CHAPTER 12 VECTORS AND THE GEOMETRY OF SPACE

12.1 THREE-DIMENSIONAL COORDINATE SYSTEMS

1. The line through the point $(2, 3, 0)$ parallel to the z-axis

2. The line through the point $(-1, 0, 0)$ parallel to the y-axis

3. The x-axis

4. The line through the point $(1, 0, 0)$ parallel to the z-axis

5. The circle $x^2 + y^2 = 4$ in the xy-plane

6. The circle $x^2 + y^2 = 4$ in the plane $z = -2$

7. The circle $x^2 + z^2 = 4$ in the xz-plane

8. The circle $y^2 + z^2 = 1$ in the yz-plane

9. The circle $y^2 + z^2 = 1$ in the yz-plane

10. The circle $x^2 + z^2 = 9$ in the plane $y = -4$

11. The circle $x^2 + y^2 = 16$ in the xy-plane

12. The circle $x^2 + z^2 = 3$ in the xz-plane

13. (a) The first quadrant of the xy-plane (b) The fourth quadrant of the xy-plane

14. (a) The slab bounded by the planes $x = 0$ and $x = 1$
    (b) The square column bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$
    (c) The unit cube in the first octant having one vertex at the origin

15. (a) The solid ball of radius 1 centered at the origin
    (b) The exterior of the sphere of radius 1 centered at the origin

16. (a) The circumference and interior of the circle $x^2 + y^2 = 1$ in the xy-plane
    (b) The circumference and interior of the circle $x^2 + y^2 = 1$ in the plane $z = 3$
    (c) A solid cylindrical column of radius 1 whose axis is the z-axis

17. (a) The closed upper hemisphere of radius 1 centered at the origin
    (b) The solid upper hemisphere of radius 1 centered at the origin

18. (a) The line $y = x$ in the xy-plane
    (b) The plane $y = x$ consisting of all points of the form (x, x, z)

19. (a) $x = 3$ (b) $y = -1$ (c) $z = -2$

20. (a) $x = 3$ (b) $y = -1$ (c) $z = 2$

21. (a) $z = 1$ (b) $x = 3$ (c) $y = -1$

22. (a) $x^2 + y^2 = 4, z = 0$ (b) $y^2 + z^2 = 4, x = 0$ (c) $x^2 + z^2 = 4, y = 0$

23. (a) $x^2 + (y-2)^2 = 4, z = 0$ (b) $(y-2)^2 + z^2 = 4, x = 0$ (c) $x^2 + z^2 = 4, y = 2$

24. (a) $(x+3)^2 + (y-4)^2 = 1, z = 1$ (b) $(y-4)^2 + (z-1)^2 = 1, x = -3$
(c) $(x+3)^2 + (z-1)^2 = 1, y = 4$

25. (a) $y = 3, z = -1$ (b) $x = 1, z = -1$ (c) $x = 1, y = 3$

26. $\sqrt{x^2 + y^2 + z^2} = \sqrt{x^2 + (y-2)^2 + z^2} \Rightarrow x^2 + y^2 + z^2 = x^2 + (y-2)^2 + z^2 \Rightarrow y^2 = y^2 - 4y + 4 \Rightarrow y = 1$

27. $x^2 + y^2 + z^2 = 25, z = 3 \Rightarrow x^2 + y^2 = 16$ in the plane $z = 3$

28. $x^2 + y^2 + (z-1)^2 = 4$ and $x^2 + y^2 + (z+1)^2 = 4 \Rightarrow x^2 + y^2 + (z-1)^2 = x^2 + y^2 + (z+1)^2 \Rightarrow z = 0, x^2 + y^2 = 3$

29. $0 \le z \le 1$

30. $0 \le x \le 2, 0 \le y \le 2, 0 \le z \le 2$

31. $z \le 0$

32. $z = \sqrt{1 - x^2 - y^2}$

33. (a) $(x-1)^2 + (y-1)^2 + (z-1)^2 < 1$ (b) $(x-1)^2 + (y-1)^2 + (z-1)^2 > 1$

34. $1 \le x^2 + y^2 + z^2 \le 4$

35. $|P_1P_2| = \sqrt{(3-1)^2 + (3-1)^2 + (0-1)^2} = \sqrt{9} = 3$

36. $|P_1P_2| = \sqrt{(2+1)^2 + (5-1)^2 + (0-5)^2} = \sqrt{50} = 5\sqrt{2}$

37. $|P_1P_2| = \sqrt{(4-1)^2 + (-2-4)^2 + (7-5)^2} = \sqrt{49} = 7$

38. $|P_1P_2| = \sqrt{(2-3)^2 + (3-4)^2 + (4-5)^2} = \sqrt{3}$

39. $|P_1P_2| = \sqrt{(2-0)^2 + (-2-0)^2 + (-2-0)^2} = \sqrt{3 \cdot 4} = 2\sqrt{3}$

40. $|P_1P_2| = \sqrt{(0-5)^2 + (0-3)^2 + (0+2)^2} = \sqrt{38}$

41. center $(-2, 0, 2)$, radius $2\sqrt{2}$

42. center $\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$, radius $\frac{\sqrt{21}}{2}$

43. center $\left(\sqrt{2}, \sqrt{2}, -\sqrt{2}\right)$, radius $\sqrt{2}$

44. center $\left(0, -\frac{1}{3}, \frac{1}{3}\right)$, radius $\frac{\sqrt{29}}{3}$

45. $(x-1)^2 + (y-2)^2 + (z-3)^2 = 14$

46. $x^2 + (y+1)^2 + (z-5)^2 = 4$

47. $(x+2)^2 + y^2 + z^2 = 3$

48. $x^2 + (y+7)^2 + z^2 = 49$

49. $x^2 + y^2 + z^2 + 4x - 4z = 0 \Rightarrow (x^2 + 4x + 4) + y^2 + (z^2 - 4z + 4) = 4 + 4$
$\Rightarrow (x+2)^2 + (y-0)^2 + (z-2)^2 = \left(\sqrt{8}\right)^2 \Rightarrow$ the center is at $(-2, 0, 2)$ and the radius is $\sqrt{8}$

50. $x^2 + y^2 + z^2 - 6y + 8z = 0 \Rightarrow x^2 + (y^2 - 6y + 9) + (z^2 + 8z + 16) = 9 + 16 \Rightarrow (x-0)^2 + (y-3)^2 + (z+4)^2 = 5^2$
$\Rightarrow$ the center is at $(0, 3, -4)$ and the radius is 5

51. $2x^2 + 2y^2 + 2z^2 + x + y + z = 9 \Rightarrow x^2 + \frac{1}{2}x + y^2 + \frac{1}{2}y + z^2 + \frac{1}{2}z = \frac{9}{2}$
$\Rightarrow \left(x^2 + \frac{1}{2}x + \frac{1}{16}\right) + \left(y^2 + \frac{1}{2}y + \frac{1}{16}\right) + \left(z^2 + \frac{1}{2}z + \frac{1}{16}\right) = \frac{9}{2} + \frac{3}{16} \Rightarrow \left(x + \frac{1}{4}\right)^2 + \left(y + \frac{1}{4}\right)^2 + \left(z + \frac{1}{4}\right)^2 = \left(\frac{5\sqrt{3}}{4}\right)^2$
$\Rightarrow$ the center is at $\left(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$ and the radius is $\frac{5\sqrt{3}}{4}$

52. $3x^2 + 3y^2 + 3z^2 + 2y - 2z = 9 \Rightarrow x^2 + y^2 + \frac{2}{3}y + z^2 - \frac{2}{3}z = 3 \Rightarrow x^2 + \left(y^2 + \frac{2}{3}y + \frac{1}{9}\right) + \left(z^2 - \frac{2}{3}z + \frac{1}{9}\right) = 3 + \frac{2}{9}$
$\Rightarrow (x-0)^2 + \left(y + \frac{1}{3}\right)^2 + \left(z - \frac{1}{3}\right)^2 = \left(\frac{\sqrt{29}}{3}\right)^2 \Rightarrow$ the center is at $\left(0, -\frac{1}{3}, \frac{1}{3}\right)$ and the radius is $\frac{\sqrt{29}}{3}$

53. (a) the distance between (x, y, z) and $(x, 0, 0)$ is $\sqrt{y^2 + z^2}$
(b) the distance between (x, y, z) and $(0, y, 0)$ is $\sqrt{x^2 + z^2}$
(c) the distance between (x, y, z) and $(0, 0, z)$ is $\sqrt{x^2 + y^2}$

54. (a) the distance between (x, y, z) and $(x, y, 0)$ is z
(b) the distance between (x, y, z) and $(0, y, z)$ is x
(c) the distance between (x, y, z) and $(x, 0, z)$ is y

55. $|AB| = \sqrt{(1-(-1))^2 + (-1-2)^2 + (3-1)^2} = \sqrt{4+9+4} = \sqrt{17}$
$|BC| = \sqrt{(3-1)^2 + (4-(-1))^2 + (5-3)^2} = \sqrt{4+25+4} = \sqrt{33}$
$|CA| = \sqrt{(-1-3)^2 + (2-4)^2 + (1-5)^2} = \sqrt{16+4+16} = \sqrt{36} = 6$
Thus the perimeter of triangle ABC is $\sqrt{17} + \sqrt{33} + 6$.

56. $|PA| = \sqrt{(2-3)^2 + (-1-1)^2 + (3-2)^2} = \sqrt{1+4+1} = \sqrt{6}$
$|PB| = \sqrt{(4-3)^2 + (3-1)^2 + (1-2)^2} = \sqrt{1+4+1} = \sqrt{6}$
Thus P is equidistant from A and B.

12.2 VECTORS

1. (a) $\langle 3(3), 3(-2) \rangle = \langle 9, -6 \rangle$
(b) $\sqrt{9^2 + (-6)^2} = \sqrt{117} = 3\sqrt{13}$

2. (a) $\langle -2(-2), -2(5) \rangle = \langle 4, -10 \rangle$
(b) $\sqrt{4^2 + (-10)^2} = \sqrt{116} = 2\sqrt{29}$

3. (a) $\langle 3 + (-2), -2 + 5 \rangle = \langle 1, 3 \rangle$
(b) $\sqrt{1^2 + 3^2} = \sqrt{10}$

4. (a) $\langle 3 - (-2), -2 - 5 \rangle = \langle 5, -7 \rangle$
(b) $\sqrt{5^2 + (-7)^2} = \sqrt{74}$

5. (a) $2\mathbf{u} = \langle 2(3), 2(-2) \rangle = \langle 6, -4 \rangle$
$3\mathbf{v} = \langle 3(-2), 3(5) \rangle = \langle -6, 15 \rangle$
$2\mathbf{u} - 3\mathbf{v} = \langle 6 - (-4), -4 - 15 \rangle = \langle 12, -19 \rangle$
(b) $\sqrt{12^2 + (-19)^2} = \sqrt{505}$

6. (a) $-2\mathbf{u} = \langle -2(3), -2(-2) \rangle = \langle -6, 4 \rangle$
$5\mathbf{v} = \langle 5(-2), 5(5) \rangle = \langle -10, 25 \rangle$
$-2\mathbf{u} + 5\mathbf{v} = \langle -6 + (-10), 4 + 25 \rangle = \langle -16, 29 \rangle$
(b) $\sqrt{(-16)^2 + 29^2} = \sqrt{1097}$

7. (a) $\frac{3}{5}\mathbf{u} = \left\langle \frac{3}{5}(3), \frac{3}{5}(-2) \right\rangle = \left\langle \frac{9}{5}, -\frac{6}{5} \right\rangle$
$\frac{4}{5}\mathbf{v} = \left\langle \frac{4}{5}(-2), \frac{4}{5}(5) \right\rangle = \left\langle -\frac{8}{5}, 4 \right\rangle$
$\frac{3}{5}\mathbf{u} + \frac{4}{5}\mathbf{v} = \left\langle \frac{9}{5} + \left(-\frac{8}{5}\right), -\frac{6}{5} + 4 \right\rangle = \left\langle \frac{1}{5}, \frac{14}{5} \right\rangle$
(b) $\sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{14}{5}\right)^2} = \frac{\sqrt{197}}{5}$

8. (a) $-\frac{5}{13}\mathbf{u} = \left\langle -\frac{5}{13}(3), -\frac{5}{13}(-2) \right\rangle = \left\langle -\frac{15}{13}, \frac{10}{13} \right\rangle$
$\frac{12}{13}\mathbf{v} = \left\langle \frac{12}{13}(-2), \frac{12}{13}(5) \right\rangle = \left\langle -\frac{24}{13}, \frac{60}{13} \right\rangle$
$-\frac{5}{13}\mathbf{u} + \frac{12}{13}\mathbf{v} = \left\langle -\frac{15}{13} + \left(-\frac{24}{13}\right), \frac{10}{13} + \frac{60}{13} \right\rangle = \left\langle -3, \frac{70}{13} \right\rangle$
(b) $\sqrt{(-3)^2 + \left(\frac{70}{13}\right)^2} = \frac{\sqrt{6421}}{13}$

9. $\langle 2 - 1, -1 - 3 \rangle = \langle 1, -4 \rangle$

10. $\left\langle \frac{2+(-4)}{2} - 0, \frac{-1+3}{2} - 0 \right\rangle = \langle -1, 1 \rangle$

