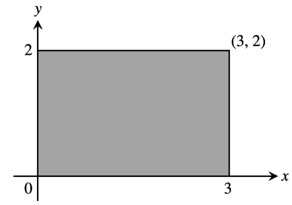
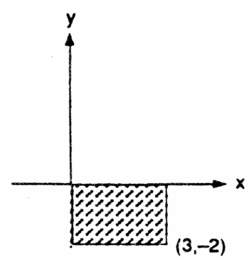
CHAPTER 15 MULTIPLE INTEGRALS

15.1 DOUBLE INTEGRALS

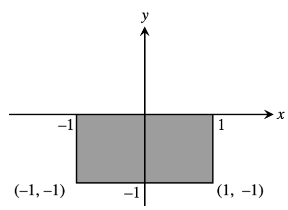
1. $\int_0^3 \int_0^2 (4 - y^2)\, dy\, dx = \int_0^3 \left[4y - \frac{y^3}{3}\right]_0^2 dx = \frac{16}{3} \int_0^3 dx = 16$



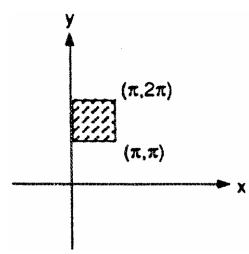
2. $\int_0^3 \int_{-2}^0 ((x^2 y - 2xy)\, dy\, dx = \int_0^3 \left[\frac{x^2 y^2}{2} - xy^2\right]_{-2}^0 dx$
$= \int_0^3 (4x - 2x^2)\, dx = \left[2x^2 - \frac{2x^3}{3}\right]_0^3 = 0$



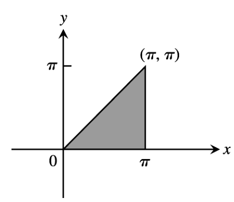
3. $\int_{-1}^0 \int_{-1}^1 (x + y + 1)\, dx\, dy = \int_{-1}^0 \left[\frac{x^2}{2} + yx + x\right]_{-1}^1 dy$
$= \int_{-1}^0 (2y + 2)\, dy = \left[y^2 + 2y\right]_{-1}^0 = 1$



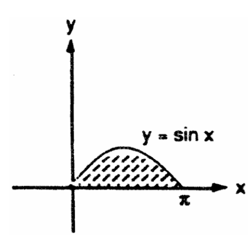
4. $\int_\pi^{2\pi} \int_0^\pi (\sin x + \cos y)\, dx\, dy = \int_\pi^{2\pi} [(-\cos x) + (\cos y)x]_0^\pi\, dy$
$= \int_\pi^{2\pi} (\pi \cos y + 2)\, dy = [\pi \sin y + 2y]_\pi^{2\pi} = 2\pi$



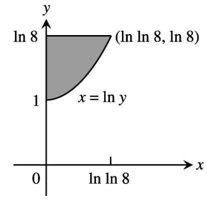
5. $\int_0^\pi \int_0^x (x \sin y)\, dy\, dx = \int_0^\pi [-x \cos y]_0^x\, dx$
$= \int_0^\pi (x - x \cos x)\, dx = \left[\frac{x^2}{2} - (\cos x + x \sin x)\right]_0^\pi$
$= \frac{\pi^2}{2} + 2$



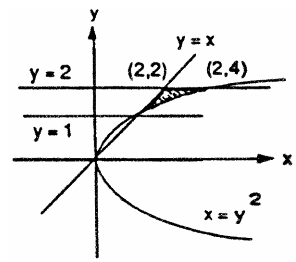
6. $\int_0^\pi \int_0^{\sin x} y\, dy\, dx = \int_0^\pi \left[\frac{y^2}{2}\right]_0^{\sin x} dx = \int_0^\pi \frac{1}{2} \sin^2 x\, dx$
$= \frac{1}{4} \int_0^\pi (1 - \cos 2x)\, dx = \frac{1}{4} \left[x - \frac{1}{2} \sin 2x\right]_0^\pi = \frac{\pi}{4}$



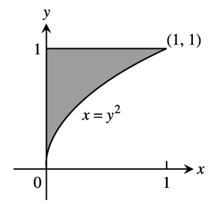
7. $\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\,dx\,dy = \int_1^{\ln 8}\left[e^{x+y}\right]_0^{\ln y}\,dy = \int_1^{\ln 8}\left(ye^y - e^y\right)dy$
$= \left[(y-1)e^y - e^y\right]_1^{\ln 8} = 8(\ln 8 - 1) - 8 + e$
$= 8\ln 8 - 16 + e$



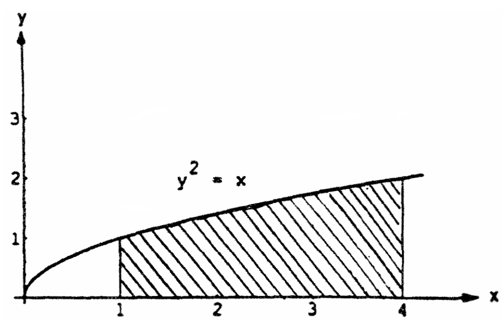
8. $\int_1^2\int_y^{y^2} dx\,dy = \int_1^2 (y^2 - y)\,dy = \left[\frac{y^3}{3} - \frac{y^2}{2}\right]_1^2$
$= \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - \frac{1}{2}\right) = \frac{7}{3} - \frac{3}{2} = \frac{5}{6}$



