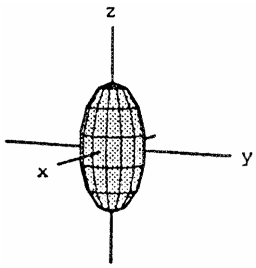
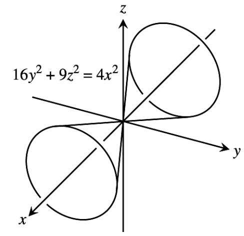
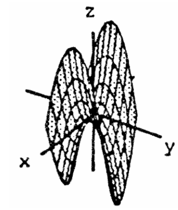
58. $4x^2 + 4y^2 + z^2 = 4$



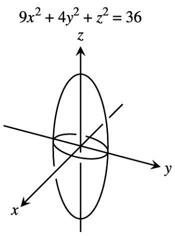
59. $16y^2 + 9z^2 = 4x^2$



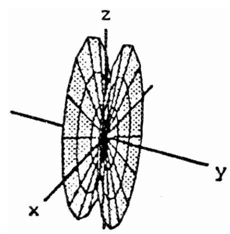
60. $z = x^2 - y^2 - 1$



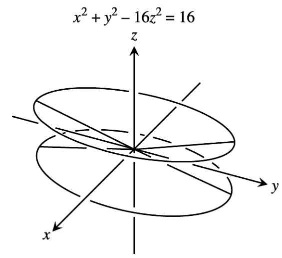
61. $9x^2 + 4y^2 + z^2 = 36$



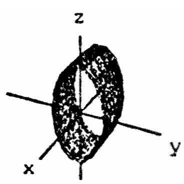
62. $4x^2 + 9z^2 = y^2$



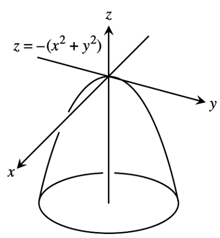
63. $x^2 + y^2 - 16z^2 = 16$



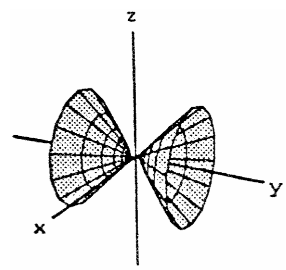
64. $z^2 + 4y^2 = 9$



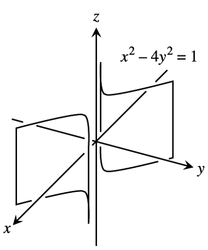
65. $z = -(x^2 + y^2)$



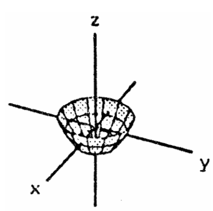
66. $y^2 - x^2 - z^2 = 1$



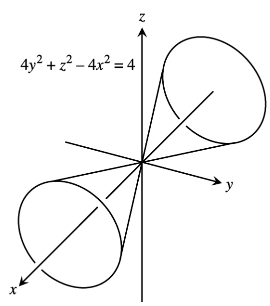
67. $x^2 - 4y^2 = 1$



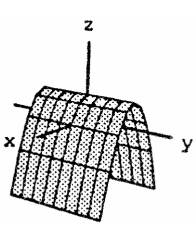
68. $z = 4x^2 + y^2 - 4$



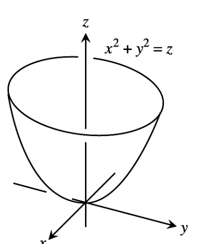
69. $4y^2 + z^2 - 4x^2 = 4$



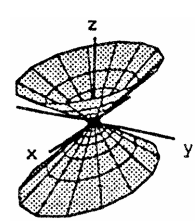
70. $z = 1 - x^2$



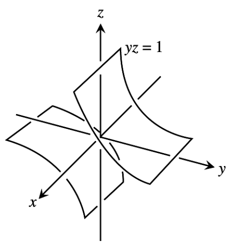
71. $x^2 + y^2 = z$



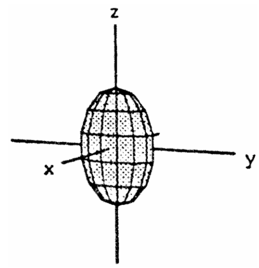
72. $\frac{x^2}{4} + y^2 - z^2 = 1$



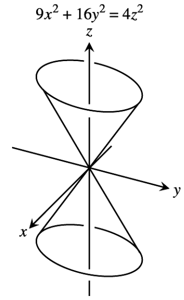
73. $yz = 1$



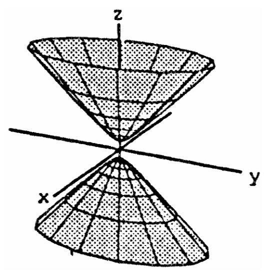
74. $36x^2 + 9y^2 + 4z^2 = 36$



75. $9x^2 + 16y^2 = 4z^2$



76. $4z^2 - x^2 - y^2 = 4$



77. (a) If $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$ and $z = c$, then $x^2 + \frac{y^2}{4} = \frac{9-c^2}{9} \Rightarrow \frac{x^2}{\left(\frac{9-c^2}{9}\right)} + \frac{y^2}{\left[\frac{4(9-c^2)}{9}\right]} = 1 \Rightarrow A = ab\pi$
$= \pi\left(\frac{\sqrt{9-c^2}}{3}\right)\left(\frac{2\sqrt{9-c^2}}{3}\right) = \frac{2\pi(9-c^2)}{9}$

(b) From part (a), each slice has the area $\frac{2\pi(9-z^2)}{9}$, where $-3 \le z \le 3$. Thus $V = 2\int_0^3 \frac{2\pi}{9}(9-z^2)\,dz$
$= \frac{4\pi}{9}\int_0^3 (9-z^2)\,dz = \frac{4\pi}{9}\left[9z - \frac{z^3}{3}\right]_0^3 = \frac{4\pi}{9}(27-9) = 8\pi$

(c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow \frac{x^2}{\left[\frac{a^2(c^2-z^2)}{c^2}\right]} + \frac{y^2}{\left[\frac{b^2(c^2-z^2)}{c^2}\right]} = 1 \Rightarrow A = \pi\left(\frac{a\sqrt{c^2-z^2}}{c}\right)\left(\frac{b\sqrt{c^2-z^2}}{c}\right)$
$\Rightarrow V = 2\int_0^c \frac{\pi ab}{c^2}(c^2-z^2)\,dz = \frac{2\pi ab}{c^2}\left[c^2z - \frac{z^3}{3}\right]_0^c = \frac{2\pi ab}{c^2}\left(\frac{2}{3}c^3\right) = \frac{4\pi abc}{3}$. Note that if $r = a = b = c$, then $V = \frac{4\pi r^3}{3}$, which is the volume of a sphere.

78. The ellipsoid has the form $\frac{x^2}{R^2} + \frac{y^2}{R^2} + \frac{z^2}{c^2} = 1$. To determine c^2 we note that the point $(0, r, h)$ lies on the surface of the barrel. Thus, $\frac{r^2}{R^2} + \frac{h^2}{c^2} = 1 \Rightarrow c^2 = \frac{h^2R^2}{R^2-r^2}$. We calculate the volume by the disk method:
$V = \pi\int_{-h}^{h} y^2\,dz$. Now, $\frac{y^2}{R^2} + \frac{z^2}{c^2} = 1 \Rightarrow y^2 = R^2\left(1 - \frac{z^2}{c^2}\right) = R^2\left[1 - \frac{z^2(R^2-r^2)}{h^2R^2}\right] = R^2 - \left(\frac{R^2-r^2}{h^2}\right)z^2$
$\Rightarrow V = \pi\int_{-h}^{h}\left[R^2 - \left(\frac{R^2-r^2}{h^2}\right)z^2\right]dz = \pi\left[R^2z - \frac{1}{3}\left(\frac{R^2-r^2}{h^2}\right)z^3\right]_{-h}^{h} = 2\pi\left[R^2h - \frac{1}{3}(R^2-r^2)h\right] = 2\pi\left(\frac{2R^2h}{3} + \frac{r^2h}{3}\right)$
$= \frac{4}{3}\pi R^2h + \frac{2}{3}\pi r^2h$, the volume of the barrel. If $r = R$, then $V = 2\pi R^2h$ which is the volume of a cylinder of radius R and height 2h. If $r = 0$ and $h = R$, then $V = \frac{4}{3}\pi R^3$ which is the volume of a sphere.

79. We calculate the volume by the slicing method, taking slices parallel to the xy-plane. For fixed z, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$ gives the ellipse $\frac{x^2}{\left(\frac{za^2}{c}\right)} + \frac{y^2}{\left(\frac{zb^2}{c}\right)} = 1$. The area of this ellipse is $\pi\left(a\sqrt{\frac{z}{c}}\right)\left(b\sqrt{\frac{z}{c}}\right) = \frac{\pi abz}{c}$ (see Exercise 77a). Hence the volume is given by $V = \int_0^h \frac{\pi abz}{c}\,dz = \left[\frac{\pi abz^2}{2c}\right]_0^h = \frac{\pi abh^2}{c}$. Now the area of the elliptic base when $z = h$ is $A = \frac{\pi abh}{c}$, as determined previously. Thus, $V = \frac{\pi abh^2}{c} = \frac{1}{2}\left(\frac{\pi abh}{c}\right)h = \frac{1}{2}$(base)(altitude), as claimed.

