CHAPTER 14 PARTIAL DERIVATIVES

14.1 FUNCTIONS OF SEVERAL VARIABLES

1. (a) Domain: all points in the xy-plane
   (b) Range: all real numbers
   (c) level curves are straight lines $y - x = c$ parallel to the line $y = x$
   (d) no boundary points
   (e) both open and closed
   (f) unbounded

2. (a) Domain: set of all (x, y) so that $y - x \geq 0 \Rightarrow y \geq x$
   (b) Range: $z \geq 0$
   (c) level curves are straight lines of the form $y - x = c$ where $c \geq 0$
   (d) boundary is $\sqrt{y - x} = 0 \Rightarrow y = x$, a straight line
   (e) closed
   (f) unbounded

3. (a) Domain: all points in the xy-plane
   (b) Range: $z \geq 0$
   (c) level curves: for $f(x, y) = 0$, the origin; for $f(x, y) = c > 0$, ellipses with center $(0, 0)$ and major and minor axes along the x- and y-axes, respectively
   (d) no boundary points
   (e) both open and closed
   (f) unbounded

4. (a) Domain: all points in the xy-plane
   (b) Range: all real numbers
   (c) level curves: for $f(x, y) = 0$, the union of the lines $y = \pm x$; for $f(x, y) = c \neq 0$, hyperbolas centered at $(0, 0)$ with foci on the x-axis if $c > 0$ and on the y-axis if $c < 0$
   (d) no boundary points
   (e) both open and closed
   (f) unbounded

5. (a) Domain: all points in the xy-plane
   (b) Range: all real numbers
   (c) level curves are hyperbolas with the x- and y-axes as asymptotes when $f(x, y) \neq 0$, and the x- and y-axes when $f(x, y) = 0$
   (d) no boundary points
   (e) both open and closed
   (f) unbounded

6. (a) Domain: all $(x, y) \neq (0, y)$
   (b) Range: all real numbers
   (c) level curves: for $f(x, y) = 0$, the x-axis minus the origin; for $f(x, y) = c \neq 0$, the parabolas $y = cx^2$ minus the origin
   (d) boundary is the line $x = 0$

(e) open
(f) unbounded

7. (a) Domain: all (x, y) satisfying $x^2 + y^2 < 16$
(b) Range: $z \geq \frac{1}{4}$
(c) level curves are circles centered at the origin with radii $r < 4$
(d) boundary is the circle $x^2 + y^2 = 16$
(e) open
(f) bounded

8. (a) Domain: all (x, y) satisfying $x^2 + y^2 \leq 9$
(b) Range: $0 \leq z \leq 3$
(c) level curves are circles centered at the origin with radii $r \leq 3$
(d) boundary is the circle $x^2 + y^2 = 9$
(e) closed
(f) bounded

9. (a) Domain: $(x, y) \neq (0, 0)$
(b) Range: all real numbers
(c) level curves are circles with center $(0, 0)$ and radii $r > 0$
(d) boundary is the single point $(0, 0)$
(e) open
(f) unbounded

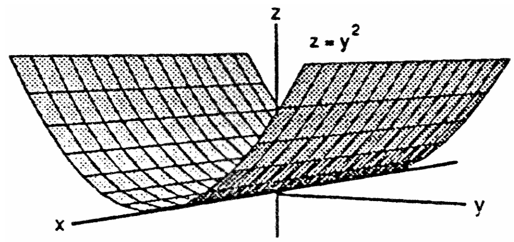
10. (a) Domain: all points in the xy-plane
(b) Range: $0 < z \leq 1$
(c) level curves are the origin itself and the circles with center $(0, 0)$ and radii $r > 0$
(d) no boundary points
(e) both open and closed
(f) unbounded

11. (a) Domain: all (x, y) satisfying $-1 \leq y - x \leq 1$
(b) Range: $-\frac{\pi}{2} \leq z \leq \frac{\pi}{2}$
(c) level curves are straight lines of the form $y - x = c$ where $-1 \leq c \leq 1$
(d) boundary is the two straight lines $y = 1 + x$ and $y = -1 + x$
(e) closed
(f) unbounded

