15.5 MASSES AND MOMENTS IN THREE DIMENSIONS

1. $I_x = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}(y^2+z^2)\,dx\,dy\,dz = a\int_{-c/2}^{c/2}\int_{-b/2}^{b/2}(y^2+z^2)\,dy\,dz = a\int_{-c/2}^{c/2}\left[\frac{y^3}{3}+yz^2\right]_{-b/2}^{b/2}dz$
$= a\int_{-c/2}^{c/2}\left(\frac{b^3}{12}+bz^2\right)dz = ab\left[\frac{b^2}{12}z+\frac{z^3}{3}\right]_{-c/2}^{c/2} = ab\left(\frac{b^2c}{12}+\frac{c^3}{12}\right) = \frac{abc}{12}(b^2+c^2) = \frac{M}{12}(b^2+c^2)$;
$R_x = \sqrt{\frac{b^2+c^2}{12}}$; likewise $R_y = \sqrt{\frac{a^2+c^2}{12}}$ and $R_z = \sqrt{\frac{a^2+b^2}{12}}$, by symmetry

2. The plane $z = \frac{4-2y}{3}$ is the top of the wedge $\Rightarrow I_x = \int_{-3}^{3}\int_{-2}^{4}\int_{-4/3}^{(4-2y)/3}(y^2+z^2)\,dz\,dy\,dx$
$= \int_{-3}^{3}\int_{-2}^{4}\left[\frac{8y^2}{3}-\frac{2y^3}{3}+\frac{8(2-y)^3}{81}+\frac{64}{81}\right]dy\,dx = \int_{-3}^{3}\frac{104}{3}\,dx = 208$; $I_y = \int_{-3}^{3}\int_{-2}^{4}\int_{-4/3}^{(4-2y)/3}(x^2+z^2)\,dz\,dy\,dx$
$= \int_{-3}^{3}\int_{-2}^{4}\left[\frac{(4-2y)^3}{81}+\frac{x^2(4-2y)}{3}+\frac{4x^2}{3}+\frac{64}{81}\right]dy\,dx = \int_{-3}^{3}\left(12x^2+\frac{32}{3}\right)dx = 280$;
$I_z = \int_{-3}^{3}\int_{-2}^{4}\int_{-4/3}^{(4-2y)/3}(x^2+y^2)\,dz\,dy\,dx = \int_{-3}^{3}\int_{-2}^{4}(x^2+y^2)\left(\frac{8}{3}-\frac{2y}{3}\right)dy\,dx = 12\int_{-3}^{3}(x^2+2)\,dx = 360$

3. $I_x = \int_0^a\int_0^b\int_0^c(y^2+z^2)\,dz\,dy\,dx = \int_0^a\int_0^b\left(cy^2+\frac{c^3}{3}\right)dy\,dx = \int_0^a\left(\frac{cb^3}{3}+\frac{c^3b}{3}\right)dx = \frac{abc\,(b^2+c^2)}{3}$
$= \frac{M}{3}(b^2+c^2)$ where $M = abc$; $I_y = \frac{M}{3}(a^2+c^2)$ and $I_z = \frac{M}{3}(a^2+b^2)$, by symmetry

4. (a) $M = \int_0^1\int_0^{1-x}\int_0^{1-x-y}dz\,dy\,dx = \int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx = \int_0^1\left(\frac{x^2}{2}-x+\frac{1}{2}\right)dx = \frac{1}{6}$;
$M_{yz} = \int_0^1\int_0^{1-x}\int_0^{1-x-y}x\,dz\,dy\,dx = \int_0^1\int_0^{1-x}x(1-x-y)\,dy\,dx = \frac{1}{2}\int_0^1(x^3-2x^2+x)\,dx = \frac{1}{24}$
$\Rightarrow \overline{x} = \overline{y} = \overline{z} = \frac{1}{4}$, by symmetry; $I_x = \int_0^1\int_0^{1-x}\int_0^{1-x-y}(y^2+z^2)\,dz\,dy\,dx$
$= \int_0^1\int_0^{1-x}\left[y^2-xy^2-y^3+\frac{(1-x-y)^3}{3}\right]dy\,dx = \frac{1}{6}\int_0^1(1-x)^4\,dx = \frac{1}{30} \Rightarrow I_y = I_x = \frac{1}{30}$, by symmetry
(b) $R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{1}{5}} = \frac{\sqrt{5}}{5} \approx 0.4472$; the distance from the centroid to the x-axis is $\sqrt{0^2+\frac{1}{16}+\frac{1}{16}} = \sqrt{\frac{1}{8}} = \frac{\sqrt{2}}{4}$
≈ 0.3536

5. $M = 4\int_0^1\int_0^1\int_{4y^2}^{4}dz\,dy\,dx = 4\int_0^1\int_0^1(4-4y^2)\,dy\,dx = 16\int_0^1\frac{2}{3}\,dx = \frac{32}{3}$; $M_{xy} = 4\int_0^1\int_0^1\int_{4y^2}^{4}z\,dz\,dy\,dx$
$= 2\int_0^1\int_0^1(16-16y^4)\,dy\,dx = \frac{128}{5}\int_0^1 dx = \frac{128}{5} \Rightarrow \overline{z} = \frac{12}{5}$, and $\overline{x} = \overline{y} = 0$, by symmetry;
$I_x = 4\int_0^1\int_0^1\int_{4y^2}^{4}(y^2+z^2)\,dz\,dy\,dx = 4\int_0^1\int_0^1\left[\left(4y^2+\frac{64}{3}\right)-\left(4y^4+\frac{64y^6}{3}\right)\right]dy\,dx = 4\int_0^1\frac{1976}{105}\,dx = \frac{7904}{105}$;
$I_y = 4\int_0^1\int_0^1\int_{4y^2}^{4}(x^2+z^2)\,dz\,dy\,dx = 4\int_0^1\int_0^1\left[\left(4x^2+\frac{64}{3}\right)-\left(4x^2y^2+\frac{64y^6}{3}\right)\right]dy\,dx = 4\int_0^1\left(\frac{8}{3}x^2+\frac{128}{7}\right)dx$
$= \frac{4832}{63}$; $I_z = 4\int_0^1\int_0^1\int_{4y^2}^{4}(x^2+y^2)\,dz\,dy\,dx = 16\int_0^1\int_0^1(x^2-x^2y^2+y^2-y^4)\,dy\,dx$
$= 16\int_0^1\left(\frac{2x^2}{3}+\frac{2}{15}\right)dx = \frac{256}{45}$

6. (a) $M = \int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2}\int_0^{2-x}dz\,dy\,dx = \int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2}(2-x)\,dy\,dx = \int_{-2}^{2}(2-x)\left(\sqrt{4-x^2}\right)dx = 4\pi$;
$M_{yz} = \int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2}\int_0^{2-x}x\,dz\,dy\,dx = \int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2}x(2-x)\,dy\,dx = \int_{-2}^{2}x(2-x)\left(\sqrt{4-x^2}\right)dx = -2\pi$;
$M_{xz} = \int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2}\int_0^{2-x}y\,dz\,dy\,dx = \int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2}y(2-x)\,dy\,dx$
$= \frac{1}{2}\int_{-2}^{2}(2-x)\left[\frac{4-x^2}{4}-\frac{4-x^2}{4}\right]dx = 0 \Rightarrow \overline{x} = -\frac{1}{2}$ and $\overline{y} = 0$

(b) $M_{xy} = \int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2}\int_{0}^{2-x} z\,dz\,dy\,dx = \frac{1}{2}\int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2} (2-x)^2\,dy\,dx = \frac{1}{2}\int_{-2}^{2}(2-x)^2\left(\sqrt{4-x^2}\right)dx$
$= 5\pi \Rightarrow \overline{z} = \frac{5}{4}$

7. (a) $M = 4\int_0^2\int_0^{\sqrt{4-x^2}}\int_{x^2+y^2}^{4} dz\,dy\,dx = 4\int_0^{\pi/2}\int_0^2\int_{r^2}^4 r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^2 (4r-r^3)\,dr\,d\theta = 4\int_0^{\pi/2} 4\,d\theta = 8\pi;$
$M_{xy} = \int_0^{2\pi}\int_0^2\int_{r^2}^4 zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \frac{r}{2}(16-r^4)\,dr\,d\theta = \frac{32}{3}\int_0^{2\pi} d\theta = \frac{64\pi}{3} \Rightarrow \overline{z} = \frac{8}{3}$, and $\overline{x} = \overline{y} = 0$, by symmetry

(b) $M = 8\pi \Rightarrow 4\pi = \int_0^{2\pi}\int_0^{\sqrt{c}}\int_{r^2}^{c} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{c}} (cr-r^3)\,dr\,d\theta = \int_0^{2\pi}\frac{c^2}{4}\,d\theta = \frac{c^2\pi}{2} \Rightarrow c^2 = 8 \Rightarrow c = 2\sqrt{2}$, since $c > 0$

8. $M = 8$; $M_{xy} = \int_{-1}^1\int_3^5\int_{-1}^1 z\,dz\,dy\,dx = \int_{-1}^1\int_3^5\left[\frac{z^2}{2}\right]_{-1}^{1} dy\,dx = 0$; $M_{yz} = \int_{-1}^1\int_3^5\int_{-1}^1 x\,dz\,dy\,dx$
$= 2\int_{-1}^1\int_3^5 x\,dy\,dx = 4\int_{-1}^1 x\,dx = 0$; $M_{xz} = \int_{-1}^1\int_3^5\int_{-1}^1 y\,dz\,dy\,dx = 2\int_{-1}^1\int_3^5 y\,dy\,dx = 16\int_{-1}^1 dx = 32$
$\Rightarrow \overline{x} = 0, \overline{y} = 4, \overline{z} = 0$; $I_x = \int_{-1}^1\int_3^5\int_{-1}^1 (y^2+z^2)\,dz\,dy\,dx = \int_{-1}^1\int_3^5\left(2y^2+\frac{2}{3}\right)dy\,dx = \frac{2}{3}\int_{-1}^1 100\,dx = \frac{400}{3}$;
$I_y = \int_{-1}^1\int_3^5\int_{-1}^1 (x^2+z^2)\,dz\,dy\,dx = \int_{-1}^1\int_3^5\left(2x^2+\frac{2}{3}\right)dy\,dx = \frac{4}{3}\int_{-1}^1 (3x^2+1)\,dx = \frac{16}{3}$;
$I_z = \int_{-1}^1\int_3^5\int_{-1}^1 (x^2+y^2)\,dz\,dy\,dx = 2\int_{-1}^1\int_3^5 (x^2+y^2)\,dy\,dx = 2\int_{-1}^1\left(2x^2+\frac{98}{3}\right)dx = \frac{400}{3} \Rightarrow R_x = R_z = \sqrt{\frac{50}{3}}$
and $R_y = \sqrt{\frac{2}{3}}$

9. The plane $y + 2z = 2$ is the top of the wedge $\Rightarrow I_L = \int_{-2}^2\int_{-2}^4\int_{-1}^{(2-y)/2}\left[(y-6)^2+z^2\right]dz\,dy\,dx$
$= \int_{-2}^2\int_{-2}^4\left[\frac{(y-6)^2(4-y)}{2} + \frac{(2-y)^3}{24} + \frac{1}{3}\right]dy\,dx$; let $t = 2 - y \Rightarrow I_L = 4\int_{-2}^4\left(\frac{13t^3}{24} + 5t^2 + 16t + \frac{49}{3}\right)dt = 1386$;
$M = \frac{1}{2}(3)(6)(4) = 36 \Rightarrow R_L = \sqrt{\frac{I_L}{M}} = \sqrt{\frac{77}{2}}$

10. The plane $y + 2z = 2$ is the top of the wedge $\Rightarrow I_L = \int_{-2}^2\int_{-2}^4\int_{-1}^{(2-y)/2}\left[(x-4)^2+y^2\right]dz\,dy\,dx$
$= \frac{1}{2}\int_{-2}^2\int_{-2}^4 (x^2-8x+16+y^2)(4-y)\,dy\,dx = \int_{-2}^2 (9x^2-72x+162)\,dx = 696$; $M = \frac{1}{2}(3)(6)(4) = 36$
$\Rightarrow R_L = \sqrt{\frac{I_L}{M}} = \sqrt{\frac{58}{3}}$

11. $M = 8$; $I_L = \int_0^4\int_0^2\int_0^1\left[z^2+(y-2)^2\right]dz\,dy\,dx = \int_0^4\int_0^2\left(y^2-4y+\frac{13}{3}\right)dy\,dx = \frac{10}{3}\int_0^4 dx = \frac{40}{3}$
$\Rightarrow R_L = \sqrt{\frac{I_L}{M}} = \sqrt{\frac{5}{3}}$

12. $M = 8$; $I_L = \int_0^4\int_0^2\int_0^1\left[(x-4)^2+y^2\right]dz\,dy\,dx = \int_0^4\int_0^2\left[(x-4)^2+y^2\right]dy\,dx = \int_0^4\left[2(x-4)^2+\frac{8}{3}\right]dx = \frac{160}{3}$
$\Rightarrow R_L = \sqrt{\frac{I_L}{M}} = \sqrt{\frac{20}{3}}$

13. (a) $M = \int_0^2\int_0^{2-x}\int_0^{2-x-y} 2x\,dz\,dy\,dx = \int_0^2\int_0^{2-x}(4x-2x^2-2xy)\,dy\,dx = \int_0^2 (x^3-4x^2+4x)\,dx = \frac{4}{3}$

(b) $M_{xy} = \int_0^2\int_0^{2-x}\int_0^{2-x-y} 2xz\,dz\,dy\,dx = \int_0^2\int_0^{2-x} x(2-x-y)^2\,dy\,dx = \int_0^2\frac{x(2-x)^3}{3}\,dx = \frac{8}{15}$; $M_{xz} = \frac{8}{15}$ by symmetry; $M_{yz} = \int_0^2\int_0^{2-x}\int_0^{2-x-y} 2x^2\,dz\,dy\,dx = \int_0^2\int_0^{2-x} 2x^2(2-x-y)\,dy\,dx = \int_0^2 (2x-x^2)^2\,dx = \frac{16}{15}$
$\Rightarrow \overline{x} = \frac{4}{5}$, and $\overline{y} = \overline{z} = \frac{2}{5}$

14. (a) $M = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} kxy\,dz\,dy\,dx = k\int_0^2 \int_0^{\sqrt{x}} xy\,(4-x^2)\,dy\,dx = \frac{k}{2}\int_0^2 (4x^2 - x^4)\,dx = \frac{32k}{15}$

(b) $M_{yz} = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} kx^2y\,dz\,dy\,dx = k\int_0^2 \int_0^{\sqrt{x}} x^2y\,(4-x^2)\,dy\,dx = \frac{k}{2}\int_0^2 (4x^3 - x^5)\,dx = \frac{8k}{3}$

$\Rightarrow\ \overline{x} = \frac{5}{4}$; $M_{xz} = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} kxy^2\,dz\,dy\,dx = k\int_0^2 \int_0^{\sqrt{x}} xy^2\,(4-x^2)\,dy\,dx = \frac{k}{3}\int_0^2 \left(4x^{5/2} - x^{9/2}\right)dx$

$= \frac{256\sqrt{2}k}{231}\ \Rightarrow\ \overline{y} = \frac{40\sqrt{2}}{77}$; $M_{xy} = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} kxyz\,dz\,dy\,dx = \int_0^2 \int_0^{\sqrt{x}} xy\,(4-x^2)^2\,dy\,dx$

$= \frac{k}{4}\int_0^2 (16x^2 - 8x^4 + x^6)\,dx = \frac{256k}{105}\ \Rightarrow\ \overline{z} = \frac{8}{7}$

15. (a) $M = \int_0^1 \int_0^1 \int_0^1 (x+y+z+1)\,dz\,dy\,dx = \int_0^1 \int_0^1 \left(x+y+\frac{3}{2}\right)dy\,dx = \int_0^1 (x+2)\,dx = \frac{5}{2}$

(b) $M_{xy} = \int_0^1 \int_0^1 \int_0^1 z(x+y+z+1)\,dz\,dy\,dx = \frac{1}{2}\int_0^1 \int_0^1 \left(x+y+\frac{5}{3}\right)dy\,dx = \frac{1}{2}\int_0^1 \left(x+\frac{13}{6}\right)dx = \frac{4}{3}$

$\Rightarrow\ M_{xy} = M_{yz} = M_{xz} = \frac{4}{3}$, by symmetry $\Rightarrow\ \overline{x} = \overline{y} = \overline{z} = \frac{8}{15}$

(c) $I_z = \int_0^1 \int_0^1 \int_0^1 (x^2+y^2)\,(x+y+z+1)\,dz\,dy\,dx = \int_0^1 \int_0^1 (x^2+y^2)\left(x+y+\frac{3}{2}\right)dy\,dx$

$= \int_0^1 \left(x^3 + 2x^2 + \frac{1}{3}x + \frac{3}{4}\right)dx = \frac{11}{6}\ \Rightarrow\ I_x = I_y = I_z = \frac{11}{6}$, by symmetry

(d) $R_x = R_y = R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{11}{15}}$

16. The plane $y + 2z = 2$ is the top of the wedge.

(a) $M = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1)\,dz\,dy\,dx = \int_{-1}^1 \int_{-2}^4 (x+1)\left(2 - \frac{y}{2}\right)dy\,dx = 18$

(b) $M_{yz} = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} x(x+1)\,dz\,dy\,dx = \int_{-1}^1 \int_{-2}^4 x(x+1)\left(2 - \frac{y}{2}\right)dy\,dx = 6;$

$M_{xz} = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} y(x+1)\,dz\,dy\,dx = \int_{-1}^1 \int_{-2}^4 y(x+1)\left(2 - \frac{y}{2}\right)dy\,dx = 0;$

$M_{xy} = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} z(x+1)\,dz\,dy\,dx = \frac{1}{2}\int_{-1}^1 \int_{-2}^4 (x+1)\left(\frac{y^2}{4} - y\right)dy\,dx = 0\ \Rightarrow\ \overline{x} = \frac{1}{3}$, and $\overline{y} = \overline{z} = 0$

(c) $I_x = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1)\,(y^2+z^2)\,dz\,dy\,dx = \int_{-1}^1 \int_{-2}^4 (x+1)\left[2y^2 + \frac{1}{3} - \frac{y^3}{2} + \frac{1}{3}\left(1 - \frac{y}{2}\right)^3\right]dy\,dx = 45;$

$I_y = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1)\,(x^2+z^2)\,dz\,dy\,dx = \int_{-1}^1 \int_{-2}^4 (x+1)\left[2x^2 + \frac{1}{3} - \frac{x^2y}{2} + \frac{1}{3}\left(1 - \frac{y}{2}\right)^3\right]dy\,dx = 15;$

$I_z = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1)\,(x^2+y^2)\,dz\,dy\,dx = \int_{-1}^1 \int_{-2}^4 (x+1)\left(2 - \frac{y}{2}\right)(x^2+y^2)\,dy\,dx = 42$

(d) $R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{5}{2}}$, $R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{5}{6}}$, and $R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{7}{3}}$

17. $M = \int_0^1 \int_{z-1}^{1-z} \int_0^{\sqrt{z}} (2y+5)\,dy\,dx\,dz = \int_0^1 \int_{z-1}^{1-z} \left(z + 5\sqrt{z}\right)dx\,dz = \int_0^1 2\left(z + 5\sqrt{z}\right)(1-z)\,dz$

$= 2\int_0^1 \left(5z^{1/2} + z - 5z^{3/2} - z^2\right)dz = 2\left[\frac{10}{3}z^{3/2} + \frac{1}{2}z^2 - 2z^{5/2} - \frac{1}{3}z^3\right]_0^1 = 2\left(\frac{9}{3} - \frac{3}{2}\right) = 3$

18. $M = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{2(x^2+y^2)}^{16-2(x^2+y^2)} \sqrt{x^2+y^2}\,dz\,dy\,dx = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,[16 - 4(x^2+y^2)]\,dy\,dx$

$= 4\int_0^{2\pi} \int_0^2 r\,(4-r^2)\,r\,dr\,d\theta = 4\int_0^{2\pi} \left[\frac{4r^3}{3} - \frac{r^5}{5}\right]_0^2 d\theta = 4\int_0^{2\pi} \frac{64}{15}\,d\theta = \frac{512\pi}{15}$

19. (a) Let ΔV_i be the volume of the ith piece, and let (x_i, y_i, z_i) be a point in the ith piece. Then the work done by gravity in moving the ith piece to the xy-plane is approximately $W_i = m_i g z_i = (x_i + y_i + z_i + 1)g\,\Delta V_i\,z_i$

$\Rightarrow$ the total work done is the triple integral $W = \int_0^1 \int_0^1 \int_0^1 (x+y+z+1)gz\,dz\,dy\,dx$

$= g\int_0^1 \int_0^1 \left[\frac{1}{2}xz^2 + \frac{1}{2}yz^2 + \frac{1}{3}z^3 + \frac{1}{2}z^2\right]_0^1 dy\,dx = g\int_0^1 \int_0^1 \left(\frac{1}{2}x + \frac{1}{2}y + \frac{5}{6}\right)dy\,dx = g\int_0^1 \left[\frac{1}{2}xy + \frac{1}{4}y^2 + \frac{5}{6}y\right]_0^1 dx$

$= g\int_0^1 \left(\frac{1}{2}x + \frac{13}{12}\right)dx = g\left[\frac{x^2}{4} + \frac{13}{12}x\right]_0^1 = g\left(\frac{16}{12}\right) = \frac{4}{3}g$

(b) From Exercise 15 the center of mass is $\left(\frac{8}{15}, \frac{8}{15}, \frac{8}{15}\right)$ and the mass of the liquid is $\frac{5}{2}$ $\Rightarrow$ the work done by gravity in moving the center of mass to the xy-plane is $W = mgd = \left(\frac{5}{2}\right)(g)\left(\frac{8}{15}\right) = \frac{4}{3}g$, which is the same as the work done in part (a).