80. (a) For each fixed value of z, the hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ results in a cross-sectional ellipse $\frac{x^2}{\left[\frac{a^2(c^2+z^2)}{c^2}\right]} + \frac{y^2}{\left[\frac{b^2(c^2+z^2)}{c^2}\right]} = 1$. The area of the cross-sectional ellipse (see Exercise 77a) is $A(z) = \pi\left(\frac{a}{c}\sqrt{c^2+z^2}\right)\left(\frac{b}{c}\sqrt{c^2+z^2}\right) = \frac{\pi ab}{c^2}(c^2+z^2)$. The volume of the solid by the method of slices is $V = \int_0^h A(z)\,dz = \int_0^h \frac{\pi ab}{c^2}(c^2+z^2)\,dz = \frac{\pi ab}{c^2}\left[c^2z + \frac{1}{3}z^3\right]_0^h = \frac{\pi ab}{c^2}\left(c^2h + \frac{1}{3}h^3\right) = \frac{\pi abh}{3c^2}(3c^2+h^2)$

(b) $A_0 = A(0) = \pi ab$ and $A_h = A(h) = \frac{\pi ab}{c^2}(c^2+h^2)$, from part (a) $\Rightarrow$ $V = \frac{\pi abh}{3c^2}(3c^2+h^2)$ $= \frac{\pi abh}{3}\left(2 + 1 + \frac{h^2}{c^2}\right) = \frac{\pi abh}{3}\left(2 + \frac{c^2+h^2}{c^2}\right) = \frac{h}{3}\left[2\pi ab + \frac{\pi ab}{c^2}(c^2+h^2)\right] = \frac{h}{3}(2A_0 + A_h)$

(c) $A_m = A\left(\frac{h}{2}\right) = \frac{\pi ab}{c^2}\left(c^2 + \frac{h^2}{4}\right) = \frac{\pi ab}{4c^2}(4c^2+h^2)$ $\Rightarrow$ $\frac{h}{6}(A_0 + 4A_m + A_h)$ $= \frac{h}{6}\left[\pi ab + \frac{\pi ab}{c^2}(4c^2+h^2) + \frac{\pi ab}{c^2}(c^2+h^2)\right] = \frac{\pi abh}{6c^2}(c^2 + 4c^2 + h^2 + c^2 + h^2) = \frac{\pi abh}{6c^2}(6c^2+2h^2)$ $= \frac{\pi abh}{3c^2}(3c^2+h^2) = V$ from part (a)

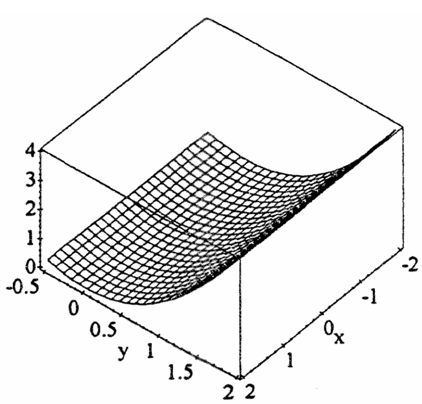
81. $y = y_1 \Rightarrow \frac{z}{c} = \frac{y_1^2}{b^2} - \frac{x^2}{a^2}$, a parabola in the plane $y = y_1$ $\Rightarrow$ vertex when $\frac{dz}{dx} = 0$ or $c\,\frac{dz}{dx} = -\frac{2x}{a^2} = 0 \Rightarrow x = 0$ $\Rightarrow$ Vertex $\left(0, y_1, \frac{cy_1^2}{b^2}\right)$; writing the parabola as $x^2 = -\frac{a^2}{c}z + \frac{a^2y_1^2}{b^2}$ we see that $4p = -\frac{a^2}{c} \Rightarrow p = -\frac{a^2}{4c}$ $\Rightarrow$ Focus $\left(0, y_1, \frac{cy_1^2}{b^2} - \frac{a^2}{4c}\right)$

82. The curve has the general form $Ax^2 + By^2 + Dxy + Gx + Hy + K = 0$ which is the same form as Eq. (1) in Section 10.3 for a conic section (including the degenerate cases) in the xy-plane.

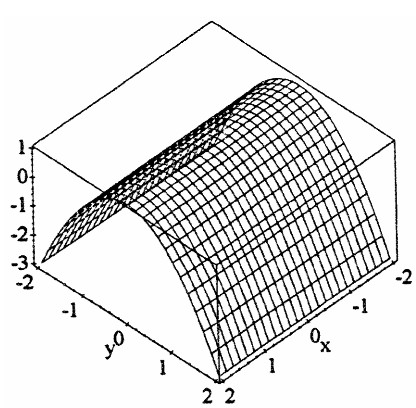
83. No, it is not mere coincidence. A plane parallel to one of the coordinate planes will set one of the variables x, y, or z equal to a constant in the general equation $Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Jz + K = 0$ for a quadric surface. The resulting equation then has the general form for a conic in that parallel plane. For example, setting $y = y_1$ results in the equation $Ax^2 + Cz^2 + D'x + E'z + Fxz + Gx + Jz + K' = 0$ where $D' = Dy_1$, $E' = Ey_1$, and $K' = K + By_1^2 + Hy_1$, which is the general form of a conic section in the plane $y = y_1$ by Section 10.3.

84. The trace will be a conic section. To see why, solve the plane's equation $Ax + By + Cz = 0$ for one of the variables in terms of the other two and substitute into the equation $Ax^2 + By^2 + Cz^2 + \ldots + K = 0$. The result will be a second degree equation in the remaining two variables. By Section 10.3, this equation will represent a conic section. (See also the discussion in Exercises 82 and 83.)

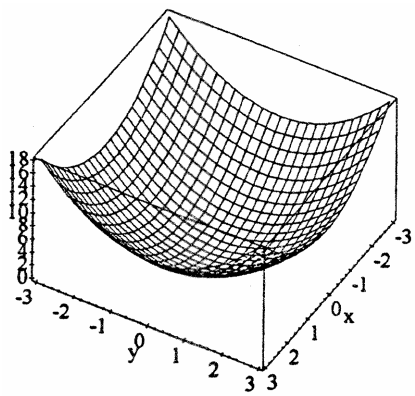
85. $z = y^2$



86. $z = 1 - y^2$

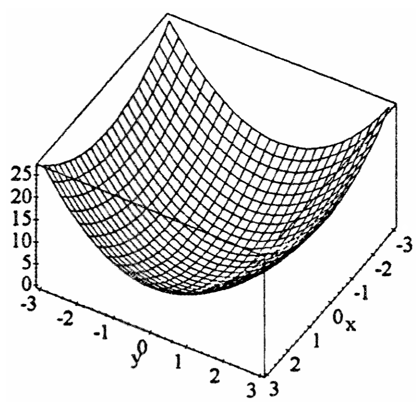


87. $z = x^2 + y^2$

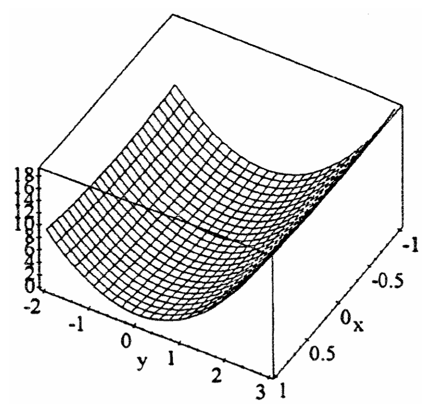


88. $z = x^2 + 2y^2$

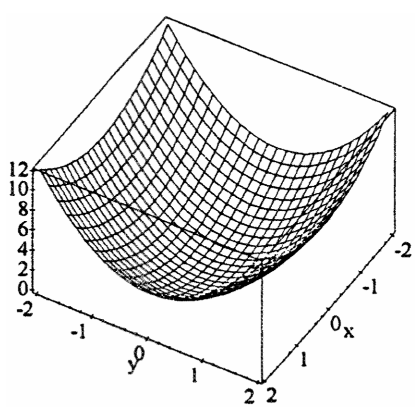
(a)



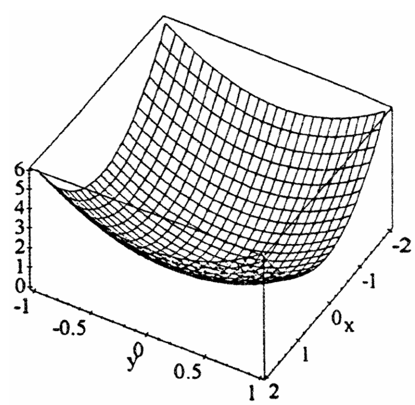
(b)



(c)



(d)



89-94. Example CAS commands:

Maple:

```
with( plots );
eq := x^2/9 + y^2/36 = 1 - z^2/25;
implicitplot3d( eq, x=-3..3, y=-6..6, z=-5..5, scaling=constrained,
                shading=zhue, axes=boxed, title="#89 (Section 12.6)" );
```

Mathematica: (functions and domains may vary):

In the following chapter, you will consider contours or level curves for surfaces in three dimensions. For the purposes of plotting the functions of two variables expressed implicitly in this section, we will call upon the function **ContourPlot3D**.

To insert the stated function, write all terms on the same side of the equal sign and the default contour equating that expression to zero will be plotted.
This built-in function requires the loading of a special graphics package.

```
<<Graphics`ContourPlot3D`
Clear[x, y, z]
ContourPlot3D[x^2/9 – y^2/16 – z^2/2 – 1, {x, –9, 9}, {y, –12, 12}, {z, –5, 5},
        Axes → True, AxesLabel → {x, y, z}, Boxed → False,
        PlotLabel → "Elliptic Hyperboloid of Two Sheets"]
```

Your identification of the plot may or may not be able to be done without considering the graph.

CHAPTER 12 PRACTICE EXERCISES

1. (a) $3\langle -3, 4\rangle - 4\langle 2, -5\rangle = \langle -9-8, 12+20\rangle = \langle -17, 32\rangle$
   (b) $\sqrt{17^2+32^2} = \sqrt{1313}$

2. (a) $\langle -3+2, 4-5\rangle = \langle -1, -1\rangle$
   (b) $\sqrt{(-1)^2+(-1)^2} = \sqrt{2}$

3. (a) $\langle -2(-3), -2(4)\rangle = \langle 6, -8\rangle$
   (b) $\sqrt{6^2+(-8)^2} = 10$

4. (a) $\langle 5(2), 5(-5)\rangle = \langle 10, -25\rangle$
   (b) $\sqrt{10^2+(-25)^2} = \sqrt{725} = 5\sqrt{29}$

5. $\frac{\pi}{6}$ radians below the negative x-axis: $\left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle$ [assuming counterclockwise].

