```
Clear[x, y, f]
f[x_, y_]:= 1 / (x y)
Integrate[f[x, y], {x, 1, 3}, {y, 1, x}]
```

To reverse the order of integration, it is best to first plot the region over which the integration extends. This can be done with ImplicitPlot and all bounds involving both x and y can be plotted. A graphics package must be loaded. Remember to use the double equal sign for the equations of the bounding curves.

```
Clear[x, y, f]
<<Graphics`ImplicitPlot`
ImplicitPlot[{x==2y, x==4, y==0, y==1},{x, 0, 4.1}, {y, 0, 1.1}];
f[x_, y_]:=Exp[x^2]
Integrate[f[x, y], {x, 0, 2}, {y, 0, x/2}] + Integrate[f[x, y], {x, 2, 4}, {y, 0, 1}]
```

To get a numerical value for the result, use the numerical integrator, **NIntegrate**. Verify that this equals the original.

```
Integrate[f[x, y], {x, 0, 2}, {y, 0, x/2}] + NIntegrate[f[x, y], {x, 2, 4}, {y, 0, 1}]
NIntegrate[f[x, y], {y, 0, 1},{x, 2y, 4}]
```

Another way to show a region is with the FilledPlot command. This assumes that functions are given as y = f(x).

```
Clear[x, y, f]
<<Graphics`FilledPlot`
FilledPlot[{x^2, 9},{x, 0,3}, AxesLabels → {x, y}];
f[x_, y_]:= x Cos[y^2]
Integrate[f[x, y], {y, 0, 9}, {x, 0, Sqrt[y]}]
```

67. $\int_1^3 \int_1^x \frac{1}{xy}\, dy\, dx \approx 0.603$

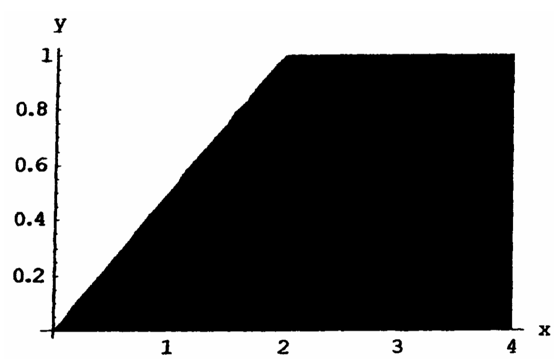
68. $\int_0^1 \int_0^1 e^{-(x^2+y^2)}\, dy\, dx \approx 0.558$

69. $\int_0^1 \int_0^1 \tan^{-1} xy\, dy\, dx \approx 0.233$

70. $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} 3\sqrt{1-x^2-y^2}\, dy\, dx \approx 3.142$

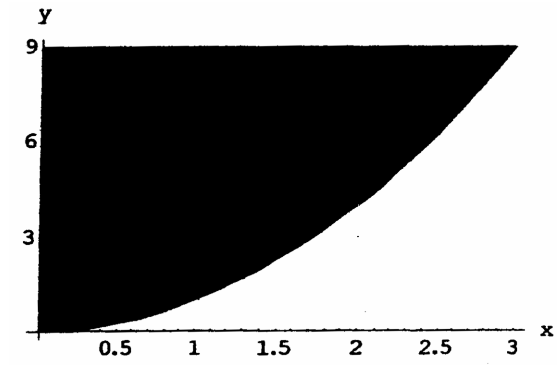
71. Evaluate the integrals:
$\int_0^1 \int_{2y}^4 e^{x^2}\, dx\, dy$
$= \int_0^2 \int_0^{x/2} e^{x^2}\, dy\, dx + \int_2^4 \int_0^1 e^{x^2}\, dy\, dx$
$= -\frac{1}{4} + \frac{1}{4}\left(e^4 - 2\sqrt{\pi}\, \text{erfi}(2) + 2\sqrt{\pi}\, \text{erfi}(4)\right)$
$\approx 1.1494 \times 10^6$

The following graphs was generated using Mathematica.



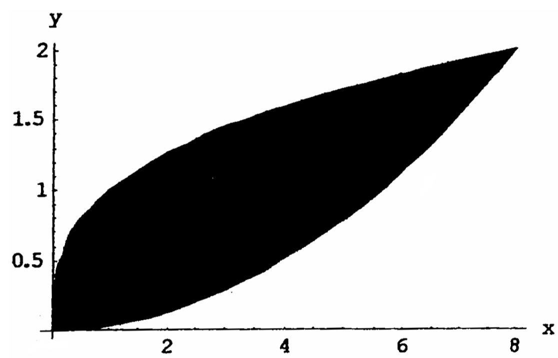
72. Evaluate the integrals:
$\int_0^3 \int_{x^2}^9 x\cos(y^2)\,dy\,dx = \int_0^9 \int_0^{\sqrt{y}} x\cos(y^2)\,dx\,dy$
$= \frac{\sin(81)}{4} \approx -0.157472$

The following graphs was generated using Mathematica.



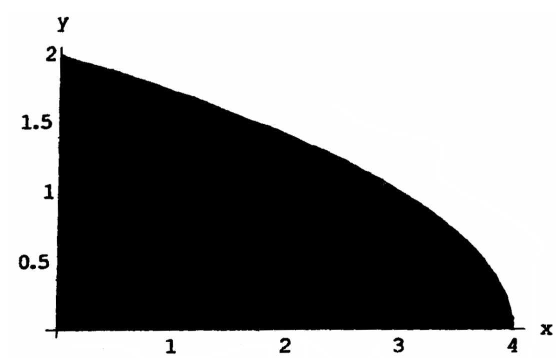

73. Evaluate the integrals:
$\int_0^2 \int_{y^3}^{4\sqrt{2y}} (x^2y - xy^2)\,dx\,dy = \int_0^8 \int_{x^2/32}^{\sqrt[3]{x}} (x^2y - xy^2)\,dy\,dx$
$= \frac{67{,}520}{693} \approx 97.4315$

The following graphs was generated using Mathematica.




74. Evaluate the integrals:
$\int_0^2 \int_0^{4-y^2} e^{xy}\,dx\,dy = \int_0^4 \int_0^{\sqrt{4-x}} e^{xy}\,dy\,dx$
$\approx 20.5648$

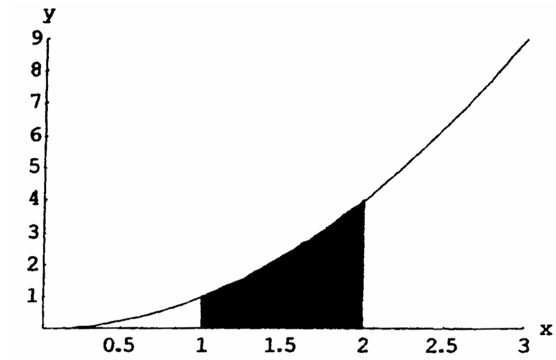
The following graphs was generated using Mathematica.



75. Evaluate the integrals:

$$\int_1^2 \int_0^{x^2} \frac{1}{x+y}\, dy\, dx$$
$$= \int_0^1 \int_1^2 \frac{1}{x+y}\, dx\, dy + \int_1^4 \int_{\sqrt{y}}^2 \frac{1}{x+y}\, dx\, dy$$
$$-1 + \ln\left(\frac{27}{4}\right) \approx 0.909543$$

The following graphs was generated using Mathematica.



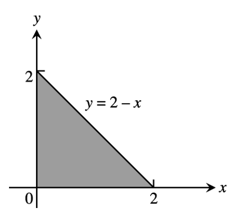

76. Evaluate the integrals:

$$\int_1^2 \int_{y^3}^8 \frac{1}{\sqrt{x^2+y^2}}\, dx\, dy = \int_1^8 \int_1^{\sqrt[3]{x}} \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx$$
$$\approx 0.866649$$

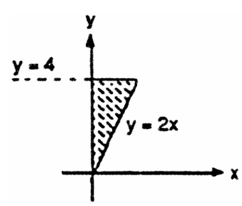
The following graphs was generated using Mathematica.


## 15.2 AREAS, MOMENTS, AND CENTERS OF MASS

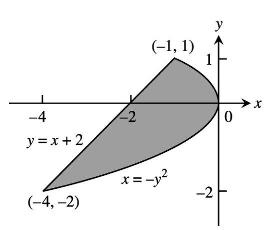
1. $\int_0^2 \int_0^{2-x} dy\, dx = \int_0^2 (2-x)\, dx = \left[2x - \frac{x^2}{2}\right]_0^2 = 2,$
or $\int_0^2 \int_0^{2-y} dx\, dy = \int_0^2 (2-y)\, dy = 2$



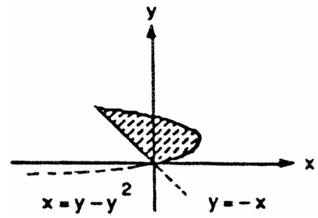

2. $\int_0^2 \int_{2x}^4 dy\, dx = \int_0^2 (4-2x)\, dx = \left[4x - x^2\right]_0^2 = 4,$
or $\int_0^4 \int_0^{y/2} dx\, dy = \int_0^4 \frac{y}{2}\, dy = 4$



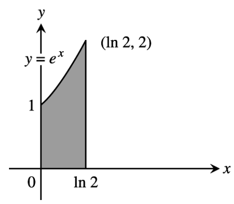

3. $\int_{-2}^1 \int_{y-2}^{-y^2} dx\, dy = \int_{-2}^1 (-y^2 - y + 2)\, dy$
$= \left[-\frac{y^3}{3} - \frac{y^2}{2} + 2y\right]_{-2}^1$
$= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) = \frac{9}{2}$



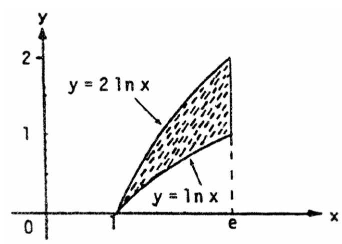
4. $\int_0^2 \int_{-y}^{y-y^2} dx\, dy = \int_0^2 (2y - y^2)\, dy = \left[y^2 - \frac{y^3}{3}\right]_0^2$
$= 4 - \frac{8}{3} = \frac{4}{3}$



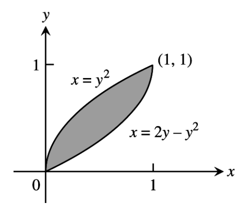

5. $\int_0^{\ln 2} \int_0^{e^x} dy\, dx = \int_0^{\ln 2} e^x\, dx = [e^x]_0^{\ln 2} = 2 - 1 = 1$



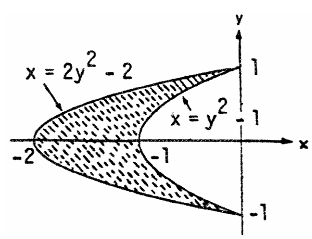

6. $\int_1^e \int_{\ln x}^{2 \ln x} dy\, dx = \int_1^e \ln x\, dx = [x \ln x - x]_1^e$
$= (e - e) - (0 - 1) = 1$



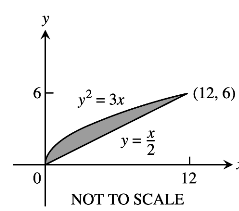

7. $\int_0^1 \int_{y^2}^{2y-y^2} dx\, dy = \int_0^1 (2y - 2y^2)\, dy = \left[y^2 - \frac{2}{3}y^3\right]_0^1$
$= \frac{1}{3}$



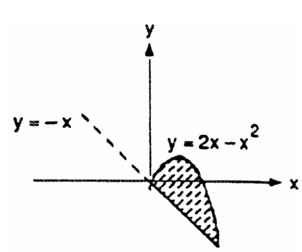

8. $\int_{-1}^1 \int_{2y^2-2}^{y^2-1} dx\, dy = \int_{-1}^1 (y^2 - 1 - 2y^2 + 2)\, dy$
$= \int_{-1}^1 (1 - y^2)\, dy = \left[y - \frac{y^3}{3}\right]_{-1}^1 = \frac{4}{3}$



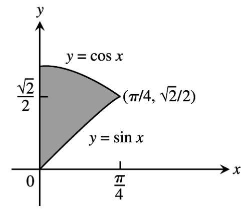

9. $\int_0^6 \int_{y^2/3}^{2y} dx\, dy = \int_0^6 \left(2y - \frac{y^2}{3}\right) dy = \left[y^2 - \frac{y^3}{9}\right]_0^6$
$= 36 - \frac{216}{9} = 12$