11. $\langle 0 - 2, 0 - 3 \rangle = \langle -2, -3 \rangle$

12. $\overrightarrow{AB} = \langle 2 - 1, 0 - (-1) \rangle = \langle 1, 1 \rangle$, $\overrightarrow{CD} = \langle -2 - (-1), 2 - 3 \rangle = \langle -1, -1 \rangle$, $\overrightarrow{AB} + \overrightarrow{CD} = \langle 0, 0 \rangle$

13. $\left\langle \cos \frac{2\pi}{3}, \sin \frac{2\pi}{3} \right\rangle = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$

14. $\left\langle \cos\left(-\frac{3\pi}{4}\right), \sin\left(-\frac{3\pi}{4}\right) \right\rangle = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$

15. This is the unit vector which makes an angle of $120^\circ + 90^\circ = 210^\circ$ with the positive x-axis;
$\langle \cos 210^\circ, \sin 210^\circ \rangle = \left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$

16. $\langle \cos 135^\circ, \sin 135^\circ \rangle = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

17. $\overrightarrow{P_1P_2} = (2 - 5)\mathbf{i} + (9 - 7)\mathbf{j} + (-2 - (-1))\mathbf{k} = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

18. $\overrightarrow{P_1P_2} = (-3 - 1)\mathbf{i} + (0 - 2)\mathbf{j} + (5 - 0)\mathbf{k} = -4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$

19. $\overrightarrow{AB} = (-10 - (-7))\mathbf{i} + (8 - (-8))\mathbf{j} + (1 - 1)\mathbf{k} = -3\mathbf{i} + 16\mathbf{j}$

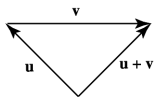
20. $\overrightarrow{AB} = (-1 - 1)\mathbf{i} + (4 - 0)\mathbf{j} + (5 - 3)\mathbf{k} = -2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

21. $5\mathbf{u} - \mathbf{v} = 5\langle 1, 1, -1 \rangle - \langle 2, 0, 3 \rangle = \langle 5, 5, -5 \rangle - \langle 2, 0, 3 \rangle = \langle 5 - 2, 5 - 0, -5 - 3 \rangle = \langle 3, 5, -8 \rangle = 3\mathbf{i} + 5\mathbf{j} - 8\mathbf{k}$

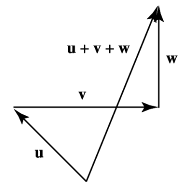
22. $-2\mathbf{u} + 3\mathbf{v} = -2\langle -1, 0, 2 \rangle + 3\langle 1, 1, 1 \rangle = \langle 2, 0, -4 \rangle + \langle 3, 3, 3 \rangle = \langle 5, 3, -1 \rangle = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

23. The vector **v** is horizontal and 1 in. long. The vectors **u** and **w** are $\frac{11}{16}$ in. long. **w** is vertical and **u** makes a 45° angle with the horizontal. All vectors must be drawn to scale.

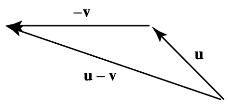
(a)



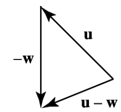
(b)



(c)

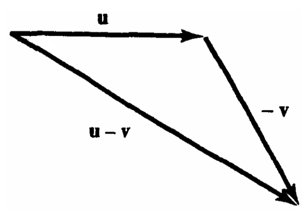


(d)

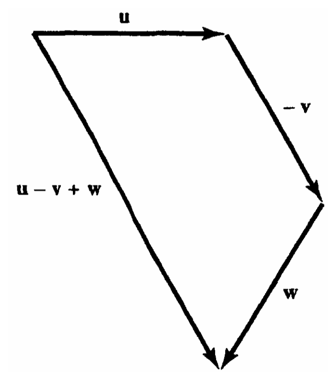


24. The angle between the vectors is 120° and vector **u** is horizontal. They are all 1 in. long. Draw to scale.

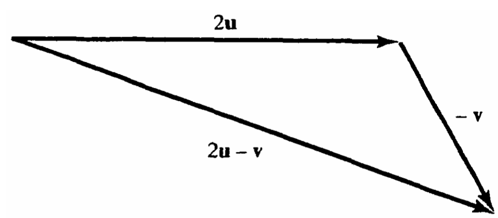
(a)



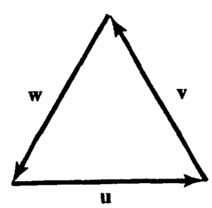
(b)



(c)



(d)



$\mathbf{u}+\mathbf{v}+\mathbf{w}=\mathbf{0}$

25. length $= |2\mathbf{i}+\mathbf{j}-2\mathbf{k}| = \sqrt{2^2+1^2+(-2)^2} = 3$, the direction is $\frac{2}{3}\,\mathbf{i}+\frac{1}{3}\,\mathbf{j}-\frac{2}{3}\,\mathbf{k} \;\Rightarrow\; 2\mathbf{i}+\mathbf{j}-2\mathbf{k} = 3\left(\frac{2}{3}\,\mathbf{i}+\frac{1}{3}\,\mathbf{j}-\frac{2}{3}\,\mathbf{k}\right)$

26. length $= |9\mathbf{i}-2\mathbf{j}+6\mathbf{k}| = \sqrt{81+4+36} = 11$, the direction is $\frac{9}{11}\,\mathbf{i}-\frac{2}{11}\,\mathbf{j}+\frac{6}{11}\,\mathbf{k} \;\Rightarrow\; 9\mathbf{i}-2\mathbf{j}+6\mathbf{k}$
$= 11\left(\frac{9}{11}\,\mathbf{i}-\frac{2}{11}\,\mathbf{j}+\frac{6}{11}\,\mathbf{k}\right)$

27. length $= |5\mathbf{k}| = \sqrt{25} = 5$, the direction is $\mathbf{k} \;\Rightarrow\; 5\mathbf{k} = 5(\mathbf{k})$

28. length $= \left|\frac{3}{5}\,\mathbf{i}+\frac{4}{5}\,\mathbf{k}\right| = \sqrt{\frac{9}{25}+\frac{16}{25}} = 1$, the direction is $\frac{3}{5}\,\mathbf{i}+\frac{4}{5}\,\mathbf{k} \;\Rightarrow\; \frac{3}{5}\,\mathbf{i}+\frac{4}{5}\,\mathbf{k} = 1\left(\frac{3}{5}\,\mathbf{i}+\frac{4}{5}\,\mathbf{k}\right)$

29. length $= \left|\frac{1}{\sqrt{6}}\,\mathbf{i}-\frac{1}{\sqrt{6}}\,\mathbf{j}-\frac{1}{\sqrt{6}}\,\mathbf{k}\right| = \sqrt{3\left(\frac{1}{\sqrt{6}}\right)^2} = \sqrt{\frac{1}{2}}$, the direction is $\frac{1}{\sqrt{3}}\,\mathbf{i}-\frac{1}{\sqrt{3}}\,\mathbf{j}-\frac{1}{\sqrt{3}}\,\mathbf{k}$
$\Rightarrow\; \frac{1}{\sqrt{6}}\,\mathbf{i}-\frac{1}{\sqrt{6}}\,\mathbf{j}-\frac{1}{\sqrt{6}}\,\mathbf{k} = \sqrt{\frac{1}{2}}\left(\frac{1}{\sqrt{3}}\,\mathbf{i}-\frac{1}{\sqrt{3}}\,\mathbf{j}-\frac{1}{\sqrt{3}}\,\mathbf{k}\right)$

30. length $= \left|\frac{1}{\sqrt{3}}\,\mathbf{i} + \frac{1}{\sqrt{3}}\,\mathbf{j} + \frac{1}{\sqrt{3}}\,\mathbf{k}\right| = \sqrt{3\left(\frac{1}{\sqrt{3}}\right)^2} = 1$, the direction is $\frac{1}{\sqrt{3}}\,\mathbf{i} + \frac{1}{\sqrt{3}}\,\mathbf{j} + \frac{1}{\sqrt{3}}\,\mathbf{k}$
$\Rightarrow\ \frac{1}{\sqrt{3}}\,\mathbf{i} + \frac{1}{\sqrt{3}}\,\mathbf{j} + \frac{1}{\sqrt{3}}\,\mathbf{k} = 1\left(\frac{1}{\sqrt{3}}\,\mathbf{i} + \frac{1}{\sqrt{3}}\,\mathbf{j} + \frac{1}{\sqrt{3}}\,\mathbf{k}\right)$

31. (a) $2\mathbf{i}$ (b) $-\sqrt{3}\mathbf{k}$ (c) $\frac{3}{10}\,\mathbf{j} + \frac{2}{5}\,\mathbf{k}$ (d) $6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

32. (a) $-7\mathbf{j}$ (b) $-\frac{3\sqrt{2}}{5}\,\mathbf{i} - \frac{4\sqrt{2}}{5}\,\mathbf{k}$ (c) $\frac{1}{4}\,\mathbf{i} - \frac{1}{3}\,\mathbf{j} - \mathbf{k}$ (d) $\frac{a}{\sqrt{2}}\,\mathbf{i} + \frac{a}{\sqrt{3}}\,\mathbf{j} - \frac{a}{\sqrt{6}}\,\mathbf{k}$

33. $|\mathbf{v}| = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$; $\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{13}\,\mathbf{v} = \frac{1}{13}\,(12\mathbf{i} - 5\mathbf{k})\ \Rightarrow$ the desired vector is $\frac{7}{13}\,(12\mathbf{i} - 5\mathbf{k})$

34. $|\mathbf{v}| = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}} = \frac{\sqrt{3}}{2}$; $\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{3}}\,\mathbf{i} - \frac{1}{\sqrt{3}}\,\mathbf{j} - \frac{1}{\sqrt{3}}\,\mathbf{k}\ \Rightarrow$ the desired vector is $-3\left(\frac{1}{\sqrt{3}}\,\mathbf{i} - \frac{1}{\sqrt{3}}\,\mathbf{j} - \frac{1}{\sqrt{3}}\,\mathbf{k}\right)$
$= -\sqrt{3}\mathbf{i} + \sqrt{3}\mathbf{j} + \sqrt{3}\mathbf{k}$

35. (a) $3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} = 5\sqrt{2}\left(\frac{3}{5\sqrt{2}}\,\mathbf{i} + \frac{4}{5\sqrt{2}}\,\mathbf{j} - \frac{1}{\sqrt{2}}\,\mathbf{k}\right)\ \Rightarrow$ the direction is $\frac{3}{5\sqrt{2}}\,\mathbf{i} + \frac{4}{5\sqrt{2}}\,\mathbf{j} - \frac{1}{\sqrt{2}}\,\mathbf{k}$
(b) the midpoint is $\left(\frac{1}{2}, 3, \frac{5}{2}\right)$

36. (a) $3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k} = 7\left(\frac{3}{7}\,\mathbf{i} - \frac{6}{7}\,\mathbf{j} + \frac{2}{7}\,\mathbf{k}\right)\ \Rightarrow$ the direction is $\frac{3}{7}\,\mathbf{i} - \frac{6}{7}\,\mathbf{j} + \frac{2}{7}\,\mathbf{k}$
(b) the midpoint is $\left(\frac{5}{2}, 1, 6\right)$

37. (a) $-\mathbf{i} - \mathbf{j} - \mathbf{k} = \sqrt{3}\left(-\frac{1}{\sqrt{3}}\,\mathbf{i} - \frac{1}{\sqrt{3}}\,\mathbf{j} - \frac{1}{\sqrt{3}}\,\mathbf{k}\right)\ \Rightarrow$ the direction is $-\frac{1}{\sqrt{3}}\,\mathbf{i} - \frac{1}{\sqrt{3}}\,\mathbf{j} - \frac{1}{\sqrt{3}}\,\mathbf{k}$
(b) the midpoint is $\left(\frac{5}{2}, \frac{7}{2}, \frac{9}{2}\right)$

38. (a) $2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} = 2\sqrt{3}\left(\frac{1}{\sqrt{3}}\,\mathbf{i} - \frac{1}{\sqrt{3}}\,\mathbf{j} - \frac{1}{\sqrt{3}}\,\mathbf{k}\right)\ \Rightarrow$ the direction is $\frac{1}{\sqrt{3}}\,\mathbf{i} - \frac{1}{\sqrt{3}}\,\mathbf{j} - \frac{1}{\sqrt{3}}\,\mathbf{k}$
(b) the midpoint is $(1, -1, -1)$

39. $\overrightarrow{AB} = (5 - a)\mathbf{i} + (1 - b)\mathbf{j} + (3 - c)\mathbf{k} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}\ \Rightarrow\ 5 - a = 1,\ 1 - b = 4$, and $3 - c = -2\ \Rightarrow\ a = 4,\ b = -3$, and $c = 5\ \Rightarrow$ A is the point $(4, -3, 5)$

40. $\overrightarrow{AB} = (a + 2)\mathbf{i} + (b + 3)\mathbf{j} + (c - 6)\mathbf{k} = -7\mathbf{i} + 3\mathbf{j} + 8\mathbf{k}\ \Rightarrow\ a + 2 = -7,\ b + 3 = 3$, and $c - 6 = 8\ \Rightarrow\ a = -9,\ b = 0$, and $c = 14\ \Rightarrow$ B is the point $(-9, 0, 14)$

41. $2\mathbf{i} + \mathbf{j} = a(\mathbf{i} + \mathbf{j}) + b(\mathbf{i} - \mathbf{j}) = (a + b)\mathbf{i} + (a - b)\mathbf{j}\ \Rightarrow\ a + b = 2$ and $a - b = 1\ \Rightarrow\ 2a = 3\ \Rightarrow\ a = \frac{3}{2}$ and $b = a - 1 = \frac{1}{2}$

42. $\mathbf{i} - 2\mathbf{j} = a(2\mathbf{i} + 3\mathbf{j}) + b(\mathbf{i} + \mathbf{j}) = (2a + b)\mathbf{i} + (3a + b)\mathbf{j}\ \Rightarrow\ 2a + b = 1$ and $3a + b = -2\ \Rightarrow\ a = -3$ and $b = 1 - 2a = 7\ \Rightarrow\ \mathbf{u}_1 = a(2\mathbf{i} + 3\mathbf{j}) = -6\mathbf{i} - 9\mathbf{j}$ and $\mathbf{u}_2 = b(\mathbf{i} + \mathbf{j}) = 7\mathbf{i} + 7\mathbf{j}$

43. If $|x|$ is the magnitude of the x-component, then $\cos 30° = \frac{|x|}{|F|}\ \Rightarrow\ |x| = |F| \cos 30° = (10)\left(\frac{\sqrt{3}}{2}\right) = 5\sqrt{3}$ lb
$\Rightarrow\ \mathbf{F}_x = 5\sqrt{3}\,\mathbf{i}$;
if $|y|$ is the magnitude of the y-component, then $\sin 30° = \frac{|y|}{|F|}\ \Rightarrow\ |y| = |F| \sin 30° = (10)\left(\frac{1}{2}\right) = 5$ lb $\Rightarrow\ \mathbf{F}_y = 5\,\mathbf{j}$.