6. $\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$

7. $2\left(\frac{1}{\sqrt{4^2+1^2}}\right)(4\mathbf{i}-\mathbf{j}) = \left(\frac{8}{\sqrt{17}}\mathbf{i} - \frac{2}{\sqrt{17}}\mathbf{j}\right)$

8. $-5\left(\frac{1}{\sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}}\right)\left(\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{j}\right) = (-3\mathbf{i}-4\mathbf{j})$

9. length $= \left|\sqrt{2}\mathbf{i}+\sqrt{2}\mathbf{j}\right| = \sqrt{2+2} = 2$, $\sqrt{2}\mathbf{i}+\sqrt{2}\mathbf{j} = 2\left(\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}\right) \Rightarrow$ the direction is $\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}$

10. length $= |-\mathbf{i}-\mathbf{j}| = \sqrt{1+1} = \sqrt{2}$, $-\mathbf{i}-\mathbf{j} = \sqrt{2}\left(-\frac{1}{\sqrt{2}}\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{j}\right) \Rightarrow$ the direction is $-\frac{1}{\sqrt{2}}\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{j}$

11. $t = \ln 2 \Rightarrow \mathbf{v} = \left(e^{\ln 2}\cos(\ln 2) - e^{\ln 2}\sin(\ln 2)\right)\mathbf{i} + \left(e^{\ln 2}\sin(\ln 2) + e^{\ln 2}\cos(\ln 2)\right)\mathbf{j}$
    $= (2\cos(\ln 2) - 2\sin(\ln 2))\mathbf{i} + (2\sin(\ln 2) + 2\cos(\ln 2))\mathbf{j} = 2[(\cos(\ln 2) - \sin(\ln 2))\mathbf{i} + (\sin(\ln 2)+\cos(\ln 2))\mathbf{j}]$
    length $= |2[(\cos(\ln 2) - \sin(\ln 2))\mathbf{i} + (\sin(\ln 2)+\cos(\ln 2))\mathbf{j}]| = 2\sqrt{(\cos(\ln 2)-\sin(\ln 2))^2 + (\cos(\ln 2)+\sin(\ln 2))^2}$
    $= 2\sqrt{2\cos^2(\ln 2) + 2\sin^2(\ln 2)} = 2\sqrt{2}$;
    $2[(\cos(\ln 2) - \sin(\ln 2))\mathbf{i} + (\sin(\ln 2)+\cos(\ln 2))\mathbf{j}] = 2\sqrt{2}\left(\frac{(\cos(\ln 2)-\sin(\ln 2))\mathbf{i} + (\sin(\ln 2)+\cos(\ln 2))\mathbf{j}}{\sqrt{2}}\right)$
    $\Rightarrow$ direction $= \frac{(\cos(\ln 2)-\sin(\ln 2))}{\sqrt{2}}\mathbf{i} + \frac{(\sin(\ln 2)+\cos(\ln 2))}{\sqrt{2}}\mathbf{j}$

12. $t = \frac{\pi}{2} \Rightarrow \mathbf{v} = \left(-2\sin\frac{\pi}{2}\right)\mathbf{i} + \left(2\cos\frac{\pi}{2}\right)\mathbf{j} = -2\mathbf{i}$; length $= |-2\mathbf{i}| = \sqrt{4+0} = 2$; $-2\mathbf{i} = 2(-\mathbf{i}) \Rightarrow$ the direction is $-\mathbf{i}$

13. length $= |2\mathbf{i}-3\mathbf{j}+6\mathbf{k}| = \sqrt{4+9+36} = 7$, $2\mathbf{i}-3\mathbf{j}+6\mathbf{k} = 7\left(\frac{2}{7}\mathbf{i}-\frac{3}{7}\mathbf{j}+\frac{6}{7}\mathbf{k}\right) \Rightarrow$ the direction is $\frac{2}{7}\mathbf{i}-\frac{3}{7}\mathbf{j}+\frac{6}{7}\mathbf{k}$

14. length $= |\mathbf{i}+2\mathbf{j}-\mathbf{k}| = \sqrt{1+4+1} = \sqrt{6}$, $\mathbf{i}+2\mathbf{j}-\mathbf{k} = \sqrt{6}\left(\frac{1}{\sqrt{6}}\,\mathbf{i}+\frac{2}{\sqrt{6}}\,\mathbf{j}-\frac{1}{\sqrt{6}}\,\mathbf{k}\right) \Rightarrow$ the direction is $\frac{1}{\sqrt{6}}\,\mathbf{i}+\frac{2}{\sqrt{6}}\,\mathbf{j}-\frac{1}{\sqrt{6}}\,\mathbf{k}$

15. $2\,\frac{\mathbf{v}}{|\mathbf{v}|} = 2\cdot\frac{4\mathbf{i}-\mathbf{j}+4\mathbf{k}}{\sqrt{4^2+(-1)^2+4^2}} = 2\cdot\frac{4\mathbf{i}-\mathbf{j}+4\mathbf{k}}{\sqrt{33}} = \frac{8}{\sqrt{33}}\,\mathbf{i}-\frac{2}{\sqrt{33}}\,\mathbf{j}+\frac{8}{\sqrt{33}}\,\mathbf{k}$

16. $-5\,\frac{\mathbf{v}}{|\mathbf{v}|} = -5\cdot\frac{\left(\frac{3}{5}\right)\mathbf{i}+\left(\frac{4}{5}\right)\mathbf{k}}{\sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}} = -5\cdot\frac{\left(\frac{3}{5}\right)\mathbf{i}+\left(\frac{4}{5}\right)\mathbf{k}}{\sqrt{\frac{9}{25}+\frac{16}{25}}} = -3\mathbf{i}-4\mathbf{k}$

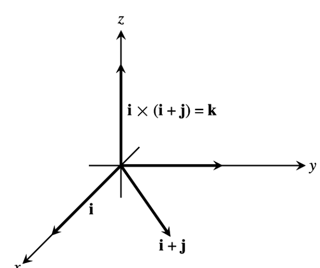
17. $|\mathbf{v}| = \sqrt{1+1} = \sqrt{2}$, $|\mathbf{u}| = \sqrt{4+1+4} = 3$, $\mathbf{v}\cdot\mathbf{u} = 3$, $\mathbf{u}\cdot\mathbf{v} = 3$, $\mathbf{v}\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 2 & 1 & -2 \end{vmatrix} = -2\mathbf{i}+2\mathbf{j}-\mathbf{k}$,
$\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u}) = 2\mathbf{i}-2\mathbf{j}+\mathbf{k}$, $|\mathbf{v}\times\mathbf{u}| = \sqrt{4+4+1} = 3$, $\theta = \cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}|\,|\mathbf{u}|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$,
$|\mathbf{u}|\cos\theta = \frac{3}{\sqrt{2}}$, $\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}||\mathbf{v}|}\right)\mathbf{v} = \frac{3}{2}(\mathbf{i}+\mathbf{j})$

18. $|\mathbf{v}| = \sqrt{1^2+1^2+2^2} = \sqrt{6}$, $|\mathbf{u}| = \sqrt{(-1)^2+(-1)^2} = \sqrt{2}$, $\mathbf{v}\cdot\mathbf{u} = (1)(-1)+(1)(0)+(2)(-1) = -3$,
$\mathbf{u}\cdot\mathbf{v} = -3$, $\mathbf{v}\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{vmatrix} = -\mathbf{i}-\mathbf{j}+\mathbf{k}$, $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u}) = \mathbf{i}+\mathbf{j}-\mathbf{k}$,
$|\mathbf{v}\times\mathbf{u}| = \sqrt{(-1)^2+(-1)^2+1^2} = \sqrt{3}$, $\theta = \cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}|\,|\mathbf{u}|}\right) = \cos^{-1}\left(\frac{-3}{\sqrt{6}\sqrt{2}}\right) = \cos^{-1}\left(\frac{-3}{\sqrt{12}}\right)$
$= \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$, $|\mathbf{u}|\cos\theta = \sqrt{2}\cdot\left(\frac{-\sqrt{3}}{2}\right) = -\frac{\sqrt{6}}{2}$, $\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}||\mathbf{v}|}\right)\mathbf{v} = \frac{-3}{6}(\mathbf{i}+\mathbf{j}+2\mathbf{k}) = -\frac{1}{2}(\mathbf{i}+\mathbf{j}+\mathbf{k})$

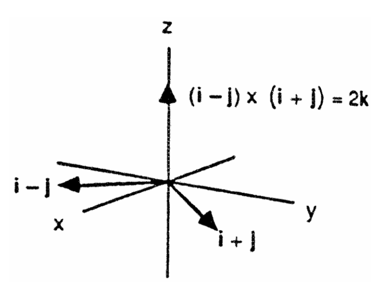
19. $\mathbf{u} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}||\mathbf{v}|}\right)\mathbf{v} + \left[\mathbf{u}-\left(\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}||\mathbf{v}|}\right)\mathbf{v}\right] = \frac{4}{3}(2\mathbf{i}+\mathbf{j}-\mathbf{k}) + \left[(\mathbf{i}+\mathbf{j}-5\mathbf{k})-\frac{4}{3}(2\mathbf{i}+\mathbf{j}-\mathbf{k})\right] = \frac{4}{3}(2\mathbf{i}+\mathbf{j}-\mathbf{k})-\frac{1}{3}(5\mathbf{i}+\mathbf{j}+11\mathbf{k})$,
where $\mathbf{v}\cdot\mathbf{u} = 8$ and $\mathbf{v}\cdot\mathbf{v} = 6$

20. $\mathbf{u} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}||\mathbf{v}|}\right)\mathbf{v} + \left[\mathbf{u}-\left(\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}||\mathbf{v}|}\right)\mathbf{v}\right] = -\frac{1}{3}(\mathbf{i}-2\mathbf{j}) + \left[(\mathbf{i}+\mathbf{j}+\mathbf{k})-\left(\frac{-1}{3}\right)(\mathbf{i}-2\mathbf{j})\right] = -\frac{1}{3}(\mathbf{i}-2\mathbf{j})+\left(\frac{4}{3}\,\mathbf{i}+\frac{5}{3}\,\mathbf{j}+\mathbf{k}\right)$,
where $\mathbf{v}\cdot\mathbf{u} = -1$ and $\mathbf{v}\cdot\mathbf{v} = 3$

21. $\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = \mathbf{k}$



22. $\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 2\mathbf{k}$



23. Let $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$. Then $|\mathbf{v} - 2\mathbf{w}|^2 = |(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) - 2(w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k})|^2$
$= |(v_1 - 2w_1)\mathbf{i} + (v_2 - 2w_2)\mathbf{j} + (v_3 - 2w_3)\mathbf{k}|^2 = \left(\sqrt{(v_1 - 2w_1)^2 + (v_2 - 2w_2)^2 + (v_3 - 2w_3)^2}\right)^2$
$= (v_1^2 + v_2^2 + v_3^2) - 4(v_1w_1 + v_2w_2 + v_3w_3) + 4\left(w_1^2 + w_2^2 + w_3^2\right) = |\mathbf{v}|^2 - 4\mathbf{v}\cdot\mathbf{w} + 4|\mathbf{w}|^2$
$= |\mathbf{v}|^2 - 4|\mathbf{v}||\mathbf{w}|\cos\theta + 4|\mathbf{w}|^2 = 4 - 4(2)(3)\left(\cos\frac{\pi}{3}\right) + 36 = 40 - 24\left(\frac{1}{2}\right) = 40 - 12 = 28 \Rightarrow |\mathbf{v} - 2\mathbf{w}| = \sqrt{28}$
$= 2\sqrt{7}$