9. $\int_0^1\int_0^{y^2} 3y^3e^{xy}\,dx\,dy = \int_0^1\left[3y^2e^{xy}\right]_0^{y^2}\,dy$
$= \int_0^1\left(3y^2e^{y^3} - 3y^2\right)dy = \left[e^{y^3} - y^3\right]_0^1 = e - 2$



10. $\int_1^4\int_0^{\sqrt{x}} \frac{3}{2}\,e^{y/\sqrt{x}}\,dy\,dx = \int_1^4\left[\frac{3}{2}\sqrt{x}\,e^{y/\sqrt{x}}\right]_0^{\sqrt{x}}\,dx$
$= \frac{3}{2}(e-1)\int_1^4\sqrt{x}\,dx = \left[\frac{3}{2}(e-1)\left(\frac{2}{3}\right)x^{3/2}\right]_1^4 = 7(e-1)$



11. $\int_1^2\int_x^{2x} \frac{x}{y}\,dy\,dx = \int_1^2\left[x\ln y\right]_x^{2x}\,dx = (\ln 2)\int_1^2 x\,dx = \frac{3}{2}\ln 2$

12. $\int_1^2\int_1^2 \frac{1}{xy}\,dy\,dx = \int_1^2 \frac{1}{x}(\ln 2 - \ln 1)\,dx = (\ln 2)\int_1^2 dx = (\ln 2)^2$

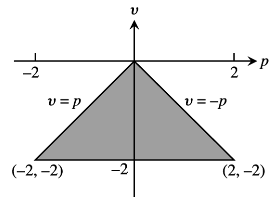
13. $\int_0^1\int_0^{1-x}(x^2+y^2)\,dy\,dx = \int_0^1\left[x^2y + \frac{y^3}{3}\right]_0^{1-x}\,dx = \int_0^1\left[x^2(1-x) + \frac{(1-x)^3}{3}\right]dx = \int_0^1\left[x^2 - x^3 + \frac{(1-x)^3}{3}\right]dx$
$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12}\right]_0^1 = \left(\frac{1}{3} - \frac{1}{4} - 0\right) - \left(0 - 0 - \frac{1}{12}\right) = \frac{1}{6}$

14. $\int_0^1\int_0^{\pi} y\cos xy\,dx\,dy = \int_0^1\left[\sin xy\right]_0^{\pi}\,dy = \int_0^1 \sin \pi y\,dy = \left[-\frac{1}{\pi}\cos \pi y\right]_0^1 = -\frac{1}{\pi}(-1-1) = \frac{2}{\pi}$

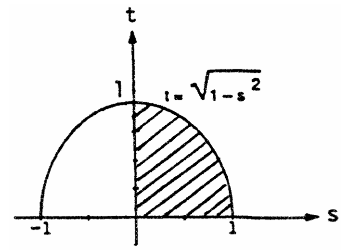
15. $\int_0^1\int_0^{1-u}\left(v - \sqrt{u}\right)dv\,du = \int_0^1\left[\frac{v^2}{2} - v\sqrt{u}\right]_0^{1-u}\,du = \int_0^1\left[\frac{1-2u+u^2}{2} - \sqrt{u}(1-u)\right]du$
$= \int_0^1\left(\frac{1}{2} - u + \frac{u^2}{2} - u^{1/2} + u^{3/2}\right)du = \left[\frac{u}{2} - \frac{u^2}{2} + \frac{u^3}{6} - \frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2}\right]_0^1 = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} - \frac{2}{3} + \frac{2}{5} = -\frac{1}{2} + \frac{2}{5} = -\frac{1}{10}$

16. $\int_1^2 \int_0^{\ln t} e^s \ln t \, ds \, dt = \int_1^2 \left[e^s \ln t\right]_0^{\ln t} dt = \int_1^2 (t \ln t - \ln t) \, dt = \left[\frac{t^2}{2} \ln t - \frac{t^2}{4} - t \ln t + t\right]_1^2$
$= (2 \ln 2 - 1 - 2 \ln 2 + 2) - \left(-\frac{1}{4} + 1\right) = \frac{1}{4}$

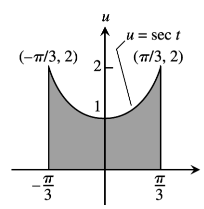
17. $\int_{-2}^{0} \int_{v}^{-v} 2 \, dp \, dv = 2\int_{-2}^{0} [p]_v^{-v} \, dv = 2\int_{-2}^{0} -2v \, dv$
$= -2\left[v^2\right]_{-2}^{0} = 8$



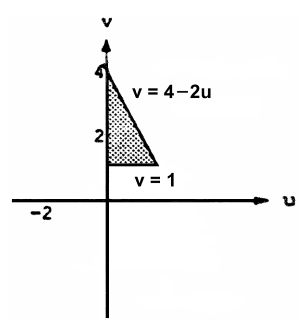
18. $\int_0^1 \int_0^{\sqrt{1-s^2}} 8t \, dt \, ds = \int_0^1 \left[4t^2\right]_0^{\sqrt{1-s^2}} ds$
$= \int_0^1 4\left(1 - s^2\right) ds = 4\left[s - \frac{s^3}{3}\right]_0^1 = \frac{8}{3}$