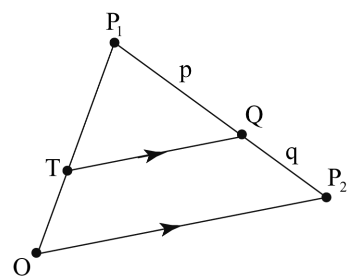
44. If $|x|$ is the magnitude of the x-component, then $\cos 45° = \frac{|x|}{|F|} \Rightarrow |x| = |F| \cos 45° = (12)\left(\frac{\sqrt{2}}{2}\right) = 6\sqrt{2}$ lb
$\Rightarrow \mathbf{F}_x = -6\sqrt{2}\,\mathbf{i}$ (he negative sign is indicated by the diagram)
if $|y|$ is the magnitude of the y-component, then $\sin 45° = \frac{|y|}{|F|} \Rightarrow |y| = |F| \sin 45° = (12)\left(\frac{\sqrt{2}}{2}\right) = 6\sqrt{2}$ lb
$\Rightarrow \mathbf{F}_y = -6\sqrt{2}\,\mathbf{j}$ (the negative sign is indicated by the diagram)

45. $25°$ west of north is $90° + 25° = 115°$ north of east. $800\langle \cos 155°, \sin 115° \rangle \approx \langle -338.095, 725.046 \rangle$

46. $10°$ east of south is $270° + 10° = 280°$ "north" of east. $600\langle \cos 280°, \sin 280° \rangle \approx \langle 104.189, -590.885 \rangle$

47. (a) The tree is located at the tip of the vector $\overrightarrow{OP} = (5\cos 60°)\mathbf{i} + (5\sin 60°)\mathbf{j} = \frac{5}{2}\,\mathbf{i} + \frac{5\sqrt{3}}{2}\,\mathbf{j} \Rightarrow P = \left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$
(b) The telephone pole is located at the point Q, which is the tip of the vector $\overrightarrow{OP} + \overrightarrow{PQ}$
$= \left(\frac{5}{2}\,\mathbf{i} + \frac{5\sqrt{3}}{2}\,\mathbf{j}\right) + (10\cos 315°)\mathbf{i} + (10\sin 315°)\mathbf{j} = \left(\frac{5}{2} + \frac{10\sqrt{2}}{2}\right)\mathbf{i} + \left(\frac{5\sqrt{3}}{2} - \frac{10\sqrt{2}}{2}\right)\mathbf{j}$
$\Rightarrow Q = \left(\frac{5+10\sqrt{2}}{2}, \frac{5\sqrt{3}-10\sqrt{2}}{2}\right)$

48. Let $t = \frac{q}{p+q}$ and $s = \frac{p}{p+q}$. Choose T on $\overline{OP_1}$ so that $\overline{TQ}$ is parallel to $\overline{OP_2}$, so that $\triangle TP_1Q$ is similar to $\triangle OP_1P_2$. Then $\frac{|OT|}{|OP_1|} = t \Rightarrow \overrightarrow{OT} = t\,\overrightarrow{OP_1}$ so that $T = (t\,x_1, t\,y_1, t\,z_1)$.
Also, $\frac{|TQ|}{|OP_2|} = s \Rightarrow \overrightarrow{TQ} = s\,\overrightarrow{OP_2} = s\langle x_2, y_2, z_2 \rangle$.
Letting $Q = (x, y, z)$, we have that
$\overrightarrow{TQ} = \langle x - t\,x_1, y - t\,y_1, z - t\,z_1 \rangle = s\langle x_2, y_2, z_2 \rangle$
Thus $x = t\,x_1 + s\,x_2$, $y = t\,y_1 + s\,y_2$, $z = t\,z_1 + s\,z_2$.
(Note that if Q is the midpoint, then $\frac{p}{q} = 1$ and $t = s = \frac{1}{2}$
so that $x = \frac{1}{2}x_1 + \frac{1}{2}x_2 = \frac{x_1+x_2}{2}$, $y = \frac{y_1+y_2}{2}$, $z = \frac{z_1+z_2}{2}$ so that this result agress with the midpoint formula.)



49. (a) the midpoint of AB is $M\left(\frac{5}{2}, \frac{5}{2}, 0\right)$ and $\overrightarrow{CM} = \left(\frac{5}{2} - 1\right)\mathbf{i} + \left(\frac{5}{2} - 1\right)\mathbf{j} + (0-3)\mathbf{k} = \frac{3}{2}\,\mathbf{i} + \frac{3}{2}\,\mathbf{j} - 3\mathbf{k}$
(b) the desired vector is $\left(\frac{2}{3}\right)\overrightarrow{CM} = \frac{2}{3}\left(\frac{3}{2}\,\mathbf{i} + \frac{3}{2}\,\mathbf{j} - 3\mathbf{k}\right) = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$
(c) the vector whose sum is the vector from the origin to C and the result of part (b) will terminate at the center of mass $\Rightarrow$ the terminal point of $(\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ is the point $(2, 2, 1)$, which is the location of the center of mass

50. The midpoint of AB is $M\left(\frac{3}{2}, 0, \frac{5}{2}\right)$ and $\left(\frac{2}{3}\right)\overrightarrow{CM} = \frac{2}{3}\left[\left(\frac{3}{2} + 1\right)\mathbf{i} + (0-2)\mathbf{j} + \left(\frac{5}{2} + 1\right)\mathbf{k}\right] = \frac{2}{3}\left(\frac{5}{2}\,\mathbf{i} - 2\mathbf{j} + \frac{7}{2}\,\mathbf{k}\right)$
$= \frac{5}{3}\,\mathbf{i} - \frac{4}{3}\,\mathbf{j} + \frac{7}{3}\,\mathbf{k}$. The terminal point of $\left(\frac{5}{3}\,\mathbf{i} - \frac{4}{3}\,\mathbf{j} + \frac{7}{3}\,\mathbf{k}\right) + \overrightarrow{OC} = \left(\frac{5}{3}\,\mathbf{i} - \frac{4}{3}\,\mathbf{j} + \frac{7}{3}\,\mathbf{k}\right) + (-\mathbf{i} + 2\mathbf{j} - \mathbf{k})$
$= \frac{2}{3}\,\mathbf{i} + \frac{2}{3}\,\mathbf{j} + \frac{4}{3}\,\mathbf{k}$ is the point $\left(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}\right)$ which is the location of the intersection of the medians.

51. Without loss of generality we identify the vertices of the quadrilateral such that $A(0,0,0)$, $B(x_b, 0, 0)$, $C(x_c, y_c, 0)$ and $D(x_d, y_d, z_d) \Rightarrow$ the midpoint of AB is $M_{AB}\left(\frac{x_b}{2}, 0, 0\right)$, the midpoint of BC is $M_{BC}\left(\frac{x_b+x_c}{2}, \frac{y_c}{2}, 0\right)$, the midpoint of CD is $M_{CD}\left(\frac{x_c+x_d}{2}, \frac{y_c+y_d}{2}, \frac{z_d}{2}\right)$ and the midpoint of AD is $M_{AD}\left(\frac{x_d}{2}, \frac{y_d}{2}, \frac{z_d}{2}\right) \Rightarrow$ the midpoint of $M_{AB}M_{CD}$ is $\left(\frac{\frac{x_b}{2} + \frac{x_c+x_d}{2}}{2}, \frac{y_c+y_d}{4}, \frac{z_d}{4}\right)$ which is the same as the midpoint of $M_{AD}M_{BC} = \left(\frac{\frac{x_b+x_c}{2} + \frac{x_d}{2}}{2}, \frac{y_c+y_d}{4}, \frac{z_d}{4}\right)$.

52. Let V_1, V_2, V_3, … , V_n be the vertices of a regular n-sided polygon and $\mathbf{v}_i$ denote the vector from the center to V_i for i = 1, 2, 3, … , n. If $\mathbf{S} = \sum_{i=1}^{n} \mathbf{v}_i$ and the polygon is rotated through an angle of $\frac{i(2\pi)}{n}$ where i = 1, 2, 3, … , n, then **S** would remain the same. Since the vector **S** does not change with these rotations we conclude that $\mathbf{S} = \mathbf{0}$.

53. Without loss of generality we can coordinatize the vertices of the triangle such that A(0, 0), B(b, 0) and $C(x_c, y_c) \Rightarrow$ a is located at $\left(\frac{b+x_c}{2}, \frac{y_c}{2}\right)$, b is at $\left(\frac{x_c}{2}, \frac{y_c}{2}\right)$ and c is at $\left(\frac{b}{2}, 0\right)$. Therefore, $\overrightarrow{Aa} = \left(\frac{b}{2} + \frac{x_c}{2}\right)\mathbf{i} + \left(\frac{y_c}{2}\right)\mathbf{j}$, $\overrightarrow{Bb} = \left(\frac{x_c}{2} - b\right)\mathbf{i} + \left(\frac{y_c}{2}\right)\mathbf{j}$, and $\overrightarrow{Cc} = \left(\frac{b}{2} - x_c\right)\mathbf{i} + (-y_c)\mathbf{j} \Rightarrow \overrightarrow{Aa} + \overrightarrow{Bb} + \overrightarrow{Cc} = \mathbf{0}$.

54. Let **u** be any unit vector in the plane. If **u** is positioned so that its initial point is at the origin and terminal point is at (x, y), then **u** makes an angle θ with **i**, measured in the counter-clockwise direction. Since $|\mathbf{u}| = 1$, we have that $x = \cos\theta$ and $y = \sin\theta$. Thus $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$. Since **u** was assumed to be any unit vector in the plane, this holds for <u>every</u> unit vector in the plane.

12.3 THE DOT PRODUCT

<u>NOTE</u>: In Exercises 1-8 below we calculate $\text{proj}_\mathbf{v}\,\mathbf{u}$ as the vector $\left(\frac{|\mathbf{u}|\cos\theta}{|\mathbf{v}|}\right)\mathbf{v}$, so the scalar multiplier of **v** is the number in column 5 divided by the number in column 2.