20. (a) From Exercise 19(a) we see that the work done is $W = g \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} kxyz\,dz\,dy\,dx$
$= kg \int_0^2 \int_0^{\sqrt{x}} \frac{1}{2}xy\left(4-x^2\right)^2 dy\,dx = \frac{kg}{4}\int_0^2 x^2\left(4-x^2\right)^2 dx = \frac{kg}{4}\int_0^2 \left(16x^2 - 8x^4 + x^6\right) dx$
$= \frac{kg}{4}\left[\frac{16}{3}x^3 - \frac{8}{5}x^5 + \frac{1}{7}x^7\right]_0^2 = \frac{256k\cdot g}{105}$

(b) From Exercise 14 the center of mass is $\left(\frac{5}{4}, \frac{40\sqrt{2}}{77}, \frac{8}{7}\right)$ and the mass of the liquid is $\frac{32k}{15}$ $\Rightarrow$ the work done by gravity in moving the center of mass to the xy-plane is $W = mgd = \left(\frac{32k}{15}\right)(g)\left(\frac{8}{7}\right) = \frac{256k\cdot g}{105}$

21. (a) $\overline{x} = \frac{M_{yz}}{M} = 0 \Rightarrow \iiint_R x\delta(x,y,z)\,dx\,dy\,dz = 0 \Rightarrow M_{yz} = 0$

(b) $I_L = \iiint_D |\mathbf{v} - h\mathbf{i}|^2\,dm = \iiint_D |(x-h)\,\mathbf{i} + y\mathbf{j}|^2\,dm = \iiint_D \left(x^2 - 2xh + h^2 + y^2\right) dm$
$= \iiint_D \left(x^2+y^2\right) dm - 2h\iiint_D x\,dm + h^2\iiint_D dm = I_x - 0 + h^2m = I_{c.m.} + h^2m$

22. $I_L = I_{c.m.} + mh^2 = \frac{2}{5}ma^2 + ma^2 = \frac{7}{5}ma^2$

23. (a) $(\overline{x},\overline{y},\overline{z}) = \left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right) \Rightarrow I_z = I_{c.m.} + abc\left(\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}\right)^2 \Rightarrow I_{c.m.} = I_z - \frac{abc\left(a^2+b^2\right)}{4}$
$= \frac{abc\left(a^2+b^2\right)}{3} - \frac{abc\left(a^2+b^2\right)}{4} = \frac{abc\left(a^2+b^2\right)}{12}$; $R_{c.m.} = \sqrt{\frac{I_{c.m.}}{M}} = \sqrt{\frac{a^2+b^2}{12}}$

(b) $I_L = I_{c.m.} + abc\left(\sqrt{\frac{a^2}{4} + \left(\frac{b}{2} - 2b\right)^2}\right)^2 = \frac{abc\left(a^2+b^2\right)}{12} + \frac{abc\left(a^2+9b^2\right)}{4} = \frac{abc\left(4a^2+28b^2\right)}{12}$
$= \frac{abc\left(a^2+7b^2\right)}{3}$; $R_L = \sqrt{\frac{I_L}{M}} = \sqrt{\frac{a^2+7b^2}{3}}$

24. $M = \int_{-3}^{3}\int_{-2}^{4}\int_{-4/3}^{(4-2y)/3} dz\,dy\,dx = \int_{-3}^{3}\int_{-2}^{4}\frac{2}{3}(4-y)\,dy\,dx = \int_{-3}^{3}\frac{2}{3}\left[4y - \frac{y^2}{2}\right]_{-2}^{4} dx = 12\int_{-3}^{3}dx = 72;$
$\overline{x} = \overline{y} = \overline{z} = 0$ from Exercise 2 $\Rightarrow I_x = I_{c.m.} + 72\left(\sqrt{0^2+0^2}\right)^2 = I_{c.m.} \Rightarrow I_L = I_{c.m.} + 72\left(\sqrt{16+\frac{16}{9}}\right)^2$
$= 208 + 72\left(\frac{160}{9}\right) = 1488$

25. $M_{yz_{B_1\cup B_2}} = \iiint_{B_1} x\,dV_1 + \iiint_{B_2} x\,dV_2 = M_{(yz)_1} + M_{(yz)_2} \Rightarrow \overline{x} = M_{(yz)_1} + M_{(yz)_2 m_1+m_2}$; similarly,
$\overline{y} = M_{(xz)_1} + M_{(xz)_2 m_1+m_2}$ and $\overline{z} = M_{(xy)_1} + M_{(xy)_2 m_1+m_2} \Rightarrow \mathbf{c} = \overline{x}\mathbf{i} + \overline{y}\mathbf{j} + \overline{z}\mathbf{k}$
$= \frac{1}{m_1+m_2}\left[\left(M_{(yz)_1} + M_{(yz)_2}\right)\mathbf{i} + \left(M_{(xz)_1} + M_{(xz)_2}\right)\mathbf{j} + \left(M_{(xy)_1} + M_{(xy)_2}\right)\mathbf{k}\right]$
$= \frac{1}{m_1+m_2}\left[\left(m_1\overline{x}_1 + m_2\overline{x}_2\right)\mathbf{i} + \left(m_1\overline{y}_1 + m_2\overline{y}_2\right)\mathbf{j} + \left(m_1\overline{z}_1 + m_2\overline{z}_2\right)\mathbf{k}\right]$
$= \frac{1}{m_1+m_2}\left[m_1\left(\overline{x}_1\mathbf{i} + \overline{y}_1\mathbf{j} + \overline{z}_1\mathbf{k}\right) + m_2\left(\overline{x}_2\mathbf{i} + \overline{y}_2\mathbf{j} + \overline{z}_2\mathbf{k}\right)\right] = \frac{m_1\mathbf{c}_1 + m_2\mathbf{c}_2}{m_1+m_2}$

26. (a) $\mathbf{c} = 12\left(\mathbf{i} + \frac{3}{2}\mathbf{j} + \mathbf{k}\right) + 2\left(\frac{1}{2}\mathbf{i} + 4\mathbf{j} + \frac{1}{2}\mathbf{k}\right)12 + 2 = \frac{\frac{13}{2}\mathbf{i} + 13\mathbf{j} + \frac{13}{2}\mathbf{k}}{7} \Rightarrow \overline{x} = \frac{13}{14}, \overline{y} = \frac{13}{7}, \overline{z} = \frac{13}{14}$

(b) $\mathbf{c} = 12\left(\mathbf{i} + \frac{3}{2}\mathbf{j} + \mathbf{k}\right) + 12\left(\mathbf{i} + \frac{11}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}\right)12 + 12 = \frac{2\mathbf{i} + 7\mathbf{j} + \frac{1}{2}\mathbf{k}}{2} \Rightarrow \overline{x} = 1, \overline{y} = \frac{7}{2}, \overline{z} = \frac{1}{4}$

(c) $\mathbf{c} = 2\left(\frac{1}{2}\mathbf{i} + 4\mathbf{j} + \frac{1}{2}\mathbf{k}\right) + 12\left(\mathbf{i} + \frac{11}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}\right)2 + 12 = \frac{13\mathbf{i} + 74\mathbf{j} - 5\mathbf{k}}{14} \Rightarrow \overline{x} = \frac{13}{14}, \overline{y} = \frac{37}{7}, \overline{z} = -\frac{5}{14}$

(d) $\mathbf{c} = 12\left(\mathbf{i} + \frac{3}{2}\mathbf{j} + \mathbf{k}\right) + 2\left(\frac{1}{2}\mathbf{i} + 4\mathbf{j} + \frac{1}{2}\mathbf{k}\right) + 12\left(\mathbf{i} + \frac{11}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}\right)12 + 2 + 12 = \frac{25\mathbf{i} + 92\mathbf{j} + 7\mathbf{k}}{26} \Rightarrow \overline{x} = \frac{25}{26}, \overline{y} = \frac{46}{13}, \overline{z} = \frac{7}{26}$

27. (a) $\mathbf{c} = \frac{\left(\frac{\pi a^2 h}{3}\right)\left(\frac{h}{4}\mathbf{k}\right) + \left(\frac{2\pi a^3}{3}\right)\left(-\frac{3a}{8}\mathbf{k}\right)}{m_1 + m_2} = \frac{\left(\frac{a^2\pi}{3}\right)\left(\frac{h^2 - 3a^2}{4}\mathbf{k}\right)}{m_1 + m_2}$, where $m_1 = \frac{\pi a^2 h}{3}$ and $m_2 = \frac{2\pi a^3}{3}$; if $\frac{h^2 - 3a^2}{4} = 0$, or $h = a\sqrt{3}$, then the centroid is on the common base

(b) See the solution to Exercise 55, Section 15.2, to see that $h = a\sqrt{2}$.

28. $\mathbf{c} = \frac{\left(\frac{s^2 h}{3}\right)\left(\frac{h}{4}\mathbf{k}\right) + s^3\left(-\frac{s}{2}\mathbf{k}\right)}{m_1 + m_2} = \frac{\left(\frac{s^2}{12}\right)\left[(h^2 - 6s^2)\,\mathbf{k}\right]}{m_1 + m_2}$, where $m_1 = \frac{s^2 h}{3}$ and $m_2 = s^3$; if $h^2 - 6s^2 = 0$, or $h = \sqrt{6}s$, then the centroid is in the base of the pyramid. The corresponding result in 15.2, Exercise 56, is $h = \sqrt{3}s$.

15.6 TRIPLE INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES

1. $\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 \left[r(2-r^2)^{1/2} - r^2\right] dr\, d\theta = \int_0^{2\pi}\left[-\frac{1}{3}(2-r^2)^{3/2} - \frac{r^3}{3}\right]_0^1 d\theta$
$= \int_0^{2\pi}\left(\frac{2^{3/2}}{3} - \frac{2}{3}\right) d\theta = \frac{4\pi\left(\sqrt{2}-1\right)}{3}$

2. $\int_0^{2\pi}\int_0^3\int_{r^2/3}^{\sqrt{18-r^2}} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^3 \left[r(18-r^2)^{1/2} - \frac{r^3}{3}\right] dr\, d\theta = \int_0^{2\pi}\left[-\frac{1}{3}(18-r^2)^{3/2} - \frac{r^4}{12}\right]_0^3 d\theta$
$= \frac{9\pi\left(8\sqrt{2}-7\right)}{2}$

3. $\int_0^{2\pi}\int_0^{\theta/2\pi}\int_0^{3+24r^2} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^{\theta/2\pi}(3r + 24r^3)\, dr\, d\theta = \int_0^{2\pi}\left[\frac{3}{2}r^2 + 6r^4\right]_0^{\theta/2\pi} d\theta = \frac{3}{2}\int_0^{2\pi}\left(\frac{\theta^2}{4\pi^2} + \frac{4\theta^4}{16\pi^4}\right) d\theta$
$= \frac{3}{2}\left[\frac{\theta^3}{12\pi^2} + \frac{\theta^5}{20\pi^4}\right]_0^{2\pi} = \frac{17\pi}{5}$

4. $\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} z\, dz\, r\, dr\, d\theta = \int_0^{\pi}\int_0^{\theta/\pi}\frac{1}{2}\left[9(4-r^2) - (4-r^2)\right] r\, dr\, d\theta = 4\int_0^{\pi}\int_0^{\theta/\pi}(4r - r^3)\, dr\, d\theta$
$= 4\int_0^{\pi}\left[2r^2 - \frac{r^4}{4}\right]_0^{\theta/\pi} = 4\int_0^{\pi}\left(\frac{2\theta^2}{\pi^2} - \frac{\theta^4}{4\pi^4}\right) d\theta = \frac{37\pi}{15}$

5. $\int_0^{2\pi}\int_0^1\int_r^{(2-r^2)^{-1/2}} 3\, dz\, r\, dr\, d\theta = 3\int_0^{2\pi}\int_0^1 \left[r(2-r^2)^{-1/2} - r^2\right] dr\, d\theta = 3\int_0^{2\pi}\left[-(2-r^2)^{1/2} - \frac{r^3}{3}\right]_0^1 d\theta$
$= 3\int_0^{2\pi}\left(\sqrt{2} - \frac{4}{3}\right) d\theta = \pi\left(6\sqrt{2} - 8\right)$

6. $\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2}(r^2\sin^2\theta + z^2)\, dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1\left(r^3\sin^2\theta + \frac{r}{12}\right) dr\, d\theta = \int_0^{2\pi}\left(\frac{\sin^2\theta}{4} + \frac{1}{24}\right) d\theta = \frac{\pi}{3}$

7. $\int_0^{2\pi}\int_0^3\int_0^{z/3} r^3\, dr\, dz\, d\theta = \int_0^{2\pi}\int_0^3 \frac{z^4}{324}\, dz\, d\theta = \int_0^{2\pi}\frac{3}{20}\, d\theta = \frac{3\pi}{10}$

8. $\int_{-1}^1\int_0^{2\pi}\int_0^{1+\cos\theta} 4r\, dr\, d\theta\, dz = \int_{-1}^1\int_0^{2\pi} 2(1+\cos\theta)^2\, d\theta\, dz = \int_{-1}^1 6\pi\, d\theta = 12\pi$

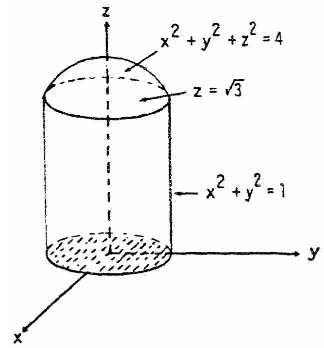
9. $\int_0^1\int_0^{\sqrt{z}}\int_0^{2\pi}(r^2\cos^2\theta + z^2)\, r\, d\theta\, dr\, dz = \int_0^1\int_0^{\sqrt{z}}\left[\frac{r^2\theta}{2} + \frac{r^2\sin 2\theta}{4} + z^2\theta\right]_0^{2\pi} r\, dr\, dz = \int_0^1\int_0^{\sqrt{z}}(\pi r^3 + 2\pi r z^2)\, dr\, dz$
$= \int_0^1\left[\frac{\pi r^4}{4} + \pi r^2 z^2\right]_0^{\sqrt{z}} dz = \int_0^1\left(\frac{\pi z^2}{4} + \pi z^3\right) dz = \left[\frac{\pi z^3}{12} + \frac{\pi z^4}{4}\right]_0^1 = \frac{\pi}{3}$

10. $\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\int_0^{2\pi}(r\sin\theta + 1)\, r\, d\theta\, dz\, dr = \int_0^2\int_{r-2}^{\sqrt{4-r^2}} 2\pi r\, dz\, dr = 2\pi\int_0^2\left[r(4-r^2)^{1/2} - r^2 + 2r\right] dr$
$= 2\pi\left[-\frac{1}{3}(4-r^2)^{3/2} - \frac{r^3}{3} + r^2\right]_0^2 = 2\pi\left[-\frac{8}{3} + 4 + \frac{1}{3}(4)^{3/2}\right] = 8\pi$

11. (a) $\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} dz\, r\, dr\, d\theta$

(b) $\int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^1 r\, dr\, dz\, d\theta + \int_0^{2\pi}\int_{\sqrt{3}}^2\int_0^{\sqrt{4-z^2}} r\, dr\, dz\, d\theta$

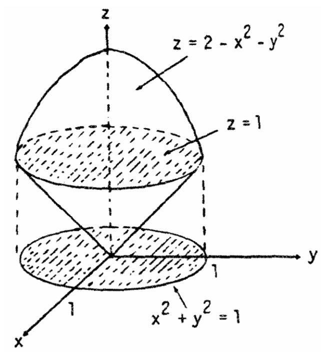
(c) $\int_0^1\int_0^{\sqrt{4-r^2}}\int_0^{2\pi} r\, d\theta\, dz\, dr$



12. (a) $\int_0^{2\pi}\int_0^1\int_r^{2-r^2} dz\, r\, dr\, d\theta$

(b) $\int_0^{2\pi}\int_0^1\int_0^z r\, dr\, dz\, d\theta + \int_0^{2\pi}\int_1^2\int_0^{\sqrt{2-z}} r\, dr\, dz\, d\theta$

(c) $\int_0^1\int_r^{2-r^2}\int_0^{2\pi} r\, d\theta\, dz\, dr$



13. $\int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}\int_0^{3r^2} f(r, \theta, z)\, dz\, r\, dr\, d\theta$

14. $\int_{-\pi/2}^{\pi/2}\int_0^1\int_0^{r\cos\theta} r^3\, dz\, dr\, d\theta = \int_{-\pi/2}^{\pi/2}\int_0^1 r^4\cos\theta\, dr\, d\theta = \frac{1}{5}\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta = \frac{2}{5}$

15. $\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{4-r\sin\theta} f(r, \theta, z)\, dz\, r\, dr\, d\theta$

16. $\int_{-\pi/2}^{\pi/2}\int_0^{3\cos\theta}\int_0^{5-r\cos\theta} f(r, \theta, z)\, dz\, r\, dr\, d\theta$

17. $\int_{-\pi/2}^{\pi/2}\int_1^{1+\cos\theta}\int_0^4 f(r, \theta, z)\, dz\, r\, dr\, d\theta$

18. $\int_{-\pi/2}^{\pi/2}\int_{\cos\theta}^{2\cos\theta}\int_0^{3-r\sin\theta} f(r, \theta, z)\, dz\, r\, dr\, d\theta$

19. $\int_0^{\pi/4}\int_0^{\sec\theta}\int_0^{2-r\sin\theta} f(r, \theta, z)\, dz\, r\, dr\, d\theta$

20. $\int_{\pi/4}^{\pi/2}\int_0^{\csc\theta}\int_0^{2-r\sin\theta} f(r, \theta, z)\, dz\, r\, dr\, d\theta$

21. $\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \frac{8}{3}\int_0^{\pi}\int_0^{\pi}\sin^4\phi\, d\phi\, d\theta = \frac{8}{3}\int_0^{\pi}\left(\left[-\frac{\sin^3\phi\cos\phi}{4}\right]_0^{\pi} + \frac{3}{4}\int_0^{\pi}\sin^2\phi\, d\phi\right) d\theta$
$= 2\int_0^{\pi}\int_0^{\pi}\sin^2\phi\, d\phi\, d\theta = \int_0^{\pi}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} d\theta = \int_0^{\pi}\pi\, d\theta = \pi^2$

22. $\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 (\rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4} 4\cos\phi\sin\phi\, d\phi\, d\theta = \int_0^{2\pi}\left[2\sin^2\phi\right]_0^{\pi/4} d\theta = \int_0^{2\pi} d\theta = 2\pi$

23. $\int_0^{2\pi}\int_0^{\pi}\int_0^{(1-\cos\phi)/2} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \frac{1}{24}\int_0^{2\pi}\int_0^{\pi}(1-\cos\phi)^3\sin\phi\, d\phi\, d\theta = \frac{1}{96}\int_0^{2\pi}\left[(1-\cos\phi)^4\right]_0^{\pi} d\theta$
$= \frac{1}{96}\int_0^{2\pi}(2^4 - 0)\, d\theta = \frac{16}{96}\int_0^{2\pi} d\theta = \frac{1}{6}(2\pi) = \frac{\pi}{3}$

24. $\int_0^{3\pi/2}\int_0^{\pi}\int_0^1 5\rho^3\sin^3\phi\, d\rho\, d\phi\, d\theta = \frac{5}{4}\int_0^{3\pi/2}\int_0^{\pi}\sin^3\phi\, d\phi\, d\theta = \frac{5}{4}\int_0^{3\pi/2}\left(\left[-\frac{\sin^2\phi\cos\phi}{3}\right]_0^{\pi} + \frac{2}{3}\int_0^{\pi}\sin\phi\, d\phi\right) d\theta$
$= \frac{5}{6}\int_0^{3\pi/2}\left[-\cos\phi\right]_0^{\pi} d\theta = \frac{5}{3}\int_0^{3\pi/2} d\theta = \frac{5\pi}{2}$

25. $\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} 3\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/3} (8-\sec^3\phi)\sin\phi \, d\phi \, d\theta = \int_0^{2\pi}\left[-8\cos\phi - \tfrac{1}{2}\sec^2\phi\right]_0^{\pi/3} d\theta$
$= \int_0^{2\pi}\left[(-4-2)-\left(-8-\tfrac{1}{2}\right)\right] d\theta = \tfrac{5}{2}\int_0^{2\pi} d\theta = 5\pi$

26. $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \rho^3 \sin\phi\cos\phi \, d\rho \, d\phi \, d\theta = \tfrac{1}{4}\int_0^{2\pi}\int_0^{\pi/4} \tan\phi\sec^2\phi \, d\phi \, d\theta = \tfrac{1}{4}\int_0^{2\pi}\left[\tfrac{1}{2}\tan^2\phi\right]_0^{\pi/4} d\theta$
$= \tfrac{1}{8}\int_0^{2\pi} d\theta = \tfrac{\pi}{4}$

27. $\int_0^2\int_{-\pi}^0\int_{\pi/4}^{\pi/2} \rho^3 \sin 2\phi \, d\phi \, d\theta \, d\rho = \int_0^2\int_{-\pi}^0 \rho^3\left[-\tfrac{\cos 2\phi}{2}\right]_{\pi/4}^{\pi/2} d\theta \, d\rho = \int_0^2\int_{-\pi}^0 \tfrac{\rho^3}{2} \, d\theta \, d\rho = \int_0^2 \tfrac{\rho^3\pi}{2} \, d\rho$
$= \left[\tfrac{\pi\rho^4}{8}\right]_0^2 = 2\pi$

28. $\int_{\pi/6}^{\pi/3}\int_{\csc\phi}^{2\csc\phi}\int_0^{2\pi} \rho^2 \sin\phi \, d\theta \, d\rho \, d\phi = 2\pi\int_{\pi/6}^{\pi/3}\int_{\csc\phi}^{2\csc\phi} \rho^2\sin\phi \, d\rho \, d\phi = \tfrac{2\pi}{3}\int_{\pi/6}^{\pi/3}\left[\rho^3\sin\phi\right]_{\csc\phi}^{2\csc\phi} d\phi$
$= \tfrac{14\pi}{3}\int_{\pi/6}^{\pi/3} \csc^2\phi \, d\phi = \tfrac{28\pi}{3\sqrt{3}}$

29. $\int_0^1\int_0^{\pi}\int_0^{\pi/4} 12\rho\sin^3\phi \, d\phi \, d\theta \, d\rho = \int_0^1\int_0^{\pi}\left(12\rho\left[\tfrac{-\sin^2\phi\cos\phi}{3}\right]_0^{\pi/4} + 8\rho\int_0^{\pi/4}\sin\phi \, d\phi\right) d\theta \, d\rho$
$= \int_0^1\int_0^{\pi}\left(-\tfrac{2\rho}{\sqrt{2}} - 8\rho\left[\cos\phi\right]_0^{\pi/4}\right) d\theta \, d\rho = \int_0^1\int_0^{\pi}\left(8\rho - \tfrac{10\rho}{\sqrt{2}}\right) d\theta \, d\rho = \pi\int_0^1\left(8\rho - \tfrac{10\rho}{\sqrt{2}}\right) d\rho = \pi\left[4\rho^2 - \tfrac{5\rho^2}{\sqrt{2}}\right]_0^1$
$= \dfrac{\left(4\sqrt{2}-5\right)\pi}{\sqrt{2}}$