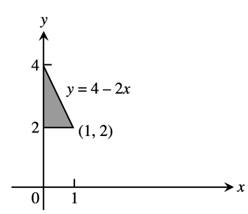
19. $\int_{-\pi/3}^{\pi/3} \int_0^{\sec t} 3 \cos t \, du \, dt = \int_{-\pi/3}^{\pi/3} \left[(3 \cos t)u\right]_0^{\sec t}$
$= \int_{-\pi/3}^{\pi/3} 3 \, dt = 2\pi$



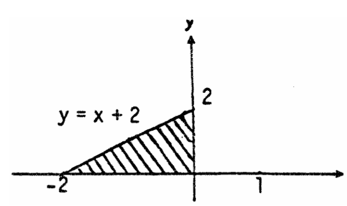
20. $\int_0^3 \int_1^{4-2u} \frac{4-2u}{v^2} \, dv \, du = \int_0^3 \left[\frac{2u-4}{v}\right]_1^{4-2u} du$
$= \int_0^3 (3 - 2u) \, du = \left[3u - u^2\right]_0^3 = 0$



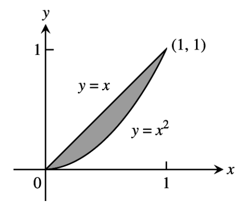
21. $\int_2^4 \int_0^{(4-y)/2} dx \, dy$



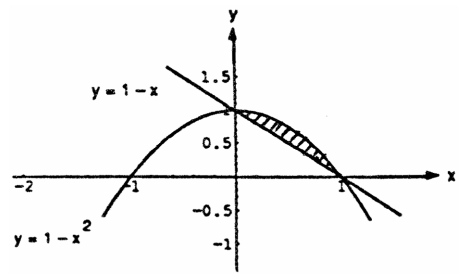
22. $\int_{-2}^{0} \int_0^{x+2} dy \, dx$



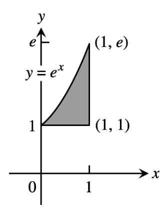
23. $\int_0^1 \int_{x^2}^{x} dy\,dx$



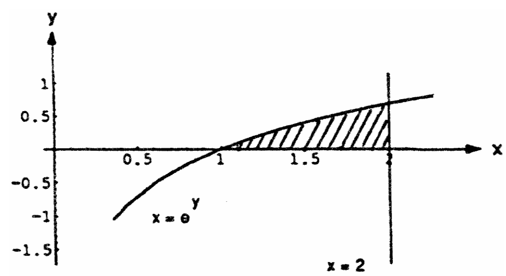
24. $\int_0^1 \int_{1-y}^{\sqrt{1-y}} dx\,dy$



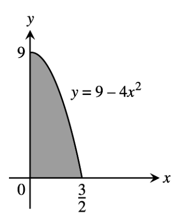
25. $\int_1^e \int_{\ln y}^{1} dx\,dy$



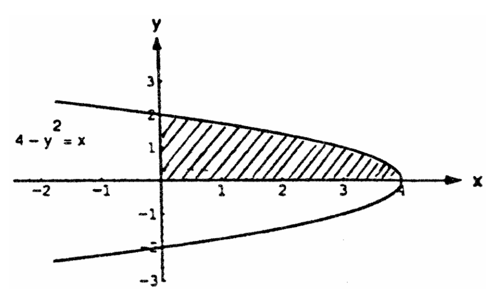
26. $\int_1^2 \int_0^{\ln x} dy\,dx$



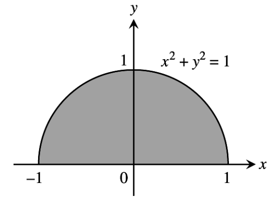
27. $\int_0^9 \int_0^{\frac{1}{2}\sqrt{9-y}} 16x\,dx\,dy$



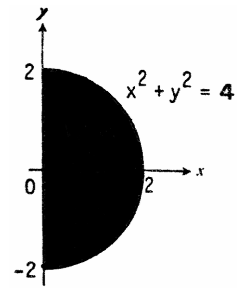
28. $\int_0^4 \int_0^{\sqrt{4-x}} y\,dy\,dx$



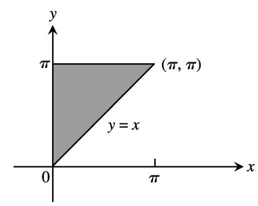
29. $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} 3y\, dy\, dx$



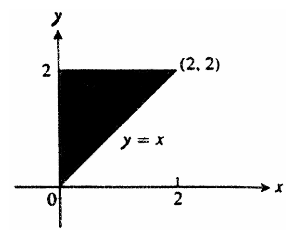
30. $\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} 6x\, dx\, dy$



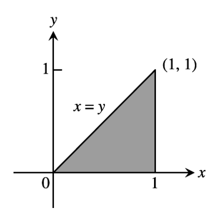
31. $\int_{0}^{\pi}\int_{x}^{\pi} \frac{\sin y}{y}\, dy\, dx = \int_{0}^{\pi}\int_{0}^{y} \frac{\sin y}{y}\, dx\, dy = \int_{0}^{\pi} \sin y\, dy = 2$