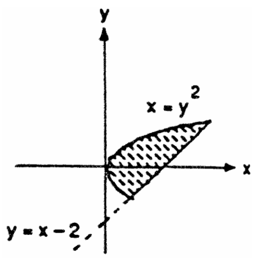

10. $\int_0^3 \int_{-x}^{2x-x^2} dy\, dx = \int_0^3 (3x - x^2)\, dx = \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3$
$= \frac{27}{2} - 9 = \frac{9}{2}$



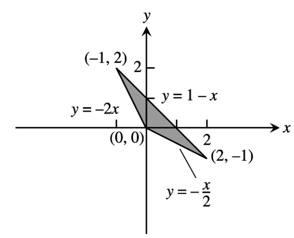
11. $\int_0^{\pi/4} \int_{\sin x}^{\cos x} dy\,dx$
$= \int_0^{\pi/4} (\cos x - \sin x)\,dx = [\sin x + \cos x]_0^{\pi/4}$
$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1) = \sqrt{2} - 1$



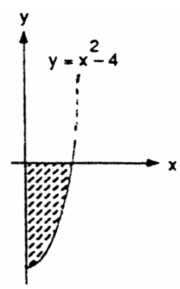

12. $\int_{-1}^{2} \int_{y^2}^{y+2} dx\,dy = \int_{-1}^{2} (y+2-y^2)\,dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2}$
$= \left(2+4-\frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = 5 - \frac{1}{2} = \frac{9}{2}$

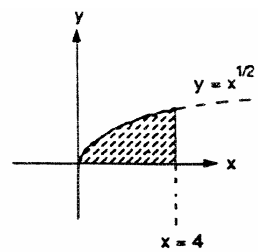



13. $\int_{-1}^{0} \int_{-2x}^{1-x} dy\,dx + \int_0^2 \int_{-x/2}^{1-x} dy\,dx$
$= \int_{-1}^{0} (1+x)\,dx + \int_0^2 \left(1 - \frac{x}{2}\right) dx$
$= \left[x + \frac{x^2}{2}\right]_{-1}^{0} + \left[x - \frac{x^2}{4}\right]_0^2 = -\left(-1 + \frac{1}{2}\right) + (2-1) = \frac{3}{2}$




14. $\int_0^2 \int_{x^2-4}^{0} dy\,dx + \int_0^4 \int_0^{\sqrt{x}} dy\,dx$
$= \int_0^2 (4-x^2)\,dx + \int_0^4 x^{1/2}\,dx$
$= \left[4x - \frac{x^3}{3}\right]_0^2 + \left[\frac{2}{3}x^{3/2}\right]_0^4 = \left(8 - \frac{8}{3}\right) + \frac{16}{3} = \frac{32}{3}$




15. (a) average $= \frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} \sin(x+y)\,dy\,dx = \frac{1}{\pi^2} \int_0^{\pi} [-\cos(x+y)]_0^{\pi}\,dx = \frac{1}{\pi^2} \int_0^{\pi} [-\cos(x+\pi) + \cos x]\,dx$
$= \frac{1}{\pi^2} [-\sin(x+\pi) + \sin x]_0^{\pi} = \frac{1}{\pi^2} [(-\sin 2\pi + \sin \pi) - (-\sin \pi + \sin 0)] = 0$

(b) average $= \frac{1}{\left(\frac{\pi^2}{2}\right)} \int_0^{\pi} \int_0^{\pi/2} \sin(x+y)\,dy\,dx = \frac{2}{\pi^2} \int_0^{\pi} [-\cos(x+y)]_0^{\pi/2}\,dx = \frac{2}{\pi^2} \int_0^{\pi} \left[-\cos\left(x + \frac{\pi}{2}\right) + \cos x\right] dx$
$= \frac{2}{\pi^2} \left[-\sin\left(x + \frac{\pi}{2}\right) + \sin x\right]_0^{\pi} = \frac{2}{\pi^2} \left[\left(-\sin \frac{3\pi}{2} + \sin \pi\right) - \left(-\sin \frac{\pi}{2} + \sin 0\right)\right] = \frac{4}{\pi^2}$

16. average value over the square $= \int_0^1 \int_0^1 xy\,dy\,dx = \int_0^1 \left[\frac{xy^2}{2}\right]_0^1 dx = \int_0^1 \frac{x}{2}\,dx = \frac{1}{4} = 0.25$;

average value over the quarter circle $= \frac{1}{\left(\frac{\pi}{4}\right)} \int_0^1 \int_0^{\sqrt{1-x^2}} xy\,dy\,dx = \frac{4}{\pi} \int_0^1 \left[\frac{xy^2}{2}\right]_0^{\sqrt{1-x^2}} dx$
$= \frac{2}{\pi} \int_0^1 (x - x^3)\,dx = \frac{2}{\pi} \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2\pi} \approx 0.159$. The average value over the square is larger.

17. average height $= \frac{1}{4} \int_0^2 \int_0^2 (x^2 + y^2)\,dy\,dx = \frac{1}{4} \int_0^2 \left[x^2 y + \frac{y^3}{3}\right]_0^2 dx = \frac{1}{4} \int_0^2 \left(2x^2 + \frac{8}{3}\right) dx = \frac{1}{2} \left[\frac{x^3}{3} + \frac{4x}{3}\right]_0^2 = \frac{8}{3}$

18. average $= \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2\ln 2} \int_{\ln 2}^{2\ln 2} \frac{1}{xy}\, dy\, dx = \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2\ln 2} \left[\frac{\ln y}{x}\right]_{\ln 2}^{2\ln 2} dx$
$= \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2\ln 2} \frac{1}{x}(\ln 2 + \ln\ln 2 - \ln\ln 2)\, dx = \left(\frac{1}{\ln 2}\right)\int_{\ln 2}^{2\ln 2} \frac{dx}{x} = \left(\frac{1}{\ln 2}\right)\, [\ln x]_{\ln 2}^{2\ln 2}$
$= \left(\frac{1}{\ln 2}\right)(\ln 2 + \ln\ln 2 - \ln\ln 2) = 1$

19. $M = \int_0^1 \int_x^{2-x^2} 3\, dy\, dx = 3\int_0^1 (2 - x^2 - x)\, dx = \frac{7}{2}$; $M_y = \int_0^1 \int_x^{2-x^2} 3x\, dy\, dx = 3\int_0^1 [xy]_x^{2-x^2} dx$
$= 3\int_0^1 (2x - x^3 - x^2)\, dx = \frac{5}{4}$; $M_x = \int_0^1 \int_x^{2-x^2} 3y\, dy\, dx = \frac{3}{2}\int_0^1 [y^2]_x^{2-x^2} dx = \frac{3}{2}\int_0^1 (4 - 5x^2 + x^4)\, dx = \frac{19}{5}$
$\Rightarrow\ \overline{x} = \frac{5}{14}$ and $\overline{y} = \frac{38}{35}$

20. $M = \delta \int_0^3 \int_0^3 dy\, dx = \delta \int_0^3 3\, dx = 9\delta$; $I_x = \delta \int_0^3 \int_0^3 y^2\, dy\, dx = \delta \int_0^3 \left[\frac{y^3}{3}\right]_0^3 dx = 27\delta$; $R_x = \sqrt{\frac{I_x}{M}} = \sqrt{3}$;
$I_y = \delta \int_0^3 \int_0^3 x^2\, dy\, dx = \delta \int_0^3 [x^2 y]_0^3\, dx = \delta \int_0^3 3x^2\, dx = 27\delta$; $R_y = \sqrt{\frac{I_y}{M}} = \sqrt{3}$

21. $M = \int_0^2 \int_{y^2/2}^{4-y} dx\, dy = \int_0^2 \left(4 - y - \frac{y^2}{2}\right) dy = \frac{14}{3}$; $M_y = \int_0^2 \int_{y^2/2}^{4-y} x\, dx\, dy = \frac{1}{2}\int_0^2 [x^2]_{y^2/2}^{4-y} dy$
$= \frac{1}{2}\int_0^2 \left(16 - 8y + y^2 - \frac{y^4}{4}\right) dy = \frac{128}{15}$; $M_x = \int_0^2 \int_{y^2/2}^{4-y} y\, dx\, dy = \int_0^2 \left(4y - y^2 - \frac{y^3}{2}\right) dy = \frac{10}{3}$
$\Rightarrow\ \overline{x} = \frac{64}{35}$ and $\overline{y} = \frac{5}{7}$

22. $M = \int_0^3 \int_0^{3-x} dy\, dx = \int_0^3 (3 - x)\, dx = \frac{9}{2}$; $M_y = \int_0^3 \int_0^{3-x} x\, dy\, dx = \int_0^3 [xy]_0^{3-x} dx = \int_0^3 (3x - x^2)\, dx = \frac{9}{2}$
$\Rightarrow\ \overline{x} = 1$ and $\overline{y} = 1$, by symmetry

23. $M = 2\int_0^1 \int_0^{\sqrt{1-x^2}} dy\, dx = 2\int_0^1 \sqrt{1-x^2}\, dx = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$; $M_x = 2\int_0^1 \int_0^{\sqrt{1-x^2}} y\, dy\, dx = \int_0^1 [y^2]_0^{\sqrt{1-x^2}} dx$
$= \int_0^1 (1 - x^2)\, dx = \left[x - \frac{x^3}{3}\right]_0^1 = \frac{2}{3}\ \Rightarrow\ \overline{y} = \frac{4}{3\pi}$ and $\overline{x} = 0$, by symmetry

24. $M = \frac{125\delta}{6}$; $M_y = \delta \int_0^5 \int_x^{6x-x^2} x\, dy\, dx = \delta \int_0^5 [xy]_x^{6x-x^2} dx = \delta \int_0^5 (5x^2 - x^3)\, dx = \frac{625\delta}{12}$;
$M_x = \delta \int_0^5 \int_x^{6x-x^2} y\, dy\, dx = \frac{\delta}{2}\int_0^5 [y^2]_x^{6x-x^2} dx = \frac{\delta}{2}\int_0^5 (35x^2 - 12x^3 + x^4)\, dx = \frac{625\delta}{6}\ \Rightarrow\ \overline{x} = \frac{5}{2}$ and $\overline{y} = 5$

25. $M = \int_0^a \int_0^{\sqrt{a^2-x^2}} dy\, dx = \frac{\pi a^2}{4}$; $M_y = \int_0^a \int_0^{\sqrt{a^2-x^2}} x\, dy\, dx = \int_0^a [xy]_0^{\sqrt{a^2-x^2}} dx = \int_0^a x\sqrt{a^2 - x^2}\, dx = \frac{a^3}{3}$
$\Rightarrow\ \overline{x} = \overline{y} = \frac{4a}{3\pi}$, by symmetry

26. $M = \int_0^\pi \int_0^{\sin x} dy\, dx = \int_0^\pi \sin x\, dx = 2$; $M_x = \int_0^\pi \int_0^{\sin x} y\, dy\, dx = \frac{1}{2}\int_0^\pi [y^2]_0^{\sin x} dx = \frac{1}{2}\int_0^\pi \sin^2 x\, dx$
$= \frac{1}{4}\int_0^\pi (1 - \cos 2x)\, dx = \frac{\pi}{4}\ \Rightarrow\ \overline{x} = \frac{\pi}{2}$ and $\overline{y} = \frac{\pi}{8}$

27. $I_x = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y^2\, dy\, dx = \int_{-2}^2 \left[\frac{y^3}{3}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx = \frac{2}{3}\int_{-2}^2 (4 - x^2)^{3/2}\, dx = 4\pi$; $I_y = 4\pi$, by symmetry;
$I_o = I_x + I_y = 8\pi$

28. $I_y = \int_\pi^{2\pi} \int_0^{(\sin^2 x)/x^2} x^2\, dy\, dx = \int_\pi^{2\pi} (\sin^2 x - 0)\, dx = \frac{1}{2}\int_\pi^{2\pi} (1 - \cos 2x)\, dx = \frac{\pi}{2}$

29. $M = \int_{-\infty}^0 \int_0^{e^x} dy\, dx = \int_{-\infty}^0 e^x\, dx = \lim_{b \to -\infty} \int_b^0 e^x\, dx = 1 - \lim_{b \to -\infty} e^b = 1$; $M_y = \int_{-\infty}^0 \int_0^{e^x} x\, dy\, dx = \int_{-\infty}^0 xe^x\, dx$
$= \lim_{b \to -\infty} \int_b^0 xe^x\, dx = \lim_{b \to -\infty} [xe^x - e^x]_b^0 = -1 - \lim_{b \to -\infty} (be^b - e^b) = -1$; $M_x = \int_{-\infty}^0 \int_0^{e^x} y\, dy\, dx$