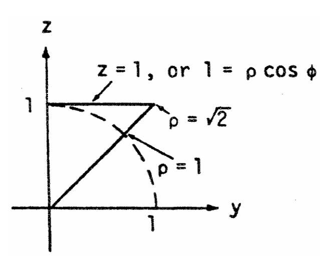
30. $\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\phi}^{2} 5\rho^4\sin^3\phi \, d\rho \, d\theta \, d\phi = \int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}(32-\csc^5\phi)\sin^3\phi \, d\theta \, d\phi = \int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}(32\sin^3\phi - \csc^2\phi) \, d\theta \, d\phi$
$= \pi\int_{\pi/6}^{\pi/2}(32\sin^3\phi - \csc^2\phi) \, d\phi = \pi\left[-\tfrac{32\sin^2\phi\cos\phi}{3}\right]_{\pi/6}^{\pi/2} + \tfrac{64\pi}{3}\int_{\pi/6}^{\pi/2}\sin\phi \, d\phi + \pi\left[\cot\phi\right]_{\pi/6}^{\pi/2}$
$= \pi\left(\tfrac{32\sqrt{3}}{24}\right) - \tfrac{64\pi}{3}\left[\cos\phi\right]_{\pi/6}^{\pi/2} - \pi\left(\sqrt{3}\right) = \tfrac{\sqrt{3}}{3}\pi + \left(\tfrac{64\pi}{3}\right)\left(\tfrac{\sqrt{3}}{2}\right) = \tfrac{33\pi\sqrt{3}}{3} = 11\pi\sqrt{3}$

31. (a) $x^2+y^2=1 \Rightarrow \rho^2\sin^2\phi = 1$, and $\rho\sin\phi = 1 \Rightarrow \rho = \csc\phi$; thus
$\int_0^{2\pi}\int_0^{\pi/6}\int_0^2 \rho^2\sin\phi \, d\rho \, d\phi \, d\theta + \int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^{\csc\phi} \rho^2\sin\phi \, d\rho \, d\phi \, d\theta$

(b) $\int_0^{2\pi}\int_1^2\int_{\pi/6}^{\sin^{-1}(1/\rho)} \rho^2\sin\phi \, d\phi \, d\rho \, d\theta + \int_0^{2\pi}\int_0^2\int_0^{\pi/6} \rho^2\sin\phi \, d\phi \, d\rho \, d\theta$

32. (a) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \rho^2\sin\phi \, d\rho \, d\phi \, d\theta$

(b) $\int_0^{2\pi}\int_0^1\int_0^{\pi/4} \rho^2\sin\phi \, d\phi \, d\rho \, d\theta$
$+ \int_0^{2\pi}\int_1^{\sqrt{2}}\int_{\cos^{-1}(1/\rho)}^{\pi/4} \rho^2\sin\phi \, d\phi \, d\rho \, d\theta$



33. $V = \int_0^{2\pi}\int_0^{\pi/2}\int_{\cos\phi}^{2} \rho^2\sin\phi \, d\rho \, d\phi \, d\theta = \tfrac{1}{3}\int_0^{2\pi}\int_0^{\pi/2}(8-\cos^3\phi)\sin\phi \, d\phi \, d\theta$
$= \tfrac{1}{3}\int_0^{2\pi}\left[-8\cos\phi + \tfrac{\cos^4\phi}{4}\right]_0^{\pi/2} d\theta = \tfrac{1}{3}\int_0^{2\pi}\left(8-\tfrac{1}{4}\right) d\theta = \left(\tfrac{31}{12}\right)(2\pi) = \tfrac{31\pi}{6}$

34. $V = \int_0^{2\pi}\int_0^{\pi/2}\int_1^{1+\cos\phi} \rho^2\sin\phi \, d\rho \, d\phi \, d\theta = \tfrac{1}{3}\int_0^{2\pi}\int_0^{\pi/2}(3\cos\phi + 3\cos^2\phi + \cos^3\phi)\sin\phi \, d\phi \, d\theta$
$= \tfrac{1}{3}\int_0^{2\pi}\left[-\tfrac{3}{2}\cos^2\phi - \cos^3\phi - \tfrac{1}{4}\cos^4\phi\right]_0^{\pi/2} d\theta = \tfrac{1}{3}\int_0^{2\pi}\left(\tfrac{3}{2}+1+\tfrac{1}{4}\right) d\theta = \tfrac{11}{12}\int_0^{2\pi} d\theta = \left(\tfrac{11}{12}\right)(2\pi) = \tfrac{11\pi}{6}$

35. $V=\int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\frac{1}{3}\int_0^{2\pi}\int_0^{\pi}(1-\cos\phi)^3\sin\phi\,d\phi\,d\theta=\frac{1}{3}\int_0^{2\pi}\left[\frac{(1-\cos\phi)^4}{4}\right]_0^{\pi}d\theta$
$=\frac{1}{12}(2)^4\int_0^{2\pi}d\theta=\frac{4}{3}(2\pi)=\frac{8\pi}{3}$

36. $V=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{1-\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2}(1-\cos\phi)^3\sin\phi\,d\phi\,d\theta=\frac{1}{3}\int_0^{2\pi}\left[\frac{(1-\cos\phi)^4}{4}\right]_0^{\pi/2}d\theta$
$=\frac{1}{12}\int_0^{2\pi}d\theta=\frac{1}{12}(2\pi)=\frac{\pi}{6}$

37. $V=\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{2\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\frac{8}{3}\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\cos^3\phi\sin\phi\,d\phi\,d\theta=\frac{8}{3}\int_0^{2\pi}\left[-\frac{\cos^4\phi}{4}\right]_{\pi/4}^{\pi/2}d\theta$
$=\left(\frac{8}{3}\right)\left(\frac{1}{16}\right)\int_0^{2\pi}d\theta=\frac{1}{6}(2\pi)=\frac{\pi}{3}$

38. $V=\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\frac{8}{3}\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\sin\phi\,d\phi\,d\theta=\frac{8}{3}\int_0^{2\pi}[-\cos\phi]_{\pi/3}^{\pi/2}d\theta=\frac{4}{3}\int_0^{2\pi}d\theta=\frac{8\pi}{3}$

39. (a) $8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$ (b) $8\int_0^{\pi/2}\int_0^{2}\int_0^{\sqrt{4-r^2}}dz\,r\,dr\,d\theta$
(c) $8\int_0^{2}\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}}dz\,dy\,dx$

40. (a) $\int_0^{\pi/2}\int_0^{3/\sqrt{2}}\int_r^{\sqrt{9-r^2}}dz\,r\,dr\,d\theta$ (b) $\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{3}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$
(c) $\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{3}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=9\int_0^{\pi/2}\int_0^{\pi/4}\sin\phi\,d\phi\,d\theta=-9\int_0^{\pi/2}\left(\frac{1}{\sqrt{2}}-1\right)d\theta=\frac{9\pi\left(2-\sqrt{2}\right)}{4}$

41. (a) $V=\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$ (b) $V=\int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}}dz\,r\,dr\,d\theta$
(c) $V=\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}}dz\,dy\,dx$
(d) $V=\int_0^{2\pi}\int_0^{\sqrt{3}}\left[r(4-r^2)^{1/2}-r\right]dr\,d\theta=\int_0^{2\pi}\left[-\frac{(4-r^2)^{3/2}}{3}-\frac{r^2}{2}\right]_0^{\sqrt{3}}d\theta=\int_0^{2\pi}\left(-\frac{1}{3}-\frac{3}{2}+\frac{4^{3/2}}{3}\right)d\theta$
$=\frac{5}{6}\int_0^{2\pi}d\theta=\frac{5\pi}{3}$

42. (a) $I_z=\int_0^{2\pi}\int_0^{1}\int_0^{\sqrt{1-r^2}}r^2\,dz\,r\,dr\,d\theta$
(b) $I_z=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{1}(\rho^2\sin^2\phi)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$, since $r^2=x^2+y^2=\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta$
$=\rho^2\sin^2\phi$
(c) $I_z=\int_0^{2\pi}\int_0^{\pi/2}\frac{1}{5}\sin^3\phi\,d\phi\,d\theta=\frac{1}{5}\int_0^{2\pi}\left(\left[-\frac{\sin^2\phi\cos\phi}{3}\right]_0^{\pi/2}+\frac{2}{3}\int_0^{\pi/2}\sin\phi\,d\phi\right)d\theta=\frac{2}{15}\int_0^{2\pi}[-\cos\phi]_0^{\pi/2}d\theta$
$=\frac{2}{15}(2\pi)=\frac{4\pi}{15}$

43. $V=4\int_0^{\pi/2}\int_0^{1}\int_{r^4-1}^{4-4r^2}dz\,r\,dr\,d\theta=4\int_0^{\pi/2}\int_0^{1}(5r-4r^3-r^5)\,dr\,d\theta=4\int_0^{\pi/2}\left(\frac{5}{2}-1-\frac{1}{6}\right)d\theta$
$=4\int_0^{\pi/2}\frac{8}{6}\,d\theta=\frac{8\pi}{3}$

44. $V=4\int_0^{\pi/2}\int_0^{1}\int_{-\sqrt{1-r^2}}^{1-r}dz\,r\,dr\,d\theta=4\int_0^{\pi/2}\int_0^{1}\left(r-r^2+r\sqrt{1-r^2}\right)dr\,d\theta=4\int_0^{\pi/2}\left[\frac{r^2}{2}-\frac{r^3}{3}-\frac{1}{3}(1-r^2)^{3/2}\right]_0^{1}d\theta$
$=4\int_0^{\pi/2}\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{3}\right)d\theta=2\int_0^{\pi/2}d\theta=2\left(\frac{\pi}{2}\right)=\pi$

45. $V = \int_{3\pi/2}^{2\pi}\int_0^{3\cos\theta}\int_0^{-r\sin\theta} dz\, r\, dr\, d\theta = \int_{3\pi/2}^{2\pi}\int_0^{3\cos\theta} -r^2\sin\theta\, dr\, d\theta = \int_{3\pi/2}^{2\pi}(-9\cos^3\theta)(\sin\theta)\, d\theta$
$= \left[\frac{9}{4}\cos^4\theta\right]_{3\pi/2}^{2\pi} = \frac{9}{4} - 0 = \frac{9}{4}$

46. $V = 2\int_{\pi/2}^{\pi}\int_0^{-3\cos\theta}\int_0^{r} dz\, r\, dr\, d\theta = 2\int_{\pi/2}^{\pi}\int_0^{-3\cos\theta} r^2\, dr\, d\theta = \frac{2}{3}\int_{\pi/2}^{\pi} -27\cos^3\theta\, d\theta$
$= -18\left(\left[\frac{\cos^2\theta\sin\theta}{3}\right]_{\pi/2}^{\pi} + \frac{2}{3}\int_{\pi/2}^{\pi}\cos\theta\, d\theta\right) = -12\left[\sin\theta\right]_{\pi/2}^{\pi} = 12$

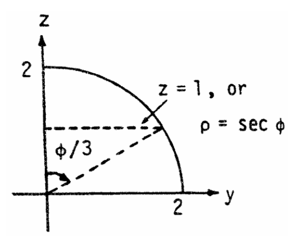
47. $V = \int_0^{\pi/2}\int_0^{\sin\theta}\int_0^{\sqrt{1-r^2}} dz\, r\, dr\, d\theta = \int_0^{\pi/2}\int_0^{\sin\theta} r\sqrt{1-r^2}\, dr\, d\theta = \int_0^{\pi/2}\left[-\frac{1}{3}\left(1-r^2\right)^{3/2}\right]_0^{\sin\theta} d\theta$
$= -\frac{1}{3}\int_0^{\pi/2}\left[\left(1-\sin^2\theta\right)^{3/2} - 1\right] d\theta = -\frac{1}{3}\int_0^{\pi/2}(\cos^3\theta - 1)\, d\theta = -\frac{1}{3}\left(\left[\frac{\cos^2\theta\sin\theta}{3}\right]_0^{\pi/2} + \frac{2}{3}\int_0^{\pi/2}\cos\theta\, d\theta\right) + \left[\frac{\theta}{3}\right]_0^{\pi/2}$
$= -\frac{2}{9}\left[\sin\theta\right]_0^{\pi/2} + \frac{\pi}{6} = \frac{-4+3\pi}{18}$

48. $V = \int_0^{\pi/2}\int_0^{\cos\theta}\int_0^{3\sqrt{1-r^2}} dz\, r\, dr\, d\theta = \int_0^{\pi/2}\int_0^{\cos\theta} 3r\sqrt{1-r^2}\, dr\, d\theta = \int_0^{\pi/2}\left[-\left(1-r^2\right)^{3/2}\right]_0^{\cos\theta} d\theta$
$= \int_0^{\pi/2}\left[-\left(1-\cos^2\theta\right)^{3/2} + 1\right] d\theta = \int_0^{\pi/2}(1-\sin^3\theta)\, d\theta = \left[\theta + \frac{\sin^2\theta\cos\theta}{3}\right]_0^{\pi/2} - \frac{2}{3}\int_0^{\pi/2}\sin\theta\, d\theta$
$= \frac{\pi}{2} + \frac{2}{3}\left[\cos\theta\right]_0^{\pi/2} = \frac{\pi}{2} - \frac{2}{3} = \frac{3\pi-4}{6}$

49. $V = \int_0^{2\pi}\int_{\pi/3}^{2\pi/3}\int_0^{a}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_{\pi/3}^{2\pi/3}\frac{a^3}{3}\sin\phi\, d\phi\, d\theta = \frac{a^3}{3}\int_0^{2\pi}\left[-\cos\phi\right]_{\pi/3}^{2\pi/3} d\theta = \frac{a^3}{3}\int_0^{2\pi}\left(\frac{1}{2}+\frac{1}{2}\right) d\theta = \frac{2\pi a^3}{3}$

50. $V = \int_0^{\pi/6}\int_0^{\pi/2}\int_0^{a}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \frac{a^3}{3}\int_0^{\pi/6}\int_0^{\pi/2}\sin\phi\, d\phi\, d\theta = \frac{a^3}{3}\int_0^{\pi/6} d\theta = \frac{a^3\pi}{18}$

51. $V = \int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$
$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/3}(8\sin\phi - \tan\phi\sec^2\phi)\, d\phi\, d\theta$
$= \frac{1}{3}\int_0^{2\pi}\left[-8\cos\phi - \frac{1}{2}\tan^2\phi\right]_0^{\pi/3} d\theta$
$= \frac{1}{3}\int_0^{2\pi}\left[-4 - \frac{1}{2}(3) + 8\right] d\theta = \frac{1}{3}\int_0^{2\pi}\frac{5}{2}\, d\theta = \frac{5}{6}(2\pi) = \frac{5\pi}{3}$



52. $V = 4\int_0^{\pi/2}\int_0^{\pi/4}\int_{\sec\phi}^{2\sec\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \frac{4}{3}\int_0^{\pi/2}\int_0^{\pi/4}(8\sec^3\phi - \sec^3\phi)\sin\phi\, d\phi\, d\theta$
$= \frac{28}{3}\int_0^{\pi/2}\int_0^{\pi/4}\sec^3\phi\sin\phi\, d\phi\, d\theta = \frac{28}{3}\int_0^{\pi/2}\int_0^{\pi/4}\tan\phi\sec^2\phi\, d\phi\, d\theta = \frac{28}{3}\int_0^{\pi/2}\left[\frac{1}{2}\tan^2\phi\right]_0^{\pi/4} d\theta$
$= \frac{14}{3}\int_0^{\pi/2} d\theta = \frac{7\pi}{3}$

53. $V = 4\int_0^{\pi/2}\int_0^{1}\int_0^{r^2} dz\, r\, dr\, d\theta = 4\int_0^{\pi/2}\int_0^{1} r^3\, dr\, d\theta = \int_0^{\pi/2} d\theta = \frac{\pi}{2}$

54. $V = 4\int_0^{\pi/2}\int_0^{1}\int_{r^2}^{r^2+1} dz\, r\, dr\, d\theta = 4\int_0^{\pi/2}\int_0^{1} r\, dr\, d\theta = 2\int_0^{\pi/2} d\theta = \pi$

55. $V = 8\int_0^{\pi/2}\int_1^{\sqrt{2}}\int_0^{r} dz\, r\, dr\, d\theta = 8\int_0^{\pi/2}\int_1^{\sqrt{2}} r^2\, dr\, d\theta = 8\left(\frac{2\sqrt{2}-1}{3}\right)\int_0^{\pi/2} d\theta = \frac{4\pi\left(2\sqrt{2}-1\right)}{3}$

56. $V = 8\int_0^{\pi/2}\int_1^{\sqrt{2}}\int_0^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta = 8\int_0^{\pi/2}\int_1^{\sqrt{2}} r\sqrt{2-r^2}\, dr\, d\theta = 8\int_0^{\pi/2}\left[-\frac{1}{3}\left(2-r^2\right)^{3/2}\right]_1^{\sqrt{2}} d\theta$
$= \frac{8}{3}\int_0^{\pi/2} d\theta = \frac{4\pi}{3}$

57. $V = \int_0^{2\pi}\int_0^2\int_0^{4-r\sin\theta} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^2 (4r - r^2\sin\theta)\, dr\, d\theta = 8\int_0^{2\pi}\left(1 - \frac{\sin\theta}{3}\right) d\theta = 16\pi$

58. $V = \int_0^{2\pi}\int_0^2\int_0^{4-r\cos\theta-r\sin\theta} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^2 [4r - r^2(\cos\theta + \sin\theta)]\, dr\, d\theta = \frac{8}{3}\int_0^{2\pi}(3 - \cos\theta - \sin\theta)\, d\theta = 16\pi$

59. The paraboloids intersect when $4x^2 + 4y^2 = 5 - x^2 - y^2 \Rightarrow x^2 + y^2 = 1$ and $z = 4$
$\Rightarrow V = 4\int_0^{\pi/2}\int_0^1\int_{4r^2}^{5-r^2} dz\, r\, dr\, d\theta = 4\int_0^{\pi/2}\int_0^1 (5r - 5r^3)\, dr\, d\theta = 20\int_0^{\pi/2}\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 d\theta = 5\int_0^{\pi/2} d\theta = \frac{5\pi}{2}$

60. The paraboloid intersects the xy-plane when $9 - x^2 - y^2 = 0 \Rightarrow x^2 + y^2 = 9 \Rightarrow$
$V = 4\int_0^{\pi/2}\int_1^3\int_0^{9-r^2} dz\, r\, dr\, d\theta = 4\int_0^{\pi/2}\int_1^3 (9r - r^3)\, dr\, d\theta = 4\int_0^{\pi/2}\left[\frac{9r^2}{2} - \frac{r^4}{4}\right]_1^3 d\theta = 4\int_0^{\pi/2}\left(\frac{81}{4} - \frac{17}{4}\right) d\theta$
$= 64\int_0^{\pi/2} d\theta = 32\pi$

61. $V = 8\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} dz\, r\, dr\, d\theta = 8\int_0^{2\pi}\int_0^1 r(4 - r^2)^{1/2}\, dr\, d\theta = 8\int_0^{2\pi}\left[-\frac{1}{3}(4 - r^2)^{3/2}\right]_0^1 d\theta$
$= -\frac{8}{3}\int_0^{2\pi}\left(3^{3/2} - 8\right) d\theta = \frac{4\pi\left(8 - 3\sqrt{3}\right)}{3}$

62. The sphere and paraboloid intersect when $x^2 + y^2 + z^2 = 2$ and $z = x^2 + y^2 \Rightarrow z^2 + z - 2 = 0$
$\Rightarrow (z + 2)(z - 1) = 0 \Rightarrow z = 1$ or $z = -2 \Rightarrow z = 1$ since $z \ge 0$. Thus, $x^2 + y^2 = 1$ and the volume is given by the triple integral $V = 4\int_0^{\pi/2}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta = 4\int_0^{\pi/2}\int_0^1\left[r(2 - r^2)^{1/2} - r^3\right] dr\, d\theta$
$= 4\int_0^{\pi/2}\left[-\frac{1}{3}(2 - r^2)^{3/2} - \frac{r^4}{4}\right]_0^1 d\theta = 4\int_0^{\pi/2}\left(\frac{2\sqrt{2}}{3} - \frac{7}{12}\right) d\theta = \frac{\pi\left(8\sqrt{2} - 7\right)}{6}$

63. $\text{average} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1\int_{-1}^1 r^2\, dz\, dr\, d\theta = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 2r^2\, dr\, d\theta = \frac{1}{3\pi}\int_0^{2\pi} d\theta = \frac{2}{3}$

64. $\text{average} = \frac{1}{\left(\frac{4\pi}{3}\right)}\int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^2\, dz\, dr\, d\theta = \frac{3}{4\pi}\int_0^{2\pi}\int_0^1 2r^2\sqrt{1 - r^2}\, dr\, d\theta$
$= \frac{3}{2\pi}\int_0^{2\pi}\left[\frac{1}{8}\sin^{-1} r - \frac{1}{8} r\sqrt{1 - r^2}\,(1 - 2r^2)\right]_0^1 d\theta = \frac{3}{16\pi}\int_0^{2\pi}\left(\frac{\pi}{2} + 0\right) d\theta = \frac{3}{32}\int_0^{2\pi} d\theta = \left(\frac{3}{32}\right)(2\pi) = \frac{3\pi}{16}$

65. $\text{average} = \frac{1}{\left(\frac{4\pi}{3}\right)}\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^3\sin\phi\, d\rho\, d\phi\, d\theta = \frac{3}{16\pi}\int_0^{2\pi}\int_0^{\pi}\sin\phi\, d\phi\, d\theta = \frac{3}{8\pi}\int_0^{2\pi} d\theta = \frac{3}{4}$

66. $\text{average} = \frac{1}{\left(\frac{2\pi}{3}\right)}\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^3\cos\phi\sin\phi\, d\rho\, d\phi\, d\theta = \frac{3}{8\pi}\int_0^{2\pi}\int_0^{\pi/2}\cos\phi\sin\phi\, d\phi\, d\theta = \frac{3}{8\pi}\int_0^{2\pi}\left[\frac{\sin^2\phi}{2}\right]_0^{\pi/2} d\theta$
$= \frac{3}{16\pi}\int_0^{2\pi} d\theta = \left(\frac{3}{16\pi}\right)(2\pi) = \frac{3}{8}$