	$\mathbf{v}\cdot\mathbf{u}$	$\lvert\mathbf{v}\rvert$	$\lvert\mathbf{u}\rvert$	$\cos\theta$	$\lvert\mathbf{u}\rvert\cos\theta$	$\text{proj}_\mathbf{v}\,\mathbf{u}$
1.	-25	5	5	-1	-5	$-2\mathbf{i} + 4\mathbf{j} - \sqrt{5}\mathbf{k}$
2.	3	1	13	$\frac{3}{13}$	3	$3\left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}\right)$
3.	25	15	5	$\frac{1}{3}$	$\frac{5}{3}$	$\frac{1}{9}(10\mathbf{i} + 11\mathbf{j} - 2\mathbf{k})$
4.	13	15	3	$\frac{13}{45}$	$\frac{13}{15}$	$\frac{13}{225}(2\mathbf{i} + 10\mathbf{j} - 11\mathbf{k})$
5.	2	$\sqrt{34}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}\sqrt{34}}$	$\frac{2}{\sqrt{34}}$	$\frac{1}{17}(5\mathbf{j} - 3\mathbf{k})$
6.	$\sqrt{3} - \sqrt{2}$	$\sqrt{2}$	3	$\frac{\sqrt{3}-\sqrt{2}}{3\sqrt{2}}$	$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}}$	$\frac{\sqrt{3}-\sqrt{2}}{2}(-\mathbf{i} + \mathbf{j})$
7.	$10 + \sqrt{17}$	$\sqrt{26}$	$\sqrt{21}$	$\frac{10+\sqrt{17}}{\sqrt{546}}$	$\frac{10+\sqrt{17}}{\sqrt{26}}$	$\frac{10+\sqrt{17}}{\sqrt{26}}(-5\mathbf{i} + \mathbf{j})$
8.	$\frac{1}{6}$	$\frac{\sqrt{30}}{6}$	$\frac{\sqrt{30}}{6}$	$\frac{1}{5}$	$\frac{1}{\sqrt{30}}$	$\frac{1}{5}\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right\rangle$

9. $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{(2)(1)+(1)(2)+(0)(-1)}{\sqrt{2^2+1^2+0^2}\sqrt{1^2+2^2+(-1)^2}}\right) = \cos^{-1}\left(\frac{4}{\sqrt{5}\sqrt{6}}\right) = \cos^{-1}\left(\frac{4}{\sqrt{30}}\right) \approx 0.75$ rad

10. $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{(2)(3)+(-2)(0)+(1)(4)}{\sqrt{2^2+(-2)^2+1^2}\sqrt{3^2+0^2+4^2}}\right) = \cos^{-1}\left(\frac{10}{\sqrt{9}\sqrt{25}}\right) = \cos^{-1}\left(\frac{2}{3}\right) \approx 0.84$ rad

11. $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{\left(\sqrt{3}\right)\left(\sqrt{3}\right)+(-7)(1)+(0)(-2)}{\sqrt{\left(\sqrt{3}\right)^2+(-7)^2+0^2}\sqrt{\left(\sqrt{3}\right)^2+(1)^2+(-2)^2}}\right) = \cos^{-1}\left(\frac{3-7}{\sqrt{52}\sqrt{8}}\right)$

$= \cos^{-1}\left(\frac{-1}{\sqrt{26}}\right) \approx 1.77$ rad

12. $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{(1)(-1)+\left(\sqrt{2}\right)(1)+\left(-\sqrt{2}\right)(1)}{\sqrt{(1)^2+\left(\sqrt{2}\right)^2+\left(-\sqrt{2}\right)^2}\,\sqrt{(-1)^2+(1)^2+(1)^2}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{5}\,\sqrt{3}}\right)$
$= \cos^{-1}\left(\frac{-1}{\sqrt{15}}\right) \approx 1.83$ rad

13. $\overrightarrow{AB} = \langle 3, 1\rangle$, $\overrightarrow{BC} = \langle -1, -3\rangle$, and $\overrightarrow{AC} = \langle 2, -2\rangle$. $\overrightarrow{BA} = \langle -3, -1\rangle$, $\overrightarrow{CB} = \langle 1, 3\rangle$, $\overrightarrow{CA} = \langle -2, 2\rangle$.
$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{BA}\right| = \sqrt{10}$, $\left|\overrightarrow{BC}\right| = \left|\overrightarrow{CB}\right| = \sqrt{10}$, $\left|\overrightarrow{AC}\right| = \left|\overrightarrow{CA}\right| = 2\sqrt{2}$,
Angle at A $= \cos^{-1}\left(\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\,\left|\overrightarrow{AC}\right|}\right) = \cos^{-1}\left(\frac{3(2)+1(-2)}{\left(\sqrt{10}\right)\left(2\sqrt{2}\right)}\right) = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63.435°$
Angle at B $= \cos^{-1}\left(\frac{\overrightarrow{BC}\cdot\overrightarrow{BA}}{\left|\overrightarrow{BC}\right|\,\left|\overrightarrow{BA}\right|}\right) = \cos^{-1}\left(\frac{(-1)(-3)+(-3)(-1)}{\left(\sqrt{10}\right)\left(\sqrt{10}\right)}\right) = \cos^{-1}\left(\frac{3}{5}\right) \approx 53.130°$, and
Angle at C $= \cos^{-1}\left(\frac{\overrightarrow{CB}\cdot\overrightarrow{CA}}{\left|\overrightarrow{CB}\right|\,\left|\overrightarrow{CA}\right|}\right) = \cos^{-1}\left(\frac{1(-2)+3(2)}{\left(\sqrt{10}\right)\left(2\sqrt{2}\right)}\right) = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63.435°$

14. $\overrightarrow{AC} = \langle 2, 4\rangle$ and $\overrightarrow{BD} = \langle 4, -2\rangle$. $\overrightarrow{AC}\cdot\overrightarrow{BD} = 2(4)+4(-2) = 0$, so the angle measures are all 90°.

15. (a) $\cos\alpha = \frac{\mathbf{i}\cdot\mathbf{v}}{|\mathbf{i}|\,|\mathbf{v}|} = \frac{a}{|\mathbf{v}|}$, $\cos\beta = \frac{\mathbf{j}\cdot\mathbf{v}}{|\mathbf{j}|\,|\mathbf{v}|} = \frac{b}{|\mathbf{v}|}$, $\cos\gamma = \frac{\mathbf{k}\cdot\mathbf{v}}{|\mathbf{k}|\,|\mathbf{v}|} = \frac{c}{|\mathbf{v}|}$ and
$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{a}{|\mathbf{v}|}\right)^2 + \left(\frac{b}{|\mathbf{v}|}\right)^2 + \left(\frac{c}{|\mathbf{v}|}\right)^2 = \frac{a^2+b^2+c^2}{|\mathbf{v}|\,|\mathbf{v}|} = \frac{|\mathbf{v}|\,|\mathbf{v}|}{|\mathbf{v}|\,|\mathbf{v}|} = 1$
(b) $|\mathbf{v}| = 1 \Rightarrow \cos\alpha = \frac{a}{|\mathbf{v}|} = a$, $\cos\beta = \frac{b}{|\mathbf{v}|} = b$ and $\cos\gamma = \frac{c}{|\mathbf{v}|} = c$ are the direction cosines of $\mathbf{v}$

16. $\mathbf{u} = 10\mathbf{i} + 2\mathbf{k}$ is parallel to the pipe in the north direction and $\mathbf{v} = 10\mathbf{j} + \mathbf{k}$ is parallel to the pipe in the east direction. The angle between the two pipes is $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{2}{\sqrt{104}\,\sqrt{101}}\right) \approx 1.55 \text{ rad} \approx 88.88°$.

17. $\mathbf{u} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right) + \left(\mathbf{u} - \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right) = \frac{3}{2}(\mathbf{i}+\mathbf{j}) + \left[(3\mathbf{j}+4\mathbf{k}) - \frac{3}{2}(\mathbf{i}+\mathbf{j})\right] = \left(\frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}\right) + \left(-\frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} + 4\mathbf{k}\right)$, where $\mathbf{v}\cdot\mathbf{u} = 3$ and $\mathbf{v}\cdot\mathbf{v} = 2$

18. $\mathbf{u} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right) + \left(\mathbf{u} - \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right) = \frac{1}{2}\mathbf{v} + \left(\mathbf{u} - \frac{1}{2}\mathbf{v}\right) = \frac{1}{2}(\mathbf{i}+\mathbf{j}) + \left[(\mathbf{j}+\mathbf{k}) - \frac{1}{2}(\mathbf{i}+\mathbf{j})\right] = \left(\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right) + \left(-\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \mathbf{k}\right)$, where $\mathbf{v}\cdot\mathbf{u} = 1$ and $\mathbf{v}\cdot\mathbf{v} = 2$

19. $\mathbf{u} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right) + \left(\mathbf{u} - \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right) = \frac{14}{3}(\mathbf{i}+2\mathbf{j}-\mathbf{k}) + \left[(8\mathbf{i}+4\mathbf{j}-12\mathbf{k}) - \left(\frac{14}{3}\mathbf{i} + \frac{28}{3}\mathbf{j} - \frac{14}{3}\mathbf{k}\right)\right]$
$= \left(\frac{14}{3}\mathbf{i} + \frac{28}{3}\mathbf{j} - \frac{14}{3}\mathbf{k}\right) + \left(\frac{10}{3}\mathbf{i} - \frac{16}{3}\mathbf{j} - \frac{22}{3}\mathbf{k}\right)$, where $\mathbf{v}\cdot\mathbf{u} = 28$ and $\mathbf{v}\cdot\mathbf{v} = 6$

20. $\mathbf{u} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right) + \left(\mathbf{u} - \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right) = \frac{1}{1}(\mathbf{A}) + \left[(\mathbf{i}+\mathbf{j}+\mathbf{k}) - \left(\frac{1}{1}\right)\mathbf{A}\right] = (\mathbf{i}) + (\mathbf{j}+\mathbf{k})$, where $\mathbf{v}\cdot\mathbf{u} = 1$ and $\mathbf{v}\cdot\mathbf{v} = 1$; yes

21. The sum of two vectors of equal length is *always* orthogonal to their difference, as we can see from the equation $(\mathbf{v}_1+\mathbf{v}_2)\cdot(\mathbf{v}_1-\mathbf{v}_2) = \mathbf{v}_1\cdot\mathbf{v}_1 + \mathbf{v}_2\cdot\mathbf{v}_1 - \mathbf{v}_1\cdot\mathbf{v}_2 - \mathbf{v}_2\cdot\mathbf{v}_2 = |\mathbf{v}_1|^2 - |\mathbf{v}_2|^2 = 0$

22. $\overrightarrow{CA}\cdot\overrightarrow{CB} = (-\mathbf{v}+(-\mathbf{u}))\cdot(-\mathbf{v}+\mathbf{u}) = \mathbf{v}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{u} = |\mathbf{v}|^2 - |\mathbf{u}|^2 = 0$ because $|\mathbf{u}| = |\mathbf{v}|$ since both equal the radius of the circle. Therefore, $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are orthogonal.

23. Let $\mathbf{u}$ and $\mathbf{v}$ be the sides of a rhombus $\Rightarrow$ the diagonals are $\mathbf{d}_1 = \mathbf{u} + \mathbf{v}$ and $\mathbf{d}_2 = -\mathbf{u} + \mathbf{v}$ $\Rightarrow$ $\mathbf{d}_1 \cdot \mathbf{d}_2 = (\mathbf{u} + \mathbf{v}) \cdot (-\mathbf{u} + \mathbf{v}) = -\mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 - |\mathbf{u}|^2 = 0$ because $|\mathbf{u}| = |\mathbf{v}|$, since a rhombus has equal sides.

24. Suppose the diagonals of a rectangle are perpendicular, and let $\mathbf{u}$ and $\mathbf{v}$ be the sides of a rectangle $\Rightarrow$ the diagonals are $\mathbf{d}_1 = \mathbf{u} + \mathbf{v}$ and $\mathbf{d}_2 = -\mathbf{u} + \mathbf{v}$. Since the diagonals are perpendicular we have $\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$ $\Leftrightarrow$ $(\mathbf{u} + \mathbf{v}) \cdot (-\mathbf{u} + \mathbf{v}) = -\mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = 0$ $\Leftrightarrow$ $|\mathbf{v}|^2 - |\mathbf{u}|^2 = 0 \Leftrightarrow (|\mathbf{v}| + |\mathbf{u}|)(|\mathbf{v}| - |\mathbf{u}|) = 0$ $\Leftrightarrow$ $(|\mathbf{v}| + |\mathbf{u}|) = 0$ which is not possible, or $(|\mathbf{v}| - |\mathbf{u}|) = 0$ which is equivalent to $|\mathbf{v}| = |\mathbf{u}|$ $\Rightarrow$ the rectangle is a square.

25. Clearly the diagonals of a rectangle are equal in length. What is not as obvious is the statement that equal diagonals happen only in a rectangle. We show this is true by letting the adjacent sides of a parallelogram be the vectors $(v_1\mathbf{i} + v_2\mathbf{j})$ and $(u_1\mathbf{i} + u_2\mathbf{j})$. The equal diagonals of the parallelogram are $\mathbf{d}_1 = (v_1\mathbf{i} + v_2\mathbf{j}) + (u_1\mathbf{i} + u_2\mathbf{j})$ and $\mathbf{d}_2 = (v_1\mathbf{i} + v_2\mathbf{j}) - (u_1\mathbf{i} + u_2\mathbf{j})$. Hence $|\mathbf{d}_1| = |\mathbf{d}_2| = |(v_1\mathbf{i} + v_2\mathbf{j}) + (u_1\mathbf{i} + u_2\mathbf{j})|$ $= |(v_1\mathbf{i} + v_2\mathbf{j}) - (u_1\mathbf{i} + u_2\mathbf{j})|$ $\Rightarrow$ $|(v_1 + u_1)\mathbf{i} + (v_2 + u_2)\mathbf{j}| = |(v_1 - u_1)\mathbf{i} + (v_2 - u_2)\mathbf{j}|$ $\Rightarrow$ $\sqrt{(v_1 + u_1)^2 + (v_2 + u_2)^2} = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2}$ $\Rightarrow$ $v_1^2 + 2v_1u_1 + u_1^2 + v_2^2 + 2v_2u_2 + u_2^2$ $= v_1^2 - 2v_1u_1 + u_1^2 + v_2^2 - 2v_2u_2 + u_2^2$ $\Rightarrow$ $2(v_1u_1 + v_2u_2) = -2(v_1u_1 + v_2u_2)$ $\Rightarrow$ $v_1u_1 + v_2u_2 = 0$ $\Rightarrow$ $(v_1\mathbf{i} + v_2\mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j}) = 0$ $\Rightarrow$ the vectors $(v_1\mathbf{i} + v_2\mathbf{j})$ and $(u_1\mathbf{i} + u_2\mathbf{j})$ are perpendicular and the parallelogram must be a rectangle.