$= \frac{1}{2}\int_{-\infty}^{0} e^{2x}\,dx = \frac{1}{2}\lim_{b\to-\infty}\int_{b}^{0} e^{2x}\,dx = \frac{1}{4} \Rightarrow \overline{x} = -1$ and $\overline{y} = \frac{1}{4}$

30. $M_y = \int_0^{\infty}\int_0^{e^{-x^2/2}} x\,dy\,dx = \lim_{b\to\infty}\int_0^b xe^{-x^2/2}\,dx = -\lim_{b\to\infty}\left[\frac{1}{e^{x^2/2}} - 1\right]_0^b = 1$

31. $M = \int_0^2\int_{-y}^{y-y^2}(x+y)\,dx\,dy = \int_0^2\left[\frac{x^2}{2} + xy\right]_{-y}^{y-y^2} dy = \int_0^2\left(\frac{y^4}{2} - 2y^3 + 2y^2\right) dy = \left[\frac{y^5}{10} - \frac{y^4}{2} + \frac{2y^3}{3}\right]_0^2 = \frac{8}{15}$;
$I_x = \int_0^2\int_{-y}^{y-y^2} y^2(x+y)\,dx\,dy = \int_0^2\left[\frac{x^2y^2}{2} + xy^3\right]_{-y}^{y-y^2} dy = \int_0^2\left(\frac{y^6}{2} - 2y^5 + 2y^4\right) dy = \frac{64}{105}$;
$R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{8}{7}} = 2\sqrt{\frac{2}{7}}$

32. $M = \int_{-\sqrt{3}/2}^{\sqrt{3}/2}\int_{4y^2}^{\sqrt{12-4y^2}} 5x\,dx\,dy = 5\int_{-\sqrt{3}/2}^{\sqrt{3}/2}\left[\frac{x^2}{2}\right]_{4y^2}^{\sqrt{12-4y^2}} dy = \frac{5}{2}\int_{-\sqrt{3}/2}^{\sqrt{3}/2}(12 - 4y^2 - 16y^4)\,dy = 23\sqrt{3}$

33. $M = \int_0^1\int_x^{2-x}(6x+3y+3)\,dy\,dx = \int_0^1\left[6xy + \frac{3}{2}y^2 + 3y\right]_x^{2-x} dx = \int_0^1(12 - 12x^2)\,dx = 8$;
$M_y = \int_0^1\int_x^{2-x} x(6x+3y+3)\,dy\,dx = \int_0^1(12x - 12x^3)\,dx = 3$; $M_x = \int_0^1\int_x^{2-x} y(6x+3y+3)\,dy\,dx$
$= \int_0^1(14 - 6x - 6x^2 - 2x^3)\,dx = \frac{17}{2} \Rightarrow \overline{x} = \frac{3}{8}$ and $\overline{y} = \frac{17}{16}$

34. $M = \int_0^1\int_{y^2}^{2y-y^2}(y+1)\,dx\,dy = \int_0^1(2y - 2y^3)\,dy = \frac{1}{2}$; $M_x = \int_0^1\int_{y^2}^{2y-y^2} y(y+1)\,dx\,dy = \int_0^1(2y^2 - 2y^4)\,dy = \frac{4}{15}$;
$M_y = \int_0^1\int_{y^2}^{2y-y^2} x(y+1)\,dx\,dy = \int_0^1(2y^2 - 2y^4)\,dy = \frac{4}{15} \Rightarrow \overline{x} = \frac{8}{15}$ and $\overline{y} = \frac{8}{15}$; $I_x = \int_0^1\int_{y^2}^{2y-y^2} y^2(y+1)\,dx\,dy$
$= 2\int_0^1(y^3 - y^5)\,dy = \frac{1}{6}$

35. $M = \int_0^1\int_0^6(x+y+1)\,dx\,dy = \int_0^1(6y+24)\,dy = 27$; $M_x = \int_0^1\int_0^6 y(x+y+1)\,dx\,dy = \int_0^1 y(6y+24)\,dy = 14$;
$M_y = \int_0^1\int_0^6 x(x+y+1)\,dx\,dy = \int_0^1(18y+90)\,dy = 99 \Rightarrow \overline{x} = \frac{11}{3}$ and $\overline{y} = \frac{14}{27}$; $I_y = \int_0^1\int_0^6 x^2(x+y+1)\,dx\,dy$
$= 216\int_0^1\left(\frac{y}{3} + \frac{11}{6}\right) dy = 432$; $R_y = \sqrt{\frac{I_y}{M}} = 4$

36. $M = \int_{-1}^1\int_{x^2}^1(y+1)\,dy\,dx = -\int_{-1}^1\left(\frac{x^4}{2} + x^2 - \frac{3}{2}\right) dx = \frac{32}{15}$; $M_x = \int_{-1}^1\int_{x^2}^1 y(y+1)\,dy\,dx = \int_{-1}^1\left(\frac{5}{6} - \frac{x^6}{3} - \frac{x^4}{2}\right) dx$
$= \frac{48}{35}$; $M_y = \int_{-1}^1\int_{x^2}^1 x(y+1)\,dy\,dx = \int_{-1}^1\left(\frac{3x}{2} - \frac{x^5}{2} - x^3\right) dx = 0 \Rightarrow \overline{x} = 0$ and $\overline{y} = \frac{9}{14}$; $I_y = \int_{-1}^1\int_{x^2}^1 x^2(y+1)\,dy\,dx$
$= \int_{-1}^1\left(\frac{3x^2}{2} - \frac{x^6}{2} - x^4\right) dx = \frac{16}{35}$; $R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{3}{14}}$

37. $M = \int_{-1}^1\int_0^{x^2}(7y+1)\,dy\,dx = \int_{-1}^1\left(\frac{7x^4}{2} + x^2\right) dx = \frac{31}{15}$; $M_x = \int_{-1}^1\int_0^{x^2} y(7y+1)\,dy\,dx = \int_{-1}^1\left(\frac{7x^6}{3} + \frac{x^4}{2}\right) dx = \frac{13}{15}$;
$M_y = \int_{-1}^1\int_0^{x^2} x(7y+1)\,dy\,dx = \int_{-1}^1\left(\frac{7x^5}{2} + x^3\right) dx = 0 \Rightarrow \overline{x} = 0$ and $\overline{y} = \frac{13}{31}$; $I_y = \int_{-1}^1\int_0^{x^2} x^2(7y+1)\,dy\,dx$
$= \int_{-1}^1\left(\frac{7x^6}{2} + x^4\right) dx = \frac{7}{5}$; $R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{21}{31}}$

38. $M = \int_0^{20}\int_{-1}^1\left(1 + \frac{x}{20}\right) dy\,dx = \int_0^{20}\left(2 + \frac{x}{10}\right) dx = 60$; $M_x = \int_0^{20}\int_{-1}^1 y\left(1 + \frac{x}{20}\right) dy\,dx = \int_0^{20}\left[\left(1 + \frac{x}{20}\right)\left(\frac{y^2}{2}\right)\right]_{-1}^1 dx = 0$;
$M_y = \int_0^{20}\int_{-1}^1 x\left(1 + \frac{x}{20}\right) dy\,dx = \int_0^{20}\left(2x + \frac{x^2}{10}\right) dx = \frac{2000}{3} \Rightarrow \overline{x} = \frac{100}{9}$ and $\overline{y} = 0$; $I_x = \int_0^{20}\int_{-1}^1 y^2\left(1 + \frac{x}{20}\right) dy\,dx$
$= \frac{2}{3}\int_0^{20}\left(1 + \frac{x}{20}\right) dx = 20$; $R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{1}{3}}$

39. $M = \int_0^1 \int_{-y}^{y} (y+1)\,dx\,dy = \int_0^1 (2y^2+2y)\,dy = \frac{5}{3}$; $M_x = \int_0^1 \int_{-y}^{y} y(y+1)\,dx\,dy = 2\int_0^1 (y^3+y^2)\,dy = \frac{7}{6}$;
$M_y = \int_0^1 \int_{-y}^{y} x(y+1)\,dx\,dy = \int_0^1 0\,dy = 0 \Rightarrow \bar{x} = 0$ and $\bar{y} = \frac{7}{10}$; $I_x = \int_0^1 \int_{-y}^{y} y^2(y+1)\,dx\,dy = \int_0^1 (2y^4+2y^3)\,dy$
$= \frac{9}{10} \Rightarrow R_x = \sqrt{\frac{I_x}{M}} = \frac{3\sqrt{6}}{10}$; $I_y = \int_0^1 \int_{-y}^{y} x^2(y+1)\,dx\,dy = \frac{1}{3}\int_0^1 (2y^4+2y^3)\,dy = \frac{3}{10} \Rightarrow R_y = \sqrt{\frac{I_y}{M}} = \frac{3\sqrt{2}}{10}$;
$I_o = I_x + I_y = \frac{6}{5} \Rightarrow R_0 = \sqrt{\frac{I_o}{M}} = \frac{3\sqrt{2}}{5}$

40. $M = \int_0^1 \int_{-y}^{y} (3x^2+1)\,dx\,dy = \int_0^1 (2y^3+2y)\,dy = \frac{3}{2}$; $M_x = \int_0^1 \int_{-y}^{y} y\,(3x^2+1)\,dx\,dy = \int_0^1 (2y^4+2y^2)\,dy = \frac{16}{15}$;
$M_y = \int_0^1 \int_{-y}^{y} x\,(3x^2+1)\,dx\,dy = 0 \Rightarrow \bar{x} = 0$ and $\bar{y} = \frac{32}{45}$; $I_x = \int_0^1 \int_{-y}^{y} y^2\,(3x^2+1)\,dx\,dy = \int_0^1 (2y^5+2y^3)\,dy = \frac{5}{6}$
$\Rightarrow R_x = \sqrt{\frac{I_x}{M}} = \frac{\sqrt{5}}{3}$; $I_y = \int_0^1 \int_{-y}^{y} x^2\,(3x^2+1)\,dx\,dy = 2\int_0^1 \left(\frac{3}{5}y^5 + \frac{1}{3}y^3\right) dy = \frac{11}{30} \Rightarrow R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{11}{45}}$;
$I_o = I_x + I_y = \frac{6}{5} \Rightarrow R_o = \sqrt{\frac{I_o}{M}} = \frac{2}{\sqrt{5}}$

41. $\int_{-5}^{5} \int_{-2}^{0} \frac{10{,}000e^y}{1+\frac{|x|}{2}}\,dy\,dx = 10{,}000\,(1-e^{-2}) \int_{-5}^{5} \frac{dx}{1+\frac{|x|}{2}} = 10{,}000\,(1-e^{-2}) \left[\int_{-5}^{0} \frac{dx}{1-\frac{x}{2}} + \int_0^5 \frac{dx}{1+\frac{x}{2}}\right]$
$= 10{,}000\,(1-e^{-2}) \left[-2\ln\left(1-\frac{x}{2}\right)\right]_{-5}^{0} + 10{,}000\,(1-e^{-2}) \left[2\ln\left(1+\frac{x}{2}\right)\right]_0^5$
$= 10{,}000\,(1-e^{-2}) \left[2\ln\left(1+\frac{5}{2}\right)\right] + 10{,}000\,(1-e^{-2}) \left[2\ln\left(1+\frac{5}{2}\right)\right] = 40{,}000\,(1-e^{-2})\ln\left(\frac{7}{2}\right) \approx 43{,}329$

42. $\int_0^1 \int_{y^2}^{2y-y^2} 100(y+1)\,dx\,dy = \int_0^1 [100(y+1)x]_{y^2}^{2y-y^2}\,dy = \int_0^1 100(y+1)\,(2y-2y^2)\,dy = 200\int_0^1 (y-y^3)\,dy$
$= 200\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = (200)\left(\frac{1}{4}\right) = 50$

43. $M = \int_{-1}^{1} \int_0^{a(1-x^2)} dy\,dx = 2a\int_0^1 (1-x^2)\,dx = 2a\left[x - \frac{x^3}{3}\right]_0^1 = \frac{4a}{3}$; $M_x = \int_{-1}^{1} \int_0^{a(1-x^2)} y\,dy\,dx$
$= \frac{2a^2}{2}\int_0^1 (1-2x^2+x^4)\,dx = a^2\left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_0^1 = \frac{8a^2}{15} \Rightarrow \bar{y} = \frac{M_x}{M} = \frac{\left(\frac{8a^2}{15}\right)}{\left(\frac{4a}{3}\right)} = \frac{2a}{5}$. The angle $\theta$ between the x-axis and the line segment from the fulcrum to the center of mass on the y-axis plus 45° must be no more than 90° if the center of mass is to lie on the left side of the line $x = 1 \Rightarrow \theta + \frac{\pi}{4} \le \frac{\pi}{2} \Rightarrow \tan^{-1}\left(\frac{2a}{5}\right) \le \frac{\pi}{4} \Rightarrow a \le \frac{5}{2}$. Thus, if $0 < a \le \frac{5}{2}$, then the appliance will have to be tipped more than 45° to fall over.