67. $M = 4\int_0^{\pi/2}\int_0^1\int_0^r dz\, r\, dr\, d\theta = 4\int_0^{\pi/2}\int_0^1 r^2\, dr\, d\theta = \frac{4}{3}\int_0^{\pi/2} d\theta = \frac{2\pi}{3}$; $M_{xy} = \int_0^{2\pi}\int_0^1\int_0^r z\, dz\, r\, dr\, d\theta$
$= \frac{1}{2}\int_0^{2\pi}\int_0^1 r^3\, dr\, d\theta = \frac{1}{8}\int_0^{2\pi} d\theta = \frac{\pi}{4} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{\pi}{4}\right)\left(\frac{3}{2\pi}\right) = \frac{3}{8}$, and $\bar{x} = \bar{y} = 0$, by symmetry

68. $M = \int_0^{\pi/2}\int_0^2\int_0^r dz\, r\, dr\, d\theta = \int_0^{\pi/2}\int_0^2 r^2\, dr\, d\theta = \frac{8}{3}\int_0^{\pi/2} d\theta = \frac{4\pi}{3}$; $M_{yz} = \int_0^{\pi/2}\int_0^2\int_0^r x\, dz\, r\, dr\, d\theta$
$= \int_0^{\pi/2}\int_0^2 r^3\cos\theta\, dr\, d\theta = 4\int_0^{\pi/2}\cos\theta\, d\theta = 4$; $M_{xz} = \int_0^{\pi/2}\int_0^2\int_0^r y\, dz\, r\, dr\, d\theta = \int_0^{\pi/2}\int_0^2 r^3\sin\theta\, dr\, d\theta$
$= 4\int_0^{\pi/2}\sin\theta\, d\theta = 4$; $M_{xy} = \int_0^{\pi/2}\int_0^2\int_0^r z\, dz\, r\, dr\, d\theta = \frac{1}{2}\int_0^{\pi/2}\int_0^2 r^3\, dr\, d\theta = 2\int_0^{\pi/2} d\theta = \pi \Rightarrow \bar{x} = \frac{M_{yz}}{M} = \frac{3}{\pi}$,
$\bar{y} = \frac{M_{xz}}{M} = \frac{3}{\pi}$, and $\bar{z} = \frac{M_{xy}}{M} = \frac{3}{4}$

69. $M = \frac{8\pi}{3}$; $M_{xy} = \int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^2 z\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^2 \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta = 4\int_0^{2\pi}\int_{\pi/3}^{\pi/2} \cos\phi \sin\phi \, d\phi \, d\theta$ $= 4\int_0^{2\pi}\left[\frac{\sin^2\phi}{2}\right]_{\pi/3}^{\pi/2} d\theta = 4\int_0^{2\pi}\left(\frac{1}{2}-\frac{3}{8}\right) d\theta = \frac{1}{2}\int_0^{2\pi} d\theta = \pi \Rightarrow \overline{z} = \frac{M_{xy}}{M} = (\pi)\left(\frac{3}{8\pi}\right) = \frac{3}{8}$, and $\overline{x} = \overline{y} = 0$, by symmetry

70. $M = \int_0^{2\pi}\int_0^{\pi/4}\int_0^a \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{a^3}{3}\int_0^{2\pi}\int_0^{\pi/4} \sin\phi \, d\phi \, d\theta = \frac{a^3}{3}\int_0^{2\pi} \frac{2-\sqrt{2}}{2} \, d\theta = \frac{\pi a^3\left(2-\sqrt{2}\right)}{3}$;
$M_{xy} = \int_0^{2\pi}\int_0^{\pi/4}\int_0^a \rho^3 \sin\phi\cos\phi \, d\rho \, d\phi \, d\theta = \frac{a^4}{4}\int_0^{2\pi}\int_0^{\pi/4} \sin\phi\cos\phi \, d\phi \, d\theta = \frac{a^4}{16}\int_0^{2\pi} d\theta = \frac{\pi a^4}{8}$
$\Rightarrow \overline{z} = \frac{M_{xy}}{M} = \left(\frac{\pi a^4}{8}\right)\left[\frac{3}{\pi a^3\left(2-\sqrt{2}\right)}\right] = \left(\frac{3a}{8}\right)\left(\frac{2+\sqrt{2}}{2}\right) = \frac{3\left(2+\sqrt{2}\right)a}{16}$, and $\overline{x} = \overline{y} = 0$, by symmetry

71. $M = \int_0^{2\pi}\int_0^4\int_0^{\sqrt{r}} dz \, r \, dr \, d\theta = \int_0^{2\pi}\int_0^4 r^{3/2} \, dr \, d\theta = \frac{64}{5}\int_0^{2\pi} d\theta = \frac{128\pi}{5}$; $M_{xy} = \int_0^{2\pi}\int_0^4\int_0^{\sqrt{r}} z \, dz \, r \, dr \, d\theta$ $= \frac{1}{2}\int_0^{2\pi}\int_0^4 r^2 \, dr \, d\theta = \frac{32}{3}\int_0^{2\pi} d\theta = \frac{64\pi}{3} \Rightarrow \overline{z} = \frac{M_{xy}}{M} = \frac{5}{6}$, and $\overline{x} = \overline{y} = 0$, by symmetry

72. $M = \int_{-\pi/3}^{\pi/3}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} dz \, r \, dr \, d\theta = \int_{-\pi/3}^{\pi/3}\int_0^1 2r\sqrt{1-r^2} \, dr \, d\theta = \int_{-\pi/3}^{\pi/3}\left[-\frac{2}{3}\left(1-r^2\right)^{3/2}\right]_0^1 d\theta$
$= \frac{2}{3}\int_{-\pi/3}^{\pi/3} d\theta = \left(\frac{2}{3}\right)\left(\frac{2\pi}{3}\right) = \frac{4\pi}{9}$; $M_{yz} = \int_{-\pi/3}^{\pi/3}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^2\cos\theta \, dz \, dr \, d\theta = 2\int_{-\pi/3}^{\pi/3}\int_0^1 r^2\sqrt{1-r^2}\cos\theta \, dr \, d\theta$
$= 2\int_{-\pi/3}^{\pi/3}\left[\frac{1}{8}\sin^{-1} r - \frac{1}{8} r\sqrt{1-r^2}\left(1-2r^2\right)\right]_0^1 \cos\theta \, d\theta = \frac{\pi}{8}\int_{-\pi/3}^{\pi/3} \cos\theta \, d\theta = \frac{\pi}{8}\left[\sin\theta\right]_{-\pi/3}^{\pi/3} = \left(\frac{\pi}{8}\right)\left(2\cdot\frac{\sqrt{3}}{2}\right) = \frac{\pi\sqrt{3}}{8}$
$\Rightarrow \overline{x} = \frac{M_{yz}}{M} = \frac{9\sqrt{3}}{32}$, and $\overline{y} = \overline{z} = 0$, by symmetry

73. $I_z = \int_0^{2\pi}\int_1^2\int_0^4 \left(x^2+y^2\right) dz \, r \, dr \, d\theta = 4\int_0^{2\pi}\int_1^2 r^3 \, dr \, d\theta = \int_0^{2\pi} 15 \, d\theta = 30\pi$; $M = \int_0^{2\pi}\int_1^2\int_0^4 dz \, r \, dr \, d\theta$ $= \int_0^{2\pi}\int_1^2 4r \, dr \, d\theta = \int_0^{2\pi} 6 \, d\theta = 12\pi \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{5}{2}}$

74. (a) $I_z = \int_0^{2\pi}\int_0^1\int_{-1}^1 r^3 \, dz \, dr \, d\theta = 2\int_0^{2\pi}\int_0^1 r^3 \, dr \, d\theta = \frac{1}{2}\int_0^{2\pi} d\theta = \pi$
(b) $I_x = \int_0^{2\pi}\int_0^1\int_{-1}^1 \left(r^2\sin^2\theta + z^2\right) dz \, r \, dr \, d\theta = 2\int_0^{2\pi}\int_0^1 \left(2r^3\sin^2\theta + \frac{2r}{3}\right) dr \, d\theta = \int_0^{2\pi}\left(\frac{\sin^2\theta}{2}+\frac{1}{3}\right) d\theta$
$= \left[\frac{\theta}{4} - \frac{\sin 2\theta}{8} + \frac{\theta}{3}\right]_0^{2\pi} = \frac{\pi}{2} + \frac{2\pi}{3} = \frac{7\pi}{6}$

75. We orient the cone with its vertex at the origin and axis along the z-axis $\Rightarrow \phi = \frac{\pi}{4}$. We use the the x-axis which is through the vertex and parallel to the base of the cone $\Rightarrow I_x = \int_0^{2\pi}\int_0^1\int_r^1 \left(r^2\sin^2\theta + z^2\right) dz \, r \, dr \, d\theta$
$= \int_0^{2\pi}\int_0^1 \left(r^3\sin^2\theta - r^4\sin^2\theta + \frac{r}{3} - \frac{r^4}{3}\right) dr \, d\theta = \int_0^{2\pi}\left(\frac{\sin^2\theta}{20} + \frac{1}{10}\right) d\theta = \left[\frac{\theta}{40} - \frac{\sin 2\theta}{80} + \frac{\theta}{10}\right]_0^{2\pi} = \frac{\pi}{20} + \frac{\pi}{5} = \frac{\pi}{4}$

76. $I_z = \int_0^{2\pi}\int_0^a\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^3 \, dz \, dr \, d\theta = \int_0^{2\pi}\int_0^a 2r^3\sqrt{a^2-r^2} \, dr \, d\theta = 2\int_0^{2\pi}\left[\left(-\frac{r^2}{5} - \frac{2a^2}{15}\right)\left(a^2-r^2\right)^{3/2}\right]_0^a d\theta$
$= 2\int_0^{2\pi} \frac{2}{15} a^5 \, d\theta = \frac{8\pi a^5}{15}$

77. $I_z = \int_0^{2\pi}\int_0^a\int_{\left(\frac{h}{a}\right)r}^h \left(x^2+y^2\right) dz \, r \, dr \, d\theta = \int_0^{2\pi}\int_0^a\int_{\frac{hr}{a}}^h r^3 \, dz \, dr \, d\theta = \int_0^{2\pi}\int_0^a \left(hr^3 - \frac{hr^4}{a}\right) dr \, d\theta$
$= \int_0^{2\pi} h\left[\frac{r^4}{4} - \frac{r^5}{5a}\right]_0^a d\theta = \int_0^{2\pi} h\left(\frac{a^4}{4} - \frac{a^5}{5a}\right) d\theta = \frac{ha^4}{20}\int_0^{2\pi} d\theta = \frac{\pi ha^4}{10}$

78. (a) $M = \int_0^{2\pi}\int_0^1\int_0^{r^2} z\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2}r^5\,dr\,d\theta = \frac{1}{12}\int_0^{2\pi} d\theta = \frac{\pi}{6}$; $M_{xy} = \int_0^{2\pi}\int_0^1\int_0^{r^2} z^2\,dz\,r\,dr\,d\theta$
$= \frac{1}{3}\int_0^{2\pi}\int_0^1 r^7\,dr\,d\theta = \frac{1}{24}\int_0^{2\pi} d\theta = \frac{\pi}{12} \Rightarrow \overline{z} = \frac{1}{2}$, and $\overline{x} = \overline{y} = 0$, by symmetry;
$I_z = \int_0^{2\pi}\int_0^1\int_0^{r^2} zr^3\,dz\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^1 r^7\,dr\,d\theta = \frac{1}{16}\int_0^{2\pi} d\theta = \frac{\pi}{8} \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \frac{\sqrt{3}}{2}$

(b) $M = \int_0^{2\pi}\int_0^1\int_0^{r^2} r^2\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r^4\,dr\,d\theta = \frac{1}{5}\int_0^{2\pi} d\theta = \frac{2\pi}{5}$; $M_{xy} = \int_0^{2\pi}\int_0^1\int_0^{r^2} zr^2\,dz\,dr\,d\theta$
$= \frac{1}{2}\int_0^{2\pi}\int_0^1 r^6\,dr\,d\theta = \frac{1}{14}\int_0^{2\pi} d\theta = \frac{\pi}{7} \Rightarrow \overline{z} = \frac{5}{14}$, and $\overline{x} = \overline{y} = 0$, by symmetry; $I_z = \int_0^{2\pi}\int_0^1\int_0^{r^2} r^4\,dz\,dr\,d\theta$
$= \int_0^{2\pi}\int_0^1 r^6\,dr\,d\theta = \frac{1}{7}\int_0^{2\pi} d\theta = \frac{2\pi}{7} \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{5}{7}}$

79. (a) $M = \int_0^{2\pi}\int_0^1\int_r^1 z\,dz\,r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^1 (r - r^3)\,dr\,d\theta = \frac{1}{8}\int_0^{2\pi} d\theta = \frac{\pi}{4}$; $M_{xy} = \int_0^{2\pi}\int_0^1\int_r^1 z^2\,dz\,r\,dr\,d\theta$
$= \frac{1}{3}\int_0^{2\pi}\int_0^1 (r - r^4)\,dr\,d\theta = \frac{1}{10}\int_0^{2\pi} d\theta = \frac{\pi}{5} \Rightarrow \overline{z} = \frac{4}{5}$, and $\overline{x} = \overline{y} = 0$, by symmetry; $I_z = \int_0^{2\pi}\int_0^1\int_r^1 zr^3\,dz\,dr\,d\theta$
$= \frac{1}{2}\int_0^{2\pi}\int_0^1 (r^3 - r^5)\,dr\,d\theta = \frac{1}{24}\int_0^{2\pi} d\theta = \frac{\pi}{12} \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{1}{3}}$

(b) $M = \int_0^{2\pi}\int_0^1\int_r^1 z^2\,dz\,r\,dr\,d\theta = \frac{\pi}{5}$ from part (a); $M_{xy} = \int_0^{2\pi}\int_0^1\int_r^1 z^3\,dz\,r\,dr\,d\theta = \frac{1}{4}\int_0^{2\pi}\int_0^1 (r - r^5)\,dr\,d\theta$
$= \frac{1}{12}\int_0^{2\pi} d\theta = \frac{\pi}{6} \Rightarrow \overline{z} = \frac{5}{6}$, and $\overline{x} = \overline{y} = 0$, by symmetry; $I_z = \int_0^{2\pi}\int_0^1\int_r^1 z^2r^3\,dz\,dr\,d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^1 (r^3 - r^6)\,dr\,d\theta$
$= \frac{1}{28}\int_0^{2\pi} d\theta = \frac{\pi}{14} \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{5}{14}}$

80. (a) $M = \int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^4 \sin\phi\,d\rho\,d\phi\,d\theta = \frac{a^5}{5}\int_0^{2\pi}\int_0^{\pi} \sin\phi\,d\phi\,d\theta = \frac{2a^5}{5}\int_0^{2\pi} d\theta = \frac{4\pi a^5}{5}$;
$I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^6 \sin^3\phi\,d\rho\,d\phi\,d\theta = \frac{a^7}{7}\int_0^{2\pi}\int_0^{\pi} (1 - \cos^2\phi)\sin\phi\,d\phi\,d\theta = \frac{a^7}{7}\int_0^{2\pi} \left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi} d\theta$
$= \frac{4a^7}{21}\int_0^{2\pi} d\theta = \frac{8a^7\pi}{21} \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{10}{21}}\,a$

(b) $M = \int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^3 \sin^2\phi\,d\rho\,d\phi\,d\theta = \frac{a^4}{4}\int_0^{2\pi}\int_0^{\pi} \frac{(1-\cos 2\phi)}{2}\,d\phi\,d\theta = \frac{\pi a^4}{8}\int_0^{2\pi} d\theta = \frac{\pi^2 a^4}{4}$;
$I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^5 \sin^4\phi\,d\rho\,d\phi\,d\theta = \frac{a^6}{6}\int_0^{2\pi}\int_0^{\pi} \sin^4\phi\,d\phi\,d\theta$
$= \frac{a^6}{6}\int_0^{2\pi}\left(\left[\frac{-\sin^3\phi\cos\phi}{4}\right]_0^{\pi} + \frac{3}{4}\int_0^{\pi} \sin^2\phi\,d\phi\right) d\theta = \frac{a^6}{8}\int_0^{2\pi}\left[\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right]_0^{\pi} d\theta = \frac{\pi a^6}{16}\int_0^{2\pi} d\theta$
$= \frac{a^6\pi^2}{8} \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \frac{a}{\sqrt{2}}$

81. $M = \int_0^{2\pi}\int_0^a\int_0^{\frac{h}{a}\sqrt{a^2-r^2}} dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^a \frac{h}{a}r\sqrt{a^2 - r^2}\,dr\,d\theta = \frac{h}{a}\int_0^{2\pi}\left[-\frac{1}{3}(a^2 - r^2)^{3/2}\right]_0^a d\theta$
$= \frac{h}{a}\int_0^{2\pi} \frac{a^3}{3}\,d\theta = \frac{2ha^2\pi}{3}$; $M_{xy} = \int_0^{2\pi}\int_0^a\int_0^{\frac{h}{a}\sqrt{a^2-r^2}} z\,dz\,r\,dr\,d\theta = \frac{h^2}{2a^2}\int_0^{2\pi}\int_0^a (a^2r - r^3)\,dr\,d\theta$
$= \frac{h^2}{2a^2}\int_0^{2\pi}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) d\theta = \frac{a^2h^2\pi}{4} \Rightarrow \overline{z} = \left(\frac{\pi a^2h^2}{4}\right)\left(\frac{3}{2ha^2\pi}\right) = \frac{3}{8}h$, and $\overline{x} = \overline{y} = 0$, by symmetry

82. Let the base radius of the cone be a and the height h, and place the cone's axis of symmetry along the z-axis with the vertex at the origin. Then $M = \frac{\pi a^2 h}{3}$ and $M_{xy} = \int_0^{2\pi}\int_0^a\int_{\left(\frac{h}{a}\right)r}^{h} z\,dz\,r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^a \left(h^2r - \frac{h^2}{a^2}r^3\right) dr\,d\theta$
$= \frac{h^2}{2}\int_0^{2\pi}\left[\frac{r^2}{2} - \frac{r^4}{4a^2}\right]_0^a d\theta = \frac{h^2}{2}\int_0^{2\pi}\left(\frac{a^2}{2} - \frac{a^2}{4}\right) d\theta = \frac{h^2a^2}{8}\int_0^{2\pi} d\theta = \frac{h^2a^2\pi}{4} \Rightarrow \overline{z} = \frac{M_{xy}}{M} = \left(\frac{h^2a^2\pi}{4}\right)\left(\frac{3}{\pi a^2 h}\right) = \frac{3}{4}h$, and
$\overline{x} = \overline{y} = 0$, by symmetry $\Rightarrow$ the centroid is one fourth of the way from the base to the vertex

83. $M = \int_0^{2\pi}\int_0^a\int_0^h (z + 1)\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^a \left(\frac{h^2}{2} + h\right) r\,dr\,d\theta = \frac{a^2(h^2+2h)}{4}\int_0^{2\pi} d\theta = \frac{\pi a^2(h^2+2h)}{2}$;
$M_{xy} = \int_0^{2\pi}\int_0^a\int_0^h (z^2 + z)\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^a \left(\frac{h^3}{3} + \frac{h^2}{2}\right) r\,dr\,d\theta = \frac{a^2(2h^3+3h^2)}{12}\int_0^{2\pi} d\theta = \frac{\pi a^2(2h^3+3h^2)}{6}$
$\Rightarrow \overline{z} = \left[\frac{\pi a^2(2h^3+3h^2)}{6}\right]\left[\frac{2}{\pi a^2(h^2+2h)}\right] = \frac{2h^2 + 3h}{3h + 6}$, and $\overline{x} = \overline{y} = 0$, by symmetry;

$I_z = \int_0^{2\pi}\int_0^a\int_0^h (z+1)r^3\,dz\,dr\,d\theta = \left(\frac{h^2+2h}{2}\right)\int_0^{2\pi}\int_0^a r^3\,dr\,d\theta = \left(\frac{h^2+2h}{2}\right)\left(\frac{a^4}{4}\right)\int_0^{2\pi} d\theta = \frac{\pi a^4\,(h^2+2h)}{4}$;
$R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{\pi a^4\,(h^2+2h)}{4}\cdot\frac{2}{\pi a^2\,(h^2+2h)}} = \frac{a}{\sqrt{2}}$

84. The mass of the plant's atmosphere to an altitude h above the surface of the planet is the triple integral
$M(h) = \int_0^{2\pi}\int_0^{\pi}\int_R^h \mu_0 e^{-c(\rho-R)}\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \int_R^h\int_0^{2\pi}\int_0^{\pi} \mu_0 e^{-c(\rho-R)}\rho^2 \sin\phi\,d\phi\,d\theta\,d\rho$
$= \int_R^h\int_0^{2\pi}\left[\mu_0 e^{-c(\rho-R)}\rho^2(-\cos\phi)\right]_0^{\pi} d\theta\,d\rho = 2\int_R^h\int_0^{2\pi} \mu_0 e^{cR}\,e^{-c\rho}\rho^2\,d\theta\,d\rho = 4\pi\mu_0 e^{cR}\int_R^h e^{-c\rho}\rho^2\,d\rho$
$= 4\pi\mu_0 e^{cR}\left[-\frac{\rho^2 e^{-c\rho}}{c} - \frac{2\rho e^{-c\rho}}{c^2} - \frac{2e^{-c\rho}}{c^3}\right]_R^h$ (by parts)
$= 4\pi\mu_0\,e^{cR}\left(-\frac{h^2 e^{-ch}}{c} - \frac{2he^{-ch}}{c^2} - \frac{2e^{-ch}}{c^3} + \frac{R^2 e^{-cR}}{c} + \frac{2Re^{-cR}}{c^2} + \frac{2e^{-cR}}{c^3}\right)$.
The mass of the planet's atmosphere is therefore $M = \lim_{h\to\infty} M(h) = 4\pi\mu_0\left(\frac{R^2}{c} + \frac{2R}{c^2} + \frac{2}{c^3}\right)$.