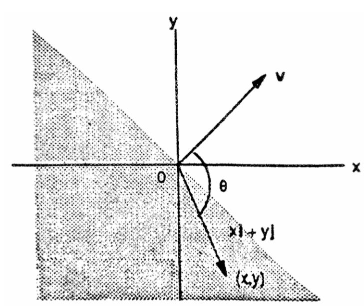
26. If $|\mathbf{u}| = |\mathbf{v}|$ and $\mathbf{u} + \mathbf{v}$ is the indicated diagonal, then $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u} = |\mathbf{u}|^2 + \mathbf{v} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2$ $= \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = (\mathbf{u} + \mathbf{v}) \cdot \mathbf{v}$ $\Rightarrow$ the angle $\cos^{-1}\left(\frac{(\mathbf{u}+\mathbf{v})\cdot\mathbf{u}}{|\mathbf{u}+\mathbf{v}|\,|\mathbf{u}|}\right)$ between the diagonal and $\mathbf{u}$ and the angle $\cos^{-1}\left(\frac{(\mathbf{u}+\mathbf{v})\cdot\mathbf{v}}{|\mathbf{u}+\mathbf{v}|\,|\mathbf{v}|}\right)$ between the diagonal and $\mathbf{v}$ are equal because the inverse cosine function is one-to-one. Therefore, the diagonal bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

27. horizontal component: $1200\cos(8^\circ) \approx 1188$ ft/s; vertical component: $1200\sin(8^\circ) \approx 167$ ft/s

28. $|\mathbf{w}|\cos(33^\circ - 15^\circ) = 2.5$ lb, so $|\mathbf{w}| = \frac{2.5\text{ lb}}{\cos 18^\circ}$. Then $\mathbf{w} = \frac{2.5\text{ lb}}{\cos 18^\circ}\langle \cos 33^\circ, \sin 33^\circ \rangle \approx \langle 2.205, 1.432 \rangle$

29. (a) Since $|\cos\theta| \le 1$, we have $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\,|\cos\theta| \le |\mathbf{u}|\,|\mathbf{v}|(1) = |\mathbf{u}|\,|\mathbf{v}|$.
(b) We have equality precisely when $|\cos\theta| = 1$ or when one or both of $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{0}$. In the case of nonzero vectors, we have equality when $\theta = 0$ or π, i.e., when the vectors are parallel.

30. $(x\mathbf{i} + y\mathbf{j}) \cdot \mathbf{v} = |x\mathbf{i} + y\mathbf{j}|\,|\mathbf{v}|\cos\theta \le 0$ when $\frac{\pi}{2} \le \theta \le \pi$. This means (x, y) has to be a point whose position vector makes an angle with $\mathbf{v}$ that is a right angle or bigger.



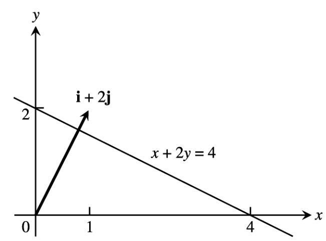
31. $\mathbf{v} \cdot \mathbf{u}_1 = (a\mathbf{u}_1 + b\mathbf{u}_2) \cdot \mathbf{u}_1 = a\mathbf{u}_1 \cdot \mathbf{u}_1 + b\mathbf{u}_2 \cdot \mathbf{u}_1 = a|\mathbf{u}_1|^2 + b(\mathbf{u}_2 \cdot \mathbf{u}_1) = a(1)^2 + b(0) = a$

32. No, $\mathbf{v}_1$ need not equal $\mathbf{v}_2$. For example, $\mathbf{i} + \mathbf{j} \ne \mathbf{i} + 2\mathbf{j}$ but $\mathbf{i} \cdot (\mathbf{i} + \mathbf{j}) = \mathbf{i} \cdot \mathbf{i} + \mathbf{i} \cdot \mathbf{j} = 1 + 0 = 1$ and $\mathbf{i} \cdot (\mathbf{i} + 2\mathbf{j}) = \mathbf{i} \cdot \mathbf{i} + 2\mathbf{i} \cdot \mathbf{j} = 1 + 2 \cdot 0 = 1$.

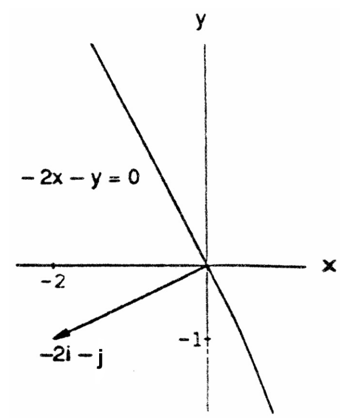
33. $P(x_1, y_1) = P\left(x_1, \frac{c}{b} - \frac{a}{b}x_1\right)$ and $Q(x_2, y_2) = Q\left(x_2, \frac{c}{b} - \frac{a}{b}x_2\right)$ are any two points P and Q on the line with $b \neq 0$ $\Rightarrow \overrightarrow{PQ} = (x_2 - x_1)\mathbf{i} + \frac{a}{b}(x_1 - x_2)\mathbf{j} \Rightarrow \overrightarrow{PQ} \cdot \mathbf{v} = \left[(x_2 - x_1)\mathbf{i} + \frac{a}{b}(x_1 - x_2)\mathbf{j}\right] \cdot (a\mathbf{i} + b\mathbf{j}) = a(x_2 - x_1) + b\left(\frac{a}{b}\right)(x_1 - x_2)$ $= 0 \Rightarrow \mathbf{v}$ is perpendicular to $\overrightarrow{PQ}$ for $b \neq 0$. If $b = 0$, then $\mathbf{v} = a\mathbf{i}$ is perpendicular to the vertical line $ax = c$. Alternatively, the slope of $\mathbf{v}$ is $\frac{b}{a}$ and the slope of the line $ax + by = c$ is $-\frac{a}{b}$, so the slopes are negative reciprocals $\Rightarrow$ the vector $\mathbf{v}$ and the line are perpendicular.

34. The slope of $\mathbf{v}$ is $\frac{b}{a}$ and the slope of $bx - ay = c$ is $\frac{b}{a}$, provided that $a \neq 0$. If $a = 0$, then $\mathbf{v} = b\mathbf{j}$ is parallel to the vertical line $bx = c$. In either case, the vector $\mathbf{v}$ is parallel to the line $ax - by = c$.

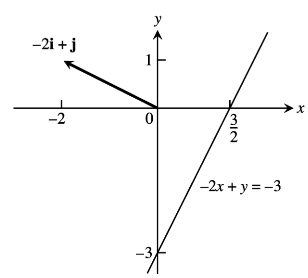
35. $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$ is perpendicular to the line $x + 2y = c$;
$P(2, 1)$ on the line $\Rightarrow 2 + 2 = c \Rightarrow x + 2y = 4$



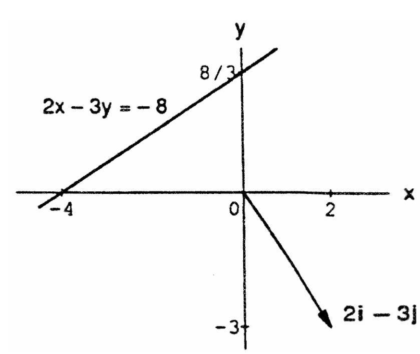
36. $\mathbf{v} = -2\mathbf{i} - \mathbf{j}$ is perpendicular to the line $-2x - y = c$;
$P(-1, 2)$ on the line $\Rightarrow (-2)(-1) - 2 = c$
$\Rightarrow -2x - y = 0$



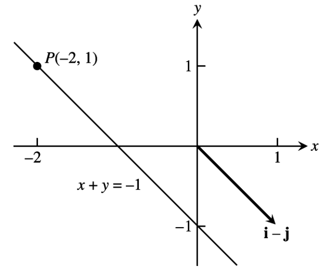
37. $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$ is perpendicular to the line $-2x + y = c$;
$P(-2, -7)$ on the line $\Rightarrow (-2)(-2) - 7 = c$
$\Rightarrow -2x + y = -3$



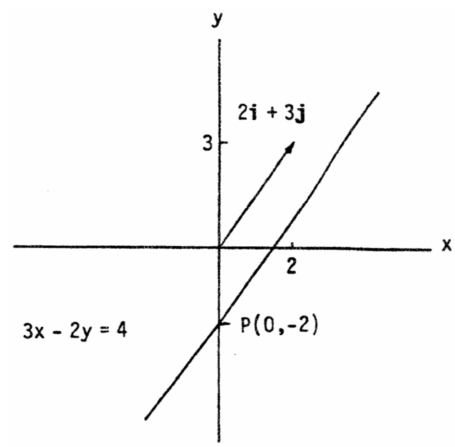
38. $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$ is perpendicular to the line $2x - 3y = c$;
$P(11, 10)$ on the line $\Rightarrow (2)(11) - (3)(10) = c$
$\Rightarrow 2x - 3y = -8$



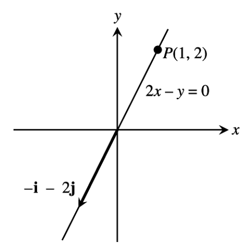
39. $\mathbf{v} = \mathbf{i} - \mathbf{j}$ is parallel to the line $-x - y = c$;
P$(-2, 1)$ on the line $\Rightarrow -(-2) - 1 = c \Rightarrow -x - y = 1$ or $x + y = -1$.



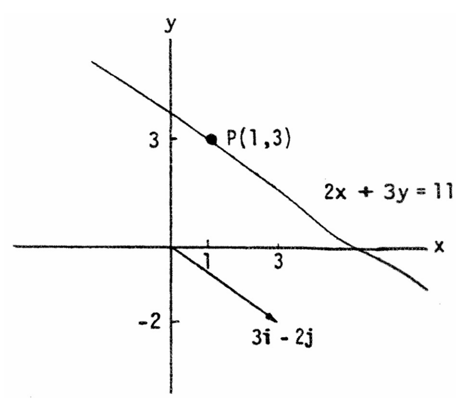
40. $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$ is parallel to the line $3x - 2y = c$;
P$(0, -2)$ on the line $\Rightarrow 0 - 2(-2) = c \Rightarrow 3x - 2y = 4$



41. $\mathbf{v} = -\mathbf{i} - 2\mathbf{j}$ is parallel to the line $-2x + y = c$;
P$(1, 2)$ on the line $\Rightarrow -2(1) + 2 = c \Rightarrow -2x - y = 0$ or $2x - y = 0$.



42. $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$ is parallel to the line $-2x - 3y = c$;
P$(1, 3)$ on the line $\Rightarrow (-2)(1) - (3)(3) = c$
$\Rightarrow -2x - 3y = -11$ or $2x + 3y = 11$



43. P$(0, 0)$, Q$(1, 1)$ and $\mathbf{F} = 5\mathbf{j} \Rightarrow \overrightarrow{PQ} = \mathbf{i} + \mathbf{j}$ and $\mathbf{W} = \mathbf{F} \cdot \overrightarrow{PQ} = (5\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j}) = 5 \text{ N} \cdot \text{m} = 5 \text{ J}$

44. $\mathbf{W} = |\mathbf{F}|$ (distance) $\cos\theta = (602{,}148 \text{ N})(605 \text{ km})(\cos 0) = 364{,}299{,}540 \text{ N} \cdot \text{km} = (364{,}299{,}540)(1000) \text{ N} \cdot \text{m}$
$= 3.6429954 \times 10^{11}$ J

45. $\mathbf{W} = |\mathbf{F}| \left|\overrightarrow{PQ}\right| \cos\theta = (200)(20)(\cos 30°) = 2000\sqrt{3} = 3464.10 \text{ N} \cdot \text{m} = 3464.10 \text{ J}$

46. $\mathbf{W} = |\mathbf{F}|\left|\overrightarrow{PQ}\right|\cos\theta = (1000)(5280)(\cos 60°) = 2{,}640{,}000$ ft · lb

In Exercises 47-52 we use the fact that $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$ is normal to the line $ax + by = c$.