44. $f(a) = I_a = \int_0^4 \int_0^2 (y-a)^2\,dy\,dx = \int_0^4 \left[\frac{(2-a)^3}{3} + \frac{a^3}{3}\right] dx = \frac{4}{3}\left[(2-a)^3 + a^3\right]$; thus $f'(a) = 0 \Rightarrow -4(2-a)^2 + 4a^2 = 0 \Rightarrow a^2 - (2-a)^2 = 0 \Rightarrow -4 + 4a = 0 \Rightarrow a = 1$. Since $f''(a) = 8(2-a) + 8a = 16 > 0$, $a = 1$ gives a minimum value of $I_a$.

45. $M = \int_0^1 \int_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}} dy\,dx = \int_0^1 \frac{2}{\sqrt{1-x^2}}\,dx = \left[2\sin^{-1}x\right]_0^1 = 2\left(\frac{\pi}{2} - 0\right) = \pi$; $M_y = \int_0^1 \int_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}} x\,dy\,dx$
$= \int_0^1 \frac{2x}{\sqrt{1-x^2}}\,dx = \left[-2\,(1-x^2)^{1/2}\right]_0^1 = 2 \Rightarrow \bar{x} = \frac{2}{\pi}$ and $\bar{y} = 0$ by symmetry

46. (a) $I = \int_{-L/2}^{L/2} \delta x^2\,dx = \frac{\delta L^3}{12} \Rightarrow R = \sqrt{\frac{\delta L^3}{12} \cdot \frac{1}{\delta L}} = \frac{L}{2\sqrt{3}}$
(b) $I = \int_0^L \delta x^2\,dx = \frac{\delta L^3}{3} \Rightarrow R = \sqrt{\frac{\delta L^3}{3} \cdot \frac{1}{\delta L}} = \frac{L}{\sqrt{3}}$

47. (a) $\frac{1}{2} = M = \int_0^1 \int_{y^2}^{2y-y^2} \delta\,dx\,dy = 2\delta\int_0^1 (y-y^2)\,dy = 2\delta\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = 2\delta\left(\frac{1}{6}\right) = \frac{\delta}{3} \Rightarrow \delta = \frac{3}{2}$

(b) average value $= \dfrac{\int_0^1 \int_{y^2}^{2y-y^2} (y+1)\,dx\,dy}{\int_0^1 \int_{y^2}^{2y-y^2} dx\,dy} = \dfrac{\left(\frac{1}{2}\right)}{\left(\frac{1}{3}\right)} = \frac{3}{2} = \delta$, so the values are the same

48. Let $(x_i, y_i)$ be the location of the weather station in county i for $i = 1, \ldots, 254$. The average temperature in Texas at time $t_0$ is approximately $\dfrac{\sum_{i=1}^{254} T(x_i, y_i)\,\Delta_i A}{A}$, where $T(x_i, y_i)$ is the temperature at time $t_0$ at the weather station in county i, $\Delta_i A$ is the area of county i, and A is the area of Texas.

49. (a) $\bar{x} = \frac{M_y}{M} = 0 \Rightarrow M_y = \iint\limits_R x\delta(x, y)\,dy\,dx = 0$

(b) $I_L = \iint\limits_R (x-h)^2\,\delta(x, y)\,dA = \iint\limits_R x^2\,\delta(x, y)\,dA - \iint\limits_R 2hx\,\delta(x, y)\,dA + \iint\limits_R h^2\,\delta(x, y)\,dA$
$= I_y - 0 + h^2 \iint\limits_R \delta(x, y)\,dA = I_{c.m.} + mh^2$

50. (a) $I_{c.m.} = I_L - mh^2 \Rightarrow I_{x=5/7} = I_y - mh^2 = \frac{39}{5} - 14\left(\frac{5}{7}\right)^2 = \frac{23}{35}$; $I_{y=11/14} = I_x - mh^2 = 12 - 14\left(\frac{11}{14}\right)^2 = \frac{47}{14}$

(b) $I_{x=1} = I_{x=5/7} + mh^2 = \frac{23}{35} + 14\left(\frac{2}{7}\right)^2 = \frac{9}{5}$; $I_{y=2} = I_{y=11/14} + mh^2 = \frac{47}{14} + 14\left(\frac{17}{14}\right)^2 = 24$

51. $M_{x_{p_1 \cup p_2}} = \iint\limits_{R_1} y\,dA_1 + \iint\limits_{R_2} y\,dA_2 = M_{x_1} + M_{x_2} \Rightarrow \bar{x} = \frac{M_{x_1} + M_{x_2}}{m_1 + m_2}$; likewise, $\bar{y} = \frac{M_{y_1} + M_{y_2}}{m_1 + m_2}$;
thus $\mathbf{c} = \bar{x}\,\mathbf{i} + \bar{y}\,\mathbf{j} = \frac{1}{m_1 + m_2}\left[(M_{x_1} + M_{x_2})\,\mathbf{i} + (M_{y_1} + M_{y_2})\,\mathbf{j}\right] = \frac{1}{m_1 + m_2}\left[(m_1\bar{x}_1 + m_2\bar{x}_2)\,\mathbf{i} + (m_1\bar{y}_1 + m_2\bar{y}_2)\,\mathbf{j}\right]$
$= \frac{1}{m_1 + m_2}\left[m_1(\bar{x}_1\mathbf{i} + \bar{y}_1\mathbf{j}) + m_2(\bar{x}_2\mathbf{i} + \bar{y}_2\mathbf{j})\right] = \frac{m_1\mathbf{c}_1 + m_2\mathbf{c}_2}{m_1 + m_2}$

52. From Exercise 51 we have that Pappus's formula is true for $n = 2$. Assume that Pappus's formula is true for $n = k - 1$, i.e., that $\mathbf{c}(k-1) = \dfrac{\sum_{i=1}^{k-1} m_i\mathbf{c}_i}{\sum_{i=1}^{k-1} m_i}$. The first moment about x of k nonoverlapping plates is
$\sum_{i=1}^{k-1}\left(\iint\limits_{R_i} y\,dA_i\right) + \iint\limits_{R_k} y\,dA_k = M_{x_{\mathbf{c}(k-1)}} + M_{x_k} \Rightarrow \bar{x} = \dfrac{M_{x_{\mathbf{c}(k-1)}} + M_{x_k}}{\left(\sum_{i=1}^{k-1} m_i\right) + m_k}$; similarly, $\bar{y} = \dfrac{M_{y_{\mathbf{c}(k-1)}} + M_{y_k}}{\left(\sum_{i=1}^{k-1} m_i\right) + m_k}$;
thus $\mathbf{c}(k) = \bar{x}\,\mathbf{i} + \bar{y}\,\mathbf{j} = \dfrac{1}{\sum_{i=1}^{k} m_i}\left[\left(M_{x_{\mathbf{c}(k-1)}} + M_{x_k}\right)\mathbf{i} + \left(M_{y_{\mathbf{c}(k-1)}} + M_{y_k}\right)\mathbf{j}\right]$
$= \dfrac{1}{\sum_{i=1}^{k} m_i}\left[\left(\left(\sum_{i=1}^{k-1} m_i\right)\bar{x}_{\mathbf{c}} + m_k\bar{x}_k\right)\mathbf{i} + \left(\left(\sum_{i=1}^{k-1} m_i\right)\bar{y}_{\mathbf{c}} + m_k\bar{y}_k\right)\mathbf{j}\right]$
$= \dfrac{1}{\sum_{i=1}^{k} m_i}\left[\left(\sum_{i=1}^{k-1} m_i\right)(\bar{x}_{\mathbf{c}}\,\mathbf{i} + \bar{y}_{\mathbf{c}}\,\mathbf{j}) + m_k(\bar{x}_k\,\mathbf{i} + \bar{y}_k\,\mathbf{j})\right] = \dfrac{\left(\sum_{i=1}^{k-1} m_i\right)\mathbf{c}(k-1) + m_k\mathbf{c}_k}{\sum_{i=1}^{k-1} m_i}$
$= \dfrac{m_1\mathbf{c}_1 + m_2\mathbf{c}_2 + \ldots + m_{k-1}\mathbf{c}_{k-1} + m_k\mathbf{c}_k}{m_1 + m_2 + \ldots + m_{k-1} + m_k}$, and by mathematical induction the statement follows.

53. (a) $\mathbf{c} = \frac{8(\mathbf{i} + 3\mathbf{j}) + 2(3\mathbf{i} + 3.5\,\mathbf{j})}{8+2} = \frac{14\mathbf{i} + 31\mathbf{j}}{10} \Rightarrow \bar{x} = \frac{7}{5}$ and $\bar{y} = \frac{31}{10}$

(b) $\mathbf{c} = \frac{8(\mathbf{i} + 3\mathbf{j}) + 6(5\mathbf{i} + 2\,\mathbf{j})}{14} = \frac{38\mathbf{i} + 36\mathbf{j}}{14} \Rightarrow \bar{x} = \frac{19}{7}$ and $\bar{y} = \frac{18}{7}$

(c) $\mathbf{c} = \frac{2(3\mathbf{i} + 3.5\mathbf{j}) + 6(5\mathbf{i} + 2\,\mathbf{j})}{8} = \frac{36\mathbf{i} + 19\mathbf{j}}{8} \Rightarrow \bar{x} = \frac{9}{2}$ and $\bar{y} = \frac{19}{8}$

(d) $\mathbf{c} = \frac{8(\mathbf{i} + 3\mathbf{j}) + 2(3\mathbf{i} + 3.5\,\mathbf{j}) + 6(5\mathbf{i} + 2\mathbf{j})}{16} = \frac{44\mathbf{i} + 43\mathbf{j}}{16} \Rightarrow \bar{x} = \frac{11}{4}$ and $\bar{y} = \frac{43}{16}$

54. $\mathbf{c} = \frac{15\left(\frac{3}{4}\,\mathbf{i} + 7\mathbf{j}\right) + 48(12\mathbf{i} + \mathbf{j})}{15+48} = \frac{15(3\mathbf{i} + 28\mathbf{j}) + 48(48\mathbf{i} + 4\mathbf{j})}{4\cdot 63} = \frac{2349\,\mathbf{i} + 612\,\mathbf{j}}{4\cdot 63} = \frac{261\,\mathbf{i} + 68\mathbf{j}}{4\cdot 7}$
$\Rightarrow \bar{x} = \frac{261}{28}$ and $\bar{y} = \frac{17}{7}$

55. Place the midpoint of the triangle's base at the origin and above the semicircle. Then the center of mass of the triangle is $\left(0, \frac{h}{3}\right)$, and the center of mass of the disk is $\left(0, -\frac{4a}{3\pi}\right)$ from Exercise 25. From Pappus's formula, $\mathbf{c} = \frac{(ah)\left(\frac{h}{3}\mathbf{j}\right)+\left(\frac{\pi a^2}{2}\right)\left(-\frac{4a}{3\pi}\mathbf{j}\right)}{\left(ah+\frac{\pi a^2}{2}\right)} = \frac{\left(\frac{ah^2-2a^3}{3}\right)\mathbf{j}}{\left(ah+\frac{\pi a^2}{2}\right)}$, so the centroid is on the boundary if $ah^2 - 2a^3 = 0 \Rightarrow h^2 = 2a^2 \Rightarrow h = a\sqrt{2}$. In order for the center of mass to be inside T we must have $ah^2 - 2a^3 > 0$ or $h > a\sqrt{2}$.