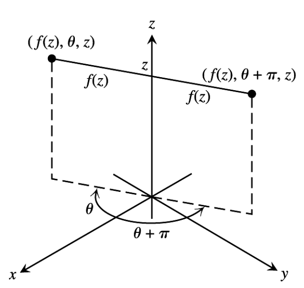
85. The density distribution function is linear so it has the form $\delta(\rho) = k\rho + C$, where ρ is the distance from the center of the planet. Now, $\delta(R) = 0 \Rightarrow kR + C = 0$, and $\delta(\rho) = k\rho - kR$. It remains to determine the constant k: $M = \int_0^{2\pi}\int_0^{\pi}\int_0^R (k\rho - kR)\,\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\left[k\,\frac{\rho^4}{4} - kR\,\frac{\rho^3}{3}\right]_0^R \sin\phi\,d\phi\,d\theta$
$= \int_0^{2\pi}\int_0^{\pi} k\left(\frac{R^4}{4} - \frac{R^4}{3}\right)\sin\phi\,d\phi\,d\theta = \int_0^{2\pi} -\frac{k}{12}R^4\left[-\cos\phi\right]_0^{\pi} d\theta = \int_0^{2\pi} -\frac{k}{6}R^4\,d\theta = -\frac{k\pi R^4}{3} \Rightarrow k = -\frac{3M}{\pi R^4}$
$\Rightarrow \delta(\rho) = -\frac{3M}{\pi R^4}\,\rho + \frac{3M}{\pi R^4}\,R$. At the center of the planet $\rho = 0 \Rightarrow \delta(0) = \left(\frac{3M}{\pi R^4}\right)R = \frac{3M}{\pi R^3}$.

86. $x^2 + y^2 = a^2 \Rightarrow (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 = a^2 \Rightarrow (\rho^2\sin^2\phi)(\cos^2\theta + \sin^2\theta) = a^2 \Rightarrow \rho^2\sin^2\phi = a^2$
$\Rightarrow \rho\sin\phi = a$ or $\rho\sin\phi = -a \Rightarrow \rho\sin\phi = a$ or $\rho = a\csc\phi$, since $0 \le \phi \le \pi$ and $\rho \ge 0$.

87. (a) A plane perpendicular to the x-axis has the form $x = a$ in rectangular coordinates $\Rightarrow r\cos\theta = a \Rightarrow r = \frac{a}{\cos\theta}$ $\Rightarrow r = a\sec\theta$, in cylindrical coordinates.
(b) A plane perpendicular to the y-axis has the form $y = b$ in rectangular coordinates $\Rightarrow r\sin\theta = b \Rightarrow r = \frac{b}{\sin\theta}$ $\Rightarrow r = b\csc\theta$, in cylindrical coordinates.

88. $ax + by = c \Rightarrow a(r\cos\theta) + b(r\sin\theta) = c \Rightarrow r(a\cos\theta + b\sin\theta) = c \Rightarrow r = \frac{c}{a\cos\theta + b\sin\theta}$.

89. The equation $r = f(z)$ implies that the point $(r, \theta, z) = (f(z), \theta, z)$ will lie on the surface for all θ. In particular $(f(z), \theta + \pi, z)$ lies on the surface whenever $(f(z), \theta, z)$ does $\Rightarrow$ the surface is symmetric with respect to the z-axis.

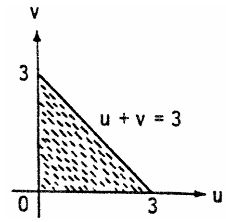


90. The equation $\rho = f(\phi)$ implies that the point $(\rho, \phi, \theta) = (f(\phi), \phi, \theta)$ lies on the surface for all θ. In particular, if $(f(\phi), \phi, \theta)$ lies on the surface, then $(f(\phi), \phi, \theta + \pi)$ lies on the surface, so the surface is symmetric wiith respect to the z-axis.

15.7 SUBSTITUTIONS IN MULTIPLE INTEGRALS

1. (a) $x - y = u$ and $2x + y = v \Rightarrow 3x = u + v$ and $y = x - u \Rightarrow x = \frac{1}{3}(u+v)$ and $y = \frac{1}{3}(-2u+v)$;
$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$

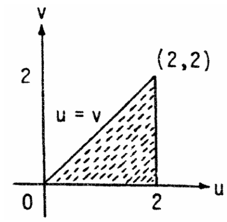
(b) The line segment $y = x$ from $(0, 0)$ to $(1, 1)$ is $x - y = 0$ $\Rightarrow u = 0$; the line segment $y = -2x$ from $(0, 0)$ to $(1, -2)$ is $2x + y = 0 \Rightarrow v = 0$; the line segment $x = 1$ from $(1, 1)$ to $(1, -2)$ is $(x - y) + (2x + y) = 3$ $\Rightarrow u + v = 3$. The transformed region is sketched at the right.



2. (a) $x + 2y = u$ and $x - y = v \Rightarrow 3y = u - v$ and $x = v + y \Rightarrow y = \frac{1}{3}(u - v)$ and $x = \frac{1}{3}(u + 2v)$;

$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$

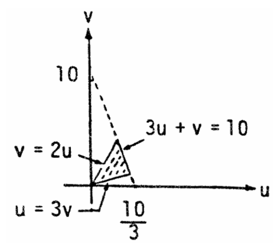
(b) The triangular region in the xy-plane has vertices $(0, 0)$, $(2, 0)$, and $\left(\frac{2}{3}, \frac{2}{3}\right)$. The line segment $y = x$ from $(0, 0)$ to $\left(\frac{2}{3}, \frac{2}{3}\right)$ is $x - y = 0 \Rightarrow v = 0$; the line segment $y = 0$ from $(0, 0)$ to $(2, 0) \Rightarrow u = v$; the line segment $x + 2y = 2$ from $\left(\frac{2}{3}, \frac{2}{3}\right)$ to $(2, 0) \Rightarrow u = 2$. The transformed region is sketched at the right.



3. (a) $3x + 2y = u$ and $x + 4y = v \Rightarrow -5x = -2u + v$ and $y = \frac{1}{2}(u - 3x) \Rightarrow x = \frac{1}{5}(2u - v)$ and $y = \frac{1}{10}(3v - u)$;

$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{vmatrix} = \frac{6}{50} - \frac{1}{50} = \frac{1}{10}$

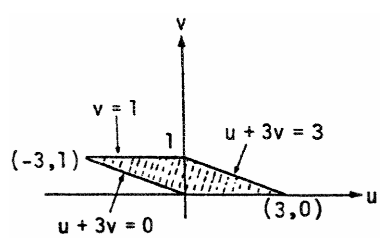
(b) The x-axis $y = 0 \Rightarrow u = 3v$; the y-axis $x = 0$ $\Rightarrow v = 2u$; the line $x + y = 1$ $\Rightarrow \frac{1}{5}(2u - v) + \frac{1}{10}(3v - u) = 1$ $\Rightarrow 2(2u - v) + (3v - u) = 10 \Rightarrow 3u + v = 10$. The transformed region is sketched at the right.



4. (a) $2x - 3y = u$ and $-x + y = v \Rightarrow -x = u + 3v$ and $y = v + x \Rightarrow x = -u - 3v$ and $y = -u - 2v$;

$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1 & -3 \\ -1 & -2 \end{vmatrix} = 2 - 3 = -1$

(b) The line $x = -3 \Rightarrow -u - 3v = -3$ or $u + 3v = 3$; $x = 0 \Rightarrow u + 3v = 0$; $y = x \Rightarrow v = 0$; $y = x + 1$ $\Rightarrow v = 1$. The transformed region is the parallelogram sketched at the right.

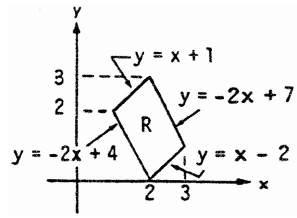


5. $\int_0^4 \int_{y/2}^{(y/2)+1} \left(x - \frac{y}{2}\right) dx\, dy = \int_0^4 \left[\frac{x^2}{2} - \frac{xy}{2}\right]_{\frac{y}{2}}^{\frac{y}{2}+1} dy = \frac{1}{2}\int_0^4 \left[\left(\frac{y}{2} + 1\right)^2 - \left(\frac{y}{2}\right)^2 - \left(\frac{y}{2} + 1\right)y + \left(\frac{y}{2}\right)y\right] dy$
$= \frac{1}{2}\int_0^4 (y + 1 - y)\, dy = \frac{1}{2}\int_0^4 dy = \frac{1}{2}(4) = 2$

6. $\iint\limits_R (2x^2 - xy - y^2)\,dx\,dy = \iint\limits_R (x-y)(2x+y)\,dx\,dy$

$= \iint\limits_G uv \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \frac{1}{3}\iint\limits_G uv\,du\,dv;$

We find the boundaries of G from the boundaries of R, shown in the accompanying figure:



xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$y = -2x + 4$	$\frac{1}{3}(-2u+v) = -\frac{2}{3}(u+v) + 4$	$v = 4$
$y = -2x + 7$	$\frac{1}{3}(-2u+v) = -\frac{2}{3}(u+v) + 7$	$v = 7$
$y = x - 2$	$\frac{1}{3}(-2u+v) = \frac{1}{3}(u+v) - 2$	$u = 2$
$y = x + 1$	$\frac{1}{3}(-2u+v) = \frac{1}{3}(u+v) + 1$	$u = -1$

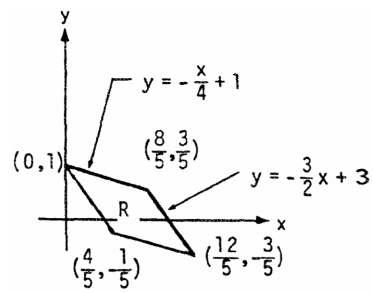
$\Rightarrow \frac{1}{3}\iint\limits_G uv\,du\,dv = \frac{1}{3}\int_{-1}^{2}\int_{4}^{7} uv\,dv\,du = \frac{1}{3}\int_{-1}^{2} u\left[\frac{v^2}{2}\right]_4^7 du = \frac{11}{2}\int_{-1}^{2} u\,du = \left(\frac{11}{2}\right)\left[\frac{u^2}{2}\right]_{-1}^{2} = \left(\frac{11}{4}\right)(4-1) = \frac{33}{4}$

7. $\iint\limits_R (3x^2 + 14xy + 8y^2)\,dx\,dy$

$= \iint\limits_R (3x+2y)(x+4y)\,dx\,dy$

$= \iint\limits_G uv \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \frac{1}{10}\iint\limits_G uv\,du\,dv;$

We find the boundaries of G from the boundaries of R, shown in the accompanying figure:



xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$y = -\frac{3}{2}x + 1$	$\frac{1}{10}(3v-u) = -\frac{3}{10}(2u-v) + 1$	$u = 2$
$y = -\frac{3}{2}x + 3$	$\frac{1}{10}(3v-u) = -\frac{3}{10}(2u-v) + 3$	$u = 6$
$y = -\frac{1}{4}x$	$\frac{1}{10}(3v-u) = -\frac{1}{20}(2u-v)$	$v = 0$
$y = -\frac{1}{4}x + 1$	$\frac{1}{10}(3v-u) = -\frac{1}{20}(2u-v) + 1$	$v = 4$

$\Rightarrow \frac{1}{10}\iint\limits_G uv\,du\,dv = \frac{1}{10}\int_2^6\int_0^4 uv\,dv\,du = \frac{1}{10}\int_2^6 u\left[\frac{v^2}{2}\right]_0^4 du = \frac{4}{5}\int_2^6 u\,du = \left(\frac{4}{5}\right)\left[\frac{u^2}{2}\right]_2^6 = \left(\frac{4}{5}\right)(18-2) = \frac{64}{5}$

8. $\iint\limits_R 2(x-y)\,dx\,dy = \iint\limits_G -2v\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint\limits_G -2v\,du\,dv$; the region G is sketched in Exercise 4

$\Rightarrow \iint\limits_G -2v\,du\,dv = \int_0^1\int_{-3v}^{3-3v} -2v\,du\,dv = \int_0^1 -2v(3-3v+3v)\,dv = \int_0^1 -6v\,dv = \left[-3v^2\right]_0^1 = -3$

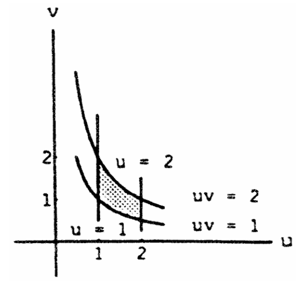
9. $x = \frac{u}{v}$ and $y = uv \Rightarrow \frac{y}{x} = v^2$ and $xy = u^2$; $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} v^{-1} & -uv^{-2} \\ v & u \end{vmatrix} = v^{-1}u + v^{-1}u = \frac{2u}{v}$;

$y = x \Rightarrow uv = \frac{u}{v} \Rightarrow v = 1$, and $y = 4x \Rightarrow v = 2$; $xy = 1 \Rightarrow u = 1$, and $xy = 9 \Rightarrow u = 3$; thus

$\iint\limits_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\,dy = \int_1^3\int_1^2 (v+u)\left(\frac{2u}{v}\right) dv\,du = \int_1^3\int_1^2 \left(2u + \frac{2u^2}{v}\right) dv\,du = \int_1^3 \left[2uv + 2u^2\ln v\right]_1^2 du$

$= \int_1^3 (2u + 2u^2\ln 2)\,du = \left[u^2 + \frac{2}{3}u^2\ln 2\right]_1^3 = 8 + \frac{2}{3}(26)(\ln 2) = 8 + \frac{52}{3}(\ln 2)$

10. (a) $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix} = u$, and the region G is sketched at the right



(b) $x = 1 \Rightarrow u = 1$, and $x = 2 \Rightarrow u = 2$; $y = 1 \Rightarrow uv = 1 \Rightarrow v = \frac{1}{u}$, and $y = 2 \Rightarrow uv = 2 \Rightarrow v = \frac{2}{u}$; thus,

$$\int_1^2 \int_1^2 \frac{y}{x}\, dy\, dx = \int_1^2 \int_{1/u}^{2/u} \left(\frac{uv}{u}\right) u\, dv\, du = \int_1^2 \int_{1/u}^{2/u} uv\, dv\, du = \int_1^2 u \left[\frac{v^2}{2}\right]_{1/u}^{2/u} du = \int_1^2 u \left(\frac{2}{u^2} - \frac{1}{2u^2}\right) du$$

$$= \frac{3}{2} \int_1^2 u \left(\frac{1}{u^2}\right) du = \frac{3}{2}\left[\ln u\right]_1^2 = \frac{3}{2}\ln 2;\ \int_1^2 \int_1^2 \frac{y}{x}\, dy\, dx = \int_1^2 \left[\frac{1}{x} \cdot \frac{y^2}{2}\right]_1^2 dx = \frac{3}{2}\int_1^2 \frac{dx}{x} = \frac{3}{2}\left[\ln x\right]_1^2 = \frac{3}{2}\ln 2$$

11. $x = ar\cos\theta$ and $y = ar\sin\theta \Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = J(r,\theta) = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr\cos^2\theta + abr\sin^2\theta = abr$;

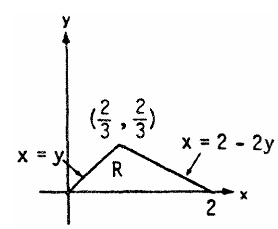
$$I_0 = \iint_R (x^2 + y^2)\, dA = \int_0^{2\pi} \int_0^1 r^2 \left(a^2\cos^2\theta + b^2\sin^2\theta\right) |J(r,\theta)|\, dr\, d\theta = \int_0^{2\pi} \int_0^1 abr^3 \left(a^2\cos^2\theta + b^2\sin^2\theta\right) dr\, d\theta$$

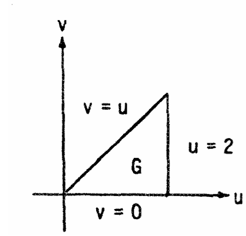
$$= \frac{ab}{4} \int_0^{2\pi} \left(a^2\cos^2\theta + b^2\sin^2\theta\right) d\theta = \frac{ab}{4}\left[\frac{a^2\theta}{2} + \frac{a^2\sin 2\theta}{4} + \frac{b^2\theta}{2} - \frac{b^2\sin 2\theta}{4}\right]_0^{2\pi} = \frac{ab\pi\left(a^2 + b^2\right)}{4}$$

12. $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$; $A = \iint_R dy\, dx = \iint_G ab\, du\, dv = \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} ab\, dv\, du$

$$= 2ab \int_{-1}^1 \sqrt{1-u^2}\, du = 2ab\left[\frac{u}{2}\sqrt{1-u^2} + \frac{1}{2}\sin^{-1} u\right]_{-1}^1 = ab\left[\sin^{-1} 1 - \sin^{-1}(-1)\right] = ab\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = ab\pi$$

13. The region of integration R in the xy-plane is sketched in the figure at the right. The boundaries of the image G are obtained as follows, with G sketched at the right:





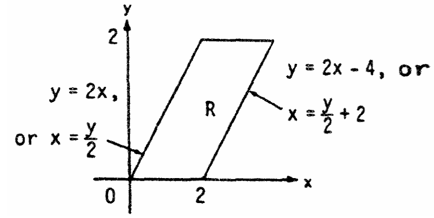
xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$x = y$	$\frac{1}{3}(u + 2v) = \frac{1}{3}(u - v)$	$v = 0$
$x = 2 - 2y$	$\frac{1}{3}(u + 2v) = 2 - \frac{2}{3}(u - v)$	$u = 2$
$y = 0$	$0 = \frac{1}{3}(u - v)$	$v = u$

Also, from Exercise 2, $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = -\frac{1}{3} \Rightarrow \int_0^{2/3} \int_y^{2-2y} (x + 2y)\, e^{(y-x)}\, dx\, dy = \int_0^2 \int_0^u ue^{-v} \left|-\frac{1}{3}\right| dv\, du$

$$= \frac{1}{3} \int_0^2 u\left[-e^{-v}\right]_0^u du = \frac{1}{3} \int_0^2 u\left(1 - e^{-u}\right) du = \frac{1}{3}\left[u\left(u + e^{-u}\right) - \frac{u^2}{2} + e^{-u}\right]_0^2 = \frac{1}{3}\left[2\left(2 + e^{-2}\right) - 2 + e^{-2} - 1\right]$$

$$= \frac{1}{3}\left(3e^{-2} + 1\right) \approx 0.4687$$

14. $x = u + \frac{v}{2}$ and $y = v \Rightarrow 2x - y = (2u + v) - v = 2u$ and $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = 1$; next, $u = x - \frac{v}{2}$ $= x - \frac{y}{2}$ and $v = y$, so the boundaries of the region of integration R in the xy-plane are transformed to the boundaries of G:



xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$x = \frac{y}{2}$	$u + \frac{v}{2} = \frac{v}{2}$	$u = 0$
$x = \frac{y}{2} + 2$	$u + \frac{v}{2} = \frac{v}{2} + 2$	$u = 2$
$y = 0$	$v = 0$	$v = 0$
$y = 2$	$v = 2$	$v = 2$

$\Rightarrow \int_0^2 \int_{y/2}^{(y/2)+2} y^3(2x - y)\, e^{(2x-y)^2}\, dx\, dy = \int_0^2 \int_0^2 v^3(2u)\, e^{4u^2}\, du\, dv = \int_0^2 v^3 \left[\frac{1}{4} e^{4u^2}\right]_0^2 dv = \frac{1}{4} \int_0^2 v^3 \left(e^{16} - 1\right) dv$
$= \frac{1}{4}\left(e^{16} - 1\right)\left[\frac{v^4}{4}\right]_0^2 = e^{16} - 1$

15. (a) $x = u \cos v$ and $y = u \sin v \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \end{vmatrix} = u \cos^2 v + u \sin^2 v = u$

(b) $x = u \sin v$ and $y = u \cos v \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sin v & u \cos v \\ \cos v & -u \sin v \end{vmatrix} = -u \sin^2 v - u \cos^2 v = -u$

16. (a) $x = u \cos v,\ y = u \sin v,\ z = w \Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \cos v & -u \sin v & 0 \\ \sin v & u \cos v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u \cos^2 v + u \sin^2 v = u$

(b) $x = 2u - 1,\ y = 3v - 4,\ z = \frac{1}{2}(w - 4) \Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{vmatrix} = (2)(3)\left(\frac{1}{2}\right) = 3$

17. $\begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$
$= (\cos\phi)\begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} + (\rho\sin\phi)\begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$
$= (\rho^2\cos\phi)(\sin\phi\cos\phi\cos^2\theta + \sin\phi\cos\phi\sin^2\theta) + (\rho^2\sin\phi)(\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta)$
$= \rho^2\sin\phi\cos^2\phi + \rho^2\sin^3\phi = (\rho^2\sin\phi)(\cos^2\phi + \sin^2\phi) = \rho^2\sin\phi$

18. Let $u = g(x) \Rightarrow J(x) = \frac{du}{dx} = g'(x) \Rightarrow \int_a^b f(u)\, du = \int_{g(a)}^{g(b)} f(g(x))g'(x)\, dx$ in accordance with Theorem 6 in Section 5.6. Note that $g'(x)$ represents the Jacobian of the transformation $u = g(x)$ or $x = g^{-1}(u)$.

