \tan^{-1} t + 1$
   (b) $\frac{dw}{dt}(1) = (4)(1)\left(\frac{\pi}{4}\right) + 1 = \pi + 1$

6. (a) $\frac{\partial w}{\partial x} = -y \cos xy,\ \frac{\partial w}{\partial y} = -x \cos xy,\ \frac{\partial w}{\partial z} = 1,\ \frac{dx}{dt} = 1,\ \frac{dy}{dt} = \frac{1}{t},\ \frac{dz}{dt} = e^{t-1} \Rightarrow \frac{dw}{dt} = -y \cos xy - \frac{x \cos xy}{t} + e^{t-1}$
   $= -(\ln t)[\cos(t \ln t)] - \frac{t \cos(t \ln t)}{t} + e^{t-1} = -(\ln t)[\cos(t \ln t)] - \cos(t \ln t) + e^{t-1}$; $w = z - \sin xy$
   $= e^{t-1} - \sin(t \ln t) \Rightarrow \frac{dw}{dt} = e^{t-1} - [\cos(t \ln t)]\left[\ln t + t\left(\frac{1}{t}\right)\right] = e^{t-1} - (1 + \ln t)\cos(t \ln t)$
   (b) $\frac{dw}{dt}(1) = 1 - (1 + 0)(1) = 0$

7. (a) $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (4e^x \ln y)\left(\frac{\cos v}{u \cos v}\right) + \left(\frac{4e^x}{y}\right)(\sin v) = \frac{4e^x \ln y}{u} + \frac{4e^x \sin v}{y}$
$= \frac{4(u \cos v) \ln(u \sin v)}{u} + \frac{4(u \cos v)(\sin v)}{u \sin v} = (4 \cos v) \ln(u \sin v) + 4 \cos v;$
$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (4e^x \ln y)\left(\frac{-u \sin v}{u \cos v}\right) + \left(\frac{4e^x}{y}\right)(u \cos v) = -(4e^x \ln y)(\tan v) + \frac{4e^x u \cos v}{y}$
$= [-4(u \cos v) \ln(u \sin v)](\tan v) + \frac{4(u \cos v)(u \cos v)}{u \sin v} = (-4u \sin v) \ln(u \sin v) + \frac{4u \cos^2 v}{\sin v};$
$z = 4e^x \ln y = 4(u \cos v) \ln(u \sin v) \Rightarrow \frac{\partial z}{\partial u} = (4 \cos v) \ln(u \sin v) + 4(u \cos v)\left(\frac{\sin v}{u \sin v}\right)$
$= (4 \cos v) \ln(u \sin v) + 4 \cos v$; also $\frac{\partial z}{\partial v} = (-4u \sin v) \ln(u \sin v) + 4(u \cos v)\left(\frac{u \cos v}{u \sin v}\right)$
$= (-4u \sin v) \ln(u \sin v) + \frac{4u \cos^2 v}{\sin v}$

(b) At $\left(2, \frac{\pi}{4}\right)$: $\frac{\partial z}{\partial u} = 4 \cos \frac{\pi}{4} \ln\left(2 \sin \frac{\pi}{4}\right) + 4 \cos \frac{\pi}{4} = 2\sqrt{2} \ln \sqrt{2} + 2\sqrt{2} = \sqrt{2}(\ln 2 + 2);$
$\frac{\partial z}{\partial v} = (-4)(2) \sin \frac{\pi}{4} \ln\left(2 \sin \frac{\pi}{4}\right) + \frac{(4)(2)\left(\cos^2 \frac{\pi}{4}\right)}{\left(\sin \frac{\pi}{4}\right)} = -4\sqrt{2} \ln \sqrt{2} + 4\sqrt{2} = -2\sqrt{2} \ln 2 + 4\sqrt{2}$

8. (a) $\frac{\partial z}{\partial u} = \left[\frac{\left(\frac{1}{y}\right)}{\left(\frac{x}{y}\right)^2 + 1}\right] \cos v + \left[\frac{\left(\frac{-x}{y^2}\right)}{\left(\frac{x}{y}\right)^2 + 1}\right] \sin v = \frac{y \cos v}{x^2 + y^2} - \frac{x \sin v}{x^2 + y^2} = \frac{(u \sin v)(\cos v) - (u \cos v)(\sin v)}{u^2} = 0;$
$\frac{\partial z}{\partial v} = \left[\frac{\left(\frac{1}{y}\right)}{\left(\frac{x}{y}\right)^2 + 1}\right](-u \sin v) + \left[\frac{\left(\frac{-x}{y^2}\right)}{\left(\frac{x}{y}\right)^2 + 1}\right] u \cos v = -\frac{yu \sin v}{x^2 + y^2} - \frac{xu \cos v}{x^2 + y^2} = \frac{-(u \sin v)(u \sin v) - (u \cos v)(u \cos v)}{u^2}$
$= -\sin^2 v - \cos^2 v = -1; z = \tan^{-1}\left(\frac{x}{y}\right) = \tan^{-1}(\cot v) \Rightarrow \frac{\partial z}{\partial u} = 0$ and $\frac{\partial z}{\partial v} = \left(\frac{1}{1 + \cot^2 v}\right)(-\csc^2 v)$
$= \frac{-1}{\sin^2 v + \cos^2 v} = -1$

(b) At $\left(1.3, \frac{\pi}{6}\right)$: $\frac{\partial z}{\partial u} = 0$ and $\frac{\partial z}{\partial v} = -1$

9. (a) $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} = (y + z)(1) + (x + z)(1) + (y + x)(v) = x + y + 2z + v(y + x)$
$= (u + v) + (u - v) + 2uv + v(2u) = 2u + 4uv; \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$
$= (y + z)(1) + (x + z)(-1) + (y + x)(u) = y - x + (y + x)u = -2v + (2u)u = -2v + 2u^2;$
$w = xy + yz + xz = (u^2 - v^2) + (u^2 v - uv^2) + (u^2 v + uv^2) = u^2 - v^2 + 2u^2 v \Rightarrow \frac{\partial w}{\partial u} = 2u + 4uv$ and
$\frac{\partial w}{\partial v} = -2v + 2u^2$

(b) At $\left(\frac{1}{2}, 1\right)$: $\frac{\partial w}{\partial u} = 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right)(1) = 3$ and $\frac{\partial w}{\partial v} = -2(1) + 2\left(\frac{1}{2}\right)^2 = -\frac{3}{2}$

10. (a) $\frac{\partial w}{\partial u} = \left(\frac{2x}{x^2 + y^2 + z^2}\right)(e^v \sin u + ue^v \cos u) + \left(\frac{2y}{x^2 + y^2 + z^2}\right)(e^v \cos u - ue^v \sin u) + \left(\frac{2z}{x^2 + y^2 + z^2}\right)(e^v)$
$= \left(\frac{2ue^v \sin u}{u^2 e^{2v} \sin^2 u + u^2 e^{2v} \cos^2 u + u^2 e^{2v}}\right)(e^v \sin u + ue^v \cos u)$
$+ \left(\frac{2ue^v \cos u}{u^2 e^{2v} \sin^2 u + u^2 e^{2v} \cos^2 u + u^2 e^{2v}}\right)(e^v \cos u - ue^v \sin u)$
$+ \left(\frac{2ue^v}{u^2 e^{2v} \sin^2 u + u^2 e^{2v} \cos^2 u + u^2 e^{2v}}\right)(e^v) = \frac{2}{u};$
$\frac{\partial w}{\partial v} = \left(\frac{2x}{x^2 + y^2 + z^2}\right)(ue^v \sin u) + \left(\frac{2y}{x^2 + y^2 + z^2}\right)(ue^v \cos u) + \left(\frac{2z}{x^2 + y^2 + z^2}\right)(ue^v)$
$= \left(\frac{2ue^v \sin u}{u^2 e^{2v} \sin^2 u + u^2 e^{2v} \cos^2 u + u^2 e^{2v}}\right)(ue^v \sin u)$
$+ \left(\frac{2ue^v \cos u}{u^2 e^{2v} \sin^2 u + u^2 e^{2v} \cos^2 u + u^2 e^{2v}}\right)(ue^v \cos u)$
$+ \left(\frac{2ue^v}{u^2 e^{2v} \sin^2 u + u^2 e^{2v} \cos^2 u + u^2 e^{2v}}\right)(ue^v) = 2; w = \ln(u^2 e^{2v} \sin^2 u + u^2 e^{2v} \cos^2 u + u^2 e^{2v}) = \ln(2u^2 e^{2v})$
$= \ln 2 + 2 \ln u + 2v \Rightarrow \frac{\partial w}{\partial u} = \frac{2}{u}$ and $\frac{\partial w}{\partial v} = 2$

(b) At $(-2, 0)$: $\frac{\partial w}{\partial u} = \frac{2}{-2} = -1$ and $\frac{\partial w}{\partial v} = 2$

11. (a) $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = \frac{1}{q - r} + \frac{r - p}{(q - r)^2} + \frac{p - q}{(q - r)^2} = \frac{q - r + r - p + p - q}{(q - r)^2} = 0;$
$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} = \frac{1}{q - r} - \frac{r - p}{(q - r)^2} + \frac{p - q}{(q - r)^2} = \frac{q - r - r + p + p - q}{(q - r)^2} = \frac{2p - 2r}{(q - r)^2}$
$= \frac{(2x + 2y + 2z) - (2x + 2y - 2z)}{(2z - 2y)^2} = \frac{z}{(z - y)^2}; \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$
$= \frac{1}{q - r} + \frac{r - p}{(q - r)^2} - \frac{p - q}{(q - r)^2} = \frac{q - r + r - p - p + q}{(q - r)^2} = \frac{2q - 2p}{(q - r)^2} = \frac{-4y}{(2z - 2y)^2} = -\frac{y}{(z - y)^2};$

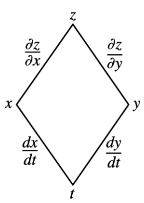
$u = \frac{p-q}{q-r} = \frac{2y}{2z-2y} = \frac{y}{z-y} \Rightarrow \frac{\partial u}{\partial x} = 0,\ \frac{\partial u}{\partial y} = \frac{(z-y)-y(-1)}{(z-y)^2} = \frac{z}{(z-y)^2}$, and $\frac{\partial u}{\partial z} = \frac{(z-y)(0)-y(1)}{(z-y)^2}$
$= -\frac{y}{(z-y)^2}$

(b) At $\left(\sqrt{3}, 2, 1\right)$: $\frac{\partial u}{\partial x} = 0,\ \frac{\partial u}{\partial y} = \frac{1}{(1-2)^2} = 1$, and $\frac{\partial u}{\partial z} = \frac{-2}{(1-2)^2} = -2$

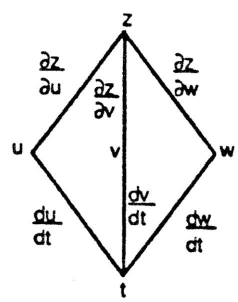
12. (a) $\frac{\partial u}{\partial x} = \frac{e^{qr}}{\sqrt{1-p^2}}(\cos x) + (re^{qr}\sin^{-1}p)(0) + (qe^{qr}\sin^{-1}p)(0) = \frac{e^{qr}\cos x}{\sqrt{1-p^2}} = \frac{e^{z\ln y}\cos x}{\sqrt{1-\sin^2 x}} = y^z$ if $-\frac{\pi}{2} < x < \frac{\pi}{2}$;
$\frac{\partial u}{\partial y} = \frac{e^{qr}}{\sqrt{1-p^2}}(0) + (re^{qr}\sin^{-1}p)\left(\frac{z^2}{y}\right) + (qe^{qr}\sin^{-1}p)(0) = \frac{z^2\, re^{qr}\sin^{-1}p}{y} = \frac{z^2\left(\frac{1}{z}\right)y^z x}{y} = xzy^{z-1}$;
$\frac{\partial u}{\partial z} = \frac{e^{qr}}{\sqrt{1-p^2}}(0) + (re^{qr}\sin^{-1}p)(2z\ln y) + (qe^{qr}\sin^{-1}p)\left(-\frac{1}{z^2}\right) = (2zre^{qr}\sin^{-1}p)(\ln y) - \frac{qe^{qr}\sin^{-1}p}{z^2}$
$= (2z)\left(\frac{1}{z}\right)(y^z x\ln y) - \frac{(z^2\ln y)(y^z)x}{z^2} = xy^z\ln y$; $u = e^{z\ln y}\sin^{-1}(\sin x) = xy^z$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2} \Rightarrow \frac{\partial u}{\partial x} = y^z$,
$\frac{\partial u}{\partial y} = xzy^{z-1}$, and $\frac{\partial u}{\partial z} = \ = xy^z\ln y$ from direct calculations

(b) At $\left(\frac{\pi}{4}, \frac{1}{2}, -\frac{1}{2}\right)$: $\frac{\partial u}{\partial x} = \left(\frac{1}{2}\right)^{-1/2} = \sqrt{2},\ \frac{\partial u}{\partial y} = \left(\frac{\pi}{4}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)^{(-1/2)-1} = -\frac{\pi\sqrt{2}}{4},\ \frac{\partial u}{\partial z} = \left(\frac{\pi}{4}\right)\left(\frac{1}{2}\right)^{-1/2}\ln\left(\frac{1}{2}\right)$
$= -\frac{\pi\sqrt{2}\ln 2}{4}$

13. $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

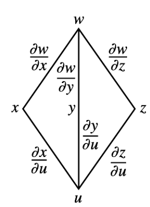


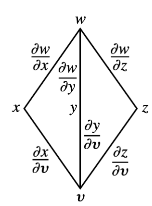
14. $\frac{dz}{dt} = \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt} + \frac{\partial x}{\partial w}\frac{dw}{dt}$



15. $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$

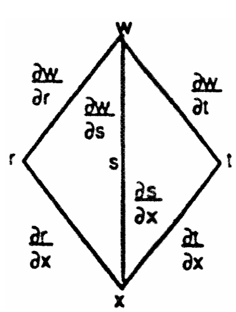
$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$

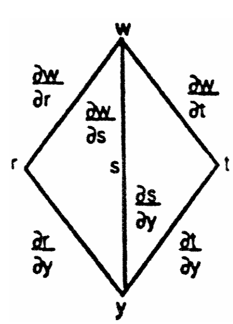




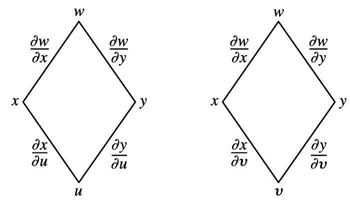
16. $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x}$

$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y}$

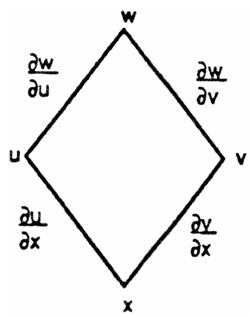


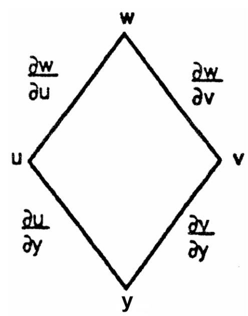


17. $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial u}$ $\qquad$ $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial v}$

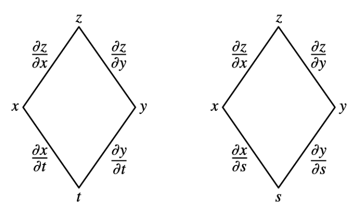


18. $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\,\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\,\frac{\partial v}{\partial x}$ $\qquad$ $\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\,\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\,\frac{\partial v}{\partial y}$

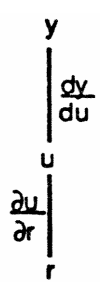




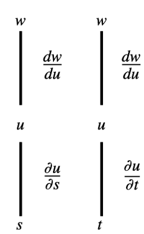
19. $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\,\frac{\partial y}{\partial t}$ $\qquad$ $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\,\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\,\frac{\partial y}{\partial s}$



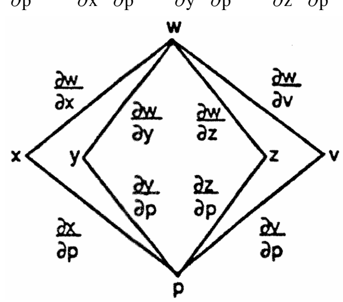
20. $\frac{\partial y}{\partial r} = \frac{dy}{du}\,\frac{\partial u}{\partial r}$



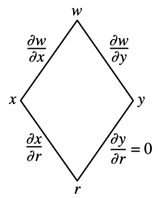
21. $\frac{\partial w}{\partial s} = \frac{dw}{du}\,\frac{\partial u}{\partial s}$ $\qquad$ $\frac{\partial w}{\partial t} = \frac{dw}{du}\,\frac{\partial u}{\partial t}$

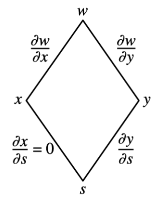


22. $\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial p} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial p} + \frac{\partial w}{\partial z}\,\frac{\partial z}{\partial p} + \frac{\partial w}{\partial v}\,\frac{\partial v}{\partial p}$

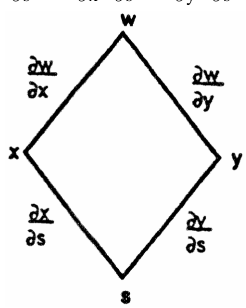


23. $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{dx}{dr} + \frac{\partial w}{\partial y}\frac{dy}{dr} = \frac{\partial w}{\partial x}\frac{dx}{dr}$ since $\frac{dy}{dr} = 0$ $\qquad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{dx}{ds} + \frac{\partial w}{\partial y}\frac{dy}{ds} = \frac{\partial w}{\partial y}\frac{dy}{ds}$ since $\frac{dx}{ds} = 0$





24. $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$



25. Let $F(x, y) = x^3 - 2y^2 + xy = 0 \Rightarrow F_x(x, y) = 3x^2 + y$ and $F_y(x, y) = -4y + x \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2+y}{(-4y+x)}$ $\Rightarrow \frac{dy}{dx}(1, 1) = \frac{4}{3}$

26. Let $F(x, y) = xy + y^2 - 3x - 3 = 0 \Rightarrow F_x(x, y) = y - 3$ and $F_y(x, y) = x + 2y \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{y-3}{x+2y}$ $\Rightarrow \frac{dy}{dx}(-1, 1) = 2$

27. Let $F(x, y) = x^2 + xy + y^2 - 7 = 0 \Rightarrow F_x(x, y) = 2x + y$ and $F_y(x, y) = x + 2y \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x+y}{x+2y}$ $\Rightarrow \frac{dy}{dx}(1, 2) = -\frac{4}{5}$

28. Let $F(x, y) = xe^y + \sin xy + y - \ln 2 = 0 \Rightarrow F_x(x, y) = e^y + y\cos xy$ and $F_y(x, y) = xe^y + x\sin xy + 1$ $\Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{e^y + y\cos xy}{xe^y + x\sin xy + 1} \Rightarrow \frac{dy}{dx}(0, \ln 2) = -(2 + \ln 2)$

29. Let $F(x, y, z) = z^3 - xy + yz + y^3 - 2 = 0 \Rightarrow F_x(x, y, z) = -y, F_y(x, y, z) = -x + z + 3y^2, F_z(x, y, z) = 3z^2 + y$ $\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-y}{3z^2+y} = \frac{y}{3z^2+y} \Rightarrow \frac{\partial z}{\partial x}(1, 1, 1) = \frac{1}{4}$; $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-x+z+3y^2}{3z^2+y} = \frac{x-z-3y^2}{3z^2+y}$ $\Rightarrow \frac{\partial z}{\partial y}(1, 1, 1) = -\frac{3}{4}$

30. Let $F(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0 \Rightarrow F_x(x, y, z) = -\frac{1}{x^2}, F_y(x, y, z) = -\frac{1}{y^2}, F_z(x, y, z) = -\frac{1}{z^2}$ $\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{z^2}\right)} = -\frac{z^2}{x^2} \Rightarrow \frac{\partial z}{\partial x}(2, 3, 6) = -9$; $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\left(-\frac{1}{y^2}\right)}{\left(-\frac{1}{z^2}\right)} = -\frac{z^2}{y^2} \Rightarrow \frac{\partial z}{\partial y}(2, 3, 6) = -4$

31. Let $F(x, y, z) = \sin(x + y) + \sin(y + z) + \sin(x + z) = 0 \Rightarrow F_x(x, y, z) = \cos(x + y) + \cos(x + z)$, $F_y(x, y, z) = \cos(x + y) + \cos(y + z), F_z(x, y, z) = \cos(y + z) + \cos(x + z) \Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ $= -\frac{\cos(x+y) + \cos(x+z)}{\cos(y+z) + \cos(x+z)} \Rightarrow \frac{\partial z}{\partial x}(\pi, \pi, \pi) = -1$; $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\cos(x+y) + \cos(y+z)}{\cos(y+z) + \cos(x+z)} \Rightarrow \frac{\partial z}{\partial y}(\pi, \pi, \pi) = -1$

32. Let $F(x, y, z) = xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0 \Rightarrow F_x(x, y, z) = e^y + \frac{2}{x}, F_y(x, y, z) = xe^y + e^z, F_z(x, y, z) = ye^z$ $\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\left(e^y + \frac{2}{x}\right)}{ye^z} \Rightarrow \frac{\partial z}{\partial x}(1, \ln 2, \ln 3) = -\frac{4}{3\ln 2}$; $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xe^y + e^z}{ye^z} \Rightarrow \frac{\partial z}{\partial y}(1, \ln 2, \ln 3) = -\frac{5}{3\ln 2}$

33. $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = 2(x + y + z)(1) + 2(x + y + z)[-\sin(r + s)] + 2(x + y + z)[\cos(r + s)]$ $= 2(x + y + z)[1 - \sin(r + s) + \cos(r + s)] = 2[r - s + \cos(r + s) + \sin(r + s)][1 - \sin(r + s) + \cos(r + s)]$

$\Rightarrow \left.\frac{\partial w}{\partial r}\right|_{r=1,s=-1} = 2(3)(2) = 12$

34. $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} = y\left(\frac{2v}{u}\right) + x(1) + \left(\frac{1}{z}\right)(0) = (u+v)\left(\frac{2v}{u}\right) + \frac{v^2}{u} \Rightarrow \left.\frac{\partial w}{\partial v}\right|_{u=-1,v=2} = (1)\left(\frac{4}{-1}\right) + \left(\frac{4}{-1}\right)$
$= -8$

35. $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} = \left(2x - \frac{y}{x^2}\right)(-2) + \left(\frac{1}{x}\right)(1) = \left[2(u-2v+1) - \frac{2u+v-2}{(u-2v+1)^2}\right](-2) + \frac{1}{u-2v+1}$
$\Rightarrow \left.\frac{\partial w}{\partial v}\right|_{u=0,v=0} = -7$

36. $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (y\cos xy + \sin y)(2u) + (x\cos xy + x\cos y)(v)$
$= [uv\cos(u^3v + uv^3) + \sin uv](2u) + [(u^2+v^2)\cos(u^3v+uv^3) + (u^2+v^2)\cos uv](v)$
$\Rightarrow \left.\frac{\partial z}{\partial u}\right|_{u=0,v=1} = 0 + (\cos 0 + \cos 0)(1) = 2$

37. $\frac{\partial z}{\partial u} = \frac{dz}{dx}\frac{\partial x}{\partial u} = \left(\frac{5}{1+x^2}\right)e^u = \left[\frac{5}{1+(e^u+\ln v)^2}\right]e^u \Rightarrow \left.\frac{\partial z}{\partial u}\right|_{u=\ln 2,v=1} = \left[\frac{5}{1+(2)^2}\right](2) = 2;$
$\frac{\partial z}{\partial v} = \frac{dz}{dx}\frac{\partial x}{\partial v} = \left(\frac{5}{1+x^2}\right)\left(\frac{1}{v}\right) = \left[\frac{5}{1+(e^u+\ln v)^2}\right]\left(\frac{1}{v}\right) \Rightarrow \left.\frac{\partial z}{\partial v}\right|_{u=\ln 2,v=1} = \left[\frac{5}{1+(2)^2}\right](1) = 1$

38. $\frac{\partial z}{\partial u} = \frac{dz}{dq}\frac{\partial q}{\partial u} = \left(\frac{1}{q}\right)\left(\frac{\sqrt{v+3}}{1+u^2}\right) = \left(\frac{1}{\sqrt{v+3}\tan^{-1}u}\right)\left(\frac{\sqrt{v+3}}{1+u^2}\right) = \frac{1}{(\tan^{-1}u)(1+u^2)}$
$\Rightarrow \left.\frac{\partial z}{\partial u}\right|_{u=1,v=-2} = \frac{1}{(\tan^{-1}1)(1+1^2)} = \frac{2}{\pi}$; $\frac{\partial z}{\partial v} = \frac{dz}{dq}\frac{\partial q}{\partial v} = \left(\frac{1}{q}\right)\left(\frac{\tan^{-1}u}{2\sqrt{v+3}}\right)$
$= \left(\frac{1}{\sqrt{v+3}\tan^{-1}u}\right)\left(\frac{\tan^{-1}u}{2\sqrt{v+3}}\right) = \frac{1}{2(v+3)} \Rightarrow \left.\frac{\partial z}{\partial v}\right|_{u=1,v=-2} = \frac{1}{2}$

39. $V = IR \Rightarrow \frac{\partial V}{\partial I} = R$ and $\frac{\partial V}{\partial R} = I$; $\frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt} = R\frac{dI}{dt} + I\frac{dR}{dt} \Rightarrow -0.01$ volts/sec
$= (600 \text{ ohms})\frac{dI}{dt} + (0.04 \text{ amps})(0.5 \text{ ohms/sec}) \Rightarrow \frac{dI}{dt} = -0.00005$ amps/sec

40. $V = abc \Rightarrow \frac{dV}{dt} = \frac{\partial V}{\partial a}\frac{da}{dt} + \frac{\partial V}{\partial b}\frac{db}{dt} + \frac{\partial V}{\partial c}\frac{dc}{dt} = (bc)\frac{da}{dt} + (ac)\frac{db}{dt} + (ab)\frac{dc}{dt}$
$\Rightarrow \left.\frac{dV}{dt}\right|_{a=1,b=2,c=3} = (2\text{ m})(3\text{ m})(1\text{ m/sec}) + (1\text{ m})(3\text{ m})(1\text{ m/sec}) + (1\text{ m})(2\text{ m})(-3\text{ m/sec}) = 3\text{ m}^3\text{/sec}$
and the volume is increasing; $S = 2ab + 2ac + 2bc \Rightarrow \frac{dS}{dt} = \frac{\partial S}{\partial a}\frac{da}{dt} + \frac{\partial S}{\partial b}\frac{db}{dt} + \frac{\partial S}{\partial c}\frac{dc}{dt}$
$= 2(b+c)\frac{da}{dt} + 2(a+c)\frac{db}{dt} + 2(a+b)\frac{dc}{dt} \Rightarrow \left.\frac{dS}{dt}\right|_{a=1,b=2,c=3}$
$= 2(5\text{ m})(1\text{ m/sec}) + 2(4\text{ m})(1\text{ m/sec}) + 2(3\text{ m})(-3\text{ m/sec}) = 0\text{ m}^2\text{/sec}$ and the surface area is not changing;
$D = \sqrt{a^2+b^2+c^2} \Rightarrow \frac{dD}{dt} = \frac{\partial D}{\partial a}\frac{da}{dt} + \frac{\partial D}{\partial b}\frac{db}{dt} + \frac{\partial D}{\partial c}\frac{dc}{dt} = \frac{1}{\sqrt{a^2+b^2+c^2}}\left(a\frac{da}{dt} + b\frac{db}{dt} + c\frac{dc}{dt}\right) \Rightarrow \left.\frac{dD}{dt}\right|_{a=1,b=2,c=3}$
$= \left(\frac{1}{\sqrt{14}\text{ m}}\right)[(1\text{ m})(1\text{ m/sec}) + (2\text{ m})(1\text{ m/sec}) + (3\text{ m})(-3\text{ m/sec})] = -\frac{6}{\sqrt{14}}\text{ m/sec} < 0 \Rightarrow$ the diagonals are decreasing in length

41. $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u}(1) + \frac{\partial f}{\partial v}(0) + \frac{\partial f}{\partial w}(-1) = \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}$,
$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} = \frac{\partial f}{\partial u}(-1) + \frac{\partial f}{\partial v}(1) + \frac{\partial f}{\partial w}(0) = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$, and
$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z} = \frac{\partial f}{\partial u}(0) + \frac{\partial f}{\partial v}(-1) + \frac{\partial f}{\partial w}(1) = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w} \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$

42. (a) $\frac{\partial w}{\partial r} = f_x\frac{\partial x}{\partial r} + f_y\frac{\partial y}{\partial r} = f_x\cos\theta + f_y\sin\theta$ and $\frac{\partial w}{\partial \theta} = f_x(-r\sin\theta) + f_y(r\cos\theta) \Rightarrow \frac{1}{r}\frac{\partial w}{\partial \theta} = -f_x\sin\theta + f_y\cos\theta$

(b) $\frac{\partial w}{\partial r}\sin\theta = f_x\sin\theta\cos\theta + f_y\sin^2\theta$ and $\left(\frac{\cos\theta}{r}\right)\frac{\partial w}{\partial \theta} = -f_x\sin\theta\cos\theta + f_y\cos^2\theta$
$\Rightarrow f_y = (\sin\theta)\frac{\partial w}{\partial r} + \left(\frac{\cos\theta}{r}\right)\frac{\partial w}{\partial \theta}$; then $\frac{\partial w}{\partial r} = f_x\cos\theta + \left[(\sin\theta)\frac{\partial w}{\partial r} + \left(\frac{\cos\theta}{r}\right)\frac{\partial w}{\partial \theta}\right](\sin\theta) \Rightarrow f_x\cos\theta$
$= \frac{\partial w}{\partial r} - (\sin^2\theta)\frac{\partial w}{\partial r} - \left(\frac{\sin\theta\cos\theta}{r}\right)\frac{\partial w}{\partial \theta} = (1-\sin^2\theta)\frac{\partial w}{\partial r} - \left(\frac{\sin\theta\cos\theta}{r}\right)\frac{\partial w}{\partial \theta} \Rightarrow f_x = (\cos\theta)\frac{\partial w}{\partial r} - \left(\frac{\sin\theta}{r}\right)\frac{\partial w}{\partial \theta}$

(c) $(f_x)^2 = (\cos^2\theta)\left(\frac{\partial w}{\partial r}\right)^2 - \left(\frac{2\sin\theta\cos\theta}{r}\right)\left(\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta}\right) + \left(\frac{\sin^2\theta}{r^2}\right)\left(\frac{\partial w}{\partial \theta}\right)^2$ and
$(f_y)^2 = (\sin^2\theta)\left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{2\sin\theta\cos\theta}{r}\right)\left(\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta}\right) + \left(\frac{\cos^2\theta}{r^2}\right)\left(\frac{\partial w}{\partial \theta}\right)^2 \Rightarrow (f_x)^2 + (f_y)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2$

43. $w_x = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = x\frac{\partial w}{\partial u} + y\frac{\partial w}{\partial v} \Rightarrow w_{xx} = \frac{\partial w}{\partial u} + x\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u}\right) + y\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial v}\right)$
$= \frac{\partial w}{\partial u} + x\left(\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v \partial u}\frac{\partial v}{\partial x}\right) + y\left(\frac{\partial^2 w}{\partial u \partial v}\frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial x}\right) = \frac{\partial w}{\partial u} + x\left(x\frac{\partial^2 w}{\partial u^2} + y\frac{\partial^2 w}{\partial v \partial u}\right) + y\left(x\frac{\partial^2 w}{\partial u \partial v} + y\frac{\partial^2 w}{\partial v^2}\right)$
$= \frac{\partial w}{\partial u} + x^2\frac{\partial^2 w}{\partial u^2} + 2xy\frac{\partial^2 w}{\partial v \partial u} + y^2\frac{\partial^2 w}{\partial v^2}$; $w_y = \frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = -y\frac{\partial w}{\partial u} + x\frac{\partial w}{\partial v}$
$\Rightarrow w_{yy} = -\frac{\partial w}{\partial u} - y\left(\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v \partial u}\frac{\partial v}{\partial y}\right) + x\left(\frac{\partial^2 w}{\partial u \partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial y}\right)$
$= -\frac{\partial w}{\partial u} - y\left(-y\frac{\partial^2 w}{\partial u^2} + x\frac{\partial^2 w}{\partial v \partial u}\right) + x\left(-y\frac{\partial^2 w}{\partial u \partial v} + x\frac{\partial^2 w}{\partial v^2}\right) = -\frac{\partial w}{\partial u} + y^2\frac{\partial^2 w}{\partial u^2} - 2xy\frac{\partial^2 w}{\partial v \partial u} + x^2\frac{\partial^2 w}{\partial v^2}$; thus
$w_{xx} + w_{yy} = (x^2+y^2)\frac{\partial^2 w}{\partial u^2} + (x^2+y^2)\frac{\partial^2 w}{\partial v^2} = (x^2+y^2)(w_{uu}+w_{vv}) = 0$, since $w_{uu} + w_{vv} = 0$

44. $\frac{\partial w}{\partial x} = f'(u)(1) + g'(v)(1) = f'(u) + g'(v) \Rightarrow w_{xx} = f''(u)(1) + g''(v)(1) = f''(u) + g''(v)$;
$\frac{\partial w}{\partial y} = f'(u)(i) + g'(v)(-i) \Rightarrow w_{yy} = f''(u)(i^2) + g''(v)(i^2) = -f''(u) - g''(v) \Rightarrow w_{xx} + w_{yy} = 0$