56. Place the midpoint of the triangle's base at the origin and above the square. From Pappus's formula, $\mathbf{c} = \frac{\left(\frac{sh}{2}\right)\left(\frac{h}{3}\mathbf{j}\right)+s^2\left(-\frac{s}{2}\mathbf{j}\right)}{\left(\frac{sh}{2}+s^2\right)}$, so the centroid is on the boundary if $\frac{sh^2}{6} - \frac{s^3}{2} = 0 \Rightarrow h^2 - 3s^2 = 0 \Rightarrow h = s\sqrt{3}$.

## 15.3 DOUBLE INTEGRALS IN POLAR FORM

1. $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\,dx = \int_{0}^{\pi}\int_{0}^{1} r\,dr\,d\theta = \frac{1}{2}\int_{0}^{\pi} d\theta = \frac{\pi}{2}$

2. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx = \int_{0}^{2\pi}\int_{0}^{1} r\,dr\,d\theta = \frac{1}{2}\int_{0}^{2\pi} d\theta = \pi$

3. $\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} (x^2+y^2)\,dx\,dy = \int_{0}^{\pi/2}\int_{0}^{1} r^3\,dr\,d\theta = \frac{1}{4}\int_{0}^{\pi/2} d\theta = \frac{\pi}{8}$

4. $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2+y^2)\,dx\,dy = \int_{0}^{2\pi}\int_{0}^{1} r^3\,dr\,d\theta = \frac{1}{4}\int_{0}^{2\pi} d\theta = \frac{\pi}{2}$

5. $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx = \int_{0}^{2\pi}\int_{0}^{a} r\,dr\,d\theta = \frac{a^2}{2}\int_{0}^{2\pi} d\theta = \pi a^2$

6. $\int_{0}^{2}\int_{0}^{\sqrt{4-y^2}} (x^2+y^2)\,dx\,dy = \int_{0}^{\pi/2}\int_{0}^{2} r^3\,dr\,d\theta = 4\int_{0}^{\pi/2} d\theta = 2\pi$

7. $\int_{0}^{6}\int_{0}^{y} x\,dx\,dy = \int_{\pi/4}^{\pi/2}\int_{0}^{6\csc\theta} r^2\cos\theta\,dr\,d\theta = 72\int_{\pi/4}^{\pi/2}\cot\theta\csc^2\theta\,d\theta = -36\left[\cot^2\theta\right]_{\pi/4}^{\pi/2} = 36$

8. $\int_{0}^{2}\int_{0}^{x} y\,dy\,dx = \int_{0}^{\pi/4}\int_{0}^{2\sec\theta} r^2\sin\theta\,dr\,d\theta = \frac{8}{3}\int_{0}^{\pi/4}\tan\theta\sec^2\theta\,d\theta = \frac{4}{3}$

9. $\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\,dy\,dx = \int_{\pi}^{3\pi/2}\int_{0}^{1} \frac{2r}{1+r}\,dr\,d\theta = 2\int_{\pi}^{3\pi/2}\int_{0}^{1}\left(1-\frac{1}{1+r}\right)dr\,d\theta = 2\int_{\pi}^{3\pi/2}(1-\ln 2)\,d\theta$
$= (1-\ln 2)\pi$

10. $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{0} \frac{4\sqrt{x^2+y^2}}{1+x^2+y^2}\,dx\,dy = \int_{\pi/2}^{3\pi/2}\int_{0}^{1} \frac{4r^2}{1+r^2}\,dr\,d\theta = 4\int_{\pi/2}^{3\pi/2}\int_{0}^{1}\left(1-\frac{1}{1+r^2}\right)dr\,d\theta = 4\int_{\pi/2}^{3\pi/2}\left(1-\frac{\pi}{4}\right)d\theta$
$= 4\pi - \pi^2$

11. $\int_{0}^{\ln 2}\int_{0}^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\,dx\,dy = \int_{0}^{\pi/2}\int_{0}^{\ln 2} re^r\,dr\,d\theta = \int_{0}^{\pi/2}(2\ln 2 - 1)\,d\theta = \frac{\pi}{2}(2\ln 2 - 1)$

12. $\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} e^{-(x^2+y^2)}\,dy\,dx = \int_{0}^{\pi/2}\int_{0}^{1} re^{-r^2}\,dr\,d\theta = -\frac{1}{2}\int_{0}^{\pi/2}\left(\frac{1}{e}-1\right)d\theta = \frac{\pi(e-1)}{4e}$

13. $\int_{0}^{2}\int_{0}^{\sqrt{1-(x-1)^2}} \frac{x+y}{x^2+y^2}\,dy\,dx = \int_{0}^{\pi/2}\int_{0}^{2\cos\theta} \frac{r(\cos\theta+\sin\theta)}{r^2}\,r\,dr\,d\theta = \int_{0}^{\pi/2}(2\cos^2\theta + 2\sin\theta\cos\theta)\,d\theta$
$= \left[\theta + \frac{\sin 2\theta}{2} + \sin^2\theta\right]_{0}^{\pi/2} = \frac{\pi+2}{2} = \frac{\pi}{2} + 1$

14. $\int_0^2 \int_{-\sqrt{1-(y-1)^2}}^{0} xy^2\,dx\,dy = \int_{\pi/2}^{\pi} \int_0^{2\sin\theta} \sin^2\theta\cos\theta\, r^4\,dr\,d\theta = \frac{32}{5}\int_{\pi/2}^{\pi} \sin^7\theta\cos\theta\,d\theta = \frac{4}{5}\left[\sin^8\theta\right]_{\pi/2}^{\pi} = -\frac{4}{5}$

15. $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1)\,dx\,dy = 4\int_0^{\pi/2}\int_0^1 \ln(r^2+1)\,r\,dr\,d\theta = 2\int_0^{\pi/2}(\ln 4 - 1)\,d\theta = \pi(\ln 4 - 1)$

16. $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\,dy\,dx = 4\int_0^{\pi/2}\int_0^1 \frac{2r}{(1+r^2)^2}\,dr\,d\theta = 4\int_0^{\pi/2}\left[-\frac{1}{1+r^2}\right]_0^1 d\theta = 2\int_0^{\pi/2} d\theta = \pi$

17. $\int_0^{\pi/2}\int_0^{2\sqrt{2-\sin 2\theta}} r\,dr\,d\theta = 2\int_0^{\pi/2}(2-\sin 2\theta)\,d\theta = 2(\pi - 1)$

18. $A = 2\int_0^{\pi/2}\int_1^{1+\cos\theta} r\,dr\,d\theta = \int_0^{\pi/2}(2\cos\theta + \cos^2\theta)\,d\theta = \frac{8+\pi}{4}$

19. $A = 2\int_0^{\pi/6}\int_0^{12\cos 3\theta} r\,dr\,d\theta = 144\int_0^{\pi/6}\cos^2 3\theta\,d\theta = 12\pi$

20. $A = \int_0^{2\pi}\int_0^{4\theta/3} r\,dr\,d\theta = \frac{8}{9}\int_0^{2\pi}\theta^2\,d\theta = \frac{64\pi^3}{27}$

21. $A = \int_0^{\pi/2}\int_0^{1+\sin\theta} r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left(\frac{3}{2} + 2\sin\theta - \frac{\cos 2\theta}{2}\right)d\theta = \frac{3\pi}{8} + 1$

22. $A = 4\int_0^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi/2}\left(\frac{3}{2} - 2\cos\theta + \frac{\cos 2\theta}{2}\right)d\theta = \frac{3\pi}{2} - 4$

23. $M_x = \int_0^{\pi}\int_0^{1-\cos\theta} 3r^2\sin\theta\,dr\,d\theta = \int_0^{\pi}(1-\cos\theta)^3\sin\theta\,d\theta = 4$

24. $I_x = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2[k(x^2+y^2)]\,dy\,dx = k\int_0^{2\pi}\int_0^a r^5\sin^2\theta\,dr\,d\theta = \frac{ka^6}{6}\int_0^{2\pi}\frac{1-\cos 2\theta}{2}\,d\theta = \frac{ka^6\pi}{6}$;

$I_o = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} k(x^2+y^2)^2\,dy\,dx = k\int_0^{2\pi}\int_0^a r^5\,dr\,d\theta = \frac{ka^6}{6}\int_0^{2\pi}d\theta = \frac{ka^6\pi}{3}$

25. $M = 2\int_{\pi/6}^{\pi/2}\int_3^{6\sin\theta} dr\,d\theta = 2\int_{\pi/6}^{\pi/2}(6\sin\theta - 3)\,d\theta = 6\left[-2\cos\theta - \theta\right]_{\pi/6}^{\pi/2} = 6\sqrt{3} - 2\pi$

26. $I_o = \int_{\pi/2}^{3\pi/2}\int_1^{1-\cos\theta} r\,dr\,d\theta = \frac{1}{2}\int_{\pi/2}^{3\pi/2}(\cos^2\theta - 2\cos\theta)\,d\theta = \frac{1}{2}\left[\frac{\sin 2\theta}{4} + \frac{\theta}{2} - 2\sin\theta\right]_{\pi/2}^{3\pi/2} = 2 + \frac{\pi}{4}$

27. $M = 2\int_0^{\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta = \int_0^{\pi}(1+\cos\theta)^2\,d\theta = \frac{3\pi}{2}$; $M_y = 2\int_0^{\pi}\int_0^{1+\cos\theta} r^2\cos\theta\,dr\,d\theta$

$= 2\int_0^{\pi}\left(\frac{4\cos\theta}{3} + \frac{15}{24} + \cos 2\theta - \sin^2\theta\cos\theta + \frac{\cos 4\theta}{4}\right)d\theta = \frac{5\pi}{4} \Rightarrow \bar{x} = \frac{5}{6}$ and $\bar{y} = 0$, by symmetry

28. $I_o = \int_0^{2\pi}\int_0^{1+\cos\theta} r^3\,dr\,d\theta = \frac{1}{4}\int_0^{2\pi}(1+\cos\theta)^4\,d\theta = \frac{35\pi}{16}$

29. average $= \frac{4}{\pi a^2}\int_0^{\pi/2}\int_0^a r\sqrt{a^2-r^2}\,dr\,d\theta = \frac{4}{3\pi a^2}\int_0^{\pi/2} a^3\,d\theta = \frac{2a}{3}$

30. average $= \frac{4}{\pi a^2}\int_0^{\pi/2}\int_0^a r^2\,dr\,d\theta = \frac{4}{3\pi a^2}\int_0^{\pi/2} a^3\,d\theta = \frac{2a}{3}$

31. average $= \frac{1}{\pi a^2}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a r^2\,dr\,d\theta = \frac{a}{3\pi}\int_0^{2\pi}d\theta = \frac{2a}{3}$

32. average $= \frac{1}{\pi} \iint\limits_R [(1-x)^2 + y^2]\, dy\, dx = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 [(1 - r\cos\theta)^2 + r^2 \sin^2\theta]\, r\, dr\, d\theta$
$= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 (r^3 - 2r^2 \cos\theta + r)\, dr\, d\theta = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{3}{4} - \frac{2\cos\theta}{3}\right) d\theta = \frac{1}{\pi} \left[\frac{3}{4}\theta - \frac{2\sin\theta}{3}\right]_0^{2\pi} = \frac{3}{2}$

33. $\int_0^{2\pi} \int_1^{\sqrt{e}} \left(\frac{\ln r^2}{r}\right) r\, dr\, d\theta = \int_0^{2\pi} \int_1^{\sqrt{e}} 2\ln r\, dr\, d\theta = 2\int_0^{2\pi} [r\ln r - r]_1^{e^{1/2}}\, d\theta = 2\int_0^{2\pi} \sqrt{e}\left[\left(\frac{1}{2} - 1\right) + 1\right] d\theta = 2\pi\left(2 - \sqrt{e}\right)$

34. $\int_0^{2\pi} \int_1^{e} \left(\frac{\ln r^2}{r}\right) dr\, d\theta = \int_0^{2\pi} \int_1^{e} \left(\frac{2\ln r}{r}\right) dr\, d\theta = \int_0^{2\pi} [(\ln r)^2]_1^e\, d\theta = \int_0^{2\pi} d\theta = 2\pi$

35. $V = 2\int_0^{\pi/2} \int_1^{1+\cos\theta} r^2 \cos\theta\, dr\, d\theta = \frac{2}{3} \int_0^{\pi/2} (3\cos^2\theta + 3\cos^3\theta + \cos^4\theta)\, d\theta$
$= \frac{2}{3}\left[\frac{15\theta}{8} + \sin 2\theta + 3\sin\theta - \sin^3\theta + \frac{\sin 4\theta}{32}\right]_0^{\pi/2} = \frac{4}{3} + \frac{5\pi}{8}$