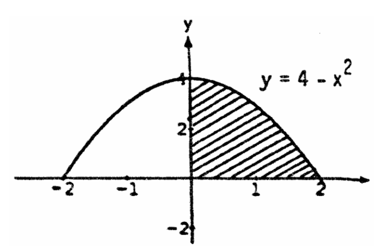
32. $\int_{0}^{2}\int_{x}^{2} 2y^2 \sin xy\, dy\, dx = \int_{0}^{2}\int_{0}^{y} 2y^2 \sin xy\, dx\, dy$
$= \int_{0}^{2} [-2y \cos xy]_{0}^{y}\, dy = \int_{0}^{2} (-2y \cos y^2 + 2y)\, dy$
$= [-\sin y^2 + y^2]_{0}^{2} = 4 - \sin 4$



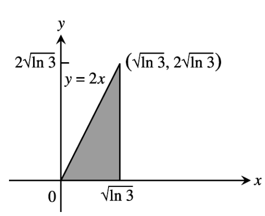
33. $\int_{0}^{1}\int_{y}^{1} x^2 e^{xy}\, dx\, dy = \int_{0}^{1}\int_{0}^{x} x^2 e^{xy}\, dy\, dx = \int_{0}^{1} [xe^{xy}]_{0}^{x}\, dx$
$= \int_{0}^{1} (xe^{x^2} - x)\, dx = \left[\frac{1}{2} e^{x^2} - \frac{x^2}{2}\right]_{0}^{1} = \frac{e-2}{2}$



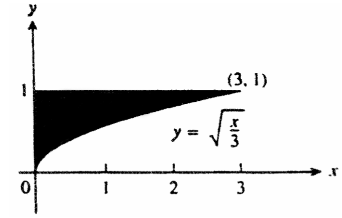
34. $\int_{0}^{2}\int_{0}^{4-x^2} \frac{xe^{2y}}{4-y}\, dy\, dx = \int_{0}^{4}\int_{0}^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\, dx\, dy$
$= \int_{0}^{4} \left[\frac{x^2 e^{2y}}{2(4-y)}\right]_{0}^{\sqrt{4-y}}\, dy = \int_{0}^{4} \frac{e^{2y}}{2}\, dy = \left[\frac{e^{2y}}{4}\right]_{0}^{4} = \frac{e^8 - 1}{4}$



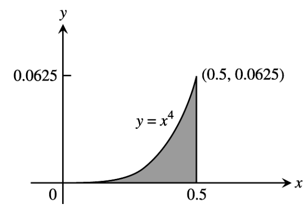
35. $\int_{0}^{2\sqrt{\ln 3}}\int_{y/2}^{\sqrt{\ln 3}} e^{x^2}\, dx\, dy = \int_{0}^{\sqrt{\ln 3}}\int_{0}^{2x} e^{x^2}\, dy\, dx$
$= \int_{0}^{\sqrt{\ln 3}} 2xe^{x^2}\, dx = [e^{x^2}]_{0}^{\sqrt{\ln 3}} = e^{\ln 3} - 1 = 2$



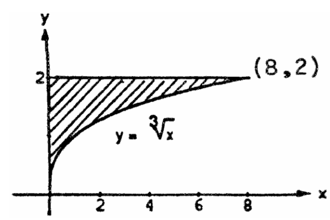
36. $\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\,dy\,dx = \int_0^1 \int_0^{3y^2} e^{y^3}\,dx\,dy$
$= \int_0^1 3y^2 e^{y^3}\,dy = \left[e^{y^3}\right]_0^1 = e - 1$



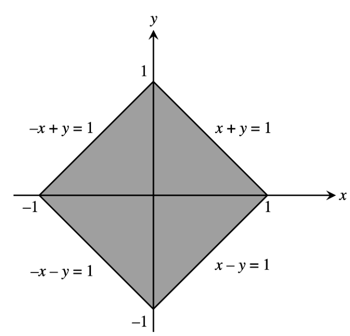
37. $\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos(16\pi x^5)\,dx\,dy = \int_0^{1/2} \int_0^{x^4} \cos(16\pi x^5)\,dy\,dx$
$= \int_0^{1/2} x^4 \cos(16\pi x^5)\,dx = \left[\frac{\sin(16\pi x^5)}{80\pi}\right]_0^{1/2} = \frac{1}{80\pi}$



38. $\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{y^4+1}\,dy\,dx = \int_0^2 \int_0^{y^3} \frac{1}{y^4+1}\,dx\,dy$
$= \int_0^2 \frac{y^3}{y^4+1}\,dy = \frac{1}{4}\left[\ln(y^4+1)\right]_0^2 = \frac{\ln 17}{4}$