19. $\int_0^3 \int_0^4 \int_{y/2}^{1+(y/2)} \left(\frac{2x-y}{2} + \frac{z}{3}\right) dx\, dy\, dz = \int_0^3 \int_0^4 \left[\frac{x^2}{2} - \frac{xy}{2} + \frac{xz}{3}\right]_{y/2}^{1+(y/2)} dy\, dz = \int_0^3 \int_0^4 \left[\frac{1}{2}(y+1) - \frac{y}{2} + \frac{z}{3}\right] dy\, dz$
$= \int_0^3 \left[\frac{(y+1)^2}{4} - \frac{y^2}{4} + \frac{yz}{3}\right]_0^4 dz = \int_0^3 \left(\frac{9}{4} + \frac{4z}{3} - \frac{1}{4}\right) dz = \int_0^3 \left(2 + \frac{4z}{3}\right) dz = \left[2z + \frac{2z^2}{3}\right]_0^3 = 12$

20. $J(u,v,w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$; the transformation takes the ellipsoid region $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$ in xyz-space into the spherical region $u^2 + v^2 + w^2 \le 1$ in uvw-space $\left(\text{which has volume } V = \frac{4}{3}\pi\right)$

$\Rightarrow \; V = \iiint\limits_{R} dx\,dy\,dz = \iiint\limits_{G} abc\,du\,dv\,dw = \frac{4\pi abc}{3}$

21. $J(u,v,w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$; for R and G as in Exercise 19, $\iiint\limits_{R} |xyz|\,dx\,dy\,dz$

$= \iiint\limits_{G} a^2b^2c^2uvw\,dw\,dv\,du = 8a^2b^2c^2 \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\,(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$

$= \frac{4a^2b^2c^2}{3}\int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\cos\theta\sin^3\phi\cos\phi\,d\phi\,d\theta = \frac{a^2b^2c^2}{3}\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = \frac{a^2b^2c^2}{6}$

22. $u = x,\ v = xy$, and $w = 3z \;\Rightarrow\; x = u,\ y = \frac{v}{u}$, and $z = \frac{1}{3}w \;\Rightarrow\; J(u,v,w) = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{3u}$;

$\iiint\limits_{R} (x^2y + 3xyz)\,dx\,dy\,dz = \iiint\limits_{G} \left[u^2\left(\frac{v}{u}\right) + 3u\left(\frac{v}{u}\right)\left(\frac{w}{3}\right)\right] |J(u,v,w)|\,du\,dv\,dw = \frac{1}{3}\int_0^3\int_0^2\int_1^2 \left(v + \frac{vw}{u}\right) du\,dv\,dw$

$= \frac{1}{3}\int_0^3\int_0^2 (v + vw\ln 2)\,dv\,dw = \frac{1}{3}\int_0^3 (1 + w\ln 2)\left[\frac{v^2}{2}\right]_0^2 dw = \frac{2}{3}\int_0^3 (1 + w\ln 2)\,dw = \frac{2}{3}\left[w + \frac{w^2}{2}\ln 2\right]_0^3$

$= \frac{2}{3}\left(3 + \frac{9}{2}\ln 2\right) = 2 + 3\ln 2 = 2 + \ln 8$

23. The first moment about the xy-coordinate plane for the semi-ellipsoid, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ using the transformation in Exercise 21 is, $M_{xy} = \iiint\limits_{D} z\,dz\,dy\,dx = \iiint\limits_{G} cw\,|J(u,v,w)|\,du\,dv\,dw$

$= abc^2 \iiint\limits_{G} w\,du\,dv\,dw = (abc^2)\cdot(M_{xy}$ of the hemisphere $x^2 + y^2 + z^2 = 1,\ z \ge 0) = \frac{abc^2\pi}{4}$;

the mass of the semi-ellipsoid is $\frac{2abc\pi}{3} \;\Rightarrow\; \bar{z} = \left(\frac{abc^2\pi}{4}\right)\left(\frac{3}{2abc\pi}\right) = \frac{3}{8}c$

24. A solid of revolutions is symmetric about the axis of revolution, therefore, the height of the solid is solely a function of r. That is, $y = f(x) = f(r)$. Using cylindrical coordinates with $x = r\cos\theta$, $y = y$ and $z = r\sin\theta$, we have

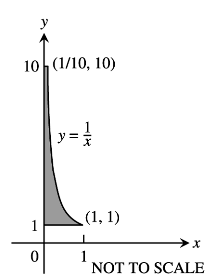
$V = \iiint\limits_{G} r\,dy\,d\theta\,dr = \int_a^b\int_0^{2\pi}\int_0^{f(r)} r\,dy\,d\theta\,dr = \int_a^b\int_0^{2\pi}[r\,y]_0^{f(r)}\,d\theta\,dr = \int_a^b\int_0^{2\pi} r\,f(r)\,d\theta\,dr = \int_a^b [r\theta f(r)]_0^{2\pi}\,dr$

$\int_a^b 2\pi r f(r)dr$. In the last integral, r is a dummy or stand-in variable and as such it can be replaced by any variable name. Choosing x instead of r we have $V = \int_a^b 2\pi x f(x)dx$, which is the same result obtained using the shell method.

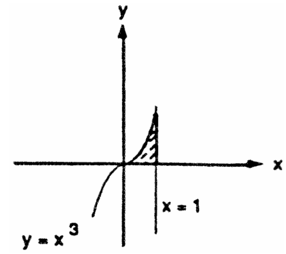
CHAPTER 15 PRACTICE EXERCISES

1. $\int_1^{10}\int_0^{1/y} ye^{xy}\,dx\,dy = \int_1^{10}\left[e^{xy}\right]_0^{1/y} dy$

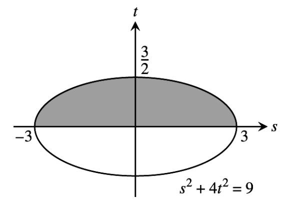
$= \int_1^{10}(e - 1)\,dy = 9e - 9$



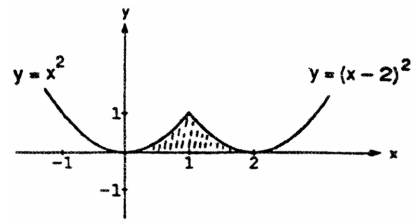
2. $\int_0^1 \int_0^{x^3} e^{y/x}\, dy\, dx = \int_0^1 x \left[e^{y/x}\right]_0^{x^3} dx$
$= \int_0^1 \left(xe^{x^2} - x\right) dx = \left[\frac{1}{2} e^{x^2} - \frac{x^2}{2}\right]_0^1 = \frac{e-2}{2}$



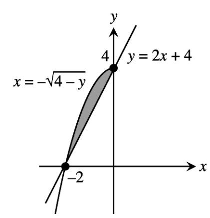
3. $\int_0^{3/2} \int_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}} t\, ds\, dt = \int_0^{3/2} [ts]_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}} dt$
$= \int_0^{3/2} 2t\sqrt{9-4t^2}\, dt = \left[-\frac{1}{6}\left(9-4t^2\right)^{3/2}\right]_0^{3/2}$
$= -\frac{1}{6}\left(0^{3/2} - 9^{3/2}\right) = \frac{27}{6} = \frac{9}{2}$



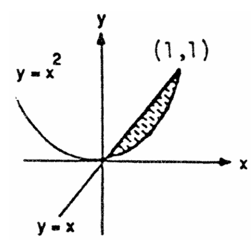
4. $\int_0^1 \int_{\sqrt{y}}^{2-\sqrt{y}} xy\, dx\, dy = \int_0^1 y \left[\frac{x^2}{2}\right]_{\sqrt{y}}^{2-\sqrt{y}} dy$
$= \frac{1}{2} \int_0^1 y\left(4 - 4\sqrt{y} + y - y\right) dy$
$= \int_0^1 \left(2y - 2y^{3/2}\right) dy = \left[y^2 - \frac{4y^{5/2}}{5}\right]_0^1 = \frac{1}{5}$



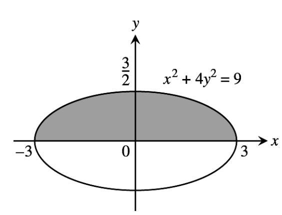
5. $\int_{-2}^0 \int_{2x+4}^{4-x^2} dy\, dx = \int_{-2}^0 \left(-x^2 - 2x\right) dx$
$= \left[-\frac{x^3}{3} - x^2\right]_{-2}^0 = -\left(\frac{8}{3} - 4\right) = \frac{4}{3}$
$\int_0^4 \int_{-\sqrt{4-y}}^{(y-4)/2} dx\, dy = \int_0^4 \left(\frac{y-4}{2} + \sqrt{4-y}\right) dy$
$= \left[\frac{y^2}{2} - 2y - \frac{2}{3}(4-y)^{3/2}\right]_0^4 = 4 - 8 + \frac{2}{3} \cdot 4^{3/2}$
$= -4 + \frac{16}{3} = \frac{4}{3}$



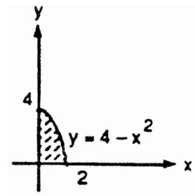
6. $\int_0^1 \int_y^{\sqrt{y}} \sqrt{x}\, dx\, dy = \int_0^1 \left[\frac{2}{3} x^{3/2}\right]_y^{\sqrt{y}} dy$
$= \frac{2}{3} \int_0^1 \left(y^{3/4} - y^{3/2}\right) dy = \frac{2}{3}\left[\frac{4}{7} y^{7/4} - \frac{2}{5} y^{5/2}\right]_0^1$
$= \frac{2}{3}\left(\frac{4}{7} - \frac{2}{5}\right) = \frac{4}{35}$
$\int_0^1 \int_{x^2}^x \sqrt{x}\, dy\, dx = \int_0^1 x^{1/2}\left(x - x^2\right) dx = \int_0^1 \left(x^{3/2} - x^{5/2}\right) dx$
$= \left[\frac{2}{5}x^{5/2} - \frac{2}{7}x^{7/2}\right]_0^1 = \frac{2}{5} - \frac{2}{7} = \frac{4}{35}$



7. $\int_{-3}^3 \int_0^{(1/2)\sqrt{9-x^2}} y\, dy\, dx = \int_{-3}^3 \left[\frac{y^2}{2}\right]_0^{(1/2)\sqrt{9-x^2}} dx$
$= \int_{-3}^3 \frac{1}{8}\left(9 - x^2\right) dx = \left[\frac{9x}{8} - \frac{x^3}{24}\right]_{-3}^3$
$= \left(\frac{27}{8} - \frac{27}{24}\right) - \left(-\frac{27}{8} + \frac{27}{24}\right) = \frac{27}{6} = \frac{9}{2}$
$\int_0^{3/2} \int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y\, dx\, dy = \int_0^{3/2} 2y\sqrt{9-4y^2}\, dy$
$= -\frac{1}{4} \cdot \frac{2}{3}\left(9-4y^2\right)^{3/2}\Big|_0^{3/2} = \frac{1}{6} \cdot 9^{3/2} = \frac{27}{6} = \frac{9}{2}$



8. $\int_0^2\int_0^{4-x^2} 2x\,dy\,dx = \int_0^2 [2xy]_0^{4-x^2}\,dx$
$= \int_0^2 (2x(4-x^2))\,dx = \int_0^2 (8x-2x^3)\,dx$
$= \left[4x^2 - \frac{x^4}{2}\right]_0^2 = 16 - \frac{16}{2} = 8$
$\int_0^4\int_0^{\sqrt{4-y}} 2x\,dx\,dy = \int_0^4 [x^2]_0^{\sqrt{4-y}}\,dy$
$= \int_0^4 (4-y)\,dy = \left[4y - \frac{y^2}{2}\right]_0^4 = 16 - \frac{16}{2} = 8$



9. $\int_0^1\int_{2y}^2 4\cos(x^2)\,dx\,dy = \int_0^2\int_0^{x/2} 4\cos(x^2)\,dy\,dx = \int_0^2 2x\cos(x^2)\,dx = [\sin(x^2)]_0^2 = \sin 4$

10. $\int_0^2\int_{y/2}^1 e^{x^2}\,dx\,dy = \int_0^1\int_0^{2x} e^{x^2}\,dy\,dx = \int_0^1 2xe^{x^2}\,dx = [e^{x^2}]_0^1 = e-1$

11. $\int_0^8\int_{\sqrt[3]{x}}^2 \frac{1}{y^4+1}\,dy\,dx = \int_0^2\int_0^{y^3} \frac{1}{y^4+1}\,dx\,dy = \frac{1}{4}\int_0^2 \frac{4y^3}{y^4+1}\,dy = \frac{\ln 17}{4}$

12. $\int_0^1\int_{\sqrt[3]{y}}^1 \frac{2\pi\sin(\pi x^2)}{x^2}\,dx\,dy = \int_0^1\int_0^{x^3} \frac{2\pi\sin(\pi x^2)}{x^2}\,dy\,dx = \int_0^1 2\pi x\sin(\pi x^2)\,dx = [-\cos(\pi x^2)]_0^1 = -(-1)-(-1) = 2$

13. $A = \int_{-2}^0\int_{2x+4}^{4-x^2} dy\,dx = \int_{-2}^0 (-x^2-2x)\,dx = \frac{4}{3}$

14. $A = \int_1^4\int_{2-y}^{\sqrt{y}} dx\,dy = \int_1^4 \left(\sqrt{y} - 2 + y\right)dy = \frac{37}{6}$

15. $V = \int_0^1\int_x^{2-x} (x^2+y^2)\,dy\,dx = \int_0^1 \left[x^2y + \frac{y^3}{3}\right]_x^{2-x} dx = \int_0^1 \left[2x^2 + \frac{(2-x)^3}{3} - \frac{7x^3}{3}\right] dx = \left[\frac{2x^3}{3} - \frac{(2-x)^4}{12} - \frac{7x^4}{12}\right]_0^1$
$= \left(\frac{2}{3} - \frac{1}{12} - \frac{7}{12}\right) + \frac{2^4}{12} = \frac{4}{3}$

16. $V = \int_{-3}^2\int_x^{6-x^2} x^2\,dy\,dx = \int_{-3}^2 [x^2y]_x^{6-x^2}\,dx = \int_{-3}^2 (6x^2-x^4-x^3)\,dx = \frac{125}{4}$

17. average value $= \int_0^1\int_0^1 xy\,dy\,dx = \int_0^1 \left[\frac{xy^2}{2}\right]_0^1 dx = \int_0^1 \frac{x}{2}\,dx = \frac{1}{4}$

18. average value $= \frac{1}{\left(\frac{\pi}{4}\right)}\int_0^1\int_0^{\sqrt{1-x^2}} xy\,dy\,dx = \frac{4}{\pi}\int_0^1 \left[\frac{xy^2}{2}\right]_0^{\sqrt{1-x^2}} dx = \frac{2}{\pi}\int_0^1 (x-x^3)\,dx = \frac{1}{2\pi}$

19. $M = \int_1^2\int_{2/x}^2 dy\,dx = \int_1^2 \left(2 - \frac{2}{x}\right)dx = 2 - \ln 4$; $M_y = \int_1^2\int_{2/x}^2 x\,dy\,dx = \int_1^2 x\left(2-\frac{2}{x}\right)dx = 1$;
$M_x = \int_1^2\int_{2/x}^2 y\,dy\,dx = \int_1^2 \left(2 - \frac{2}{x^2}\right)dx = 1 \Rightarrow \overline{x} = \overline{y} = \frac{1}{2-\ln 4}$

20. $M = \int_0^4\int_{-2y}^{2y-y^2} dx\,dy = \int_0^4 (4y-y^2)\,dy = \frac{32}{3}$; $M_x = \int_0^4\int_{-2y}^{2y-y^2} y\,dx\,dy = \int_0^4 (4y^2-y^3)\,dy = \left[\frac{4y^3}{3} - \frac{y^4}{4}\right]_0^4 = \frac{64}{3}$;
$M_y = \int_0^4\int_{-2y}^{2y-y^2} x\,dx\,dy = \int_0^4 \left[\frac{(2y-y^2)^2}{2} - 2y^2\right] dy = \left[\frac{y^5}{10} - \frac{y^4}{2}\right]_0^4 = -\frac{128}{5} \Rightarrow \overline{x} = \frac{M_y}{M} = -\frac{12}{5}$ and $\overline{y} = \frac{M_x}{M} = 2$

21. $I_o = \int_0^2\int_{2x}^4 (x^2+y^2)(3)\,dy\,dx = 3\int_0^2 \left(4x^2 + \frac{64}{3} - \frac{14x^3}{3}\right)dx = 104$

22. (a) $I_o = \int_{-2}^2\int_{-1}^1 (x^2+y^2)\,dy\,dx = \int_{-2}^2 \left(2x^2 + \frac{2}{3}\right)dx = \frac{40}{3}$
(b) $I_x = \int_{-a}^a\int_{-b}^b y^2\,dy\,dx = \int_{-a}^a \frac{2b^3}{3}\,dx = \frac{4ab^3}{3}$; $I_y = \int_{-b}^b\int_{-a}^a x^2\,dx\,dy = \int_{-b}^b \frac{2a^3}{3}\,dy = \frac{4a^3b}{3} \Rightarrow I_o = I_x + I_y$
$= \frac{4ab^3}{3} + \frac{4a^3b}{3} = \frac{4ab(b^2+a^2)}{3}$

23. $M = \delta \int_0^3 \int_0^{2x/3} dy\,dx = \delta \int_0^3 \frac{2x}{3}\,dx = 3\delta$; $I_x = \delta \int_0^3 \int_0^{2x/3} y^2\,dy\,dx = \frac{8\delta}{81} \int_0^3 x^3\,dx = \left(\frac{8\delta}{81}\right)\left(\frac{3^4}{4}\right) = 2\delta \Rightarrow R_x = \sqrt{\frac{2}{3}}$

24. $M = \int_0^1 \int_{x^2}^{x} (x+1)\,dy\,dx = \int_0^1 (x - x^3)\,dx = \frac{1}{4}$; $M_x = \int_0^1 \int_{x^2}^{x} y(x+1)\,dy\,dx = \frac{1}{2} \int_0^1 (x^3 - x^5 + x^2 - x^4)\,dx = \frac{13}{120}$; $M_y = \int_0^1 \int_{x^2}^{x} x(x+1)\,dy\,dx = \int_0^1 (x^2 - x^4)\,dx = \frac{2}{15} \Rightarrow \overline{x} = \frac{8}{15}$ and $\overline{y} = \frac{13}{30}$; $I_x = \int_0^1 \int_{x^2}^{x} y^2(x+1)\,dy\,dx$
$= \frac{1}{3} \int_0^1 (x^4 - x^7 + x^3 - x^6)\,dx = \frac{17}{280} \Rightarrow R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{17}{70}}$; $I_y = \int_0^1 \int_{x^2}^{x} x^2(x+1)\,dy\,dx = \int_0^1 (x^3 - x^5)\,dx$
$= \frac{1}{12} \Rightarrow R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{1}{3}}$

25. $M = \int_{-1}^1 \int_{-1}^1 \left(x^2 + y^2 + \frac{1}{3}\right) dy\,dx = \int_{-1}^1 \left(2x^2 + \frac{4}{3}\right) dx = 4$; $M_x = \int_{-1}^1 \int_{-1}^1 y\left(x^2 + y^2 + \frac{1}{3}\right) dy\,dx = \int_{-1}^1 0\,dx = 0$; $M_y = \int_{-1}^1 \int_{-1}^1 x\left(x^2 + y^2 + \frac{1}{3}\right) dy\,dx = \int_{-1}^1 \left(2x^3 + \frac{4}{3}x\right) dx = 0$

26. Place the ΔABC with its vertices at $A(0,0)$, $B(b,0)$ and $C(a,h)$. The line through the points A and C is $y = \frac{h}{a}x$; the line through the points C and B is $y = \frac{h}{a-b}(x - b)$. Thus, $M = \int_0^h \int_{ay/h}^{(a-b)y/h+b} \delta\,dx\,dy$
$= b\delta \int_0^h \left(1 - \frac{y}{h}\right) dy = \frac{\delta bh}{2}$; $I_x = \int_0^h \int_{ay/h}^{(a-b)y/h+b} y^2\delta\,dx\,dy = b\delta \int_0^h \left(y^2 - \frac{y^3}{h}\right) dy = \frac{\delta bh^3}{12}$; $R_x = \sqrt{\frac{I_x}{M}} = \frac{h}{\sqrt{6}}$

27. $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\,dy\,dx = \int_0^{2\pi} \int_0^1 \frac{2r}{(1+r^2)^2}\,dr\,d\theta = \int_0^{2\pi} \left[-\frac{1}{1+r^2}\right]_0^1 d\theta = \frac{1}{2} \int_0^{2\pi} d\theta = \pi$

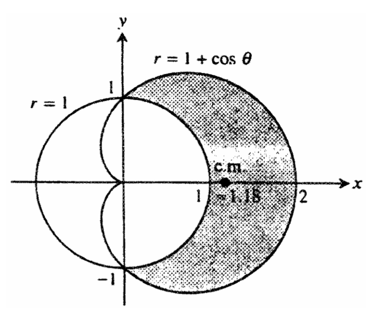
28. $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2 + y^2 + 1)\,dx\,dy = \int_0^{2\pi} \int_0^1 r \ln(r^2+1)\,dr\,d\theta = \int_0^{2\pi} \int_1^2 \frac{1}{2} \ln u\,du\,d\theta = \frac{1}{2} \int_0^{2\pi} [u \ln u - u]_1^2\,d\theta$
$= \frac{1}{2} \int_0^{2\pi} (2 \ln 2 - 1)\,d\theta = [\ln(4) - 1]\,\pi$

29. $M = \int_{-\pi/3}^{\pi/3} \int_0^3 r\,dr\,d\theta = \frac{9}{2} \int_{-\pi/3}^{\pi/3} d\theta = 3\pi$; $M_y = \int_{-\pi/3}^{\pi/3} \int_0^3 r^2 \cos\theta\,dr\,d\theta = 9 \int_{-\pi/3}^{\pi/3} \cos\theta\,d\theta = 9\sqrt{3} \Rightarrow \overline{x} = \frac{3\sqrt{3}}{\pi}$, and $\overline{y} = 0$ by symmetry

30. $M = \int_0^{\pi/2} \int_1^3 r\,dr\,d\theta = 4 \int_0^{\pi/2} d\theta = 2\pi$; $M_y = \int_0^{\pi/2} \int_1^3 r^2 \cos\theta\,dr\,d\theta = \frac{26}{3} \int_0^{\pi/2} \cos\theta\,d\theta = \frac{26}{3} \Rightarrow \overline{x} = \frac{13}{3\pi}$, and $\overline{y} = \frac{13}{3\pi}$ by symmetry

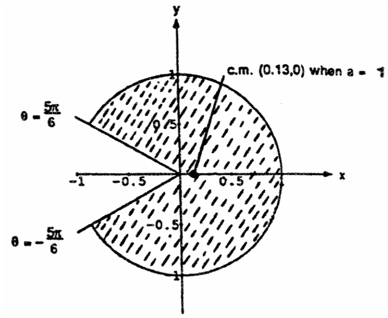
31. (a) $M = 2 \int_0^{\pi/2} \int_1^{1+\cos\theta} r\,dr\,d\theta$
$= \int_0^{\pi/2} \left(2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta = \frac{8+\pi}{4}$;
$M_y = \int_{-\pi/2}^{\pi/2} \int_1^{1+\cos\theta} (r\cos\theta)\,r\,dr\,d\theta$
$= \int_{-\pi/2}^{\pi/2} \left(\cos^2\theta + \cos^3\theta + \frac{\cos^4\theta}{3}\right) d\theta$
$= \frac{32+15\pi}{24} \Rightarrow \overline{x} = \frac{15\pi+32}{6\pi+48}$, and
$\overline{y} = 0$ by symmetry

(b)



32. (a) $M = \int_{-\alpha}^{\alpha} \int_0^a r\,dr\,d\theta = \int_{-\alpha}^{\alpha} \frac{a^2}{2}\,d\theta = a^2\alpha$; $M_y = \int_{-\alpha}^{\alpha} \int_0^a (r\cos\theta)\,r\,dr\,d\theta = \int_{-\alpha}^{\alpha} \frac{a^3\cos\theta}{3}\,d\theta = \frac{2a^3\sin\alpha}{3}$
$\Rightarrow \overline{x} = \frac{2a\sin\alpha}{3\alpha}$, and $\overline{y} = 0$ by symmetry; $\lim_{\alpha \to \pi^-} \overline{x} = \lim_{\alpha \to \pi^-} \frac{2a\sin\alpha}{3\alpha} = 0$