47. $\mathbf{n}_1 = 3\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = 2\mathbf{i} - \mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{6-1}{\sqrt{10}\sqrt{5}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

48. $\mathbf{n}_1 = -\sqrt{3}\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = \sqrt{3}\mathbf{i} + \mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{-3+1}{\sqrt{4}\sqrt{4}}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

49. $\mathbf{n}_1 = \sqrt{3}\mathbf{i} - \mathbf{j}$ and $\mathbf{n}_2 = \mathbf{i} - \sqrt{3}\mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{\sqrt{3}+\sqrt{3}}{\sqrt{4}\sqrt{4}}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

50. $\mathbf{n}_1 = \mathbf{i} + \sqrt{3}\mathbf{j}$ and $\mathbf{n}_2 = \left(1-\sqrt{3}\right)\mathbf{i} + \left(1+\sqrt{3}\right)\mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right)$
$= \cos^{-1}\left(\frac{1-\sqrt{3}+\sqrt{3}+3}{\sqrt{1+3}\sqrt{1-2\sqrt{3}+3+1+2\sqrt{3}+3}}\right) = \cos^{-1}\left(\frac{4}{2\sqrt{8}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

51. $\mathbf{n}_1 = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{n}_2 = \mathbf{i} - \mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{3+4}{\sqrt{25}\sqrt{2}}\right) = \cos^{-1}\left(\frac{7}{5\sqrt{2}}\right) \approx 0.14$ rad

52. $\mathbf{n}_1 = 12\mathbf{i} + 5\mathbf{j}$ and $\mathbf{n}_2 = 2\mathbf{i} - 2\mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{24-10}{\sqrt{169}\sqrt{8}}\right) = \cos^{-1}\left(\frac{14}{26\sqrt{2}}\right) \approx 1.18$ rad

53. The angle between the corresponding normals is equal to the angle between the corresponding tangents. The points of intersection are $\left(-\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$. At $\left(-\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$ the tangent line for $f(x) = x^2$ is $y - \frac{3}{4} = f'\left(-\frac{\sqrt{3}}{2}\right)\left(x - \left(-\frac{\sqrt{3}}{2}\right)\right) \Rightarrow y = -\sqrt{3}\left(x + \frac{\sqrt{3}}{2}\right) + \frac{3}{4} \Rightarrow y = -\sqrt{3}x - \frac{3}{4}$, and the tangent line for $f(x) = \left(\frac{3}{2}\right) - x^2$ is $y - \frac{3}{4} = f'\left(-\frac{\sqrt{3}}{2}\right)\left(x - \left(-\frac{\sqrt{3}}{2}\right)\right) \Rightarrow y = \sqrt{3}\left(x + \frac{\sqrt{3}}{2}\right) + \frac{3}{4} = \sqrt{3}x + \frac{9}{4}$. The corresponding normals are $\mathbf{n}_1 = \sqrt{3}\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = -\sqrt{3}\mathbf{i} + \mathbf{j}$. The angle at $\left(-\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$ is $\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right)$
$= \cos^{-1}\left(\frac{-3+1}{\sqrt{4}\sqrt{4}}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$, the angle is $\frac{\pi}{3}$ and $\frac{2\pi}{3}$. At $\left(\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$ the tangent line for $f(x) = x^2$ is $y = \sqrt{3}\left(x + \frac{\sqrt{3}}{2}\right) + \frac{3}{4} = \sqrt{3}x + \frac{9}{4}$ and the tangent line for $f(x) = \frac{3}{2} - x^2$ is $y = -\sqrt{3}\left(x + \frac{\sqrt{3}}{2}\right) + \frac{3}{4}$
$= -\sqrt{3}x - \frac{3}{4}$. The corresponding normals are $\mathbf{n}_1 = -\sqrt{3}\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = \sqrt{3}\mathbf{i} + \mathbf{j}$. The angle at $\left(\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$ is $\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{-3+1}{\sqrt{4}\sqrt{4}}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$, the angle is $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

54. The points of intersection are $\left(0, \frac{\sqrt{3}}{2}\right)$ and $\left(0, -\frac{\sqrt{3}}{2}\right)$. The curve $x = \frac{3}{4} - y^2$ has derivative $\frac{dy}{dx} = -\frac{1}{2y} \Rightarrow$ the tangent line at $\left(0, \frac{\sqrt{3}}{2}\right)$ is $y - \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}}(x - 0) \Rightarrow \mathbf{n}_1 = \frac{1}{\sqrt{3}}\mathbf{i} + \mathbf{j}$ is normal to the curve at that point. The curve $x = y^2 - \frac{3}{4}$ has derivative $\frac{dy}{dx} = \frac{1}{2y} \Rightarrow$ the tangent line at $\left(0, \frac{\sqrt{3}}{2}\right)$ is $y - \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}(x - 0)$
$\Rightarrow \mathbf{n}_2 = -\frac{1}{\sqrt{3}}\mathbf{i} + \mathbf{j}$ is normal to the curve. The angle between the curves is $\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right)$
$= \cos^{-1}\left(\frac{-\frac{1}{3}+1}{\sqrt{\frac{1}{3}+1}\sqrt{\frac{1}{3}+1}}\right) = \cos^{-1}\left(\frac{\left(\frac{2}{3}\right)}{\left(\frac{4}{3}\right)}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ and $\frac{2\pi}{3}$. Because of symmetry the angles between the curves at the two points of intersection are the same.

55. The curves intersect when $y = x^3 = \left(y^2\right)^3 = y^6 \Rightarrow y = 0$ or $y = 1$. The points of intersection are $(0,0)$ and $(1,1)$. Note that $y \ge 0$ since $y = y^6$. At $(0,0)$ the tangent line for $y = x^3$ is $y = 0$ and the tangent line for

$y = \sqrt{x}$ is $x = 0$. Therefore, the angle of intersection at $(0,0)$ is $\frac{\pi}{2}$. At $(1,1)$ the tangent line for $y = x^3$ is $y = 3x - 2$ and the tangent line for $y = \sqrt{x}$ is $y = \frac{1}{2}x + \frac{1}{2}$. The corresponding normal vectors are $\mathbf{n}_1 = -3\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = -\frac{1}{2}\mathbf{i} + \mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$, the angle is $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

56. The points of intersection for the curves $y = -x^2$ and $y = \sqrt[3]{x}$ are $(0,0)$ and $(-1,-1)$. At $(0,0)$ the tangent line for $y = -x^2$ is $y = 0$ and the tangent line for $y = \sqrt[3]{x}$ is $x = 0$. Therefore, the angle of intersection at $(0,0)$ is $\frac{\pi}{2}$. At $(-1,-1)$ the tangent line for $y = -x^2$ is $y = 2x + 1$ and the tangent line for $y = \sqrt[3]{x}$ is $y = \frac{1}{3}x - \frac{2}{3}$. The corresponding normal vectors are $\mathbf{n}_1 = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{n}_2 = \frac{1}{3}\mathbf{i} - \mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right)$

$$= \cos^{-1}\left(\frac{\frac{2}{3}+1}{\sqrt{5}\sqrt{\frac{1}{9}+1}}\right) = \cos^{-1}\left(\frac{\left(\frac{5}{3}\right)}{\frac{\sqrt{5}\sqrt{10}}{3}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}, \text{ the angle is } \frac{\pi}{4} \text{ and } \frac{3\pi}{4}.$$

12.4 THE CROSS PRODUCT

1. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & -1 \\ 1 & 0 & -1 \end{vmatrix} = 3\left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right) \Rightarrow$ length $= 3$ and the direction is $\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$;

   $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -3\left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right) \Rightarrow$ length $= 3$ and the direction is $-\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$

2. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ -1 & 1 & 0 \end{vmatrix} = 5(\mathbf{k}) \Rightarrow$ length $= 5$ and the direction is $\mathbf{k}$

   $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -5(\mathbf{k}) \Rightarrow$ length $= 5$ and the direction is $-\mathbf{k}$

3. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{vmatrix} = \mathbf{0} \Rightarrow$ length $= 0$ and has no direction

   $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = \mathbf{0} \Rightarrow$ length $= 0$ and has no direction

4. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix} = \mathbf{0} \Rightarrow$ length $= 0$ and has no direction

   $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = \mathbf{0} \Rightarrow$ length $= 0$ and has no direction

5. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 0 \\ 0 & -3 & 0 \end{vmatrix} = -6(\mathbf{k}) \Rightarrow$ length $= 6$ and the direction is $-\mathbf{k}$

   $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 6(\mathbf{k}) \Rightarrow$ length $= 6$ and the direction is $\mathbf{k}$

6. $\mathbf{u} \times \mathbf{v} = (\mathbf{i} \times \mathbf{j}) \times (\mathbf{j} \times \mathbf{k}) = \mathbf{k} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j} \Rightarrow$ length $= 1$ and the direction is $\mathbf{j}$

   $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -\mathbf{j} \Rightarrow$ length $= 1$ and the direction is $-\mathbf{j}$

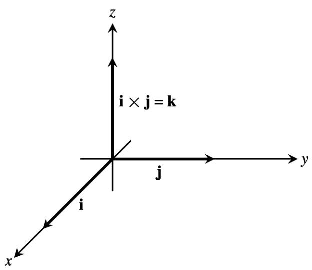
7. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & -2 & -4 \\ 2 & 2 & 1 \end{vmatrix} = 6\mathbf{i} - 12\mathbf{k} \Rightarrow$ length $= 6\sqrt{5}$ and the direction is $\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{k}$

   $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -(6\mathbf{i} - 12\mathbf{k}) \Rightarrow$ length $= 6\sqrt{5}$ and the direction is $-\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k}$

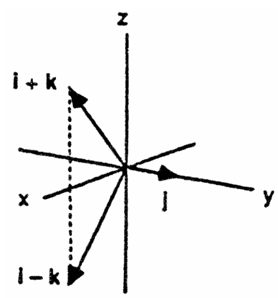
8. $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{3}{2} & -\frac{1}{2} & 1\\ 1 & 1 & 2\end{vmatrix}=-2\mathbf{i}-2\mathbf{j}+2\mathbf{k} \Rightarrow$ length $=2\sqrt{3}$ and the direction is $-\frac{1}{\sqrt{3}}\mathbf{i}-\frac{1}{\sqrt{3}}\mathbf{j}+\frac{1}{\sqrt{3}}\mathbf{k}$

$\mathbf{v}\times\mathbf{u}=-(\mathbf{u}\times\mathbf{v})=-(-2\mathbf{i}-2\mathbf{j}+2\mathbf{k}) \Rightarrow$ length $=2\sqrt{3}$ and the direction is $\frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{\sqrt{3}}\mathbf{j}-\frac{1}{\sqrt{3}}\mathbf{k}$

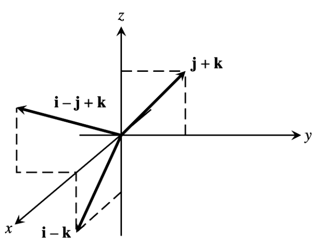
9. $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & 0\\ 0 & 1 & 0\end{vmatrix}=\mathbf{k}$



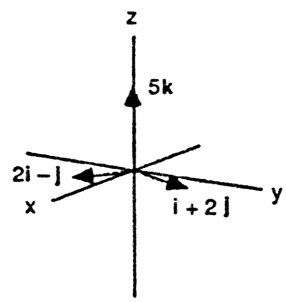
10. $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & -1\\ 0 & 1 & 0\end{vmatrix}=\mathbf{i}+\mathbf{k}$



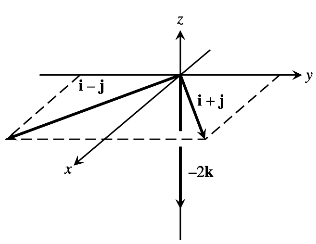
11. $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & -1\\ 0 & 1 & 1\end{vmatrix}=\mathbf{i}-\mathbf{j}+\mathbf{k}$



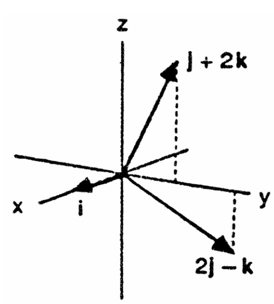
12. $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -1 & 0\\ 1 & 2 & 0\end{vmatrix}=5\mathbf{k}$



13. $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & 0\\ 1 & -1 & 0\end{vmatrix}=-2\mathbf{k}$



14. $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 1 & 2\\ 1 & 0 & 0\end{vmatrix}=2\mathbf{j}-\mathbf{k}$



15. (a) $\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & -3\\ -1 & 3 & -1\end{vmatrix}=8\mathbf{i}+4\mathbf{j}+4\mathbf{k} \Rightarrow \text{Area}=\frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|=\frac{1}{2}\sqrt{64+16+16}=2\sqrt{6}$

(b) $\mathbf{u}=\pm\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|}=\pm\frac{1}{\sqrt{6}}(2\mathbf{i}+\mathbf{j}+\mathbf{k})$

16. (a) $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & -2 & 0 \end{vmatrix} = 4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \Rightarrow \text{Area} = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{1}{2}\sqrt{16+16+4} = 3$

(b) $\mathbf{u} = \pm \frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|} = \pm \frac{1}{3}(2\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

17. (a) $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\mathbf{i} + \mathbf{j} \Rightarrow \text{Area} = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{1}{2}\sqrt{1+1} = \frac{\sqrt{2}}{2}$

(b) $\mathbf{u} = \pm \frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|} = \pm \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}) = \pm \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$

18. (a) $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -1 \\ 1 & 0 & -2 \end{vmatrix} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow \text{Area} = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{1}{2}\sqrt{4+9+1} = \frac{\sqrt{14}}{2}$

(b) $\mathbf{u} = \pm \frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|} = \pm \frac{1}{\sqrt{14}}(2\mathbf{i} + 3\mathbf{j} + \mathbf{k})$

19. If $\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and $\mathbf{w} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, then $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$,

$\mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$ and $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ which all have the same value, since the interchanging of two pair of rows in a determinant does not change its value $\Rightarrow$ the volume is

$|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \text{abs}\begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8$

20. $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \text{abs}\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = 4$ (for details about verification, see Exercise 19)

21. $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \text{abs}\begin{vmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = |-7| = 7$ (for details about verification, see Exercise 19)

22. $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \text{abs}\begin{vmatrix} 1 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 4 & -2 \end{vmatrix} = 8$ (for details about verification, see Exercise 19)

23. (a) $\mathbf{u} \cdot \mathbf{v} = -6$, $\mathbf{u} \cdot \mathbf{w} = -81$, $\mathbf{v} \cdot \mathbf{w} = 18 \Rightarrow$ none

(b) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 1 \\ 0 & 1 & -5 \end{vmatrix} \neq \mathbf{0}$, $\mathbf{u} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 1 \\ -15 & 3 & -3 \end{vmatrix} = \mathbf{0}$, $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -5 \\ -15 & 3 & -3 \end{vmatrix} \neq \mathbf{0}$