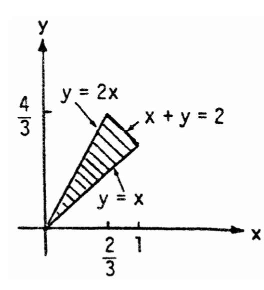


39. $\iint_R (y - 2x^2)\,dA$
$= \int_{-1}^0 \int_{-x-1}^{x+1} (y - 2x^2)\,dy\,dx + \int_0^1 \int_{x-1}^{1-x} (y - 2x^2)\,dy\,dx$
$= \int_{-1}^0 \left[\frac{1}{2}y^2 - 2x^2 y\right]_{-x-1}^{x+1} dx + \int_0^1 \left[\frac{1}{2}y^2 - 2x^2 y\right]_{x-1}^{1-x} dx$
$= \int_{-1}^0 \left[\frac{1}{2}(x+1)^2 - 2x^2(x+1) - \frac{1}{2}(-x-1)^2 + 2x^2(-x-1)\right] dx$
$+ \int_0^1 \left[\frac{1}{2}(1-x)^2 - 2x^2(1-x) - \frac{1}{2}(x-1)^2 + 2x^2(x-1)\right] dx$



$= -4\int_{-1}^0 (x^3 + x^2)\,dx + 4\int_0^1 (x^3 - x^2)\,dx = -4\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^0 + 4\left[\frac{x^4}{4} - \frac{x^3}{3}\right]_0^1$
$= 4\left[\frac{(-1)^4}{4} + \frac{(-1)^3}{3}\right] + 4\left(\frac{1}{4} - \frac{1}{3}\right) = 8\left(\frac{3}{12} - \frac{4}{12}\right) = -\frac{8}{12} = -\frac{2}{3}$

40. $\iint_R xy\,dA = \int_0^{2/3} \int_x^{2x} xy\,dy\,dx + \int_{2/3}^1 \int_x^{2-x} xy\,dy\,dx$
$= \int_0^{2/3} \left[\frac{1}{2}xy^2\right]_x^{2x} dx + \int_{2/3}^1 \left[\frac{1}{2}xy^2\right]_x^{2-x} dx$
$= \int_0^{2/3} \left(2x^3 - \frac{1}{2}x^3\right) dx + \int_{2/3}^1 \left[\frac{1}{2}x(2-x)^2 - \frac{1}{2}x^3\right] dx$
$= \int_0^{2/3} \frac{3}{2}x^3\,dx + \int_{2/3}^1 (2x - x^2)\,dx$



$= \left[\frac{3}{8}x^4\right]_0^{2/3} + \left[x^2 - \frac{2}{3}x^3\right]_{2/3}^1 = \left(\frac{3}{8}\right)\left(\frac{16}{81}\right) + \left(1 - \frac{2}{3}\right) - \left[\frac{4}{9} - \left(\frac{2}{3}\right)\left(\frac{8}{27}\right)\right] = \frac{6}{81} + \frac{27}{81} - \left(\frac{36}{81} - \frac{16}{81}\right) = \frac{13}{81}$

41. $V = \int_0^1 \int_x^{2-x} (x^2 + y^2)\,dy\,dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{2-x} dx = \int_0^1 \left[2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3}\right] dx = \left[\frac{2x^3}{3} - \frac{7x^4}{12} - \frac{(2-x)^4}{12}\right]_0^1$
$= \left(\frac{2}{3} - \frac{7}{12} - \frac{1}{12}\right) - \left(0 - 0 - \frac{16}{12}\right) = \frac{4}{3}$

42. $V=\int_{-2}^{1}\int_{x}^{2-x^2} x^2\,dy\,dx=\int_{-2}^{1}\left[x^2y\right]_x^{2-x^2}dx=\int_{-2}^{1}(2x^2-x^4-x^3)\,dx=\left[\frac{2}{3}x^3-\frac{1}{5}x^5-\frac{1}{4}x^4\right]_{-2}^{1}$
$=\left(\frac{2}{3}-\frac{1}{5}-\frac{1}{4}\right)-\left(-\frac{16}{3}+\frac{32}{5}-\frac{16}{4}\right)=\left(\frac{40}{60}-\frac{12}{60}-\frac{15}{60}\right)-\left(-\frac{320}{60}+\frac{384}{60}-\frac{240}{60}\right)=\frac{189}{60}=\frac{63}{20}$

43. $V=\int_{-4}^{1}\int_{3x}^{4-x^2}(x+4)\,dy\,dx=\int_{-4}^{1}\left[xy+4y\right]_{3x}^{4-x^2}dx=\int_{-4}^{1}\left[x\left(4-x^2\right)+4\left(4-x^2\right)-3x^2-12x\right]dx$
$=\int_{-4}^{1}(-x^3-7x^2-8x+16)\,dx=\left[-\frac{1}{4}x^4-\frac{7}{3}x^3-4x^2+16x\right]_{-4}^{1}=\left(-\frac{1}{4}-\frac{7}{3}+12\right)-\left(\frac{64}{3}-64\right)$
$=\frac{157}{3}-\frac{1}{4}=\frac{625}{12}$

44. $V=\int_{0}^{2}\int_{0}^{\sqrt{4-x^2}}(3-y)\,dy\,dx=\int_{0}^{2}\left[3y-\frac{y^2}{2}\right]_{0}^{\sqrt{4-x^2}}dx=\int_{0}^{2}\left[3\sqrt{4-x^2}-\left(\frac{4-x^2}{2}\right)\right]dx$
$=\left[\frac{3}{2}x\sqrt{4-x^2}+6\sin^{-1}\left(\frac{x}{2}\right)-2x+\frac{x^3}{6}\right]_{0}^{2}=6\left(\frac{\pi}{2}\right)-4+\frac{8}{6}=3\pi-\frac{16}{6}=\frac{9\pi-8}{3}$