24. $\mathbf{u}$ and $\mathbf{v}$ are parallel when $\mathbf{u} \times \mathbf{v} = \mathbf{0} \Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -5 \\ -4 & -8 & a \end{vmatrix} = \mathbf{0} \Rightarrow (4a - 40)\mathbf{i} + (20 - 2a)\mathbf{j} + (0)\mathbf{k} = \mathbf{0}$
$\Rightarrow 4a - 40 = 0$ and $20 - 2a = 0 \Rightarrow a = 10$

25. (a) area $= |\mathbf{u} \times \mathbf{v}| = \text{abs}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{vmatrix} = |2\mathbf{i} - 3\mathbf{j} - \mathbf{k}| = \sqrt{4+9+1} = \sqrt{14}$

(b) volume $= \mathbf{u}\cdot(\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ -1 & -2 & 3 \end{vmatrix} = 1(3+2) + 1(-1-6) - 1(-4+1) = 1$

26. (a) area $= |\mathbf{u} \times \mathbf{v}| = \text{abs}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = |\mathbf{k}| = 1$

(b) volume $= \mathbf{u}\cdot(\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1(1-0) + 1(0-0) + 0 = 1$

27. The desired vector is $\mathbf{n} \times \mathbf{v}$ or $\mathbf{v} \times \mathbf{n}$ since $\mathbf{n} \times \mathbf{v}$ is perpendicular to both $\mathbf{n}$ and $\mathbf{v}$ and, therefore, also parallel to the plane.

28. If $a = 0$ and $b \neq 0$, then the line $by = c$ and $\mathbf{i}$ are parallel. If $a \neq 0$ and $b = 0$, then the line $ax = c$ and $\mathbf{j}$ are parallel. If a and b are both $\neq 0$, then $ax + by = c$ contains the points $\left(\frac{c}{a}, 0\right)$ and $\left(0, \frac{c}{b}\right) \Rightarrow$ the vector $ab\left(\frac{c}{a}\mathbf{i} - \frac{c}{b}\mathbf{j}\right) = c(b\mathbf{i} - a\mathbf{j})$ and the line are parallel. Therefore, the vector $b\mathbf{i} - a\mathbf{j}$ is parallel to the line $ax + by = c$ in every case.

29. The line L passes through the point $P(0, 0, -1)$ parallel to $\mathbf{v} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$. With $\overrightarrow{PS} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ -1 & 1 & 1 \end{vmatrix} = (2-1)\mathbf{i} + (-1-2)\mathbf{j} + (2+2)\mathbf{k} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, we find the distance $d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1+9+16}}{\sqrt{1+1+1}} = \frac{\sqrt{26}}{\sqrt{3}} = \frac{\sqrt{78}}{3}$.

30. The line L passes through the point $P(2, 2, 0)$ parallel to $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. With $\overrightarrow{PS} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = (2-1)\mathbf{i} + (1+2)\mathbf{j} + (-2-2)\mathbf{k} = \mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, we find the distance $d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1+9+16}}{\sqrt{1+1+1}} = \frac{\sqrt{26}}{\sqrt{3}} = \frac{\sqrt{78}}{3}$.

31. Parametric equations for the line are $x = 1 - 3t$, $y = 2$, $z = 3 + 7t$.

32. The line is parallel to $\overrightarrow{PQ} = 0\mathbf{i} + \mathbf{j} - \mathbf{k}$ and contains the point $P(1, 2, 0) \Rightarrow$ parametric equations are $x = 1, y = 2 + t, z = -t$ for $0 \le t \le 1$.

33. The point $P(4, 0, 0)$ lies on the plane $x - y = 4$, and $\overrightarrow{PS} = (6-4)\mathbf{i} + 0\mathbf{j} + (-6+0)\mathbf{k} = 2\mathbf{i} - 6\mathbf{k}$ with $\mathbf{n} = \mathbf{i} - \mathbf{j}$ $\Rightarrow d = \frac{\left|\mathbf{n}\cdot\overrightarrow{PS}\right|}{|\mathbf{n}|} = \left|\frac{2+0+0}{\sqrt{1+1+0}}\right| = \frac{2}{\sqrt{2}} = \sqrt{2}$.

34. The point $P(0, 0, 2)$ lies on the plane $2x + 3y + z = 2$, and $\overrightarrow{PS} = (3-0)\mathbf{i} + (0-0)\mathbf{j} + (10+2)\mathbf{k} = 3\mathbf{i} + 8\mathbf{k}$ with $\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow d = \frac{\left|\mathbf{n}\cdot\overrightarrow{PS}\right|}{|\mathbf{n}|} = \left|\frac{6+0+8}{\sqrt{4+9+1}}\right| = \frac{14}{\sqrt{14}} = \sqrt{14}$.

35. $P(3, -2, 1)$ and $\mathbf{n} = 2\mathbf{i} + \mathbf{j} - \mathbf{k} \Rightarrow (2)(x-3) + (1)(y-(-2)) + (1)(z-1) = 0 \Rightarrow 2x + y + z = 5$

36. $P(-1, 6, 0)$ and $\mathbf{n} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \Rightarrow (1)(x-(-1)) + (-2)(y-6) + (3)(z-0) = 0 \Rightarrow x - 2y + 3z = -13$

37. $P(1, -1, 2)$, $Q(2, 1, 3)$ and $R(-1, 2, -1) \Rightarrow \overrightarrow{PQ} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\overrightarrow{PR} = -2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$ and $\overrightarrow{PQ} \times \overrightarrow{PR}$ $= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ -2 & 3 & -3 \end{vmatrix} = -9\mathbf{i} + \mathbf{j} + 7\mathbf{k}$ is normal to the plane $\Rightarrow (-9)(x-1) + (1)(y+1) + (7)(z-2) = 0$ $\Rightarrow -9x + y + 7z = 4$

38. $P(1, 0, 0)$, $Q(0, 1, 0)$ and $R(0, 0, 1) \Rightarrow \overrightarrow{PQ} = -\mathbf{i} + \mathbf{j}$, $\overrightarrow{PR} = -\mathbf{i} + \mathbf{k}$ and $\overrightarrow{PQ} \times \overrightarrow{PR}$ $= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ is normal to the plane $\Rightarrow (1)(x-1) + (1)(y-0) + (1)(z-0) = 0$ $\Rightarrow x + y + z = 1$

39. $\left(0, -\frac{1}{2}, -\frac{3}{2}\right)$, since $t = -\frac{1}{2}$, $y = -\frac{1}{2}$ and $z = -\frac{3}{2}$ when $x = 0$; $(-1, 0, -3)$, since $t = -1$, $x = -1$ and $z = -3$ when $y = 0$; $(1, -1, 0)$, since $t = 0$, $x = 1$ and $y = -1$ when $z = 0$

40. $x = 2t, y = -t, z = -t$ represents a line containing the origin and perpendicular to the plane $2x - y - z = 4$; this line intersects the plane $3x - 5y + 2z = 6$ when t is the solution of $3(2t) - 5(-t) + 2(-t) = 6$ $\Rightarrow t = \frac{2}{3} \Rightarrow \left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$ is the point of intersection

41. $\mathbf{n}_1 = \mathbf{i}$ and $\mathbf{n}_2 = \mathbf{i} + \mathbf{j} + \sqrt{2}\mathbf{k} \Rightarrow$ the desired angle is $\cos^{-1}\left(\frac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

42. $\mathbf{n}_1 = \mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = \mathbf{j} + \mathbf{k} \Rightarrow$ the desired angle is $\cos^{-1}\left(\frac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

43. The direction of the line is $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$. Since the point $(-5, 3, 0)$ is on both planes, the desired line is $x = -5 + 5t, y = 3 - t, z = -3t$.

44. The direction of the intersection is $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 5 & -2 & -1 \end{vmatrix} = -6\mathbf{i} - 9\mathbf{j} - 12\mathbf{k} = -3(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$ and is the same as the direction of the given line.

45. (a) The corresponding normals are $\mathbf{n}_1 = 3\mathbf{i} + 6\mathbf{k}$ and $\mathbf{n}_2 = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and since $\mathbf{n}_1 \cdot \mathbf{n}_2 = (3)(2) + (0)(2) + (6)(-1) = 6 + 0 - 6 = 0$, we have that the planes are orthogonal

(b) The line of intersection is parallel to $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 6 \\ 2 & 2 & -1 \end{vmatrix} = -12\mathbf{i} + 15\mathbf{j} + 6\mathbf{k}$. Now to find a point in the intersection, solve $\begin{cases} 3x + 6z = 1 \\ 2x + 2y - z = 3 \end{cases} \Rightarrow \begin{cases} 3x + 6z = 1 \\ 12x + 12y - 6z = 18 \end{cases} \Rightarrow 15x + 12y = 19 \Rightarrow x = 0$ and $y = \frac{19}{12}$ $\Rightarrow \left(0, \frac{19}{12}, \frac{1}{6}\right)$ is a point on the line we seek. Therefore, the line is $x = -12t$, $y = \frac{19}{12} + 15t$ and $z = \frac{1}{6} + 6t$.

46. A vector in the direction of the plane's normal is $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 7\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$ and $P(1, 2, 3)$ on the plane $\Rightarrow 7(x - 1) - 3(y - 2) - 5(z - 3) = 0 \Rightarrow 7x - 3y - 5z = -14$.