36. $V = 4\int_0^{\pi/4} \int_0^{\sqrt{2\cos 2\theta}} r\sqrt{2 - r^2}\, dr\, d\theta = -\frac{4}{3}\int_0^{\pi/4} \left[(2 - 2\cos 2\theta)^{3/2} - 2^{3/2}\right] d\theta$
$= \frac{2\pi\sqrt{2}}{3} - \frac{32}{3}\int_0^{\pi/4} (1 - \cos^2\theta)\sin\theta\, d\theta = \frac{2\pi\sqrt{2}}{3} - \frac{32}{3}\left[\frac{\cos^3\theta}{3} - \cos\theta\right]_0^{\pi/4} = \frac{6\pi\sqrt{2} + 40\sqrt{2} - 64}{9}$

37. (a) $I^2 = \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)}\, dx\, dy = \int_0^{\pi/2} \int_0^{\infty} \left(e^{-r^2}\right) r\, dr\, d\theta = \int_0^{\pi/2} \left[\lim_{b\to\infty} \int_0^b re^{-r^2}\, dr\right] d\theta$
$= -\frac{1}{2}\int_0^{\pi/2} \lim_{b\to\infty} \left(e^{-b^2} - 1\right) d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2}$
(b) $\lim_{x\to\infty} \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}}\, dt = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2}\, dt = \left(\frac{2}{\sqrt{\pi}}\right)\left(\frac{\sqrt{\pi}}{2}\right) = 1$, from part (a)

38. $\int_0^{\infty} \int_0^{\infty} \frac{1}{(1+x^2+y^2)^2}\, dx\, dy = \int_0^{\pi/2} \int_0^{\infty} \frac{r}{(1+r^2)^2}\, dr\, d\theta = \frac{\pi}{2} \lim_{b\to\infty} \int_0^b \frac{r}{(1+r^2)^2}\, dr = \frac{\pi}{4} \lim_{b\to\infty} \left[-\frac{1}{1+r^2}\right]_0^b$
$= \frac{\pi}{4} \lim_{b\to\infty} \left(1 - \frac{1}{1+b^2}\right) = \frac{\pi}{4}$

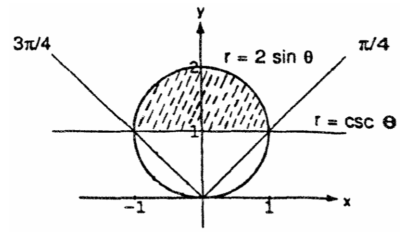
39. Over the disk $x^2 + y^2 \le \frac{3}{4}$: $\iint\limits_R \frac{1}{1-x^2-y^2}\, dA = \int_0^{2\pi} \int_0^{\sqrt{3}/2} \frac{r}{1-r^2}\, dr\, d\theta = \int_0^{2\pi} \left[-\frac{1}{2}\ln(1-r^2)\right]_0^{\sqrt{3}/2} d\theta$
$= \int_0^{2\pi} \left(-\frac{1}{2}\ln\frac{1}{4}\right) d\theta = (\ln 2)\int_0^{2\pi} d\theta = \pi\ln 4$
Over the disk $x^2 + y^2 \le 1$: $\iint\limits_R \frac{1}{1-x^2-y^2}\, dA = \int_0^{2\pi} \int_0^1 \frac{r}{1-r^2}\, dr\, d\theta = \int_0^{2\pi} \left[\lim_{a\to 1^-} \int_0^a \frac{r}{1-r^2}\, dr\right] d\theta$
$= \int_0^{2\pi} \lim_{a\to 1^-} \left[-\frac{1}{2}\ln(1-a^2)\right] d\theta = 2\pi \cdot \lim_{a\to 1^-} \left[-\frac{1}{2}\ln(1-a^2)\right] = 2\pi \cdot \infty$, so the integral does not exist over $x^2 + y^2 \le 1$

40. The area in polar coordinates is given by $A = \int_{\alpha}^{\beta} \int_0^{f(\theta)} r\, dr\, d\theta = \int_{\alpha}^{\beta} \left[\frac{r^2}{2}\right]_0^{f(\theta)} d\theta = \frac{1}{2}\int_{\alpha}^{\beta} f^2(\theta)\, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} r^2\, d\theta$,
where $r = f(\theta)$

41. average $= \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a [(r\cos\theta - h)^2 + r^2\sin^2\theta]\, r\, dr\, d\theta = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a (r^3 - 2r^2 h\cos\theta + rh^2)\, dr\, d\theta$
$= \frac{1}{\pi a^2} \int_0^{2\pi} \left(\frac{a^4}{4} - \frac{2a^3 h\cos\theta}{3} + \frac{a^2 h^2}{2}\right) d\theta = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{a^2}{4} - \frac{2ah\cos\theta}{3} + \frac{h^2}{2}\right) d\theta = \frac{1}{\pi} \left[\frac{a^2\theta}{4} - \frac{2ah\sin\theta}{3} + \frac{h^2\theta}{2}\right]_0^{2\pi}$
$= \frac{1}{2}(a^2 + 2h^2)$

42. (a) $A = \int_{\pi/4}^{3\pi/4} \int_{\csc\theta}^{2\sin\theta} r\,dr\,d\theta = \frac{1}{2}\int_{\pi/4}^{3\pi/4} (4\sin^2\theta - \csc^2\theta)\,d\theta$
$= \frac{1}{2}\left[2\theta - \sin 2\theta + \cot\theta\right]_{\pi/4}^{3\pi/4} = \frac{\pi}{2}$

(b) $V = 2\pi\overline{y}A = 2\pi\left(\frac{3\pi+4}{3\pi}\right)\left(\frac{\pi}{2}\right) = \pi^2 + \frac{4\pi}{3}$




44-46. Example CAS commands:

Maple:

```
f := (x,y) -> y/(x^2+y^2);
    a,b := 0,1;
    f1 := x -> x;
    f2 := x -> 1;
    plot3d( f(x,y), y=f1(x)..f2(x), x=a..b, axes=boxed, style=patchnogrid, shading=zhue, orientation=[0,180], title="#43(a)
        (Section 15.3)" );                                             # (a)
    q1 := eval( x=a, [x=r*cos(theta),y=r*sin(theta)] );               # (b)
    q2 := eval( x=b, [x=r*cos(theta),y=r*sin(theta)] );
    q3 := eval( y=f1(x), [x=r*cos(theta),y=r*sin(theta)] );
    q4 := eval( y=f2(x), [x=r*cos(theta),y=r*sin(theta)] );
    theta1 := solve( q3, theta );
    theta2 := solve( q1, theta );
    r1 := 0;
    r2 := solve( q4, r );
    plot3d(0,r=r1..r2, theta=theta1..theta2, axes=boxed, style=patchnogrid, shading=zhue, orientation=[-90,0],
        title="#43(c) (Section 15.3)" );
    fP := simplify(eval( f(x,y), [x=r*cos(theta),y=r*sin(theta)] ));  # (d)
    q5 := Int( Int( fP*r, r=r1..r2 ), theta=theta1..theta2 );
    value( q5 );
```

Mathematica: (functions and bounds will vary)

For 43 and 44, begin by drawing the region of integration with the **FilledPlot** command.

```
Clear[x, y, r, t]
<<Graphics`FilledPlot`
FilledPlot[{x, 1}, {x, 0, 1}, AspectRatio → 1, AxesLabel → {x,y}];
```

The picture demonstrates that r goes from 0 to the line y=1 or r = 1/ Sin[t], while t goes from $\pi$/4 to $\pi$/2.

```
f:= y / (x^2 + y^2)
topolar={x → r Cos[t], y → r Sin[t]};
fp= f/.topolar //Simplify
Integrate[r fp, {t, π/4, π/2}, {r, 0, 1/Sin[t]}]
```

For 45 and 46, drawing the region of integration with the ImplicitPlot command.

```
Clear[x, y]
<<Graphics`ImplicitPlot`
ImplicitPlot[{x==y, x==2 − y, y==0, y==1}, {x, 0, 2.1}, {y, 0, 1.1}];
```

The picture shows that as t goes from 0 to $\pi$/4, r goes from 0 to the line x=2 − y. **Solve** will find the bound for r.

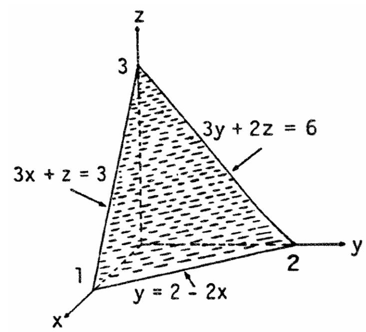
```
bdr=Solve[r Cos[t]==2 − r Sin[t], r]//Simplify
f:=Sqrt[x + y]
topolar={x → r Cos[t], y → r Sin[t]};
fp= f/.topolar //Simplify
Integrate[r fp, {t, 0, π/4}, {r, 0, bdr[[1, 1, 2]]}]
```

**15.4 TRIPLE INTEGRALS IN RECTANGULAR COORDINATES**

1. $\int_0^1 \int_0^{1-x} \int_{x+z}^1 F(x, y, z)\, dy\, dz\, dx = \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy\, dz\, dx = \int_0^1 \int_0^{1-x} (1 - x - z)\, dz\, dx$
$= \int_0^1 \left[(1 - x) - x(1 - x) - \frac{(1-x)^2}{2}\right] dx = \int_0^1 \frac{(1-x)^2}{2} dx = \left[-\frac{(1-x)^3}{6}\right]_0^1 = \frac{1}{6}$

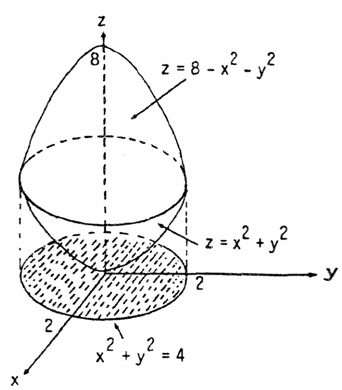
2. $\int_0^1 \int_0^2 \int_0^3 dz\, dy\, dx = \int_0^1 \int_0^2 3\, dy\, dx = \int_0^1 6\, dx = 6$, $\int_0^2 \int_0^1 \int_0^3 dz\, dx\, dy$, $\int_0^3 \int_0^2 \int_0^1 dx\, dy\, dz$, $\int_0^2 \int_0^3 \int_0^1 dx\, dz\, dy$,
$\int_0^3 \int_0^1 \int_0^2 dy\, dx\, dz$, $\int_0^1 \int_0^3 \int_0^2 dy\, dz\, dx$

3. $\int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} dz\, dy\, dx$
$= \int_0^1 \int_0^{2-2x} \left(3 - 3x - \frac{3}{2} y\right) dy\, dx$
$= \int_0^1 \left[3(1 - x) \cdot 2(1 - x) - \frac{3}{4} \cdot 4(1 - x)^2\right] dx$
$= 3 \int_0^1 (1 - x)^2\, dx = \left[-(1 - x)^3\right]_0^1 = 1,$
$\int_0^2 \int_0^{1-y/2} \int_0^{3-3x-3y/2} dz\, dx\, dy$, $\int_0^1 \int_0^{3-3x} \int_0^{2-2x-2z/3} dy\, dz\, dx$,
$\int_0^3 \int_0^{1-z/3} \int_0^{2-2x-2z/3} dy\, dx\, dz$, $\int_0^2 \int_0^{3-3y/2} \int_0^{1-y/2-z/3} dx\, dz\, dy$,
$\int_0^3 \int_0^{2-2z/3} \int_0^{1-y/2-z/3} dx\, dy\, dz$

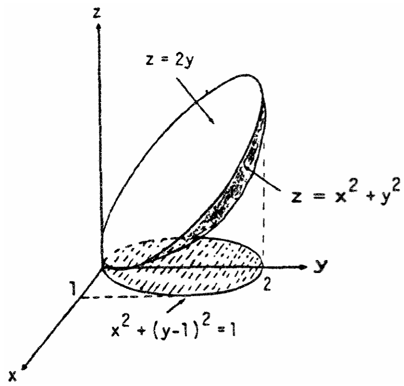



4. $\int_0^2 \int_0^3 \int_0^{\sqrt{4-x^2}} dz\, dy\, dx = \int_0^2 \int_0^3 \sqrt{4 - x^2}\, dy\, dx = \int_0^2 3\sqrt{4 - x^2}\, dx = \frac{3}{2}\left[x\sqrt{4 - x^2} + 4 \sin^{-1} \frac{x}{2}\right]_0^2 = 6 \sin^{-1} 1 = 3\pi,$
$\int_0^3 \int_0^2 \int_0^{\sqrt{4-x^2}} dz\, dx\, dy$, $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^3 dy\, dz\, dx$, $\int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^3 dy\, dx\, dz$, $\int_0^2 \int_0^3 \int_0^{\sqrt{4-z^2}} dx\, dy\, dz$, $\int_0^3 \int_0^2 \int_0^{\sqrt{4-z^2}} dx\, dz\, dy$

5. $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{8-x^2-y^2} dz\, dy\, dx = 4 \int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+y^2}^{8-x^2-y^2} dz\, dy\, dx$
$= 4 \int_0^2 \int_0^{\sqrt{4-x^2}} \left[8 - 2\left(x^2 + y^2\right)\right] dy\, dx$
$= 8 \int_0^2 \int_0^{\sqrt{4-x^2}} \left(4 - x^2 - y^2\right) dy\, dx$
$= 8 \int_0^{\pi/2} \int_0^2 \left(4 - r^2\right) r\, dr\, d\theta = 8 \int_0^{\pi/2} \left[2r^2 - \frac{r^4}{4}\right]_0^2 d\theta$
$= 32 \int_0^{\pi/2} d\theta = 32\left(\frac{\pi}{2}\right) = 16\pi,$
$\int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{x^2+y^2}^{8-x^2-y^2} dz\, dx\, dy,$
$\int_{-2}^2 \int_{y^2}^4 \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\, dz\, dy + \int_{-2}^2 \int_4^{8-y^2} \int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} dx\, dz\, dy,$
$\int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\, dy\, dz + \int_4^8 \int_{-\sqrt{8-z}}^{\sqrt{8-z}} \int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} dx\, dy\, dz,$ $\int_{-2}^2 \int_{x^2}^4 \int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dy\, dz\, dx + \int_{-2}^2 \int_4^{8-x^2} \int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dy\, dz\, dx,$
$\int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dy\, dx\, dz + \int_4^8 \int_{-\sqrt{8-z}}^{\sqrt{8-z}} \int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dy\, dx\, dz$



6. The projection of D onto the xy-plane has the boundary $x^2 + y^2 = 2y \Rightarrow x^2 + (y-1)^2 = 1$, which is a circle. Therefore the two integrals are:
$\int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \int_{x^2+y^2}^{2y} dz\,dx\,dy$ and $\int_{-1}^{1} \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} \int_{x^2+y^2}^{2y} dz\,dy\,dx$




7. $\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2)\,dz\,dy\,dx = \int_0^1 \int_0^1 \left(x^2 + y^2 + \frac{1}{3}\right) dy\,dx = \int_0^1 \left(x^2 + \frac{2}{3}\right) dx = 1$

8. $\int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy = \int_0^{\sqrt{2}} \int_0^{3y} (8 - 2x^2 - 4y^2)\,dx\,dy = \int_0^{\sqrt{2}} \left[8x - \frac{2}{3}x^3 - 4xy^2\right]_0^{3y} dy$
$= \int_0^{\sqrt{2}} (24y - 18y^3 - 12y^3)\,dy = \left[12y^2 - \frac{15}{2}y^4\right]_0^{\sqrt{2}} = 24 - 30 = -6$

9. $\int_1^e \int_1^e \int_1^e \frac{1}{xyz}\,dx\,dy\,dz = \int_1^e \int_1^e \left[\frac{\ln x}{yz}\right]_1^e dy\,dz = \int_1^e \int_1^e \frac{1}{yz}\,dy\,dz = \int_1^e \left[\frac{\ln y}{z}\right]_1^e dz = \int_1^e \frac{1}{z}\,dz = 1$

10. $\int_0^1 \int_0^{3-3x} \int_0^{3-3x-y} dz\,dy\,dx = \int_0^1 \int_0^{3-3x} (3 - 3x - y)\,dy\,dx = \int_0^1 \left[(3-3x)^2 - \frac{1}{2}(3-3x)^2\right] dx = \frac{9}{2}\int_0^1 (1-x)^2\,dx$
$= -\frac{3}{2}\left[(1-x)^3\right]_0^1 = \frac{3}{2}$

11. $\int_0^1 \int_0^{\pi} \int_0^{\pi} y \sin z\,dx\,dy\,dz = \int_0^1 \int_0^{\pi} \pi y \sin z\,dy\,dz = \frac{\pi^3}{2} \int_0^1 \sin z\,dz = \frac{\pi^3}{2}(1 - \cos 1)$

12. $\int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (x + y + z)\,dy\,dx\,dz = \int_{-1}^1 \int_{-1}^1 \left[xy + \frac{1}{2}y^2 + zy\right]_{-1}^1 dx\,dz = \int_{-1}^1 \int_{-1}^1 (2x + 2z)\,dx\,dz = \int_{-1}^1 \left[x^2 + 2zx\right]_{-1}^1 dz$
$= \int_{-1}^1 4z\,dz = 0$

13. $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} dz\,dy\,dx = \int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{9-x^2}\,dy\,dx = \int_0^3 (9 - x^2)\,dx = \left[9x - \frac{x^3}{3}\right]_0^3 = 18$

14. $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_0^{2x+y} dz\,dx\,dy = \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (2x + y)\,dx\,dy = \int_0^2 \left[x^2 + xy\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy = \int_0^2 (4 - y^2)^{1/2}(2y)\,dy$
$= \left[-\frac{2}{3}(4 - y^2)^{3/2}\right]_0^2 = \frac{2}{3}(4)^{3/2} = \frac{16}{3}$

15. $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} dz\,dy\,dx = \int_0^1 \int_0^{2-x} (2 - x - y)\,dy\,dx = \int_0^1 \left[(2-x)^2 - \frac{1}{2}(2-x)^2\right] dx = \frac{1}{2}\int_0^1 (2-x)^2\,dx$
$= \left[-\frac{1}{6}(2-x)^3\right]_0^1 = -\frac{1}{6} + \frac{8}{6} = \frac{7}{6}$

16. $\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} x\,dz\,dy\,dx = \int_0^1 \int_0^{1-x^2} x(1 - x^2 - y)\,dy\,dx = \int_0^1 x\left[(1-x^2)^2 - \frac{1}{2}(1-x^2)\right] dx = \int_0^1 \frac{1}{2}x(1-x^2)^2\,dx$
$= \left[-\frac{1}{12}(1-x^2)^3\right]_0^1 = \frac{1}{12}$

17. $\int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \cos(u + v + w)\,du\,dv\,dw = \int_0^{\pi} \int_0^{\pi} \left[\sin(w + v + \pi) - \sin(w + v)\right] dv\,dw$
$= \int_0^{\pi} \left[(-\cos(w + 2\pi) + \cos(w + \pi)) + (\cos(w + \pi) - \cos w)\right] dw$

$= \left[-\sin(w+2\pi) + \sin(w+\pi) - \sin w + \sin(w+\pi)\right]_0^{\pi} = 0$

18. $\int_1^e \int_1^e \int_1^e \ln r \ln s \ln t \, dt\, dr\, ds = \int_1^e \int_1^e (\ln r \ln s)\left[t \ln t - t\right]_1^e dr\, ds = \int_1^e (\ln s)\left[r \ln r - r\right]_1^e ds = \left[s \ln s - s\right]_1^e = 1$

19. $\int_0^{\pi/4} \int_0^{\ln \sec v} \int_{-\infty}^{2t} e^x \, dx\, dt\, dv = \int_0^{\pi/4} \int_0^{\ln \sec v} \lim_{b \to -\infty} \left(e^{2t} - e^b\right) dt\, dv = \int_0^{\pi/4} \int_0^{\ln \sec v} e^{2t}\, dt\, dv = \int_0^{\pi/4} \left(\frac{1}{2} e^{2 \ln \sec v} - \frac{1}{2}\right) dv$
$= \int_0^{\pi/4} \left(\frac{\sec^2 v}{2} - \frac{1}{2}\right) dv = \left[\frac{\tan v}{2} - \frac{v}{2}\right]_0^{\pi/4} = \frac{1}{2} - \frac{\pi}{8}$

20. $\int_0^7 \int_0^2 \int_0^{\sqrt{4-q^2}} \frac{q}{r+1}\, dp\, dq\, dr = \int_0^7 \int_0^2 \frac{q\sqrt{4-q^2}}{r+1}\, dq\, dr = \int_0^7 \frac{1}{3(r+1)} \left[-\left(4-q^2\right)^{3/2}\right]_0^2 dr = \frac{8}{3} \int_0^7 \frac{1}{r+1}\, dr$
$= \frac{8 \ln 8}{3} = 8 \ln 2$

21. (a) $\int_{-1}^1 \int_0^{1-x^2} \int_{x^2}^{1-z} dy\, dz\, dx$ (b) $\int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{x^2}^{1-z} dy\, dx\, dz$ (c) $\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy\, dz$
(d) $\int_0^1 \int_0^{1-y} \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dz\, dy$ (e) $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{1-y} dz\, dx\, dy$

22. (a) $\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy\, dz\, dx$ (b) $\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy\, dx\, dz$ (c) $\int_0^1 \int_{-1}^{-\sqrt{z}} \int_0^1 dx\, dy\, dz$
(d) $\int_{-1}^0 \int_0^{y^2} \int_0^1 dx\, dz\, dy$ (e) $\int_{-1}^0 \int_0^1 \int_0^{y^2} dz\, dx\, dy$

23. $V = \int_0^1 \int_{-1}^1 \int_0^{y^2} dz\, dy\, dx = \int_0^1 \int_{-1}^1 y^2\, dy\, dx = \frac{2}{3} \int_0^1 dx = \frac{2}{3}$

24. $V = \int_0^1 \int_0^{1-x} \int_0^{2-2z} dy\, dz\, dx = \int_0^1 \int_0^{1-x} (2-2z)\, dz\, dx = \int_0^1 \left[2z - z^2\right]_0^{1-x} dx = \int_0^1 \left(1-x^2\right) dx = \left[x - \frac{x^3}{3}\right]_0^1 = \frac{2}{3}$

25. $V = \int_0^4 \int_0^{\sqrt{4-x}} \int_0^{2-y} dz\, dy\, dx = \int_0^4 \int_0^{\sqrt{4-x}} (2-y)\, dy\, dx = \int_0^4 \left[2\sqrt{4-x} - \left(\frac{4-x}{2}\right)\right] dx$
$= \left[-\frac{4}{3}(4-x)^{3/2} + \frac{1}{4}(4-x)^2\right]_0^4 = \frac{4}{3}(4)^{3/2} - \frac{1}{4}(16) = \frac{32}{3} - 4 = \frac{20}{3}$

26. $V = 2 \int_0^1 \int_{-\sqrt{1-x^2}}^0 \int_0^{-y} dz\, dy\, dx = -2 \int_0^1 \int_{-\sqrt{1-x^2}}^0 y\, dy\, dx = \int_0^1 \left(1-x^2\right) dx = \frac{2}{3}$

27. $V = \int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} dz\, dy\, dx = \int_0^1 \int_0^{2-2x} \left(3 - 3x - \frac{3}{2}y\right) dy\, dx = \int_0^1 \left[6(1-x)^2 - \frac{3}{4} \cdot 4(1-x)^2\right] dx$
$= \int_0^1 3(1-x)^2\, dx = \left[-(1-x)^3\right]_0^1 = 1$

28. $V = \int_0^1 \int_0^{1-x} \int_0^{\cos(\pi x/2)} dz\, dy\, dx = \int_0^1 \int_0^{1-x} \cos\left(\frac{\pi x}{2}\right) dy\, dx = \int_0^1 \left(\cos \frac{\pi x}{2}\right)(1-x)\, dx$
$= \int_0^1 \cos\left(\frac{\pi x}{2}\right) dx - \int_0^1 x \cos\left(\frac{\pi x}{2}\right) dx = \left[\frac{2}{\pi} \sin \frac{\pi x}{2}\right]_0^1 - \frac{4}{\pi^2} \int_0^{\pi/2} u \cos u\, du = \frac{2}{\pi} - \frac{4}{\pi^2}\left[\cos u + u \sin u\right]_0^{\pi/2}$
$= \frac{2}{\pi} - \frac{4}{\pi^2}\left(\frac{\pi}{2} - 1\right) = \frac{4}{\pi^2}$

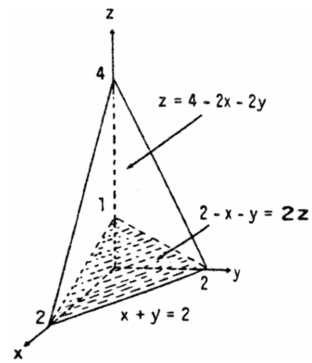
29. $V = 8 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz\, dy\, dx = 8 \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\, dy\, dx = 8 \int_0^1 \left(1-x^2\right) dx = \frac{16}{3}$

30. $V = \int_0^2 \int_0^{4-x^2} \int_0^{4-x^2-y} dz\, dy\, dx = \int_0^2 \int_0^{4-x^2} \left(4 - x^2 - y\right) dy\, dx = \int_0^2 \left[\left(4-x^2\right)^2 - \frac{1}{2}\left(4-x^2\right)^2\right] dx$
$= \frac{1}{2} \int_0^2 \left(4-x^2\right)^2 dx = \int_0^2 \left(8 - 4x^2 + \frac{x^4}{2}\right) dx = \frac{128}{15}$

31. $V=\int_0^4\int_0^{\left(\sqrt{16-y^2}\right)/2}\int_0^{4-y} dx\,dz\,dy=\int_0^4\int_0^{\left(\sqrt{16-y^2}\right)/2}(4-y)\,dz\,dy=\int_0^4 \frac{\sqrt{16-y^2}}{2}(4-y)\,dy$
$=\int_0^4 2\sqrt{16-y^2}\,dy-\frac{1}{2}\int_0^4 y\sqrt{16-y^2}\,dy=\left[y\sqrt{16-y^2}+16\sin^{-1}\frac{y}{4}\right]_0^4+\left[\frac{1}{6}\left(16-y^2\right)^{3/2}\right]_0^4$
$=16\left(\frac{\pi}{2}\right)-\frac{1}{6}(16)^{3/2}=8\pi-\frac{32}{3}$

32. $V=\int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{3-x} dz\,dy\,dx=\int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(3-x)\,dy\,dx=2\int_{-2}^2(3-x)\sqrt{4-x^2}\,dx$
$=3\int_{-2}^2 2\sqrt{4-x^2}\,dx-2\int_{-2}^2 x\sqrt{4-x^2}\,dx=3\left[x\sqrt{4-x^2}+4\sin^{-1}\frac{x}{2}\right]_{-2}^2+\left[\frac{2}{3}\left(4-x^2\right)^{3/2}\right]_{-2}^2$
$=12\sin^{-1}1-12\sin^{-1}(-1)=12\left(\frac{\pi}{2}\right)-12\left(-\frac{\pi}{2}\right)=12\pi$

33. $\int_0^2\int_0^{2-x}\int_{(2-x-y)/2}^{4-2x-2y} dz\,dy\,dx=\int_0^2\int_0^{2-x}\left(3-\frac{3x}{2}-\frac{3y}{2}\right)dy\,dx$
$=\int_0^2\left[3\left(1-\frac{x}{2}\right)(2-x)-\frac{3}{4}(2-x)^2\right]dx$
$=\int_0^2\left[6-6x+\frac{3x^2}{2}-\frac{3(2-x)^2}{4}\right]dx$
$=\left[6x-3x^2+\frac{x^3}{2}+\frac{(2-x)^3}{4}\right]_0^2=(12-12+4+0)-\frac{2^3}{4}=2$




34. $V=\int_0^4\int_z^8\int_z^{8-z} dx\,dy\,dz=\int_0^4\int_z^8(8-2z)\,dy\,dz=\int_0^4(8-2z)(8-z)\,dz=\int_0^4\left(64-24z+2z^2\right)dz$
$=\left[64z-12z^2+\frac{2}{3}z^3\right]_0^4=\frac{320}{3}$

35. $V=2\int_{-2}^2\int_0^{\sqrt{4-x^2}/2}\int_0^{x+2} dz\,dy\,dx=2\int_{-2}^2\int_0^{\sqrt{4-x^2}/2}(x+2)\,dy\,dx=\int_{-2}^2(x+2)\sqrt{4-x^2}\,dx$
$=\int_{-2}^2 2\sqrt{4-x^2}\,dx+\int_{-2}^2 x\sqrt{4-x^2}\,dx=\left[x\sqrt{4-x^2}+4\sin^{-1}\frac{x}{2}\right]_{-2}^2+\left[-\frac{1}{3}\left(4-x^2\right)^{3/2}\right]_{-2}^2$
$=4\left(\frac{\pi}{2}\right)-4\left(-\frac{\pi}{2}\right)=4\pi$

36. $V=2\int_0^1\int_0^{1-y^2}\int_0^{x^2+y^2} dz\,dx\,dy=2\int_0^1\int_0^{1-y^2}\left(x^2+y^2\right)dx\,dy=2\int_0^1\left[\frac{x^3}{3}+xy^2\right]_0^{1-y^2}dy$
$=2\int_0^1\left(1-y^2\right)\left[\frac{1}{3}\left(1-y^2\right)^2+y^2\right]dy=2\int_0^1\left(1-y^2\right)\left(\frac{1}{3}+\frac{1}{3}y^2+\frac{1}{3}y^4\right)dy=\frac{2}{3}\int_0^1\left(1-y^6\right)dy$
$=\frac{2}{3}\left[y-\frac{y^7}{7}\right]_0^1=\left(\frac{2}{3}\right)\left(\frac{6}{7}\right)=\frac{4}{7}$

37. average $=\frac{1}{8}\int_0^2\int_0^2\int_0^2\left(x^2+9\right)dz\,dy\,dx=\frac{1}{8}\int_0^2\int_0^2\left(2x^2+18\right)dy\,dx=\frac{1}{8}\int_0^2\left(4x^2+36\right)dx=\frac{31}{3}$

38. average $=\frac{1}{2}\int_0^1\int_0^1\int_0^2(x+y-z)\,dz\,dy\,dx=\frac{1}{2}\int_0^1\int_0^1(2x+2y-2)\,dy\,dx=\frac{1}{2}\int_0^1(2x-1)\,dx=0$

39. average $=\int_0^1\int_0^1\int_0^1\left(x^2+y^2+z^2\right)dz\,dy\,dx=\int_0^1\int_0^1\left(x^2+y^2+\frac{1}{3}\right)dy\,dx=\int_0^1\left(x^2+\frac{2}{3}\right)dx=1$

40. average $=\frac{1}{8}\int_0^2\int_0^2\int_0^2 xyz\,dz\,dy\,dx=\frac{1}{4}\int_0^2\int_0^2 xy\,dy\,dx=\frac{1}{2}\int_0^2 x\,dx=1$

41. $\int_0^4\int_0^1\int_{2y}^2\frac{4\cos\left(x^2\right)}{2\sqrt{z}}\,dx\,dy\,dz=\int_0^4\int_0^2\int_0^{x/2}\frac{4\cos\left(x^2\right)}{2\sqrt{z}}\,dy\,dx\,dz=\int_0^4\int_0^2\frac{x\cos\left(x^2\right)}{\sqrt{z}}\,dx\,dz=\int_0^4\left(\frac{\sin 4}{2}\right)z^{-1/2}\,dz$
$=\left[(\sin 4)z^{1/2}\right]_0^4=2\sin 4$

42. $\int_0^1 \int_0^1 \int_{x^2}^1 12xz\,e^{zy^2}\,dy\,dx\,dz = \int_0^1 \int_0^1 \int_0^{\sqrt{y}} 12xz\,e^{zy^2}\,dx\,dy\,dz = \int_0^1 \int_0^1 6yz\,e^{zy^2}\,dy\,dz = \int_0^1 \left[3e^{zy^2}\right]_0^1 dz$
$= 3\int_0^1 (e^z - z)\,dz = 3\left[e^z - 1\right]_0^1 = 3e - 6$

43. $\int_0^1 \int_{\sqrt[3]{z}}^1 \int_0^{\ln 3} \frac{\pi e^{2x} \sin(\pi y^2)}{y^2}\,dx\,dy\,dz = \int_0^1 \int_{\sqrt[3]{z}}^1 \frac{4\pi \sin(\pi y^2)}{y^2}\,dy\,dz = \int_0^1 \int_0^{y^3} \frac{4\pi \sin(\pi y^2)}{y^2}\,dz\,dy$
$= \int_0^1 4\pi y \sin(\pi y^2)\,dy = \left[-2\cos(\pi y^2)\right]_0^1 = -2(-1) + 2(1) = 4$

44. $\int_0^2 \int_0^{4-x^2} \int_0^x \frac{\sin 2z}{4-z}\,dy\,dz\,dx = \int_0^2 \int_0^{4-x^2} \frac{x \sin 2z}{4-z}\,dz\,dx = \int_0^4 \int_0^{\sqrt{4-z}} \left(\frac{\sin 2z}{4-z}\right) x\,dx\,dz = \int_0^4 \left(\frac{\sin 2z}{4-z}\right) \frac{1}{2}(4-z)\,dz$
$= \left[-\frac{1}{4}\cos 2z\right]_0^4 = \left[-\frac{1}{4} + \frac{1}{2}\sin^2 z\right]_0^4 = \frac{\sin^2 4}{2}$

45. $\int_0^1 \int_0^{4-a-x^2} \int_a^{4-x^2-y} dz\,dy\,dx = \frac{4}{15} \Rightarrow \int_0^1 \int_0^{4-a-x^2} (4 - x^2 - y - a)\,dy\,dx = \frac{4}{15}$
$\Rightarrow \int_0^1 \left[(4-a-x^2)^2 - \frac{1}{2}(4-a-x^2)^2\right] dx = \frac{4}{15} \Rightarrow \frac{1}{2}\int_0^1 (4-a-x^2)^2\,dx = \frac{4}{15} \Rightarrow \int_0^1 \left[(4-a)^2 - 2x^2(4-a) + x^4\right] dx$
$= \frac{8}{15} \Rightarrow \left[(4-a)^2 x - \frac{2}{3}x^3(4-a) + \frac{x^5}{5}\right]_0^1 = \frac{8}{15} \Rightarrow (4-a)^2 - \frac{2}{3}(4-a) + \frac{1}{5} = \frac{8}{15} \Rightarrow 15(4-a)^2 - 10(4-a) - 5 = 0$
$\Rightarrow 3(4-a)^2 - 2(4-a) - 1 = 0 \Rightarrow [3(4-a)+1][(4-a)-1] = 0 \Rightarrow 4 - a = -\frac{1}{3}$ or $4 - a = 1 \Rightarrow a = \frac{13}{3}$ or $a = 3$

46. The volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{4abc\pi}{3}$ so that $\frac{4(1)(2)(c)\pi}{3} = 8\pi \Rightarrow c = 3$.

47. To minimize the integral, we want the domain to include all points where the integrand is negative and to exclude all points where it is positive. These criteria are met by the points $(x, y, z)$ such that $4x^2 + 4y^2 + z^2 - 4 \le 0$ or $4x^2 + 4y^2 + z^2 \le 4$, which is a solid ellipsoid centered at the origin.

48. To maximize the integral, we want the domain to include all points where the integrand is positive and to exclude all points where it is negative. These criteria are met by the points $(x, y, z)$ such that $1 - x^2 - y^2 - z^2 \ge 0$ or $x^2 + y^2 + z^2 \le 1$, which is a solid sphere of radius 1 centered at the origin.

49-52. Example CAS commands:

Maple:

```
F := (x,y,z) -> x^2*y^2*z;
q1 := Int( Int( Int( F(x,y,z), y=-sqrt(1-x^2)..sqrt(1-x^2) ), x=-1..1 ), z=0..1 );
value( q1 );
```

Mathematica: (functions and bounds will vary)

Due to the nature of the bounds, cylindrical coordinates are appropriate, although Mathematica can do it as is also.

```
Clear[f, x, y, z];
f:= x^2 y^2 z
Integrate[f, {x,–1,1}, {y,–Sqrt[1 – x^2], Sqrt[1 – x^2]}, {z, 0, 1}]
N[%]
topolar={x → r Cos[t], y → r Sin[t]};
fp= f/.topolar //Simplify
Integrate[r fp, {t, 0, 2π}, {r, 0, 1},{z, 0, 1}]
N[%]
```