45. $V=\int_{0}^{2}\int_{0}^{3}(4-y^2)\,dx\,dy=\int_{0}^{2}\left[4x-y^2x\right]_{0}^{3}dy=\int_{0}^{2}(12-3y^2)\,dy=\left[12y-y^3\right]_{0}^{2}=24-8=16$

46. $V=\int_{0}^{2}\int_{0}^{4-x^2}(4-x^2-y)\,dy\,dx=\int_{0}^{2}\left[\left(4-x^2\right)y-\frac{y^2}{2}\right]_{0}^{4-x^2}dx=\int_{0}^{2}\frac{1}{2}\left(4-x^2\right)^2dx=\int_{0}^{2}\left(8-4x^2+\frac{x^4}{2}\right)dx$
$=\left[8x-\frac{4}{3}x^3+\frac{1}{10}x^5\right]_{0}^{2}=16-\frac{32}{3}+\frac{32}{10}=\frac{480-320+96}{30}=\frac{128}{15}$

47. $V=\int_{0}^{2}\int_{0}^{2-x}(12-3y^2)\,dy\,dx=\int_{0}^{2}\left[12y-y^3\right]_{0}^{2-x}dx=\int_{0}^{2}\left[24-12x-(2-x)^3\right]dx$
$=\left[24x-6x^2+\frac{(2-x)^4}{4}\right]_{0}^{2}=20$

48. $V=\int_{-1}^{0}\int_{-x-1}^{x+1}(3-3x)\,dy\,dx+\int_{0}^{1}\int_{x-1}^{1-x}(3-3x)\,dy\,dx=6\int_{-1}^{0}(1-x^2)\,dx+6\int_{0}^{1}(1-x)^2\,dx=4+2=6$

49. $V=\int_{1}^{2}\int_{-1/x}^{1/x}(x+1)\,dy\,dx=\int_{1}^{2}\left[xy+y\right]_{-1/x}^{1/x}dx=\int_{1}^{2}\left[1+\frac{1}{x}-\left(-1-\frac{1}{x}\right)\right]=2\int_{1}^{2}\left(1+\frac{1}{x}\right)dx$
$=2\left[x+\ln x\right]_{1}^{2}=2(1+\ln 2)$

50. $V=4\int_{0}^{\pi/3}\int_{0}^{\sec x}(1+y^2)\,dy\,dx=4\int_{0}^{\pi/3}\left[y+\frac{y^3}{3}\right]_{0}^{\sec x}dx=4\int_{0}^{\pi/3}\left(\sec x+\frac{\sec^3 x}{3}\right)dx$
$=\frac{2}{3}\left[7\ln\left|\sec x+\tan x\right|+\sec x\tan x\right]_{0}^{\pi/3}=\frac{2}{3}\left[7\ln\left(2+\sqrt{3}\right)+2\sqrt{3}\right]$

51. $\int_{1}^{\infty}\int_{e^{-x}}^{1}\frac{1}{x^3y}\,dy\,dx=\int_{1}^{\infty}\left[\frac{\ln y}{x^3}\right]_{e^{-x}}^{1}dx=\int_{1}^{\infty}-\left(\frac{-x}{x^3}\right)dx=-\lim_{b\to\infty}\left[\frac{1}{x}\right]_{1}^{b}=-\lim_{b\to\infty}\left(\frac{1}{b}-1\right)=1$

52. $\int_{-1}^{1}\int_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}}(2y+1)\,dy\,dx=\int_{-1}^{1}\left[y^2+y\right]\Big|_{-1/\left(1-x^2\right)^{1/2}}^{1/\left(1-x^2\right)^{1/2}}dx=\int_{-1}^{1}\frac{2}{\sqrt{1-x^2}}\,dx=4\lim_{b\to 1^-}\left[\sin^{-1}x\right]_{0}^{b}$
$=4\lim_{b\to 1^-}\left[\sin^{-1}b-0\right]=2\pi$

53. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(x^2+1)\,(y^2+1)}\,dx\,dy=2\int_{0}^{\infty}\left(\frac{2}{y^2+1}\right)\left(\lim_{b\to\infty}\tan^{-1}b-\tan^{-1}0\right)dy=2\pi\lim_{b\to\infty}\int_{0}^{b}\frac{1}{y^2+1}\,dy$
$=2\pi\left(\lim_{b\to\infty}\tan^{-1}b-\tan^{-1}0\right)=(2\pi)\left(\frac{\pi}{2}\right)=\pi^2$

54. $\int_{0}^{\infty}\int_{0}^{\infty}xe^{-(x+2y)}\,dx\,dy=\int_{0}^{\infty}e^{-2y}\lim_{b\to\infty}\left[-xe^{-x}-e^{-x}\right]_{0}^{b}dy=\int_{0}^{\infty}e^{-2y}\lim_{b\to\infty}\left(-be^{-b}-e^{-b}+1\right)dy$
$=\int_{0}^{\infty}e^{-2y}\,dy=\frac{1}{2}\lim_{b\to\infty}\left(-e^{-2b}+1\right)=\frac{1}{2}$