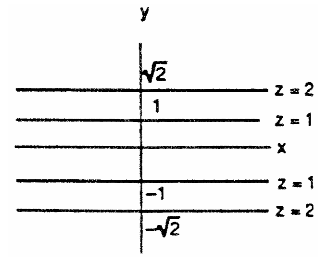
12. (a) Domain: all (x, y), $x \neq 0$
(b) Range: $-\frac{\pi}{2} < z < \frac{\pi}{2}$
(c) level curves are the straight lines of the form $y = cx$, c any real number and $x \neq 0$
(d) boundary is the line $x = 0$
(e) open
(f) unbounded

13. f
14. e
15. a
16. c
17. d
18. b

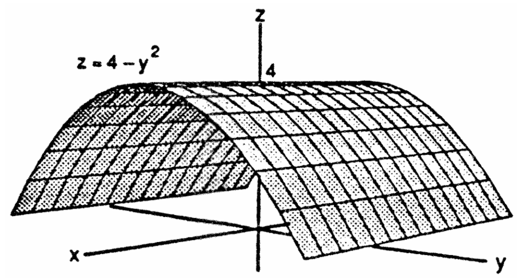
19. (a)



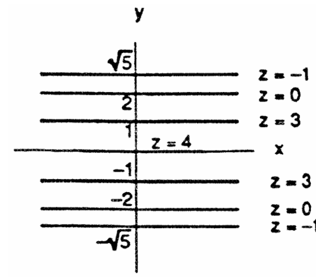
(b)



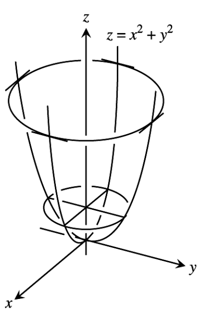
20. (a)



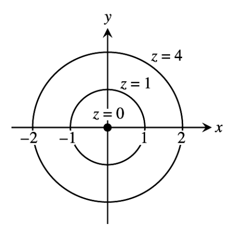
(b)



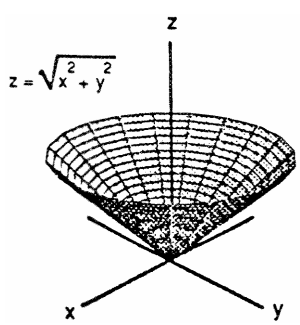
21. (a)



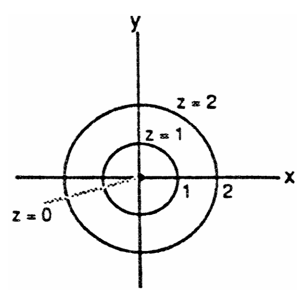
(b)



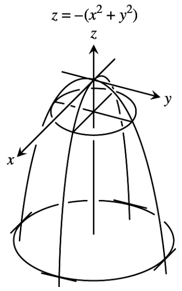
22. (a)



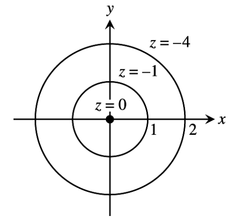
(b)



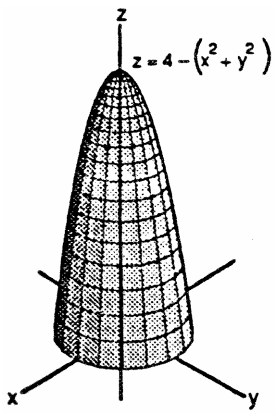
23. (a)



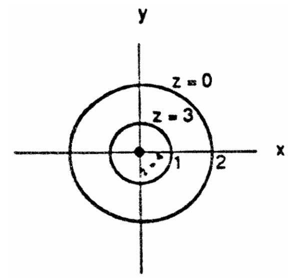
(b)



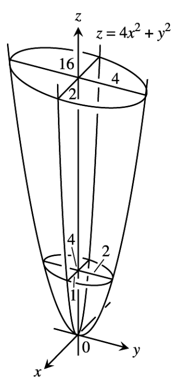
24. (a)



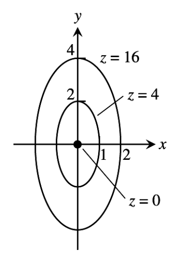
(b)