(b) $\overline{x} = \frac{2a}{5\pi}$ and $\overline{y} = 0$



33. $(x^2+y^2)^2 - (x^2-y^2) = 0 \Rightarrow r^4 - r^2 \cos 2\theta = 0 \Rightarrow r^2 = \cos 2\theta$ so the integral is $\int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\, dr\, d\theta$

$= \int_{-\pi/4}^{\pi/4} \left[-\frac{1}{2(1+r^2)}\right]_0^{\sqrt{\cos 2\theta}} d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \left(1 - \frac{1}{1+\cos 2\theta}\right) d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \left(1 - \frac{1}{2\cos^2\theta}\right) d\theta$

$= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \left(1 - \frac{\sec^2\theta}{2}\right) d\theta = \frac{1}{2}\left[\theta - \frac{\tan\theta}{2}\right]_{-\pi/4}^{\pi/4} = \frac{\pi-2}{4}$

34. (a) $\iint\limits_R \frac{1}{(1+x^2+y^2)^2}\, dx\, dy = \int_0^{\pi/3} \int_0^{\sec\theta} \frac{r}{(1+r^2)^2}\, dr\, d\theta = \int_0^{\pi/3} \left[-\frac{1}{2(1+r^2)}\right]_0^{\sec\theta} d\theta$

$= \int_0^{\pi/3} \left[\frac{1}{2} - \frac{1}{2(1+\sec^2\theta)}\right] d\theta = \frac{1}{2} \int_0^{\pi/3} \frac{\sec^2\theta}{1+\sec^2\theta}\, d\theta; \begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta\, d\theta \end{bmatrix} \to \frac{1}{2} \int_0^{\sqrt{3}} \frac{du}{2+u^2}$

$= \frac{1}{2}\left[\frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}}\right]_0^{\sqrt{3}} = \frac{\sqrt{2}}{4} \tan^{-1} \sqrt{\frac{3}{2}}$

(b) $\iint\limits_R \frac{1}{(1+x^2+y^2)^2}\, dx\, dy = \int_0^{\pi/2} \int_0^{\infty} \frac{r}{(1+r^2)^2}\, dr\, d\theta = \int_0^{\pi/2} \lim\limits_{b\to\infty} \left[-\frac{1}{2(1+r^2)}\right]_0^{b} d\theta$

$= \int_0^{\pi/2} \lim\limits_{b\to\infty} \left[\frac{1}{2} - \frac{1}{2(1+b^2)}\right] d\theta = \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{\pi}{4}$

35. $\int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \cos(x+y+z)\, dx\, dy\, dz = \int_0^{\pi} \int_0^{\pi} [\sin(z+y+\pi) - \sin(z+y)]\, dy\, dz$

$= \int_0^{\pi} [-\cos(z+2\pi) + \cos(z+\pi) - \cos z + \cos(z+\pi)]\, dz = 0$

36. $\int_{\ln 6}^{\ln 7} \int_0^{\ln 2} \int_{\ln 4}^{\ln 5} e^{(x+y+z)}\, dz\, dy\, dx = \int_{\ln 6}^{\ln 7} \int_0^{\ln 2} e^{(x+y)}\, dy\, dx = \int_{\ln 6}^{\ln 7} e^x\, dx = 1$

37. $\int_0^1 \int_0^{x^2} \int_0^{x+y} (2x - y - z)\, dz\, dy\, dx = \int_0^1 \int_0^{x^2} \left(\frac{3x^2}{2} - \frac{3y^2}{2}\right) dy\, dx = \int_0^1 \left(\frac{3x^4}{2} - \frac{x^6}{2}\right) dx = \frac{8}{35}$

38. $\int_1^e \int_1^x \int_0^z \frac{2y}{z^3}\, dy\, dz\, dx = \int_1^e \int_1^x \frac{1}{z}\, dz\, dx = \int_1^e \ln x\, dx = [x \ln x - x]_1^e = 1$

39. $V = 2 \int_0^{\pi/2} \int_{-\cos y}^{0} \int_0^{-2x} dz\, dx\, dy = 2 \int_0^{\pi/2} \int_{-\cos y}^{0} -2x\, dx\, dy = 2 \int_0^{\pi/2} \cos^2 y\, dy = 2\left[\frac{y}{2} + \frac{\sin 2y}{4}\right]_0^{\pi/2} = \frac{\pi}{2}$

40. $V = 4 \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{4-x^2} dz\, dy\, dx = 4 \int_0^2 \int_0^{\sqrt{4-x^2}} (4-x^2)\, dy\, dx = 4 \int_0^2 (4-x^2)^{3/2}\, dx$

$= \left[x(4-x^2)^{3/2} + 6x\sqrt{4-x^2} + 24 \sin^{-1} \frac{x}{2}\right]_0^2 = 24 \sin^{-1} 1 = 12\pi$

41. average $= \frac{1}{3} \int_0^1 \int_0^3 \int_0^1 30xz\sqrt{x^2+y}\, dz\, dy\, dx = \frac{1}{3} \int_0^1 \int_0^3 15x\sqrt{x^2+y}\, dy\, dx = \frac{1}{3} \int_0^3 \int_0^1 15x\sqrt{x^2+y}\, dx\, dy$

$= \frac{1}{3} \int_0^3 \left[5(x^2+y)^{3/2}\right]_0^1 dy = \frac{1}{3} \int_0^3 \left[5(1+y)^{3/2} - 5y^{3/2}\right] dy = \frac{1}{3}\left[2(1+y)^{5/2} - 2y^{5/2}\right]_0^3 = \frac{1}{3}\left[2(4)^{5/2} - 2(3)^{5/2} - 2\right]$

$= \frac{1}{3}\left[2\left(31 - 3^{5/2}\right)\right]$

42. average $= \frac{3}{4\pi a^3} \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^3 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{3a}{16\pi} \int_0^{2\pi}\int_0^{\pi} \sin\phi \, d\phi \, d\theta = \frac{3a}{8\pi} \int_0^{2\pi} d\theta = \frac{3a}{4}$

43. (a) $\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} 3 \, dz \, dx \, dy$

(b) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} 3\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

(c) $\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} 3 \, dz \, r \, dr \, d\theta = 3\int_0^{2\pi}\int_0^{\sqrt{2}} \left[r\left(4-r^2\right)^{1/2} - r^2\right] dr \, d\theta = 3\int_0^{2\pi} \left[-\frac{1}{3}\left(4-r^2\right)^{3/2} - \frac{r^3}{3}\right]_0^{\sqrt{2}} d\theta$

$= \int_0^{2\pi} \left(-2^{3/2} - 2^{3/2} + 4^{3/2}\right) d\theta = \left(8 - 4\sqrt{2}\right)\int_0^{2\pi} d\theta = 2\pi\left(8 - 4\sqrt{2}\right)$

44. (a) $\int_{-\pi/2}^{\pi/2}\int_0^{1}\int_{-r^2}^{r^2} 21(r\cos\theta)(r\sin\theta)^2 \, dz \, r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2}\int_0^{1}\int_{-r^2}^{r^2} 21r^3 \cos\theta \sin^2\theta \, dz \, r \, dr \, d\theta$

(b) $\int_{-\pi/2}^{\pi/2}\int_0^{1}\int_{-r^2}^{r^2} 21r^3 \cos\theta \sin^2\theta \, dz \, r \, dr \, d\theta = 84\int_0^{\pi/2}\int_0^{1} r^6 \sin^2\theta \cos\theta \, dr \, d\theta = 12\int_0^{\pi/2} \sin^2\theta \cos\theta \, d\theta = 4$

45. (a) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

(b) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4} (\sec\phi)(\sec\phi\tan\phi) \, d\phi \, d\theta = \frac{1}{3}\int_0^{2\pi} \left[\frac{1}{2}\tan^2\phi\right]_0^{\pi/4} d\theta = \frac{1}{6}\int_0^{2\pi} d\theta = \frac{\pi}{3}$

46. (a) $\int_0^{1}\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{x^2+y^2}} (6+4y) \, dz \, dy \, dx$ (b) $\int_0^{\pi/2}\int_0^{1}\int_0^{r} (6+4r\sin\theta) \, dz \, r \, dr \, d\theta$

(c) $\int_0^{\pi/2}\int_{\pi/4}^{\pi/2}\int_0^{\csc\phi} (6+4\rho\sin\phi\sin\theta)\left(\rho^2\sin\phi\right) d\rho \, d\phi \, d\theta$

(d) $\int_0^{\pi/2}\int_0^{1}\int_0^{r} (6+4r\sin\theta) \, dz \, r \, dr \, d\theta = \int_0^{\pi/2}\int_0^{1} \left(6r^2 + 4r^3\sin\theta\right) dr \, d\theta = \int_0^{\pi/2} \left[2r^3 + r^4\sin\theta\right]_0^1 d\theta$

$= \int_0^{\pi/2} (2+\sin\theta) \, d\theta = \left[2\theta - \cos\theta\right]_0^{\pi/2} = \pi + 1$

47. $\int_0^{1}\int_{\sqrt{1-x^2}}^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}} z^2yx \, dz \, dy \, dx + \int_1^{\sqrt{3}}\int_0^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}} z^2yx \, dz \, dy \, dx$

48. (a) Bounded on the top and bottom by the sphere $x^2 + y^2 + z^2 = 4$, on the right by the right circular cylinder $(x-1)^2 + y^2 = 1$, on the left by the plane $y = 0$

(b) $\int_0^{\pi/2}\int_0^{2\cos\theta}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dz \, r \, dr \, d\theta$

49. (a) $V = \int_0^{2\pi}\int_0^{2}\int_2^{\sqrt{8-r^2}} dz \, r \, dr \, d\theta = \int_0^{2\pi}\int_0^{2} \left(r\sqrt{8-r^2} - 2r\right) dr \, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}\left(8-r^2\right)^{3/2} - r^2\right]_0^2 d\theta$

$= \int_0^{2\pi} \left[-\frac{1}{3}(4)^{3/2} - 4 + \frac{1}{3}(8)^{3/2}\right] d\theta = \int_0^{2\pi} \frac{4}{3}\left(-2 - 3 + 2\sqrt{8}\right) d\theta = \frac{4}{3}\left(4\sqrt{2} - 5\right)\int_0^{2\pi} d\theta = \frac{8\pi\left(4\sqrt{2}-5\right)}{3}$

(b) $V = \int_0^{2\pi}\int_0^{\pi/4}\int_{2\sec\phi}^{\sqrt{8}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{8}{3}\int_0^{2\pi}\int_0^{\pi/4} \left(2\sqrt{2}\sin\phi - \sec^3\phi\sin\phi\right) d\phi \, d\theta$

$= \frac{8}{3}\int_0^{2\pi}\int_0^{\pi/4} \left(2\sqrt{2}\sin\phi - \tan\phi\sec^2\phi\right) d\phi \, d\theta = \frac{8}{3}\int_0^{2\pi} \left[-2\sqrt{2}\cos\phi - \frac{1}{2}\tan^2\phi\right]_0^{\pi/4} d\theta$

$= \frac{8}{3}\int_0^{2\pi} \left(-2 - \frac{1}{2} + 2\sqrt{2}\right) d\theta = \frac{8}{3}\int_0^{2\pi} \left(\frac{-5+4\sqrt{2}}{2}\right) d\theta = \frac{8\pi\left(4\sqrt{2}-5\right)}{3}$

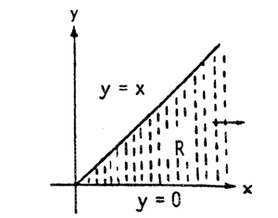
50. $I_z = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{2} (\rho\sin\phi)^2\left(\rho^2\sin\phi\right) d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{2} \rho^4 \sin^3\phi \, d\rho \, d\phi \, d\theta$

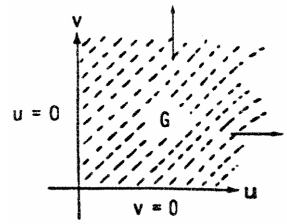
$= \frac{32}{5}\int_0^{2\pi}\int_0^{\pi/3} \left(\sin\phi - \cos^2\phi\sin\phi\right) d\phi \, d\theta = \frac{32}{5}\int_0^{2\pi} \left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi/3} d\theta = \frac{8\pi}{3}$

51. With the centers of the spheres at the origin, $I_z = \int_0^{2\pi}\int_0^{\pi}\int_a^b \delta(\rho \sin\phi)^2\,(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$
$= \frac{\delta\,(b^5-a^5)}{5}\int_0^{2\pi}\int_0^{\pi}\sin^3\phi\,d\phi\,d\theta = \frac{\delta\,(b^5-a^5)}{5}\int_0^{2\pi}\int_0^{\pi}(\sin\phi-\cos^2\phi\sin\phi)\,d\phi\,d\theta$
$= \frac{\delta\,(b^5-a^5)}{5}\int_0^{2\pi}\left[-\cos\phi+\frac{\cos^3\phi}{3}\right]_0^{\pi}d\theta = \frac{4\delta\,(b^5-a^5)}{15}\int_0^{2\pi}d\theta = \frac{8\pi\delta\,(b^5-a^5)}{15}$

52. $I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\theta}(\rho\sin\phi)^2\,(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\theta}\rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta$
$= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi}(1-\cos\phi)^5\sin^3\phi\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}(1-\cos\phi)^6(1+\cos\phi)\sin\phi\,d\phi\,d\theta;$
$\begin{bmatrix} u = 1-\cos\phi \\ du = \sin\phi\,d\phi \end{bmatrix} \to \frac{1}{5}\int_0^{2\pi}\int_0^{2}u^6(2-u)\,du\,d\theta = \frac{1}{5}\int_0^{2\pi}\left[\frac{2u^7}{7}-\frac{u^8}{8}\right]_0^2 d\theta = \frac{1}{5}\int_0^{2\pi}\left(\frac{1}{7}-\frac{1}{8}\right)2^8\,d\theta$
$= \frac{1}{5}\int_0^{2\pi}\frac{2^3\cdot 2^5}{56}\,d\theta = \frac{32}{35}\int_0^{2\pi}d\theta = \frac{64\pi}{35}$

53. $x = u + y$ and $y = v \Rightarrow x = u + v$ and $y = v$
$\Rightarrow J(u,v) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$; the boundary of the image G is obtained from the boundary of R as follows:





xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$y = x$	$v = u + v$	$u = 0$
$y = 0$	$v = 0$	$v = 0$

$\Rightarrow \int_0^{\infty}\int_0^{x}e^{-sx}f(x-y,y)\,dy\,dx = \int_0^{\infty}\int_0^{\infty}e^{-s(u+v)}f(u,v)\,du\,dv$

54. If $s = \alpha x + \beta y$ and $t = \gamma x + \delta y$ where $(\alpha\delta-\beta\gamma)^2 = ac - b^2$, then $x = \frac{\delta s - \beta t}{\alpha\delta-\beta\gamma}$, $y = \frac{-\gamma s + \alpha t}{\alpha\delta - \beta\gamma}$,
and $J(s,t) = \frac{1}{(\alpha\delta-\beta\gamma)^2}\begin{vmatrix} \delta & -\beta \\ -\gamma & \alpha \end{vmatrix} = \frac{1}{\alpha\delta-\beta\gamma} \Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(s^2+t^2)}\frac{1}{\sqrt{ac-b^2}}\,ds\,dt$
$= \frac{1}{\sqrt{ac-b^2}}\int_0^{2\pi}\int_0^{\infty}re^{-r^2}\,dr\,d\theta = \frac{1}{2\sqrt{ac-b^2}}\int_0^{2\pi}d\theta = \frac{\pi}{\sqrt{ac-b^2}}$. Therefore, $\frac{\pi}{\sqrt{ac-b^2}} = 1 \Rightarrow ac - b^2 = \pi^2$.

CHAPTER 15 ADDITIONAL AND ADVANCED EXERCISES

1. (a) $V = \int_{-3}^{2}\int_{x}^{6-x^2}x^2\,dy\,dx$ (b) $V = \int_{-3}^{2}\int_{x}^{6-x^2}\int_0^{x^2}dz\,dy\,dx$
(c) $V = \int_{-3}^{2}\int_{x}^{6-x^2}x^2\,dy\,dx = \int_{-3}^{2}\int_{x}^{6-x^2}(6x^2-x^4-x^3)\,dx = \left[2x^3-\frac{x^5}{5}-\frac{x^4}{4}\right]_{-3}^{2} = \frac{125}{4}$

2. Place the sphere's center at the origin with the surface of the water at $z = -3$. Then
$9 = 25 - x^2 - y^2 \Rightarrow x^2 + y^2 = 16$ is the projection of the volume of water onto the xy-plane

$\Rightarrow\ V = \int_0^{2\pi}\int_0^4\int_{-\sqrt{25-r^2}}^{-3} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^4 \left(r\sqrt{25-r^2} - 3r\right) dr\, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}\left(25-r^2\right)^{3/2} - \frac{3}{2}r^2\right]_0^4 d\theta$
$= \int_0^{2\pi} \left[-\frac{1}{3}(9)^{3/2} - 24 + \frac{1}{3}(25)^{3/2}\right] d\theta = \int_0^{2\pi} \frac{26}{3}\, d\theta = \frac{52\pi}{3}$

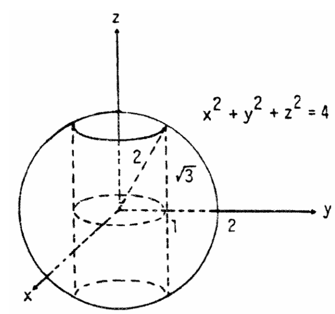
3. Using cylindrical coordinates, $V = \int_0^{2\pi}\int_0^1\int_0^{2-r(\cos\theta+\sin\theta)} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 (2r - r^2\cos\theta - r^2\sin\theta)\, dr\, d\theta$
$= \int_0^{2\pi}\left(1 - \frac{1}{3}\cos\theta - \frac{1}{3}\sin\theta\right) d\theta = \left[\theta - \frac{1}{3}\sin\theta + \frac{1}{3}\cos\theta\right]_0^{2\pi} = 2\pi$

4. $V = 4\int_0^{\pi/2}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta = 4\int_0^{\pi/2}\int_0^1 \left(r\sqrt{2-r^2} - r^3\right) dr\, d\theta = 4\int_0^{\pi/2}\left[-\frac{1}{3}\left(2-r^2\right)^{3/2} - \frac{r^4}{4}\right]_0^1 d\theta$
$= 4\int_0^{\pi/2}\left(-\frac{1}{3} - \frac{1}{4} + \frac{2\sqrt{2}}{3}\right) d\theta = \left(\frac{8\sqrt{2}-7}{3}\right)\int_0^{\pi/2} d\theta = \frac{\pi\left(8\sqrt{2}-7\right)}{6}$

5. The surfaces intersect when $3 - x^2 - y^2 = 2x^2 + 2y^2 \ \Rightarrow\ x^2 + y^2 = 1$. Thus the volume is
$V = 4\int_0^1\int_0^{\sqrt{1-x^2}}\int_{2x^2+2y^2}^{3-x^2-y^2} dz\, dy\, dx = 4\int_0^{\pi/2}\int_0^1\int_{2r^2}^{3-r^2} dz\, r\, dr\, d\theta = 4\int_0^{\pi/2}\int_0^1 (3r - 3r^3)\, dr\, d\theta = 3\int_0^{\pi/2} d\theta = \frac{3\pi}{2}$

6. $V = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2\sin\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{64}{3}\int_0^{\pi/2}\int_0^{\pi/2} \sin^4\phi\, d\phi\, d\theta$
$= \frac{64}{3}\int_0^{\pi/2}\left[-\frac{\sin^3\phi\cos\phi}{4}\Big|_0^{\pi/2} + \frac{3}{4}\int_0^{\pi/2}\sin^2\phi\, d\phi\right] d\theta = 16\int_0^{\pi/2}\left[\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right]_0^{\pi/2} d\theta = 4\pi\int_0^{\pi/2} d\theta = 2\pi^2$

7. (a) The radius of the hole is 1, and the radius of the sphere is 2.