45. $f_x(x,y,z) = \cos t$, $f_y(x,y,z) = \sin t$, and $f_z(x,y,z) = t^2 + t - 2 \Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$
$= (\cos t)(-\sin t) + (\sin t)(\cos t) + (t^2+t-2)(1) = t^2 + t - 2$; $\frac{df}{dt} = 0 \Rightarrow t^2 + t - 2 = 0 \Rightarrow t = -2$
or $t = 1$; $t = -2 \Rightarrow x = \cos(-2)$, $y = \sin(-2)$, $z = -2$ for the point $(\cos(-2), \sin(-2), -2)$; $t = 1 \Rightarrow x = \cos 1$,
$y = \sin 1$, $z = 1$ for the point $(\cos 1, \sin 1, 1)$

46. $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = (2xe^{2y}\cos 3z)(-\sin t) + (2x^2e^{2y}\cos 3z)\left(\frac{1}{t+2}\right) + (-3x^2e^{2y}\sin 3z)(1)$
$= -2xe^{2y}\cos 3z \sin t + \frac{2x^2e^{2y}\cos 3z}{t+2} - 3x^2e^{2y}\sin 3z$; at the point on the curve $z = 0 \Rightarrow t = z = 0$
$\Rightarrow \left.\frac{dw}{dt}\right|_{(1,\ln 2,0)} = 0 + \frac{2(1)^2(4)(1)}{2} - 0 = 4$

47. (a) $\frac{\partial T}{\partial x} = 8x - 4y$ and $\frac{\partial T}{\partial y} = 8y - 4x \Rightarrow \frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = (8x-4y)(-\sin t) + (8y-4x)(\cos t)$
$= (8\cos t - 4\sin t)(-\sin t) + (8\sin t - 4\cos t)(\cos t) = 4\sin^2 t - 4\cos^2 t \Rightarrow \frac{d^2T}{dt^2} = 16\sin t\cos t$;
$\frac{dT}{dt} = 0 \Rightarrow 4\sin^2 t - 4\cos^2 t = 0 \Rightarrow \sin^2 t = \cos^2 t \Rightarrow \sin t = \cos t$ or $\sin t = -\cos t \Rightarrow t = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$ on the interval $0 \le t \le 2\pi$;
$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{\pi}{4}} = 16\sin\frac{\pi}{4}\cos\frac{\pi}{4} > 0 \Rightarrow$ T has a minimum at $(x,y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$;
$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{3\pi}{4}} = 16\sin\frac{3\pi}{4}\cos\frac{3\pi}{4} < 0 \Rightarrow$ T has a maximum at $(x,y) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$;
$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{5\pi}{4}} = 16\sin\frac{5\pi}{4}\cos\frac{5\pi}{4} > 0 \Rightarrow$ T has a minimum at $(x,y) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$;
$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{7\pi}{4}} = 16\sin\frac{7\pi}{4}\cos\frac{7\pi}{4} < 0 \Rightarrow$ T has a maximum at $(x,y) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

(b) $T = 4x^2 - 4xy + 4y^2 \Rightarrow \frac{\partial T}{\partial x} = 8x - 4y$, and $\frac{\partial T}{\partial y} = 8y - 4x$ so the extreme values occur at the four points found in part (a): $T\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = T\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 4\left(\frac{1}{2}\right) - 4\left(-\frac{1}{2}\right) + 4\left(\frac{1}{2}\right) = 6$, the maximum and
$T\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = T\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 4\left(\frac{1}{2}\right) - 4\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right) = 2$, the minimum

48. (a) $\frac{\partial T}{\partial x} = y$ and $\frac{\partial T}{\partial y} = x \Rightarrow \frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = y\left(-2\sqrt{2}\sin t\right) + x\left(\sqrt{2}\cos t\right)$
$= \left(\sqrt{2}\sin t\right)\left(-2\sqrt{2}\sin t\right) + \left(2\sqrt{2}\cos t\right)\left(\sqrt{2}\cos t\right) = -4\sin^2 t + 4\cos^2 t = -4\sin^2 t + 4(1 - \sin^2 t)$
$= 4 - 8\sin^2 t \Rightarrow \frac{d^2T}{dt^2} = -16\sin t \operatorname{cost} t$; $\frac{dT}{dt} = 0 \Rightarrow 4 - 8\sin^2 t = 0 \Rightarrow \sin^2 t = \frac{1}{2} \Rightarrow \sin t = \pm\frac{1}{\sqrt{2}} \Rightarrow t = \frac{\pi}{4}$,
$\frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ on the interval $0 \le t \le 2\pi$;
$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{\pi}{4}} = -8\sin 2\left(\frac{\pi}{4}\right) = -8 \Rightarrow$ T has a maximum at $(x,y) = (2,1)$;
$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{3\pi}{4}} = -8\sin 2\left(\frac{3\pi}{4}\right) = 8 \Rightarrow$ T has a minimum at $(x,y) = (-2,1)$;

$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{5\pi}{4}} = -8 \sin 2\left(\frac{5\pi}{4}\right) = -8 \Rightarrow$ T has a maximum at $(x, y) = (-2, -1)$;

$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{7\pi}{4}} = -8 \sin 2\left(\frac{7\pi}{4}\right) = 8 \Rightarrow$ T has a minimum at $(x, y) = (2, -1)$

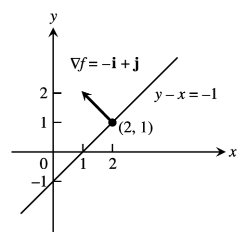
(b) $T = xy - 2 \Rightarrow \frac{\partial T}{\partial x} = y$ and $\frac{\partial T}{\partial y} = x$ so the extreme values occur at the four points found in part (a): $T(2, 1) = T(-2, -1) = 0$, the maximum and $T(-2, 1) = T(2, -1) = -4$, the minimum

49. $G(u, x) = \int_a^u g(t, x)\, dt$ where $u = f(x) \Rightarrow \frac{dG}{dx} = \frac{\partial G}{\partial u}\frac{du}{dx} + \frac{\partial G}{\partial x}\frac{dx}{dx} = g(u, x)f'(x) + \int_a^u g_x(t, x)\, dt$; thus $F(x) = \int_0^{x^2} \sqrt{t^4 + x^3}\, dt \Rightarrow F'(x) = \sqrt{(x^2)^4 + x^3}\,(2x) + \int_0^{x^2} \frac{\partial}{\partial x}\sqrt{t^4 + x^3}\, dt = 2x\sqrt{x^8 + x^3} + \int_0^{x^2} \frac{3x^2}{2\sqrt{t^4 + x^3}}\, dt$

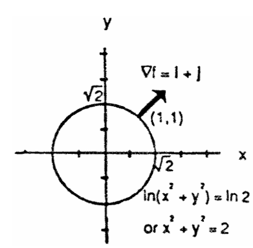
50. Using the result in Exercise 49, $F(x) = \int_{x^2}^1 \sqrt{t^3 + x^2}\, dt = -\int_1^{x^2} \sqrt{t^3 + x^2}\, dt \Rightarrow F'(x)$ $= \left[-\sqrt{(x^2)^3 + x^2}\, x^2 - \int_1^{x^2} \frac{\partial}{\partial x}\sqrt{t^3 + x^2}\, dt\right] = -x^2\sqrt{x^6 + x^2} + \int_{x^2}^1 \frac{x}{\sqrt{t^3 + x^2}}\, dt$

14.5 DIRECTIONAL DERIVATIVES AND GRADIENT VECTORS

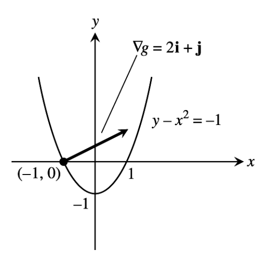
1. $\frac{\partial f}{\partial x} = -1$, $\frac{\partial f}{\partial y} = 1 \Rightarrow \nabla f = -\mathbf{i} + \mathbf{j}$; $f(2, 1) = -1$ $\Rightarrow -1 = y - x$ is the level curve



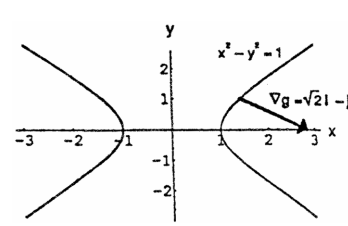
2. $\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2} \Rightarrow \frac{\partial f}{\partial x}(1, 1) = 1$; $\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2}$ $\Rightarrow \frac{\partial f}{\partial y}(1, 1) = 1 \Rightarrow \nabla f = \mathbf{i} + \mathbf{j}$; $f(1, 1) = \ln 2 \Rightarrow \ln 2$ $= \ln\left(x^2 + y^2\right) \Rightarrow 2 = x^2 + y^2$ is the level curve



3. $\frac{\partial g}{\partial x} = -2x \Rightarrow \frac{\partial g}{\partial x}(-1, 0) = 2$; $\frac{\partial g}{\partial y} = 1$ $\Rightarrow \nabla g = 2\mathbf{i} + \mathbf{j}$; $g(-1, 0) = -1$ $\Rightarrow -1 = y - x^2$ is the level curve



4. $\frac{\partial g}{\partial x} = x \Rightarrow \frac{\partial g}{\partial x}\left(\sqrt{2}, 1\right) = \sqrt{2}$; $\frac{\partial g}{\partial y} = -y$ $\Rightarrow \frac{\partial g}{\partial y}\left(\sqrt{2}, 1\right) = -1 \Rightarrow \nabla g = \sqrt{2}\,\mathbf{i} - \mathbf{j}$; $g\left(\sqrt{2}, 1\right) = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{x^2}{2} - \frac{y^2}{2}$ or $1 = x^2 - y^2$ is the level curve



5. $\frac{\partial f}{\partial x} = 2x + \frac{z}{x} \Rightarrow \frac{\partial f}{\partial x}(1, 1, 1) = 3$; $\frac{\partial f}{\partial y} = 2y \Rightarrow \frac{\partial f}{\partial y}(1, 1, 1) = 2$; $\frac{\partial f}{\partial z} = -4z + \ln x \Rightarrow \frac{\partial f}{\partial z}(1, 1, 1) = -4$; thus $\nabla f = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$

6. $\frac{\partial f}{\partial x} = -6xz + \frac{z}{x^2z^2+1} \Rightarrow \frac{\partial f}{\partial x}(1,1,1) = -\frac{11}{2}$; $\frac{\partial f}{\partial y} = -6yz \Rightarrow \frac{\partial f}{\partial y}(1,1,1) = -6$; $\frac{\partial f}{\partial z} = 6z^2 - 3(x^2+y^2) + \frac{x}{x^2z^2+1}$ $\Rightarrow \frac{\partial f}{\partial z}(1,1,1) = \frac{1}{2}$; thus $\nabla f = -\frac{11}{2}\mathbf{i} - 6\mathbf{j} + \frac{1}{2}\mathbf{k}$

7. $\frac{\partial f}{\partial x} = -\frac{x}{(x^2+y^2+z^2)^{3/2}} + \frac{1}{x} \Rightarrow \frac{\partial f}{\partial x}(-1,2,-2) = -\frac{26}{27}$; $\frac{\partial f}{\partial y} = -\frac{y}{(x^2+y^2+z^2)^{3/2}} + \frac{1}{y} \Rightarrow \frac{\partial f}{\partial y}(-1,2,-2) = \frac{23}{54}$; $\frac{\partial f}{\partial z} = -\frac{z}{(x^2+y^2+z^2)^{3/2}} + \frac{1}{z} \Rightarrow \frac{\partial f}{\partial z}(-1,2,-2) = -\frac{23}{54}$; thus $\nabla f = -\frac{26}{27}\mathbf{i} + \frac{23}{54}\mathbf{j} - \frac{23}{54}\mathbf{k}$

8. $\frac{\partial f}{\partial x} = e^{x+y}\cos z + \frac{y+1}{\sqrt{1-x^2}} \Rightarrow \frac{\partial f}{\partial x}\left(0,0,\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + 1$; $\frac{\partial f}{\partial y} = e^{x+y}\cos z + \sin^{-1}x \Rightarrow \frac{\partial f}{\partial y}\left(0,0,\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$; $\frac{\partial f}{\partial z} = -e^{x+y}\sin z \Rightarrow \frac{\partial f}{\partial z}\left(0,0,\frac{\pi}{6}\right) = -\frac{1}{2}$; thus $\nabla f = \left(\frac{\sqrt{3}+2}{2}\right)\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$

9. $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{4\mathbf{i}+3\mathbf{j}}{\sqrt{4^2+3^2}} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$; $f_x(x,y) = 2y \Rightarrow f_x(5,5) = 10$; $f_y(x,y) = 2x - 6y \Rightarrow f_y(5,5) = -20$ $\Rightarrow \nabla f = 10\mathbf{i} - 20\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = 10\left(\frac{4}{5}\right) - 20\left(\frac{3}{5}\right) = -4$

10. $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i}-4\mathbf{j}}{\sqrt{3^2+(-4)^2}} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$; $f_x(x,y) = 4x \Rightarrow f_x(-1,1) = -4$; $f_y(x,y) = 2y \Rightarrow f_y(-1,1) = 2$ $\Rightarrow \nabla f = -4\mathbf{i} + 2\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = -\frac{12}{5} - \frac{8}{5} = -4$

11. $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{12\mathbf{i}+5\mathbf{j}}{\sqrt{12^2+5^2}} = \frac{12}{13}\mathbf{i} + \frac{5}{13}\mathbf{j}$; $g_x(x,y) = 1 + \frac{y^2}{x^2} + \frac{2y\sqrt{3}}{2xy\sqrt{4x^2y^2-1}} \Rightarrow g_x(1,1) = 3$; $g_y(x,y)$ $= -\frac{2y}{x} + \frac{2x\sqrt{3}}{2xy\sqrt{4x^2y^2-1}} \Rightarrow g_y(1,1) = -1 \Rightarrow \nabla g = 3\mathbf{i} - \mathbf{j} \Rightarrow (D_{\mathbf{u}}g)_{P_0} = \nabla g \cdot \mathbf{u} = \frac{36}{13} - \frac{5}{13} = \frac{31}{13}$

12. $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i}-2\mathbf{j}}{\sqrt{3^2+(-2)^2}} = \frac{3}{\sqrt{13}}\mathbf{i} - \frac{2}{\sqrt{13}}\mathbf{j}$; $h_x(x,y) = \frac{\left(\frac{-y}{x^2}\right)}{\left(\frac{y}{x}\right)^2+1} + \frac{\left(\frac{y}{2}\right)\sqrt{3}}{\sqrt{1-\left(\frac{x^2y^2}{4}\right)}} \Rightarrow h_x(1,1) = \frac{1}{2}$;

$h_y(x,y) = \frac{\left(\frac{1}{x}\right)}{\left(\frac{y}{x}\right)^2+1} + \frac{\left(\frac{x}{2}\right)\sqrt{3}}{\sqrt{1-\left(\frac{x^2y^2}{4}\right)}} \Rightarrow h_y(1,1) = \frac{3}{2} \Rightarrow \nabla h = \frac{1}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} \Rightarrow (D_{\mathbf{u}}h)_{P_0} = \nabla h \cdot \mathbf{u} = \frac{3}{2\sqrt{13}} - \frac{6}{2\sqrt{13}}$ $= -\frac{3}{2\sqrt{13}}$

13. $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i}+6\mathbf{j}-2\mathbf{k}}{\sqrt{3^2+6^2+(-2)^2}} = \frac{3}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{2}{7}\mathbf{k}$; $f_x(x,y,z) = y + z \Rightarrow f_x(1,-1,2) = 1$; $f_y(x,y,z) = x + z$ $\Rightarrow f_y(1,-1,2) = 3$; $f_z(x,y,z) = y + x \Rightarrow f_z(1,-1,2) = 0 \Rightarrow \nabla f = \mathbf{i} + 3\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = \frac{3}{7} + \frac{18}{7} = 3$

14. $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$; $f_x(x,y,z) = 2x \Rightarrow f_x(1,1,1) = 2$; $f_y(x,y,z) = 4y$ $\Rightarrow f_y(1,1,1) = 4$; $f_z(x,y,z) = -6z \Rightarrow f_z(1,1,1) = -6 \Rightarrow \nabla f = 2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u}$ $= 2\left(\frac{1}{\sqrt{3}}\right) + 4\left(\frac{1}{\sqrt{3}}\right) - 6\left(\frac{1}{\sqrt{3}}\right) = 0$

15. $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2\mathbf{i}+\mathbf{j}-2\mathbf{k}}{\sqrt{2^2+1^2+(-2)^2}} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$; $g_x(x,y,z) = 3e^x\cos yz \Rightarrow g_x(0,0,0) = 3$; $g_y(x,y,z) = -3ze^x\sin yz$ $\Rightarrow g_y(0,0,0) = 0$; $g_z(x,y,z) = -3ye^x\sin yz \Rightarrow g_z(0,0,0) = 0 \Rightarrow \nabla g = 3\mathbf{i} \Rightarrow (D_{\mathbf{u}}g)_{P_0} = \nabla g \cdot \mathbf{u} = 2$

16. $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i}+2\mathbf{j}+2\mathbf{k}}{\sqrt{1^2+2^2+2^2}} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$; $h_x(x,y,z) = -y\sin xy + \frac{1}{x} \Rightarrow h_x\left(1,0,\frac{1}{2}\right) = 1$; $h_y(x,y,z) = -x\sin xy + ze^{yz} \Rightarrow h_y\left(1,0,\frac{1}{2}\right) = \frac{1}{2}$; $h_z(x,y,z) = ye^{yz} + \frac{1}{z} \Rightarrow h_z\left(1,0,\frac{1}{2}\right) = 2 \Rightarrow \nabla h = \mathbf{i} + \frac{1}{2}\mathbf{j} + 2\mathbf{k}$ $\Rightarrow (D_{\mathbf{u}}h)_{P_0} = \nabla h \cdot \mathbf{u} = \frac{1}{3} + \frac{1}{3} + \frac{4}{3} = 2$

17. $\nabla f = (2x+y)\mathbf{i} + (x+2y)\mathbf{j} \Rightarrow \nabla f(-1,1) = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{-\mathbf{i}+\mathbf{j}}{\sqrt{(-1)^2+1^2}} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$; f increases most rapidly in the direction $\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ and decreases most rapidly in the direction $-\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$; $(D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = |\nabla f| = \sqrt{2}$ and $(D_{-\mathbf{u}}f)_{P_0} = -\sqrt{2}$

18. $\nabla f = (2xy + ye^{xy} \sin y)\,\mathbf{i} + (x^2 + xe^{xy} \sin y + e^{xy} \cos y)\,\mathbf{j} \Rightarrow \nabla f(1,0) = 2\mathbf{j} \Rightarrow \mathbf{u} = \frac{\nabla f}{|\nabla f|} = \mathbf{j}$; f increases most rapidly in the direction $\mathbf{u} = \mathbf{j}$ and decreases most rapidly in the direction $-\mathbf{u} = -\mathbf{j}$; $(D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = |\nabla f| = 2$ and $(D_{-\mathbf{u}}f)_{P_0} = -2$

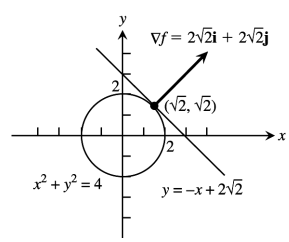
19. $\nabla f = \frac{1}{y}\,\mathbf{i} - \left(\frac{x}{y^2} + z\right)\mathbf{j} - y\mathbf{k} \Rightarrow \nabla f(4,1,1) = \mathbf{i} - 5\mathbf{j} - \mathbf{k} \Rightarrow \mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{\mathbf{i} - 5\mathbf{j} - \mathbf{k}}{\sqrt{1^2 + (-5)^2 + (-1)^2}}$ $= \frac{1}{3\sqrt{3}}\,\mathbf{i} - \frac{5}{3\sqrt{3}}\,\mathbf{j} - \frac{1}{3\sqrt{3}}\,\mathbf{k}$; f increases most rapidly in the direction of $\mathbf{u} = \frac{1}{3\sqrt{3}}\,\mathbf{i} - \frac{5}{3\sqrt{3}}\,\mathbf{j} - \frac{1}{3\sqrt{3}}\,\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{1}{3\sqrt{3}}\,\mathbf{i} + \frac{5}{3\sqrt{3}}\,\mathbf{j} + \frac{1}{3\sqrt{3}}\,\mathbf{k}$; $(D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = |\nabla f| = 3\sqrt{3}$ and $(D_{-\mathbf{u}}f)_{P_0} = -3\sqrt{3}$

20. $\nabla g = e^y\mathbf{i} + xe^y\mathbf{j} + 2z\mathbf{k} \Rightarrow \nabla g\left(1, \ln 2, \frac{1}{2}\right) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{u} = \frac{\nabla g}{|\nabla g|} = \frac{2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{2^2 + 2^2 + 1^2}} = \frac{2}{3}\,\mathbf{i} + \frac{2}{3}\,\mathbf{j} + \frac{1}{3}\,\mathbf{k}$; g increases most rapidly in the direction $\mathbf{u} = \frac{2}{3}\,\mathbf{i} + \frac{2}{3}\,\mathbf{j} + \frac{1}{3}\,\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{2}{3}\,\mathbf{i} - \frac{2}{3}\,\mathbf{j} - \frac{1}{3}\,\mathbf{k}$; $(D_{\mathbf{u}}g)_{P_0} = \nabla g \cdot \mathbf{u} = |\nabla g| = 3$ and $(D_{-\mathbf{u}}g)_{P_0} = -3$

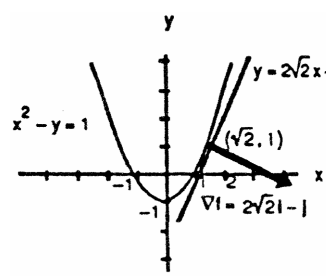
21. $\nabla f = \left(\frac{1}{x} + \frac{1}{x}\right)\mathbf{i} + \left(\frac{1}{y} + \frac{1}{y}\right)\mathbf{j} + \left(\frac{1}{z} + \frac{1}{z}\right)\mathbf{k} \Rightarrow \nabla f(1,1,1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{3}}\,\mathbf{i} + \frac{1}{\sqrt{3}}\,\mathbf{j} + \frac{1}{\sqrt{3}}\,\mathbf{k}$; f increases most rapidly in the direction $\mathbf{u} = \frac{1}{\sqrt{3}}\,\mathbf{i} + \frac{1}{\sqrt{3}}\,\mathbf{j} + \frac{1}{\sqrt{3}}\,\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{1}{\sqrt{3}}\,\mathbf{i} - \frac{1}{\sqrt{3}}\,\mathbf{j} - \frac{1}{\sqrt{3}}\,\mathbf{k}$; $(D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = |\nabla f| = 2\sqrt{3}$ and $(D_{-\mathbf{u}}f)_{P_0} = -2\sqrt{3}$

22. $\nabla h = \left(\frac{2x}{x^2 + y^2 - 1}\right)\mathbf{i} + \left(\frac{2y}{x^2 + y^2 - 1} + 1\right)\mathbf{j} + 6\mathbf{k} \Rightarrow \nabla h(1,1,0) = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k} \Rightarrow \mathbf{u} = \frac{\nabla h}{|\nabla h|} = \frac{2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}{\sqrt{2^2 + 3^2 + 6^2}}$ $= \frac{2}{7}\,\mathbf{i} + \frac{3}{7}\,\mathbf{j} + \frac{6}{7}\,\mathbf{k}$; h increases most rapidly in the direction $\mathbf{u} = \frac{2}{7}\,\mathbf{i} + \frac{3}{7}\,\mathbf{j} + \frac{6}{7}\,\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{2}{7}\,\mathbf{i} - \frac{3}{7}\,\mathbf{j} - \frac{6}{7}\,\mathbf{k}$; $(D_{\mathbf{u}}h)_{P_0} = \nabla h \cdot \mathbf{u} = |\nabla h| = 7$ and $(D_{-\mathbf{u}}h)_{P_0} = -7$

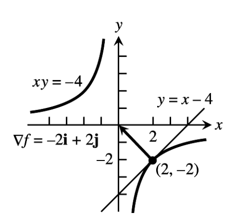
23. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} \Rightarrow \nabla f\left(\sqrt{2}, \sqrt{2}\right) = 2\sqrt{2}\,\mathbf{i} + 2\sqrt{2}\,\mathbf{j}$
$\Rightarrow$ Tangent line: $2\sqrt{2}\left(x - \sqrt{2}\right) + 2\sqrt{2}\left(y - \sqrt{2}\right) = 0$
$\Rightarrow \sqrt{2}x + \sqrt{2}y = 4$



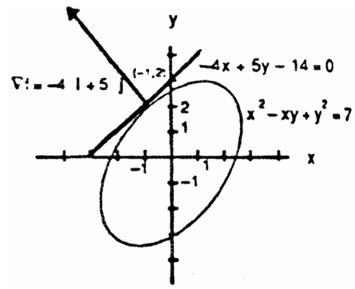
24. $\nabla f = 2x\mathbf{i} - \mathbf{j} \Rightarrow \nabla f\left(\sqrt{2}, 1\right) = 2\sqrt{2}\,\mathbf{i} - \mathbf{j}$
$\Rightarrow$ Tangent line: $2\sqrt{2}\left(x - \sqrt{2}\right) - (y - 1) = 0$
$\Rightarrow y = 2\sqrt{2}x - 3$



25. $\nabla f = y\mathbf{i} + x\mathbf{j} \Rightarrow \nabla f(2,-2) = -2\mathbf{i} + 2\mathbf{j}$
$\Rightarrow$ Tangent line: $-2(x - 2) + 2(y + 2) = 0$
$\Rightarrow y = x - 4$



26. $\nabla f = (2x - y)\mathbf{i} + (2y - x)\mathbf{j} \Rightarrow \nabla f(-1, 2) = -4\mathbf{i} + 5\mathbf{j}$
$\Rightarrow$ Tangent line: $-4(x + 1) + 5(y - 2) = 0$
$\Rightarrow -4x + 5y - 14 = 0$



27. $\nabla f = y\mathbf{i} + (x + 2y)\mathbf{j} \Rightarrow \nabla f(3, 2) = 2\mathbf{i} + 7\mathbf{j}$; a vector orthogonal to ∇f is $\mathbf{v} = 7\mathbf{i} - 2\mathbf{j} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{7\mathbf{i} - 2\mathbf{j}}{\sqrt{7^2 + (-2)^2}}$
$= \frac{7}{\sqrt{53}}\mathbf{i} - \frac{2}{\sqrt{53}}\mathbf{j}$ and $-\mathbf{u} = -\frac{7}{\sqrt{53}}\mathbf{i} + \frac{2}{\sqrt{53}}\mathbf{j}$ are the directions where the derivative is zero

28. $\nabla f = \frac{4xy^2}{(x^2 + y^2)^2}\mathbf{i} - \frac{4x^2y}{(x^2 + y^2)^2}\mathbf{j} \Rightarrow \nabla f(1, 1) = \mathbf{i} - \mathbf{j}$; a vector orthogonal to ∇f is $\mathbf{v} = \mathbf{i} + \mathbf{j}$
$\Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ and $-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ are the directions where the derivative is zero

29. $\nabla f = (2x - 3y)\mathbf{i} + (-3x + 8y)\mathbf{j} \Rightarrow \nabla f(1, 2) = -4\mathbf{i} + 13\mathbf{j} \Rightarrow |\nabla f(1, 2)| = \sqrt{(-4)^2 + (13)^2} = \sqrt{185}$; no, the maximum rate of change is $\sqrt{185} < 14$

30. $\nabla T = 2y\mathbf{i} + (2x - z)\mathbf{j} - y\mathbf{k} \Rightarrow \nabla T(1, -1, 1) = -2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow |\nabla T(1, -1, 1)| = \sqrt{(-2)^2 + 1^2 + 1^2} = \sqrt{6}$; no, the minimum rate of change is $-\sqrt{6} > -3$

31. $\nabla f = f_x(1, 2)\mathbf{i} + f_y(1, 2)\mathbf{j}$ and $\mathbf{u}_1 = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \Rightarrow (D_{\mathbf{u}_1}f)(1, 2) = f_x(1, 2)\left(\frac{1}{\sqrt{2}}\right) + f_y(1, 2)\left(\frac{1}{\sqrt{2}}\right)$
$= 2\sqrt{2} \Rightarrow f_x(1, 2) + f_y(1, 2) = 4$; $\mathbf{u}_2 = -\mathbf{j} \Rightarrow (D_{\mathbf{u}_2}f)(1, 2) = f_x(1, 2)(0) + f_y(1, 2)(-1) = -3 \Rightarrow -f_y(1, 2) = -3$
$\Rightarrow f_y(1, 2) = 3$; then $f_x(1, 2) + 3 = 4 \Rightarrow f_x(1, 2) = 1$; thus $\nabla f(1, 2) = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-\mathbf{i} - 2\mathbf{j}}{\sqrt{(-1)^2 + (-2)^2}}$
$= -\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = -\frac{1}{\sqrt{5}} - \frac{6}{\sqrt{5}} = -\frac{7}{\sqrt{5}}$

32. (a) $(D_{\mathbf{u}}f)_P = 2\sqrt{3} \Rightarrow |\nabla f| = 2\sqrt{3}$; $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j} - \mathbf{k}}{\sqrt{1^2 + 1^2 + (-1)^2}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$; thus $\mathbf{u} = \frac{\nabla f}{|\nabla f|}$
$\Rightarrow \nabla f = |\nabla f|\mathbf{u} \Rightarrow \nabla f = 2\sqrt{3}\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right) = 2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

(b) $\mathbf{v} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = 2\left(\frac{1}{\sqrt{2}}\right) + 2\left(\frac{1}{\sqrt{2}}\right) - 2(0) = 2\sqrt{2}$

33. The directional derivative is the scalar component. With ∇f evaluated at P_0, the scalar component of ∇f in the direction of $\mathbf{u}$ is $\nabla f \cdot \mathbf{u} = (D_{\mathbf{u}}f)_{P_0}$.

34. $D_{\mathbf{i}}f = \nabla f \cdot \mathbf{i} = (f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}) \cdot \mathbf{i} = f_x$; similarly, $D_{\mathbf{j}}f = \nabla f \cdot \mathbf{j} = f_y$ and $D_{\mathbf{k}}f = \nabla f \cdot \mathbf{k} = f_z$

35. If (x, y) is a point on the line, then $\mathbf{T}(x, y) = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j}$ is a vector parallel to the line $\Rightarrow \mathbf{T} \cdot \mathbf{N} = 0$
$\Rightarrow A(x - x_0) + B(y - y_0) = 0$, as claimed.