$\Rightarrow$ $\mathbf{u}$ and $\mathbf{w}$ are parallel

24. (a) $\mathbf{u} \cdot \mathbf{v} = 0$, $\mathbf{u} \times \mathbf{w} = 0$, $\mathbf{u} \cdot \mathbf{r} = -3\pi$, $\mathbf{v} \cdot \mathbf{w} = 0$, $\mathbf{v} \cdot \mathbf{r} = 0$, $\mathbf{w} \cdot \mathbf{r} = 0 \Rightarrow \mathbf{u} \perp \mathbf{v}$, $\mathbf{u} \perp \mathbf{w}$, $\mathbf{v} \perp \mathbf{w}$, $\mathbf{v} \perp \mathbf{r}$ and $\mathbf{w} \perp \mathbf{r}$

(b) $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&-1\\-1&1&1\end{vmatrix}\neq\mathbf{0}$, $\mathbf{u}\times\mathbf{w}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&-1\\1&0&1\end{vmatrix}\neq\mathbf{0}$, $\mathbf{u}\times\mathbf{r}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&-1\\-\frac{\pi}{2}&-\pi&\frac{\pi}{2}\end{vmatrix}=\mathbf{0}$

$\mathbf{v}\times\mathbf{w}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&1&1\\1&0&1\end{vmatrix}\neq\mathbf{0}$, $\mathbf{v}\times\mathbf{r}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&1&1\\-\frac{\pi}{2}&-\pi&\frac{\pi}{2}\end{vmatrix}\neq\mathbf{0}$, $\mathbf{w}\times\mathbf{r}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&1\\-\frac{\pi}{2}&-\pi&\frac{\pi}{2}\end{vmatrix}\neq\mathbf{0}$

$\Rightarrow$ **u** and **r** are parallel

25. $\left|\overrightarrow{PQ}\times\mathbf{F}\right|=\left|\overrightarrow{PQ}\right|\,|\mathbf{F}|\sin(60°)=\frac{2}{3}\cdot 30\cdot\frac{\sqrt{3}}{2}\text{ ft}\cdot\text{lb}=10\sqrt{3}\text{ ft}\cdot\text{lb}$

26. $\left|\overrightarrow{PQ}\times\mathbf{F}\right|=\left|\overrightarrow{PQ}\right|\,|\mathbf{F}|\sin(135°)=\frac{2}{3}\cdot 30\cdot\frac{\sqrt{2}}{2}\text{ ft}\cdot\text{lb}=10\sqrt{2}\text{ ft}\cdot\text{lb}$

27. (a) true, $|\mathbf{u}|=\sqrt{a_1^2+a_2^2+a_3^2}=\sqrt{\mathbf{u}\cdot\mathbf{u}}$

(b) not always true, $\mathbf{u}\cdot\mathbf{u}=|\mathbf{u}|^2$

(c) true, $\mathbf{u}\times\mathbf{0}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\0&0&0\end{vmatrix}=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}$ and $\mathbf{0}\times\mathbf{u}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&0&0\\a_1&a_2&a_3\end{vmatrix}=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}$

(d) true, $\mathbf{u}\times(-\mathbf{u})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\-a_1&-a_2&-a_3\end{vmatrix}=(-a_2a_3+a_2a_3)\mathbf{i}-(-a_1a_3+a_1a_3)\mathbf{j}+(-a_1a_2+a_1a_2)\mathbf{k}=\mathbf{0}$

(e) not always true, $\mathbf{i}\times\mathbf{j}=\mathbf{k}\neq-\mathbf{k}=\mathbf{j}\times\mathbf{i}$ for example

(f) true, distributive property of the cross product

(g) true, $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v}=\mathbf{u}\cdot(\mathbf{v}\times\mathbf{v})=\mathbf{u}\cdot\mathbf{0}=0$

(h) true, the volume of a parallelpiped with **u**, **v**, and **w** along the three edges is $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}=(\mathbf{v}\times\mathbf{w})\cdot\mathbf{u}=\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$, since the dot product is commutative.

28. (a) true, $\mathbf{u}\cdot\mathbf{v}=a_1b_1+a_2b_2+a_3b_3=b_1a_1+b_2a_2+b_3a_3=\mathbf{v}\cdot\mathbf{u}$

(b) true, $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}=-\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\b_1&b_2&b_3\\a_1&a_2&a_3\end{vmatrix}=-(\mathbf{v}\times\mathbf{u})$

(c) true, $(-\mathbf{u})\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-a_1&-a_2&-a_3\\b_1&b_2&b_3\end{vmatrix}=-\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}=-(\mathbf{u}\times\mathbf{v})$

(d) true, $(c\mathbf{u})\cdot\mathbf{v}=(ca_1)b_1+(ca_2)b_2+(ca_3)b_3=a_1(cb_1)+a_2(cb_2)+a_3(cb_3)=\mathbf{u}\cdot(c\mathbf{v})=c(a_1b_1+a_2b_2+a_3b_3)$
$=c(\mathbf{u}\cdot\mathbf{v})$

(e) true, $c(\mathbf{u}\times\mathbf{v})=c\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ca_1&ca_2&ca_3\\b_1&b_2&b_3\end{vmatrix}=(c\mathbf{u})\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\cb_1&cb_2&cb_3\end{vmatrix}=\mathbf{u}\times(c\mathbf{v})$

(f) true, $\mathbf{u}\cdot\mathbf{u}=a_1^2+a_2^2+a_3^2=\left(\sqrt{a_1^2+a_2^2+a_3^2}\right)^2=|\mathbf{u}|^2$

(g) true, $(\mathbf{u}\times\mathbf{u})\cdot\mathbf{u}=\mathbf{0}\cdot\mathbf{u}=0$

(h) true, $\mathbf{u}\times\mathbf{v}\perp\mathbf{u}$ and $\mathbf{u}\times\mathbf{v}\perp\mathbf{v}\Rightarrow(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u}=\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v})=0$

29. (a) $\text{proj}_{\mathbf{v}}\,\mathbf{u}=\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}||\mathbf{v}|}\right)\mathbf{v}$ (b) $\pm(\mathbf{u}\times\mathbf{v})$ (c) $\pm((\mathbf{u}\times\mathbf{v})\times\mathbf{w})$ (d) $|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|$

30. (a) $(\mathbf{u}\times\mathbf{v})\times(\mathbf{u}\times\mathbf{w})$

(b) $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})=(\mathbf{u}+\mathbf{v})\times\mathbf{u}-(\mathbf{u}+\mathbf{v})\times\mathbf{v}=\mathbf{u}\times\mathbf{u}+\mathbf{v}\times\mathbf{u}-\mathbf{u}\times\mathbf{v}-\mathbf{v}\times\mathbf{v}$
$=\mathbf{0}+\mathbf{v}\times\mathbf{u}-\mathbf{u}\times\mathbf{v}-\mathbf{0}=2(\mathbf{v}\times\mathbf{u})$, or simply $\mathbf{u}\times\mathbf{v}$

(c) $|\mathbf{u}|\,\frac{\mathbf{v}}{|\mathbf{v}|}$ (d) $|\mathbf{u}\times\mathbf{w}|$

31. (a) yes, $\mathbf{u} \times \mathbf{v}$ and $\mathbf{w}$ are both vectors
(b) no, $\mathbf{u}$ is a vector but $\mathbf{v} \cdot \mathbf{w}$ is a scalar
(c) yes, $\mathbf{u}$ and $\mathbf{u} \times \mathbf{w}$ are both vectors
(d) no, $\mathbf{u}$ is a vector but $\mathbf{v} \cdot \mathbf{w}$ is a scalar

32. $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ is perpendicular to $\mathbf{u} \times \mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ $\Rightarrow$ $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ is parallel to a vector in the plane of $\mathbf{u}$ and $\mathbf{v}$ which means it lies in the plane determined by $\mathbf{u}$ and $\mathbf{v}$. The situation is degenerate if $\mathbf{u}$ and $\mathbf{v}$ are parallel so $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ and the vectors do not determine a plane. Similar reasoning shows that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ lies in the plane of $\mathbf{v}$ and $\mathbf{w}$ provided $\mathbf{v}$ and $\mathbf{w}$ are nonparallel.

33. No, $\mathbf{v}$ need not equal $\mathbf{w}$. For example, $\mathbf{i} + \mathbf{j} \neq -\mathbf{i} + \mathbf{j}$, but $\mathbf{i} \times (\mathbf{i} + \mathbf{j}) = \mathbf{i} \times \mathbf{i} + \mathbf{i} \times \mathbf{j} = \mathbf{0} + \mathbf{k} = \mathbf{k}$ and $\mathbf{i} \times (-\mathbf{i} + \mathbf{j}) = -\mathbf{i} \times \mathbf{i} + \mathbf{i} \times \mathbf{j} = \mathbf{0} + \mathbf{k} = \mathbf{k}$.

34. Yes. If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then $\mathbf{u} \times (\mathbf{v} - \mathbf{w}) = \mathbf{0}$ and $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = 0$. Suppose now that $\mathbf{v} \neq \mathbf{w}$. Then $\mathbf{u} \times (\mathbf{v} - \mathbf{w}) = \mathbf{0}$ implies that $\mathbf{v} - \mathbf{w} = \mathrm{k}\mathbf{u}$ for some real number $\mathrm{k} \neq 0$. This in turn implies that $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot (\mathrm{k}\mathbf{u}) = \mathrm{k}\,|\mathbf{u}|^2 = 0$, which implies that $\mathbf{u} = \mathbf{0}$. Since $\mathbf{u} \neq \mathbf{0}$, it cannot be true that $\mathbf{v} \neq \mathbf{w}$, so $\mathbf{v} = \mathbf{w}$.

35. $\overrightarrow{AB} = -\mathbf{i} + \mathbf{j}$ and $\overrightarrow{AD} = -\mathbf{i} - \mathbf{j}$ $\Rightarrow$ $\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & -1 & 0 \end{vmatrix} = 2\mathbf{k}$ $\Rightarrow$ area $= \left|\overrightarrow{AB} \times \overrightarrow{AD}\right| = 2$

36. $\overrightarrow{AB} = 7\mathbf{i} + 3\mathbf{j}$ and $\overrightarrow{AD} = 2\mathbf{i} + 5\mathbf{j}$ $\Rightarrow$ $\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 3 & 0 \\ 2 & 5 & 0 \end{vmatrix} = 29\mathbf{k}$ $\Rightarrow$ area $= \left|\overrightarrow{AB} \times \overrightarrow{AD}\right| = 29$

37. $\overrightarrow{AB} = 3\mathbf{i} - 2\mathbf{j}$ and $\overrightarrow{AD} = 5\mathbf{i} + \mathbf{j}$ $\Rightarrow$ $\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 0 \\ 5 & 1 & 0 \end{vmatrix} = 13\mathbf{k}$ $\Rightarrow$ area $= \left|\overrightarrow{AB} \times \overrightarrow{AD}\right| = 13$

38. $\overrightarrow{AB} = 7\mathbf{i} - 4\mathbf{j}$ and $\overrightarrow{AD} = 2\mathbf{i} + 5\mathbf{j}$ $\Rightarrow$ $\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -4 & 0 \\ 2 & 5 & 0 \end{vmatrix} = 43\mathbf{k}$ $\Rightarrow$ area $= \left|\overrightarrow{AB} \times \overrightarrow{AD}\right| = 43$

39. $\overrightarrow{AB} = -2\mathbf{i} + 3\mathbf{j}$ and $\overrightarrow{AC} = 3\mathbf{i} + \mathbf{j}$ $\Rightarrow$ $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 0 \\ 3 & 1 & 0 \end{vmatrix} = -11\mathbf{k}$ $\Rightarrow$ area $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{11}{2}$

40. $\overrightarrow{AB} = 4\mathbf{i} + 4\mathbf{j}$ and $\overrightarrow{AC} = 3\mathbf{i} + 2\mathbf{j}$ $\Rightarrow$ $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 4 & 0 \\ 3 & 2 & 0 \end{vmatrix} = -4\mathbf{k}$ $\Rightarrow$ area $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = 2$

41. $\overrightarrow{AB} = 6\mathbf{i} - 5\mathbf{j}$ and $\overrightarrow{AC} = 11\mathbf{i} - 5\mathbf{j}$ $\Rightarrow$ $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -5 & 0 \\ 11 & -5 & 0 \end{vmatrix} = 25\mathbf{k}$ $\Rightarrow$ area $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{25}{2}$

42. $\overrightarrow{AB} = 16\mathbf{i} - 5\mathbf{j}$ and $\overrightarrow{AC} = 4\mathbf{i} + 4\mathbf{j}$ $\Rightarrow$ $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 16 & -5 & 0 \\ 4 & 4 & 0 \end{vmatrix} = 84\mathbf{k}$ $\Rightarrow$ area $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = 42$

43. If $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j}$, then $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$ and the triangle's area is $\frac{1}{2}|\mathbf{A} \times \mathbf{B}| = \pm \frac{1}{2} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$. The applicable sign is $(+)$ if the acute angle from $\mathbf{A}$ to $\mathbf{B}$ runs counterclockwise in the xy-plane, and $(-)$ if it runs clockwise, because the area must be a nonnegative number.