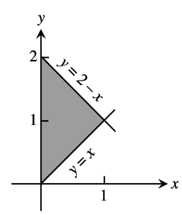
55. $\iint\limits_R f(x,y)\,dA \approx \frac{1}{4}\,f\left(-\frac{1}{2},0\right) + \frac{1}{8}\,f(0,0) + \frac{1}{8}\,f\left(\frac{1}{4},0\right) + \frac{1}{4}\,f\left(\frac{1}{2},0\right) + \frac{1}{4}\,f\left(-\frac{1}{2},\frac{1}{2}\right) + \frac{1}{8}\,f\left(0,\frac{1}{2}\right) + \frac{1}{8}\,f\left(\frac{1}{4},\frac{1}{2}\right)$
$= \frac{1}{4}\left(-\frac{1}{2}+\frac{1}{2}+0\right) + \frac{1}{8}\left(0+\frac{1}{4}+\frac{1}{2}+\frac{3}{4}\right) = \frac{3}{16}$

56. $\iint\limits_R f(x,y)\,dA \approx \frac{1}{4}\left[f\left(\frac{7}{4},\frac{9}{4}\right) + f\left(\frac{9}{4},\frac{9}{4}\right) + f\left(\frac{5}{4},\frac{11}{4}\right) + f\left(\frac{7}{4},\frac{11}{4}\right) + f\left(\frac{9}{4},\frac{11}{4}\right) + f\left(\frac{11}{4},\frac{11}{4}\right) + f\left(\frac{5}{4},\frac{13}{4}\right) + f\left(\frac{7}{4},\frac{13}{4}\right)\right.$
$\left. + f\left(\frac{9}{4},\frac{13}{4}\right) + f\left(\frac{11}{4},\frac{13}{4}\right) + f\left(\frac{7}{4},\frac{15}{4}\right) + f\left(\frac{9}{4},\frac{15}{4}\right)\right]$
$= \frac{1}{16}(25+27+27+29+31+33+31+33+35+37+37+39) = \frac{384}{16} = 24$

57. The ray $\theta = \frac{\pi}{6}$ meets the circle $x^2+y^2=4$ at the point $\left(\sqrt{3},1\right) \Rightarrow$ the ray is represented by the line $y = \frac{x}{\sqrt{3}}$.
Thus, $\iint\limits_R f(x,y)\,dA = \int_0^{\sqrt{3}}\int_{x/\sqrt{3}}^{\sqrt{4-x^2}} \sqrt{4-x^2}\,dy\,dx = \int_0^{\sqrt{3}}\left[(4-x^2) - \frac{x}{\sqrt{3}}\sqrt{4-x^2}\right]dx = \left[4x - \frac{x^3}{3} + \frac{(4-x^2)^{3/2}}{3\sqrt{3}}\right]_0^{\sqrt{3}}$
$= \frac{20\sqrt{3}}{9}$

58. $\int_2^{\infty}\int_0^2 \frac{1}{(x^2-x)(y-1)^{2/3}}\,dy\,dx = \int_2^{\infty}\left[\frac{3(y-1)^{1/3}}{(x^2-x)}\right]_0^2 dx = \int_2^{\infty}\left(\frac{3}{x^2-x}+\frac{3}{x^2-x}\right)dx = 6\int_2^{\infty}\frac{dx}{x(x-1)}$
$= 6\lim\limits_{b\to\infty}\int_2^b\left(\frac{1}{x-1}-\frac{1}{x}\right)dx = 6\lim\limits_{b\to\infty}\left[\ln(x-1)-\ln x\right]_2^b = 6\lim\limits_{b\to\infty}\left[\ln(b-1)-\ln b-\ln 1+\ln 2\right]$
$= 6\left[\lim\limits_{b\to\infty}\ln\left(1-\frac{1}{b}\right)+\ln 2\right] = 6\ln 2$

59. $V = \int_0^1\int_x^{2-x}(x^2+y^2)\,dy\,dx = \int_0^1\left[x^2y+\frac{y^3}{3}\right]_x^{2-x}dx$
$= \int_0^1\left[2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3}\right]dx = \left[\frac{2x^3}{3} - \frac{7x^4}{12} - \frac{(2-x)^4}{12}\right]_0^1$
$= \left(\frac{2}{3}-\frac{7}{12}-\frac{1}{12}\right) - \left(0-0-\frac{16}{12}\right) = \frac{4}{3}$



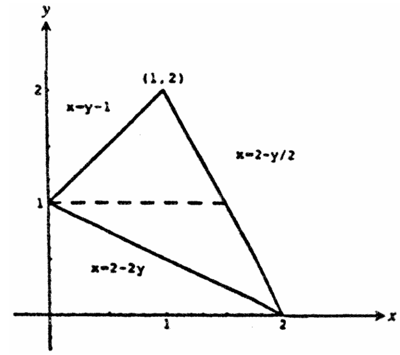
60. $\int_0^2(\tan^{-1}\pi x - \tan^{-1}x)\,dx = \int_0^2\int_x^{\pi x}\frac{1}{1+y^2}\,dy\,dx = \int_0^2\int_{y/\pi}^{y}\frac{1}{1+y^2}\,dx\,dy + \int_2^{2\pi}\int_{y/\pi}^{2}\frac{1}{1+y^2}\,dx\,dy$
$= \int_0^2\frac{\left(1-\frac{1}{\pi}\right)y}{1+y^2}\,dy + \int_2^{2\pi}\frac{\left(2-\frac{y}{\pi}\right)}{1+y^2}\,dy = \left(\frac{\pi-1}{2\pi}\right)\left[\ln(1+y^2)\right]_0^2 + \left[2\tan^{-1}y + \frac{1}{2\pi}\ln(1+y^2)\right]_2^{2\pi}$
$= \left(\frac{\pi-1}{2\pi}\right)\ln 5 + 2\tan^{-1}2\pi - \frac{1}{2\pi}\ln(1+4\pi^2) - 2\tan^{-1}2 + \frac{1}{2\pi}\ln 5$
$= 2\tan^{-1}2\pi - 2\tan^{-1}2 - \frac{1}{2\pi}\ln(1+4\pi^2) + \frac{\ln 5}{2}$