47. Yes; $\mathbf{v} \cdot \mathbf{n} = (2\mathbf{i} - 4\mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} + 0\mathbf{k}) = 2 \cdot 2 - 4 \cdot 1 + 1 \cdot 0 = 0 \Rightarrow$ the vector is orthogonal to the plane's normal $\Rightarrow \mathbf{v}$ is parallel to the plane

48. $\mathbf{n} \cdot \overrightarrow{PP_0} > 0$ represents the half-space of points lying on one side of the plane in the direction which the normal $\mathbf{n}$ points

49. A normal to the plane is $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 2 & -1 & 0 \end{vmatrix} = -\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} \Rightarrow$ the distance is $d = \left| \frac{\overrightarrow{AP} \cdot \mathbf{n}}{\mathbf{n}} \right|$ $= \left| \frac{(\mathbf{i} + 4\mathbf{j}) \cdot (-\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})}{\sqrt{1 + 4 + 4}} \right| = \left| \frac{-1 - 8 + 0}{3} \right| = 3$

50. $P(0, 0, 0)$ lies on the plane $2x + 3y + 5z = 0$, and $\overrightarrow{PS} = 2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ with $\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} \Rightarrow$ $d = \left| \frac{\mathbf{n} \cdot \overrightarrow{PS}}{|\mathbf{n}|} \right| = \left| \frac{4 + 6 + 15}{\sqrt{4 + 9 + 25}} \right| = \frac{25}{\sqrt{38}}$

51. $\mathbf{n} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$ is normal to the plane $\Rightarrow \mathbf{n} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 0\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} = -3\mathbf{j} + 3\mathbf{k}$ is orthogonal to $\mathbf{v}$ and parallel to the plane

52. The vector $\mathbf{B} \times \mathbf{C}$ is normal to the plane of $\mathbf{B}$ and $\mathbf{C} \Rightarrow \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is orthogonal to $\mathbf{A}$ and parallel to the plane of $\mathbf{B}$ and $\mathbf{C}$:

$\mathbf{B} \times \mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = -5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ -5 & 3 & -1 \end{vmatrix} = -2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

$\Rightarrow |\mathbf{A} \times (\mathbf{B} \times \mathbf{C})| = \sqrt{4 + 9 + 1} = \sqrt{14}$ and $\mathbf{u} = \frac{1}{\sqrt{14}}(-2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$ is the desired unit vector.

53. A vector parallel to the line of intersection is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{25 + 1 + 9} = \sqrt{35} \Rightarrow 2\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \frac{2}{\sqrt{35}}(5\mathbf{i} - \mathbf{j} - 3\mathbf{k})$ is the desired vector.

54. The line containing $(0, 0, 0)$ normal to the plane is represented by $x = 2t$, $y = -t$, and $z = -t$. This line intersects the plane $3x - 5y + 2z = 6$ when $3(2t) - 5(-t) + 2(-t) = 6 \Rightarrow t = \frac{2}{3} \Rightarrow$ the point is $\left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$.

55. The line is represented by $x = 3 + 2t$, $y = 2 - t$, and $z = 1 + 2t$. It meets the plane $2x - y + 2z = -2$ when $2(3 + 2t) - (2 - t) + 2(1 + 2t) = -2 \Rightarrow t = -\frac{8}{9} \Rightarrow$ the point is $\left(\frac{11}{9}, \frac{26}{9}, -\frac{7}{9}\right)$.

56. The direction of the intersection is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 1 & 2 \end{vmatrix} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{i}}{|\mathbf{v}|\,|\mathbf{i}|}\right)$
$= \cos^{-1}\left(\frac{3}{\sqrt{35}}\right) \approx 59.5°$

57. The intersection occurs when $(3 + 2t) + 3(2t) - t = -4 \Rightarrow t = -1 \Rightarrow$ the point is $(1, -2, -1)$. The required line must be perpendicular to both the given line and to the normal, and hence is parallel to $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ 1 & 3 & -1 \end{vmatrix}$
$= -5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} \Rightarrow$ the line is represented by $x = 1 - 5t$, $y = -2 + 3t$, and $z = -1 + 4t$.

58. If $P(a, b, c)$ is a point on the line of intersection, then P lies in both planes $\Rightarrow a - 2b + c + 3 = 0$ and $2a - b - c + 1 = 0 \Rightarrow (a - 2b + c + 3) + k(2a - b - c + 1) = 0$ for all k.

59. The vector $\overrightarrow{AB} \times \overrightarrow{CD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 4 \\ \frac{26}{5} & 0 & -\frac{26}{5} \end{vmatrix} = \frac{26}{5}(2\mathbf{i} + 7\mathbf{j} + 2\mathbf{k})$ is normal to the plane and $A(-2, 0, -3)$ lies on the plane $\Rightarrow 2(x + 2) + 7(y - 0) + 2(z - (-3)) = 0 \Rightarrow 2x + 7y + 2z + 10 = 0$ is an equation of the plane.

60. Yes; the line's direction vector is $2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ which is parallel to the line and also parallel to the normal $-4\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$ to the plane $\Rightarrow$ the line is orthogonal to the plane.

61. The vector $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -3 & 0 & 1 \end{vmatrix} = -\mathbf{i} - 11\mathbf{j} - 3\mathbf{k}$ is normal to the plane.

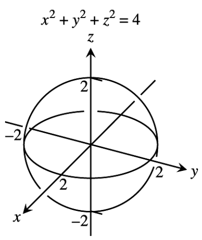
(a) No, the plane is not orthogonal to $\overrightarrow{PQ} \times \overrightarrow{PR}$.
(b) No, these equations represent a line, not a plane.
(c) No, the plane $(x + 2) + 11(y - 1) - 3z = 0$ has normal $\mathbf{i} + 11\mathbf{j} - 3\mathbf{k}$ which is not parallel to $\overrightarrow{PQ} \times \overrightarrow{PR}$.
(d) No, this vector equation is equivalent to the equations $3y + 3z = 3$, $3x - 2z = -6$, and $3x + 2y = -4$ $\Rightarrow x = -\frac{4}{3} - \frac{2}{3}t$, $y = t$, $z = 1 - t$, which represents a line, not a plane.
(e) Yes, this is a plane containing the point $R(-2, 1, 0)$ with normal $\overrightarrow{PQ} \times \overrightarrow{PR}$.

62. (a) The line through A and B is $x = 1 + t$, $y = -t$, $z = -1 + 5t$; the line through C and D must be parallel and is L_1: $x = 1 + t$, $y = 2 - t$, $z = 3 + 5t$. The line through B and C is $x = 1$, $y = 2 + 2s$, $z = 3 + 4s$; the line through A and D must be parallel and is L_2: $x = 2$, $y = -1 + 2s$, $z = 4 + 4s$. The lines L_1 and L_2 intersect at $D(2, 1, 8)$ where $t = 1$ and $s = 1$.
(b) $\cos\theta = \frac{(2\mathbf{j} + 4\mathbf{k})\cdot(\mathbf{i} - \mathbf{j} + 5\mathbf{k})}{\sqrt{20}\,\sqrt{27}} = \frac{3}{\sqrt{15}}$
(c) $\left(\frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\overrightarrow{BC}\cdot\overrightarrow{BC}}\right)\overrightarrow{BC} = \frac{18}{20}\overrightarrow{BC} = \frac{9}{5}(\mathbf{j} + 2\mathbf{k})$ where $\overrightarrow{BA} = \mathbf{i} - \mathbf{j} + 5\mathbf{k}$ and $\overrightarrow{BC} = 2\mathbf{j} + 4\mathbf{k}$
(d) area $= |(2\mathbf{j} + 4\mathbf{k}) \times (\mathbf{i} - \mathbf{j} + 5\mathbf{k})| = |14\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}| = 6\sqrt{6}$
(e) From part (d), $\mathbf{n} = 14\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ is normal to the plane $\Rightarrow 14(x - 1) + 4(y - 0) - 2(z + 1) = 0$ $\Rightarrow 7x + 2y - z = 8$.
(f) From part (d), $\mathbf{n} = 14\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \Rightarrow$ the area of the projection on the yz-plane is $|\mathbf{n}\cdot\mathbf{i}| = 14$; the area of the projection on the xy-plane is $|\mathbf{n}\cdot\mathbf{j}| = 4$; and the area of the projection on the xy-plane is $|\mathbf{n}\cdot\mathbf{k}| = 2$.

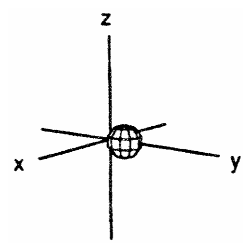
63. $\overrightarrow{AB} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\overrightarrow{CD} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$, and $\overrightarrow{AC} = 2\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ 1 & 4 & -1 \end{vmatrix} = -5\mathbf{i} - \mathbf{j} - 9\mathbf{k} \Rightarrow$ the distance is $d = \left| \frac{(2\mathbf{i}+\mathbf{j})\cdot(-5\mathbf{i}-\mathbf{j}-9\mathbf{k})}{\sqrt{25+1+81}} \right| = \frac{11}{\sqrt{107}}$

64. $\overrightarrow{AB} = -2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, $\overrightarrow{CD} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, and $\overrightarrow{AC} = -3\mathbf{i} + 3\mathbf{j} \Rightarrow \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 4 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 7\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} \Rightarrow$ the distance is $d = \left| \frac{(-3\mathbf{i}+3\mathbf{j})\cdot(7\mathbf{i}+3\mathbf{j}-2\mathbf{k})}{\sqrt{49+9+4}} \right| = \frac{12}{\sqrt{62}}$