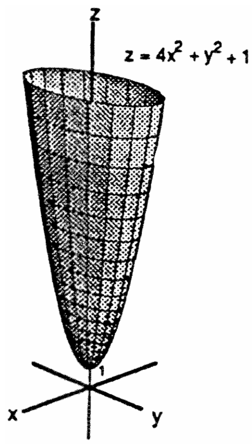
25. (a)



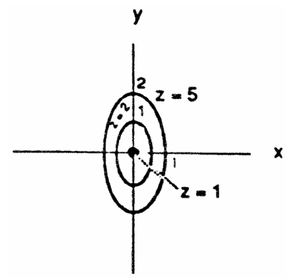
(b)



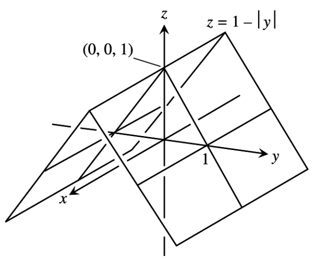
26. (a)



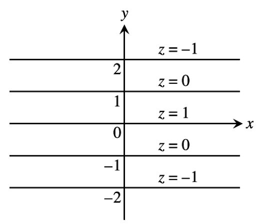
(b)



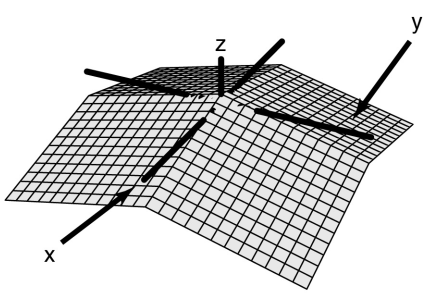
27. (a)



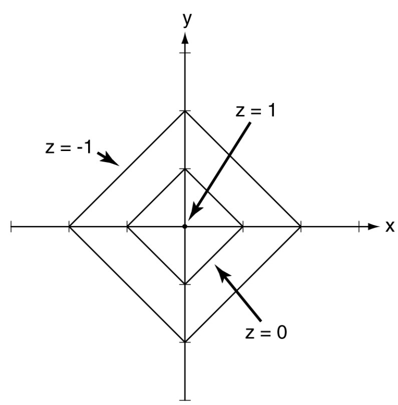
(b)



28. (a)



(b)



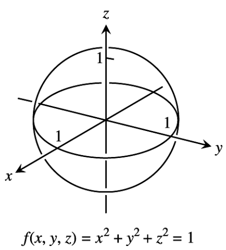
29. $f(x, y) = 16 - x^2 - y^2$ and $\left(2\sqrt{2}, \sqrt{2}\right) \Rightarrow z = 16 - \left(2\sqrt{2}\right)^2 - \left(\sqrt{2}\right)^2 = 6 \Rightarrow 6 = 16 - x^2 - y^2 \Rightarrow x^2 + y^2 = 10$

30. $f(x, y) = \sqrt{x^2 - 1}$ and $(1, 0) \Rightarrow z = \sqrt{1^2 - 1} = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = 1$ or $x = -1$

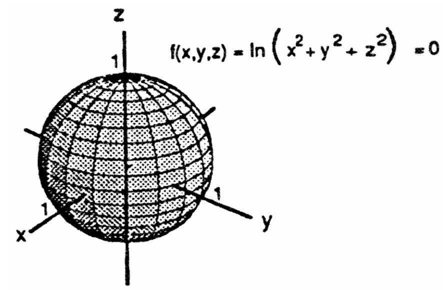
31. $f(x, y) = \int_x^y \frac{1}{1+t^2}\, dt$ at $\left(-\sqrt{2}, \sqrt{2}\right) \Rightarrow z = \tan^{-1} y - \tan^{-1} x$; at $\left(-\sqrt{2}, \sqrt{2}\right) \Rightarrow z = \tan^{-1} \sqrt{2} - \tan^{-1}\left(-\sqrt{2}\right)$
$= 2 \tan^{-1} \sqrt{2} \Rightarrow \tan^{-1} y - \tan^{-1} x = 2 \tan^{-1} \sqrt{2}$

32. $f(x, y) = \sum_{n=0}^{\infty} \left(\frac{x}{y}\right)^n$ at $(1, 2) \Rightarrow z = \frac{1}{1 - \left(\frac{x}{y}\right)} = \frac{y}{y - x}$; at $(1, 2) \Rightarrow z = \frac{2}{2 - 1} = 2 \Rightarrow 2 = \frac{y}{y - x} \Rightarrow 2y - 2x = y$
$\Rightarrow y = 2x$

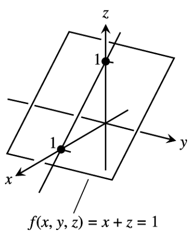
33.