61. To maximize the integral, we want the domain to include all points where the integrand is positive and to exclude all points where the integrand is negative. These criteria are met by the points (x,y) such that $4-x^2-2y^2 \ge 0$ or $x^2+2y^2 \le 4$, which is the ellipse $x^2+2y^2=4$ together with its interior.

62. To minimize the integral, we want the domain to include all points where the integrand is negative and to exclude all points where the integrand is positive. These criteria are met by the points (x,y) such that $x^2+y^2-9 \le 0$ or $x^2+y^2 \le 9$, which is the closed disk of radius 3 centered at the origin.

63. No, it is not possible By Fubini's theorem, the two orders of integration must give the same result.

64. One way would be to partition R into two triangles with the line $y = 1$. The integral of f over R could then be written as a sum of integrals that could be evaluated by integrating first with respect to x and then with respect to y:
$$\iint_R f(x,y)\,dA$$
$$= \int_0^1 \int_{2-2y}^{2-(y/2)} f(x,y)\,dx\,dy + \int_1^2 \int_{y-1}^{2-(y/2)} f(x,y)\,dx\,dy.$$
Partitioning R with the line $x = 1$ would let us write the integral of f over R as a sum of iterated integrals with order dy dx.



65. $$\int_{-b}^{b}\int_{-b}^{b} e^{-x^2-y^2}\,dx\,dy = \int_{-b}^{b}\int_{-b}^{b} e^{-y^2}e^{-x^2}\,dx\,dy = \int_{-b}^{b} e^{-y^2}\left(\int_{-b}^{b} e^{-x^2}\,dx\right)dy = \left(\int_{-b}^{b} e^{-x^2}\,dx\right)\left(\int_{-b}^{b} e^{-y^2}\,dy\right)$$
$$= \left(\int_{-b}^{b} e^{-x^2}\,dx\right)^2 = \left(2\int_0^{b} e^{-x^2}\,dx\right)^2 = 4\left(\int_0^{b} e^{-x^2}\,dx\right)^2$$; taking limits as $b \to \infty$ gives the stated result.

66. $$\int_0^1\int_0^3 \frac{x^2}{(y-1)^{2/3}}\,dy\,dx = \int_0^3\int_0^1 \frac{x^2}{(y-1)^{2/3}}\,dx\,dy = \int_0^3 \frac{1}{(y-1)^{2/3}}\left[\frac{x^3}{3}\right]_0^1 dy = \frac{1}{3}\int_0^3 \frac{dy}{(y-1)^{2/3}}$$
$$= \frac{1}{3}\lim_{b\to 1^-}\int_0^b \frac{dy}{(y-1)^{2/3}} + \frac{1}{3}\lim_{b\to 1^+}\int_b^3 \frac{dy}{(y-1)^{2/3}} = \lim_{b\to 1^-}\left[(y-1)^{1/3}\right]_0^b + \lim_{b\to 1^+}\left[(y-1)^{1/3}\right]_b^3$$
$$= \left[\lim_{b\to 1^-}(b-1)^{1/3} - (-1)^{1/3}\right] - \left[\lim_{b\to 1^+}(b-1)^{1/3} - (2)^{1/3}\right] = (0+1) - \left(0 - \sqrt[3]{2}\right) = 1 + \sqrt[3]{2}$$

67-70. Example CAS commands:

<u>Maple</u>:

```
f := (x,y) -> 1/x/y;
q1 := Int( Int( f(x,y), y=1..x ), x=1..3 );
evalf( q1 );
value( q1 );
evalf( value(q1) );
```

71-76. Example CAS commands:

<u>Maple</u>:

```
f := (x,y) -> exp(x^2);
c,d := 0,1;
g1 := y ->2*y;
g2 := y -> 4;
q5 := Int( Int( f(x,y), x=g1(y)..g2(y) ), y=c..d );
value( q5 );
plot3d( 0, x=g1(y)..g2(y), y=c..d, color=pink, style=patchnogrid, axes=boxed, orientation=[-90,0],
        scaling=constrained, title="#71 (Section 15.1)" );
r5 := Int( Int( f(x,y), y=0..x/2 ), x=0..2 ) + Int( Int( f(x,y), y=0..1 ), x=2..4 );
value( r5);
value( q5-r5 );
```

67-76. Example CAS commands:

<u>Mathematica</u>: (functions and bounds will vary)

You can integrate using the built-in integral signs or with the command **Integrate**. In the **Integrate** command, the integration begins with the variable on the right. (In this case, y going from 1 to x).