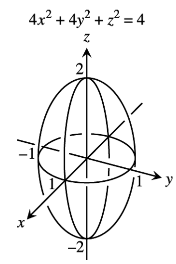
65. $x^2 + y^2 + z^2 = 4$



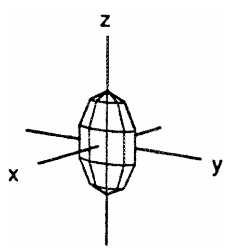
66. $x^2 + (y-1)^2 + z^2 = 1$



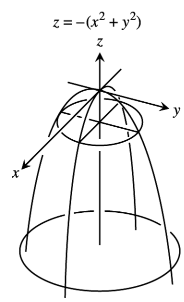
67. $4x^2 + 4y^2 + z^2 = 4$



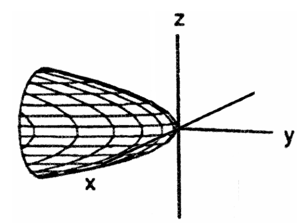
68. $36x^2 + 9y^2 + 4z^2 = 36$



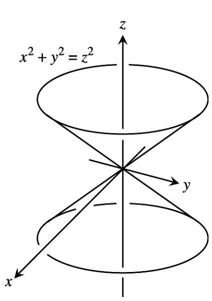
69. $z = -(x^2 + y^2)$



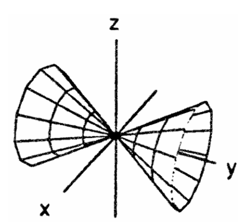
70. $y = -(x^2 + z^2)$



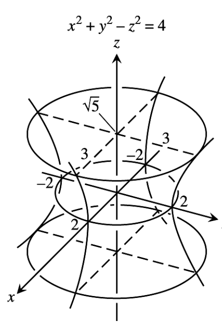
71. $x^2 + y^2 = z^2$



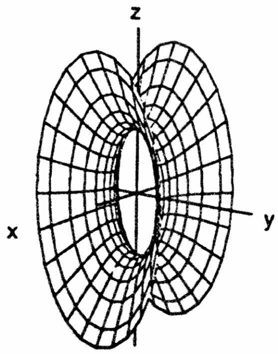
72. $x^2 + z^2 = y^2$



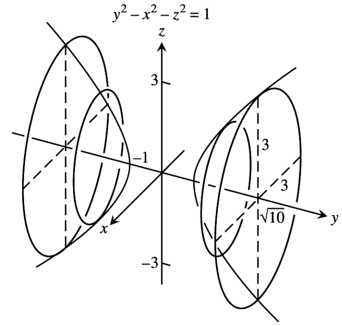
73. $x^2 + y^2 - z^2 = 4$



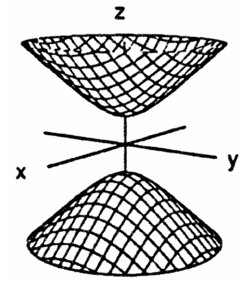
74. $4y^2 + z^2 - 4x^2 = 4$



75. $y^2 - x^2 - z^2 = 1$



76. $z^2 - x^2 - y^2 = 1$



CHAPTER 12 ADDITIONAL AND ADVANCED EXERCISES

1. Information from ship A indicates the submarine is now on the line L_1: $x = 4 + 2t$, $y = 3t$, $z = -\frac{1}{3}t$; information from ship B indicates the submarine is now on the line L_2: $x = 18s$, $y = 5 - 6s$, $z = -s$. The current position of the sub is $\left(6, 3, -\frac{1}{3}\right)$ and occurs when the lines intersect at $t = 1$ and $s = \frac{1}{3}$. The straight line path of the submarine contains both points $P\left(2, -1, -\frac{1}{3}\right)$ and $Q\left(6, 3, -\frac{1}{3}\right)$; the line representing this path is L: $x = 2 + 4t$, $y = -1 + 4t$, $z = -\frac{1}{3}$. The submarine traveled the distance between P and Q in 4 minutes $\Rightarrow$ a speed of $\frac{\left|\overrightarrow{PQ}\right|}{4} = \frac{\sqrt{32}}{4} = \sqrt{2}$ thousand ft/min. In 20 minutes the submarine will move $20\sqrt{2}$ thousand ft from Q along the line L $\Rightarrow$ $20\sqrt{2} = \sqrt{(2 + 4t - 6)^2 + (-1 + 4t - 3)^2 + 0^2}$ $\Rightarrow$ $800 = 16(t-1)^2 + 16(t-1)^2 = 32(t-1)^2$ $\Rightarrow$ $(t-1)^2 = \frac{800}{32} = 25$ $\Rightarrow$ $t = 6$ $\Rightarrow$ the submarine will be located at $\left(26, 23, -\frac{1}{3}\right)$ in 20 minutes.

2. H_2 stops its flight when $6 + 110t = 446$ $\Rightarrow$ $t = 4$ hours. After 6 hours, H_1 is at $P(246, 57, 9)$ while H_2 is at $(446, 13, 0)$. The distance between P and Q is $\sqrt{(246 - 446)^2 + (57 - 13)^2 + (9 - 0)^2} \approx 204.98$ miles. At 150 mph, it would take about 1.37 hours for H_1 to reach H_2.

3. Torque $= \left|\overrightarrow{PQ} \times \mathbf{F}\right|$ $\Rightarrow$ 15 ft-lb $= \left|\overrightarrow{PQ}\right| |\mathbf{F}| \sin\frac{\pi}{2} = \frac{3}{4}$ ft $\cdot |\mathbf{F}|$ $\Rightarrow$ $|\mathbf{F}| = 20$ lb

4. Let $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ be the vector from O to A and $\mathbf{b} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ be the vector from O to B. The vector $\mathbf{v}$ orthogonal to $\mathbf{a}$ and $\mathbf{b}$ $\Rightarrow$ $\mathbf{v}$ is parallel to $\mathbf{b} \times \mathbf{a}$ (since the rotation is closkwise). Now $\mathbf{b} \times \mathbf{a} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$; $\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ $\Rightarrow$ (2, 2, 2) is the center of the circular path (1, 3, 2) takes $\Rightarrow$ radius $= \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}$ $\Rightarrow$ arc length per second covered by the point is $\frac{3}{2}\sqrt{2}$ units/sec $= |\mathbf{v}|$ (velocity is constant). A unit vector in the direction of $\mathbf{v}$ is $\frac{\mathbf{b}\times\mathbf{a}}{|\mathbf{b}\times\mathbf{a}|}$ $= \frac{1}{\sqrt{6}}\mathbf{i} + \frac{1}{\sqrt{6}}\mathbf{j} - \frac{2}{\sqrt{6}}\mathbf{k}$ $\Rightarrow$ $\mathbf{v} = |\mathbf{v}|\left(\frac{\mathbf{b}\times\mathbf{a}}{|\mathbf{b}\times\mathbf{a}|}\right) = \frac{3}{2}\sqrt{2}\left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{1}{\sqrt{6}}\mathbf{j} - \frac{2}{\sqrt{6}}\mathbf{k}\right) = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} - \sqrt{3}\mathbf{k}$

5. (a) If $P(x, y, z)$ is a point in the plane determined by the three points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ and $P_3(x_3, y_3, z_3)$, then the vectors $\overrightarrow{PP_1}$, $\overrightarrow{PP_2}$ and $\overrightarrow{PP_3}$ all lie in the plane. Thus $\overrightarrow{PP_1} \cdot \left(\overrightarrow{PP_2} \times \overrightarrow{PP_3}\right) = 0$

$$\Rightarrow \begin{vmatrix} x_1 - x & y_1 - y & z_1 - z \\ x_2 - x & y_2 - y & z_2 - z \\ x_3 - x & y_3 - y & z_3 - z \end{vmatrix} = 0$$ by the determinant formula for the triple scalar product in Section 10.4.

(b) Subtract row 1 from rows 2, 3, and 4 and evaluate the resulting determinant (which has the same value as the given determinant) by cofactor expansion about column 4. This expansion is exactly the determinant in part (a) so we have all points $P(x, y, z)$ in the plane determined by $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$.

6. Let L_1: $x = a_1 s + b_1$, $y = a_2 s + b_2$, $z = a_3 s + b_3$ and L_2: $x = c_1 t + d_1$, $y = c_2 t + d_2$, $z = c_3 t + d_3$. If $L_1 \parallel L_2$, then for some k, $a_i = kc_i$, $i = 1, 2, 3$ and the determinant $\begin{vmatrix} a_1 & c_1 & b_1 - d_1 \\ a_2 & c_2 & b_2 - d_2 \\ a_3 & c_3 & b_3 - d_3 \end{vmatrix} = \begin{vmatrix} kc_1 & c_1 & b_1 - d_1 \\ kc_2 & c_2 & b_2 - d_2 \\ kc_3 & c_3 & b_3 - d_3 \end{vmatrix} = 0$, since the first column is a multiple of the second column. The lines L_1 and L_2 intersect if and only if the system $\begin{cases} a_1 s - c_1 t + (b_1 - d_1) = 0 \\ a_2 s - c_2 t + (b_2 - d_2) = 0 \\ a_3 s - c_3 t + (b_3 - d_3) = 0 \end{cases}$ has a nontrivial solution $\Leftrightarrow$ the determinant of the coefficients is zero.

7. (a) $\overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB}$
   (b) $\overrightarrow{AP} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BD} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{AD}\right)$
   (c) $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$, so by part (b), $\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AC}$

8. Extend $\overrightarrow{CD}$ to $\overrightarrow{CG}$ so that $\overrightarrow{CD} = \overrightarrow{DG}$. Then $\overrightarrow{CG} = t\overrightarrow{CF} = \overrightarrow{CB} + \overrightarrow{BG}$ and $t\overrightarrow{CF} = 3\overrightarrow{CE} + \overrightarrow{CA}$, since ACBG is a parallelogram. If $t\overrightarrow{CF} - 3\overrightarrow{CE} - \overrightarrow{CA} = \mathbf{0}$, then $t - 3 - 1 = 0 \Rightarrow t = 4$, since F, E, and A are collinear. Therefore, $\overrightarrow{CG} = 4\overrightarrow{CF} \Rightarrow \overrightarrow{CD} = 2\overrightarrow{CF} \Rightarrow$ F is the midpoint of $\overline{CD}$.

9. If $Q(x, y)$ is a point on the line $ax + by = c$, then $\overrightarrow{P_1Q} = (x - x_1)\mathbf{i} + (y - y_1)\mathbf{j}$, and $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$ is normal to the line. The distance is $\left|\text{proj}_{\mathbf{n}} \overrightarrow{P_1Q}\right| = \left|\frac{[(x - x_1)\mathbf{i} + (y - y_1)\mathbf{j}]\cdot(a\mathbf{i} + b\mathbf{j})}{\sqrt{a^2 + b^2}}\right| = \frac{|a(x - x_1) + b(y - y_1)|}{\sqrt{a^2 + b^2}}$
$= \frac{|ax_1 + by_1 - c|}{\sqrt{a^2 + b^2}}$, since $c = ax + by$.

10. (a) Let $Q(x, y, z)$ be any point on $Ax + By + Cz - D = 0$. Let $\overrightarrow{QP_1} = (x - x_1)\mathbf{i} + (y - y_1)\mathbf{j} + (z - z_1)\mathbf{k}$, and $\mathbf{n} = \frac{A\mathbf{i} + B\mathbf{j} + C\mathbf{k}}{\sqrt{A^2 + B^2 + C^2}}$. The distance is $\left|\text{proj}_{\mathbf{n}} \overrightarrow{QP_1}\right| = \left|((x - x_1)\mathbf{i} + (y - y_1)\mathbf{j} + (z - z_1)\mathbf{k}) \cdot \left(\frac{A\mathbf{i} + B\mathbf{j} + C\mathbf{k}}{\sqrt{A^2 + B^2 + C^2}}\right)\right|$
$= \frac{|Ax_1 + By_1 + Cz_1 - (Ax + By + Cz)|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|Ax_1 + By_1 + Cz_1 - D|}{\sqrt{A^2 + B^2 + C^2}}$.
   (b) Since both tangent planes are parallel, one-half of the distance between them is equal to the radius of the sphere, i.e., $r = \frac{1}{2}\frac{|3 - 9|}{\sqrt{1 + 1 + 1}} = \sqrt{3}$ (see also Exercise 17a). Clearly, the points $(1, 2, 3)$ and $(-1, -2, -3)$ are on the line containing the sphere's center. Hence, the line containing the center is $x = 1 + 2t$, $y = 2 + 4t$, $z = 3 + 6t$. The distance from the plane $x + y + z - 3 = 0$ to the center is $\sqrt{3}$
$\Rightarrow \frac{|(1 + 2t) + (2 + 4t) + (3 + 6t) - 3|}{\sqrt{1 + 1 + 1}} = \sqrt{3}$ from part (a) $\Rightarrow t = 0 \Rightarrow$ the center is at $(1, 2, 3)$. Therefore an equation of the sphere is $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 3$.