(b) $V = 2\int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-z^2}} r\, dr\, dz\, d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} (3 - z^2)\, dz\, d\theta = 2\sqrt{3}\int_0^{2\pi} d\theta = 4\sqrt{3}\pi$

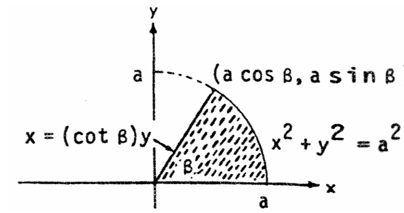
8. $V = \int_0^{\pi}\int_0^{3\sin\theta}\int_0^{\sqrt{9-r^2}} dz\, r\, dr\, d\theta = \int_0^{\pi}\int_0^{3\sin\theta} r\sqrt{9-r^2}\, dr\, d\theta = \int_0^{\pi}\left[-\frac{1}{3}\left(9-r^2\right)^{3/2}\right]_0^{3\sin\theta} d\theta$
$= \int_0^{\pi}\left[-\frac{1}{3}\left(9 - 9\sin^2\theta\right)^{3/2} + \frac{1}{3}(9)^{3/2}\right] d\theta = 9\int_0^{\pi}\left[1 - \left(1 - \sin^2\theta\right)^{3/2}\right] d\theta = 9\int_0^{\pi}(1 - \cos^3\theta)\, d\theta$
$= \int_0^{\pi}(1 - \cos\theta + \sin^2\theta\cos\theta)\, d\theta = 9\left[\theta - \sin\theta + \frac{\sin^3\theta}{3}\right]_0^{\pi} = 9\pi$

9. The surfaces intersect when $x^2 + y^2 = \frac{x^2+y^2+1}{2} \ \Rightarrow\ x^2 + y^2 = 1$. Thus the volume in cylindrical coordinates is $V = 4\int_0^{\pi/2}\int_0^1\int_{r^2}^{(r^2+1)/2} dz\, r\, dr\, d\theta = 4\int_0^{\pi/2}\int_0^1\left(\frac{r}{2} - \frac{r^3}{2}\right) dr\, d\theta = 4\int_0^{\pi/2}\left[\frac{r^2}{4} - \frac{r^4}{8}\right]_0^1 d\theta$
$= \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}$

10. $V = \int_0^{\pi/2}\int_1^2\int_0^{r^2\sin\theta\cos\theta} dz\, r\, dr\, d\theta = \int_0^{\pi/2}\int_1^2 r^3\sin\theta\cos\theta\, dr\, d\theta = \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_1^2 \sin\theta\cos\theta\, d\theta$
$= \frac{15}{4}\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta = \frac{15}{4}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{15}{8}$

11. $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx = \int_0^\infty \int_a^b e^{-xy}\,dy\,dx = \int_a^b \int_0^\infty e^{-xy}\,dx\,dy = \int_a^b \left(\lim\limits_{t\to\infty} \int_0^t e^{-xy}\,dx \right) dy$
$= \int_a^b \lim\limits_{t\to\infty} \left[-\frac{e^{-xy}}{y} \right]_0^t dy = \int_a^b \lim\limits_{t\to\infty} \left(\frac{1}{y} - \frac{e^{-yt}}{y} \right) dy = \int_a^b \frac{1}{y}\,dy = [\ln y]_a^b = \ln\left(\frac{b}{a}\right)$

12. (a) The region of integration is sketched at the right



$\Rightarrow \int_0^{a\sin\beta} \int_{y\cot\beta}^{\sqrt{a^2-y^2}} \ln(x^2+y^2)\,dx\,dy$
$= \int_0^\beta \int_0^a r\ln(r^2)\,dr\,d\theta;$
$\begin{bmatrix} u = r^2 \\ du = 2r\,dr \end{bmatrix} \to \frac{1}{2}\int_0^\beta \int_0^{a^2} \ln u\,du\,d\theta$
$= \frac{1}{2}\int_0^\beta [u\ln u - u]_0^{a^2}\,d\theta$
$= \frac{1}{2}\int_0^\beta \left[2a^2\ln a - a^2 - \lim\limits_{t\to 0} t\ln t\right] d\theta = \frac{a^2}{2}\int_0^\beta (2\ln a - 1)\,d\theta = a^2\beta\left(\ln a - \frac{1}{2}\right)$

(b) $\int_0^{a\cos\beta} \int_0^{(\tan\beta)x} \ln(x^2+y^2)\,dy\,dx + \int_{a\cos\beta}^a \int_0^{\sqrt{a^2-x^2}} \ln(x^2+y^2)\,dy\,dx$

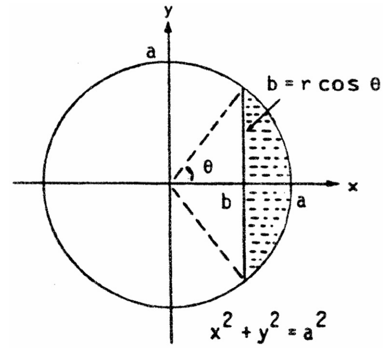
13. $\int_0^x \int_0^u e^{m(x-t)} f(t)\,dt\,du = \int_0^x \int_t^x e^{m(x-t)} f(t)\,du\,dt = \int_0^x (x-t)e^{m(x-t)} f(t)\,dt$; also
$\int_0^x \int_0^v \int_0^u e^{m(x-t)} f(t)\,dt\,du\,dv = \int_0^x \int_t^x \int_t^v e^{m(x-t)} f(t)\,du\,dv\,dt = \int_0^x \int_t^x (v-t)e^{m(x-t)} f(t)\,dv\,dt$
$= \int_0^x \left[\frac{1}{2}(v-t)^2 e^{m(x-t)} f(t)\right]_t^x dt = \int_0^x \frac{(x-t)^2}{2} e^{m(x-t)} f(t)\,dt$

14. $\int_0^1 f(x)\left(\int_0^x g(x-y)f(y)\,dy\right) dx = \int_0^1 \int_0^x g(x-y)f(x)f(y)\,dy\,dx$
$= \int_0^1 \int_y^1 g(x-y)f(x)f(y)\,dx\,dy = \int_0^1 f(y)\left(\int_y^1 g(x-y)f(x)\,dx\right) dy;$
$\int_0^1 \int_0^1 g(|x-y|)f(x)f(y)\,dx\,dy = \int_0^1 \int_0^x g(x-y)f(x)f(y)\,dy\,dx + \int_0^1 \int_x^1 g(y-x)f(x)f(y)\,dy\,dx$
$= \int_0^1 \int_y^1 g(x-y)f(x)f(y)\,dx\,dy + \int_0^1 \int_x^1 g(y-x)f(x)f(y)\,dy\,dx$
$= \int_0^1 \int_y^1 g(x-y)f(x)f(y)\,dx\,dy + \underbrace{\int_0^1 \int_y^1 g(x-y)f(y)f(x)\,dx\,dy}_{\text{simply interchange x and y variable names}}$
$= 2\int_0^1 \int_y^1 g(x-y)f(x)f(y)\,dx\,dy$, and the statement now follows.

15. $I_o(a) = \int_0^a \int_0^{x/a^2} (x^2+y^2)\,dy\,dx = \int_0^a \left[x^2y + \frac{y^3}{3}\right]_0^{x/a^2} dx = \int_0^a \left(\frac{x^3}{a^2} + \frac{x^3}{3a^6}\right) dx = \left[\frac{x^4}{4a^2} + \frac{x^4}{12a^6}\right]_0^a$
$= \frac{a^2}{4} + \frac{1}{12}a^{-2}$; $I_o'(a) = \frac{1}{2}a - \frac{1}{6}a^{-3} = 0 \Rightarrow a^4 = \frac{1}{3} \Rightarrow a = \sqrt[4]{\frac{1}{3}} = \frac{1}{\sqrt[4]{3}}$. Since $I_o''(a) = \frac{1}{2} + \frac{1}{2}a^{-4} > 0$, the value of a does provide a <u>minimum</u> for the polar moment of inertia $I_o(a)$.

16. $I_o = \int_0^2 \int_{2x}^4 (x^2+y^2)(3)\,dy\,dx = 3\int_0^2 \left(4x^2 - \frac{14x^3}{3} + \frac{64}{3}\right) dx = 104$

17. $M = \int_{-\theta}^{\theta}\int_{b\sec\theta}^{a} r\,dr\,d\theta = \int_{-\theta}^{\theta}\left(\frac{a^2}{2} - \frac{b^2}{2}\sec^2\theta\right)d\theta$
$= a^2\theta - b^2\tan\theta = a^2\cos^{-1}\left(\frac{b}{a}\right) - b^2\left(\frac{\sqrt{a^2-b^2}}{b}\right)$
$= a^2\cos^{-1}\left(\frac{b}{a}\right) - b\sqrt{a^2-b^2}$; $I_o = \int_{-\theta}^{\theta}\int_{b\sec\theta}^{a} r^3\,dr\,d\theta$
$= \frac{1}{4}\int_{-\theta}^{\theta}(a^4 + b^4\sec^4\theta)\,d\theta$
$= \frac{1}{4}\int_{-\theta}^{\theta}[a^4 + b^4(1+\tan^2\theta)(\sec^2\theta)]\,d\theta$
$= \frac{1}{4}\left[a^4\theta - b^4\tan\theta - \frac{b^4\tan^3\theta}{3}\right]_{-\theta}^{\theta}$
$= \frac{a^4\theta}{2} - \frac{b^4\tan\theta}{2} - \frac{b^4\tan^3\theta}{6}$
$= \frac{1}{2}a^4\cos^{-1}\left(\frac{b}{a}\right) - \frac{1}{2}b^3\sqrt{a^2-b^2} - \frac{1}{6}b^3(a^2-b^2)^{3/2}$



18. $M = \int_{-2}^{2}\int_{1-(y^2/4)}^{2-(y^2/2)} dx\,dy = \int_{-2}^{2}\left(1 - \frac{y^2}{4}\right)dy = \left[y - \frac{y^3}{12}\right]_{-2}^{2} = \frac{8}{3}$; $M_y = \int_{-2}^{2}\int_{1-(y^2/4)}^{2-(y^2/2)} x\,dx\,dy$
$= \int_{-2}^{2}\left[\frac{x^2}{2}\right]_{1-(y^2/4)}^{2-(y^2/2)} dy = \int_{-2}^{2}\frac{3}{32}(4-y^2)\,dy = \frac{3}{32}\int_{-2}^{2}(16 - 8y^2 + y^4)\,dy = \frac{3}{16}\left[16y - \frac{8y^3}{3} + \frac{y^5}{5}\right]_0^2$
$= \frac{3}{16}\left(32 - \frac{64}{3} + \frac{32}{5}\right) = \left(\frac{3}{16}\right)\left(\frac{32\cdot 8}{15}\right) = \frac{48}{15} \Rightarrow \overline{x} = \frac{M_y}{M} = \left(\frac{48}{15}\right)\left(\frac{3}{8}\right) = \frac{6}{5}$, and $\overline{y} = 0$ by symmetry

19. $\int_0^a\int_0^b e^{\max(b^2x^2, a^2y^2)}\,dy\,dx = \int_0^a\int_0^{bx/a} e^{b^2x^2}\,dy\,dx + \int_0^b\int_0^{ay/b} e^{a^2y^2}\,dx\,dy$
$= \int_0^a\left(\frac{b}{a}x\right)e^{b^2x^2}\,dx + \int_0^b\left(\frac{a}{b}y\right)e^{a^2y^2}\,dy = \left[\frac{1}{2ab}e^{b^2x^2}\right]_0^a + \left[\frac{1}{2ba}e^{a^2y^2}\right]_0^b = \frac{1}{2ab}\left(e^{b^2a^2} - 1\right) + \frac{1}{2ab}\left(e^{a^2b^2} - 1\right)$
$= \frac{1}{ab}\left(e^{a^2b^2} - 1\right)$

20. $\int_{y_0}^{y_1}\int_{x_0}^{x_1}\frac{\partial^2 F(x,y)}{\partial x\,\partial y}\,dx\,dy = \int_{y_0}^{y_1}\left[\frac{\partial F(x,y)}{\partial y}\right]_{x_0}^{x_1} dy = \int_{y_0}^{y_1}\left[\frac{\partial F(x_1,y)}{\partial y} - \frac{\partial F(x_0,y)}{\partial y}\right]dx = [F(x_1,y) - F(x_0,y)]_{y_0}^{y_1}$
$= F(x_1,y_1) - F(x_0,y_1) - F(x_1,y_0) + F(x_0,y_0)$

21. (a) (i) Fubini's Theorem
(ii) Treating G(y) as a constant
(iii) Algebraic rearrangement
(iv) The definite integral is a constant number

(b) $\int_0^{\ln 2}\int_0^{\pi/2} e^x\cos y\,dy\,dx = \left(\int_0^{\ln 2} e^x\,dx\right)\left(\int_0^{\pi/2}\cos y\,dy\right) = (e^{\ln 2} - e^0)\left(\sin\frac{\pi}{2} - \sin 0\right) = (1)(1) = 1$

(c) $\int_1^2\int_{-1}^1 \frac{x}{y^2}\,dx\,dy = \left(\int_1^2 \frac{1}{y^2}\,dy\right)\left(\int_{-1}^1 x\,dx\right) = \left[-\frac{1}{y}\right]_1^2\left[\frac{x^2}{2}\right]_{-1}^1 = \left(-\frac{1}{2} + 1\right)\left(\frac{1}{2} - \frac{1}{2}\right) = 0$

22. (a) $\nabla f = x\mathbf{i} + y\mathbf{j} \Rightarrow D_u f = u_1x + u_2y$; the area of the region of integration is $\frac{1}{2}$
$\Rightarrow$ average $= 2\int_0^1\int_0^{1-x}(u_1x + u_2y)\,dy\,dx = 2\int_0^1\left[u_1x(1-x) + \frac{1}{2}u_2(1-x)^2\right]dx$
$= 2\left[u_1\left(\frac{x^2}{2} - \frac{x^3}{3}\right) - \left(\frac{1}{2}u_2\right)\frac{(1-x)^3}{3}\right]_0^1 = 2\left(\frac{1}{6}u_1 + \frac{1}{6}u_2\right) = \frac{1}{3}(u_1 + u_2)$

(b) average $= \frac{1}{\text{area}}\iint_R (u_1x + u_2y)\,dA = \frac{u_1}{\text{area}}\iint_R x\,dA + \frac{u_2}{\text{area}}\iint_R y\,dA = u_1\left(\frac{M_y}{M}\right) + u_2\left(\frac{M_x}{M}\right) = u_1\overline{x} + u_2\overline{y}$

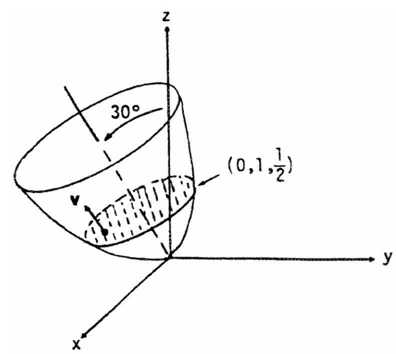
23. (a) $I^2 = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^\infty (e^{-r^2})\,r\,dr\,d\theta = \int_0^{\pi/2}\left[\lim_{b\to\infty}\int_0^b re^{-r^2}\,dr\right]d\theta$
$= -\frac{1}{2}\int_0^{\pi/2}\lim_{b\to\infty}(e^{-b^2} - 1)\,d\theta = \frac{1}{2}\int_0^{\pi/2}d\theta = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2}$

(b) $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2}e^{-t}\,dt = \int_0^\infty (y^2)^{-1/2}e^{-y^2}(2y)\,dy = 2\int_0^\infty e^{-y^2}\,dy = 2\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}$, where $y = \sqrt{t}$

24. $Q = \int_0^{2\pi}\int_0^R kr^2(1-\sin\theta)\,dr\,d\theta = \frac{kR^3}{3}\int_0^{2\pi}(1-\sin\theta)\,d\theta = \frac{kR^3}{3}\left[\theta+\cos\theta\right]_0^{2\pi} = \frac{2\pi kR^3}{3}$

25. For a height h in the bowl the volume of water is $V = \int_{-\sqrt{h}}^{\sqrt{h}}\int_{-\sqrt{h-x^2}}^{\sqrt{h-x^2}}\int_{x^2+y^2}^{h} dz\,dy\,dx$
$= \int_{-\sqrt{h}}^{\sqrt{h}}\int_{-\sqrt{h-x^2}}^{\sqrt{h-x^2}}(h-x^2-y^2)\,dy\,dx = \int_0^{2\pi}\int_0^{\sqrt{h}}(h-r^2)\,r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{hr^2}{2}-\frac{r^4}{4}\right]_0^{\sqrt{h}} d\theta = \int_0^{2\pi}\frac{h^2}{4}\,d\theta = \frac{h^2\pi}{2}.$
Since the top of the bowl has area 10π, then we calibrate the bowl by comparing it to a right circular cylinder whose cross sectional area is 10π from $z=0$ to $z=10$. If such a cylinder contains $\frac{h^2\pi}{2}$ cubic inches of water to a depth w then we have $10\pi w = \frac{h^2\pi}{2} \Rightarrow w = \frac{h^2}{20}$. So for 1 inch of rain, $w=1$ and $h=\sqrt{20}$; for 3 inches of rain, $w=3$ and $h=\sqrt{60}$.

26. (a) An equation for the satellite dish in standard position is $z = \frac{1}{2}x^2 + \frac{1}{2}y^2$. Since the axis is tilted 30°, a unit vector $\mathbf{v} = 0\mathbf{i} + a\mathbf{j} + b\mathbf{k}$ normal to the plane of the water level satisfies $b = \mathbf{v}\cdot\mathbf{k} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$
$\Rightarrow a = -\sqrt{1-b^2} = -\frac{1}{2} \Rightarrow \mathbf{v} = -\frac{1}{2}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}$
$\Rightarrow -\frac{1}{2}(y-1) + \frac{\sqrt{3}}{2}\left(z-\frac{1}{2}\right) = 0$
$\Rightarrow z = \frac{1}{\sqrt{3}}y + \left(\frac{1}{2}-\frac{1}{\sqrt{3}}\right)$
is an equation of the plane of the water level. Therefore the volume of water is $V = \iint_R \int_{\frac{1}{2}x^2+\frac{1}{2}y^2}^{\frac{1}{\sqrt{3}}y+\frac{1}{2}-\frac{1}{\sqrt{3}}} dz\,dy\,dx$, where R is the interior of the ellipse



$x^2 + y^2 - \frac{2}{3}y - 1 + \frac{2}{\sqrt{3}} = 0$. When $x=0$, then $y=\alpha$ or $y=\beta$, where $\alpha = \frac{\frac{2}{3}+\sqrt{\frac{4}{9}-4\left(\frac{2}{\sqrt{3}}-1\right)}}{2}$
and $\beta = \frac{\frac{2}{3}-\sqrt{\frac{4}{9}-4\left(\frac{2}{\sqrt{3}}-1\right)}}{2} \Rightarrow V = \int_\alpha^\beta \int_{-\left(\frac{2}{3}y+1-\frac{2}{\sqrt{3}}-y^2\right)^{1/2}}^{\left(\frac{2}{3}y+1-\frac{2}{\sqrt{3}}-y^2\right)^{1/2}} \int_{\frac{1}{2}x^2+\frac{1}{2}y^2}^{\frac{1}{\sqrt{3}}y+\frac{1}{2}-\frac{1}{\sqrt{3}}} 1\,dz\,dx\,dy$

(b) $x=0 \Rightarrow z=\frac{1}{2}y^2$ and $\frac{dz}{dy}=y$; $y=1 \Rightarrow \frac{dz}{dy}=1 \Rightarrow$ the tangent line has slope 1 or a 45° slant $\Rightarrow$ at 45° and thereafter, the dish will not hold water.

27. The cylinder is given by $x^2+y^2=1$ from $z=1$ to $\infty \Rightarrow \iiint_D z\left(r^2+z^2\right)^{-5/2} dV$
$= \int_0^{2\pi}\int_0^1\int_1^\infty \frac{z}{(r^2+z^2)^{5/2}}\,dz\,r\,dr\,d\theta = \lim_{a\to\infty}\int_0^{2\pi}\int_0^1\int_1^a \frac{rz}{(r^2+z^2)^{5/2}}\,dz\,dr\,d\theta$
$= \lim_{a\to\infty}\int_0^{2\pi}\int_0^1\left[\left(-\frac{1}{3}\right)\frac{r}{(r^2+z^2)^{3/2}}\right]_1^a dr\,d\theta = \lim_{a\to\infty}\int_0^{2\pi}\int_0^1\left[\left(-\frac{1}{3}\right)\frac{r}{(r^2+a^2)^{3/2}} + \left(\frac{1}{3}\right)\frac{r}{(r^2+1)^{3/2}}\right] dr\,d\theta$
$= \lim_{a\to\infty}\int_0^{2\pi}\left[\frac{1}{3}\left(r^2+a^2\right)^{-1/2} - \frac{1}{3}\left(r^2+1\right)^{-1/2}\right]_0^1 d\theta = \lim_{a\to\infty}\int_0^{2\pi}\left[\frac{1}{3}\left(1+a^2\right)^{-1/2} - \frac{1}{3}\left(2^{-1/2}\right) - \frac{1}{3}\left(a^2\right)^{-1/2} + \frac{1}{3}\right] d\theta$
$= \lim_{a\to\infty} 2\pi\left[\frac{1}{3}\left(1+a^2\right)^{-1/2} - \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{3}\left(\frac{1}{a}\right) + \frac{1}{3}\right] = 2\pi\left[\frac{1}{3} - \left(\frac{1}{3}\right)\frac{\sqrt{2}}{2}\right].$

28. Let's see?
The length of the "unit" line segment is: $L = 2\int_0^1 dx = 2.$
The area of the unit circle is: $A = 4\int_0^1\int_0^{\sqrt{1-x^2}} dy\,dx = \pi.$
The volume of the unit sphere is: $V = 8\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} dz\,dy\,dx = \frac{4}{3}\pi.$
Therefore, the hypervolume of the unit 4-sphere should be:
$$V_{\text{hyper}} = 16\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\int_0^{\sqrt{1-x^2-y^2-z^2}} dw\,dz\,dy\,dx.$$
Mathematica is able to handle this integral, but we'll use the brute force approach.

$$
\begin{aligned}
V_{\text{hyper}} &= 16\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\int_0^{\sqrt{1-x^2-y^2-z^2}} dw\,dz\,dy\,dx = 16\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \sqrt{1-x^2-y^2-z^2}\,dz\,dy\,dx \\
&= 16\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \sqrt{1-x^2-y^2}\sqrt{1-\tfrac{z^2}{1-x^2-y^2}}\,dz\,dy\,dx = \begin{bmatrix} \frac{z}{\sqrt{1-x^2-y^2}} = \cos\theta \\ dz = -\sqrt{1-x^2-y^2}\,\sin\theta\,d\theta \end{bmatrix} \\
&= 16\int_0^1\int_0^{\sqrt{1-x^2}} (1-x^2-y^2)\int_{\pi/2}^0 -\sqrt{1-\cos^2\theta}\,\sin\theta\,d\theta\,dy\,dx = 16\int_0^1\int_0^{\sqrt{1-x^2}} (1-x^2-y^2)\int_{\pi/2}^0 -\sin^2\theta\,d\theta\,dy\,dx \\
&= 16\int_0^1\int_0^{\sqrt{1-x^2}} \tfrac{\pi}{4}(1-x^2-y^2)\,dy\,dx = 4\pi\int_0^1 \left(\sqrt{1-x^2} - x^2\sqrt{1-x^2} - \tfrac{1}{3}\left(1-x^2\right)^{3/2}\right) dx \\
&= 4\pi\int_0^1 \sqrt{1-x^2}\left[(1-x^2) - \tfrac{1-x^3}{3}\right] dx = \tfrac{8}{3}\pi\int_0^1 \left(1-x^2\right)^{3/2} dx = \begin{bmatrix} x = \cos\theta \\ dx = -\sin\theta\,d\theta \end{bmatrix} = -\tfrac{8}{3}\pi\int_{\pi/2}^0 \sin^4\theta\,d\theta \\
&= -\tfrac{8}{3}\pi\int_{\pi/2}^0 \left(\tfrac{1-\cos 2\theta}{2}\right)^2 d\theta = -\tfrac{2}{3}\pi\int_{\pi/2}^0 (1 - 2\cos 2\theta + \cos^2 2\theta)\,d\theta = -\tfrac{2}{3}\pi\int_{\pi/2}^0 \left(\tfrac{3}{2} - 2\cos 2\theta + \tfrac{\cos 4\theta}{2}\right) d\theta = \tfrac{\pi^2}{2}
\end{aligned}
$$