36. (a) $\nabla(kf) = \frac{\partial(kf)}{\partial x}\mathbf{i} + \frac{\partial(kf)}{\partial y}\mathbf{j} + \frac{\partial(kf)}{\partial z}\mathbf{k} = k\left(\frac{\partial f}{\partial x}\right)\mathbf{i} + k\left(\frac{\partial f}{\partial y}\right)\mathbf{j} + k\left(\frac{\partial f}{\partial z}\right)\mathbf{k} = k\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) = k\nabla f$

(b) $\nabla(f + g) = \frac{\partial(f+g)}{\partial x}\mathbf{i} + \frac{\partial(f+g)}{\partial y}\mathbf{j} + \frac{\partial(f+g)}{\partial z}\mathbf{k} = \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right)\mathbf{i} + \left(\frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}\right)\mathbf{j} + \left(\frac{\partial f}{\partial z} + \frac{\partial g}{\partial z}\right)\mathbf{k}$
$= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} + \frac{\partial g}{\partial z}\mathbf{k} = \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) + \left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\right) = \nabla f + \nabla g$

(c) $\nabla(f - g) = \nabla f - \nabla g$ (Substitute $-g$ for g in part (b) above)

(d) $\nabla(fg) = \frac{\partial(fg)}{\partial x}\mathbf{i} + \frac{\partial(fg)}{\partial y}\mathbf{j} + \frac{\partial(fg)}{\partial z}\mathbf{k} = \left(\frac{\partial f}{\partial x}g + \frac{\partial g}{\partial x}f\right)\mathbf{i} + \left(\frac{\partial f}{\partial y}g + \frac{\partial g}{\partial y}f\right)\mathbf{j} + \left(\frac{\partial f}{\partial z}g + \frac{\partial g}{\partial z}f\right)\mathbf{k}$

$= \left(\frac{\partial f}{\partial x}g\right)\mathbf{i} + \left(\frac{\partial g}{\partial x}f\right)\mathbf{i} + \left(\frac{\partial f}{\partial y}g\right)\mathbf{j} + \left(\frac{\partial g}{\partial y}f\right)\mathbf{j} + \left(\frac{\partial f}{\partial z}g\right)\mathbf{k} + \left(\frac{\partial g}{\partial z}f\right)\mathbf{k}$

$= f\left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\right) + g\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) = f\nabla g + g\nabla f$

(e) $\nabla\left(\frac{f}{g}\right) = \frac{\partial\left(\frac{f}{g}\right)}{\partial x}\mathbf{i} + \frac{\partial\left(\frac{f}{g}\right)}{\partial y}\mathbf{j} + \frac{\partial\left(\frac{f}{g}\right)}{\partial z}\mathbf{k} = \left(\frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}\right)\mathbf{i} + \left(\frac{g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}}{g^2}\right)\mathbf{j} + \left(\frac{g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}}{g^2}\right)\mathbf{k}$

$= \left(\frac{g\frac{\partial f}{\partial x}\mathbf{i} + g\frac{\partial f}{\partial y}\mathbf{j} + g\frac{\partial f}{\partial z}\mathbf{k}}{g^2}\right) - \left(\frac{f\frac{\partial g}{\partial x}\mathbf{i} + f\frac{\partial g}{\partial y}\mathbf{j} + f\frac{\partial g}{\partial z}\mathbf{k}}{g^2}\right) = \frac{g\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right)}{g^2} - \frac{f\left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\right)}{g^2}$

$= \frac{g\nabla f}{g^2} - \frac{f\nabla g}{g^2} = \frac{g\nabla f - f\nabla g}{g^2}$

14.6 TANTGENT PLANES AND DIFFERENTIALS

1. (a) $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow \nabla f(1,1,1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow$ Tangent plane: $2(x-1) + 2(y-1) + 2(z-1) = 0$ $\Rightarrow x + y + z = 3$;
   (b) Normal line: $x = 1 + 2t,\ y = 1 + 2t,\ z = 1 + 2t$

2. (a) $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k} \Rightarrow \nabla f(3,5,-4) = 6\mathbf{i} + 10\mathbf{j} + 8\mathbf{k} \Rightarrow$ Tangent plane: $6(x-3) + 10(y-5) + 8(z+4) = 0$ $\Rightarrow 3x + 5y + 4z = 18$;
   (b) Normal line: $x = 3 + 6t,\ y = 5 + 10t,\ z = -4 + 8t$

3. (a) $\nabla f = -2x\mathbf{i} + 2\mathbf{k} \Rightarrow \nabla f(2,0,2) = -4\mathbf{i} + 2\mathbf{k} \Rightarrow$ Tangent plane: $-4(x-2) + 2(z-2) = 0$ $\Rightarrow -4x + 2z + 4 = 0 \Rightarrow -2x + z + 2 = 0$;
   (b) Normal line: $x = 2 - 4t,\ y = 0,\ z = 2 + 2t$

4. (a) $\nabla f = (2x + 2y)\mathbf{i} + (2x - 2y)\mathbf{j} + 2z\mathbf{k} \Rightarrow \nabla f(1,-1,3) = 4\mathbf{j} + 6\mathbf{k} \Rightarrow$ Tangent plane: $4(y+1) + 6(z-3) = 0$ $\Rightarrow 2y + 3z = 7$;
   (b) Normal line: $x = 1,\ y = -1 + 4t,\ z = 3 + 6t$

5. (a) $\nabla f = (-\pi\sin\pi x - 2xy + ze^{xz})\,\mathbf{i} + (-x^2 + z)\,\mathbf{j} + (xe^{xz} + y)\,\mathbf{k} \Rightarrow \nabla f(0,1,2) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \Rightarrow$ Tangent plane: $2(x-0) + 2(y-1) + 1(z-2) = 0 \Rightarrow 2x + 2y + z - 4 = 0$;
   (b) Normal line: $x = 2t,\ y = 1 + 2t,\ z = 2 + t$

6. (a) $\nabla f = (2x - y)\mathbf{i} - (x + 2y)\mathbf{j} - \mathbf{k} \Rightarrow \nabla f(1,1,-1) = \mathbf{i} - 3\mathbf{j} - \mathbf{k} \Rightarrow$ Tangent plane: $1(x-1) - 3(y-1) - 1(z+1) = 0 \Rightarrow x - 3y - z = -1$;
   (b) Normal line: $x = 1 + t,\ y = 1 - 3t,\ z = -1 - t$

7. (a) $\nabla f = \mathbf{i} + \mathbf{j} + \mathbf{k}$ for all points $\Rightarrow \nabla f(0,1,0) = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow$ Tangent plane: $1(x-0) + 1(y-1) + 1(z-0) = 0$ $\Rightarrow x + y + z - 1 = 0$;
   (b) Normal line: $x = t,\ y = 1 + t,\ z = t$

8. (a) $\nabla f = (2x - 2y - 1)\mathbf{i} + (2y - 2x + 3)\mathbf{j} - \mathbf{k} \Rightarrow \nabla f(2,-3,18) = 9\mathbf{i} - 7\mathbf{j} - \mathbf{k} \Rightarrow$ Tangent plane: $9(x-2) - 7(y+3) - 1(z-18) = 0 \Rightarrow 9x - 7y - z = 21$;
   (b) Normal line: $x = 2 + 9t,\ y = -3 - 7t,\ z = 18 - t$

9. $z = f(x,y) = \ln(x^2 + y^2) \Rightarrow f_x(x,y) = \frac{2x}{x^2+y^2}$ and $f_y(x,y) = \frac{2y}{x^2+y^2} \Rightarrow f_x(1,0) = 2$ and $f_y(1,0) = 0 \Rightarrow$ from Eq. (4) the tangent plane at $(1,0,0)$ is $2(x-1) - z = 0$ or $2x - z - 2 = 0$

10. $z = f(x, y) = e^{-(x^2+y^2)} \Rightarrow f_x(x, y) = -2xe^{-(x^2+y^2)}$ and $f_y(x, y) = -2ye^{-(x^2+y^2)} \Rightarrow f_x(0, 0) = 0$ and $f_y(0, 0) = 0$ $\Rightarrow$ from Eq. (4) the tangent plane at $(0, 0, 1)$ is $z - 1 = 0$ or $z = 1$

11. $z = f(x, y) = \sqrt{y - x} \Rightarrow f_x(x, y) = -\frac{1}{2}(y - x)^{-1/2}$ and $f_y(x, y) = \frac{1}{2}(y - x)^{-1/2} \Rightarrow f_x(1, 2) = -\frac{1}{2}$ and $f_y(1, 2) = \frac{1}{2}$ $\Rightarrow$ from Eq. (4) the tangent plane at $(1, 2, 1)$ is $-\frac{1}{2}(x - 1) + \frac{1}{2}(y - 2) - (z - 1) = 0 \Rightarrow x - y + 2z - 1 = 0$

12. $z = f(x, y) = 4x^2 + y^2 \Rightarrow f_x(x, y) = 8x$ and $f_y(x, y) = 2y \Rightarrow f_x(1, 1) = 8$ and $f_y(1, 1) = 2 \Rightarrow$ from Eq. (4) the tangent plane at $(1, 1, 5)$ is $8(x - 1) + 2(y - 1) - (z - 5) = 0$ or $8x + 2y - z - 5 = 0$

13. $\nabla f = \mathbf{i} + 2y\mathbf{j} + 2\mathbf{k} \Rightarrow \nabla f(1, 1, 1) = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\nabla g = \mathbf{i}$ for all points; $\mathbf{v} = \nabla f \times \nabla g$
$$\Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 2\mathbf{j} - 2\mathbf{k} \Rightarrow \text{Tangent line: } x = 1,\ y = 1 + 2t,\ z = 1 - 2t$$

14. $\nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} \Rightarrow \nabla f(1, 1, 1) = \mathbf{i} + \mathbf{j} + \mathbf{k}$; $\nabla g = 2x\mathbf{i} + 4y\mathbf{j} + 6z\mathbf{k} \Rightarrow \nabla g(1, 1, 1) = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$;
$$\Rightarrow \mathbf{v} = \nabla f \times \nabla g \Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 4 & 6 \end{vmatrix} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} \Rightarrow \text{Tangent line: } x = 1 + 2t,\ y = 1 - 4t,\ z = 1 + 2t$$

15. $\nabla f = 2x\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow \nabla f\left(1, 1, \frac{1}{2}\right) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\nabla g = \mathbf{j}$ for all points; $\mathbf{v} = \nabla f \times \nabla g$
$$\Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{k} \Rightarrow \text{Tangent line: } x = 1 - 2t,\ y = 1,\ z = \tfrac{1}{2} + 2t$$

16. $\nabla f = \mathbf{i} + 2y\mathbf{j} + \mathbf{k} \Rightarrow \nabla f\left(\frac{1}{2}, 1, \frac{1}{2}\right) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\nabla g = \mathbf{j}$ for all points; $\mathbf{v} = \nabla f \times \nabla g$
$$\Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i} + \mathbf{k} \Rightarrow \text{Tangent line: } x = \tfrac{1}{2} - t,\ y = 1,\ z = \tfrac{1}{2} + t$$

17. $\nabla f = (3x^2 + 6xy^2 + 4y)\,\mathbf{i} + (6x^2y + 3y^2 + 4x)\,\mathbf{j} - 2z\mathbf{k} \Rightarrow \nabla f(1, 1, 3) = 13\mathbf{i} + 13\mathbf{j} - 6\mathbf{k}$; $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$
$$\Rightarrow \nabla g(1, 1, 3) = 2\mathbf{i} + 2\mathbf{j} + 6\mathbf{k};\ \mathbf{v} = \nabla f \times \nabla g \Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 13 & 13 & -6 \\ 2 & 2 & 6 \end{vmatrix} = 90\mathbf{i} - 90\mathbf{j} \Rightarrow \text{Tangent line:}$$
$x = 1 + 90t,\ y = 1 - 90t,\ z = 3$

18. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} \Rightarrow \nabla f\left(\sqrt{2}, \sqrt{2}, 4\right) = 2\sqrt{2}\,\mathbf{i} + 2\sqrt{2}\,\mathbf{j}$; $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow \nabla g\left(\sqrt{2}, \sqrt{2}, 4\right)$
$$= 2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} - \mathbf{k};\ \mathbf{v} = \nabla f \times \nabla g \Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\sqrt{2} & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 2\sqrt{2} & -1 \end{vmatrix} = -2\sqrt{2}\,\mathbf{i} + 2\sqrt{2}\,\mathbf{j} \Rightarrow \text{Tangent line:}$$
$x = \sqrt{2} - 2\sqrt{2}\,t,\ y = \sqrt{2} + 2\sqrt{2}\,t,\ z = 4$

19. $\nabla f = \left(\frac{x}{x^2+y^2+z^2}\right)\mathbf{i} + \left(\frac{y}{x^2+y^2+z^2}\right)\mathbf{j} + \left(\frac{z}{x^2+y^2+z^2}\right)\mathbf{k} \Rightarrow \nabla f(3, 4, 12) = \frac{3}{169}\,\mathbf{i} + \frac{4}{169}\,\mathbf{j} + \frac{12}{169}\,\mathbf{k}$;
$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i}+6\mathbf{j}-2\mathbf{k}}{\sqrt{3^2+6^2+(-2)^2}} = \frac{3}{7}\,\mathbf{i} + \frac{6}{7}\,\mathbf{j} - \frac{2}{7}\,\mathbf{k} \Rightarrow \nabla f \cdot \mathbf{u} = \frac{9}{1183}$ and $df = (\nabla f \cdot \mathbf{u})\,ds = \left(\frac{9}{1183}\right)(0.1) \approx 0.0008$

20. $\nabla f = (e^x \cos yz)\,\mathbf{i} - (ze^x \sin yz)\,\mathbf{j} - (ye^x \sin yz)\,\mathbf{k} \Rightarrow \nabla f(0, 0, 0) = \mathbf{i}$; $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2\mathbf{i}+2\mathbf{j}-2\mathbf{k}}{\sqrt{2^2+2^2+(-2)^2}}$
$= \frac{1}{\sqrt{3}}\,\mathbf{i} + \frac{1}{\sqrt{3}}\,\mathbf{j} - \frac{1}{\sqrt{3}}\,\mathbf{k} \Rightarrow \nabla f \cdot \mathbf{u} = \frac{1}{\sqrt{3}}$ and $df = (\nabla f \cdot \mathbf{u})\,ds = \frac{1}{\sqrt{3}}(0.1) \approx 0.0577$

21. $\nabla g = (1 + \cos z)\mathbf{i} + (1 - \sin z)\mathbf{j} + (-x \sin z - y \cos z)\mathbf{k} \Rightarrow \nabla g(2, -1, 0) = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$; $\mathbf{A} = \overrightarrow{P_0P_1} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
$\Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-2\mathbf{i}+2\mathbf{j}+2\mathbf{k}}{\sqrt{(-2)^2+2^2+2^2}} = -\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k} \Rightarrow \nabla g \cdot \mathbf{u} = 0$ and $dg = (\nabla g \cdot \mathbf{u})\, ds = (0)(0.2) = 0$

22. $\nabla h = [-\pi y \sin(\pi xy) + z^2]\,\mathbf{i} - [\pi x \sin(\pi xy)]\,\mathbf{j} + 2xz\mathbf{k} \Rightarrow \nabla h(-1, -1, -1) = (\pi \sin \pi + 1)\mathbf{i} + (\pi \sin \pi)\mathbf{j} + 2\mathbf{k}$
$= \mathbf{i} + 2\mathbf{k}$; $\mathbf{v} = \overrightarrow{P_0P_1} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ where $P_1 = (0, 0, 0) \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$
$\Rightarrow \nabla h \cdot \mathbf{u} = \frac{3}{\sqrt{3}} = \sqrt{3}$ and $dh = (\nabla h \cdot \mathbf{u})\, ds = \sqrt{3}(0.1) \approx 0.1732$

23. (a) The unit tangent vector at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ in the direction of motion is $\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$;
$\nabla T = (\sin 2y)\mathbf{i} + (2x \cos 2y)\mathbf{j} \Rightarrow \nabla T\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \left(\sin \sqrt{3}\right)\mathbf{i} + \left(\cos \sqrt{3}\right)\mathbf{j} \Rightarrow D_{\mathbf{u}}T\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \nabla T \cdot \mathbf{u}$
$= \frac{\sqrt{3}}{2} \sin \sqrt{3} - \frac{1}{2} \cos \sqrt{3} \approx 0.935°$ C/ft

(b) $\mathbf{r}(t) = (\sin 2t)\mathbf{i} + (\cos 2t)\mathbf{j} \Rightarrow \mathbf{v}(t) = (2 \cos 2t)\mathbf{i} - (2 \sin 2t)\mathbf{j}$ and $|\mathbf{v}| = 2$; $\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$
$= \nabla T \cdot \mathbf{v} = \left(\nabla T \cdot \frac{\mathbf{v}}{|\mathbf{v}|}\right) |\mathbf{v}| = (D_{\mathbf{u}}T)\, |\mathbf{v}|$, where $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$; at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ we have $\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$ from part (a)
$\Rightarrow \frac{dT}{dt} = \left(\frac{\sqrt{3}}{2} \sin \sqrt{3} - \frac{1}{2} \cos \sqrt{3}\right) \cdot 2 = \sqrt{3} \sin \sqrt{3} - \cos \sqrt{3} \approx 1.87°$ C/sec

24. (a) $\nabla T = (4x - yz)\mathbf{i} - xz\mathbf{j} - xy\mathbf{k} \Rightarrow \nabla T(8, 6, -4) = 56\mathbf{i} + 32\mathbf{j} - 48\mathbf{k}$; $\mathbf{r}(t) = 2t^2\mathbf{i} + 3t\mathbf{j} - t^2\mathbf{k} \Rightarrow$ the particle is at the point $P(8, 6, -4)$ when $t = 2$; $\mathbf{v}(t) = 4t\mathbf{i} + 3\mathbf{j} - 2t\mathbf{k} \Rightarrow \mathbf{v}(2) = 8\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= \frac{8}{\sqrt{89}}\mathbf{i} + \frac{3}{\sqrt{89}}\mathbf{j} - \frac{4}{\sqrt{89}}\mathbf{k} \Rightarrow D_{\mathbf{u}}T(8, 6, -4) = \nabla T \cdot \mathbf{u} = \frac{1}{\sqrt{89}}[56 \cdot 8 + 32 \cdot 3 - 48 \cdot (-4)] = \frac{736}{\sqrt{89}}$ ° C/m

(b) $\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = \nabla T \cdot \mathbf{v} = (\nabla T \cdot \mathbf{u})\, |\mathbf{v}| \Rightarrow$ at $t = 2$, $\frac{dT}{dt} = D_{\mathbf{u}}T\big|_{t=2}\, \mathbf{v}(2) = \left(\frac{736}{\sqrt{89}}\right)\sqrt{89} = 736°$ C/sec

25. (a) $f(0, 0) = 1$, $f_x(x, y) = 2x \Rightarrow f_x(0, 0) = 0$, $f_y(x, y) = 2y \Rightarrow f_y(0, 0) = 0 \Rightarrow L(x, y) = 1 + 0(x - 0) + 0(y - 0) = 1$

(b) $f(1, 1) = 3$, $f_x(1, 1) = 2$, $f_y(1, 1) = 2 \Rightarrow L(x, y) = 3 + 2(x - 1) + 2(y - 1) = 2x + 2y - 1$

26. (a) $f(0, 0) = 4$, $f_x(x, y) = 2(x + y + 2) \Rightarrow f_x(0, 0) = 4$, $f_y(x, y) = 2(x + y + 2) \Rightarrow f_y(0, 0) = 4$
$\Rightarrow L(x, y) = 4 + 4(x - 0) + 4(y - 0) = 4x + 4y + 4$

(b) $f(1, 2) = 25$, $f_x(1, 2) = 10$, $f_y(1, 2) = 10 \Rightarrow L(x, y) = 25 + 10(x - 1) + 10(y - 2) = 10x + 10y - 5$

27. (a) $f(0, 0) = 5$, $f_x(x, y) = 3$ for all (x, y), $f_y(x, y) = -4$ for all $(x, y) \Rightarrow L(x, y) = 5 + 3(x - 0) - 4(y - 0)$
$= 3x - 4y + 5$

(b) $f(1, 1) = 4$, $f_x(1, 1) = 3$, $f_y(1, 1) = -4 \Rightarrow L(x, y) = 4 + 3(x - 1) - 4(y - 1) = 3x - 4y + 5$

28. (a) $f(1, 1) = 1$, $f_x(x, y) = 3x^2y^4 \Rightarrow f_x(1, 1) = 3$, $f_y(x, y) = 4x^3y^3 \Rightarrow f_y(1, 1) = 4$
$\Rightarrow L(x, y) = 1 + 3(x - 1) + 4(y - 1) = 3x + 4y - 6$

(b) $f(0, 0) = 0$, $f_x(0, 0) = 0$, $f_y(0, 0) = 0 \Rightarrow L(x, y) = 0$

29. (a) $f(0, 0) = 1$, $f_x(x, y) = e^x \cos y \Rightarrow f_x(0, 0) = 1$, $f_y(x, y) = -e^x \sin y \Rightarrow f_y(0, 0) = 0$
$\Rightarrow L(x, y) = 1 + 1(x - 0) + 0(y - 0) = x + 1$

(b) $f\left(0, \frac{\pi}{2}\right) = 0$, $f_x\left(0, \frac{\pi}{2}\right) = 0$, $f_y\left(0, \frac{\pi}{2}\right) = -1 \Rightarrow L(x, y) = 0 + 0(x - 0) - 1\left(y - \frac{\pi}{2}\right) = -y + \frac{\pi}{2}$

30. (a) $f(0, 0) = 1$, $f_x(x, y) = -e^{2y-x} \Rightarrow f_x(0, 0) = -1$, $f_y(x, y) = 2e^{2y-x} \Rightarrow f_y(0, 0) = 2$
$\Rightarrow L(x, y) = 1 - 1(x - 0) + 2(y - 0) = -x + 2y + 1$

(b) $f(1, 2) = e^3$, $f_x(1, 2) = -e^3$, $f_y(1, 2) = 2e^3 \Rightarrow L(x, y) = e^3 - e^3(x - 1) + 2e^3(y - 2)$
$= -e^3x + 2e^3y - 2e^3$

31. $f(2,1)=3,\ f_x(x,y)=2x-3y \Rightarrow f_x(2,1)=1,\ f_y(x,y)=-3x \Rightarrow f_y(2,1)=-6 \Rightarrow L(x,y)=3+1(x-2)-6(y-1)$ $=7+x-6y;\ f_{xx}(x,y)=2,\ f_{yy}(x,y)=0,\ f_{xy}(x,y)=-3 \Rightarrow M=3$; thus $|E(x,y)| \le \left(\frac{1}{2}\right)(3)\left(|x-2|+|y-1|\right)^2$ $\le \left(\frac{3}{2}\right)(0.1+0.1)^2=0.06$

32. $f(2,2)=11,\ f_x(x,y)=x+y+3 \Rightarrow f_x(2,2)=7,\ f_y(x,y)=x+\frac{y}{2}-3 \Rightarrow f_y(2,2)=0$ $\Rightarrow L(x,y)=11+7(x-2)+0(y-2)=7x-3;\ f_{xx}(x,y)=1,\ f_{yy}(x,y)=\frac{1}{2},\ f_{xy}(x,y)=1$ $\Rightarrow M=1$; thus $|E(x,y)| \le \left(\frac{1}{2}\right)(1)\left(|x-2|+|y-2|\right)^2 \le \left(\frac{1}{2}\right)(0.1+0.1)^2=0.02$

33. $f(0,0)=1,\ f_x(x,y)=\cos y \Rightarrow f_x(0,0)=1,\ f_y(x,y)=1-x\sin y \Rightarrow f_y(0,0)=1$ $\Rightarrow L(x,y)=1+1(x-0)+1(y-0)=x+y+1;\ f_{xx}(x,y)=0,\ f_{yy}(x,y)=-x\cos y,\ f_{xy}(x,y)=-\sin y \Rightarrow M=1$; thus $|E(x,y)| \le \left(\frac{1}{2}\right)(1)\left(|x|+|y|\right)^2 \le \left(\frac{1}{2}\right)(0.2+0.2)^2=0.08$

34. $f(1,2)=6,\ f_x(x,y)=y^2-y\sin(x-1) \Rightarrow f_x(1,2)=4,\ f_y(x,y)=2xy+\cos(x-1) \Rightarrow f_y(1,2)=5$ $\Rightarrow L(x,y)=6+4(x-1)+5(y-2)=4x+5y-8;\ f_{xx}(x,y)=-y\cos(x-1),\ f_{yy}(x,y)=2x,$ $f_{xy}(x,y)=2y-\sin(x-1);\ |x-1|\le 0.1 \Rightarrow 0.9\le x\le 1.1$ and $|y-2|\le 0.1 \Rightarrow 1.9\le y\le 2.1$; thus the max of $|f_{xx}(x,y)|$ on R is 2.1, the max of $|f_{yy}(x,y)|$ on R is 2.2, and the max of $|f_{xy}(x,y)|$ on R is $2(2.1)-\sin(0.9-1)$ $\le 4.3 \Rightarrow M=4.3$; thus $|E(x,y)| \le \left(\frac{1}{2}\right)(4.3)\left(|x-1|+|y-2|\right)^2 \le (2.15)(0.1+0.1)^2=0.086$

35. $f(0,0)=1,\ f_x(x,y)=e^x\cos y \Rightarrow f_x(0,0)=1,\ f_y(x,y)=-e^x\sin y \Rightarrow f_y(0,0)=0$ $\Rightarrow L(x,y)=1+1(x-0)+0(y-0)=1+x;\ f_{xx}(x,y)=e^x\cos y,\ f_{yy}(x,y)=-e^x\cos y,\ f_{xy}(x,y)=-e^x\sin y;$ $|x|\le 0.1 \Rightarrow -0.1\le x\le 0.1$ and $|y|\le 0.1 \Rightarrow -0.1\le y\le 0.1$; thus the max of $|f_{xx}(x,y)|$ on R is $e^{0.1}\cos(0.1)$ ≤ 1.11, the max of $|f_{yy}(x,y)|$ on R is $e^{0.1}\cos(0.1)\le 1.11$, and the max of $|f_{xy}(x,y)|$ on R is $e^{0.1}\sin(0.1)$ $\le 0.12 \Rightarrow M=1.11$; thus $|E(x,y)| \le \left(\frac{1}{2}\right)(1.11)\left(|x|+|y|\right)^2 \le (0.555)(0.1+0.1)^2=0.0222$

36. $f(1,1)=0,\ f_x(x,y)=\frac{1}{x} \Rightarrow f_x(1,1)=1,\ f_y(x,y)=\frac{1}{y} \Rightarrow f_y(1,1)=1 \Rightarrow L(x,y)=0+1(x-1)+1(y-1)$ $=x+y-2;\ f_{xx}(x,y)=-\frac{1}{x^2},\ f_{yy}(x,y)=-\frac{1}{y^2},\ f_{xy}(x,y)=0;\ |x-1|\le 0.2 \Rightarrow 0.98\le x\le 1.2$ so the max of $|f_{xx}(x,y)|$ on R is $\frac{1}{(0.98)^2}\le 1.04;\ |y-1|\le 0.2 \Rightarrow 0.98\le y\le 1.2$ so the max of $|f_{yy}(x,y)|$ on R is $\frac{1}{(0.98)^2}\le 1.04 \Rightarrow M=1.04$; thus $|E(x,y)| \le \left(\frac{1}{2}\right)(1.04)\left(|x-1|+|y-1|\right)^2 \le (0.52)(0.2+0.2)^2=0.0832$

37. (a) $f(1,1,1)=3,\ f_x(1,1,1)=y+z|_{(1,1,1)}=2,\ f_y(1,1,1)=x+z|_{(1,1,1)}=2,\ f_z(1,1,1)=y+x|_{(1,1,1)}=2$ $\Rightarrow L(x,y,z)=3+2(x-1)+2(y-1)+2(z-1)=2x+2y+2z-3$
(b) $f(1,0,0)=0,\ f_x(1,0,0)=0,\ f_y(1,0,0)=1,\ f_z(1,0,0)=1 \Rightarrow L(x,y,z)=0+0(x-1)+(y-0)+(z-0)$ $=y+z$
(c) $f(0,0,0)=0,\ f_x(0,0,0)=0,\ f_y(0,0,0)=0,\ f_z(0,0,0)=0 \Rightarrow L(x,y,z)=0$

38. (a) $f(1,1,1)=3,\ f_x(1,1,1)=2x|_{(1,1,1)}=2,\ f_y(1,1,1)=2y|_{(1,1,1)}=2,\ f_z(1,1,1)=2z|_{(1,1,1)}=2$ $\Rightarrow L(x,y,z)=3+2(x-1)+2(y-1)+2(z-1)=2x+2y+2z-3$
(b) $f(0,1,0)=1,\ f_x(0,1,0)=0,\ f_y(0,1,0)=2,\ f_z(0,1,0)=0 \Rightarrow L(x,y,z)=1+0(x-0)+2(y-1)+0(z-0)$ $=2y-1$
(c) $f(1,0,0)=1,\ f_x(1,0,0)=2,\ f_y(1,0,0)=0,\ f_z(1,0,0)=0 \Rightarrow L(x,y,z)=1+2(x-1)+0(y-0)+0(z-0)$ $=2x-1$

39. (a) $f(1,0,0)=1,\ f_x(1,0,0)=\left.\frac{x}{\sqrt{x^2+y^2+z^2}}\right|_{(1,0,0)}=1,\ f_y(1,0,0)=\left.\frac{y}{\sqrt{x^2+y^2+z^2}}\right|_{(1,0,0)}=0,$ $f_z(1,0,0)=\left.\frac{z}{\sqrt{x^2+y^2+z^2}}\right|_{(1,0,0)}=0 \Rightarrow L(x,y,z)=1+1(x-1)+0(y-0)+0(z-0)=x$

(b) $f(1,1,0) = \sqrt{2}$, $f_x(1,1,0) = \frac{1}{\sqrt{2}}$, $f_y(1,1,0) = \frac{1}{\sqrt{2}}$, $f_z(1,1,0) = 0$
$\Rightarrow L(x,y,z) = \sqrt{2} + \frac{1}{\sqrt{2}}(x-1) + \frac{1}{\sqrt{2}}(y-1) + 0(z-0) = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y$

(c) $f(1,2,2) = 3$, $f_x(1,2,2) = \frac{1}{3}$, $f_y(1,2,2) = \frac{2}{3}$, $f_z(1,2,2) = \frac{2}{3} \Rightarrow L(x,y,z) = 3 + \frac{1}{3}(x-1) + \frac{2}{3}(y-2) + \frac{2}{3}(z-2)$
$= \frac{1}{3}x + \frac{2}{3}y + \frac{2}{3}z$

40. (a) $f\left(\frac{\pi}{2},1,1\right) = 1$, $f_x\left(\frac{\pi}{2},1,1\right) = \left.\frac{y\cos xy}{z}\right|_{\left(\frac{\pi}{2},1,1\right)} = 0$, $f_y\left(\frac{\pi}{2},1,1\right) = \left.\frac{x\cos xy}{z}\right|_{\left(\frac{\pi}{2},1,1\right)} = 0$,
$f_z\left(\frac{\pi}{2},1,1\right) = \left.\frac{-\sin xy}{z^2}\right|_{\left(\frac{\pi}{2},1,1\right)} = -1 \Rightarrow L(x,y,z) = 1 + 0\left(x-\frac{\pi}{2}\right) + 0(y-1) - 1(z-1) = 2 - z$

(b) $f(2,0,1) = 0$, $f_x(2,0,1) = 0$, $f_y(2,0,1) = 2$, $f_z(2,0,1) = 0 \Rightarrow L(x,y,z) = 0 + 0(x-2) + 2(y-0) + 0(z-1) = 2y$

41. (a) $f(0,0,0) = 2$, $f_x(0,0,0) = \left.e^x\right|_{(0,0,0)} = 1$, $f_y(0,0,0) = \left.-\sin(y+z)\right|_{(0,0,0)} = 0$,
$f_z(0,0,0) = \left.-\sin(y+z)\right|_{(0,0,0)} = 0 \Rightarrow L(x,y,z) = 2 + 1(x-0) + 0(y-0) + 0(z-0) = 2 + x$

(b) $f\left(0,\frac{\pi}{2},0\right) = 1$, $f_x\left(0,\frac{\pi}{2},0\right) = 1$, $f_y\left(0,\frac{\pi}{2},0\right) = -1$, $f_z\left(0,\frac{\pi}{2},0\right) = -1 \Rightarrow L(x,y,z)$
$= 1 + 1(x-0) - 1\left(y-\frac{\pi}{2}\right) - 1(z-0) = x - y - z + \frac{\pi}{2} + 1$

(c) $f\left(0,\frac{\pi}{4},\frac{\pi}{4}\right) = 1$, $f_x\left(0,\frac{\pi}{4},\frac{\pi}{4}\right) = 1$, $f_y\left(0,\frac{\pi}{4},\frac{\pi}{4}\right) = -1$, $f_z\left(0,\frac{\pi}{4},\frac{\pi}{4}\right) = -1 \Rightarrow L(x,y,z)$
$= 1 + 1(x-0) - 1\left(y-\frac{\pi}{4}\right) - 1\left(z-\frac{\pi}{4}\right) = x - y - z + \frac{\pi}{2} + 1$

42. (a) $f(1,0,0) = 0$, $f_x(1,0,0) = \left.\frac{yz}{(xyz)^2+1}\right|_{(1,0,0)} = 0$, $f_y(1,0,0) = \left.\frac{xz}{(xyz)^2+1}\right|_{(1,0,0)} = 0$,
$f_z(1,0,0) = \left.\frac{xy}{(xyz)^2+1}\right|_{(1,0,0)} = 0 \Rightarrow L(x,y,z) = 0$

(b) $f(1,1,0) = 0$, $f_x(1,1,0) = 0$, $f_y(1,1,0) = 0$, $f_z(1,1,0) = 1 \Rightarrow L(x,y,z) = 0 + 0(x-1) + 0(y-1) + 1(z-0) = z$

(c) $f(1,1,1) = \frac{\pi}{4}$, $f_x(1,1,1) = \frac{1}{2}$, $f_y(1,1,1) = \frac{1}{2}$, $f_z(1,1,1) = \frac{1}{2} \Rightarrow L(x,y,z) = \frac{\pi}{4} + \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{2}(z-1)$
$= \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z + \frac{\pi}{4} - \frac{3}{2}$

43. $f(x,y,z) = xz - 3yz + 2$ at $P_0(1,1,2) \Rightarrow f(1,1,2) = -2$; $f_x = z$, $f_y = -3z$, $f_z = x - 3y \Rightarrow L(x,y,z)$
$= -2 + 2(x-1) - 6(y-1) - 2(z-2) = 2x - 6y - 2z + 6$; $f_{xx} = 0$, $f_{yy} = 0$, $f_{zz} = 0$, $f_{xy} = 0$, $f_{yz} = -3$
$\Rightarrow M = 3$; thus, $|E(x,y,z)| \le \left(\frac{1}{2}\right)(3)(0.01 + 0.01 + 0.02)^2 = 0.0024$

44. $f(x,y,z) = x^2 + xy + yz + \frac{1}{4}z^2$ at $P_0(1,1,2) \Rightarrow f(1,1,2) = 5$; $f_x = 2x + y$, $f_y = x + z$, $f_z = y + \frac{1}{2}z$
$\Rightarrow L(x,y,z) = 5 + 3(x-1) + 3(y-1) + 2(z-2) = 3x + 3y + 2z - 5$; $f_{xx} = 2$, $f_{yy} = 0$, $f_{zz} = \frac{1}{2}$, $f_{xy} = 1$, $f_{xz} = 0$,
$f_{yz} = 1 \Rightarrow M = 2$; thus $|E(x,y,z)| \le \left(\frac{1}{2}\right)(2)(0.01 + 0.01 + 0.08)^2 = 0.01$

45. $f(x,y,z) = xy + 2yz - 3xz$ at $P_0(1,1,0) \Rightarrow f(1,1,0) = 1$; $f_x = y - 3z$, $f_y = x + 2z$, $f_z = 2y - 3x$
$\Rightarrow L(x,y,z) = 1 + (x-1) + (y-1) - (z-0) = x + y - z - 1$; $f_{xx} = 0$, $f_{yy} = 0$, $f_{zz} = 0$, $f_{xy} = 1$, $f_{xz} = -3$,
$f_{yz} = 2 \Rightarrow M = 3$; thus $|E(x,y,z)| \le \left(\frac{1}{2}\right)(3)(0.01 + 0.01 + 0.01)^2 = 0.00135$

46. $f(x,y,z) = \sqrt{2}\cos x \sin(y+z)$ at $P_0\left(0,0,\frac{\pi}{4}\right) \Rightarrow f\left(0,0,\frac{\pi}{4}\right) = 1$; $f_x = -\sqrt{2}\sin x \sin(y+z)$,
$f_y = \sqrt{2}\cos x \cos(y+z)$, $f_z = \sqrt{2}\cos x \cos(y+z) \Rightarrow L(x,y,z) = 1 - 0(x-0) + (y-0) + \left(z-\frac{\pi}{4}\right)$
$= y + z - \frac{\pi}{4} + 1$; $f_{xx} = -\sqrt{2}\cos x \sin(y+z)$, $f_{yy} = -\sqrt{2}\cos x \sin(y+z)$, $f_{zz} = -\sqrt{2}\cos x \sin(y+z)$,
$f_{xy} = -\sqrt{2}\sin x \cos(y+z)$, $f_{xz} = -\sqrt{2}\sin x \cos(y+z)$, $f_{yz} = -\sqrt{2}\cos x \sin(y+z)$. The absolute value of each of these second partial derivatives is bounded above by $\sqrt{2} \Rightarrow M = \sqrt{2}$; thus $|E(x,y,z)|$
$\le \left(\frac{1}{2}\right)\left(\sqrt{2}\right)(0.01 + 0.01 + 0.01)^2 = 0.000636.$

47. $T_x(x, y) = e^y + e^{-y}$ and $T_y(x, y) = x(e^y - e^{-y}) \Rightarrow dT = T_x(x, y)\,dx + T_y(x, y)\,dy$ $= (e^y + e^{-y})\,dx + x(e^y - e^{-y})\,dy \Rightarrow dT|_{(2,\ln 2)} = 2.5\,dx + 3.0\,dy$. If $|dx| \le 0.1$ and $|dy| \le 0.02$, then the maximum possible error in the computed value of T is $(2.5)(0.1) + (3.0)(0.02) = 0.31$ in magnitude.