44. If $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}$, $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j}$, and $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j}$, then the area of the triangle is $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$. Now,

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 - a_1 & b_2 - a_2 & 0 \\ c_1 - a_1 & c_2 - a_2 & 0 \end{vmatrix} = \begin{vmatrix} b_1 - a_1 & b_2 - a_2 \\ c_1 - a_1 & c_2 - a_2 \end{vmatrix} \mathbf{k} \Rightarrow \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

$$= \frac{1}{2}|(b_1 - a_1)(c_2 - a_2) - (c_1 - a_1)(b_2 - a_2)| = \frac{1}{2}|a_1(b_2 - c_2) + a_2(c_1 - b_1) + (b_1c_2 - c_1b_2)|$$

$$= \pm \frac{1}{2}\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}.$$ The applicable sign ensures the area formula gives a nonnegative number.

12.5 LINES AND PLANES IN SPACE

1. The direction $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $P(3, -4, -1) \Rightarrow x = 3 + t, y = -4 + t, z = -1 + t$

2. The direction $\overrightarrow{PQ} = -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $P(1, 2, -1) \Rightarrow x = 1 - 2t, y = 2 - 2t, z = -1 + 2t$

3. The direction $\overrightarrow{PQ} = 5\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}$ and $P(-2, 0, 3) \Rightarrow x = -2 + 5t, y = 5t, z = 3 - 5t$

4. The direction $\overrightarrow{PQ} = -\mathbf{j} - \mathbf{k}$ and $P(1, 2, 0) \Rightarrow x = 1, y = 2 - t, z = -t$

5. The direction $2\mathbf{j} + \mathbf{k}$ and $P(0, 0, 0) \Rightarrow x = 0, y = 2t, z = t$

6. The direction $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $P(3, -2, 1) \Rightarrow x = 3 + 2t, y = -2 - t, z = 1 + 3t$

7. The direction $\mathbf{k}$ and $P(1, 1, 1) \Rightarrow x = 1, y = 1, z = 1 + t$

8. The direction $3\mathbf{i} + 7\mathbf{j} - 5\mathbf{k}$ and $P(2, 4, 5) \Rightarrow x = 2 + 3t, y = 4 + 7t, z = 5 - 5t$

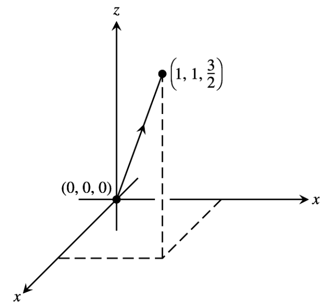
9. The direction $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $P(0, -7, 0) \Rightarrow x = t, y = -7 + 2t, z = 2t$

10. The direction is $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} = -2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $P(2, 3, 0) \Rightarrow x = 2 - 2t, y = 3 + 4t, z = -2t$

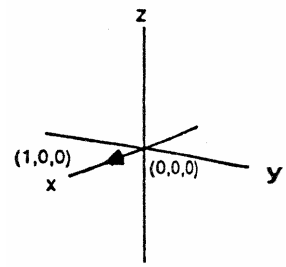
11. The direction $\mathbf{i}$ and $P(0, 0, 0) \Rightarrow x = t, y = 0, z = 0$

12. The direction $\mathbf{k}$ and $P(0, 0, 0) \Rightarrow x = 0, y = 0, z = t$

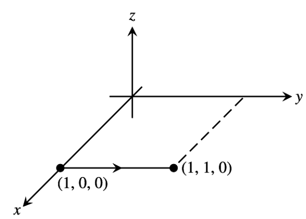
13. The direction $\overrightarrow{PQ} = \mathbf{i} + \mathbf{j} + \frac{3}{2}\mathbf{k}$ and $P(0,0,0) \Rightarrow x = t$, $y = t$, $z = \frac{3}{2}t$, where $0 \le t \le 1$



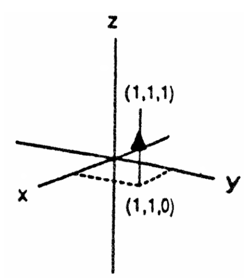
14. The direction $\overrightarrow{PQ} = \mathbf{i}$ and $P(0,0,0) \Rightarrow x = t$, $y = 0$, $z = 0$, where $0 \le t \le 1$



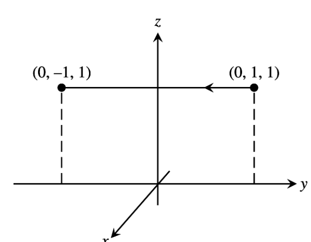
15. The direction $\overrightarrow{PQ} = \mathbf{j}$ and $P(1,1,0) \Rightarrow x = 1$, $y = 1 + t$, $z = 0$, where $-1 \le t \le 0$



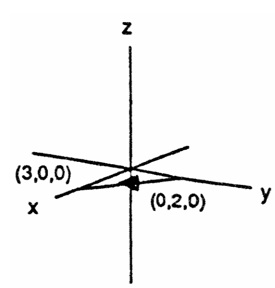
16. The direction $\overrightarrow{PQ} = \mathbf{k}$ and $P(1,1,0) \Rightarrow x = 1$, $y = 1$, $z = t$, where $0 \le t \le 1$



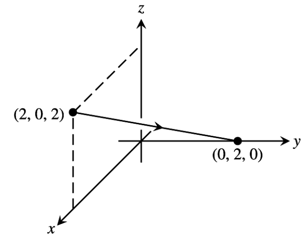
17. The direction $\overrightarrow{PQ} = -2\mathbf{j}$ and $P(0,1,1) \Rightarrow x = 0$, $y = 1 - 2t$, $z = 1$, where $0 \le t \le 1$



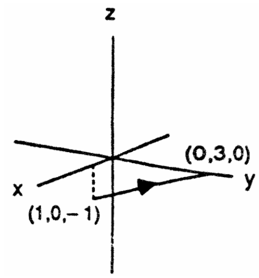
18. The direction $\overrightarrow{PQ} = 3\mathbf{i} - 2\mathbf{j}$ and $P(0,2,0) \Rightarrow x = 3t$, $y = 2 - 2t$, $z = 0$, where $0 \le t \le 1$



19. The direction $\overrightarrow{PQ} = -2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $P(2, 0, 2)$
$\Rightarrow x = 2 - 2t,\ y = 2t,\ z = 2 - 2t$, where $0 \le t \le 1$



20. The direction $\overrightarrow{PQ} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $P(1, 0, -1)$
$\Rightarrow x = 1 - t,\ y = 3t,\ z = -1 + t$, where $0 \le t \le 1$



21. $3(x - 0) + (-2)(y - 2) + (-1)(z + 1) = 0 \Rightarrow 3x - 2y - z = -3$

22. $3(x - 1) + (1)(y + 1) + (1)(z - 3) = 0 \Rightarrow 3x + y + z = 5$

23. $\overrightarrow{PQ} = \mathbf{i} - \mathbf{j} + 3\mathbf{k},\ \overrightarrow{PS} = -\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} \Rightarrow \overrightarrow{PQ} \times \overrightarrow{PS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix} = 7\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$ is normal to the plane
$\Rightarrow 7(x - 2) + (-5)(y - 0) + (-4)(z - 2) = 0 \Rightarrow 7x - 5y - 4z = 6$

24. $\overrightarrow{PQ} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k},\ \overrightarrow{PS} = -3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \Rightarrow \overrightarrow{PQ} \times \overrightarrow{PS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ -3 & 2 & 3 \end{vmatrix} = -\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is normal to the plane
$\Rightarrow (-1)(x - 1) + (-3)(y - 5) + (1)(z - 7) = 0 \Rightarrow x + 3y - z = 9$

25. $\mathbf{n} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k},\ P(2, 4, 5) = (1)(x - 2) + (3)(y - 4) + (4)(z - 5) = 0 \Rightarrow x + 3y + 4z = 34$

26. $\mathbf{n} = \mathbf{i} - 2\mathbf{j} + \mathbf{k},\ P(1, -2, 1) = (1)(x - 1) + (-2)(y + 2) + (1)(z - 1) = 0 \Rightarrow x - 2y + z = 6$

27. $\begin{cases} x = 2t + 1 = s + 2 \\ y = 3t + 2 = 2s + 4 \end{cases} \Rightarrow \begin{cases} 2t - s = 1 \\ 3t - 2s = 2 \end{cases} \Rightarrow \begin{cases} 4t - 2s = 2 \\ 3t - 2s = 2 \end{cases} \Rightarrow t = 0$ and $s = -1$; then $z = 4t + 3 = -4s - 1$
$\Rightarrow 4(0) + 3 = (-4)(-1) - 1$ is satisfied $\Rightarrow$ the lines do intersect when $t = 0$ and $s = -1 \Rightarrow$ the point of intersection is $x = 1$, $y = 2$, and $z = 3$ or $P(1, 2, 3)$. A vector normal to the plane determined by these lines is
$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ 1 & 2 & -4 \end{vmatrix} = -20\mathbf{i} + 12\mathbf{j} + \mathbf{k}$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are directions of the lines $\Rightarrow$ the plane containing the lines is represented by$(-20)(x - 1) + (12)(y - 2) + (1)(z - 3) = 0 \Rightarrow -20x + 12y + z = 7$.

28. $\begin{cases} x = t = 2s + 2 \\ y = -t + 2 = s + 3 \end{cases} \Rightarrow \begin{cases} t - 2s = 2 \\ -t - s = 1 \end{cases} \Rightarrow s = -1$ and $t = 0$; then $z = t + 1 = 5s + 6 \Rightarrow 0 + 1 = 5(-1) + 6$ is satisfied $\Rightarrow$ the lines do intersect when $s = -1$ and $t = 0 \Rightarrow$ the point of intersection is $x = 0$, $y = 2$ and $z = 1$ or $P(0, 2, 1)$. A vector normal to the plane determined by these lines is $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 1 & 5 \end{vmatrix}$

$= -6\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are directions of the lines $\Rightarrow$ the plane containing the lines is represented by $(-6)(x - 0) + (-3)(y - 2) + (3)(z - 1) = 0 \Rightarrow 6x + 3y - 3z = 3$.

29. The cross product of $\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $-4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ has the same direction as the normal to the plane $\Rightarrow \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ -4 & 2 & -2 \end{vmatrix} = 6\mathbf{j} + 6\mathbf{k}$. Select a point on either line, such as $P(-1, 2, 1)$. Since the lines are given to intersect, the desired plane is $0(x + 1) + 6(y - 2) + 6(z - 1) = 0 \Rightarrow 6y + 6z = 18 \Rightarrow y + z = 3$.

30. The cross product of $\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + \mathbf{j} + \mathbf{k}$ has the same direction as the normal to the plane $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$. Select a point on either line, such as $P(0, 3, -2)$. Since the lines are given to intersect, the desired plane is $(-2)(x - 0) + (-2)(y - 3) + (4)(z + 2) = 0 \Rightarrow -2x - 2y + 4z = -14$ $\Rightarrow x + y - 2z = 7$.

31. $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$ is a vector in the direction of the line of intersection of the planes $\Rightarrow 3(x - 2) + (-3)(y - 1) + 3(z + 1) = 0 \Rightarrow 3x - 3y + 3z = 0 \Rightarrow x - y + z = 0$ is the desired plane containing $P_0(2, 1, -1)$

32. A vector normal to the desired plane is $\overrightarrow{P_1P_2} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -2 \\ 4 & -1 & 2 \end{vmatrix} = -2\mathbf{i} - 12\mathbf{j} - 2\mathbf{k}$; choosing $P_1(1, 2, 3)$ as a point on the plane $\Rightarrow (-2)(x - 1) + (-12)(y - 2) + (-2)(z - 3) = 0 \Rightarrow -2x - 12y - 2z = -32 \Rightarrow x + 6y + z = 16$ is the desired plane

33. $S(0, 0, 12)$, $P(0, 0, 0)$ and $\mathbf{v} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \Rightarrow \overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 12 \\ 4 & -2 & 2 \end{vmatrix} = 24\mathbf{i} + 48\mathbf{j} = 24(\mathbf{i} + 2\mathbf{j})$

$\Rightarrow d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{24\sqrt{1+4}}{\sqrt{16+4+4}} = \frac{24\sqrt{5}}{\sqrt{24}} = \sqrt{5 \cdot 24} = 2\sqrt{30}$ is the distance from S to the line

34. $S(0, 0, 0)$, $P(5, 5, -3)$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} \Rightarrow \overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & -5 & 3 \\ 3 & 4 & -5 \end{vmatrix} = 13\mathbf{i} - 16\mathbf{j} - 5\mathbf{k}$

$\Rightarrow d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{169+256+25}}{\sqrt{9+16+25}} = \frac{\sqrt{450}}{\sqrt{50}} = \sqrt{9} = 3$ is the distance from S to the line

35. $S(2, 1, 3)$, $P(2, 1, 3)$ and $\mathbf{v} = 2\mathbf{i} + 6\mathbf{j} \Rightarrow \overrightarrow{PS} \times \mathbf{v} = \mathbf{0} \Rightarrow d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{0}{\sqrt{40}} = 0$ is the distance from S to the line (i.e., the point S lies on the line)

36. $S(2, 1, -1)$, $P(0, 1, 0)$ and $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow \overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 2 & 2 & 2 \end{vmatrix} = 2\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}$

$\Rightarrow d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{4+36+16}}{\sqrt{4+4+4}} = \frac{\sqrt{56}}{\sqrt{12}} = \sqrt{\frac{14}{3}}$ is the distance from S to the line