11. (a) If (x_1, y_1, z_1) is on the plane $Ax + By + Cz = D_1$, then the distance d between the planes is $d = \frac{|Ax_1 + By_1 + Cz_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|D_1 - D_2|}{|A\mathbf{i} + B\mathbf{j} + C\mathbf{k}|}$, since $Ax_1 + By_1 + Cz_1 = D_1$, by Exercise 10(a).
   (b) $d = \frac{|12 - 6|}{\sqrt{4 + 9 + 1}} = \frac{6}{\sqrt{14}}$
   (c) $\frac{|2(3) + (-1)(2) + 2(-1) + 4|}{\sqrt{14}} = \frac{|2(3) + (-1)(2) + 2(-1) - D|}{\sqrt{14}} \Rightarrow D = 8$ or $-4 \Rightarrow$ the desired plane is $2x - y + 2x = 8$
   (d) Choose the point $(2, 0, 1)$ on the plane. Then $\frac{|3 - D|}{\sqrt{6}} = 5 \Rightarrow D = 3 \pm 5\sqrt{6} \Rightarrow$ the desired planes are $x - 2y + z = 3 + 5\sqrt{6}$ and $x - 2y + z = 3 - 5\sqrt{6}$.

12. Let $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{BC}$ and $D(x, y, z)$ be any point in the plane determined by A, B and C. Then the point D lies in this plane if and only if $\overrightarrow{AD} \cdot \mathbf{n} = 0 \Leftrightarrow \overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{BC}) = 0$.

13. $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ is normal to the plane $x + 2y + 6z = 6$; $\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 2 & 6 \end{vmatrix} = 4\mathbf{i} - 5\mathbf{j} + \mathbf{k}$ is parallel to the plane and perpendicular to the plane of $\mathbf{v}$ and $\mathbf{n}$ $\Rightarrow$ $\mathbf{w} = \mathbf{n} \times (\mathbf{v} \times \mathbf{n}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 6 \\ 4 & -5 & 1 \end{vmatrix} = 32\mathbf{i} + 23\mathbf{j} - 13\mathbf{k}$ is a vector parallel to the plane $x + 2y + 6z = 6$ in the direction of the projection vector $\text{proj}_P\, \mathbf{v}$. Therefore,
$\text{proj}_P\, \mathbf{v} = \text{proj}_{\mathbf{w}}\, \mathbf{v} = \left(\mathbf{v} \cdot \frac{\mathbf{w}}{|\mathbf{w}|}\right) \frac{\mathbf{w}}{|\mathbf{w}|} = \left(\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\right)\mathbf{w} = \left(\frac{32 + 23 - 13}{32^2 + 23^2 + 13^2}\right)\mathbf{w} = \frac{42}{1722}\,\mathbf{w} = \frac{1}{41}\,\mathbf{w} = \frac{32}{41}\,\mathbf{i} + \frac{23}{41}\,\mathbf{j} - \frac{13}{41}\,\mathbf{k}$

14. $\text{proj}_{\mathbf{z}}\, \mathbf{w} = -\text{proj}_{\mathbf{z}}\, \mathbf{v}$ and $\mathbf{w} - \text{proj}_{\mathbf{z}}\, \mathbf{w} = \mathbf{v} - \text{proj}_{\mathbf{z}}\, \mathbf{v}$ $\Rightarrow$ $\mathbf{w} = (\mathbf{w} - \text{proj}_{\mathbf{z}}\, \mathbf{w}) + \text{proj}_{\mathbf{z}}\, \mathbf{w} = (\mathbf{v} - \text{proj}_{\mathbf{z}}\, \mathbf{v}) + \text{proj}_{\mathbf{z}}\, \mathbf{w}$
$= \mathbf{v} - 2\,\text{proj}_{\mathbf{z}}\, \mathbf{v} = \mathbf{v} - 2\left(\frac{\mathbf{v}\cdot\mathbf{z}}{|\mathbf{z}|^2}\right)\mathbf{z}$

15. (a) $\mathbf{u} \times \mathbf{v} = 2\mathbf{i} \times 2\mathbf{j} = 4\mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{C} = \mathbf{0}$; $(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = 0\mathbf{v} - 0\mathbf{u} = \mathbf{0}$; $\mathbf{v} \times \mathbf{w} = 4\mathbf{i} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{0}$;
$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = 0\mathbf{v} - 0\mathbf{w} = \mathbf{0}$

(b) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 3 \\ -1 & 2 & -1 \end{vmatrix} = -10\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$;
$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = -4(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) - 2(\mathbf{i} - \mathbf{j} + \mathbf{k}) = -10\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$;
$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 3 & 4 & 5 \end{vmatrix} = -9\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$;
$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = -4(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) - (-1)(-\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = -9\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$

(c) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{vmatrix} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -4 \\ 1 & 0 & 2 \end{vmatrix} = -4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$;
$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = 2(2\mathbf{i} - \mathbf{j} + \mathbf{k}) - 4(2\mathbf{i} + \mathbf{j}) = -4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$;
$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = -2\mathbf{i} - 3\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ -2 & -3 & 1 \end{vmatrix} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$;
$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = 2(2\mathbf{i} - \mathbf{j} + \mathbf{k}) - 3(\mathbf{i} + 2\mathbf{k}) = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$

(d) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ -1 & 0 & -1 \end{vmatrix} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 1 \\ 2 & 4 & -2 \end{vmatrix} = -10\mathbf{i} - 10\mathbf{k}$;
$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = 10(-\mathbf{i} - \mathbf{k}) - 0(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = -10\mathbf{i} - 10\mathbf{k}$;
$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & -1 \\ 2 & 4 & -2 \end{vmatrix} = 4\mathbf{i} - 4\mathbf{j} - 4\mathbf{k} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 4 & -4 & -4 \end{vmatrix} = -12\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$;
$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = 10(-\mathbf{i} - \mathbf{k}) - 1(2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = -12\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$

16. (a) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} + (\mathbf{v} \cdot \mathbf{u})\mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} + (\mathbf{w} \cdot \mathbf{v})\mathbf{u} - (\mathbf{w} \cdot \mathbf{u})\mathbf{v} = \mathbf{0}$

(b) $[\mathbf{u} \cdot (\mathbf{v} \times \mathbf{i})]\mathbf{i} + [(\mathbf{u} \cdot (\mathbf{v} \times \mathbf{j})]\mathbf{j} + [(\mathbf{u} \cdot (\mathbf{v} \times \mathbf{k})]\mathbf{k} = [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{i}]\mathbf{i} + [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{j}]\mathbf{j} + [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{k}]\mathbf{k} = \mathbf{u} \times \mathbf{v}$

(c) $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{r}) = \mathbf{u} \cdot [\mathbf{v} \times (\mathbf{w} \times \mathbf{r})] = \mathbf{u} \cdot [(\mathbf{v} \cdot \mathbf{r})\mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\mathbf{r}] = (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{r}) - (\mathbf{u} \cdot \mathbf{r})(\mathbf{v} \cdot \mathbf{w})$
$= \begin{vmatrix} \mathbf{u} \cdot \mathbf{w} & \mathbf{v} \cdot \mathbf{w} \\ \mathbf{u} \cdot \mathbf{r} & \mathbf{v} \cdot \mathbf{r} \end{vmatrix}$

17. The formula is always true; $\mathbf{u} \times [\mathbf{u} \times (\mathbf{u} \times \mathbf{v})] \cdot \mathbf{w} = \mathbf{u} \times [(\mathbf{u} \cdot \mathbf{v})\mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{v}] \cdot \mathbf{w}$
$= [(\mathbf{u} \cdot \mathbf{v})\mathbf{u} \times \mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{u} \times \mathbf{v}] \cdot \mathbf{w} = -|\mathbf{u}|^2 \mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = -|\mathbf{u}|^2 \mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$

18. If $\mathbf{A} = (\cos\alpha)\mathbf{i} + (\sin\alpha)\mathbf{j}$ and $\mathbf{B} = (\cos\beta)\mathbf{i} + (\sin\beta)\mathbf{j}$, where $\beta > \alpha$, then $\mathbf{A}\times\mathbf{B} = [|\mathbf{A}|\,|\mathbf{B}|\sin(\beta-\alpha)]\,\mathbf{k}$ $= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \end{vmatrix} = (\cos\alpha\sin\beta - \sin\alpha\cos\beta)\mathbf{k} \Rightarrow \sin(\beta-\alpha) = \cos\alpha\sin\beta - \sin\alpha\cos\beta$, since $|\mathbf{A}| = 1$ and $|\mathbf{B}| = 1$.

19. If $\mathbf{A} = a\mathbf{i} + b\mathbf{j}$ and $\mathbf{B} = c\mathbf{i} + d\mathbf{j}$, then $\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\cos\theta \Rightarrow ac + bd = \sqrt{a^2+b^2}\,\sqrt{c^2+d^2}\cos\theta$ $\Rightarrow (ac+bd)^2 = (a^2+b^2)(c^2+d^2)\cos^2\theta \Rightarrow (ac+bd)^2 \le (a^2+b^2)(c^2+d^2)$, since $\cos^2\theta \le 1$.