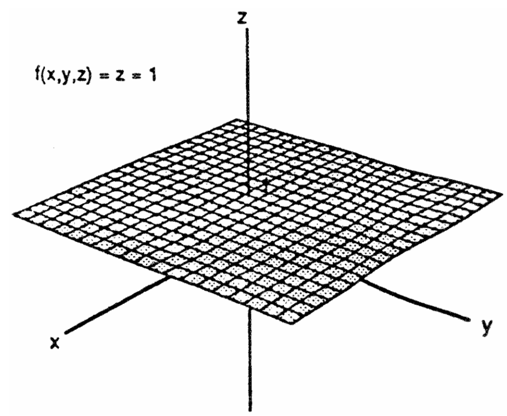
34.



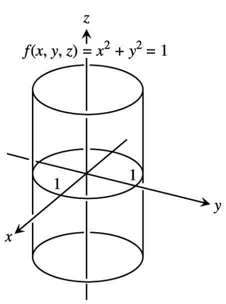
35.



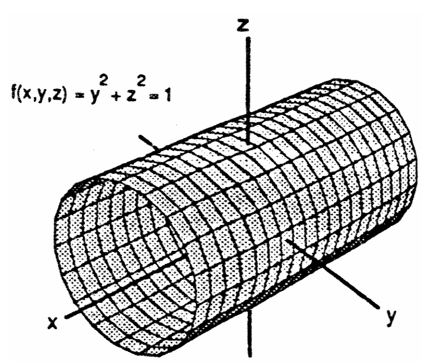
36.



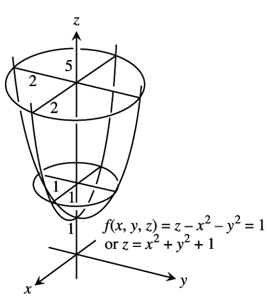
37.



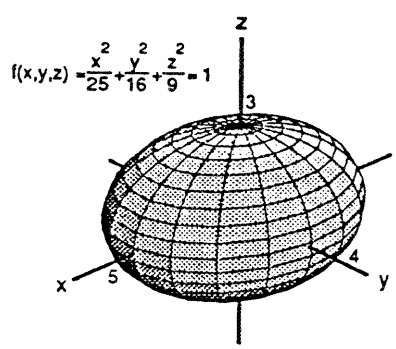
38.



39.



40.



41. $f(x, y, z) = \sqrt{x - y} - \ln z$ at $(3, -1, 1) \Rightarrow w = \sqrt{x - y} - \ln z$; at $(3, -1, 1) \Rightarrow w = \sqrt{3 - (-1)} - \ln 1 = 2$
$\Rightarrow \sqrt{x - y} - \ln z = 2$

42. $f(x, y, z) = \ln(x^2 + y + z^2)$ at $(-1, 2, 1) \Rightarrow w = \ln(x^2 + y + z^2)$; at $(-1, 2, 1) \Rightarrow w = \ln(1 + 2 + 1) = \ln 4$
$\Rightarrow \ln 4 = \ln(x^2 + y + z^2) \Rightarrow x^2 + y + z^2 = 4$

43. $g(x, y, z) = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!\, z^n}$ at $(\ln 2, \ln 4, 3) \Rightarrow w = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!\, z^n} = e^{(x+y)/z}$; at $(\ln 2, \ln 4, 3) \Rightarrow w = e^{(\ln 2 + \ln 4)/3}$
$= e^{(\ln 8)/3} = e^{\ln 2} = 2 \Rightarrow 2 = e^{(x+y)/z} \Rightarrow \frac{x+y}{z} = \ln 2$

44. $g(x, y, z) = \int_x^y \frac{d\theta}{\sqrt{1-\theta^2}} + \int_{\sqrt{2}}^z \frac{dt}{t\sqrt{t^2-1}}$ at $\left(0, \frac{1}{2}, 2\right) \Rightarrow w = \left[\sin^{-1}\theta\right]_x^y + \left[\sec^{-1} t\right]_{\sqrt{2}}^z$
$= \sin^{-1} y - \sin^{-1} x + \sec^{-1} z - \sec^{-1}\left(\sqrt{2}\right) \Rightarrow w = \sin^{-1} y - \sin^{-1} x + \sec^{-1} z - \frac{\pi}{4}$; at $\left(0, \frac{1}{2}, 2\right)$
$\Rightarrow w = \sin^{-1}\frac{1}{2} - \sin^{-1} 0 + \sec^{-1} 2 - \frac{\pi}{4} = \frac{\pi}{4} \Rightarrow \frac{\pi}{2} = \sin^{-1} y - \sin^{-1} x + \sec^{-1} z$

45. $f(x, y, z) = xyz$ and $x = 20 - t$, $y = t$, $z = 20 \Rightarrow w = (20 - t)(t)(20)$ along the line $\Rightarrow w = 400t - 20t^2$
$\Rightarrow \frac{dw}{dt} = 400 - 40t$; $\frac{dw}{dt} = 0 \Rightarrow 400 - 40t = 0 \Rightarrow t = 10$ and $\frac{d^2w}{dt^2} = -40$ for all t $\Rightarrow$ yes, maximum at $t = 10$
$\Rightarrow x = 20 - 10 = 10$, $y = 10$, $z = 20 \Rightarrow$ maximum of f along the line is $f(10, 10, 20) = (10)(10)(20) = 2000$

46. $f(x, y, z) = xy - z$ and $x = t - 1$, $y = t - 2$, $z = t + 7 \Rightarrow w = (t - 1)(t - 2) - (t + 7) = t^2 - 4t - 5$ along the line
$\Rightarrow \frac{dw}{dt} = 2t - 4$; $\frac{dw}{dt} = 0 \Rightarrow 2t - 4 = 0 \Rightarrow t = 2$ and $\frac{d^2w}{dt^2} = 2$ for all t $\Rightarrow$ yes, minimum at $t = 2 \Rightarrow x = 2 - 1 = 1$,
$y = 2 - 2 = 0$, and $z = 2 + 7 = 9 \Rightarrow$ minimum of f along the line is $f(1, 0, 9) = (1)(0) - 9 = -9$

47. $w = 4\left(\frac{Th}{d}\right)^{1/2} = 4\left[\frac{(290\text{ K})(16.8\text{ km})}{5\text{ K/km}}\right]^{1/2} \approx 124.86$ km $\Rightarrow$ must be $\frac{1}{2}(124.86) \approx 63$ km south of Nantucket

48. The graph of $f(x_1, x_2, x_3, x_4)$ is a set in a five-dimensional space. It is the set of points $(x_1, x_2, x_3, x_4, f(x_1, x_2, x_3, x_4))$ for (x_1, x_2, x_3, x_4) in the domain of f. The graph of $f(x_1, x_2, x_3, \ldots, x_n)$ is a set in an $(n + 1)$-dimensional space. It is the set of points $(x_1, x_2, x_3, \ldots, x_n, f(x_1, x_2, x_3, \ldots, x_n))$ for $(x_1, x_2, x_3, \ldots, x_n)$ in the domain of f.

49-52. Example CAS commands:

<u>Maple</u>:

```
with( plots );
f := (x,y) -> x*sin(y/2) + y*sin(2*x);
xdomain := x=0..5*Pi;
ydomain := y=0..5*Pi;
x0,y0 := 3*Pi,3*Pi;
plot3d( f(x,y), xdomain, ydomain, axes=boxed, style=patch, shading=zhue, title="#49(a) (Section 14.1)" );
plot3d( f(x,y), xdomain, ydomain, grid=[50,50], axes=boxed, shading=zhue, style=patchcontour, orientation=[-90,0],
        title="#49(b) (Section 14.1)" );                                    # (b)
L := evalf( f(x0,y0) );                                  # (c)
plot3d( f(x,y), xdomain, ydomain, grid=[50,50], axes=boxed, shading=zhue, style=patchcontour, contours=[L],
        orientation=[-90,0], title="#49(c) (Section 14.1)" );
```

53-56. Example CAS commands:

<u>Maple</u>:

```
eq := 4*ln(x^2+y^2+z^2)=1;
implicitplot3d( eq, x=-2..2, y=-2..2, z=-2..2, grid=[30,30,30], axes=boxed, title="#53 (Section 14.1)" );
```

57-60. Example CAS commands:

Maple:

```
x := (u,v) -> u*cos(v);
y := (u,v) ->u*sin(v);
z := (u,v) -> u;
plot3d( [x(u,v),y(u,v),z(u,v)], u=0..2, v=0..2*Pi, axes=boxed, style=patchcontour, contours=[($0..4)/2], shading=zhue,
    title="#57 (Section 14.1)" );
```

49-60. Example CAS commands:

Mathematica: (assigned functions and bounds will vary)

For 49 - 52, the command **ContourPlot** draws 2-dimensional contours that are z-level curves of surfaces z = f(x,y).

```
Clear[x, y, f]
f[x_, y_]:= x Sin[y/2] + y Sin[2x]
xmin= 0; xmax= 5π; ymin= 0; ymax= 5π; {x0, y0}={3π, 3π};
cp= ContourPlot[f[x,y], {x, xmin, xmax}, {y, ymin, ymax}, ContourShading → False];
cp0= ContourPlot[[f[x,y], {x, xmin, xmax}, {y, ymin, ymax}, Contours → {f[x0,y0]}, ContourShading → False,
        PlotStyle → {RGBColor[1,0,0]}];
Show[cp, cp0]
```

For 53 - 56, the command **ContourPlot3D** will be used and requires loading a package. Write the function f[x, y, z] so that when it is equated to zero, it represents the level surface given.

For 53, the problem associated with Log[0] can be avoided by rewriting the function as x2 + y2 +z2 - e1/4

```
<<Graphics`ContourPlot3D`
Clear[x, y, z, f]
f[x_, y_, z_]:= x^2 + y^2 + z^2 − Exp[1/4]
ContourPlot3D[f[x, y, z], {x, −5, 5}, {y, −5, 5}, {z, −5, 5}, PlotPoints → {7, 7}];
```

For 57 - 60, the command ParametricPlot3D will be used and requires loading a package. To get the z-level curves here, we solve x and y in terms of z and either u or v (v here), create a table of level curves, then plot that table.

```
<<Graphics`ParametricPlot3D`
Clear[x, y, z, u, v]
ParametricPlot3D[{u Cos[v], u Sin[v], u}, {u, 0, 2}, {v, 0, 2p}];
zlevel= Table[{z Cos[v], z sin[v]}, {z, 0, 2, .1}];
ParametricPlot[Evaluate[zlevel],{v, 0, 2π}];
```

14.2 LIMITS AND CONTINUITY

1. $\lim\limits_{(x,y)\to(0,0)} \frac{3x^2-y^2+5}{x^2+y^2+2} = \frac{3(0)^2-0^2+5}{0^2+0^2+2} = \frac{5}{2}$

2. $\lim\limits_{(x,y)\to(0,4)} \frac{x}{\sqrt{y}} = \frac{0}{\sqrt{4}} = 0$

3. $\lim\limits_{(x,y)\to(3,4)} \sqrt{x^2+y^2-1} = \sqrt{3^2+4^2-1} = \sqrt{24} = 2\sqrt{6}$

4. $\lim\limits_{(x,y)\to(2,-3)} \left(\frac{1}{x}+\frac{1}{y}\right)^2 = \left[\frac{1}{2}+\left(\frac{1}{-3}\right)\right]^2 = \left(\frac{1}{6}\right)^2 = \frac{1}{36}$

5. $\lim\limits_{(x,y)\to\left(0,\frac{\pi}{4}\right)} \sec x \tan y = (\sec 0)\left(\tan\frac{\pi}{4}\right) = (1)(1) = 1$