48. $V_r = 2\pi rh$ and $V_h = \pi r^2 \Rightarrow dV = V_r\,dr + V_h\,dh \Rightarrow \frac{dV}{V} = \frac{2\pi rh\,dr + \pi r^2\,dh}{\pi r^2 h} = \frac{2}{r}\,dr + \frac{1}{h}\,dh$; now $\left|\frac{dr}{r} \cdot 100\right| \le 1$ and $\left|\frac{dh}{h} \cdot 100\right| \le 1 \Rightarrow \left|\frac{dV}{V} \cdot 100\right| \le \left|\left(2\,\frac{dr}{r}\right)(100) + \left(\frac{dh}{h}\right)(100)\right| \le 2\left|\frac{dr}{r} \cdot 100\right| + \left|\frac{dh}{h} \cdot 100\right| \le 2(1) + 1 = 3 \Rightarrow 3\%$

49. $V_r = 2\pi rh$ and $V_h = \pi r^2 \Rightarrow dV = V_r\,dr + V_h\,dh \Rightarrow dV = 2\pi rh\,dr + \pi r^2\,dh \Rightarrow dV|_{(5,12)} = 120\pi\,dr + 25\pi\,dh$; $|dr| \le 0.1$ cm and $|dh| \le 0.1$ cm $\Rightarrow dV \le (120\pi)(0.1) + (25\pi)(0.1) = 14.5\pi\ \text{cm}^3$; $V(5, 12) = 300\pi\ \text{cm}^3$ $\Rightarrow$ maximum percentage error is $\pm\frac{14.5\pi}{300\pi} \times 100 = \pm 4.83\%$

50. (a) $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow -\frac{1}{R^2}\,dR = -\frac{1}{R_1^2}\,dR_1 - \frac{1}{R_2^2}\,dR_2 \Rightarrow dR = \left(\frac{R}{R_1}\right)^2 dR_1 + \left(\frac{R}{R_2}\right)^2 dR_2$

(b) $dR = R^2\left[\left(\frac{1}{R_1^2}\right)dR_1 + \left(\frac{1}{R_2^2}\right)dR_2\right] \Rightarrow dR|_{(100,400)} = R^2\left[\frac{1}{(100)^2}\,dR_1 + \frac{1}{(400)^2}\,dR_2\right] \Rightarrow$ R will be more sensitive to a variation in R_1 since $\frac{1}{(100)^2} > \frac{1}{(400)^2}$

(c) From part (a), $dR = \left(\frac{R}{R_1}\right)^2 dR_1 + \left(\frac{R}{R_2}\right)^2 dR_2$ so that R_1 changing from 20 to 20.1 ohms $\Rightarrow dR_1 = 0.1$ ohm and R_2 changing from 25 to 24.9 ohms $\Rightarrow dR_2 = -0.1$ ohms; $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow R = \frac{100}{9}$ ohms $\Rightarrow dR|_{(20,25)} = \frac{\left(\frac{100}{9}\right)^2}{(20)^2}(0.1) + \frac{\left(\frac{100}{9}\right)^2}{(25)^2}(-0.1) \approx 0.011$ ohms $\Rightarrow$ percentage change is $\left.\frac{dR}{R}\right|_{(20,25)} \times 100$ $= \frac{0.011}{\left(\frac{100}{9}\right)} \times 100 \approx 0.1\%$

51. $A = xy \Rightarrow dA = x\,dy + y\,dx$; if $x > y$ then a 1-unit change in y gives a greater change in dA than a 1-unit change in x. Thus, pay more attention to y which is the smaller of the two dimensions.

52. (a) $f_x(x, y) = 2x(y + 1) \Rightarrow f_x(1, 0) = 2$ and $f_y(x, y) = x^2 \Rightarrow f_y(1, 0) = 1 \Rightarrow df = 2\,dx + 1\,dy \Rightarrow$ df is more sensitive to changes in x

(b) $df = 0 \Rightarrow 2\,dx + dy = 0 \Rightarrow 2\,\frac{dx}{dy} + 1 = 0 \Rightarrow \frac{dx}{dy} = -\frac{1}{2}$

53. (a) $r^2 = x^2 + y^2 \Rightarrow 2r\,dr = 2x\,dx + 2y\,dy \Rightarrow dr = \frac{x}{r}\,dx + \frac{y}{r}\,dy \Rightarrow dr|_{(3,4)} = \left(\frac{3}{5}\right)(\pm 0.01) + \left(\frac{4}{5}\right)(\pm 0.01)$ $= \pm\frac{0.07}{5} = \pm 0.014 \Rightarrow \left|\frac{dr}{r} \times 100\right| = \left|\pm\frac{0.014}{5} \times 100\right| = 0.28\%$; $d\theta = \frac{\left(-\frac{y}{x^2}\right)}{\left(\frac{y}{x}\right)^2 + 1}\,dx + \frac{\left(\frac{1}{x}\right)}{\left(\frac{y}{x}\right)^2 + 1}\,dy$ $= \frac{-y}{y^2 + x^2}\,dx + \frac{x}{y^2 + x^2}\,dy \Rightarrow d\theta|_{(3,4)} = \left(\frac{-4}{25}\right)(\pm 0.01) + \left(\frac{3}{25}\right)(\pm 0.01) = \frac{\mp 0.04}{25} + \frac{\pm 0.03}{25}$ $\Rightarrow$ maximum change in $d\theta$ occurs when dx and dy have opposite signs ($dx = 0.01$ and $dy = -0.01$ or vice versa) $\Rightarrow d\theta = \frac{\pm 0.07}{25} \approx \pm 0.0028$; $\theta = \tan^{-1}\left(\frac{4}{3}\right) \approx 0.927255218 \Rightarrow \left|\frac{d\theta}{\theta} \times 100\right| = \left|\frac{\pm 0.0028}{0.927255218} \times 100\right|$ $\approx 0.30\%$

(b) the radius r is more sensitive to changes in y, and the angle θ is more sensitive to changes in x

54. (a) $V = \pi r^2 h \Rightarrow dV = 2\pi rh\,dr + \pi r^2\,dh \Rightarrow$ at $r = 1$ and $h = 5$ we have $dV = 10\pi\,dr + \pi\,dh \Rightarrow$ the volume is about 10 times more sensitive to a change in r

(b) $dV = 0 \Rightarrow 0 = 2\pi rh\,dr + \pi r^2\,dh = 2h\,dr + r\,dh = 10\,dr + dh \Rightarrow dr = -\frac{1}{10}\,dh$; choose $dh = 1.5$ $\Rightarrow dr = -0.15 \Rightarrow h = 6.5$ in. and $r = 0.85$ in. is one solution for $\Delta V \approx dV = 0$

55. $f(a, b, c, d) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \Rightarrow f_a = d,\ f_b = -c,\ f_c = -b,\ f_d = a \Rightarrow df = d\,da - c\,db - b\,dc + a\,dd$; since $|a|$ is much greater than $|b|$, $|c|$, and $|d|$, the function f is most sensitive to a change in d.

56. $u_x = e^y$, $u_y = xe^y + \sin z$, $u_z = y \cos z \Rightarrow du = e^y\,dx + (xe^y + \sin z)\,dy + (y \cos z)\,dz$
$\Rightarrow du|_{(2,\ln 3,\frac{\pi}{2})} = 3\,dx + 7\,dy + 0\,dz = 3\,dx + 7\,dy \Rightarrow$ magnitude of the maximum possible error
$\le 3(0.2) + 7(0.6) = 4.8$

57. $Q_K = \frac{1}{2}\left(\frac{2KM}{h}\right)^{-1/2}\left(\frac{2M}{h}\right)$, $Q_M = \frac{1}{2}\left(\frac{2KM}{h}\right)^{-1/2}\left(\frac{2K}{h}\right)$, and $Q_h = \frac{1}{2}\left(\frac{2KM}{h}\right)^{-1/2}\left(\frac{-2KM}{h^2}\right)$
$\Rightarrow dQ = \frac{1}{2}\left(\frac{2KM}{h}\right)^{-1/2}\left(\frac{2M}{h}\right)dK + \frac{1}{2}\left(\frac{2KM}{h}\right)^{-1/2}\left(\frac{2K}{h}\right)dM + \frac{1}{2}\left(\frac{2KM}{h}\right)^{-1/2}\left(\frac{-2KM}{h^2}\right)dh$
$= \frac{1}{2}\left(\frac{2KM}{h}\right)^{-1/2}\left[\frac{2M}{h}\,dK + \frac{2K}{h}\,dM - \frac{2KM}{h^2}\,dh\right] \Rightarrow dQ|_{(2,20,0.05)}$
$= \frac{1}{2}\left[\frac{(2)(2)(20)}{0.05}\right]^{-1/2}\left[\frac{(2)(20)}{0.05}\,dK + \frac{(2)(2)}{0.05}\,dM - \frac{(2)(2)(20)}{(0.05)^2}\,dh\right] = (0.0125)(800\,dK + 80\,dM - 32{,}000\,dh)$
$\Rightarrow$ Q is most sensitive to changes in h

58. $A = \frac{1}{2}ab \sin C \Rightarrow A_a = \frac{1}{2}b \sin C$, $A_b = \frac{1}{2}a \sin C$, $A_c = \frac{1}{2}ab \cos C$
$\Rightarrow dA = \left(\frac{1}{2}b \sin C\right)da + \left(\frac{1}{2}a \sin C\right)db + \left(\frac{1}{2}ab \cos C\right)dC$; $dC = |2°| = |0.0349|$ radians, $da = |0.5|$ ft,
$db = |0.5|$ ft; at $a = 150$ ft, $b = 200$ ft, and $C = 60°$, we see that the change is approximately
$dA = \frac{1}{2}(200)(\sin 60°)\,|0.5| + \frac{1}{2}(150)(\sin 60°)\,|0.5| + \frac{1}{2}(200)(150)(\cos 60°)\,|0.0349| = \pm 338\text{ ft}^2$

59. $z = f(x,y) \Rightarrow g(x,y,z) = f(x,y) - z = 0 \Rightarrow g_x(x,y,z) = f_x(x,y)$, $g_y(x,y,z) = f_y(x,y)$ and $g_z(x,y,z) = -1$
$\Rightarrow g_x(x_0,y_0,f(x_0,y_0)) = f_x(x_0,y_0)$, $g_y(x_0,y_0,f(x_0,y_0)) = f_y(x_0,y_0)$ and $g_z(x_0,y_0,f(x_0,y_0)) = -1 \Rightarrow$ the tangent plane at the point P_0 is $f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - [z - f(x_0,y_0)] = 0$ or
$z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f(x_0,y_0)$

60. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} = 2(\cos t + t \sin t)\mathbf{i} + 2(\sin t - t \cos t)\mathbf{j}$ and $\mathbf{v} = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= \frac{(t \cos t)\mathbf{i} + (t \sin t)\mathbf{j}}{\sqrt{(t \cos t)^2 + (t \sin t)^2}} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ since $t > 0 \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u}$
$= 2(\cos t + t \sin t)(\cos t) + 2(\sin t - t \cos t)(\sin t) = 2$

61. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + 2t\mathbf{k}$ and $\mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= \frac{(-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}}{\sqrt{(\sin t)^2 + (\cos t)^2 + 1^2}} = \left(\frac{-\sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{\cos t}{\sqrt{2}}\right)\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u}$
$= (2 \cos t)\left(\frac{-\sin t}{\sqrt{2}}\right) + (2 \sin t)\left(\frac{\cos t}{\sqrt{2}}\right) + (2t)\left(\frac{1}{\sqrt{2}}\right) = \frac{2t}{\sqrt{2}} \Rightarrow (D_{\mathbf{u}}f)\left(\frac{-\pi}{4}\right) = \frac{-\pi}{2\sqrt{2}}$, $(D_{\mathbf{u}}f)(0) = 0$ and
$(D_{\mathbf{u}}f)\left(\frac{\pi}{4}\right) = \frac{\pi}{2\sqrt{2}}$

62. $\mathbf{r} = \sqrt{t}\mathbf{i} + \sqrt{t}\mathbf{j} - \frac{1}{4}(t+3)\mathbf{k} \Rightarrow \mathbf{v} = \frac{1}{2}t^{-1/2}\mathbf{i} + \frac{1}{2}t^{-1/2}\mathbf{j} - \frac{1}{4}\mathbf{k}$; $t = 1 \Rightarrow x = 1, y = 1, z = -1 \Rightarrow P_0 = (1,1,-1)$
and $\mathbf{v}(1) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \frac{1}{4}\mathbf{k}$; $f(x,y,z) = x^2 + y^2 - z - 3 = 0 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$
$\Rightarrow \nabla f(1,1,-1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$; therefore $\mathbf{v} = \frac{1}{4}(\nabla f) \Rightarrow$ the curve is normal to the surface

63. $\mathbf{r} = \sqrt{t}\mathbf{i} + \sqrt{t}\mathbf{j} + (2t-1)\mathbf{k} \Rightarrow \mathbf{v} = \frac{1}{2}t^{-1/2}\mathbf{i} + \frac{1}{2}t^{-1/2}\mathbf{j} + 2\mathbf{k}$; $t = 1 \Rightarrow x = 1, y = 1, z = 1 \Rightarrow P_0 = (1,1,1)$ and
$\mathbf{v}(1) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + 2\mathbf{k}$; $f(x,y,z) = x^2 + y^2 - z - 1 = 0 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow \nabla f(1,1,1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$;
now $\mathbf{v}(1) \cdot \nabla f(1,1,1) = 0$, thus the curve is tangent to the surface when $t = 1$

14.7 EXTREME VALUES AND SADDLE POINTS

1. $f_x(x,y) = 2x + y + 3 = 0$ and $f_y(x,y) = x + 2y - 3 = 0 \Rightarrow x = -3$ and $y = 3 \Rightarrow$ critical point is $(-3,3)$;
$f_{xx}(-3,3) = 2$, $f_{yy}(-3,3) = 2$, $f_{xy}(-3,3) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 3 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of
$f(-3,3) = -5$

2. $f_x(x,y) = 2x + 3y - 6 = 0$ and $f_y(x,y) = 3x + 6y + 3 = 0 \Rightarrow x = 15$ and $y = -8 \Rightarrow$ critical point is $(15,-8)$; $f_{xx}(15,-8) = 2$, $f_{yy}(15,-8) = 6$, $f_{xy}(15,-8) = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 3 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(15,-8) = -63$

3. $f_x(x,y) = 2y - 10x + 4 = 0$ and $f_y(x,y) = 2x - 4y + 4 = 0 \Rightarrow x = \frac{2}{3}$ and $y = \frac{4}{3} \Rightarrow$ critical point is $\left(\frac{2}{3},\frac{4}{3}\right)$; $f_{xx}\left(\frac{2}{3},\frac{4}{3}\right) = -10$, $f_{yy}\left(\frac{2}{3},\frac{4}{3}\right) = -4$, $f_{xy}\left(\frac{2}{3},\frac{4}{3}\right) = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f\left(\frac{2}{3},\frac{4}{3}\right) = 0$

4. $f_x(x,y) = 2y - 10x + 4 = 0$ and $f_y(x,y) = 2x - 4y = 0 \Rightarrow x = \frac{4}{9}$ and $y = \frac{2}{9} \Rightarrow$ critical point is $\left(\frac{4}{9},\frac{2}{9}\right)$; $f_{xx}\left(\frac{4}{9},\frac{2}{9}\right) = -10$, $f_{yy}\left(\frac{4}{9},\frac{2}{9}\right) = -4$, $f_{xy}\left(\frac{4}{9},\frac{2}{9}\right) = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f\left(\frac{4}{9},\frac{2}{9}\right) = -\frac{28}{9}$

5. $f_x(x,y) = 2x + y + 3 = 0$ and $f_y(x,y) = x + 2 = 0 \Rightarrow x = -2$ and $y = 1 \Rightarrow$ critical point is $(-2,1)$; $f_{xx}(-2,1) = 2$, $f_{yy}(-2,1) = 0$, $f_{xy}(-2,1) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -1 < 0 \Rightarrow$ saddle point

6. $f_x(x,y) = y - 2 = 0$ and $f_y(x,y) = 2y + x - 2 = 0 \Rightarrow x = -2$ and $y = 2 \Rightarrow$ critical point is $(-2,2)$; $f_{xx}(-2,2) = 0$, $f_{yy}(-2,2) = 2$, $f_{xy}(-2,2) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -1 < 0 \Rightarrow$ saddle point

7. $f_x(x,y) = 5y - 14x + 3 = 0$ and $f_y(x,y) = 5x - 6 = 0 \Rightarrow x = \frac{6}{5}$ and $y = \frac{69}{25} \Rightarrow$ critical point is $\left(\frac{6}{5},\frac{69}{25}\right)$; $f_{xx}\left(\frac{6}{5},\frac{69}{25}\right) = -14$, $f_{yy}\left(\frac{6}{5},\frac{69}{25}\right) = 0$, $f_{xy}\left(\frac{6}{5},\frac{69}{25}\right) = 5 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -25 < 0 \Rightarrow$ saddle point

8. $f_x(x,y) = 2y - 2x + 3 = 0$ and $f_y(x,y) = 2x - 4y = 0 \Rightarrow x = 3$ and $y = \frac{3}{2} \Rightarrow$ critical point is $\left(3,\frac{3}{2}\right)$; $f_{xx}\left(3,\frac{3}{2}\right) = -2$, $f_{yy}\left(3,\frac{3}{2}\right) = -4$, $f_{xy}\left(3,\frac{3}{2}\right) = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f\left(3,\frac{3}{2}\right) = \frac{17}{2}$

9. $f_x(x,y) = 2x - 4y = 0$ and $f_y(x,y) = -4x + 2y + 6 = 0 \Rightarrow x = 2$ and $y = 1 \Rightarrow$ critical point is $(2,1)$; $f_{xx}(2,1) = 2$, $f_{yy}(2,1) = 2$, $f_{xy}(2,1) = -4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -12 < 0 \Rightarrow$ saddle point

10. $f_x(x,y) = 6x + 6y - 2 = 0$ and $f_y(x,y) = 6x + 14y + 4 = 0 \Rightarrow x = \frac{13}{12}$ and $y = -\frac{3}{4} \Rightarrow$ critical point is $\left(\frac{13}{12},-\frac{3}{4}\right)$; $f_{xx}\left(\frac{13}{12},-\frac{3}{4}\right) = 6$, $f_{yy}\left(\frac{13}{12},-\frac{3}{4}\right) = 14$, $f_{xy}\left(\frac{13}{12},-\frac{3}{4}\right) = 6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 48 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f\left(\frac{13}{12},-\frac{3}{4}\right) = -\frac{31}{12}$

11. $f_x(x,y) = 4x + 3y - 5 = 0$ and $f_y(x,y) = 3x + 8y + 2 = 0 \Rightarrow x = 2$ and $y = -1 \Rightarrow$ critical point is $(2,-1)$; $f_{xx}(2,-1) = 4$, $f_{yy}(2,-1) = 8$, $f_{xy}(2,-1) = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 23 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(2,-1) = -6$

12. $f_x(x,y) = 8x - 6y - 20 = 0$ and $f_y(x,y) = -6x + 10y + 26 = 0 \Rightarrow x = 1$ and $y = -2 \Rightarrow$ critical point is $(1,-2)$; $f_{xx}(1,-2) = 8$, $f_{yy}(1,-2) = 10$, $f_{xy}(1,-2) = -6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 44 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(1,-2) = -36$

13. $f_x(x,y) = 2x - 2 = 0$ and $f_y(x,y) = -2y + 4 = 0 \Rightarrow x = 1$ and $y = 2 \Rightarrow$ critical point is $(1,2)$; $f_{xx}(1,2) = 2$, $f_{yy}(1,2) = -2$, $f_{xy}(1,2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -4 < 0 \Rightarrow$ saddle point

14. $f_x(x,y) = 2x - 2y - 2 = 0$ and $f_y(x,y) = -2x + 4y + 2 = 0 \Rightarrow x = 1$ and $y = 0 \Rightarrow$ critical point is $(1,0)$; $f_{xx}(1,0) = 2$, $f_{yy}(1,0) = 4$, $f_{xy}(1,0) = -2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(1,0) = 0$

15. $f_x(x,y) = 2x + 2y = 0$ and $f_y(x,y) = 2x = 0 \Rightarrow x = 0$ and $y = 0 \Rightarrow$ critical point is $(0,0)$; $f_{xx}(0,0) = 2$, $f_{yy}(0,0) = 0$, $f_{xy}(0,0) = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -4 < 0 \Rightarrow$ saddle point

16. $f_x(x,y) = 2 - 4x - 2y = 0$ and $f_y(x,y) = 2 - 2x - 2y = 0 \Rightarrow x = 0$ and $y = 1 \Rightarrow$ critical point is $(0,1)$; $f_{xx}(0,1) = -4$, $f_{yy}(0,1) = -2$, $f_{xy}(0,1) = -2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(0,1) = 4$

17. $f_x(x,y) = 3x^2 - 2y = 0$ and $f_y(x,y) = -3y^2 - 2x = 0 \Rightarrow x = 0$ and $y = 0$, or $x = -\frac{2}{3}$ and $y = \frac{2}{3} \Rightarrow$ critical points are $(0,0)$ and $\left(-\frac{2}{3}, \frac{2}{3}\right)$; for $(0,0)$: $f_{xx}(0,0) = 6x|_{(0,0)} = 0$, $f_{yy}(0,0) = -6y|_{(0,0)} = 0$, $f_{xy}(0,0) = -2$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -4 < 0 \Rightarrow$ saddle point; for $\left(-\frac{2}{3}, \frac{2}{3}\right)$: $f_{xx}\left(-\frac{2}{3}, \frac{2}{3}\right) = -4$, $f_{yy}\left(-\frac{2}{3}, \frac{2}{3}\right) = -4$, $f_{xy}\left(-\frac{2}{3}, \frac{2}{3}\right) = -2$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 12 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f\left(-\frac{2}{3}, \frac{2}{3}\right) = \frac{170}{27}$

18. $f_x(x,y) = 3x^2 + 3y = 0$ and $f_y(x,y) = 3x + 3y^2 = 0 \Rightarrow x = 0$ and $y = 0$, or $x = -1$ and $y = -1 \Rightarrow$ critical points are $(0,0)$ and $(-1,-1)$; for $(0,0)$: $f_{xx}(0,0) = 6x|_{(0,0)} = 0$, $f_{yy}(0,0) = 6y|_{(0,0)} = 0$, $f_{xy}(0,0) = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2$ $= -9 < 0 \Rightarrow$ saddle point; for $(-1,-1)$: $f_{xx}(-1,-1) = -6$, $f_{yy}(-1,-1) = -6$, $f_{xy}(-1,-1) = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2$ $= 27 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-1,-1) = 1$

19. $f_x(x,y) = 12x - 6x^2 + 6y = 0$ and $f_y(x,y) = 6y + 6x = 0 \Rightarrow x = 0$ and $y = 0$, or $x = 1$ and $y = -1 \Rightarrow$ critical points are $(0,0)$ and $(1,-1)$; for $(0,0)$: $f_{xx}(0,0) = 12 - 12x|_{(0,0)} = 12$, $f_{yy}(0,0) = 6$, $f_{xy}(0,0) = 6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2$ $= 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(0,0) = 0$; for $(1,-1)$: $f_{xx}(1,-1) = 0$, $f_{yy}(1,-1) = 6$, $f_{xy}(1,-1) = 6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point

20. $f_x(x,y) = -6x + 6y = 0 \Rightarrow x = y$; $f_y(x,y) = 6y - 6y^2 + 6x = 0 \Rightarrow 12y - 6y^2 = 0 \Rightarrow 6y(2-y) = 0 \Rightarrow y = 0$ or $y = 2 \Rightarrow (0,0)$ and $(2,2)$ are the critical points; $f_{xx}(x,y) = -6$, $f_{yy}(x,y) = 6 - 12y$, $f_{xy}(x,y) = 6$; for $(0,0)$: $f_{xx}(0,0) = -6$, $f_{yy}(0,0) = 6$, $f_{xy}(0,0) = 6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -72 < 0 \Rightarrow$ saddle point; for $(2,2)$: $f_{xx}(2,2) = -6$, $f_{yy}(2,2) = -18$, $f_{xy}(2,2) = 6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 72 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(2,2) = 8$

21. $f_x(x,y) = 27x^2 - 4y = 0$ and $f_y(x,y) = y^2 - 4x = 0 \Rightarrow x = 0$ and $y = 0$, or $x = \frac{4}{9}$ and $y = \frac{4}{3} \Rightarrow$ critical points are $(0,0)$ and $\left(\frac{4}{9}, \frac{4}{3}\right)$; for $(0,0)$: $f_{xx}(0,0) = 54x|_{(0,0)} = 0$, $f_{yy}(0,0) = 2y|_{(0,0)} = 0$, $f_{xy}(0,0) = -4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2$ $= -16 < 0 \Rightarrow$ saddle point; for $\left(\frac{4}{9}, \frac{4}{3}\right)$: $f_{xx}\left(\frac{4}{9}, \frac{4}{3}\right) = 24$, $f_{yy}\left(\frac{4}{9}, \frac{4}{3}\right) = \frac{8}{3}$, $f_{xy}\left(\frac{4}{9}, \frac{4}{3}\right) = -4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 48 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f\left(\frac{4}{9}, \frac{4}{3}\right) = -\frac{64}{81}$

22. $f_x(x,y) = 24x^2 + 6y = 0 \Rightarrow y = -4x^2$; $f_y(x,y) = 3y^2 + 6x = 0 \Rightarrow 3\left(-4x^2\right)^2 + 6x = 0 \Rightarrow 16x^4 + 2x = 0$ $\Rightarrow 2x\left(8x^3 + 1\right) = 0 \Rightarrow x = 0$ or $x = -\frac{1}{2} \Rightarrow (0,0)$ and $\left(-\frac{1}{2}, -1\right)$ are the critical points; $f_{xx}(x,y) = 48x$, $f_{yy}(x,y) = 6y$, and $f_{xy}(x,y) = 6$; for $(0,0)$: $f_{xx}(0,0) = 0$, $f_{yy}(0,0) = 0$, $f_{xy}(0,0) = 6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0$ $\Rightarrow$ saddle point; for $\left(-\frac{1}{2}, -1\right)$: $f_{xx}\left(-\frac{1}{2}, -1\right) = -24$, $f_{yy}\left(-\frac{1}{2}, -1\right) = -6$, $f_{xy}\left(-\frac{1}{2}, -1\right) = 6$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 108 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f\left(-\frac{1}{2}, -1\right) = 1$

23. $f_x(x,y) = 3x^2 + 6x = 0 \Rightarrow x = 0$ or $x = -2$; $f_y(x,y) = 3y^2 - 6y = 0 \Rightarrow y = 0$ or $y = 2 \Rightarrow$ the critical points are $(0,0)$, $(0,2)$, $(-2,0)$, and $(-2,2)$; for $(0,0)$: $f_{xx}(0,0) = 6x + 6|_{(0,0)} = 6$, $f_{yy}(0,0) = 6y - 6|_{(0,0)} = -6$, $f_{xy}(0,0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point; for $(0,2)$: $f_{xx}(0,2) = 6$, $f_{yy}(0,2) = 6$, $f_{xy}(0,2) = 0$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(0,2) = -12$; for $(-2,0)$: $f_{xx}(-2,0) = -6$, $f_{yy}(-2,0) = -6$, $f_{xy}(-2,0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-2,0) = -4$; for $(-2,2)$: $f_{xx}(-2,2) = -6$, $f_{yy}(-2,2) = 6$, $f_{xy}(-2,2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point

24. $f_x(x, y) = 6x^2 - 18x = 0 \Rightarrow 6x(x - 3) = 0 \Rightarrow x = 0$ or $x = 3$; $f_y(x, y) = 6y^2 + 6y - 12 = 0 \Rightarrow 6(y + 2)(y - 1) = 0$ $\Rightarrow y = -2$ or $y = 1 \Rightarrow$ the critical points are $(0, -2)$, $(0, 1)$, $(3, -2)$, and $(3, 1)$; $f_{xx}(x, y) = 12x - 18$, $f_{yy}(x, y) = 12y + 6$, and $f_{xy}(x, y) = 0$; for $(0, -2)$: $f_{xx}(0, -2) = -18$, $f_{yy}(0, -2) = -18$, $f_{xy}(0, -2) = 0$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 324 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(0, -2) = 20$; for $(0, 1)$: $f_{xx}(0, 1) = -18$, $f_{yy}(0, 1) = 18$, $f_{xy}(0, 1) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -324 < 0 \Rightarrow$ saddle point; for $(3, -2)$: $f_{xx}(3, -2) = 18$, $f_{yy}(3, -2) = -18$, $f_{xy}(3, -2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -324 < 0 \Rightarrow$ saddle point; for $(3, 1)$: $f_{xx}(3, 1) = 18$, $f_{yy}(3, 1) = 18$, $f_{xy}(3, 1) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 324 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(3, 1) = -34$

25. $f_x(x, y) = 4y - 4x^3 = 0$ and $f_y(x, y) = 4x - 4y^3 = 0 \Rightarrow x = y \Rightarrow x(1 - x^2) = 0 \Rightarrow x = 0, 1, -1 \Rightarrow$ the critical points are $(0, 0)$, $(1, 1)$, and $(-1, -1)$; for $(0, 0)$: $f_{xx}(0, 0) = -12x^2|_{(0,0)} = 0$, $f_{yy}(0, 0) = -12y^2|_{(0,0)} = 0$, $f_{xy}(0, 0) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -16 < 0 \Rightarrow$ saddle point; for $(1, 1)$: $f_{xx}(1, 1) = -12$, $f_{yy}(1, 1) = -12$, $f_{xy}(1, 1) = 4$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 128 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(1, 1) = 2$; for $(-1, -1)$: $f_{xx}(-1, -1) = -12$, $f_{yy}(-1, -1) = -12$, $f_{xy}(-1, -1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 128 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-1, -1) = 2$

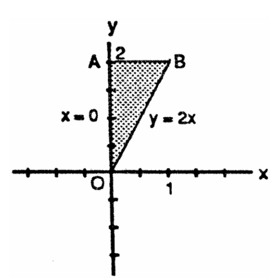
26. $f_x(x, y) = 4x^3 + 4y = 0$ and $f_y(x, y) = 4y^3 + 4x = 0 \Rightarrow x = -y \Rightarrow -x^3 + x = 0 \Rightarrow x(1 - x^2) = 0 \Rightarrow x = 0, 1, -1$ $\Rightarrow$ the critical points are $(0, 0)$, $(1, -1)$, and $(-1, 1)$; $f_{xx}(x, y) = 12x^2$, $f_{yy}(x, y) = 12y^2$, and $f_{xy}(x, y) = 4$; for $(0, 0)$: $f_{xx}(0, 0) = 0$, $f_{yy}(0, 0) = 0$, $f_{xy}(0, 0) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -16 < 0 \Rightarrow$ saddle point; for $(1, -1)$: $f_{xx}(1, -1) = 12$, $f_{yy}(1, -1) = 12$, $f_{xy}(1, -1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 128 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(1, -1) = -2$; for $(-1, 1)$: $f_{xx}(-1, 1) = 12$, $f_{yy}(-1, 1) = 12$, $f_{xy}(-1, 1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 128 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(-1, 1) = -2$

27. $f_x(x, y) = \frac{-2x}{(x^2 + y^2 - 1)^2} = 0$ and $f_y(x, y) = \frac{-2y}{(x^2 + y^2 - 1)^2} = 0 \Rightarrow x = 0$ and $y = 0 \Rightarrow$ the critical point is $(0, 0)$; $f_{xx} = \frac{4x^2 - 2y^2 + 2}{(x^2 + y^2 - 1)^3}$, $f_{yy} = \frac{-2x^2 + 4y^2 + 2}{(x^2 + y^2 - 1)^3}$, $f_{xy} = \frac{8xy}{(x^2 + y^2 - 1)^3}$; $f_{xx}(0, 0) = -2$, $f_{yy}(0, 0) = -2$, $f_{xy}(0, 0) = 0$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(0, 0) = -1$

28. $f_x(x, y) = -\frac{1}{x^2} + y = 0$ and $f_y(x, y) = x - \frac{1}{y^2} = 0 \Rightarrow x = 1$ and $y = 1 \Rightarrow$ the critical point is $(1, 1)$; $f_{xx} = \frac{2}{x^3}$, $f_{yy} = \frac{2}{y^3}$, $f_{xy} = 1$; $f_{xx}(1, 1) = 2$, $f_{yy}(1, 1) = 2$, $f_{xy}(1, 1) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 3 > 0$ and $f_{xx} > 2 \Rightarrow$ local minimum of $f(1, 1) = 3$

29. $f_x(x, y) = y \cos x = 0$ and $f_y(x, y) = \sin x = 0 \Rightarrow x = n\pi$, n an integer, and $y = 0 \Rightarrow$ the critical points are $(n\pi, 0)$, n an integer (Note: cos x and sin x cannot both be 0 for the same x, so sin x must be 0 and $y = 0$); $f_{xx} = -y \sin x$, $f_{yy} = 0$, $f_{xy} = \cos x$; $f_{xx}(n\pi, 0) = 0$, $f_{yy}(n\pi, 0) = 0$, $f_{xy}(n\pi, 0) = 1$ if n is even and $f_{xy}(n\pi, 0) = -1$ if n is odd $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -1 < 0 \Rightarrow$ saddle point.

30. $f_x(x, y) = 2e^{2x} \cos y = 0$ and $f_y(x, y) = -e^{2x} \sin y = 0 \Rightarrow$ no solution since $e^{2x} \neq 0$ for any x and the functions cos y and sin y cannot equal 0 for the same y $\Rightarrow$ no critical points $\Rightarrow$ no extrema and no saddle points



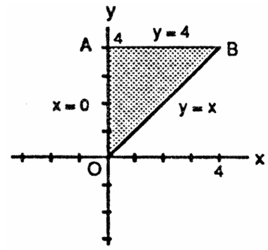
31. (i) On OA, $f(x, y) = f(0, y) = y^2 - 4y + 1$ on $0 \le y \le 2$;
$f'(0, y) = 2y - 4 = 0 \Rightarrow y = 2$;
$f(0, 0) = 1$ and $f(0, 2) = -3$

(ii) On AB, $f(x, y) = f(x, 2) = 2x^2 - 4x - 3$ on $0 \le x \le 1$;
$f'(x, 2) = 4x - 4 = 0 \Rightarrow x = 1$;
$f(0, 2) = -3$ and $f(1, 2) = -5$

(iii) On OB, $f(x, y) = f(x, 2x) = 6x^2 - 12x + 1$ on $0 \le x \le 1$; endpoint values have been found above;
$f'(x, 2x) = 12x - 12 = 0 \Rightarrow x = 1$ and $y = 2$, but $(1, 2)$ is not an interior point of OB

(iv) For interior points of the triangular region, $f_x(x, y) = 4x - 4 = 0$ and $f_y(x, y) = 2y - 4 = 0$ $\Rightarrow$ $x = 1$ and $y = 2$, but $(1, 2)$ is not an interior point of the region. Therefore, the absolute maximum is 1 at $(0, 0)$ and the absolute minimum is -5 at $(1, 2)$.

32\. (i) On OA, $D(x, y) = D(0, y) = y^2 + 1$ on $0 \le y \le 4$; $D'(0, y) = 2y = 0 \Rightarrow y = 0$; $D(0, 0) = 1$ and $D(0, 4) = 17$

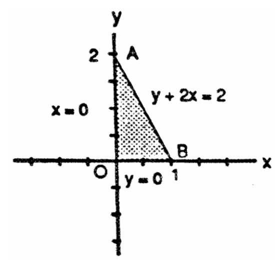


(ii) On AB, $D(x, y) = D(x, 4) = x^2 - 4x + 17$ on $0 \le x \le 4$; $D'(x, 4) = 2x - 4 = 0 \Rightarrow x = 2$ and $(2, 4)$ is an interior point of AB; $D(2, 4) = 13$ and $D(4, 4) = D(0, 4) = 17$

(iii) On OB, $D(x, y) = D(x, x) = x^2 + 1$ on $0 \le x \le 4$; $D'(x, x) = 2x = 0 \Rightarrow x = 0$ and $y = 0$, which is not an interior point of OB; endpoint values have been found above

(iv) For interior points of the triangular region, $f_x(x, y) = 2x - y = 0$ and $f_y(x, y) = -x + 2y = 0 \Rightarrow x = 0$ and $y = 0$, which is not an interior point of the region. Therefore, the absolute maximum is 17 at $(0, 4)$ and $(4, 4)$, and the absolute minimum is 1 at $(0, 0)$.

33\. (i) On OA, $f(x, y) = f(0, y) = y^2$ on $0 \le y \le 2$; $f'(0, y) = 2y = 0 \Rightarrow y = 0$ and $x = 0$; $f(0, 0) = 0$ and $f(0, 2) = 4$

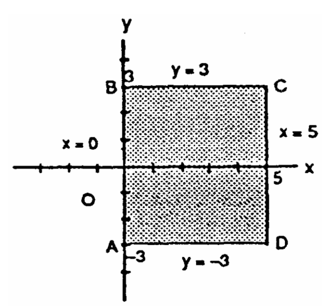


(ii) On OB, $f(x, y) = f(x, 0) = x^2$ on $0 \le x \le 1$; $f'(x, 0) = 2x = 0 \Rightarrow x = 0$ and $y = 0$; $f(0, 0) = 0$ and $f(1, 0) = 1$

(iii) On AB, $f(x, y) = f(x, -2x + 2) = 5x^2 - 8x + 4$ on $0 \le x \le 1$; $f'(x, -2x + 2) = 10x - 8 = 0 \Rightarrow x = \frac{4}{5}$ and $y = \frac{2}{5}$; $f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{4}{5}$; endpoint values have been found above.

(iv) For interior points of the triangular region, $f_x(x, y) = 2x = 0$ and $f_y(x, y) = 2y = 0 \Rightarrow x = 0$ and $y = 0$, but $(0, 0)$ is not an interior point of the region. Therefore the absolute maximum is 4 at $(0, 2)$ and the absolute minimum is 0 at $(0, 0)$.

34\. (i) On AB, $T(x, y) = T(0, y) = y^2$ on $-3 \le y \le 3$; $T'(0, y) = 2y = 0 \Rightarrow y = 0$ and $x = 0$; $T(0, 0) = 0$, $T(0, -3) = 9$, and $T(0, 3) = 9$

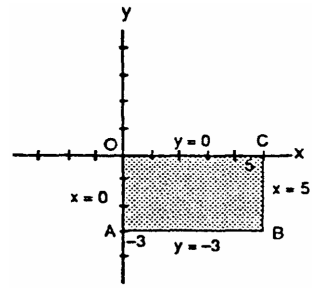


(ii) On BC, $T(x, y) = T(x, 3) = x^2 - 3x + 9$ on $0 \le x \le 5$; $T'(x, 3) = 2x - 3 = 0 \Rightarrow x = \frac{3}{2}$ and $y = 3$; $T\left(\frac{3}{2}, 3\right) = \frac{27}{4}$ and $T(5, 3) = 19$

(iii) On CD, $T(x, y) = T(5, y) = y^2 + 5y - 5$ on $-3 \le y \le 3$; $T'(5, y) = 2y + 5 = 0 \Rightarrow y = -\frac{5}{2}$ and $x = 5$; $T\left(5, -\frac{5}{2}\right) = -\frac{45}{4}$, $T(5, -3) = -11$ and $T(5, 3) = 19$

(iv) On AD, $T(x, y) = T(x, -3) = x^2 - 9x + 9$ on $0 \le x \le 5$; $T'(x, -3) = 2x - 9 = 0 \Rightarrow x = \frac{9}{2}$ and $y = -3$; $T\left(\frac{9}{2}, -3\right) = -\frac{45}{4}$, $T(0, -3) = 9$ and $T(5, -3) = -11$

(v) For interior points of the rectangular region, $T_x(x, y) = 2x + y - 6 = 0$ and $T_y(x, y) = x + 2y = 0 \Rightarrow x = 4$ and $y = -2 \Rightarrow (4, -2)$ is an interior critical point with $T(4, -2) = -12$. Therefore the absolute maximum is 19 at $(5, 3)$ and the absolute minimum is -12 at $(4, -2)$.



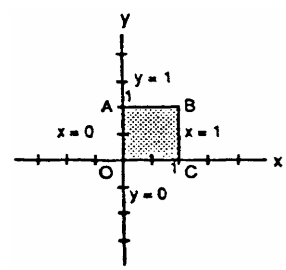
35. (i) On OC, $T(x, y) = T(x, 0) = x^2 - 6x + 2$ on $0 \le x \le 5$; $T'(x, 0) = 2x - 6 = 0 \Rightarrow x = 3$ and $y = 0$; $T(3, 0) = -7$, $T(0, 0) = 2$, and $T(5, 0) = -3$

(ii) On CB, $T(x, y) = T(5, y) = y^2 + 5y - 3$ on $-3 \le y \le 0$; $T'(5, y) = 2y + 5 = 0 \Rightarrow y = -\frac{5}{2}$ and $x = 5$; $T\left(5, -\frac{5}{2}\right) = -\frac{37}{4}$ and $T(5, -3) = -9$

(iii) On AB, $T(x, y) = T(x, -3) = x^2 - 9x + 11$ on $0 \le x \le 5$; $T'(x, -3) = 2x - 9 = 0 \Rightarrow x = \frac{9}{2}$ and $y = -3$; $T\left(\frac{9}{2}, -3\right) = -\frac{37}{4}$ and $T(0, -3) = 11$

(iv) On AO, $T(x, y) = T(0, y) = y^2 + 2$ on $-3 \le y \le 0$; $T'(0, y) = 2y = 0 \Rightarrow y = 0$ and $x = 0$, but $(0, 0)$ is not an interior point of AO

(v) For interior points of the rectangular region, $T_x(x, y) = 2x + y - 6 = 0$ and $T_y(x, y) = x + 2y = 0 \Rightarrow x = 4$ and $y = -2$, an interior critical point with $T(4, -2) = -10$. Therefore the absolute maximum is 11 at $(0, -3)$ and the absolute minimum is -10 at $(4, -2)$.



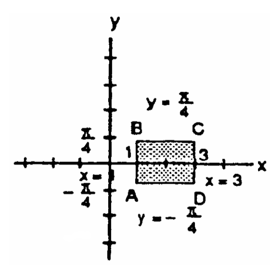
36. (i) On OA, $f(x, y) = f(0, y) = -24y^2$ on $0 \le y \le 1$; $f'(0, y) = -48y = 0 \Rightarrow y = 0$ and $x = 0$, but $(0, 0)$ is not an interior point of OA; $f(0, 0) = 0$ and $f(0, 1) = -24$

(ii) On AB, $f(x, y) = f(x, 1) = 48x - 32x^3 - 24$ on $0 \le x \le 1$; $f'(x, 1) = 48 - 96x^2 = 0 \Rightarrow x = \frac{1}{\sqrt{2}}$ and $y = 1$, or $x = -\frac{1}{\sqrt{2}}$ and $y = 1$, but $\left(-\frac{1}{\sqrt{2}}, 1\right)$ is not in the interior of AB; $f\left(\frac{1}{\sqrt{2}}, 1\right) = 16\sqrt{2} - 24$ and $f(1, 1) = -8$

(iii) On BC, $f(x, y) = f(1, y) = 48y - 32 - 24y^2$ on $0 \le y \le 1$; $f'(1, y) = 48 - 48y = 0 \Rightarrow y = 1$ and $x = 1$, but $(1, 1)$ is not an interior point of BC; $f(1, 0) = -32$ and $f(1, 1) = -8$

(iv) On OC, $f(x, y) = f(x, 0) = -32x^3$ on $0 \le x \le 1$; $f'(x, 0) = -96x^2 = 0 \Rightarrow x = 0$ and $y = 0$, but $(0, 0)$ is not an interior point of OC; $f(0, 0) = 0$ and $f(1, 0) = -32$

(v) For interior points of the rectangular region, $f_x(x, y) = 48y - 96x^2 = 0$ and $f_y(x, y) = 48x - 48y = 0$ $\Rightarrow x = 0$ and $y = 0$, or $x = \frac{1}{2}$ and $y = \frac{1}{2}$, but $(0, 0)$ is not an interior point of the region; $f\left(\frac{1}{2}, \frac{1}{2}\right) = 2$. Therefore the absolute maximum is 2 at $\left(\frac{1}{2}, \frac{1}{2}\right)$ and the absolute minimum is -32 at $(1, 0)$.



37. (i) On AB, $f(x, y) = f(1, y) = 3\cos y$ on $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$; $f'(1, y) = -3\sin y = 0 \Rightarrow y = 0$ and $x = 1$; $f(1, 0) = 3$, $f\left(1, -\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$, and $f\left(1, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$

(ii) On CD, $f(x, y) = f(3, y) = 3\cos y$ on $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$; $f'(3, y) = -3\sin y = 0 \Rightarrow y = 0$ and $x = 3$; $f(3, 0) = 3$, $f\left(3, -\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$ and $f\left(3, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$

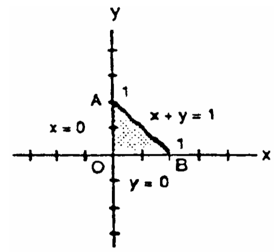
(iii) On BC, $f(x, y) = f\left(x, \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(4x - x^2)$ on $1 \le x \le 3$; $f'\left(x, \frac{\pi}{4}\right) = \sqrt{2}(2 - x) = 0 \Rightarrow x = 2$ and $y = \frac{\pi}{4}$; $f\left(2, \frac{\pi}{4}\right) = 2\sqrt{2}$, $f\left(1, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$, and $f\left(3, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$

(iv) On AD, $f(x, y) = f\left(x, -\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(4x - x^2)$ on $1 \le x \le 3$; $f'\left(x, -\frac{\pi}{4}\right) = \sqrt{2}(2 - x) = 0 \Rightarrow x = 2$ and $y = -\frac{\pi}{4}$; $f\left(2, -\frac{\pi}{4}\right) = 2\sqrt{2}$, $f\left(1, -\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$, and $f\left(3, -\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$

(v) For interior points of the region, $f_x(x, y) = (4 - 2x)\cos y = 0$ and $f_y(x, y) = -(4x - x^2)\sin y = 0 \Rightarrow x = 2$ and $y = 0$, which is an interior critical point with $f(2, 0) = 4$. Therefore the absolute maximum is 4 at

$(2,0)$ and the absolute minimum is $\frac{3\sqrt{2}}{2}$ at $\left(3,-\frac{\pi}{4}\right)$, $\left(3,\frac{\pi}{4}\right)$, $\left(1,-\frac{\pi}{4}\right)$, and $\left(1,\frac{\pi}{4}\right)$.

38. (i) On OA, $f(x,y)=f(0,y)=2y+1$ on $0\le y\le 1$; $f'(0,y)=2 \Rightarrow$ no interior critical points; $f(0,0)=1$ and $f(0,1)=3$



(ii) On OB, $f(x,y)=f(x,0)=4x+1$ on $0\le x\le 1$; $f'(x,0)=4 \Rightarrow$ no interior critical points; $f(1,0)=5$

(iii) On AB, $f(x,y)=f(x,-x+1)=8x^2-6x+3$ on $0\le x\le 1$; $f'(x,-x+1)=16x-6=0 \Rightarrow x=\frac{3}{8}$ and $y=\frac{5}{8}$; $f\left(\frac{3}{8},\frac{5}{8}\right)=\frac{15}{8}$, $f(0,1)=3$, and $f(1,0)=5$

(iv) For interior points of the triangular region, $f_x(x,y)=4-8y=0$ and $f_y(x,y)=-8x+2=0$ $\Rightarrow y=\frac{1}{2}$ and $x=\frac{1}{4}$ which is an interior critical point with $f\left(\frac{1}{4},\frac{1}{2}\right)=2$. Therefore the absolute maximum is 5 at $(1,0)$ and the absolute minimum is 1 at $(0,0)$.

39. Let $F(a,b)=\int_a^b (6-x-x^2)\,dx$ where $a\le b$. The boundary of the domain of F is the line $a=b$ in the ab-plane, and $F(a,a)=0$, so F is identically 0 on the boundary of its domain. For interior critical points we have: $\frac{\partial F}{\partial a}=-(6-a-a^2)=0 \Rightarrow a=-3, 2$ and $\frac{\partial F}{\partial b}=(6-b-b^2)=0 \Rightarrow b=-3, 2$. Since $a\le b$, there is only one interior critical point $(-3,2)$ and $F(-3,2)=\int_{-3}^{2}(6-x-x^2)\,dx$ gives the area under the parabola $y=6-x-x^2$ that is above the x-axis. Therefore, $a=-3$ and $b=2$.

40. Let $F(a,b)=\int_a^b (24-2x-x^2)^{1/3}\,dx$ where $a\le b$. The boundary of the domain of F is the line $a=b$ and on this line F is identically 0. For interior critical points we have: $\frac{\partial F}{\partial a}=-(24-2a-a^2)^{1/3}=0 \Rightarrow a=4,-6$ and $\frac{\partial F}{\partial b}=(24-2b-b^2)^{1/3}=0 \Rightarrow b=4,-6$. Since $a\le b$, there is only one critical point $(-6,4)$ and $F(-6,4)=\int_{-6}^{4}(24-2x-x^2)\,dx$ gives the area under the curve $y=(24-2x-x^2)^{1/3}$ that is above the x-axis. Therefore, $a=-6$ and $b=4$.

41. $T_x(x,y)=2x-1=0$ and $T_y(x,y)=4y=0 \Rightarrow x=\frac{1}{2}$ and $y=0$ with $T\left(\frac{1}{2},0\right)=-\frac{1}{4}$; on the boundary $x^2+y^2=1$: $T(x,y)=-x^2-x+2$ for $-1\le x\le 1 \Rightarrow T'(x,y)=-2x-1=0 \Rightarrow x=-\frac{1}{2}$ and $y=\pm\frac{\sqrt{3}}{2}$; $T\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)=\frac{9}{4}$, $T\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)=\frac{9}{4}$, $T(-1,0)=2$, and $T(1,0)=0 \Rightarrow$ the hottest is $2\frac{1}{4}{}^\circ$ at $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$; the coldest is $-\frac{1}{4}{}^\circ$ at $\left(\frac{1}{2},0\right)$.

42. $f_x(x,y)=y+2-\frac{2}{x}=0$ and $f_y(x,y)=x-\frac{1}{y}=0 \Rightarrow x=\frac{1}{2}$ and $y=2$; $f_{xx}\left(\frac{1}{2},2\right)=\frac{2}{x^2}\Big|_{\left(\frac{1}{2},2\right)}=8$, $f_{yy}\left(\frac{1}{2},2\right)=\frac{1}{y^2}\Big|_{\left(\frac{1}{2},2\right)}=\frac{1}{4}$, $f_{xy}\left(\frac{1}{2},2\right)=1 \Rightarrow f_{xx}f_{yy}-f_{xy}^2=1>0$ and $f_{xx}>0 \Rightarrow$ a local minimum of $f\left(\frac{1}{2},2\right)$ $=2-\ln\frac{1}{2}=2+\ln 2$

43. (a) $f_x(x,y)=2x-4y=0$ and $f_y(x,y)=2y-4x=0 \Rightarrow x=0$ and $y=0$; $f_{xx}(0,0)=2$, $f_{yy}(0,0)=2$, $f_{xy}(0,0)=-4 \Rightarrow f_{xx}f_{yy}-f_{xy}^2=-12<0 \Rightarrow$ saddle point at $(0,0)$

(b) $f_x(x,y)=2x-2=0$ and $f_y(x,y)=2y-4=0 \Rightarrow x=1$ and $y=2$; $f_{xx}(1,2)=2$, $f_{yy}(1,2)=2$, $f_{xy}(1,2)=0 \Rightarrow f_{xx}f_{yy}-f_{xy}^2=4>0$ and $f_{xx}>0 \Rightarrow$ local minimum at $(1,2)$

(c) $f_x(x,y)=9x^2-9=0$ and $f_y(x,y)=2y+4=0 \Rightarrow x=\pm 1$ and $y=-2$; $f_{xx}(1,-2)=18x|_{(1,-2)}=18$, $f_{yy}(1,-2)=2$, $f_{xy}(1,-2)=0 \Rightarrow f_{xx}f_{yy}-f_{xy}^2=36>0$ and $f_{xx}>0 \Rightarrow$ local minimum at $(1,-2)$; $f_{xx}(-1,-2)=-18$, $f_{yy}(-1,-2)=2$, $f_{xy}(-1,-2)=0 \Rightarrow f_{xx}f_{yy}-f_{xy}^2=-36<0 \Rightarrow$ saddle point at $(-1,-2)$

44. (a) Minimum at $(0,0)$ since $f(x,y) > 0$ for all other (x,y)
(b) Maximum of 1 at $(0,0)$ since $f(x,y) < 1$ for all other (x,y)
(c) Neither since $f(x,y) < 0$ for $x < 0$ and $f(x,y) > 0$ for $x > 0$
(d) Neither since $f(x,y) < 0$ for $x < 0$ and $f(x,y) > 0$ for $x > 0$
(e) Neither since $f(x,y) < 0$ for $x < 0$ and $y > 0$, but $f(x,y) > 0$ for $x > 0$ and $y > 0$
(f) Minimum at $(0,0)$ since $f(x,y) > 0$ for all other (x,y)

45. If $k = 0$, then $f(x,y) = x^2 + y^2 \Rightarrow f_x(x,y) = 2x = 0$ and $f_y(x,y) = 2y = 0 \Rightarrow x = 0$ and $y = 0 \Rightarrow (0,0)$ is the only critical point. If $k \neq 0$, $f_x(x,y) = 2x + ky = 0 \Rightarrow y = -\frac{2}{k}x$; $f_y(x,y) = kx + 2y = 0 \Rightarrow kx + 2\left(-\frac{2}{k}x\right) = 0$ $\Rightarrow kx - \frac{4x}{k} = 0 \Rightarrow \left(k - \frac{4}{k}\right)x = 0 \Rightarrow x = 0$ or $k = \pm 2 \Rightarrow y = \left(-\frac{2}{k}\right)(0) = 0$ or $y = \pm x$; in any case $(0,0)$ is a critical point.

46. (See Exercise 45 above): $f_{xx}(x,y) = 2$, $f_{yy}(x,y) = 2$, and $f_{xy}(x,y) = k \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 - k^2$; f will have a saddle point at $(0,0)$ if $4 - k^2 < 0 \Rightarrow k > 2$ or $k < -2$; f will have a local minimum at $(0,0)$ if $4 - k^2 > 0$ $\Rightarrow -2 < k < 2$; the test is inconclusive if $4 - k^2 = 0 \Rightarrow k = \pm 2$.

47. No; for example $f(x,y) = xy$ has a saddle point at $(a,b) = (0,0)$ where $f_x = f_y = 0$.

48. If $f_{xx}(a,b)$ and $f_{yy}(a,b)$ differ in sign, then $f_{xx}(a,b)\,f_{yy}(a,b) < 0$ so $f_{xx}f_{yy} - f_{xy}^2 < 0$. The surface must therefore have a saddle point at (a,b) by the second derivative test.

49. We want the point on $z = 10 - x^2 - y^2$ where the tangent plane is parallel to the plane $x + 2y + 3z = 0$. To find a normal vector to $z = 10 - x^2 - y^2$ let $w = z + x^2 + y^2 - 10$. Then $\nabla w = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$ is normal to $z = 10 - x^2 - y^2$ at (x,y). The vector ∇w is parallel to $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ which is normal to the plane $x + 2y + 3z = 0$ if $6x\mathbf{i} + 6y\mathbf{j} + 3\mathbf{k} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ or $x = \frac{1}{6}$ and $y = \frac{1}{3}$. Thus the point is $\left(\frac{1}{6}, \frac{1}{3}, 10 - \frac{1}{36} - \frac{1}{9}\right)$ or $\left(\frac{1}{6}, \frac{1}{3}, \frac{355}{36}\right)$.

50. We want the point on $z = x^2 + y^2 + 10$ where the tangent plane is parallel to the plane $x + 2y - z = 0$. Let $w = z - x^2 - y^2 - 10$, then $\nabla w = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$ is normal to $z = x^2 + y^2 + 10$ at (x,y). The vector ∇w is parallel to $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ which is normal to the plane if $x = \frac{1}{2}$ and $y = 1$. Thus the point $\left(\frac{1}{2}, 1, \frac{1}{4} + 1 + 10\right)$ or $\left(\frac{1}{2}, 1, \frac{45}{4}\right)$ is the point on the surface $z = x^2 + y^2 + 10$ nearest the plane $x + 2y - z = 0$.

51. No, because the domain $x \geq 0$ and $y \geq 0$ is unbounded since x and y can be as large as we please. Absolute extrema are guaranteed for continuous functions defined over closed <u>and</u> <u>bounded</u> domains in the plane. Since the domain is unbounded, the continuous function $f(x,y) = x + y$ need not have an absolute maximum (although, in this case, it does have an absolute minimum value of $f(0,0) = 0$).

52. (a) (i) On $x = 0$, $f(x,y) = f(0,y) = y^2 - y + 1$ for $0 \leq y \leq 1$; $f'(0,y) = 2y - 1 = 0 \Rightarrow y = \frac{1}{2}$ and $x = 0$; $f\left(0, \frac{1}{2}\right) = \frac{3}{4}$, $f(0,0) = 1$, and $f(0,1) = 1$
(ii) On $y = 1$, $f(x,y) = f(x,1) = x^2 + x + 1$ for $0 \leq x \leq 1$; $f'(x,1) = 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$ and $y = 1$, but $\left(-\frac{1}{2}, 1\right)$ is outside the domain; $f(0,1) = 1$ and $f(1,1) = 3$
(iii) On $x = 1$, $f(x,y) = f(1,y) = y^2 + y + 1$ for $0 \leq y \leq 1$; $f'(1,y) = 2y + 1 = 0 \Rightarrow y = -\frac{1}{2}$ and $x = 1$, but $\left(1, -\frac{1}{2}\right)$ is outside the domain; $f(1,0) = 1$ and $f(1,1) = 3$
(iv) On $y = 0$, $f(x,y) = f(x,0) = x^2 - x + 1$ for $0 \leq x \leq 1$; $f'(x,0) = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$ and $y = 0$; $f\left(\frac{1}{2}, 0\right) = \frac{3}{4}$; $f(0,0) = 1$, and $f(1,0) = 1$
(v) On the interior of the square, $f_x(x,y) = 2x + 2y - 1 = 0$ and $f_y(x,y) = 2y + 2x - 1 = 0 \Rightarrow 2x + 2y = 1$ $\Rightarrow (x + y) = \frac{1}{2}$. Then $f(x,y) = x^2 + y^2 + 2xy - x - y + 1 = (x+y)^2 - (x+y) + 1 = \frac{3}{4}$ is the absolute minimum value when $2x + 2y = 1$.

(b) The absolute maximum is $f(1,1) = 3$.

53. (a) $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = -2\sin t + 2\cos t = 0 \Rightarrow \cos t = \sin t \Rightarrow x = y$

(i) On the semicircle $x^2 + y^2 = 4$, $y \ge 0$, we have $t = \frac{\pi}{4}$ and $x = y = \sqrt{2} \Rightarrow f\left(\sqrt{2}, \sqrt{2}\right) = 2\sqrt{2}$. At the endpoints, $f(-2,0) = -2$ and $f(2,0) = 2$. Therefore the absolute minimum is $f(-2,0) = -2$ when $t = \pi$; the absolute maximum is $f\left(\sqrt{2}, \sqrt{2}\right) = 2\sqrt{2}$ when $t = \frac{\pi}{4}$.

(ii) On the quartercircle $x^2 + y^2 = 4$, $x \ge 0$ and $y \ge 0$, the endpoints give $f(0,2) = 2$ and $f(2,0) = 2$. Therefore the absolute minimum is $f(2,0) = 2$ and $f(0,2) = 2$ when $t = 0, \frac{\pi}{2}$ respectively; the absolute maximum is $f\left(\sqrt{2}, \sqrt{2}\right) = 2\sqrt{2}$ when $t = \frac{\pi}{4}$.

(b) $\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt} = -4\sin^2 t + 4\cos^2 t = 0 \Rightarrow \cos t = \pm\sin t \Rightarrow x = \pm y$.

(i) On the semicircle $x^2 + y^2 = 4$, $y \ge 0$, we obtain $x = y = \sqrt{2}$ at $t = \frac{\pi}{4}$ and $x = -\sqrt{2}$, $y = \sqrt{2}$ at $t = \frac{3\pi}{4}$. Then $g\left(\sqrt{2}, \sqrt{2}\right) = 2$ and $g\left(-\sqrt{2}, \sqrt{2}\right) = -2$. At the endpoints, $g(-2,0) = g(2,0) = 0$. Therefore the absolute minimum is $g\left(-\sqrt{2}, \sqrt{2}\right) = -2$ when $t = \frac{3\pi}{4}$; the absolute maximum is $g\left(\sqrt{2}, \sqrt{2}\right) = 2$ when $t = \frac{\pi}{4}$.

(ii) On the quartercircle $x^2 + y^2 = 4$, $x \ge 0$ and $y \ge 0$, the endpoints give $g(0,2) = 0$ and $g(2,0) = 0$. Therefore the absolute minimum is $g(2,0) = 0$ and $g(0,2) = 0$ when $t = 0, \frac{\pi}{2}$ respectively; the absolute maximum is $g\left(\sqrt{2}, \sqrt{2}\right) = 2$ when $t = \frac{\pi}{4}$.

(c) $\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt} = 4x\frac{dx}{dt} + 2y\frac{dy}{dt} = (8\cos t)(-2\sin t) + (4\sin t)(2\cos t) = -8\cos t\sin t = 0$ $\Rightarrow t = 0, \frac{\pi}{2}, \pi$ yielding the points $(2,0)$, $(0,2)$ for $0 \le t \le \pi$.

(i) On the semicircle $x^2 + y^2 = 4$, $y \ge 0$ we have $h(2,0) = 8$, $h(0,2) = 4$, and $h(-2,0) = 8$. Therefore, the absolute minimum is $h(0,2) = 4$ when $t = \frac{\pi}{2}$; the absolute maximum is $h(2,0) = 8$ and $h(-2,0) = 8$ when $t = 0, \pi$ respectively.

(ii) On the quartercircle $x^2 + y^2 = 4$, $x \ge 0$ and $y \ge 0$ the absolute minimum is $h(0,2) = 4$ when $t = \frac{\pi}{2}$; the absolute maximum is $h(2,0) = 8$ when $t = 0$.

54. (a) $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 2\frac{dx}{dt} + 3\frac{dy}{dt} = -6\sin t + 6\cos t = 0 \Rightarrow \sin t = \cos t \Rightarrow t = \frac{\pi}{4}$ for $0 \le t \le \pi$.

(i) On the semi-ellipse, $\frac{x^2}{9} + \frac{y^2}{4} = 1$, $y \ge 0$, $f(x,y) = 2x + 3y = 6\cos t + 6\sin t = 6\left(\frac{\sqrt{2}}{2}\right) + 6\left(\frac{\sqrt{2}}{2}\right) = 6\sqrt{2}$ at $t = \frac{\pi}{4}$. At the endpoints, $f(-3,0) = -6$ and $f(3,0) = 6$. The absolute minimum is $f(-3,0) = -6$ when $t = \pi$; the absolute maximum is $f\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 6\sqrt{2}$ when $t = \frac{\pi}{4}$.

(ii) On the quarter ellipse, at the endpoints $f(0,2) = 6$ and $f(3,0) = 6$. The absolute minimum is $f(3,0) = 6$ and $f(0,2) = 6$ when $t = 0, \frac{\pi}{2}$ respectively; the absolute maximum is $f\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 6\sqrt{2}$ when $t = \frac{\pi}{4}$.

(b) $\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt} = (2\sin t)(-3\sin t) + (3\cos t)(2\cos t) = 6\left(\cos^2 t - \sin^2 t\right) = 6\cos 2t = 0$ $\Rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4}$ for $0 \le t \le \pi$.

(i) On the semi-ellipse, $g(x,y) = xy = 6\sin t\cos t$. Then $g\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 3$ when $t = \frac{\pi}{4}$, and $g\left(-\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = -3$ when $t = \frac{3\pi}{4}$. At the endpoints, $g(-3,0) = g(3,0) = 0$. The absolute minimum is $g\left(-\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = -3$ when $t = \frac{3\pi}{4}$; the absolute maximum is $g\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 3$ when $t = \frac{\pi}{4}$.

(ii) On the quarter ellipse, at the endpoints $g(0,2) = 0$ and $g(3,0) = 0$. The absolute minimum is $g(3,0) = 0$ and $g(0,2) = 0$ at $t = 0, \frac{\pi}{2}$ respectively; the absolute maximum is $g\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 3$ when $t = \frac{\pi}{4}$.

(c) $\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt} = 2x\frac{dx}{dt} + 6y\frac{dy}{dt} = (6\cos t)(-3\sin t) + (12\sin t)(2\cos t) = 6\sin t\cos t = 0$ $\Rightarrow\ t = 0, \frac{\pi}{2}, \pi$ for $0 \le t \le \pi$, yielding the points $(3,0)$, $(0,2)$, and $(-3,0)$.

(i) On the semi-ellipse, $y \ge 0$ so that $h(3,0) = 9$, $h(0,2) = 12$, and $h(-3,0) = 9$. The absolute minimum is $h(3,0) = 9$ and $h(-3,0) = 9$ when $t = 0, \pi$ respectively; the absolute maximum is $h(0,2) = 12$ when $t = \frac{\pi}{2}$.

(ii) On the quarter ellipse, the absolute minimum is $h(3,0) = 9$ when $t = 0$; the absolute maximum is $h(0,2) = 12$ when $t = \frac{\pi}{2}$.

55. $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt}$

(i) $x = 2t$ and $y = t + 1\ \Rightarrow\ \frac{df}{dt} = (t+1)(2) + (2t)(1) = 4t + 2 = 0\ \Rightarrow\ t = -\frac{1}{2}\ \Rightarrow\ x = -1$ and $y = \frac{1}{2}$ with $f\left(-1, \frac{1}{2}\right) = -\frac{1}{2}$. The absolute minimum is $f\left(-1, \frac{1}{2}\right) = -\frac{1}{2}$ when $t = -\frac{1}{2}$; there is no absolute maximum.

(ii) For the endpoints: $t = -1\ \Rightarrow\ x = -2$ and $y = 0$ with $f(-2,0) = 0$; $t = 0\ \Rightarrow\ x = 0$ and $y = 1$ with $f(0,1) = 0$. The absolute minimum is $f\left(-1, \frac{1}{2}\right) = -\frac{1}{2}$ when $t = -\frac{1}{2}$; the absolute maximum is $f(0,1) = 0$ and $f(-2,0) = 0$ when $t = -1, 0$ respectively.

(iii) There are no interior critical points. For the endpoints: $t = 0\ \Rightarrow\ x = 0$ and $y = 1$ with $f(0,1) = 0$; $t = 1\ \Rightarrow\ x = 2$ and $y = 2$ with $f(2,2) = 4$. The absolute minimum is $f(0,1) = 0$ when $t = 0$; the absolute maximum is $f(2,2) = 4$ when $t = 1$.

56. (a) $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

(i) $x = t$ and $y = 2 - 2t\ \Rightarrow\ \frac{df}{dt} = (2t)(1) + 2(2-2t)(-2) = 10t - 8 = 0\ \Rightarrow\ t = \frac{4}{5}\ \Rightarrow\ x = \frac{4}{5}$ and $y = \frac{2}{5}$ with $f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{16}{25} + \frac{4}{25} = \frac{4}{5}$. The absolute minimum is $f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{4}{5}$ when $t = \frac{4}{5}$; there is no absolute maximum along the line.

(ii) For the endpoints: $t = 0\ \Rightarrow\ x = 0$ and $y = 2$ with $f(0,2) = 4$; $t = 1\ \Rightarrow\ x = 1$ and $y = 0$ with $f(1,0) = 1$. The absolute minimum is $f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{4}{5}$ at the interior critical point when $t = \frac{4}{5}$; the absolute maximum is $f(0,2) = 4$ at the endpoint when $t = 0$.

(b) $\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} = \left[\frac{-2x}{(x^2+y^2)^2}\right]\frac{dx}{dt} + \left[\frac{-2y}{(x^2+y^2)^2}\right]\frac{dy}{dt}$

(i) $x = t$ and $y = 2 - 2t\ \Rightarrow\ x^2 + y^2 = 5t^2 - 8t + 4\ \Rightarrow\ \frac{dg}{dt} = -\left(5t^2 - 8t + 4\right)^{-2}[(-2t)(1) + (-2)(2-2t)(-2)]$ $= -\left(5t^2 - 8t + 4\right)^{-2}(-10t + 8) = 0\ \Rightarrow\ t = \frac{4}{5}\ \Rightarrow\ x = \frac{4}{5}$ and $y = \frac{2}{5}$ with $g\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{1}{\left(\frac{4}{5}\right)} = \frac{5}{4}$. The absolute maximum is $g\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{5}{4}$ when $t = \frac{4}{5}$; there is no absolute minimum along the line since x and y can be as large as we please.

(ii) For the endpoints: $t = 0\ \Rightarrow\ x = 0$ and $y = 2$ with $g(0,2) = \frac{1}{4}$; $t = 1\ \Rightarrow\ x = 1$ and $y = 0$ with $g(1,0) = 1$. The absolute minimum is $g(0,2) = \frac{1}{4}$ when $t = 0$; the absolute maximum is $g\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{5}{4}$ when $t = \frac{4}{5}$.

57. $m = \frac{(2)(-1) - 3(-14)}{(2)^2 - 3(10)} = -\frac{20}{13}$ and $b = \frac{1}{3}\left[-1 - \left(-\frac{20}{13}\right)(2)\right] = \frac{9}{13}$ $\Rightarrow\ y = -\frac{20}{13}x + \frac{9}{13}$; $y\big|_{x=4} = -\frac{71}{13}$

k	x_k	y_k	x_k^2	$x_k y_k$
1	−1	2	1	−2
2	0	1	0	0
3	3	−4	9	−12
Σ	2	−1	10	−14

58. $m = \frac{(0)(5) - 3(6)}{(0)^2 - 3(8)} = \frac{3}{4}$ and $b = \frac{1}{3}\left[5 - \frac{3}{4}(0)\right] = \frac{5}{3}$ $\Rightarrow\ y = \frac{3}{4}x + \frac{5}{3}$; $y\big|_{x=4} = \frac{14}{3}$

k	x_k	y_k	x_k^2	$x_k y_k$
1	−2	0	4	0
2	0	2	0	0
3	2	3	4	6
Σ	0	5	8	6

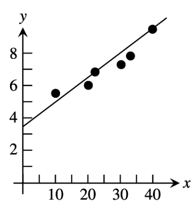
59. $m = \frac{(3)(5) - 3(8)}{(3)^2 - 3(5)} = \frac{3}{2}$ and
$b = \frac{1}{3}\left[5 - \frac{3}{2}(3)\right] = \frac{1}{6}$
$\Rightarrow\ y = \frac{3}{2}x + \frac{1}{6}$; $y\big|_{x=4} = \frac{37}{6}$

k	x_k	y_k	x_k^2	$x_k y_k$
1	0	0	0	0
2	1	2	1	2
3	2	3	4	6
Σ	3	5	5	8

60. $m = \frac{(5)(5) - 3(10)}{(5)^2 - 3(13)} = \frac{5}{14}$ and
$b = \frac{1}{3}\left[5 - \frac{5}{14}(5)\right] = \frac{15}{14}$
$\Rightarrow\ y = \frac{5}{14}x + \frac{15}{14}$; $y\big|_{x=4} = \frac{35}{14} = \frac{5}{2}$

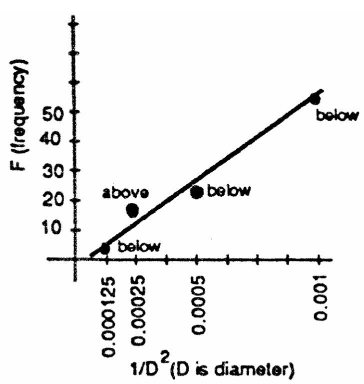
k	x_k	y_k	x_k^2	$x_k y_k$
1	0	1	0	0
2	2	2	4	4
3	3	2	9	6
Σ	5	5	13	10

61. $m = \frac{(162)(41.32) - 6(1192.8)}{(162)^2 - 6(5004)} \approx 0.122$ and
$b = \frac{1}{6}[41.32 - (0.122)(162)] \approx 3.59$
$\Rightarrow\ y = 0.122x + 3.59$



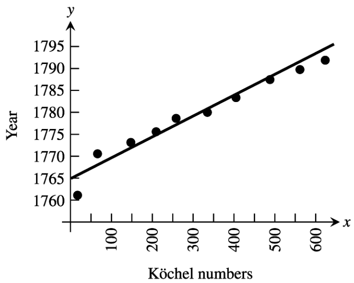
k	x_k	y_k	x_k^2	$x_k y_k$
1	12	5.27	144	63.24
2	18	5.68	324	102.24
3	24	6.25	576	150
4	30	7.21	900	216.3
5	36	8.20	1296	295.2
6	42	8.71	1764	365.82
Σ	162	41.32	5004	1192.8

62. $m = \frac{(0.001863)(91) - 4(0.065852)}{(0.001863)^2 - 4(0.000001323)} \approx 51{,}545$
and $b = \frac{1}{4}(91 - 51{,}545(0.001863)) \approx -1.26$
$\Rightarrow\ F = 51{,}545\,\frac{1}{D^2} - 1.26$



k	$\left(\frac{1}{D^2}\right)_k$	F_k	$\left(\frac{1}{D^2}\right)_k^2$	$\left(\frac{1}{D^2}\right)_k F_k$
1	0.001	51	0.000001	0.051
2	0.0005	22	0.00000025	0.011
3	0.00024	14	0.0000000576	0.00336
4	0.000123	4	0.0000000153	0.000492
Σ	0.001863	91	0.000001323	0.065852

63. (b) $m = \frac{(3201)(17{,}785) - 10(5{,}710{,}292)}{(3201)^2 - 10(1{,}430{,}389)}$
≈ 0.0427 and $b = \frac{1}{10}[17{,}785 - (0.0427)(3201)]$
$\approx 1764.8 \Rightarrow y = 0.0427K + 1764.8$



k	K_k	y_k	K^2	$K_k y_k$
1	1	1761	1	1761
2	75	1771	5625	132,825
3	155	1772	24,025	274,660
4	219	1775	47,961	388,725
5	271	1777	73,441	481,567
6	351	1780	123,201	624,780
7	425	1783	180,625	757,775
8	503	1786	253,009	898,358
9	575	1789	330,625	1,028,675
10	626	1791	391,876	1,121,166
Σ	3201	17,785	1,430,389	5,710,292

(c) $K = 364 \Rightarrow y = (0.0427)(364)$
$\Rightarrow y = (0.0427)(364) + 1764.8 \approx 1780$

64. $m = \frac{(123)(140) - 16(1431)}{(123)^2 - 16(1287)} \approx 1.04$ and
$b = \frac{1}{16}[140 - (1.04)(123)] \approx 0.755$
$\Rightarrow y = 1.04x + 0.755$

k	x_k	y_k	x_k^2	$x_k y_k$
1	3	3	9	9
2	2	2	4	4
3	4	6	16	24
4	2	3	4	6
5	5	4	25	20
6	5	3	25	15
7	9	11	81	99
8	12	9	144	108
9	8	10	64	80
10	13	16	169	208
11	14	13	196	182
12	3	5	9	15
13	4	6	16	24
14	13	19	169	247
15	10	15	100	150
16	16	15	256	240
Σ	123	140	1287	1431

65-70. Example CAS commands:

Maple:

```
f := (x,y) -> x^2+y^3-3*x*y;
x0,x1 := -5,5;
y0,y1 := -5,5;
plot3d( f(x,y), x=x0..x1, y=y0..y1, axes=boxed, shading=zhue, title="#65(a) (Section 14.7)" );
plot3d( f(x,y), x=x0..x1, y=y0..y1, grid=[40,40], axes=boxed, shading=zhue, style=patchcontour, title="#65(b)
    (Section 14.7)" );
fx := D[1](f);                                        # (c)
fy := D[2](f);
crit_pts := solve( {fx(x,y)=0,fy(x,y)=0}, {x,y} );
fxx := D[1](fx);                                      # (d)
fxy := D[2](fx);
fyy := D[2](fy);
discr := unapply( fxx(x,y)*fyy(x,y)-fxy(x,y)^2, (x,y) );
for CP in {crit_pts} do                               # (e)
 eval( [x,y,fxx(x,y),discr(x,y)], CP );
```

```
end do;
# (0,0) is a saddle point
# ( 9/4,  3/2) is a local minimum
```

Mathematica: (assigned functions and bounds will vary)

```
Clear[x,y,f]
f[x_,y_]:= x^2 + y^3 − 3x y
xmin= −5;  xmax= 5;  ymin= −5;  ymax= 5;
Plot3D[f[x,y], {x, xmin, xmax}, {y, ymin, ymax}, AxesLabel → {x, y, z}]
ContourPlot[f[x,y], {x, xmin, xmax}, {y, ymin, ymax}, ContourShading → False, Contours → 40]
fx= D[f[x,y], x];
fy= D[f[x,y], y];
critical=Solve[{fx==0, fy==0},{x, y}]
fxx= D[fx, x];
fxy= D[fx, y];
fyy= D[fy, y];
discriminant= fxx fyy − fxy^2
{{x, y}, f[x, y], discriminant, fxx} /.critical
```

14.8 LAGRANGE MULTIPLIERS

1. $\nabla f = y\mathbf{i} + x\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 4y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow y\mathbf{i} + x\mathbf{j} = \lambda(2x\mathbf{i} + 4y\mathbf{j}) \Rightarrow y = 2x\lambda$ and $x = 4y\lambda$ $\Rightarrow x = 8x\lambda^2 \Rightarrow \lambda = \pm\frac{\sqrt{2}}{4}$ or $x = 0$.
   CASE 1: If $x = 0$, then $y = 0$. But $(0,0)$ is not on the ellipse so $x \neq 0$.
   CASE 2: $x \neq 0 \Rightarrow \lambda = \pm\frac{\sqrt{2}}{4} \Rightarrow x = \pm\sqrt{2}y \Rightarrow \left(\pm\sqrt{2}y\right)^2 + 2y^2 = 1 \Rightarrow y = \pm\frac{1}{2}$.
   Therefore f takes on its extreme values at $\left(\pm\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$ and $\left(\pm\frac{\sqrt{2}}{2}, -\frac{1}{2}\right)$. The extreme values of f on the ellipse are $\pm\frac{\sqrt{2}}{2}$.

2. $\nabla f = y\mathbf{i} + x\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow y\mathbf{i} + x\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow y = 2x\lambda$ and $x = 2y\lambda$ $\Rightarrow x = 4x\lambda^2 \Rightarrow x = 0$ or $\lambda = \pm\frac{1}{2}$.
   CASE 1: If $x = 0$, then $y = 0$. But $(0,0)$ is not on the circle $x^2 + y^2 - 10 = 0$ so $x \neq 0$.
   CASE 2: $x \neq 0 \Rightarrow \lambda = \pm\frac{1}{2} \Rightarrow y = 2x\left(\pm\frac{1}{2}\right) = \pm x \Rightarrow x^2 + (\pm x)^2 - 10 = 0 \Rightarrow x = \pm\sqrt{5} \Rightarrow y = \pm\sqrt{5}$.
   Therefore f takes on its extreme values at $\left(\pm\sqrt{5}, \sqrt{5}\right)$ and $\left(\pm\sqrt{5}, -\sqrt{5}\right)$. The extreme values of f on the circle are 5 and -5.

3. $\nabla f = -2x\mathbf{i} - 2y\mathbf{j}$ and $\nabla g = \mathbf{i} + 3\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow -2x\mathbf{i} - 2y\mathbf{j} = \lambda(\mathbf{i} + 3\mathbf{j}) \Rightarrow x = -\frac{\lambda}{2}$ and $y = -\frac{3\lambda}{2}$ $\Rightarrow \left(-\frac{\lambda}{2}\right) + 3\left(-\frac{3\lambda}{2}\right) = 10 \Rightarrow \lambda = -2 \Rightarrow x = 1$ and $y = 3 \Rightarrow$ f takes on its extreme value at $(1,3)$ on the line. The extreme value is $f(1,3) = 49 - 1 - 9 = 39$.

4. $\nabla f = 2xy\mathbf{i} + x^2\mathbf{j}$ and $\nabla g = \mathbf{i} + \mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2xy\mathbf{i} + x^2\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j}) \Rightarrow 2xy = \lambda$ and $x^2 = \lambda$ $\Rightarrow 2xy = x^2 \Rightarrow x = 0$ or $2y = x$.
   CASE 1: If $x = 0$, then $x + y = 3 \Rightarrow y = 3$.
   CASE 2: If $x \neq 0$, then $2y = x$ so that $x + y = 3 \Rightarrow 2y + y = 3 \Rightarrow y = 1 \Rightarrow x = 2$.
   Therefore f takes on its extreme values at $(0,3)$ and $(2,1)$. The extreme values of f are $f(0,3) = 0$ and $f(2,1) = 4$.

5. We optimize $f(x,y) = x^2 + y^2$, the square of the distance to the origin, subject to the constraint $g(x,y) = xy^2 - 54 = 0$. Thus $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = y^2\mathbf{i} + 2xy\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} = \lambda\left(y^2\mathbf{i} + 2xy\mathbf{j}\right) \Rightarrow 2x = \lambda y^2$ and $2y = 2\lambda xy$.
CASE 1: If $y = 0$, then $x = 0$. But $(0,0)$ does not satisfy the constraint $xy^2 = 54$ so $y \neq 0$.
CASE 2: If $y \neq 0$, then $2 = 2\lambda x \Rightarrow x = \frac{1}{\lambda} \Rightarrow 2\left(\frac{1}{\lambda}\right) = \lambda y^2 \Rightarrow y^2 = \frac{2}{\lambda^2}$. Then $xy^2 = 54 \Rightarrow \left(\frac{1}{\lambda}\right)\left(\frac{2}{\lambda^2}\right) = 54$ $\Rightarrow \lambda^3 = \frac{1}{27} \Rightarrow \lambda = \frac{1}{3} \Rightarrow x = 3$ and $y^2 = 18 \Rightarrow x = 3$ and $y = \pm 3\sqrt{2}$.
Therefore $\left(3, \pm 3\sqrt{2}\right)$ are the points on the curve $xy^2 = 54$ nearest the origin (since $xy^2 = 54$ has points increasingly far away as y gets close to 0, no points are farthest away).

6. We optimize $f(x,y) = x^2 + y^2$, the square of the distance to the origin subject to the constraint $g(x,y) = x^2y - 2 = 0$. Thus $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = 2xy\mathbf{i} + x^2\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x = 2xy\lambda$ and $2y = x^2\lambda$ $\Rightarrow \lambda = \frac{2y}{x^2}$, since $x = 0 \Rightarrow y = 0$ (but $g(0,0) \neq 0$). Thus $x \neq 0$ and $2x = 2xy\left(\frac{2y}{x^2}\right) \Rightarrow x^2 = 2y^2$ $\Rightarrow \left(2y^2\right)y - 2 = 0 \Rightarrow y = 1$ (since $y > 0$) $\Rightarrow x = \pm\sqrt{2}$. Therefore $\left(\pm\sqrt{2}, 1\right)$ are the points on the curve $x^2y = 2$ nearest the origin (since $x^2y = 2$ has points increasingly far away as x gets close to 0, no points are farthest away).

7. (a) $\nabla f = \mathbf{i} + \mathbf{j}$ and $\nabla g = y\mathbf{i} + x\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} + \mathbf{j} = \lambda(y\mathbf{i} + x\mathbf{j}) \Rightarrow 1 = \lambda y$ and $1 = \lambda x \Rightarrow y = \frac{1}{\lambda}$ and $x = \frac{1}{\lambda} \Rightarrow \frac{1}{\lambda^2} = 16 \Rightarrow \lambda = \pm\frac{1}{4}$. Use $\lambda = \frac{1}{4}$ since $x > 0$ and $y > 0$. Then $x = 4$ and $y = 4 \Rightarrow$ the minimum value is 8 at the point $(4,4)$. Now, $xy = 16$, $x > 0$, $y > 0$ is a branch of a hyperbola in the first quadrant with the x-and y-axes as asymptotes. The equations $x + y = c$ give a family of parallel lines with $m = -1$. As these lines move away from the origin, the number c increases. Thus the minimum value of c occurs where $x + y = c$ is tangent to the hyperbola's branch.

   (b) $\nabla f = y\mathbf{i} + x\mathbf{j}$ and $\nabla g = \mathbf{i} + \mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow y\mathbf{i} + x\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j}) \Rightarrow y = \lambda = x\ y + y = 16 \Rightarrow y = 8$ $\Rightarrow x = 8 \Rightarrow f(8,8) = 64$ is the maximum value. The equations $xy = c$ ($x > 0$ and $y > 0$ or $x < 0$ and $y < 0$ to get a maximum value) give a family of hyperbolas in the first and third quadrants with the x- and y-axes as asymptotes. The maximum value of c occurs where the hyperbola $xy = c$ is tangent to the line $x + y = 16$.

8. Let $f(x,y) = x^2 + y^2$ be the square of the distance from the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = (2x + y)\mathbf{i} + (2y + x)\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x = \lambda(2x + y)$ and $2y = \lambda(2y + x) \Rightarrow \frac{2y}{2y+x} = \lambda$ $\Rightarrow 2x = \left(\frac{2y}{2y+x}\right)(2x + y) \Rightarrow x(2y + x) = y(2x + y) \Rightarrow x^2 = y^2 \Rightarrow y = \pm x$.
CASE 1: $y = x \Rightarrow x^2 + x(x) + x^2 - 1 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$ and $y = x$.
CASE 2: $y = -x \Rightarrow x^2 + x(-x) + (-x)^2 - 1 = 0 \Rightarrow x = \pm 1$ and $y = -x$. Thus $f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{3}$ $= f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ and $f(1,-1) = 2 = f(-1,1)$.
Therefore the points $(1,-1)$ and $(-1,1)$ are the farthest away; $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ are the closest points to the origin.

9. $V = \pi r^2 h \Rightarrow 16\pi = \pi r^2 h \Rightarrow 16 = r^2 h \Rightarrow g(r,h) = r^2h - 16$; $S = 2\pi rh + 2\pi r^2 \Rightarrow \nabla S = (2\pi h + 4\pi r)\mathbf{i} + 2\pi r\mathbf{j}$ and $\nabla g = 2rh\mathbf{i} + r^2\mathbf{j}$ so that $\nabla S = \lambda \nabla g \Rightarrow (2\pi rh + 4\pi r)\mathbf{i} + 2\pi r\mathbf{j} = \lambda\left(2rh\mathbf{i} + r^2\mathbf{j}\right) \Rightarrow 2\pi rh + 4\pi r = 2rh\lambda$ and $2\pi r = \lambda r^2 \Rightarrow r = 0$ or $\lambda = \frac{2\pi}{r}$. But $r = 0$ gives no physical can, so $r \neq 0 \Rightarrow \lambda = \frac{2\pi}{r} \Rightarrow 2\pi h + 4\pi r = 2rh\left(\frac{2\pi}{r}\right) \Rightarrow 2r = h \Rightarrow 16 = r^2(2r) \Rightarrow r = 2 \Rightarrow h = 4$; thus $r = 2$ cm and $h = 4$ cm give the only extreme surface area of 24π cm^2. Since $r = 4$ cm and $h = 1$ cm $\Rightarrow V = 16\pi$ cm^3 and $S = 40\pi$ cm^2, which is a larger surface area, then 24π cm^2 must be the minimum surface area.

10. For a cylinder of radius r and height h we want to maximize the surface area $S = 2\pi rh$ subject to the constraint $g(r,h) = r^2 + \left(\frac{h}{2}\right)^2 - a^2 = 0$. Thus $\nabla S = 2\pi h\mathbf{i} + 2\pi r\mathbf{j}$ and $\nabla g = 2r\mathbf{i} + \frac{h}{2}\mathbf{j}$ so that $\nabla S = \lambda \nabla g \Rightarrow 2\pi h = 2\lambda r$ and $2\pi r = \frac{\lambda h}{2} \Rightarrow \frac{\pi h}{r} = \lambda$ and $2\pi r = \left(\frac{\pi h}{r}\right)\left(\frac{h}{2}\right) \Rightarrow 4r^2 = h^2 \Rightarrow h = 2r \Rightarrow r^2 + \frac{4r^2}{4} = a^2 \Rightarrow 2r^2 = a^2 \Rightarrow r = \frac{a}{\sqrt{2}}$ $\Rightarrow h = a\sqrt{2} \Rightarrow S = 2\pi\left(\frac{a}{\sqrt{2}}\right)\left(a\sqrt{2}\right) = 2\pi a^2$.

11. $A = (2x)(2y) = 4xy$ subject to $g(x,y) = \frac{x^2}{16} + \frac{y^2}{9} - 1 = 0$; $\nabla A = 4y\mathbf{i} + 4x\mathbf{j}$ and $\nabla g = \frac{x}{8}\mathbf{i} + \frac{2y}{9}\mathbf{j}$ so that $\nabla A = \lambda \nabla g \Rightarrow 4y\mathbf{i} + 4x\mathbf{j} = \lambda\left(\frac{x}{8}\mathbf{i} + \frac{2y}{9}\mathbf{j}\right) \Rightarrow 4y = \left(\frac{x}{8}\right)\lambda$ and $4x = \left(\frac{2y}{9}\right)\lambda \Rightarrow \lambda = \frac{32y}{x}$ and $4x = \left(\frac{2y}{9}\right)\left(\frac{32y}{x}\right)$ $\Rightarrow y = \pm\frac{3}{4}x \Rightarrow \frac{x^2}{16} + \frac{\left(\frac{\pm 3}{4}x\right)^2}{9} = 1 \Rightarrow x^2 = 8 \Rightarrow x = \pm 2\sqrt{2}$. We use $x = 2\sqrt{2}$ since x represents distance. Then $y = \frac{3}{4}\left(2\sqrt{2}\right) = \frac{3\sqrt{2}}{2}$, so the length is $2x = 4\sqrt{2}$ and the width is $2y = 3\sqrt{2}$.

12. $P = 4x + 4y$ subject to $g(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$; $\nabla P = 4\mathbf{i} + 4\mathbf{j}$ and $\nabla g = \frac{2x}{a^2}\mathbf{i} + \frac{2y}{b^2}\mathbf{j}$ so that $\nabla P = \lambda \nabla g$ $\Rightarrow 4 = \left(\frac{2x}{a^2}\right)\lambda$ and $4 = \left(\frac{2y}{b^2}\right)\lambda \Rightarrow \lambda = \frac{2a^2}{x}$ and $4 = \left(\frac{2y}{b^2}\right)\left(\frac{2a^2}{x}\right) \Rightarrow y = \left(\frac{b^2}{a^2}\right)x \Rightarrow \frac{x^2}{a^2} + \frac{\left(\frac{b^2}{a^2}\right)^2 x^2}{b^2} = 1 \Rightarrow \frac{x^2}{a^2} + \frac{b^2x^2}{a^4}$ $= 1 \Rightarrow (a^2 + b^2)x^2 = a^4 \Rightarrow x = \frac{a^2}{\sqrt{a^2+b^2}}$, since $x > 0 \Rightarrow y = \left(\frac{b^2}{a^2}\right)x = \frac{b^2}{\sqrt{a^2+b^2}} \Rightarrow$ width $= 2x = \frac{2a^2}{\sqrt{a^2+b^2}}$ and height $= 2y = \frac{2b^2}{\sqrt{a^2+b^2}} \Rightarrow$ perimeter is $P = 4x + 4y = \frac{4a^2+4b^2}{\sqrt{a^2+b^2}} = 4\sqrt{a^2+b^2}$

13. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = (2x-2)\mathbf{i} + (2y-4)\mathbf{j}$ so that $\nabla f = \lambda \nabla g = 2x\mathbf{i} + 2y\mathbf{j} = \lambda[(2x-2)\mathbf{i} + (2y-4)\mathbf{j}]$ $\Rightarrow 2x = \lambda(2x-2)$ and $2y = \lambda(2y-4) \Rightarrow x = \frac{\lambda}{\lambda-1}$ and $y = \frac{2\lambda}{\lambda-1}$, $\lambda \neq 1 \Rightarrow y = 2x \Rightarrow x^2 - 2x + (2x)^2 - 4(2x)$ $= 0 \Rightarrow x = 0$ and $y = 0$, or $x = 2$ and $y = 4$. Therefore $f(0,0) = 0$ is the minimum value and $f(2,4) = 20$ is the maximum value. (Note that $\lambda = 1$ gives $2x = 2x - 2$ or $0 = -2$, which is impossible.)

14. $\nabla f = 3\mathbf{i} - \mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 3 = 2\lambda x$ and $-1 = 2\lambda y \Rightarrow \lambda = \frac{3}{2x}$ and $-1 = 2\left(\frac{3}{2x}\right)y$ $\Rightarrow y = -\frac{x}{3} \Rightarrow x^2 + \left(-\frac{x}{3}\right)^2 = 4 \Rightarrow 10x^2 = 36 \Rightarrow x = \pm\frac{6}{\sqrt{10}} \Rightarrow x = \frac{6}{\sqrt{10}}$ and $y = -\frac{2}{\sqrt{10}}$, or $x = -\frac{6}{\sqrt{10}}$ and $y = \frac{2}{\sqrt{10}}$. Therefore $f\left(\frac{6}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right) = \frac{20}{\sqrt{10}} + 6 = 2\sqrt{10} + 6 \approx 12.325$ is the maximum value, and $f\left(-\frac{6}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right) = -2\sqrt{10} + 6 \approx -0.325$ is the minimum value.

15. $\nabla T = (8x - 4y)\mathbf{i} + (-4x + 2y)\mathbf{j}$ and $g(x,y) = x^2 + y^2 - 25 = 0 \Rightarrow \nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla T = \lambda \nabla g$ $\Rightarrow (8x - 4y)\mathbf{i} + (-4x + 2y)\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow 8x - 4y = 2\lambda x$ and $-4x + 2y = 2\lambda y \Rightarrow y = \frac{-2x}{\lambda-1}$, $\lambda \neq 1$ $\Rightarrow 8x - 4\left(\frac{-2x}{\lambda-1}\right) = 2\lambda x \Rightarrow x = 0$, or $\lambda = 0$, or $\lambda = 5$.

CASE 1: $x = 0 \Rightarrow y = 0$; but $(0,0)$ is not on $x^2 + y^2 = 25$ so $x \neq 0$.

CASE 2: $\lambda = 0 \Rightarrow y = 2x \Rightarrow x^2 + (2x)^2 = 25 \Rightarrow x = \pm\sqrt{5}$ and $y = 2x$.

CASE 3: $\lambda = 5 \Rightarrow y = \frac{-2x}{4} = -\frac{x}{2} \Rightarrow x^2 + \left(-\frac{x}{2}\right)^2 = 25 \Rightarrow x = \pm 2\sqrt{5} \Rightarrow x = 2\sqrt{5}$ and $y = -\sqrt{5}$, or $x = -2\sqrt{5}$ and $y = \sqrt{5}$.

Therefore $T\left(\sqrt{5}, 2\sqrt{5}\right) = 0° = T\left(-\sqrt{5}, -2\sqrt{5}\right)$ is the minimum value and $T\left(2\sqrt{5}, -\sqrt{5}\right) = 125°$ $= T\left(-2\sqrt{5}, \sqrt{5}\right)$ is the maximum value. (Note: $\lambda = 1 \Rightarrow x = 0$ from the equation $-4x + 2y = 2\lambda y$; but we found $x \neq 0$ in CASE 1.)

16. The surface area is given by $S = 4\pi r^2 + 2\pi rh$ subject to the constraint $V(r,h) = \frac{4}{3}\pi r^3 + \pi r^2 h = 8000$. Thus $\nabla S = (8\pi r + 2\pi h)\mathbf{i} + 2\pi r\mathbf{j}$ and $\nabla V = (4\pi r^2 + 2\pi rh)\mathbf{i} + \pi r^2\mathbf{j}$ so that $\nabla S = \lambda \nabla V = (8\pi r + 2\pi h)\mathbf{i} + 2\pi r\mathbf{j}$ $= \lambda\left[(4\pi r^2 + 2\pi rh)\mathbf{i} + \pi r^2\mathbf{j}\right] \Rightarrow 8\pi r + 2\pi h = \lambda(4\pi r^2 + 2\pi rh)$ and $2\pi r = \lambda\pi r^2 \Rightarrow r = 0$ or $2 = r\lambda$. But $r \neq 0$ so $2 = r\lambda \Rightarrow \lambda = \frac{2}{r} \Rightarrow 4r + h = \frac{2}{r}(2r^2 + rh) \Rightarrow h = 0 \Rightarrow$ the tank is a sphere (there is no cylindrical part) and $\frac{4}{3}\pi r^3 = 8000 \Rightarrow r = 10\left(\frac{6}{\pi}\right)^{1/3}$.

17. Let $f(x, y, z) = (x-1)^2 + (y-1)^2 + (z-1)^2$ be the square of the distance from $(1, 1, 1)$. Then $\nabla f = 2(x-1)\mathbf{i} + 2(y-1)\mathbf{j} + 2(z-1)\mathbf{k}$ and $\nabla g = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ so that $\nabla f = \lambda \nabla g$ $\Rightarrow\ 2(x-1)\mathbf{i} + 2(y-1)\mathbf{j} + 2(z-1)\mathbf{k} = \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})\ \Rightarrow\ 2(x-1) = \lambda,\ 2(y-1) = 2\lambda,\ 2(z-1) = 3\lambda$ $\Rightarrow\ 2(y-1) = 2[2(x-1)]$ and $2(z-1) = 3[2(x-1)]\ \Rightarrow\ x = \frac{y+1}{2}\ \Rightarrow\ z + 2 = 3\left(\frac{y+1}{2}\right)$ or $z = \frac{3y-1}{2}$; thus $\frac{y+1}{2} + 2y + 3\left(\frac{3y-1}{2}\right) - 13 = 0\ \Rightarrow\ y = 2\ \Rightarrow\ x = \frac{3}{2}$ and $z = \frac{5}{2}$. Therefore the point $\left(\frac{3}{2}, 2, \frac{5}{2}\right)$ is closest (since no point on the plane is farthest from the point $(1, 1, 1)$).

18. Let $f(x, y, z) = (x-1)^2 + (y+1)^2 + (z-1)^2$ be the square of the distance from $(1, -1, 1)$. Then $\nabla f = 2(x-1)\mathbf{i} + 2(y+1)\mathbf{j} + 2(z-1)\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow x - 1 = \lambda x,\ y + 1 = \lambda y$ and $z - 1 = \lambda z\ \Rightarrow\ x = \frac{1}{1-\lambda},\ y = -\frac{1}{1-\lambda}$, and $z = \frac{1}{1-\lambda}$ for $\lambda \neq 1\ \Rightarrow\ \left(\frac{1}{1-\lambda}\right)^2 + \left(\frac{-1}{1-\lambda}\right)^2 + \left(\frac{1}{1-\lambda}\right)^2 = 4$ $\Rightarrow\ \frac{1}{1-\lambda} = \pm\frac{2}{\sqrt{3}}\ \Rightarrow\ x = \frac{2}{\sqrt{3}},\ y = -\frac{2}{\sqrt{3}},\ z = \frac{2}{\sqrt{3}}$ or $x = -\frac{2}{\sqrt{3}},\ y = \frac{2}{\sqrt{3}},\ z = -\frac{2}{\sqrt{3}}$. The largest value of f occurs where $x < 0$, $y > 0$, and $z < 0$ or at the point $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$ on the sphere.

19. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance from the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = 2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g\ \Rightarrow\ 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k})\ \Rightarrow\ 2x = 2x\lambda,\ 2y = -2y\lambda$, and $2z = -2z\lambda\ \Rightarrow\ x = 0$ or $\lambda = 1$.
CASE 1: $\lambda = 1\ \Rightarrow\ 2y = -2y\ \Rightarrow\ y = 0;\ 2z = -2z\ \Rightarrow\ z = 0\ \Rightarrow\ x^2 - 1 = 0\ \Rightarrow\ x = \pm 1$ and $y = z = 0$.
CASE 2: $x = 0\ \Rightarrow\ -y^2 - z^2 = 1$, which has no solution.
Therefore the points on the unit circle $x^2 + y^2 = 1$, are the points on the surface $x^2 + y^2 - z^2 = 1$ closest to the origin. The minimum distance is 1.

20. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance to the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = y\mathbf{i} + x\mathbf{j} - \mathbf{k}$ so that $\nabla f = \lambda \nabla g\ \Rightarrow\ 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(y\mathbf{i} + x\mathbf{j} - \mathbf{k})\ \Rightarrow\ 2x = \lambda y,\ 2y = \lambda x$, and $2z = -\lambda$ $\Rightarrow\ x = \frac{\lambda y}{2}\ \Rightarrow\ 2y = \lambda\left(\frac{\lambda y}{2}\right)\ \Rightarrow\ y = 0$ or $\lambda = \pm 2$.
CASE 1: $y = 0\ \Rightarrow\ x = 0\ \Rightarrow\ -z + 1 = 0\ \Rightarrow\ z = 1$.
CASE 2: $\lambda = 2\ \Rightarrow\ x = y$ and $z = -1\ \Rightarrow\ x^2 - (-1) + 1 = 0\ \Rightarrow\ x^2 + 2 = 0$, so no solution.
CASE 3: $\lambda = -2\ \Rightarrow\ x = -y$ and $z = 1\ \Rightarrow\ (-y)y - 1 + 1 = 0\ \Rightarrow\ y = 0$, again.
Therefore $(0, 0, 1)$ is the point on the surface closest to the origin since this point gives the only extreme value and there is no maximum distance from the surface to the origin.

21. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance to the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = -y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g\ \Rightarrow\ 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(-y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k})\ \Rightarrow\ 2x = -y\lambda,\ 2y = -x\lambda$, and $2z = 2z\lambda\ \Rightarrow\ \lambda = 1$ or $z = 0$.
CASE 1: $\lambda = 1\ \Rightarrow\ 2x = -y$ and $2y = -x\ \Rightarrow\ y = 0$ and $x = 0\ \Rightarrow\ z^2 - 4 = 0\ \Rightarrow\ z = \pm 2$ and $x = y = 0$.
CASE 2: $z = 0\ \Rightarrow -xy - 4 = 0\ \Rightarrow\ y = -\frac{4}{x}$. Then $2x = \frac{4}{x}\lambda\ \Rightarrow\ \lambda = \frac{x^2}{2}$, and $-\frac{8}{x} = -x\lambda\ \Rightarrow\ -\frac{8}{x} = -x\left(\frac{x^2}{2}\right)$
$\Rightarrow\ x^4 = 16\ \Rightarrow\ x = \pm 2$. Thus, $x = 2$ and $y = -2$, or $x = -2$ and $y = 2$.
Therefore we get four points: $(2, -2, 0)$, $(-2, 2, 0)$, $(0, 0, 2)$ and $(0, 0, -2)$. But the points $(0, 0, 2)$ and $(0, 0, -2)$ are closest to the origin since they are 2 units away and the others are $2\sqrt{2}$ units away.

22. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance to the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ so that $\nabla f = \lambda \nabla g\ \Rightarrow\ 2x = \lambda yz,\ 2y = \lambda xz$, and $2z = \lambda xy\ \Rightarrow\ 2x^2 = \lambda xyz$ and $2y^2 = \lambda yxz$ $\Rightarrow\ x^2 = y^2\ \Rightarrow\ y = \pm x\ \Rightarrow\ z = \pm x\ \Rightarrow\ x(\pm x)(\pm x) = 1\ \Rightarrow\ x = \pm 1\ \Rightarrow$ the points are $(1, 1, 1)$, $(1, -1, -1)$, $(-1, -1, 1)$, and $(-1, 1, -1)$.

23. $\nabla f = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} - 2\mathbf{j} + 5\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 1 = 2x\lambda$, $-2 = 2y\lambda$, and $5 = 2z\lambda\ \Rightarrow\ x = \frac{1}{2\lambda},\ y = -\frac{1}{\lambda} = -2x$, and $z = \frac{5}{2\lambda} = 5x\ \Rightarrow\ x^2 + (-2x)^2 + (5x)^2 = 30\ \Rightarrow\ x = \pm 1$.

Thus, $x = 1$, $y = -2$, $z = 5$ or $x = -1$, $y = 2$, $z = -5$. Therefore $f(1, -2, 5) = 30$ is the maximum value and $f(-1, 2, -5) = -30$ is the minimum value.

24. $\nabla f = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 1 = 2x\lambda$, $2 = 2y\lambda$, and $3 = 2z\lambda \Rightarrow x = \frac{1}{2\lambda}$, $y = \frac{1}{\lambda} = 2x$, and $z = \frac{3}{2\lambda} = 3x \Rightarrow x^2 + (2x)^2 + (3x)^2 = 25 \Rightarrow x = \pm \frac{5}{\sqrt{14}}$. Thus, $x = \frac{5}{\sqrt{14}}$, $y = \frac{10}{\sqrt{14}}$, $z = \frac{15}{\sqrt{14}}$ or $x = -\frac{5}{\sqrt{14}}$, $y = -\frac{10}{\sqrt{14}}$, $z = -\frac{15}{\sqrt{14}}$. Therefore $f\left(\frac{5}{\sqrt{14}}, \frac{10}{\sqrt{14}}, \frac{15}{\sqrt{14}}\right) = 5\sqrt{14}$ is the maximum value and $f\left(-\frac{5}{\sqrt{14}}, -\frac{10}{\sqrt{14}}, -\frac{15}{\sqrt{14}}\right) = -5\sqrt{14}$ is the minimum value.

25. $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = x + y + z - 9 = 0 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = \mathbf{i} + \mathbf{j} + \mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) \Rightarrow 2x = \lambda$, $2y = \lambda$, and $2z = \lambda \Rightarrow x = y = z \Rightarrow x + x + x - 9 = 0 \Rightarrow x = 3$, $y = 3$, and $z = 3$.

26. $f(x, y, z) = xyz$ and $g(x, y, z) = x + y + z^2 - 16 = 0 \Rightarrow \nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and $\nabla g = \mathbf{i} + \mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + 2z\mathbf{k}) \Rightarrow yz = \lambda$, $xz = \lambda$, and $xy = 2z\lambda \Rightarrow yz = xz \Rightarrow z = 0$ or $y = x$. But $z > 0$ so that $y = x \Rightarrow x^2 = 2z\lambda$ and $xz = \lambda$. Then $x^2 = 2z(xz) \Rightarrow x = 0$ or $x = 2z^2$. But $x > 0$ so that $x = 2z^2 \Rightarrow y = 2z^2 \Rightarrow 2z^2 + 2z^2 + z^2 = 16 \Rightarrow z = \pm \frac{4}{\sqrt{5}}$. We use $z = \frac{4}{\sqrt{5}}$ since $z > 0$. Then $x = \frac{32}{5}$ and $y = \frac{32}{5}$ which yields $f\left(\frac{32}{5}, \frac{32}{5}, \frac{4}{\sqrt{5}}\right) = \frac{4096}{25\sqrt{5}}$.

27. $V = 6xyz$ and $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0 \Rightarrow \nabla V = 6yz\mathbf{i} + 6xz\mathbf{j} + 6xy\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla V = \lambda \nabla g \Rightarrow 3yz = \lambda x$, $3xz = \lambda y$, and $3xy = \lambda z \Rightarrow 3xyz = \lambda x^2$ and $3xyz = \lambda y^2 \Rightarrow y = \pm x \Rightarrow z = \pm x \Rightarrow x^2 + x^2 + x^2 = 1 \Rightarrow x = \frac{1}{\sqrt{3}}$ since $x > 0 \Rightarrow$ the dimensions of the box are $\frac{2}{\sqrt{3}}$ by $\frac{2}{\sqrt{3}}$ by $\frac{2}{\sqrt{3}}$ for maximum volume. (Note that there is no minimum volume since the box could be made arbitrarily thin.)

28. $V = xyz$ with x, y, z all positive and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$; thus $V = xyz$ and $g(x, y, z) = bcx + acy + abz - abc = 0 \Rightarrow \nabla V = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and $\nabla g = bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k}$ so that $\nabla V = \lambda \nabla g \Rightarrow yz = \lambda bc$, $xz = \lambda ac$, and $xy = \lambda ab \Rightarrow xyz = \lambda bcx$, $xyz = \lambda acy$, and $xyz = \lambda abz \Rightarrow \lambda \neq 0$. Also, $\lambda bcx = \lambda acy = \lambda abz \Rightarrow bx = ay$, $cy = bz$, and $cx = az \Rightarrow y = \frac{b}{a}x$ and $z = \frac{c}{a}x$. Then $\frac{x}{a} + \frac{y}{b} + \frac{c}{z} = 1 \Rightarrow \frac{x}{a} + \frac{1}{b}\left(\frac{b}{a}x\right) + \frac{1}{c}\left(\frac{c}{a}x\right) = 1 \Rightarrow \frac{3x}{a} = 1 \Rightarrow x = \frac{a}{3} \Rightarrow y = \left(\frac{b}{a}\right)\left(\frac{a}{3}\right) = \frac{b}{3}$ and $z = \left(\frac{c}{a}\right)\left(\frac{a}{3}\right) = \frac{c}{3} \Rightarrow V = xyz = \left(\frac{a}{3}\right)\left(\frac{b}{3}\right)\left(\frac{c}{3}\right) = \frac{abc}{27}$ is the maximum volume. (Note that there is no minimum volume since the box could be made arbitrarily thin.)

29. $\nabla T = 16x\mathbf{i} + 4z\mathbf{j} + (4y - 16)\mathbf{k}$ and $\nabla g = 8x\mathbf{i} + 2y\mathbf{j} + 8z\mathbf{k}$ so that $\nabla T = \lambda \nabla g \Rightarrow 16x\mathbf{i} + 4z\mathbf{j} + (4y - 16)\mathbf{k} = \lambda(8x\mathbf{i} + 2y\mathbf{j} + 8z\mathbf{k}) \Rightarrow 16x = 8x\lambda$, $4z = 2y\lambda$, and $4y - 16 = 8z\lambda \Rightarrow \lambda = 2$ or $x = 0$.

CASE 1: $\lambda = 2 \Rightarrow 4z = 2y(2) \Rightarrow z = y$. Then $4z - 16 = 16z \Rightarrow z = -\frac{4}{3} \Rightarrow y = -\frac{4}{3}$. Then $4x^2 + \left(-\frac{4}{3}\right)^2 + 4\left(-\frac{4}{3}\right)^2 = 16 \Rightarrow x = \pm \frac{4}{3}$.

CASE 2: $x = 0 \Rightarrow \lambda = \frac{2z}{y} \Rightarrow 4y - 16 = 8z\left(\frac{2z}{y}\right) \Rightarrow y^2 - 4y = 4z^2 \Rightarrow 4(0)^2 + y^2 + (y^2 - 4y) - 16 = 0 \Rightarrow y^2 - 2y - 8 = 0 \Rightarrow (y - 4)(y + 2) = 0 \Rightarrow y = 4$ or $y = -2$. Now $y = 4 \Rightarrow 4z^2 = 4^2 - 4(4) \Rightarrow z = 0$ and $y = -2 \Rightarrow 4z^2 = (-2)^2 - 4(-2) \Rightarrow z = \pm \sqrt{3}$.

The temperatures are $T\left(\pm \frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right) = 642\frac{2}{3}^\circ$, $T(0, 4, 0) = 600^\circ$, $T\left(0, -2, \sqrt{3}\right) = \left(600 - 24\sqrt{3}\right)^\circ$, and $T\left(0, -2, -\sqrt{3}\right) = \left(600 + 24\sqrt{3}\right)^\circ \approx 641.6^\circ$. Therefore $\left(\pm \frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$ are the hottest points on the space probe.

30. $\nabla T = 400yz^2\mathbf{i} + 400xz^2\mathbf{j} + 800xyz\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla T = \lambda \nabla g \Rightarrow 400yz^2\mathbf{i} + 400xz^2\mathbf{j} + 800xyz\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 400yz^2 = 2x\lambda$, $400xz^2 = 2y\lambda$, and $800xyz = 2z\lambda$. Solving this system yields the points $(0, \pm 1, 0)$, $(\pm 1, 0, 0)$, and $\left(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{\sqrt{2}}{2}\right)$. The corresponding

temperatures are $T(0, \pm 1, 0) = 0$, $T(\pm 1, 0, 0) = 0$, and $T\left(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{\sqrt{2}}{2}\right) = \pm 50$. Therefore 50 is the maximum temperature at $\left(\frac{1}{2}, \frac{1}{2}, \pm \frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{1}{2}, \pm \frac{\sqrt{2}}{2}\right)$; -50 is the minimum temperature at $\left(\frac{1}{2}, -\frac{1}{2}, \pm \frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}, \pm \frac{\sqrt{2}}{2}\right)$.

31. $\nabla U = (y+2)\mathbf{i} + x\mathbf{j}$ and $\nabla g = 2\mathbf{i} + \mathbf{j}$ so that $\nabla U = \lambda \nabla g \Rightarrow (y+2)\mathbf{i} + x\mathbf{j} = \lambda(2\mathbf{i} + \mathbf{j}) \Rightarrow y + 2 = 2\lambda$ and $x = \lambda \Rightarrow y + 2 = 2x \Rightarrow y = 2x - 2 \Rightarrow 2x + (2x - 2) = 30 \Rightarrow x = 8$ and $y = 14$. Therefore $U(8, 14) = \$128$ is the maximum value of U under the constraint.

32. $\nabla M = (6+z)\mathbf{i} - 2y\mathbf{j} + x\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla M = \lambda \nabla g \Rightarrow (6+z)\mathbf{i} - 2y\mathbf{j} + x\mathbf{k}$ $= \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 6 + z = 2x\lambda, -2y = 2y\lambda, x = 2z\lambda \Rightarrow \lambda = -1$ or $y = 0$.

CASE 1: $\lambda = -1 \Rightarrow 6 + z = -2x$ and $x = -2z \Rightarrow 6 + z = -2(-2z) \Rightarrow z = 2$ and $x = -4$. Then $(-4)^2 + y^2 + 2^2 - 36 = 0 \Rightarrow y = \pm 4$.

CASE 2: $y = 0, 6 + z = 2x\lambda$, and $x = 2z\lambda \Rightarrow \lambda = \frac{x}{2z} \Rightarrow 6 + z = 2x\left(\frac{x}{2z}\right) \Rightarrow 6z + z^2 = x^2$ $\Rightarrow (6z + z^2) + 0^2 + z^2 = 36 \Rightarrow z = -6$ or $z = 3$. Now $z = -6 \Rightarrow x^2 = 0 \Rightarrow x = 0$; $z = 3$ $\Rightarrow x^2 = 27 \Rightarrow x = \pm 3\sqrt{3}$.

Therefore we have the points $\left(\pm 3\sqrt{3}, 0, 3\right)$, $(0, 0, -6)$, and $(-4, \pm 4, 2)$. Then $M\left(3\sqrt{3}, 0, 3\right)$ $= 27\sqrt{3} + 60 \approx 106.8$, $M\left(-3\sqrt{3}, 0, 3\right) = 60 - 27\sqrt{3} \approx 13.2$, $M(0, 0, -6) = 60$, and $M(-4, 4, 2) = 12$ $= M(-4, -4, 2)$. Therefore, the weakest field is at $(-4, \pm 4, 2)$.

33. Let $g_1(x, y, z) = 2x - y = 0$ and $g_2(x, y, z) = y + z = 0 \Rightarrow \nabla g_1 = 2\mathbf{i} - \mathbf{j}$, $\nabla g_2 = \mathbf{j} + \mathbf{k}$, and $\nabla f = 2x\mathbf{i} + 2\mathbf{j} - 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2x\mathbf{i} + 2\mathbf{j} - 2z\mathbf{k} = \lambda(2\mathbf{i} - \mathbf{j}) + \mu(\mathbf{j} + \mathbf{k}) \Rightarrow 2x\mathbf{i} + 2\mathbf{j} - 2z\mathbf{k} = 2\lambda\mathbf{i} + (\mu - \lambda)\mathbf{j} + \mu\mathbf{k}$ $\Rightarrow 2x = 2\lambda, 2 = \mu - \lambda$, and $-2z = \mu \Rightarrow x = \lambda$. Then $2 = -2z - x \Rightarrow x = -2z - 2$ so that $2x - y = 0$ $\Rightarrow 2(-2z - 2) - y = 0 \Rightarrow -4z - 4 - y = 0$. This equation coupled with $y + z = 0$ implies $z = -\frac{4}{3}$ and $y = \frac{4}{3}$. Then $x = \frac{2}{3}$ so that $\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right)$ is the point that gives the maximum value $f\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right) = \left(\frac{2}{3}\right)^2 + 2\left(\frac{4}{3}\right) - \left(-\frac{4}{3}\right)^2$ $= \frac{4}{3}$.

34. Let $g_1(x, y, z) = x + 2y + 3z - 6 = 0$ and $g_2(x, y, z) = x + 3y + 9z - 9 = 0 \Rightarrow \nabla g_1 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\nabla g_2 = \mathbf{i} + 3\mathbf{j} + 9\mathbf{k}$, and $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ $= \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \mu(\mathbf{i} + 3\mathbf{j} + 9\mathbf{k}) \Rightarrow 2x = \lambda + \mu, 2y = 2\lambda + 3\mu$, and $2z = 3\lambda + 9\mu$. Then $0 = x + 2y + 3z - 6$ $= \frac{1}{2}(\lambda + \mu) + (2\lambda + 3\mu) + \left(\frac{9}{2}\lambda + \frac{27}{2}\mu\right) - 6 \Rightarrow 7\lambda + 17\mu = 6$; $0 = x + 3y + 9z - 9$ $\Rightarrow \frac{1}{2}(\lambda + \mu) + \left(3\lambda + \frac{9}{2}\mu\right) + \left(\frac{27}{2}\lambda + \frac{81}{2}\mu\right) - 9 \Rightarrow 34\lambda + 91\mu = 18$. Solving these two equations for λ and μ gives $\lambda = \frac{240}{59}$ and $\mu = -\frac{78}{59} \Rightarrow x = \frac{\lambda + \mu}{2} = \frac{81}{59}$, $y = \frac{2\lambda + 3\mu}{2} = \frac{123}{59}$, and $z = \frac{3\lambda + 9\mu}{2} = \frac{9}{59}$. The minimum value is $f\left(\frac{81}{59}, \frac{123}{59}, \frac{9}{59}\right) = \frac{21{,}771}{59^2} = \frac{369}{59}$. (Note that there is no <u>maximum</u> value of f subject to the constraints because at least one of the variables x, y, or z can be made arbitrary and assume a value as large as we please.)

35. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance from the origin. We want to minimize $f(x, y, z)$ subject to the constraints $g_1(x, y, z) = y + 2z - 12 = 0$ and $g_2(x, y, z) = x + y - 6 = 0$. Thus $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\nabla g_1 = \mathbf{j} + 2\mathbf{k}$, and $\nabla g_2 = \mathbf{i} + \mathbf{j}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2x = \mu, 2y = \lambda + \mu$, and $2z = 2\lambda$. Then $0 = y + 2z - 12 = \left(\frac{\lambda}{2} + \frac{\mu}{2}\right) + 2\lambda - 12 \Rightarrow \frac{5}{2}\lambda + \frac{1}{2}\mu = 12 \Rightarrow 5\lambda + \mu = 24$; $0 = x + y - 6 = \frac{\mu}{2} + \left(\frac{\lambda}{2} + \frac{\mu}{2}\right) - 6$ $\Rightarrow \frac{1}{2}\lambda + \mu = 6 \Rightarrow \lambda + 2\mu = 12$. Solving these two equations for λ and μ gives $\lambda = 4$ and $\mu = 4 \Rightarrow x = \frac{\mu}{2} = 2$, $y = \frac{\lambda + \mu}{2} = 4$, and $z = \lambda = 4$. The point $(2, 4, 4)$ on the line of intersection is closest to the origin. (There is no maximum distance from the origin since points on the line can be arbitrarily far away.)

36. The maximum value is $f\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right) = \frac{4}{3}$ from Exercise 33 above.

37. Let $g_1(x,y,z) = z - 1 = 0$ and $g_2(x,y,z) = x^2 + y^2 + z^2 - 10 = 0 \Rightarrow \nabla g_1 = \mathbf{k}$, $\nabla g_2 = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, and $\nabla f = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k} = \lambda(\mathbf{k}) + \mu(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})$ $\Rightarrow 2xyz = 2x\mu$, $x^2z = 2y\mu$, and $x^2y = 2z\mu + \lambda \Rightarrow xyz = x\mu \Rightarrow x = 0$ or $yz = \mu \Rightarrow \mu = y$ since $z = 1$.
CASE 1: $x = 0$ and $z = 1 \Rightarrow y^2 - 9 = 0$ (from g_2) $\Rightarrow y = \pm 3$ yielding the points $(0, \pm 3, 1)$.
CASE 2: $\mu = y \Rightarrow x^2z = 2y^2 \Rightarrow x^2 = 2y^2$ (since $z = 1$) $\Rightarrow 2y^2 + y^2 + 1 - 10 = 0$ (from g_2) $\Rightarrow 3y^2 - 9 = 0$ $\Rightarrow y = \pm\sqrt{3} \Rightarrow x^2 = 2\left(\pm\sqrt{3}\right)^2 \Rightarrow x = \pm\sqrt{6}$ yielding the points $\left(\pm\sqrt{6}, \pm\sqrt{3}, 1\right)$.
Now $f(0, \pm 3, 1) = 1$ and $f\left(\pm\sqrt{6}, \pm\sqrt{3}, 1\right) = 6\left(\pm\sqrt{3}\right) + 1 = 1 \pm 6\sqrt{3}$. Therefore the maximum of f is $1 + 6\sqrt{3}$ at $\left(\pm\sqrt{6}, \sqrt{3}, 1\right)$, and the minimum of f is $1 - 6\sqrt{3}$ at $\left(\pm\sqrt{6}, -\sqrt{3}, 1\right)$.

38. (a) Let $g_1(x,y,z) = x + y + z - 40 = 0$ and $g_2(x,y,z) = x + y - z = 0 \Rightarrow \nabla g_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\nabla g_2 = \mathbf{i} + \mathbf{j} - \mathbf{k}$, and $\nabla w = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ so that $\nabla w = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k})$ $\Rightarrow yz = \lambda + \mu$, $xz = \lambda + \mu$, and $xy = \lambda - \mu \Rightarrow yz = xz \Rightarrow z = 0$ or $y = x$.
CASE 1: $z = 0 \Rightarrow x + y = 40$ and $x + y = 0 \Rightarrow$ no solution.
CASE 2: $x = y \Rightarrow 2x + z - 40 = 0$ and $2x - z = 0 \Rightarrow z = 20 \Rightarrow x = 10$ and $y = 10 \Rightarrow w = (10)(10)(20) = 2000$

(b) $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{j}$ is parallel to the line of intersection $\Rightarrow$ the line is $x = -2t + 10$, $y = 2t + 10$, $z = 20$. Since $z = 20$, we see that $w = xyz = (-2t + 10)(2t + 10)(20) = (-4t^2 + 100)(20)$ which has its maximum when $t = 0 \Rightarrow x = 10$, $y = 10$, and $z = 20$.

39. Let $g_1(x,y,z) = y - x = 0$ and $g_2(x,y,z) = x^2 + y^2 + z^2 - 4 = 0$. Then $\nabla f = y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$, $\nabla g_1 = -\mathbf{i} + \mathbf{j}$, and $\nabla g_2 = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k} = \lambda(-\mathbf{i} + \mathbf{j}) + \mu(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})$ $\Rightarrow y = -\lambda + 2x\mu$, $x = \lambda + 2y\mu$, and $2z = 2z\mu \Rightarrow z = 0$ or $\mu = 1$.
CASE 1: $z = 0 \Rightarrow x^2 + y^2 - 4 = 0 \Rightarrow 2x^2 - 4 = 0$ (since $x = y$) $\Rightarrow x = \pm\sqrt{2}$ and $y = \pm\sqrt{2}$ yielding the points $\left(\pm\sqrt{2}, \pm\sqrt{2}, 0\right)$.
CASE 2: $\mu = 1 \Rightarrow y = -\lambda + 2x$ and $x = \lambda + 2y \Rightarrow x + y = 2(x + y) \Rightarrow 2x = 2(2x)$ since $x = y \Rightarrow x = 0 \Rightarrow y = 0$ $\Rightarrow z^2 - 4 = 0 \Rightarrow z = \pm 2$ yielding the points $(0, 0, \pm 2)$.
Now, $f(0, 0, \pm 2) = 4$ and $f\left(\pm\sqrt{2}, \pm\sqrt{2}, 0\right) = 2$. Therefore the maximum value of f is 4 at $(0, 0, \pm 2)$ and the minimum value of f is 2 at $\left(\pm\sqrt{2}, \pm\sqrt{2}, 0\right)$.

40. Let $f(x,y,z) = x^2 + y^2 + z^2$ be the square of the distance from the origin. We want to minimize $f(x,y,z)$ subject to the constraints $g_1(x,y,z) = 2y + 4z - 5 = 0$ and $g_2(x,y,z) = 4x^2 + 4y^2 - z^2 = 0$. Thus $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\nabla g_1 = 2\mathbf{j} + 4\mathbf{k}$, and $\nabla g_2 = 8x\mathbf{i} + 8y\mathbf{j} - 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ $= \lambda(2\mathbf{j} + 4\mathbf{k}) + \mu(8x\mathbf{i} + 8y\mathbf{j} - 2z\mathbf{k}) \Rightarrow 2x = 8x\mu$, $2y = 2\lambda + 8y\mu$, and $2z = 4\lambda - 2z\mu \Rightarrow x = 0$ or $\mu = \frac{1}{4}$.
CASE 1: $x = 0 \Rightarrow 4(0)^2 + 4y^2 - z^2 = 0 \Rightarrow z = \pm 2y \Rightarrow 2y + 4(2y) - 5 = 0 \Rightarrow y = \frac{1}{2}$, or $2y + 4(-2y) - 5 = 0$ $\Rightarrow y = -\frac{5}{6}$ yielding the points $\left(0, \frac{1}{2}, 1\right)$ and $\left(0, -\frac{5}{6}, \frac{5}{3}\right)$.
CASE 2: $\mu = \frac{1}{4} \Rightarrow y = \lambda + y \Rightarrow \lambda = 0 \Rightarrow 2z = 4(0) - 2z\left(\frac{1}{4}\right) \Rightarrow z = 0 \Rightarrow 2y + 4(0) = 5 \Rightarrow y = \frac{5}{2}$ and $(0)^2 = 4x^2 + 4\left(\frac{5}{2}\right)^2 \Rightarrow$ no solution.
Then $f\left(0, \frac{1}{2}, 1\right) = \frac{5}{4}$ and $f\left(0, -\frac{5}{6}, \frac{5}{3}\right) = 25\left(\frac{1}{36} + \frac{1}{9}\right) = \frac{125}{36} \Rightarrow$ the point $\left(0, \frac{1}{2}, 1\right)$ is closest to the origin.

41. $\nabla f = \mathbf{i} + \mathbf{j}$ and $\nabla g = y\mathbf{i} + x\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} + \mathbf{j} = \lambda(y\mathbf{i} + x\mathbf{j}) \Rightarrow 1 = y\lambda$ and $1 = x\lambda \Rightarrow y = x$ $\Rightarrow y^2 = 16 \Rightarrow y = \pm 4 \Rightarrow (4, 4)$ and $(-4, -4)$ are candidates for the location of extreme values. But as $x \to \infty$, $y \to \infty$ and $f(x,y) \to \infty$; as $x \to -\infty$, $y \to 0$ and $f(x,y) \to -\infty$. Therefore no maximum or minimum value exists subject to the constraint.

42. Let $f(A,B,C) = \sum_{k=1}^{4} (Ax_k + By_k + C - z_k)^2 = C^2 + (B+C-1)^2 + (A+B+C-1)^2 + (A+C+1)^2$. We want to minimize f. Then $f_A(A,B,C) = 4A + 2B + 4C$, $f_B(A,B,C) = 2A + 4B + 4C - 4$, and $f_C(A,B,C) = 4A + 4B + 8C - 2$. Set each partial derivative equal to 0 and solve the system to get $A = -\frac{1}{2}$, $B = \frac{3}{2}$, and $C = -\frac{1}{4}$ or the critical point of f is $\left(-\frac{1}{2}, \frac{3}{2}, -\frac{1}{4}\right)$.

43. (a) Maximize $f(a,b,c) = a^2b^2c^2$ subject to $a^2 + b^2 + c^2 = r^2$. Thus $\nabla f = 2ab^2c^2\mathbf{i} + 2a^2bc^2\mathbf{j} + 2a^2b^2c\mathbf{k}$ and $\nabla g = 2a\mathbf{i} + 2b\mathbf{j} + 2c\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2ab^2c^2 = 2a\lambda$, $2a^2bc^2 = 2b\lambda$, and $2a^2b^2c = 2c\lambda$ $\Rightarrow 2a^2b^2c^2 = 2a^2\lambda = 2b^2\lambda = 2c^2\lambda \Rightarrow \lambda = 0$ or $a^2 = b^2 = c^2$.

CASE 1: $\lambda = 0 \Rightarrow a^2b^2c^2 = 0$.

CASE 2: $a^2 = b^2 = c^2 \Rightarrow f(a,b,c) = a^2a^2a^2$ and $3a^2 = r^2 \Rightarrow f(a,b,c) = \left(\frac{r^2}{3}\right)^3$ is the maximum value.

(b) The point $\left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right)$ is on the sphere if $a + b + c = r^2$. Moreover, by part (a), $abc = f\left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right) \le \left(\frac{r^2}{3}\right)^3 \Rightarrow (abc)^{1/3} \le \frac{r^2}{3} = \frac{a+b+c}{3}$, as claimed.

44. Let $f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_ix_i = a_1x_1 + a_2x_2 + \ldots + a_nx_n$ and $g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \ldots + x_n^2 - 1$. Then we want $\nabla f = \lambda \nabla g \Rightarrow a_1 = \lambda(2x_1)$, $a_2 = \lambda(2x_2)$, …, $a_n = \lambda(2x_n)$, $\lambda \ne 0 \Rightarrow x_i = \frac{a_i}{2\lambda} \Rightarrow \frac{a_1^2}{4\lambda^2} + \frac{a_2^2}{4\lambda^2} + \ldots + \frac{a_n^2}{4\lambda^2} = 1$ $\Rightarrow 4\lambda^2 = \sum_{i=1}^{n} a_i^2 \Rightarrow 2\lambda = \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \Rightarrow f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_ix_i = \sum_{i=1}^{n} a_i\left(\frac{a_i}{2\lambda}\right) = \frac{1}{2\lambda}\sum_{i=1}^{n} a_i^2 = \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}$ is the maximum value.

45-50. Example CAS commands:

Maple:

```
f := (x,y,z) -> x*y+y*z;
g1 := (x,y,z) -> x^2+y^2-2;
g2 := (x,y,z) -> x^2+z^2-2;
h := unapply( f(x,y,z)-lambda[1]*g1(x,y,z)-lambda[2]*g2(x,y,z), (x,y,z,lambda[1],lambda[2]) );   # (a)
hx := diff( h(x,y,z,lambda[1],lambda[2]), x );                                                     #(b)
hy := diff( h(x,y,z,lambda[1],lambda[2]), y );
hz := diff( h(x,y,z,lambda[1],lambda[2]), z );
hl1 := diff( h(x,y,z,lambda[1],lambda[2]), lambda[1] );
hl2 := diff( h(x,y,z,lambda[1],lambda[2]), lambda[2] );
sys := { hx=0, hy=0, hz=0, hl1=0, hl2=0 };
q1 := solve( sys, {x,y,z,lambda[1],lambda[2]} );                                                   # (c)
q2 := map(allvalues,{q1});
for p in q2 do                                                                                     # (d)
  eval( [x,y,z,f(x,y,z)], p );
  ``=evalf(eval( [x,y,z,f(x,y,z)], p ));
end do;
```

Mathematica: (assigned functions will vary)

```
Clear[x, y, z, lambda1, lambda2]
f[x_,y_,z_]:= x y + y z
g1[x_,y_,z_]:= x^2 + y^2 - 2
g2[x_,y_,z_]:= x^2 + z^2 - 2
h = f[x, y, z] - lambda1 g1[x, y, z] - lambda2 g2[x, y, z];
hx= D[h, x];  hy= D[h, y];  hz= D[h,z]; hL1=D[h, lambda1];  hL2= D[h, lambda2];
critical=Solve[{hx==0, hy==0, hz==0, hL1==0, hL2==0, g1[x,y,z]==0, g2[x,y,z]==0},
```

{x, y, z, lambda1, lambda2}]//N
{{x, y, z}, f[x, y, z]}/.critical

14.9 PARTIAL DERIVATIVES WITH CONSTRAINED VARIABLES

1. $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$:

(a) $\begin{pmatrix} y \\ z \end{pmatrix} \to \begin{pmatrix} x = x(y,z) \\ y = y \\ z = z \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial y}\right)_z = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}$; $\frac{\partial z}{\partial y} = 0$ and $\frac{\partial z}{\partial y} = 2x\frac{\partial x}{\partial y} + 2y\frac{\partial y}{\partial y}$

$= 2x\frac{\partial x}{\partial y} + 2y \Rightarrow 0 = 2x\frac{\partial x}{\partial y} + 2y \Rightarrow \frac{\partial x}{\partial y} = -\frac{y}{x} \Rightarrow \left(\frac{\partial w}{\partial y}\right)_z = (2x)\left(-\frac{y}{x}\right) + (2y)(1) + (2z)(0) = -2y + 2y = 0$

(b) $\begin{pmatrix} x \\ z \end{pmatrix} \to \begin{pmatrix} x = x \\ y = y(x,z) \\ z = z \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_x = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z}$; $\frac{\partial x}{\partial z} = 0$ and $\frac{\partial z}{\partial z} = 2x\frac{\partial x}{\partial z} + 2y\frac{\partial y}{\partial z}$

$\Rightarrow 1 = 2y\frac{\partial y}{\partial z} \Rightarrow \frac{\partial y}{\partial z} = \frac{1}{2y} \Rightarrow \left(\frac{\partial w}{\partial z}\right)_x = (2x)(0) + (2y)\left(\frac{1}{2y}\right) + (2z)(1) = 1 + 2z$

(c) $\begin{pmatrix} y \\ z \end{pmatrix} \to \begin{pmatrix} x = x(y,z) \\ y = y \\ z = z \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_y = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z}$; $\frac{\partial y}{\partial z} = 0$ and $\frac{\partial z}{\partial z} = 2x\frac{\partial x}{\partial z} + 2y\frac{\partial y}{\partial z}$

$\Rightarrow 1 = 2x\frac{\partial x}{\partial z} \Rightarrow \frac{\partial x}{\partial z} = \frac{1}{2x} \Rightarrow \left(\frac{\partial w}{\partial z}\right)_y = (2x)\left(\frac{1}{2x}\right) + (2y)(0) + (2z)(1) = 1 + 2z$

2. $w = x^2 + y - z + \sin t$ and $x + y = t$:

(a) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x = x \\ y = y \\ z = z \\ t = x + y \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial y}\right)_{x,z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y}$; $\frac{\partial x}{\partial y} = 0$, $\frac{\partial z}{\partial y} = 0$, and

$\frac{\partial t}{\partial y} = 1 \Rightarrow \left(\frac{\partial w}{\partial y}\right)_{x,t} = (2x)(0) + (1)(1) + (-1)(0) + (\cos t)(1) = 1 + \cos t = 1 + \cos(x + y)$

(b) $\begin{pmatrix} y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} x = t - y \\ y = y \\ z = z \\ t = t \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial y}\right)_{z,t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y}$; $\frac{\partial z}{\partial y} = 0$ and $\frac{\partial t}{\partial y} = 0$

$\Rightarrow \frac{\partial x}{\partial y} = \frac{\partial t}{\partial y} - \frac{\partial y}{\partial y} = -1 \Rightarrow \left(\frac{\partial w}{\partial y}\right)_{z,t} = (2x)(-1) + (1)(1) + (-1)(0) + (\cos t)(0) = 1 - 2(t - y) = 1 + 2y - 2t$

(c) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x = x \\ y = y \\ z = z \\ t = x + y \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_{x,y} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial z}$; $\frac{\partial x}{\partial z} = 0$ and $\frac{\partial y}{\partial z} = 0$

$\Rightarrow \left(\frac{\partial w}{\partial z}\right)_{x,y} = (2x)(0) + (1)(0) + (-1)(1) + (\cos t)(0) = -1$

(d) $\begin{pmatrix} y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} x = t - y \\ y = y \\ z = z \\ t = t \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_{y,t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial z}$; $\frac{\partial y}{\partial z} = 0$ and $\frac{\partial t}{\partial z} = 0$

$\Rightarrow \left(\frac{\partial w}{\partial z}\right)_{y,t} = (2x)(0) + (1)(0) + (-1)(1) + (\cos t)(0) = -1$

(e) $\begin{pmatrix} x \\ z \\ t \end{pmatrix} \to \begin{pmatrix} x = x \\ y = t - x \\ z = z \\ t = t \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial t}\right)_{x,z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial t}$; $\frac{\partial x}{\partial t} = 0$ and $\frac{\partial z}{\partial t} = 0$

$\Rightarrow \left(\frac{\partial w}{\partial t}\right)_{x,z} = (2x)(0) + (1)(1) + (-1)(0) + (\cos t)(1) = 1 + \cos t$

(f) $\begin{pmatrix} y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} x = t - y \\ y = y \\ z = z \\ t = t \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial t}\right)_{y,z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial t}$; $\frac{\partial y}{\partial t} = 0$ and $\frac{\partial z}{\partial t} = 0$

$\Rightarrow \left(\frac{\partial w}{\partial t}\right)_{y,z} = (2x)(1) + (1)(0) + (-1)(0) + (\cos t)(1) = \cos t + 2x = \cos t + 2(t - y)$

3. $U = f(P, V, T)$ and $PV = nRT$

(a) $\begin{pmatrix} P \\ V \end{pmatrix} \to \begin{pmatrix} P = P \\ V = V \\ T = \frac{PV}{nR} \end{pmatrix} \to U \Rightarrow \left(\frac{\partial U}{\partial P}\right)_V = \frac{\partial U}{\partial P}\frac{\partial P}{\partial P} + \frac{\partial U}{\partial V}\frac{\partial V}{\partial P} + \frac{\partial U}{\partial T}\frac{\partial T}{\partial P} = \frac{\partial U}{\partial P} + \left(\frac{\partial U}{\partial V}\right)(0) + \left(\frac{\partial U}{\partial T}\right)\left(\frac{V}{nR}\right)$

$= \frac{\partial U}{\partial P} + \left(\frac{\partial U}{\partial T}\right)\left(\frac{V}{nR}\right)$

(b) $\begin{pmatrix} V \\ T \end{pmatrix} \to \begin{pmatrix} P = \frac{nRT}{V} \\ V = V \\ T = T \end{pmatrix} \to U \Rightarrow \left(\frac{\partial U}{\partial T}\right)_V = \frac{\partial U}{\partial P}\frac{\partial P}{\partial T} + \frac{\partial U}{\partial V}\frac{\partial V}{\partial T} + \frac{\partial U}{\partial T}\frac{\partial T}{\partial T} = \left(\frac{\partial U}{\partial P}\right)\left(\frac{nR}{V}\right) + \left(\frac{\partial U}{\partial V}\right)(0) + \frac{\partial U}{\partial T}$

$= \left(\frac{\partial U}{\partial P}\right)\left(\frac{nR}{V}\right) + \frac{\partial U}{\partial T}$

4. $w = x^2 + y^2 + z^2$ and $y \sin z + z \sin x = 0$

(a) $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x = x \\ y = y \\ z = z(x, y) \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial x}\right)_y = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x}$; $\frac{\partial y}{\partial x} = 0$ and

$(y \cos z)\frac{\partial z}{\partial x} + (\sin x)\frac{\partial z}{\partial x} + z \cos x = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-z \cos x}{y \cos z + \sin x}$. At $(0, 1, \pi)$, $\frac{\partial z}{\partial x} = \frac{-\pi}{-1} = \pi$

$\Rightarrow \left(\frac{\partial w}{\partial x}\right)_{y}\Big|_{(0,1,\pi)} = (2x)(1) + (2y)(0) + (2z)(\pi)\Big|_{(0,1,\pi)} = 2\pi^2$

(b) $\begin{pmatrix} y \\ z \end{pmatrix} \to \begin{pmatrix} x = x(y, z) \\ y = y \\ z = z \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_y = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z} = (2x)\frac{\partial x}{\partial z} + (2y)(0) + (2z)(1)$

$= (2x)\frac{\partial x}{\partial z} + 2z$. Now $(\sin z)\frac{\partial y}{\partial z} + y \cos z + \sin x + (z \cos x)\frac{\partial x}{\partial z} = 0$ and $\frac{\partial y}{\partial z} = 0$

$\Rightarrow y \cos z + \sin x + (z \cos x)\frac{\partial x}{\partial z} = 0 \Rightarrow \frac{\partial x}{\partial z} = \frac{-y \cos z - \sin x}{z \cos x}$. At $(0, 1, \pi)$, $\frac{\partial x}{\partial z} = \frac{1 - 0}{(\pi)(1)} = \frac{1}{\pi}$

$\Rightarrow \left(\frac{\partial w}{\partial z}\right)_{y}\Big|_{(0,1,\pi)} = 2(0)\left(\frac{1}{\pi}\right) + 2\pi = 2\pi$

5. $w = x^2y^2 + yz - z^3$ and $x^2 + y^2 + z^2 = 6$

(a) $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x = x \\ y = y \\ z = z(x, y) \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial y}\right)_x = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}$

$= (2xy^2)(0) + (2x^2y + z)(1) + (y - 3z^2)\frac{\partial z}{\partial y} = 2x^2y + z + (y - 3z^2)\frac{\partial z}{\partial y}$. Now $(2x)\frac{\partial x}{\partial y} + 2y + (2z)\frac{\partial z}{\partial y} = 0$ and $\frac{\partial x}{\partial y} = 0 \Rightarrow 2y + (2z)\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{y}{z}$. At $(w, x, y, z) = (4, 2, 1, -1)$, $\frac{\partial z}{\partial y} = -\frac{1}{-1} = 1 \Rightarrow \left(\frac{\partial w}{\partial y}\right)_x\Big|_{(4,2,1,-1)}$

$= \left[(2)(2)^2(1) + (-1)\right] + \left[1 - 3(-1)^2\right](1) = 5$

(b) $\begin{pmatrix} y \\ z \end{pmatrix} \to \begin{pmatrix} x = x(y, z) \\ y = y \\ z = z \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial y}\right)_z = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}$

$= (2xy^2)\frac{\partial x}{\partial y} + (2x^2y + z)(1) + (y - 3z^2)(0) = (2x^2y)\frac{\partial x}{\partial y} + 2x^2y + z$. Now $(2x)\frac{\partial x}{\partial y} + 2y + (2z)\frac{\partial z}{\partial y} = 0$ and $\frac{\partial z}{\partial y} = 0 \Rightarrow (2x)\frac{\partial x}{\partial y} + 2y = 0 \Rightarrow \frac{\partial x}{\partial y} = -\frac{y}{x}$. At $(w, x, y, z) = (4, 2, 1, -1)$, $\frac{\partial x}{\partial y} = -\frac{1}{2} \Rightarrow \left(\frac{\partial w}{\partial y}\right)_z\Big|_{(4,2,1,-1)}$

$= (2)(2)(1)^2\left(-\frac{1}{2}\right) + (2)(2)^2(1) + (-1) = 5$

6. $y = uv \Rightarrow 1 = v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y}$; $x = u^2 + v^2$ and $\frac{\partial x}{\partial y} = 0 \Rightarrow 0 = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} \Rightarrow \frac{\partial v}{\partial y} = \left(-\frac{u}{v}\right)\frac{\partial u}{\partial y} \Rightarrow 1$
$= v\frac{\partial u}{\partial y} + u\left(-\frac{u}{v}\frac{\partial u}{\partial y}\right) = \left(\frac{v^2 - u^2}{v}\right)\frac{\partial u}{\partial y} \Rightarrow \frac{\partial u}{\partial y} = \frac{v}{v^2 - u^2}$. At $(u, v) = \left(\sqrt{2}, 1\right)$, $\frac{\partial u}{\partial y} = \frac{1}{1^2 - \left(\sqrt{2}\right)^2} = -1$

$\Rightarrow \left(\frac{\partial u}{\partial y}\right)_x = -1$

7. $\begin{pmatrix} r \\ \theta \end{pmatrix} \to \begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix} \Rightarrow \left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta$; $x^2 + y^2 = r^2 \Rightarrow 2x + 2y\,\frac{\partial y}{\partial x} = 2r\,\frac{\partial r}{\partial x}$ and $\frac{\partial y}{\partial x} = 0 \Rightarrow 2x = 2r\,\frac{\partial r}{\partial x}$
$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \Rightarrow \left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{\sqrt{x^2+y^2}}$

8. If x, y, and z are independent, then $\left(\frac{\partial w}{\partial x}\right)_{y,z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x}$
$= (2x)(1) + (-2y)(0) + (4)(0) + (1)\left(\frac{\partial t}{\partial x}\right) = 2x + \frac{\partial t}{\partial x}$. Thus $x + 2z + t = 25 \Rightarrow 1 + 0 + \frac{\partial t}{\partial x} = 0 \Rightarrow \frac{\partial t}{\partial x} = -1$
$\Rightarrow \left(\frac{\partial w}{\partial x}\right)_{y,z} = 2x - 1$. On the other hand, if x, y, and t are independent, then $\left(\frac{\partial w}{\partial x}\right)_{y,t}$
$= \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x} = (2x)(1) + (-2y)(0) + 4\,\frac{\partial z}{\partial x} + (1)(0) = 2x + 4\,\frac{\partial z}{\partial x}$. Thus, $x + 2z + t = 25$
$\Rightarrow 1 + 2\,\frac{\partial z}{\partial x} + 0 = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{1}{2} \Rightarrow \left(\frac{\partial w}{\partial x}\right)_{y,t} = 2x + 4\left(-\frac{1}{2}\right) = 2x - 2.$

9. If x is a differentiable function of y and z, then $f(x,y,z) = 0 \Rightarrow \frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} = 0$
$\Rightarrow \left(\frac{\partial x}{\partial y}\right)_z = -\frac{\partial f/\partial y}{\partial f/\partial z}$. Similarly, if y is a differentiable function of x and z, $\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\partial f/\partial z}{\partial f/\partial x}$ and if z is a differentiable function of x and y, $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\partial f/\partial x}{\partial f/\partial y}$. Then $\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y$
$= \left(-\frac{\partial f/\partial y}{\partial f/\partial z}\right)\left(-\frac{\partial f/\partial z}{\partial f/\partial x}\right)\left(-\frac{\partial f/\partial x}{\partial f/\partial y}\right) = -1.$

10. $z = z + f(u)$ and $u = xy \Rightarrow \frac{\partial z}{\partial x} = 1 + \frac{df}{du}\frac{\partial u}{\partial x} = 1 + y\,\frac{df}{du}$; also $\frac{\partial z}{\partial y} = 0 + \frac{df}{du}\frac{\partial u}{\partial y} = x\,\frac{df}{du}$ so that $x\,\frac{\partial z}{\partial x} - y\,\frac{\partial z}{\partial y}$
$= x\left(1 + y\,\frac{df}{du}\right) - y\left(x\,\frac{df}{du}\right) = x$

11. If x and y are independent, then $g(x,y,z) = 0 \Rightarrow \frac{\partial g}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial y} = 0$ and $\frac{\partial x}{\partial y} = 0 \Rightarrow \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial y} = 0$
$\Rightarrow \left(\frac{\partial z}{\partial y}\right)_x = -\frac{\partial g/\partial y}{\partial g/\partial z}$, as claimed.

12. Let x and y be independent. Then $f(x,y,z,w) = 0$, $g(x,y,z,w) = 0$ and $\frac{\partial y}{\partial x} = 0$
$\Rightarrow \frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = 0$ and
$\frac{\partial g}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial x} = 0$ imply

$$\begin{cases} \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = -\frac{\partial f}{\partial x} \\ \frac{\partial g}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial x} = -\frac{\partial g}{\partial x} \end{cases} \Rightarrow \left(\frac{\partial z}{\partial x}\right)_y = \frac{\begin{vmatrix} -\frac{\partial f}{\partial x} & \frac{\partial f}{\partial w} \\ -\frac{\partial g}{\partial x} & \frac{\partial g}{\partial w} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{vmatrix}} = \frac{-\frac{\partial f}{\partial x}\frac{\partial g}{\partial w} + \frac{\partial g}{\partial x}\frac{\partial f}{\partial w}}{\frac{\partial f}{\partial z}\frac{\partial g}{\partial w} - \frac{\partial g}{\partial z}\frac{\partial f}{\partial w}} = -\frac{\frac{\partial f}{\partial x}\frac{\partial g}{\partial w} - \frac{\partial f}{\partial w}\frac{\partial g}{\partial x}}{\frac{\partial f}{\partial z}\frac{\partial g}{\partial w} - \frac{\partial f}{\partial w}\frac{\partial g}{\partial z}}, \text{ as claimed.}$$

Likewise, $f(x,y,z,w) = 0$, $g(x,y,z,w) = 0$ and $\frac{\partial x}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y}$
$= \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} = 0$ and (similarly) $\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial y} = 0$ imply

$$\begin{cases} \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} = -\frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial y} = -\frac{\partial g}{\partial y} \end{cases} \Rightarrow \left(\frac{\partial w}{\partial y}\right)_x = \frac{\begin{vmatrix} \frac{\partial f}{\partial z} & -\frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial z} & -\frac{\partial g}{\partial y} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{vmatrix}} = \frac{-\frac{\partial f}{\partial z}\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}\frac{\partial g}{\partial w} - \frac{\partial g}{\partial z}\frac{\partial f}{\partial w}} = -\frac{\frac{\partial f}{\partial z}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial z}}{\frac{\partial f}{\partial z}\frac{\partial g}{\partial w} - \frac{\partial f}{\partial w}\frac{\partial g}{\partial z}}, \text{ as claimed.}$$

14.10 TAYLOR'S FORMULA FOR TWO VARIABLES

1. $f(x,y) = xe^y \Rightarrow f_x = e^y,\ f_y = xe^y,\ f_{xx} = 0,\ f_{xy} = e^y,\ f_{yy} = xe^y$
$\Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 0 + x\cdot 1 + y\cdot 0 + \frac{1}{2}\left(x^2\cdot 0 + 2xy\cdot 1 + y^2\cdot 0\right) = x + xy$ quadratic approximation;
$f_{xxx} = 0,\ f_{xxy} = 0,\ f_{xyy} = e^y,\ f_{yyy} = xe^y$

$\Rightarrow\ f(x, y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$
$= x + xy + \frac{1}{6}\left(x^3 \cdot 0 + 3x^2 y \cdot 0 + 3xy^2 \cdot 1 + y^3 \cdot 0\right) = x + xy + \frac{1}{2}xy^2$, cubic approximation

2. $f(x, y) = e^x \cos y \Rightarrow f_x = e^x \cos y,\ f_y = -e^x \sin y,\ f_{xx} = e^x \cos y,\ f_{xy} = -e^x \sin y,\ f_{yy} = -e^x \cos y$
$\Rightarrow\ f(x, y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 1 + x \cdot 1 + y \cdot 0 + \frac{1}{2}\left[x^2 \cdot 1 + 2xy \cdot 0 + y^2 \cdot (-1)\right] = 1 + x + \frac{1}{2}\left(x^2 - y^2\right)$, quadratic approximation;
$f_{xxx} = e^x \cos y,\ f_{xxy} = -e^x \sin y,\ f_{xyy} = -e^x \cos y,\ f_{yyy} = e^x \sin y$
$\Rightarrow\ f(x, y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$
$= 1 + x + \frac{1}{2}\left(x^2 - y^2\right) + \frac{1}{6}\left[x^3 \cdot 1 + 3x^2 y \cdot 0 + 3xy^2 \cdot (-1) + y^3 \cdot 0\right]$
$= 1 + x + \frac{1}{2}\left(x^2 - y^2\right) + \frac{1}{6}\left(x^3 - 3xy^2\right)$, cubic approximation

3. $f(x, y) = y \sin x \Rightarrow f_x = y \cos x,\ f_y = \sin x,\ f_{xx} = -y \sin x,\ f_{xy} = \cos x,\ f_{yy} = 0$
$\Rightarrow\ f(x, y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 0 + x \cdot 0 + y \cdot 0 + \frac{1}{2}\left(x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot 0\right) = xy$, quadratic approximation;
$f_{xxx} = -y \cos x,\ f_{xxy} = -\sin x,\ f_{xyy} = 0,\ f_{yyy} = 0$
$\Rightarrow\ f(x, y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$
$= xy + \frac{1}{6}\left(x^3 \cdot 0 + 3x^2 y \cdot 0 + 3xy^2 \cdot 0 + y^3 \cdot 0\right) = xy$, cubic approximation

4. $f(x, y) = \sin x \cos y \Rightarrow f_x = \cos x \cos y,\ f_y = -\sin x \sin y,\ f_{xx} = -\sin x \cos y,\ f_{xy} = -\cos x \sin y,$
$f_{yy} = -\sin x \cos y \Rightarrow f(x, y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 0 + x \cdot 1 + y \cdot 0 + \frac{1}{2}\left(x^2 \cdot 0 + 2xy \cdot 0 + y^2 \cdot 0\right) = x$, quadratic approximation;
$f_{xxx} = -\cos x \cos y,\ f_{xxy} = \sin x \sin y,\ f_{xyy} = -\cos x \cos y,\ f_{yyy} = \sin x \sin y$
$\Rightarrow\ f(x, y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$
$= x + \frac{1}{6}\left[x^3 \cdot (-1) + 3x^2 y \cdot 0 + 3xy^2 \cdot (-1) + y^3 \cdot 0\right] = x - \frac{1}{6}\left(x^3 + 3xy^2\right)$, cubic approximation

5. $f(x, y) = e^x \ln(1 + y) \Rightarrow f_x = e^x \ln(1 + y),\ f_y = \frac{e^x}{1 + y},\ f_{xx} = e^x \ln(1 + y),\ f_{xy} = \frac{e^x}{1 + y},\ f_{yy} = -\frac{e^x}{(1 + y)^2}$
$\Rightarrow\ f(x, y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 0 + x \cdot 0 + y \cdot 1 + \frac{1}{2}\left[x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot (-1)\right] = y + \frac{1}{2}\left(2xy - y^2\right)$, quadratic approximation;
$f_{xxx} = e^x \ln(1 + y),\ f_{xxy} = \frac{e^x}{1 + y},\ f_{xyy} = -\frac{e^x}{(1 + y)^2},\ f_{yyy} = \frac{2e^x}{(1 + y)^3}$
$\Rightarrow\ f(x, y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$
$= y + \frac{1}{2}\left(2xy - y^2\right) + \frac{1}{6}\left[x^3 \cdot 0 + 3x^2 y \cdot 1 + 3xy^2 \cdot (-1) + y^3 \cdot 2\right]$
$= y + \frac{1}{2}\left(2xy - y^2\right) + \frac{1}{6}\left(3x^2 y - 3xy^2 + 2y^3\right)$, cubic approximation

6. $f(x, y) = \ln(2x + y + 1) \Rightarrow f_x = \frac{2}{2x + y + 1},\ f_y = \frac{1}{2x + y + 1},\ f_{xx} = \frac{-4}{(2x + y + 1)^2},\ f_{xy} = \frac{-2}{(2x + y + 1)^2},$
$f_{yy} = \frac{-1}{(2x + y + 1)^2} \Rightarrow f(x, y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 0 + x \cdot 2 + y \cdot 1 + \frac{1}{2}\left[x^2 \cdot (-4) + 2xy \cdot (-2) + y^2 \cdot (-1)\right] = 2x + y + \frac{1}{2}\left(-4x^2 - 4xy - y^2\right)$
$= (2x + y) - \frac{1}{2}(2x + y)^2$, quadratic approximation;
$f_{xxx} = \frac{16}{(2x + y + 1)^3},\ f_{xxy} = \frac{8}{(2x + y + 1)^3},\ f_{xyy} = \frac{4}{(2x + y + 1)^3},\ f_{yyy} = \frac{2}{(2x + y + 1)^3}$
$\Rightarrow\ f(x, y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$
$= (2x + y) - \frac{1}{2}(2x + y)^2 + \frac{1}{6}\left(x^3 \cdot 16 + 3x^2 y \cdot 8 + 3xy^2 \cdot 4 + y^3 \cdot 2\right)$
$= (2x + y) - \frac{1}{2}(2x + y)^2 + \frac{1}{3}\left(8x^3 + 12x^2 y + 6xy^2 + y^2\right)$
$= (2x + y) - \frac{1}{2}(2x + y)^2 + \frac{1}{3}(2x + y)^3$, cubic approximation

7. $f(x, y) = \sin\left(x^2 + y^2\right) \Rightarrow f_x = 2x \cos\left(x^2 + y^2\right),\ f_y = 2y \cos\left(x^2 + y^2\right),\ f_{xx} = 2\cos\left(x^2 + y^2\right) - 4x^2 \sin\left(x^2 + y^2\right),$
$f_{xy} = -4xy \sin\left(x^2 + y^2\right),\ f_{yy} = 2\cos\left(x^2 + y^2\right) - 4y^2 \sin\left(x^2 + y^2\right)$

$\Rightarrow\ f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2f_{yy}(0,0)\right]$
$= 0 + x\cdot 0 + y\cdot 0 + \frac{1}{2}\left(x^2\cdot 2 + 2xy\cdot 0 + y^2\cdot 2\right) = x^2 + y^2$, quadratic approximation;
$f_{xxx} = -12x\sin(x^2+y^2) - 8x^3\cos(x^2+y^2)$, $f_{xxy} = -4y\sin(x^2+y^2) - 8x^2y\cos(x^2+y^2)$,
$f_{xyy} = -4x\sin(x^2+y^2) - 8xy^2\cos(x^2+y^2)$, $f_{yyy} = -12y\sin(x^2+y^2) - 8y^3\cos(x^2+y^2)$
$\Rightarrow\ f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3f_{xxx}(0,0) + 3x^2yf_{xxy}(0,0) + 3xy^2f_{xyy}(0,0) + y^3f_{yyy}(0,0)\right]$
$= x^2 + y^2 + \frac{1}{6}\left(x^3\cdot 0 + 3x^2y\cdot 0 + 3xy^2\cdot 0 + y^3\cdot 0\right) = x^2 + y^2$, cubic approximation

8. $f(x,y) = \cos(x^2+y^2)\ \Rightarrow\ f_x = -2x\sin(x^2+y^2)$, $f_y = -2y\sin(x^2+y^2)$,
$f_{xx} = -2\sin(x^2+y^2) - 4x^2\cos(x^2+y^2)$, $f_{xy} = -4xy\cos(x^2+y^2)$, $f_{yy} = -2\sin(x^2+y^2) - 4y^2\cos(x^2+y^2)$
$\Rightarrow\ f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2f_{yy}(0,0)\right]$
$= 1 + x\cdot 0 + y\cdot 0 + \frac{1}{2}\left[x^2\cdot 0 + 2xy\cdot 0 + y^2\cdot 0\right] = 1$, quadratic approximation;
$f_{xxx} = -12x\cos(x^2+y^2) + 8x^3\sin(x^2+y^2)$, $f_{xxy} = -4y\cos(x^2+y^2) + 8x^2y\sin(x^2+y^2)$,
$f_{xyy} = -4x\cos(x^2+y^2) + 8xy^2\sin(x^2+y^2)$, $f_{yyy} = -12y\cos(x^2+y^2) + 8y^3\sin(x^2+y^2)$
$\Rightarrow\ f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3f_{xxx}(0,0) + 3x^2yf_{xxy}(0,0) + 3xy^2f_{xyy}(0,0) + y^3f_{yyy}(0,0)\right]$
$= 1 + \frac{1}{6}\left(x^3\cdot 0 + 3x^2y\cdot 0 + 3xy^2\cdot 0 + y^3\cdot 0\right) = 1$, cubic approximation

9. $f(x,y) = \frac{1}{1-x-y}\ \Rightarrow\ f_x = \frac{1}{(1-x-y)^2} = f_y$, $f_{xx} = \frac{2}{(1-x-y)^3} = f_{xy} = f_{yy}$
$\Rightarrow\ f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2f_{yy}(0,0)\right]$
$= 1 + x\cdot 1 + y\cdot 1 + \frac{1}{2}\left(x^2\cdot 2 + 2xy\cdot 2 + y^2\cdot 2\right) = 1 + (x+y) + (x^2 + 2xy + y^2)$
$= 1 + (x+y) + (x+y)^2$, quadratic approximation; $f_{xxx} = \frac{6}{(1-x-y)^4} = f_{xxy} = f_{xyy} = f_{yyy}$
$\Rightarrow\ f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3f_{xxx}(0,0) + 3x^2yf_{xxy}(0,0) + 3xy^2f_{xyy}(0,0) + y^3f_{yyy}(0,0)\right]$
$= 1 + (x+y) + (x+y)^2 + \frac{1}{6}\left(x^3\cdot 6 + 3x^2y\cdot 6 + 3xy^2\cdot 6 + y^3\cdot 6\right)$
$= 1 + (x+y) + (x+y)^2 + (x^3 + 3x^2y + 3xy^2 + y^3) = 1 + (x+y) + (x+y)^2 + (x+y)^3$, cubic approximation

10. $f(x,y) = \frac{1}{1-x-y+xy}\ \Rightarrow\ f_x = \frac{1-y}{(1-x-y+xy)^2}$, $f_y = \frac{1-x}{(1-x-y+xy)^2}$, $f_{xx} = \frac{2(1-y)^2}{(1-x-y+xy)^3}$,
$f_{xy} = \frac{1}{(1-x-y+xy)^2}$, $f_{yy} = \frac{2(1-x)^2}{(1-x-y+xy)^3}$
$\Rightarrow\ f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2f_{yy}(0,0)\right]$
$= 1 + x\cdot 1 + y\cdot 1 + \frac{1}{2}\left(x^2\cdot 2 + 2xy\cdot 1 + y^2\cdot 2\right) = 1 + x + y + x^2 + xy + y^2$, quadratic approximation;
$f_{xxx} = \frac{6(1-y)^3}{(1-x-y+xy)^4}$, $f_{xxy} = \frac{[-4(1-x-y+xy) + 6(1-y)(1-x)](1-y)}{(1-x-y+xy)^4}$,
$f_{xyy} = \frac{[-4(1-x-y+xy) + 6(1-x)(1-y)](1-x)}{(1-x-y+xy)^4}$, $f_{yyy} = \frac{6(1-x)^3}{(1-x-y+xy)^4}$
$\Rightarrow\ f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3f_{xxx}(0,0) + 3x^2yf_{xxy}(0,0) + 3xy^2f_{xyy}(0,0) + y^3f_{yyy}(0,0)\right]$
$= 1 + x + y + x^2 + xy + y^2 + \frac{1}{6}\left(x^3\cdot 6 + 3x^2y\cdot 2 + 3xy^2\cdot 2 + y^3\cdot 6\right)$
$= 1 + x + y + x^2 + xy + y^2 + x^3 + x^2y + xy^2 + y^3$, cubic approximation

11. $f(x,y) = \cos x\cos y\ \Rightarrow\ f_x = -\sin x\cos y$, $f_y = -\cos x\sin y$, $f_{xx} = -\cos x\cos y$, $f_{xy} = \sin x\sin y$,
$f_{yy} = -\cos x\cos y\ \Rightarrow\ f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2f_{yy}(0,0)\right]$
$= 1 + x\cdot 0 + y\cdot 0 + \frac{1}{2}\left[x^2\cdot(-1) + 2xy\cdot 0 + y^2\cdot(-1)\right] = 1 - \frac{x^2}{2} - \frac{y^2}{2}$, quadratic approximation. Since all partial derivatives of f are products of sines and cosines, the absolute value of these derivatives is less than or equal to 1 $\Rightarrow\ E(x,y) \le \frac{1}{6}\left[(0.1)^3 + 3(0.1)^3 + 3(0.1)^3 + 0.1)^3\right] \le 0.00134$.

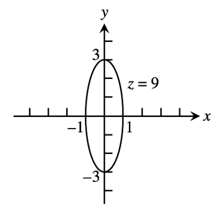
12. $f(x,y) = e^x\sin y\ \Rightarrow\ f_x = e^x\sin y$, $f_y = e^x\cos y$, $f_{xx} = e^x\sin y$, $f_{xy} = e^x\cos y$, $f_{yy} = -e^x\sin y$
$\Rightarrow\ f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2f_{yy}(0,0)\right]$
$= 0 + x\cdot 0 + y\cdot 1 + \frac{1}{2}\left(x^2\cdot 0 + 2xy\cdot 1 + y^2\cdot 0\right) = y + xy$, quadratic approximation. Now, $f_{xxx} = e^x\sin y$, $f_{xxy} = e^x\cos y$, $f_{xyy} = -e^x\sin y$, and $f_{yyy} = -e^x\cos y$. Since $|x| \le 0.1$, $|e^x\sin y| \le |e^{0.1}\sin 0.1| \approx 0.11$ and $|e^x\cos y| \le |e^{0.1}\cos 0.1| \approx 1.11$. Therefore,

$E(x, y) \le \frac{1}{6}\left[(0.11)(0.1)^3 + 3(1.11)(0.1)^3 + 3(0.11)(0.1)^3 + (1.11)(0.1)^3\right] \le 0.000814.$

CHAPTER 14 PRACTICE EXERCISES

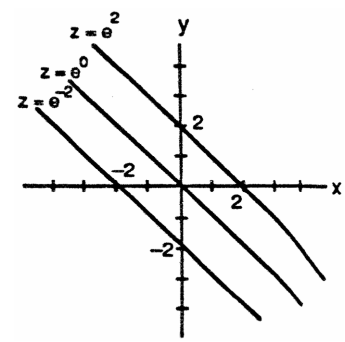
1. Domain: All points in the xy-plane
   Range: $z \ge 0$

   Level curves are ellipses with major axis along the y-axis and minor axis along the x-axis.



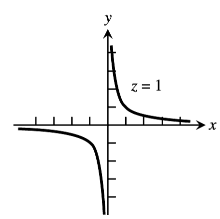
2. Domain: All points in the xy-plane
   Range: $0 < z < \infty$

   Level curves are the straight lines $x + y = \ln z$ with slope -1, and $z > 0$.



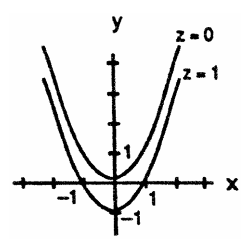
3. Domain: All (x, y) such that $x \ne 0$ and $y \ne 0$
   Range: $z \ne 0$

   Level curves are hyperbolas with the x- and y-axes as asymptotes.



4. Domain: All (x, y) so that $x^2 - y \ge 0$
   Range: $z \ge 0$

   Level curves are the parabolas $y = x^2 - c$, $c \ge 0$.



5. Domain: All points (x, y, z) in space
   Range: All real numbers

   Level surfaces are paraboloids of revolution with the z-axis as axis.