20. $\mathbf{C} = \text{proj}_{\mathbf{B}}\,\mathbf{A} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{B}||\mathbf{B}|}\right)\mathbf{B}$ and $\mathbf{D} = \mathbf{A} - \mathbf{C} = \mathbf{A} - \left(\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{B}||\mathbf{B}|}\right)\mathbf{B}$

21. $|\mathbf{A}+\mathbf{B}|^2 = (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) = \mathbf{A}\cdot\mathbf{A} + 2\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{B} \le |\mathbf{A}|^2 + 2\,|\mathbf{A}|\,|\mathbf{B}| + |\mathbf{B}|^2 = (|\mathbf{A}| + |\mathbf{B}|)^2 \Rightarrow |\mathbf{A}+\mathbf{B}|$ $\le |\mathbf{A}| + |\mathbf{B}|$

22. Let α denote the angle between $\mathbf{C}$ and $\mathbf{A}$, and β the angle between $\mathbf{C}$ and $\mathbf{B}$. Let $a = |\mathbf{A}|$ and $b = |\mathbf{B}|$. Then $\cos\alpha = \frac{\mathbf{C}\cdot\mathbf{A}}{|\mathbf{C}|\,|\mathbf{A}|} = \frac{(a\mathbf{B}+b\mathbf{A})\cdot\mathbf{A}}{|\mathbf{C}|\,|\mathbf{A}|} = \frac{(a\mathbf{B}\cdot\mathbf{A}+b\mathbf{A}\cdot\mathbf{A})}{|\mathbf{C}|\,|\mathbf{A}|} = \frac{(a\mathbf{B}\cdot\mathbf{A}+b\mathbf{A}\cdot\mathbf{A})}{|\mathbf{C}|\,|\mathbf{A}|} = \frac{(a\mathbf{B}\cdot\mathbf{A}+ba^2)}{|\mathbf{C}|\,a} = \frac{\mathbf{B}\cdot\mathbf{A}+ba}{|\mathbf{C}|}$, and likewise, $\cos\beta = \frac{\mathbf{A}\cdot\mathbf{B}+ba}{|\mathbf{C}|}$. Since the angle between A and B is always $\le \frac{\pi}{2}$ and $\cos\alpha = \cos\beta$, we have that $\alpha = \beta \Rightarrow$ $\mathbf{C}$ bisects the angle between $\mathbf{A}$ and $\mathbf{B}$.

23. $(a\mathbf{B}+b\mathbf{A})\cdot(b\mathbf{A}-a\mathbf{B}) = a\mathbf{B}\cdot b\mathbf{A} + b\mathbf{A}\cdot b\mathbf{A} - a\mathbf{B}\cdot a\mathbf{B} - b\mathbf{A}\cdot a\mathbf{B} = b\mathbf{A}\cdot a\mathbf{B} + b^2\mathbf{A}\cdot\mathbf{A} - a^2\mathbf{B}\cdot\mathbf{B} - b\mathbf{A}\cdot a\mathbf{B}$ $= b^2a^2 - a^2b^2 = 0$, where $a = |\mathbf{A}|$ and $b = |\mathbf{B}|$

24. If $\mathbf{A} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then $\mathbf{A}\cdot\mathbf{A} = a^2 + b^2 + c^2 \ge 0$ and $\mathbf{A}\cdot\mathbf{A} = 0$ iff $a = b = c = 0$.

25. (a) The vector from $(0, d)$ to $(kd, 0)$ is $\mathbf{r}_k = kd\mathbf{i} - d\mathbf{j} \Rightarrow \frac{1}{|\mathbf{r}_k|^3} = \frac{1}{d^3\,(k^2+1)^{3/2}} \Rightarrow \frac{\mathbf{r}_k}{|\mathbf{r}_k|^3} = \frac{k\mathbf{i}-\mathbf{j}}{d^2\,(k^2+1)^{3/2}}$. The total force on the mass $(0, d)$ due to the masses Q_k for $k = -n, -n+1, \ldots, n-1, n$ is

$$\mathbf{F} = \frac{GMm}{d^2}(-\mathbf{j}) + \frac{GMm}{2d^2}\left(\frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}\right) + \frac{GMm}{5d^2}\left(\frac{2\mathbf{i}-\mathbf{j}}{\sqrt{5}}\right) + \ldots + \frac{GMm}{(n^2+1)\,d^2}\left(\frac{n\mathbf{i}-\mathbf{j}}{\sqrt{n^2+1}}\right) + \frac{GMm}{2d^2}\left(\frac{-\mathbf{i}-\mathbf{j}}{\sqrt{2}}\right)$$
$$+ \frac{GMm}{5d^2}\left(\frac{-2\mathbf{i}-\mathbf{j}}{\sqrt{5}}\right) + \ldots + \frac{GMm}{(n^2+1)\,d^2}\left(\frac{-n\mathbf{i}-\mathbf{j}}{\sqrt{n^2+1}}\right)$$

The $\mathbf{i}$ components cancel, giving

$\mathbf{F} = \frac{GMm}{d^2}\left(-1 - \frac{2}{2\sqrt{2}} - \frac{2}{5\sqrt{5}} - \ldots - \frac{2}{(n^2+1)\,(n^2+1)^{1/2}}\right)\mathbf{j} \Rightarrow$ the magnitude of the force is

$$|\mathbf{F}| = \frac{GMm}{d^2}\left(1 + \sum_{i=1}^{n} \frac{2}{(i^2+1)^{3/2}}\right).$$

(b) Yes, it is finite: $\lim_{n\to\infty} |\mathbf{F}| = \frac{GMm}{d^2}\left(1 + \sum_{i=1}^{\infty} \frac{2}{(i^2+1)^{3/2}}\right)$ is finite since $\sum_{i=1}^{\infty} \frac{2}{(i^2+1)^{3/2}}$ converges.

26. (a) If $\vec{x}\cdot\vec{y} = 0$, then $\vec{x}\times(\vec{x}\times\vec{y}) = (\vec{x}\cdot\vec{y})\vec{x} - (\vec{x}\cdot\vec{x})\vec{y} = -(\vec{x}\cdot\vec{x})\vec{y}$. This means that $\vec{x}\oplus\vec{y} = \vec{x} + \vec{y} + \frac{1}{c^2}\cdot\frac{1}{1+\sqrt{1-\frac{\vec{x}\cdot\vec{x}}{c^2}}}\left(-(\vec{x}\cdot\vec{x})\right)\vec{y} = \vec{x} + \left(1 - \frac{|\vec{x}|^2}{c^2+\sqrt{c^4 - c^2\,|\vec{x}|^2}}\right)\vec{y}$. Since $\vec{x}$ and $\vec{y}$ are orthogonal, then $|\vec{x}\oplus\vec{y}|^2 = |\vec{x}|^2 + \left(1 - \frac{|\vec{x}|^2}{c^2+\sqrt{c^4-c^2\,|\vec{x}|^2}}\right)^2 |\vec{y}|^2$. A calculation will show that $|\vec{x}|^2 + \left(1 - \frac{|\vec{x}|^2}{c^2+\sqrt{c^4-c^2\,|\vec{x}|^2}}\right)^2 c^2 = c^2$. Since $|\vec{y}| < c$, then $|\vec{y}|^2 < c^2$ so $\left(1 - \frac{|\vec{x}|^2}{c^2+\sqrt{c^4-c^2\,|\vec{x}|^2}}\right)^2 |\vec{y}|^2 < \left(1 - \frac{|\vec{x}|^2}{c^2+\sqrt{c^4-c^2\,|\vec{x}|^2}}\right) c^2$. This means that

$$|\vec{x} \oplus \vec{y}|^2 = |\vec{x}|^2 + \left(1 - \frac{|\vec{x}|^2}{c^2 + \sqrt{c^4 - c^2|\vec{x}|^2}}\right)^2 |\vec{y}|^2 < |\vec{x}|^2 + \left(1 - \frac{|\vec{x}|^2}{c^2 + \sqrt{c^4 - c^2|\vec{x}|^2}}\right)^2 c^2 = c^2.$$

We now have $|\vec{x} \oplus \vec{y}|^2 < c^2$, so $|\vec{x} \oplus \vec{y}| < c$.

(b) If $\vec{x}$ and $\vec{y}$ are parallel, then $\vec{x} \times (\vec{x} \times \vec{y}) = \vec{0}$. This gives $\vec{x} \oplus \vec{y} = \frac{\vec{x} + \vec{y}}{1 + \frac{\vec{x} \cdot \vec{y}}{c^2}}$.

(i) If $\vec{x}$ and $\vec{y}$ have the same direction, then $\vec{x} \oplus \vec{y} = \frac{\vec{x} + \vec{y}}{1 + \frac{|\vec{x}|}{c} \cdot \frac{|\vec{y}|}{c}}$ and $|\vec{x} \oplus \vec{y}| = \frac{|\vec{x}| + |\vec{y}|}{1 + \frac{|\vec{x}|}{c} \cdot \frac{|\vec{y}|}{c}}$.

Since $|\vec{y}| < c$, $|\vec{x}| < c$, we have $|\vec{y}|\left(1 - \frac{|\vec{x}|}{c}\right) < c\left(1 - \frac{|\vec{x}|}{c}\right) \Rightarrow |\vec{y}| - \frac{|\vec{y}|\,|\vec{x}|}{c} < c - |\vec{x}|$

$\Rightarrow |\vec{x}| + |\vec{y}| < c + \frac{|\vec{x}|\,|\vec{y}|}{c} = c\left(1 + \frac{|\vec{x}|}{c} \cdot \frac{|\vec{y}|}{c}\right) \Rightarrow \frac{|\vec{x}| + |\vec{y}|}{1 + \frac{|\vec{x}|}{c} \cdot \frac{|\vec{y}|}{c}} < c$. This means that $|\vec{x} \oplus \vec{y}| < c$.

(ii) If $\vec{x}$ and $\vec{y}$ have opposite directions, then $\vec{x} \cdot \vec{y} = -|\vec{x}|\,|\vec{y}|$ and $\vec{x} \oplus \vec{y} = \frac{\vec{x} + \vec{y}}{1 - \frac{|\vec{x}|\,|\vec{y}|}{c^2}}$.

Assume $|\vec{x}| \geq |\vec{y}|$, then $|\vec{x} \oplus \vec{y}| = \frac{|\vec{x}| - |\vec{y}|}{1 - \frac{|\vec{x}|\,|\vec{y}|}{c^2}}$. Since $|\vec{x}| < c$, we have $|\vec{x}|\left(1 + \frac{|\vec{y}|}{c}\right) < c\left(1 + \frac{|\vec{y}|}{c}\right)$

$\Rightarrow |\vec{x}| + \frac{|\vec{x}|\,|\vec{y}|}{c} < c + |\vec{y}| \Rightarrow |\vec{x}| - |\vec{y}| < c - \frac{|\vec{x}|\,|\vec{y}|}{c} = c\left(1 - \frac{|\vec{x}|\,|\vec{y}|}{c^2}\right) \Rightarrow \frac{|\vec{x}| - |\vec{y}|}{1 - \frac{|\vec{x}|\,|\vec{y}|}{c^2}} < c.$

This means that $|\vec{x} \oplus \vec{y}| < c$. A similar argument holds if $|\vec{x}| > |\vec{y}|$.

(c) $\lim_{c \to \infty} \vec{x} \oplus \vec{y} = \vec{x} + \vec{y}$.

NOTES: