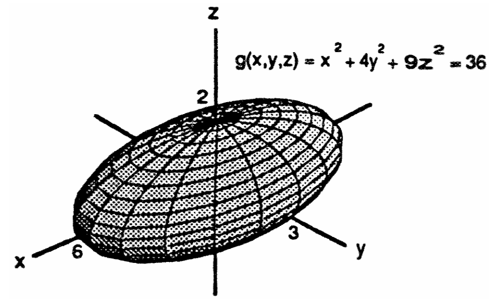
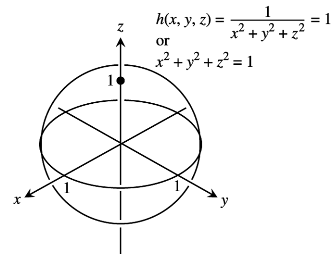
6. Domain: All points (x, y, z) in space
Range: Nonnegative real numbers

Level surfaces are ellipsoids with center $(0, 0, 0)$.



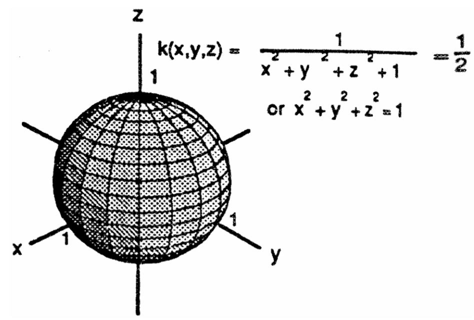
7. Domain: All (x, y, z) such that $(x, y, z) \neq (0, 0, 0)$
Range: Positive real numbers

Level surfaces are spheres with center $(0, 0, 0)$ and radius $r > 0$.



8. Domain: All points (x, y, z) in space
Range: $(0, 1]$

Level surfaces are spheres with center $(0, 0, 0)$ and radius $r > 0$.



9. $\lim\limits_{(x,y) \to (\pi, \ln 2)} e^y \cos x = e^{\ln 2} \cos \pi = (2)(-1) = -2$

10. $\lim\limits_{(x,y) \to (0,0)} \frac{2+y}{x + \cos y} = \frac{2+0}{0 + \cos 0} = 2$

11. $\lim\limits_{\substack{(x,y) \to (1,1) \\ x \neq \pm y}} \frac{x-y}{x^2 - y^2} = \lim\limits_{\substack{(x,y) \to (1,1) \\ x \neq \pm y}} \frac{x-y}{(x-y)(x+y)} = \lim\limits_{(x,y) \to (1,1)} \frac{1}{x+y} = \frac{1}{1+1} = \frac{1}{2}$

12. $\lim\limits_{(x,y) \to (1,1)} \frac{x^3y^3 - 1}{xy - 1} = \lim\limits_{(x,y) \to (1,1)} \frac{(xy-1)(x^2y^2 + xy + 1)}{xy - 1} = \lim\limits_{(x,y) \to (1,1)} (x^2y^2 + xy + 1) = 1^2 \cdot 1^2 + 1 \cdot 1 + 1 = 3$

13. $\lim\limits_{P \to (1,-1,e)} \ln|x + y + z| = \ln|1 + (-1) + e| = \ln e = 1$

14. $\lim\limits_{P \to (1,-1,-1)} \tan^{-1}(x + y + z) = \tan^{-1}(1 + (-1) + (-1)) = \tan^{-1}(-1) = -\frac{\pi}{4}$

15. Let $y = kx^2$, $k \neq 1$. Then $\lim\limits_{\substack{(x,y) \to (0,0) \\ y \neq x^2}} \frac{y}{x^2 - y} = \lim\limits_{(x, kx^2) \to (0,0)} \frac{kx^2}{x^2 - kx^2} = \frac{k}{1-k^2}$ which gives different limits for different values of k $\Rightarrow$ the limit does not exist.

16. Let $y = kx$, $k \neq 0$. Then $\lim\limits_{\substack{(x,y) \to (0,0) \\ xy \neq 0}} \frac{x^2 + y^2}{xy} = \lim\limits_{(x, kx) \to (0,0)} \frac{x^2 + (kx)^2}{x(kx)} = \frac{1+k^2}{k}$ which gives different limits for

different values of k $\Rightarrow$ the limit does not exist.

17. Let $y = kx$. Then $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \frac{x^2-k^2x^2}{x^2+k^2x^2} = \frac{1-k^2}{1+k^2}$ which gives different limits for different values of k $\Rightarrow$ the limit does not exist so $f(0,0)$ cannot be defined in a way that makes f continuous at the origin.

18. Along the x-axis, $y = 0$ and $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x-y)}{|x+y|} = \lim\limits_{x\to 0} \frac{\sin x}{|x|} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$, so the limit fails to exist $\Rightarrow$ f is not continuous at $(0,0)$.

19. $\frac{\partial g}{\partial r} = \cos\theta + \sin\theta$, $\frac{\partial g}{\partial \theta} = -r\sin\theta + r\cos\theta$

20. $\frac{\partial f}{\partial x} = \frac{1}{2}\left(\frac{2x}{x^2+y^2}\right) + \frac{\left(-\frac{y}{x^2}\right)}{1+\left(\frac{y}{x}\right)^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} = \frac{x-y}{x^2+y^2}$,

$\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{2y}{x^2+y^2}\right) + \frac{\left(\frac{1}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = \frac{y}{x^2+y^2} + \frac{x}{x^2+y^2} = \frac{x+y}{x^2+y^2}$

21. $\frac{\partial f}{\partial R_1} = -\frac{1}{R_1^2}$, $\frac{\partial f}{\partial R_2} = -\frac{1}{R_2^2}$, $\frac{\partial f}{\partial R_3} = -\frac{1}{R_3^2}$

22. $h_x(x,y,z) = 2\pi\cos(2\pi x + y - 3z)$, $h_y(x,y,z) = \cos(2\pi x + y - 3z)$, $h_z(x,y,z) = -3\cos(2\pi x + y - 3z)$

23. $\frac{\partial P}{\partial n} = \frac{RT}{V}$, $\frac{\partial P}{\partial R} = \frac{nT}{V}$, $\frac{\partial P}{\partial T} = \frac{nR}{V}$, $\frac{\partial P}{\partial V} = -\frac{nRT}{V^2}$

24. $f_r(r,\ell,T,w) = -\frac{1}{2r^2\ell}\sqrt{\frac{T}{\pi w}}$, $f_\ell(r,\ell,T,w) = -\frac{1}{2r\ell^2}\sqrt{\frac{T}{\pi w}}$, $f_T(r,\ell,T,w) = \left(\frac{1}{2r\ell}\right)\left(\frac{1}{\sqrt{\pi w}}\right)\left(\frac{1}{2\sqrt{T}}\right)$
$= \frac{1}{4r\ell}\sqrt{\frac{1}{T\pi w}} = \frac{1}{4r\ell T}\sqrt{\frac{T}{\pi w}}$, $f_w(r,\ell,T,w) = \left(\frac{1}{2r\ell}\right)\sqrt{\frac{T}{\pi}}\left(-\frac{1}{2}w^{-3/2}\right) = -\frac{1}{4r\ell w}\sqrt{\frac{T}{\pi w}}$

25. $\frac{\partial g}{\partial x} = \frac{1}{y}$, $\frac{\partial g}{\partial y} = 1 - \frac{x}{y^2} \Rightarrow \frac{\partial^2 g}{\partial x^2} = 0$, $\frac{\partial^2 g}{\partial y^2} = \frac{2x}{y^3}$, $\frac{\partial^2 g}{\partial y \partial x} = \frac{\partial^2 g}{\partial x \partial y} = -\frac{1}{y^2}$

26. $g_x(x,y) = e^x + y\cos x$, $g_y(x,y) = \sin x \Rightarrow g_{xx}(x,y) = e^x - y\sin x$, $g_{yy}(x,y) = 0$, $g_{xy}(x,y) = g_{yx}(x,y) = \cos x$

27. $\frac{\partial f}{\partial x} = 1 + y - 15x^2 + \frac{2x}{x^2+1}$, $\frac{\partial f}{\partial y} = x \Rightarrow \frac{\partial^2 f}{\partial x^2} = -30x + \frac{2-2x^2}{(x^2+1)^2}$, $\frac{\partial^2 f}{\partial y^2} = 0$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 1$

28. $f_x(x,y) = -3y$, $f_y(x,y) = 2y - 3x - \sin y + 7e^y \Rightarrow f_{xx}(x,y) = 0$, $f_{yy}(x,y) = 2 - \cos y + 7e^y$, $f_{xy}(x,y) = f_{yx}(x,y)$
$= -3$

29. $\frac{\partial w}{\partial x} = y\cos(xy+\pi)$, $\frac{\partial w}{\partial y} = x\cos(xy+\pi)$, $\frac{dx}{dt} = e^t$, $\frac{dy}{dt} = \frac{1}{t+1}$
$\Rightarrow \frac{dw}{dt} = [y\cos(xy+\pi)]e^t + [x\cos(xy+\pi)]\left(\frac{1}{t+1}\right)$; $t = 0 \Rightarrow x = 1$ and $y = 0$
$\Rightarrow \left.\frac{dw}{dt}\right|_{t=0} = 0 \cdot 1 + [1 \cdot (-1)]\left(\frac{1}{0+1}\right) = -1$

30. $\frac{\partial w}{\partial x} = e^y$, $\frac{\partial w}{\partial y} = xe^y + \sin z$, $\frac{\partial w}{\partial z} = y\cos z + \sin z$, $\frac{dx}{dt} = t^{-1/2}$, $\frac{dy}{dt} = 1 + \frac{1}{t}$, $\frac{dz}{dt} = \pi$
$\Rightarrow \frac{dw}{dt} = e^y t^{-1/2} + (xe^y + \sin z)\left(1 + \frac{1}{t}\right) + (y\cos z + \sin z)\pi$; $t = 1 \Rightarrow x = 2$, $y = 0$, and $z = \pi$
$\Rightarrow \left.\frac{dw}{dt}\right|_{t=1} = 1 \cdot 1 + (2 \cdot 1 - 0)(2) + (0+0)\pi = 5$

31. $\frac{\partial w}{\partial x} = 2\cos(2x-y)$, $\frac{\partial w}{\partial y} = -\cos(2x-y)$, $\frac{\partial x}{\partial r} = 1$, $\frac{\partial x}{\partial s} = \cos s$, $\frac{\partial y}{\partial r} = s$, $\frac{\partial y}{\partial s} = r$
$\Rightarrow \frac{\partial w}{\partial r} = [2\cos(2x-y)](1) + [-\cos(2x-y)](s)$; $r = \pi$ and $s = 0 \Rightarrow x = \pi$ and $y = 0$

$\Rightarrow \left.\frac{\partial w}{\partial r}\right|_{(\pi,0)} = (2\cos 2\pi) - (\cos 2\pi)(0) = 2;\ \frac{\partial w}{\partial s} = [2\cos(2x - y)](\cos s) + [-\cos(2x - y)](r)$
$\Rightarrow \left.\frac{\partial w}{\partial s}\right|_{(\pi,0)} = (2\cos 2\pi)(\cos 0) - (\cos 2\pi)(\pi) = 2 - \pi$

32. $\frac{\partial w}{\partial u} = \frac{dw}{dx}\frac{\partial x}{\partial u} = \left(\frac{x}{1+x^2} - \frac{1}{x^2+1}\right)(2e^u \cos v)$; $u = v = 0 \Rightarrow x = 2 \Rightarrow \left.\frac{\partial w}{\partial u}\right|_{(0,0)} = \left(\frac{2}{5} - \frac{1}{5}\right)(2) = \frac{2}{5}$;
$\frac{\partial w}{\partial v} = \frac{dw}{dx}\frac{\partial x}{\partial v} = \left(\frac{x}{1+x^2} - \frac{1}{x^2+1}\right)(-2e^u \sin v) \Rightarrow \left.\frac{\partial w}{\partial v}\right|_{(0,0)} = \left(\frac{2}{5} - \frac{1}{5}\right)(0) = 0$

33. $\frac{\partial f}{\partial x} = y + z,\ \frac{\partial f}{\partial y} = x + z,\ \frac{\partial f}{\partial z} = y + x,\ \frac{dx}{dt} = -\sin t,\ \frac{dy}{dt} = \cos t,\ \frac{dz}{dt} = -2\sin 2t$
$\Rightarrow \frac{df}{dt} = -(y+z)(\sin t) + (x+z)(\cos t) - 2(y+x)(\sin 2t)$; $t = 1 \Rightarrow x = \cos 1$, $y = \sin 1$, and $z = \cos 2$
$\Rightarrow \left.\frac{df}{dt}\right|_{t=1} = -(\sin 1 + \cos 2)(\sin 1) + (\cos 1 + \cos 2)(\cos 1) - 2(\sin 1 + \cos 1)(\sin 2)$

34. $\frac{\partial w}{\partial x} = \frac{dw}{ds}\frac{\partial s}{\partial x} = (5)\frac{dw}{ds}$ and $\frac{\partial w}{\partial y} = \frac{dw}{ds}\frac{\partial s}{\partial y} = (1)\frac{dw}{ds} = \frac{dw}{ds} \Rightarrow \frac{\partial w}{\partial x} - 5\frac{\partial w}{\partial y} = 5\frac{dw}{ds} - 5\frac{dw}{ds} = 0$

35. $F(x, y) = 1 - x - y^2 - \sin xy \Rightarrow F_x = -1 - y\cos xy$ and $F_y = -2y - x\cos xy \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-1 - y\cos xy}{-2y - x\cos xy}$
$= \frac{1 + y\cos xy}{-2y - x\cos xy} \Rightarrow$ at $(x, y) = (0, 1)$ we have $\left.\frac{dy}{dx}\right|_{(0,1)} = \frac{1+1}{-2} = -1$

36. $F(x, y) = 2xy + e^{x+y} - 2 \Rightarrow F_x = 2y + e^{x+y}$ and $F_y = 2x + e^{x+y} \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2y + e^{x+y}}{2x + e^{x+y}}$
$\Rightarrow$ at $(x, y) = (0, \ln 2)$ we have $\left.\frac{dy}{dx}\right|_{(0,\ln 2)} = -\frac{2\ln 2 + 2}{0 + 2} = -(\ln 2 + 1)$

37. $\nabla f = (-\sin x \cos y)\mathbf{i} - (\cos x \sin y)\mathbf{j} \Rightarrow \left.\nabla f\right|_{\left(\frac{\pi}{4},\frac{\pi}{4}\right)} = -\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} \Rightarrow |\nabla f| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$;
$\mathbf{u} = \frac{\nabla f}{|\nabla f|} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j} \Rightarrow$ f increases most rapidly in the direction $\mathbf{u} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$ and decreases most rapidly in the direction $-\mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$; $(D_{\mathbf{u}}f)_{P_0} = |\nabla f| = \frac{\sqrt{2}}{2}$ and $(D_{-\mathbf{u}}f)_{P_0} = -\frac{\sqrt{2}}{2}$;
$\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} + 4\mathbf{j}}{\sqrt{3^2 + 4^2}} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} \Rightarrow (D_{\mathbf{u}_1}f)_{P_0} = \nabla f \cdot \mathbf{u}_1 = \left(-\frac{1}{2}\right)\left(\frac{3}{5}\right) + \left(-\frac{1}{2}\right)\left(\frac{4}{5}\right) = -\frac{7}{10}$

38. $\nabla f = 2xe^{-2y}\mathbf{i} - 2x^2e^{-2y}\mathbf{j} \Rightarrow \left.\nabla f\right|_{(1,0)} = 2\mathbf{i} - 2\mathbf{j} \Rightarrow |\nabla f| = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$; $\mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$
$\Rightarrow$ f increases most rapidly in the direction $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$; $(D_{\mathbf{u}}f)_{P_0} = |\nabla f| = 2\sqrt{2}$ and $(D_{-\mathbf{u}}f)_{P_0} = -2\sqrt{2}$; $\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$
$\Rightarrow (D_{\mathbf{u}_1}f)_{P_0} = \nabla f \cdot \mathbf{u}_1 = (2)\left(\frac{1}{\sqrt{2}}\right) + (-2)\left(\frac{1}{\sqrt{2}}\right) = 0$

39. $\nabla f = \left(\frac{2}{2x + 3y + 6z}\right)\mathbf{i} + \left(\frac{3}{2x + 3y + 6z}\right)\mathbf{j} + \left(\frac{6}{2x + 3y + 6z}\right)\mathbf{k} \Rightarrow \left.\nabla f\right|_{(-1,-1,1)} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$;
$\mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}{\sqrt{2^2 + 3^2 + 6^2}} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \Rightarrow$ f increases most rapidly in the direction $\mathbf{u} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} - \frac{6}{7}\mathbf{k}$; $(D_{\mathbf{u}}f)_{P_0} = |\nabla f| = 7$, $(D_{-\mathbf{u}}f)_{P_0} = -7$;
$\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \Rightarrow (D_{\mathbf{u}_1}f)_{P_0} = (D_{\mathbf{u}}f)_{P_0} = 7$

40. $\nabla f = (2x + 3y)\mathbf{i} + (3x + 2)\mathbf{j} + (1 - 2z)\mathbf{k} \Rightarrow \left.\nabla f\right|_{(0,0,0)} = 2\mathbf{j} + \mathbf{k}$; $\mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{2}{\sqrt{5}}\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k} \Rightarrow$ f increases most rapidly in the direction $\mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{2}{\sqrt{5}}\mathbf{j} - \frac{1}{\sqrt{5}}\mathbf{k}$;
$(D_{\mathbf{u}}f)_{P_0} = |\nabla f| = \sqrt{5}$ and $(D_{-\mathbf{u}}f)_{P_0} = -\sqrt{5}$; $\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$
$\Rightarrow (D_{\mathbf{u}_1}f)_{P_0} = \nabla f \cdot \mathbf{u}_1 = (0)\left(\frac{1}{\sqrt{3}}\right) + (2)\left(\frac{1}{\sqrt{3}}\right) + (1)\left(\frac{1}{\sqrt{3}}\right) = \frac{3}{\sqrt{3}} = \sqrt{3}$

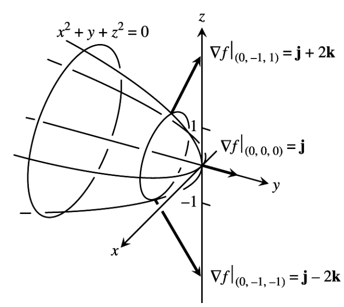
41. $\mathbf{r} = (\cos 3t)\mathbf{i} + (\sin 3t)\mathbf{j} + 3t\mathbf{k} \Rightarrow \mathbf{v}(t) = (-3\sin 3t)\mathbf{i} + (3\cos 3t)\mathbf{j} + 3\mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{3}\right) = -3\mathbf{j} + 3\mathbf{k}$
$\Rightarrow \mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$; $f(x, y, z) = xyz \Rightarrow \nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$; $t = \frac{\pi}{3}$ yields the point on the helix $(-1, 0, \pi)$
$\Rightarrow \nabla f|_{(-1,0,\pi)} = -\pi\mathbf{j} \Rightarrow \nabla f \cdot \mathbf{u} = (-\pi\mathbf{j}) \cdot \left(-\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}\right) = \frac{\pi}{\sqrt{2}}$

42. $f(x, y, z) = xyz \Rightarrow \nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$; at $(1, 1, 1)$ we get $\nabla f = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow$ the maximum value of $D_{\mathbf{u}}f|_{(1,1,1)} = |\nabla f| = \sqrt{3}$

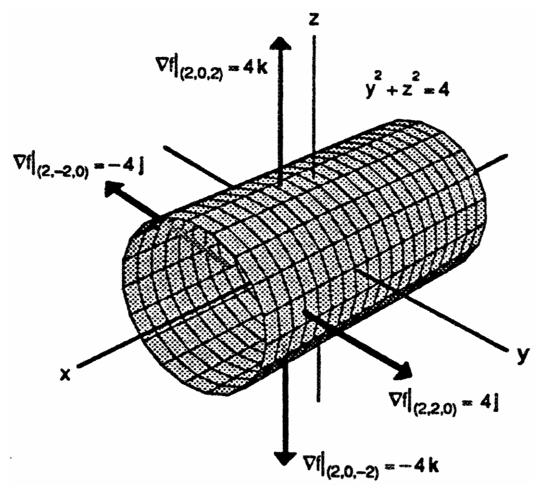
43. (a) Let $\nabla f = a\mathbf{i} + b\mathbf{j}$ at $(1, 2)$. The direction toward $(2, 2)$ is determined by $\mathbf{v}_1 = (2-1)\mathbf{i} + (2-2)\mathbf{j} = \mathbf{i} = \mathbf{u}$ so that $\nabla f \cdot \mathbf{u} = 2 \Rightarrow a = 2$. The direction toward $(1, 1)$ is determined by $\mathbf{v}_2 = (1-1)\mathbf{i} + (1-2)\mathbf{j} = -\mathbf{j} = \mathbf{u}$ so that $\nabla f \cdot \mathbf{u} = -2 \Rightarrow -b = -2 \Rightarrow b = 2$. Therefore $\nabla f = 2\mathbf{i} + 2\mathbf{j}$; $f_x(1, 2) = f_y(1, 2) = 2$.
(b) The direction toward $(4, 6)$ is determined by $\mathbf{v}_3 = (4-1)\mathbf{i} + (6-2)\mathbf{j} = 3\mathbf{i} + 4\mathbf{j} \Rightarrow \mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$
$\Rightarrow \nabla f \cdot \mathbf{u} = \frac{14}{5}$.

44. (a) True (b) False (c) True (d) True

45. $\nabla f = 2x\mathbf{i} + \mathbf{j} + 2z\mathbf{k} \Rightarrow$
$\nabla f|_{(0,-1,-1)} = \mathbf{j} - 2\mathbf{k}$,
$\nabla f|_{(0,0,0)} = \mathbf{j}$,
$\nabla f|_{(0,-1,1)} = \mathbf{j} + 2\mathbf{k}$



46. $\nabla f = 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow$
$\nabla f|_{(2,2,0)} = 4\mathbf{j}$,
$\nabla f|_{(2,-2,0)} = -4\mathbf{j}$,
$\nabla f|_{(2,0,2)} = 4\mathbf{k}$,
$\nabla f|_{(2,0,-2)} = -4\mathbf{k}$



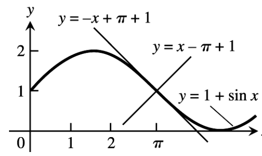
47. $\nabla f = 2x\mathbf{i} - \mathbf{j} - 5\mathbf{k} \Rightarrow \nabla f|_{(2,-1,1)} = 4\mathbf{i} - \mathbf{j} - 5\mathbf{k} \Rightarrow$ Tangent Plane: $4(x-2) - (y+1) - 5(z-1) = 0$
$\Rightarrow 4x - y - 5z = 4$; Normal Line: $x = 2 + 4t$, $y = -1 - t$, $z = 1 - 5t$

48. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k} \Rightarrow \nabla f|_{(1,1,2)} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \Rightarrow$ Tangent Plane: $2(x-1) + 2(y-1) + (z-2) = 0$
$\Rightarrow 2x + 2y + z - 6 = 0$; Normal Line: $x = 1 + 2t$, $y = 1 + 2t$, $z = 2 + t$

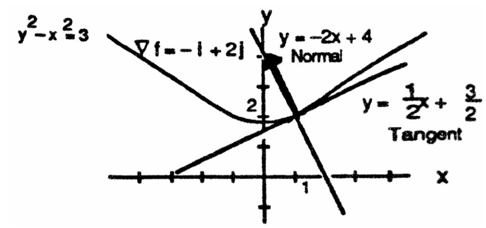
49. $\frac{\partial z}{\partial x} = \frac{2x}{x^2+y^2} \Rightarrow \frac{\partial z}{\partial x}\Big|_{(0,1,0)} = 0$ and $\frac{\partial z}{\partial y} = \frac{2y}{x^2+y^2} \Rightarrow \frac{\partial z}{\partial y}\Big|_{(0,1,0)} = 2$; thus the tangent plane is
$2(y-1) - (z-0) = 0$ or $2y - z - 2 = 0$

50. $\frac{\partial z}{\partial x} = -2x\left(x^2+y^2\right)^{-2} \Rightarrow \left.\frac{\partial z}{\partial x}\right|_{\left(1,1,\frac{1}{2}\right)} = -\frac{1}{2}$ and $\frac{\partial z}{\partial y} = -2y\left(x^2+y^2\right)^{-2} \Rightarrow \left.\frac{\partial z}{\partial y}\right|_{\left(1,1,\frac{1}{2}\right)} = -\frac{1}{2}$; thus the tangent plane is $-\frac{1}{2}(x-1) - \frac{1}{2}(y-1) - \left(z-\frac{1}{2}\right) = 0$ or $x+y+2z-3=0$

51. $\nabla f = (-\cos x)\mathbf{i} + \mathbf{j} \Rightarrow \left.\nabla f\right|_{(\pi,1)} = \mathbf{i}+\mathbf{j} \Rightarrow$ the tangent line is $(x-\pi)+(y-1)=0 \Rightarrow x+y=\pi+1$; the normal line is $y-1 = 1(x-\pi) \Rightarrow y = x-\pi+1$



52. $\nabla f = -x\mathbf{i} + y\mathbf{j} \Rightarrow \left.\nabla f\right|_{(1,2)} = -\mathbf{i}+2\mathbf{j} \Rightarrow$ the tangent line is $-(x-1)+2(y-2)=0 \Rightarrow y = \frac{1}{2}x + \frac{3}{2}$; the normal line is $y-2 = -2(x-1) \Rightarrow y = -2x+4$



53. Let $f(x,y,z) = x^2+2y+2z-4$ and $g(x,y,z) = y-1$. Then $\nabla f = 2x\mathbf{i}+2\mathbf{j}+2\mathbf{k}\big|_{\left(1,1,\frac{1}{2}\right)} = 2\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ and $\nabla g = \mathbf{j} \Rightarrow \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2\mathbf{i}+2\mathbf{k} \Rightarrow$ the line is $x = 1-2t$, $y = 1$, $z = \frac{1}{2}+2t$

54. Let $f(x,y,z) = x+y^2+z-2$ and $g(x,y,z) = y-1$. Then $\nabla f = \mathbf{i}+2y\mathbf{j}+\mathbf{k}\big|_{\left(\frac{1}{2},1,\frac{1}{2}\right)} = \mathbf{i}+2\mathbf{j}+\mathbf{k}$ and $\nabla g = \mathbf{j} \Rightarrow \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i}+\mathbf{k} \Rightarrow$ the line is $x = \frac{1}{2}-t$, $y = 1$, $z = \frac{1}{2}+t$

55. $f\left(\frac{\pi}{4},\frac{\pi}{4}\right) = \frac{1}{2}$, $f_x\left(\frac{\pi}{4},\frac{\pi}{4}\right) = \cos x\cos y\big|_{(\pi/4,\pi/4)} = \frac{1}{2}$, $f_y\left(\frac{\pi}{4},\frac{\pi}{4}\right) = -\sin x\sin y\big|_{(\pi/4,\pi/4)} = -\frac{1}{2}$ $\Rightarrow L(x,y) = \frac{1}{2}+\frac{1}{2}\left(x-\frac{\pi}{4}\right)-\frac{1}{2}\left(y-\frac{\pi}{4}\right) = \frac{1}{2}+\frac{1}{2}x-\frac{1}{2}y$; $f_{xx}(x,y) = -\sin x\cos y$, $f_{yy}(x,y) = -\sin x\cos y$, and $f_{xy}(x,y) = -\cos x\sin y$. Thus an upper bound for E depends on the bound M used for $|f_{xx}|$, $|f_{xy}|$, and $|f_{yy}|$.
With $M = \frac{\sqrt{2}}{2}$ we have $|E(x,y)| \le \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)\left(\left|x-\frac{\pi}{4}\right|+\left|y-\frac{\pi}{4}\right|\right)^2 \le \frac{\sqrt{2}}{4}(0.2)^2 \le 0.0142$;
with $M = 1$, $|E(x,y)| \le \frac{1}{2}(1)\left(\left|x-\frac{\pi}{4}\right|+\left|y-\frac{\pi}{4}\right|\right)^2 = \frac{1}{2}(0.2)^2 = 0.02$.

56. $f(1,1) = 0$, $f_x(1,1) = y\big|_{(1,1)} = 1$, $f_y(1,1) = x-6y\big|_{(1,1)} = -5 \Rightarrow L(x,y) = (x-1)-5(y-1) = x-5y+4$; $f_{xx}(x,y) = 0$, $f_{yy}(x,y) = -6$, and $f_{xy}(x,y) = 1 \Rightarrow$ maximum of $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ is 6 $\Rightarrow M = 6$ $\Rightarrow |E(x,y)| \le \frac{1}{2}(6)\left(|x-1|+|y-1|\right)^2 = \frac{1}{2}(6)(0.1+0.2)^2 = 0.27$

57. $f(1,0,0) = 0$, $f_x(1,0,0) = y-3z\big|_{(1,0,0)} = 0$, $f_y(1,0,0) = x+2z\big|_{(1,0,0)} = 1$, $f_z(1,0,0) = 2y-3x\big|_{(1,0,0)} = -3$ $\Rightarrow L(x,y,z) = 0(x-1)+(y-0)-3(z-0) = y-3z$; $f(1,1,0) = 1$, $f_x(1,1,0) = 1$, $f_y(1,1,0) = 1$, $f_z(1,1,0) = -1$ $\Rightarrow L(x,y,z) = 1+(x-1)+(y-1)-1(z-0) = x+y-z-1$

58. $f\left(0,0,\frac{\pi}{4}\right) = 1$, $f_x\left(0,0,\frac{\pi}{4}\right) = -\sqrt{2}\sin x\sin(y+z)\Big|_{\left(0,0,\frac{\pi}{4}\right)} = 0$, $f_y\left(0,0,\frac{\pi}{4}\right) = \sqrt{2}\cos x\cos(y+z)\Big|_{\left(0,0,\frac{\pi}{4}\right)} = 1$, $f_z\left(0,0,\frac{\pi}{4}\right) = \sqrt{2}\cos x\cos(y+z)\Big|_{\left(0,0,\frac{\pi}{4}\right)} = 1 \Rightarrow L(x,y,z) = 1+1(y-0)+1\left(z-\frac{\pi}{4}\right) = 1+y+z-\frac{\pi}{4}$;
$f\left(\frac{\pi}{4},\frac{\pi}{4},0\right) = \frac{\sqrt{2}}{2}$, $f_x\left(\frac{\pi}{4},\frac{\pi}{4},0\right) = -\frac{\sqrt{2}}{2}$, $f_y\left(\frac{\pi}{4},\frac{\pi}{4},0\right) = \frac{\sqrt{2}}{2}$, $f_z\left(\frac{\pi}{4},\frac{\pi}{4},0\right) = \frac{\sqrt{2}}{2}$
$\Rightarrow L(x,y,z) = \frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)+\frac{\sqrt{2}}{2}\left(y-\frac{\pi}{4}\right)+\frac{\sqrt{2}}{2}(z-0) = \frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{2}y+\frac{\sqrt{2}}{2}z$

59. $V = \pi r^2 h \Rightarrow dV = 2\pi rh\, dr + \pi r^2\, dh \Rightarrow dV|_{(1.5,5280)} = 2\pi(1.5)(5280)\, dr + \pi(1.5)^2\, dh = 15{,}840\pi\, dr + 2.25\pi\, dh$. You should be more careful with the diameter since it has a greater effect on dV.

60. $df = (2x - y)\, dx + (-x + 2y)\, dy \Rightarrow df|_{(1,2)} = 3\, dy \Rightarrow$ f is more sensitive to changes in y; in fact, near the point $(1, 2)$ a change in x does not change f.

61. $dI = \frac{1}{R}\, dV - \frac{V}{R^2}\, dR \Rightarrow dI\big|_{(24,100)} = \frac{1}{100}\, dV - \frac{24}{100^2}\, dR \Rightarrow dI\big|_{dV=-1,dR=-20} = -0.01 + (480)(.0001) = 0.038$, or increases by 0.038 amps; % change in $V = (100)\left(-\frac{1}{24}\right) \approx -4.17\%$; % change in $R = \left(-\frac{20}{100}\right)(100) = -20\%$; $I = \frac{24}{100} = 0.24 \Rightarrow$ estimated % change in $I = \frac{dI}{I} \times 100 = \frac{0.038}{0.24} \times 100 \approx 15.83\% \Rightarrow$ more sensitive to voltage change.

62. $A = \pi ab \Rightarrow dA = \pi b\, da + \pi a\, db \Rightarrow dA|_{(10,16)} = 16\pi\, da + 10\pi\, db$; $da = \pm 0.1$ and $db = \pm 0.1$ $\Rightarrow dA = \pm 26\pi(0.1) = \pm 2.6\pi$ and $A = \pi(10)(16) = 160\pi \Rightarrow \left|\frac{dA}{A} \times 100\right| = \left|\frac{2.6\pi}{160\pi} \times 100\right| \approx 1.625\%$

63. (a) $y = uv \Rightarrow dy = v\, du + u\, dv$; percentage change in $u \le 2\% \Rightarrow |du| \le 0.02$, and percentage change in $v \le 3\%$ $\Rightarrow |dv| \le 0.03$; $\frac{dy}{y} = \frac{v\,du + u\,dv}{uv} = \frac{du}{u} + \frac{dv}{v} \Rightarrow \left|\frac{dy}{y} \times 100\right| = \left|\frac{du}{u} \times 100 + \frac{dv}{v} \times 100\right| \le \left|\frac{du}{u} \times 100\right| + \left|\frac{dv}{v} \times 100\right|$ $\le 2\% + 3\% = 5\%$

(b) $z = u + v \Rightarrow \frac{dz}{z} = \frac{du + dv}{u + v} = \frac{du}{u+v} + \frac{dv}{u+v} \le \frac{du}{u} + \frac{dv}{v}$ (since $u > 0$, $v > 0$) $\Rightarrow \left|\frac{dz}{z} \times 100\right| \le \left|\frac{du}{u} \times 100 + \frac{dv}{v} \times 100\right| = \left|\frac{dy}{y} \times 100\right|$

64. $C = \frac{7}{71.84 w^{0.425} h^{0.725}} \Rightarrow C_w = \frac{(-0.425)(7)}{71.84 w^{1.425} h^{0.725}}$ and $C_h = \frac{(-0.725)(7)}{71.84 w^{0.425} h^{1.725}}$ $\Rightarrow dC = \frac{-2.975}{71.84 w^{1.425} h^{0.725}}\, dw + \frac{-5.075}{71.84 w^{0.425} h^{1.725}}\, dh$; thus when $w = 70$ and $h = 180$ we have $dC|_{(70,180)} \approx -(0.00000225)\, dw - (0.00000149)\, dh \Rightarrow$ 1 kg error in weight has more effect

65. $f_x(x, y) = 2x - y + 2 = 0$ and $f_y(x, y) = -x + 2y + 2 = 0 \Rightarrow x = -2$ and $y = -2 \Rightarrow (-2, -2)$ is the critical point; $f_{xx}(-2, -2) = 2$, $f_{yy}(-2, -2) = 2$, $f_{xy}(-2, -2) = -1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 3 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of $f(-2, -2) = -8$

66. $f_x(x, y) = 10x + 4y + 4 = 0$ and $f_y(x, y) = 4x - 4y - 4 = 0 \Rightarrow x = 0$ and $y = -1 \Rightarrow (0, -1)$ is the critical point; $f_{xx}(0, -1) = 10$, $f_{yy}(0, -1) = -4$, $f_{xy}(0, -1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -56 < 0 \Rightarrow$ saddle point with $f(0, -1) = 2$

67. $f_x(x, y) = 6x^2 + 3y = 0$ and $f_y(x, y) = 3x + 6y^2 = 0 \Rightarrow y = -2x^2$ and $3x + 6\left(4x^4\right) = 0 \Rightarrow x\left(1 + 8x^3\right) = 0$ $\Rightarrow x = 0$ and $y = 0$, or $x = -\frac{1}{2}$ and $y = -\frac{1}{2} \Rightarrow$ the critical points are $(0, 0)$ and $\left(-\frac{1}{2}, -\frac{1}{2}\right)$. For $(0, 0)$: $f_{xx}(0, 0) = 12x|_{(0,0)} = 0$, $f_{yy}(0, 0) = 12y|_{(0,0)} = 0$, $f_{xy}(0, 0) = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -9 < 0 \Rightarrow$ saddle point with $f(0, 0) = 0$. For $\left(-\frac{1}{2}, -\frac{1}{2}\right)$: $f_{xx} = -6$, $f_{yy} = -6$, $f_{xy} = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 27 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum value of $f\left(-\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{4}$

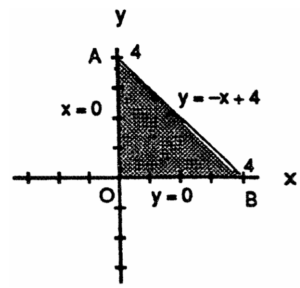
68. $f_x(x, y) = 3x^2 - 3y = 0$ and $f_y(x, y) = 3y^2 - 3x = 0 \Rightarrow y = x^2$ and $x^4 - x = 0 \Rightarrow x\left(x^3 - 1\right) = 0 \Rightarrow$ the critical points are $(0, 0)$ and $(1, 1)$. For $(0, 0)$: $f_{xx}(0, 0) = 6x|_{(0,0)} = 0$, $f_{yy}(0, 0) = 6y|_{(0,0)} = 0$, $f_{xy}(0, 0) = -3$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -9 < 0 \Rightarrow$ saddle point with $f(0, 0) = 15$. For $(1, 1)$: $f_{xx}(1, 1) = 6$, $f_{yy}(1, 1) = 6$, $f_{xy}(1, 1) = -3$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 27 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of $f(1, 1) = 14$

69. $f_x(x, y) = 3x^2 + 6x = 0$ and $f_y(x, y) = 3y^2 - 6y = 0 \Rightarrow x(x + 2) = 0$ and $y(y - 2) = 0 \Rightarrow x = 0$ or $x = -2$ and $y = 0$ or $y = 2 \Rightarrow$ the critical points are $(0, 0)$, $(0, 2)$, $(-2, 0)$, and $(-2, 2)$. For $(0, 0)$: $f_{xx}(0, 0) = 6x + 6|_{(0,0)}$ $= 6$, $f_{yy}(0, 0) = 6y - 6|_{(0,0)} = -6$, $f_{xy}(0, 0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point with $f(0, 0) = 0$. For $(0, 2)$: $f_{xx}(0, 2) = 6$, $f_{yy}(0, 2) = 6$, $f_{xy}(0, 2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of

$f(0,2) = -4$. For $(-2,0)$: $f_{xx}(-2,0) = -6$, $f_{yy}(-2,0) = -6$, $f_{xy}(-2,0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} < 0$ $\Rightarrow$ local maximum value of $f(-2,0) = 4$. For $(-2,2)$: $f_{xx}(-2,2) = -6$, $f_{yy}(-2,2) = 6$, $f_{xy}(-2,2) = 0$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point with $f(-2,2) = 0$.

70. $f_x(x,y) = 4x^3 - 16x = 0 \Rightarrow 4x\,(x^2 - 4) = 0 \Rightarrow x = 0, 2, -2$; $f_y(x,y) = 6y - 6 = 0 \Rightarrow y = 1$. Therefore the critical points are $(0,1)$, $(2,1)$, and $(-2,1)$. For $(0,1)$: $f_{xx}(0,1) = 12x^2 - 16|_{(0,1)} = -16$, $f_{yy}(0,1) = 6$, $f_{xy}(0,1) = 0$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -96 < 0 \Rightarrow$ saddle point with $f(0,1) = -3$. For $(2,1)$: $f_{xx}(2,1) = 32$, $f_{yy}(2,1) = 6$, $f_{xy}(2,1) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 192 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of $f(2,1) = -19$. For $(-2,1)$: $f_{xx}(-2,1) = 32$, $f_{yy}(-2,1) = 6$, $f_{xy}(-2,1) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 192 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of $f(-2,1) = -19$.

71. (i) On OA, $f(x,y) = f(0,y) = y^2 + 3y$ for $0 \le y \le 4$ $\Rightarrow f'(0,y) = 2y + 3 = 0 \Rightarrow y = -\frac{3}{2}$. But $\left(0, -\frac{3}{2}\right)$ is not in the region.
Endpoints: $f(0,0) = 0$ and $f(0,4) = 28$.

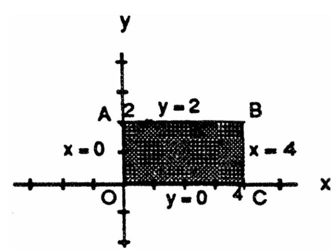


(ii) On AB, $f(x,y) = f(x,-x+4) = x^2 - 10x + 28$ for $0 \le x \le 4 \Rightarrow f'(x,-x+4) = 2x - 10 = 0$ $\Rightarrow x = 5$, $y = -1$. But $(5,-1)$ is not in the region.
Endpoints: $f(4,0) = 4$ and $f(0,4) = 28$.

(iii) On OB, $f(x,y) = f(x,0) = x^2 - 3x$ for $0 \le x \le 4 \Rightarrow f'(x,0) = 2x - 3 \Rightarrow x = \frac{3}{2}$ and $y = 0 \Rightarrow \left(\frac{3}{2}, 0\right)$ is a critical point with $f\left(\frac{3}{2}, 0\right) = -\frac{9}{4}$.
Endpoints: $f(0,0) = 0$ and $f(4,0) = 4$.

(iv) For the interior of the triangular region, $f_x(x,y) = 2x + y - 3 = 0$ and $f_y(x,y) = x + 2y + 3 = 0 \Rightarrow x = 3$ and $y = -3$. But $(3,-3)$ is not in the region. Therefore the absolute maximum is 28 at $(0,4)$ and the absolute minimum is $-\frac{9}{4}$ at $\left(\frac{3}{2}, 0\right)$.

72. (i) On OA, $f(x,y) = f(0,y) = -y^2 + 4y + 1$ for $0 \le y \le 2 \Rightarrow f'(0,y) = -2y + 4 = 0 \Rightarrow y = 2$ and $x = 0$. But $(0,2)$ is not in the interior of OA.
Endpoints: $f(0,0) = 1$ and $f(0,2) = 5$.



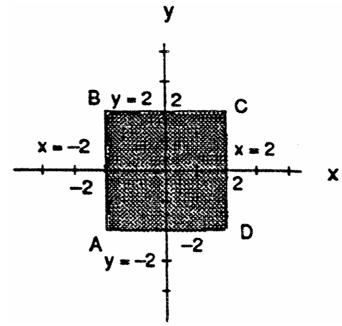
(ii) On AB, $f(x,y) = f(x,2) = x^2 - 2x + 5$ for $0 \le x \le 4$ $\Rightarrow f'(x,2) = 2x - 2 = 0 \Rightarrow x = 1$ and $y = 2$ $\Rightarrow (1,2)$ is an interior critical point of AB with $f(1,2) = 4$. Endpoints: $f(4,2) = 13$ and $f(0,2) = 5$.

(iii) On BC, $f(x,y) = f(4,y) = -y^2 + 4y + 9$ for $0 \le y \le 2 \Rightarrow f'(4,y) = -2y + 4 = 0 \Rightarrow y = 2$ and $x = 4$. But $(4,2)$ is not in the interior of BC. Endpoints: $f(4,0) = 9$ and $f(4,2) = 13$.

(iv) On OC, $f(x,y) = f(x,0) = x^2 - 2x + 1$ for $0 \le x \le 4 \Rightarrow f'(x,0) = 2x - 2 = 0 \Rightarrow x = 1$ and $y = 0 \Rightarrow (1,0)$ is an interior critical point of OC with $f(1,0) = 0$. Endpoints: $f(0,0) = 1$ and $f(4,0) = 9$.

(v) For the interior of the rectangular region, $f_x(x,y) = 2x - 2 = 0$ and $f_y(x,y) = -2y + 4 = 0 \Rightarrow x = 1$ and $y = 2$. But $(1,2)$ is not in the interior of the region. Therefore the absolute maximum is 13 at $(4,2)$ and the absolute minimum is 0 at $(1,0)$.

73. (i) On AB, $f(x,y) = f(-2,y) = y^2 - y - 4$ for $-2 \le y \le 2 \Rightarrow f'(-2,y) = 2y - 1 \Rightarrow y = \frac{1}{2}$ and $x = -2 \Rightarrow \left(-2, \frac{1}{2}\right)$ is an interior critical point in AB with $f\left(-2, \frac{1}{2}\right) = -\frac{17}{4}$. Endpoints: $f(-2,-2) = 2$ and $f(2,2) = -2$.



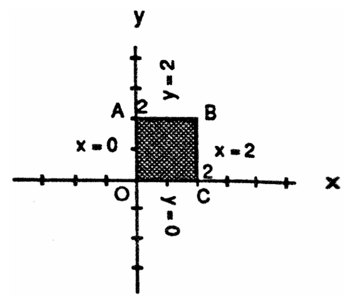
(ii) On BC, $f(x,y) = f(x,2) = -2$ for $-2 \le x \le 2$ $\Rightarrow f'(x,2) = 0 \Rightarrow$ no critical points in the interior of BC. Endpoints: $f(-2,2) = -2$ and $f(2,2) = -2$.

(iii) On CD, $f(x,y) = f(2,y) = y^2 - 5y + 4$ for $-2 \le y \le 2 \Rightarrow f'(2,y) = 2y - 5 = 0 \Rightarrow y = \frac{5}{2}$ and $x = 2$. But $\left(2, \frac{5}{2}\right)$ is not in the region. Endpoints: $f(2,-2) = 18$ and $f(2,2) = -2$.

(iv) On AD, $f(x,y) = f(x,-2) = 4x + 10$ for $-2 \le x \le 2 \Rightarrow f'(x,-2) = 4 \Rightarrow$ no critical points in the interior of AD. Endpoints: $f(-2,-2) = 2$ and $f(2,-2) = 18$.

(v) For the interior of the square, $f_x(x,y) = -y + 2 = 0$ and $f_y(x,y) = 2y - x - 3 = 0 \Rightarrow y = 2$ and $x = 1$ $\Rightarrow (1,2)$ is an interior critical point of the square with $f(1,2) = -2$. Therefore the absolute maximum is 18 at $(2,-2)$ and the absolute minimum is $-\frac{17}{4}$ at $\left(-2, \frac{1}{2}\right)$.

74. (i) On OA, $f(x,y) = f(0,y) = 2y - y^2$ for $0 \le y \le 2$ $\Rightarrow f'(0,y) = 2 - 2y = 0 \Rightarrow y = 1$ and $x = 0 \Rightarrow$ $(0,1)$ is an interior critical point of OA with $f(0,1) = 1$. Endpoints: $f(0,0) = 0$ and $f(0,2) = 0$.



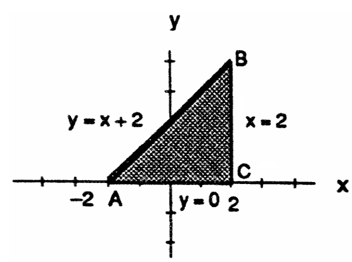
(ii) On AB, $f(x,y) = f(x,2) = 2x - x^2$ for $0 \le x \le 2$ $\Rightarrow f'(x,2) = 2 - 2x = 0 \Rightarrow x = 1$ and $y = 2$ $\Rightarrow (1,2)$ is an interior critical point of AB with $f(1,2) = 1$. Endpoints: $f(0,2) = 0$ and $f(2,2) = 0$.

(iii) On BC, $f(x,y) = f(2,y) = 2y - y^2$ for $0 \le y \le 2$ $\Rightarrow f'(2,y) = 2 - 2y = 0 \Rightarrow y = 1$ and $x = 2$ $\Rightarrow (2,1)$ is an interior critical point of BC with $f(2,1) = 1$. Endpoints: $f(2,0) = 0$ and $f(2,2) = 0$.

(iv) On OC, $f(x,y) = f(x,0) = 2x - x^2$ for $0 \le x \le 2 \Rightarrow f'(x,0) = 2 - 2x = 0 \Rightarrow x = 1$ and $y = 0 \Rightarrow (1,0)$ is an interior critical point of OC with $f(1,0) = 1$. Endpoints: $f(0,0) = 0$ and $f(0,2) = 0$.

(v) For the interior of the rectangular region, $f_x(x,y) = 2 - 2x = 0$ and $f_y(x,y) = 2 - 2y = 0 \Rightarrow x = 1$ and $y = 1 \Rightarrow (1,1)$ is an interior critical point of the square with $f(1,1) = 2$. Therefore the absolute maximum is 2 at $(1,1)$ and the absolute minimum is 0 at the four corners $(0,0)$, $(0,2)$, $(2,2)$, and $(2,0)$.

75. (i) On AB, $f(x,y) = f(x, x+2) = -2x + 4$ for $-2 \le x \le 2 \Rightarrow f'(x, x+2) = -2 = 0 \Rightarrow$ no critical points in the interior of AB. Endpoints: $f(-2,0) = 8$ and $f(2,4) = 0$.

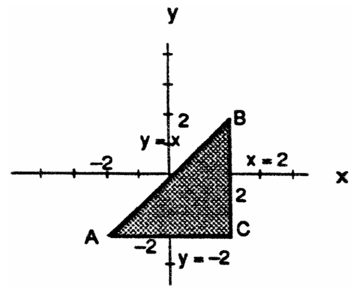


(ii) On BC, $f(x,y) = f(2,y) = -y^2 + 4y$ for $0 \le y \le 4$ $\Rightarrow f'(2,y) = -2y + 4 = 0 \Rightarrow y = 2$ and $x = 2$ $\Rightarrow (2,2)$ is an interior critical point of BC with $f(2,2) = 4$. Endpoints: $f(2,0) = 0$ and $f(2,4) = 0$.

(iii) On AC, $f(x,y) = f(x,0) = x^2 - 2x$ for $-2 \le x \le 2$ $\Rightarrow f'(x,0) = 2x - 2 \Rightarrow x = 1$ and $y = 0 \Rightarrow (1,0)$ is an interior critical point of AC with $f(1,0) = -1$. Endpoints: $f(-2,0) = 8$ and $f(2,0) = 0$.

(iv) For the interior of the triangular region, $f_x(x,y) = 2x - 2 = 0$ and $f_y(x,y) = -2y + 4 = 0 \Rightarrow x = 1$ and $y = 2 \Rightarrow (1,2)$ is an interior critical point of the region with $f(1,2) = 3$. Therefore the absolute maximum is 8 at $(-2,0)$ and the absolute minimum is -1 at $(1,0)$.

76. (i) On AB, $f(x, y) = f(x, x) = 4x^2 - 2x^4 + 16$ for $-2 \le x \le 2 \Rightarrow f'(x, x) = 8x - 8x^3 = 0 \Rightarrow x = 0$ and $y = 0$, or $x = 1$ and $y = 1$, or $x = -1$ and $y = -1$ $\Rightarrow (0, 0), (1, 1), (-1, -1)$ are all interior points of AB with $f(0, 0) = 16$, $f(1, 1) = 18$, and $f(-1, -1) = 18$. Endpoints: $f(-2, -2) = 0$ and $f(2, 2) = 0$.

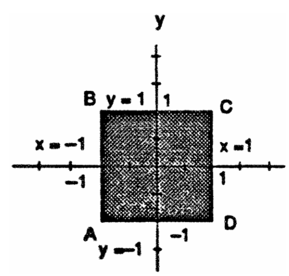


(ii) On BC, $f(x, y) = f(2, y) = 8y - y^4$ for $-2 \le y \le 2$ $\Rightarrow f'(2, y) = 8 - 4y^3 = 0 \Rightarrow y = \sqrt[3]{2}$ and $x = 2$ $\Rightarrow \left(2, \sqrt[3]{2}\right)$ is an interior critical point of BC with $f\left(2, \sqrt[3]{2}\right) = 6\sqrt[3]{2}$. Endpoints: $f(2, -2) = -32$ and $f(2, 2) = 0$.

(iii) On AC, $f(x, y) = f(x, -2) = -8x - x^4$ for $-2 \le x \le 2 \Rightarrow f'(x, -2) = -8 - 4x^3 = 0 \Rightarrow x = \sqrt[3]{-2}$ and $y = -2$ $\Rightarrow \left(\sqrt[3]{-2}, -2\right)$ is an interior critical point of AC with $f\left(\sqrt[3]{-2}, -2\right) = 6\sqrt[3]{2}$. Endpoints: $f(-2, -2) = 0$ and $f(2, -2) = -32$.

(iv) For the interior of the triangular region, $f_x(x, y) = 4y - 4x^3 = 0$ and $f_y(x, y) = 4x - 4y^3 = 0 \Rightarrow x = 0$ and $y = 0$, or $x = 1$ and $y = 1$ or $x = -1$ and $y = -1$. But neither of the points $(0, 0)$ and $(1, 1)$, or $(-1, -1)$ are interior to the region. Therefore the absolute maximum is 18 at $(1, 1)$ and $(-1, -1)$, and the absolute minimum is -32 at $(2, -2)$.

77. (i) On AB, $f(x, y) = f(-1, y) = y^3 - 3y^2 + 2$ for $-1 \le y \le 1 \Rightarrow f'(-1, y) = 3y^2 - 6y = 0 \Rightarrow y = 0$ and $x = -1$, or $y = 2$ and $x = -1 \Rightarrow (-1, 0)$ is an interior critical point of AB with $f(-1, 0) = 2$; $(-1, 2)$ is outside the boundary. Endpoints: $f(-1, -1) = -2$ and $f(-1, 1) = 0$.



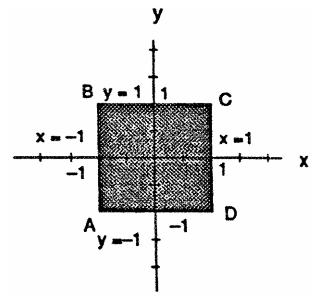
(ii) On BC, $f(x, y) = f(x, 1) = x^3 + 3x^2 - 2$ for $-1 \le x \le 1 \Rightarrow f'(x, 1) = 3x^2 + 6x = 0 \Rightarrow x = 0$ and $y = 1$, or $x = -2$ and $y = 1 \Rightarrow (0, 1)$ is an interior critical point of BC with $f(0, 1) = -2$; $(-2, 1)$ is outside the boundary. Endpoints: $f(-1, 1) = 0$ and $f(1, 1) = 2$.

(iii) On CD, $f(x, y) = f(1, y) = y^3 - 3y^2 + 4$ for $-1 \le y \le 1 \Rightarrow f'(1, y) = 3y^2 - 6y = 0 \Rightarrow y = 0$ and $x = 1$, or $y = 2$ and $x = 1 \Rightarrow (1, 0)$ is an interior critical point of CD with $f(1, 0) = 4$; $(1, 2)$ is outside the boundary. Endpoints: $f(1, 1) = 2$ and $f(1, -1) = 0$.

(iv) On AD, $f(x, y) = f(x, -1) = x^3 + 3x^2 - 4$ for $-1 \le x \le 1 \Rightarrow f'(x, -1) = 3x^2 + 6x = 0 \Rightarrow x = 0$ and $y = -1$, or $x = -2$ and $y = -1 \Rightarrow (0, -1)$ is an interior point of AD with $f(0, -1) = -4$; $(-2, -1)$ is outside the boundary. Endpoints: $f(-1, -1) = -2$ and $f(1, -1) = 0$.

(v) For the interior of the square, $f_x(x, y) = 3x^2 + 6x = 0$ and $f_y(x, y) = 3y^2 - 6y = 0 \Rightarrow x = 0$ or $x = -2$, and $y = 0$ or $y = 2 \Rightarrow (0, 0)$ is an interior critical point of the square region with $f(0, 0) = 0$; the points $(0, 2)$, $(-2, 0)$, and $(-2, 2)$ are outside the region. Therefore the absolute maximum is 4 at $(1, 0)$ and the absolute minimum is -4 at $(0, -1)$.

78. (i) On AB, $f(x,y) = f(-1,y) = y^3 - 3y$ for $-1 \le y \le 1$ $\Rightarrow f'(-1,y) = 3y^2 - 3 = 0 \Rightarrow y = \pm 1$ and $x = -1$ yielding the corner points $(-1,-1)$ and $(-1,1)$ with $f(-1,-1) = 2$ and $f(-1,1) = -2$.



(ii) On BC, $f(x,y) = f(x,1) = x^3 + 3x + 2$ for $-1 \le x \le 1 \Rightarrow f'(x,1) = 3x^2 + 3 = 0 \Rightarrow$ no solution. Endpoints: $f(-1,1) = -2$ and $f(1,1) = 6$.

(iii) On CD, $f(x,y) = f(1,y) = y^3 + 3y + 2$ for $-1 \le y \le 1 \Rightarrow f'(1,y) = 3y^2 + 3 = 0 \Rightarrow$ no solution. Endpoints: $f(1,1) = 6$ and $f(1,-1) = -2$.

(iv) On AD, $f(x,y) = f(x,-1) = x^3 - 3x$ for $-1 \le x \le 1 \Rightarrow f'(x,-1) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$ and $y = -1$ yielding the corner points $(-1,-1)$ and $(1,-1)$ with $f(-1,-1) = 2$ and $f(1,-1) = -2$

(v) For the interior of the square, $f_x(x,y) = 3x^2 + 3y = 0$ and $f_y(x,y) = 3y^2 + 3x = 0 \Rightarrow y = -x^2$ and $x^4 + x = 0 \Rightarrow x = 0$ or $x = -1 \Rightarrow y = 0$ or $y = -1 \Rightarrow (0,0)$ is an interior critical point of the square region with $f(0,0) = 1$; $(-1,-1)$ is on the boundary. Therefore the absolute maximum is 6 at $(1,1)$ and the absolute minimum is -2 at $(1,-1)$ and $(-1,1)$.

79. $\nabla f = 3x^2\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 3x^2\mathbf{i} + 2y\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow 3x^2 = 2x\lambda$ and $2y = 2y\lambda \Rightarrow \lambda = 1$ or $y = 0$.

CASE 1: $\lambda = 1 \Rightarrow 3x^2 = 2x \Rightarrow x = 0$ or $x = \frac{2}{3}$; $x = 0 \Rightarrow y = \pm 1$ yielding the points $(0,1)$ and $(0,-1)$; $x = \frac{2}{3}$ $\Rightarrow y = \pm \frac{\sqrt{5}}{3}$ yielding the points $\left(\frac{2}{3}, \frac{\sqrt{5}}{3}\right)$ and $\left(\frac{2}{3}, -\frac{\sqrt{5}}{3}\right)$.

CASE 2: $y = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$ yielding the points $(1,0)$ and $(-1,0)$.

Evaluations give $f(0, \pm 1) = 1$, $f\left(\frac{2}{3}, \pm \frac{\sqrt{5}}{3}\right) = \frac{23}{27}$, $f(1,0) = 1$, and $f(-1,0) = -1$. Therefore the absolute maximum is 1 at $(0, \pm 1)$ and $(1,0)$, and the absolute minimum is -1 at $(-1,0)$.

80. $\nabla f = y\mathbf{i} + x\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow y\mathbf{i} + x\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow y = 2\lambda x$ and $xy = 2\lambda y \Rightarrow x = 2\lambda(2\lambda x) = 4\lambda^2 x \Rightarrow x = 0$ or $4\lambda^2 = 1$.

CASE 1: $x = 0 \Rightarrow y = 0$ but $(0,0)$ does not lie on the circle, so no solution.

CASE 2: $4\lambda^2 = 1 \Rightarrow \lambda = \frac{1}{2}$ or $\lambda = -\frac{1}{2}$. For $\lambda = \frac{1}{2}$, $y = x \Rightarrow 1 = x^2 + y^2 = 2x^2 \Rightarrow x = y = \pm \frac{1}{\sqrt{2}}$ yielding the points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. For $\lambda = -\frac{1}{2}$, $y = -x \Rightarrow 1 = x^2 + y^2 = 2x^2 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$ and $y = -x$ yielding the points $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

Evaluations give the absolute maximum value $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$ and the absolute minimum value $f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$.

81. (i) $f(x,y) = x^2 + 3y^2 + 2y$ on $x^2 + y^2 = 1 \Rightarrow \nabla f = 2x\mathbf{i} + (6y+2)\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g$ $\Rightarrow 2x\mathbf{i} + (6y+2)\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow 2x = 2x\lambda$ and $6y + 2 = 2y\lambda \Rightarrow \lambda = 1$ or $x = 0$.

CASE 1: $\lambda = 1 \Rightarrow 6y + 2 = 2y \Rightarrow y = -\frac{1}{2}$ and $x = \pm \frac{\sqrt{3}}{2}$ yielding the points $\left(\pm \frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

CASE 2: $x = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$ yielding the points $(0, \pm 1)$.

Evaluations give $f\left(\pm \frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \frac{1}{2}$, $f(0,1) = 5$, and $f(0,-1) = 1$. Therefore $\frac{1}{2}$ and 5 are the extreme values on the boundary of the disk.

(ii) For the interior of the disk, $f_x(x,y) = 2x = 0$ and $f_y(x,y) = 6y + 2 = 0 \Rightarrow x = 0$ and $y = -\frac{1}{3}$ $\Rightarrow \left(0, -\frac{1}{3}\right)$ is an interior critical point with $f\left(0, -\frac{1}{3}\right) = -\frac{1}{3}$. Therefore the absolute maximum of f on the disk is 5 at $(0,1)$ and the absolute minimum of f on the disk is $-\frac{1}{3}$ at $\left(0, -\frac{1}{3}\right)$.

82. (i) $f(x, y) = x^2 + y^2 - 3x - xy$ on $x^2 + y^2 = 9 \Rightarrow \nabla f = (2x - 3 - y)\mathbf{i} + (2y - x)\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow (2x - 3 - y)\mathbf{i} + (2y - x)\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow 2x - 3 - y = 2x\lambda$ and $2y - x = 2y\lambda$ $\Rightarrow 2x(1 - \lambda) - y = 3$ and $-x + 2y(1 - \lambda) = 0 \Rightarrow 1 - \lambda = \frac{x}{2y}$ and $(2x)\left(\frac{x}{2y}\right) - y = 3 \Rightarrow x^2 - y^2 = 3y$ $\Rightarrow x^2 = y^2 + 3y$. Thus, $9 = x^2 + y^2 = y^2 + 3y + y^2 \Rightarrow 2y^2 + 3y - 9 = 0 \Rightarrow (2y - 3)(y + 3) = 0$ $\Rightarrow y = -3, \frac{3}{2}$. For $y = -3$, $x^2 + y^2 = 9 \Rightarrow x = 0$ yielding the point $(0, -3)$. For $y = \frac{3}{2}$, $x^2 + y^2 = 9$ $\Rightarrow x^2 + \frac{9}{4} = 9 \Rightarrow x^2 = \frac{27}{4} \Rightarrow x = \pm \frac{3\sqrt{3}}{2}$. Evaluations give $f(0, -3) = 9$, $f\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right) = 9 + \frac{27\sqrt{3}}{4}$ ≈ 20.691, and $f\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right) = 9 - \frac{27\sqrt{3}}{4} \approx -2.691$.

(ii) For the interior of the disk, $f_x(x, y) = 2x - 3 - y = 0$ and $f_y(x, y) = 2y - x = 0 \Rightarrow x = 2$ and $y = 1$ $\Rightarrow (2, 1)$ is an interior critical point of the disk with $f(2, 1) = -3$. Therefore, the absolute maximum of f on the disk is $9 + \frac{27\sqrt{3}}{4}$ at $\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$ and the absolute minimum of f on the disk is -3 at $(2, 1)$.

83. $\nabla f = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} - \mathbf{j} + \mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 1 = 2x\lambda$, $-1 = 2y\lambda$, $1 = 2z\lambda \Rightarrow x = -y = z = \frac{1}{\lambda}$. Thus $x^2 + y^2 + z^2 = 1 \Rightarrow 3x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{3}}$ yielding the points $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$. Evaluations give the absolute maximum value of $f\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{3}{\sqrt{3}} = \sqrt{3}$ and the absolute minimum value of $f\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = -\sqrt{3}$.

84. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance to the origin and $g(x, y, z) = z^2 - xy - 4$. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = -y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x = -\lambda y$, $2y = -\lambda x$, and $2z = 2\lambda z$ $\Rightarrow z = 0$ or $\lambda = 1$.

CASE 1: $z = 0 \Rightarrow xy = -4 \Rightarrow x = -\frac{4}{y}$ and $y = -\frac{4}{x} \Rightarrow 2\left(-\frac{4}{y}\right) = -\lambda y$ and $2\left(-\frac{4}{x}\right) = -\lambda x \Rightarrow \frac{8}{\lambda} = y^2$ and $\frac{8}{\lambda} = x^2$ $\Rightarrow y^2 = x^2 \Rightarrow y = \pm x$. But $y = x \Rightarrow x^2 = -4$ leads to no solution, so $y = -x \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$ yielding the points $(-2, 2, 0)$ and $(2, -2, 0)$.

CASE 2: $\lambda = 1 \Rightarrow 2x = -y$ and $2y = -x \Rightarrow 2y = -\left(-\frac{y}{2}\right) \Rightarrow 4y = y \Rightarrow y = 0 \Rightarrow x = 0 \Rightarrow z^2 - 4 = 0 \Rightarrow z = \pm 2$ yielding the points $(0, 0, -2)$ and $(0, 0, 2)$.

Evaluations give $f(-2, 2, 0) = f(2, -2, 0) = 8$ and $f(0, 0, -2) = f(0, 0, 2) = 4$. Thus the points $(0, 0, -2)$ and $(0, 0, 2)$ on the surface are closest to the origin.

85. The cost is $f(x, y, z) = 2axy + 2bxz + 2cyz$ subject to the constraint $xyz = V$. Then $\nabla f = \lambda \nabla g$ $\Rightarrow 2ay + 2bz = \lambda yz$, $2ax + 2cz = \lambda xz$, and $2bx + 2cy = \lambda xy \Rightarrow 2axy + 2bxz = \lambda xyz$, $2axy + 2cyz = \lambda xyz$, and $2bxz + 2cyz = \lambda xyz \Rightarrow 2axy + 2bxz = 2axy + 2cyz \Rightarrow y = \left(\frac{b}{c}\right)x$. Also $2axy + 2bxz = 2bxz + 2cyz \Rightarrow z = \left(\frac{a}{c}\right)x$. Then $x\left(\frac{b}{c}x\right)\left(\frac{a}{c}x\right) = V \Rightarrow x^3 = \frac{c^2V}{ab} \Rightarrow$ width $= x = \left(\frac{c^2V}{ab}\right)^{1/3}$, Depth $= y = \left(\frac{b}{c}\right)\left(\frac{c^2V}{ab}\right)^{1/3} = \left(\frac{b^2V}{ac}\right)^{1/3}$, and Height $= z = \left(\frac{a}{c}\right)\left(\frac{c^2V}{ab}\right)^{1/3} = \left(\frac{a^2V}{bc}\right)^{1/3}$.

86. The volume of the pyramid in the first octant formed by the plane is $V(a, b, c) = \frac{1}{3}\left(\frac{1}{2}ab\right)c = \frac{1}{6}abc$. The point $(2, 1, 2)$ on the plane $\Rightarrow \frac{2}{a} + \frac{1}{b} + \frac{2}{c} = 1$. We want to minimize V subject to the constraint $2bc + ac + 2ab = abc$. Thus, $\nabla V = \frac{bc}{6}\mathbf{i} + \frac{ac}{6}\mathbf{j} + \frac{ab}{6}\mathbf{k}$ and $\nabla g = (c + 2b - bc)\mathbf{i} + (2c + 2a - ac)\mathbf{j} + (2b + a - ab)\mathbf{k}$ so that $\nabla V = \lambda \nabla g$ $\Rightarrow \frac{bc}{6} = \lambda(c + 2b - bc)$, $\frac{ac}{6} = \lambda(2c + 2a - ac)$, and $\frac{ab}{6} = \lambda(2b + a - ab) \Rightarrow \frac{abc}{6} = \lambda(ac + 2ab - abc)$, $\frac{abc}{6} = \lambda(2bc + 2ab - abc)$, and $\frac{abc}{6} = \lambda(2bc + ac - abc) \Rightarrow \lambda ac = 2\lambda bc$ and $2\lambda ab = 2\lambda bc$. Now $\lambda \neq 0$ since $a \neq 0$, $b \neq 0$, and $c \neq 0 \Rightarrow ac = 2bc$ and $ab = bc \Rightarrow a = 2b = c$. Substituting into the constraint equation gives $\frac{2}{a} + \frac{2}{a} + \frac{2}{a} = 1 \Rightarrow a = 6 \Rightarrow b = 3$ and $c = 6$. Therefore the desired plane is $\frac{x}{6} + \frac{y}{3} + \frac{z}{6} = 1$ or $x + 2y + z = 6$.

87. $\nabla f = (y + z)\mathbf{i} + x\mathbf{j} + x\mathbf{k}$, $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$, and $\nabla h = z\mathbf{i} + x\mathbf{k}$ so that $\nabla f = \lambda \nabla g + \mu \nabla h$ $\Rightarrow (y + z)\mathbf{i} + x\mathbf{j} + x\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) + \mu(z\mathbf{i} + x\mathbf{k}) \Rightarrow y + z = 2\lambda x + \mu z$, $x = 2\lambda y$, $x = \mu x \Rightarrow x = 0$

or $\mu = 1$.

CASE 1: $x = 0$ which is impossible since $xz = 1$.

CASE 2: $\mu = 1 \Rightarrow y + z = 2\lambda x + z \Rightarrow y = 2\lambda x$ and $x = 2\lambda y \Rightarrow y = (2\lambda)(2\lambda y) \Rightarrow y = 0$ or $4\lambda^2 = 1$. If $y = 0$, then $x^2 = 1 \Rightarrow x = \pm 1$ so with $xz = 1$ we obtain the points $(1, 0, 1)$ and $(-1, 0, -1)$. If $4\lambda^2 = 1$, then $\lambda = \pm \frac{1}{2}$. For $\lambda = -\frac{1}{2}$, $y = -x$ so $x^2 + y^2 = 1 \Rightarrow x^2 = \frac{1}{2}$ $\Rightarrow x = \pm \frac{1}{\sqrt{2}}$ with $xz = 1 \Rightarrow z = \pm\sqrt{2}$, and we obtain the points $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \sqrt{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2}\right)$. For $\lambda = \frac{1}{2}$, $y = x \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$ with $xz = 1 \Rightarrow z = \pm\sqrt{2}$, and we obtain the points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$.

Evaluations give $f(1, 0, 1) = 1$, $f(-1, 0, -1) = 1$, $f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \sqrt{2}\right) = \frac{1}{2}$, $f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2}\right) = \frac{1}{2}$, $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right) = \frac{3}{2}$, and $f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}\right) = \frac{3}{2}$. Therefore the absolute maximum is $\frac{3}{2}$ at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$, and the absolute minimum is $\frac{1}{2}$ at $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2}\right)$ and $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \sqrt{2}\right)$.

88. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance to the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\nabla g = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and $\nabla h = 4x\mathbf{i} + 4y\mathbf{j} - 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g + \mu \nabla h \Rightarrow 2x = \lambda + 4x\mu$, $2y = \lambda + 4y\mu$, and $2z = \lambda - 2z\mu \Rightarrow \lambda = 2x(1 - 2\mu) = 2y(1 - 2\mu) = 2z(1 + 2\mu) \Rightarrow x = y$ or $\mu = \frac{1}{2}$.

CASE 1: $x = y \Rightarrow z^2 = 4x^2 \Rightarrow z = \pm 2x$ so that $x + y + z = 1 \Rightarrow x + x + 2x = 1$ or $x + x - 2x = 1$ (impossible) $\Rightarrow x = \frac{1}{4} \Rightarrow y = \frac{1}{4}$ and $z = \frac{1}{2}$ yielding the point $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$.

CASE 2: $\mu = \frac{1}{2} \Rightarrow \lambda = 0 \Rightarrow 0 = 2z(1 + 1) \Rightarrow z = 0$ so that $2x^2 + 2y^2 = 0 \Rightarrow x = y = 0$. But the origin $(0, 0, 0)$ fails to satisfy the first constraint $x + y + z = 1$.

Therefore, the point $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$ on the curve of intersection is closest to the origin.

89. (a) y, z are independent with $w = x^2 e^{yz}$ and $z = x^2 - y^2 \Rightarrow \frac{\partial w}{\partial y} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}$ $= (2xe^{yz})\frac{\partial x}{\partial y} + (zx^2e^{yz})(1) + (yx^2e^{yz})(0)$; $z = x^2 - y^2 \Rightarrow 0 = 2x\frac{\partial x}{\partial y} - 2y \Rightarrow \frac{\partial x}{\partial y} = \frac{y}{x}$; therefore, $\left(\frac{\partial w}{\partial y}\right)_z = (2xe^{yz})\left(\frac{y}{x}\right) + zx^2e^{yz} = (2y + zx^2)\,e^{yz}$

(b) z, x are independent with $w = x^2 e^{yz}$ and $z = x^2 - y^2 \Rightarrow \frac{\partial w}{\partial z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z}$ $= (2xe^{yz})(0) + (zx^2e^{yz})\frac{\partial y}{\partial z} + (yx^2e^{yz})(1)$; $z = x^2 - y^2 \Rightarrow 1 = 0 - 2y\frac{\partial y}{\partial z} \Rightarrow \frac{\partial y}{\partial z} = -\frac{1}{2y}$; therefore, $\left(\frac{\partial w}{\partial z}\right)_x = (zx^2e^{yz})\left(-\frac{1}{2y}\right) + yx^2e^{yz} = x^2e^{yz}\left(y - \frac{z}{2y}\right)$

(c) z, y are independent with $w = x^2 e^{yz}$ and $z = x^2 - y^2 \Rightarrow \frac{\partial w}{\partial z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z}$ $= (2xe^{yz})\frac{\partial x}{\partial z} + (zx^2e^{yz})(0) + (yx^2e^{yz})(1)$; $z = x^2 - y^2 \Rightarrow 1 = 2x\frac{\partial x}{\partial z} - 0 \Rightarrow \frac{\partial x}{\partial z} = \frac{1}{2x}$; therefore, $\left(\frac{\partial w}{\partial z}\right)_y = (2xe^{yz})\left(\frac{1}{2x}\right) + yx^2e^{yz} = (1 + x^2y)\,e^{yz}$

90. (a) T, P are independent with $U = f(P, V, T)$ and $PV = nRT \Rightarrow \frac{\partial U}{\partial T} = \frac{\partial U}{\partial P}\frac{\partial P}{\partial T} + \frac{\partial U}{\partial V}\frac{\partial V}{\partial T} + \frac{\partial U}{\partial T}\frac{\partial T}{\partial T}$ $= \left(\frac{\partial U}{\partial P}\right)(0) + \left(\frac{\partial U}{\partial V}\right)\left(\frac{\partial V}{\partial T}\right) + \left(\frac{\partial U}{\partial T}\right)(1)$; $PV = nRT \Rightarrow P\frac{\partial V}{\partial T} = nR \Rightarrow \frac{\partial V}{\partial T} = \frac{nR}{P}$; therefore, $\left(\frac{\partial U}{\partial T}\right)_P = \left(\frac{\partial U}{\partial V}\right)\left(\frac{nR}{P}\right) + \frac{\partial U}{\partial T}$

(b) V, T are independent with $U = f(P, V, T)$ and $PV = nRT \Rightarrow \frac{\partial U}{\partial V} = \frac{\partial U}{\partial P}\frac{\partial P}{\partial V} + \frac{\partial U}{\partial V}\frac{\partial V}{\partial V} + \frac{\partial U}{\partial T}\frac{\partial T}{\partial V}$ $= \left(\frac{\partial U}{\partial P}\right)\left(\frac{\partial P}{\partial V}\right) + \left(\frac{\partial U}{\partial V}\right)(1) + \left(\frac{\partial U}{\partial T}\right)(0)$; $PV = nRT \Rightarrow V\frac{\partial P}{\partial V} + P = (nR)\left(\frac{\partial T}{\partial V}\right) = 0 \Rightarrow \frac{\partial P}{\partial V} = -\frac{P}{V}$; therefore, $\left(\frac{\partial U}{\partial V}\right)_T = \left(\frac{\partial U}{\partial P}\right)\left(-\frac{P}{V}\right) + \frac{\partial U}{\partial V}$

91. Note that $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. Thus,
$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x} = \left(\frac{\partial w}{\partial r}\right)\left(\frac{x}{\sqrt{x^2 + y^2}}\right) + \left(\frac{\partial w}{\partial \theta}\right)\left(\frac{-y}{x^2 + y^2}\right) = (\cos\theta)\frac{\partial w}{\partial r} - \left(\frac{\sin\theta}{r}\right)\frac{\partial w}{\partial \theta}$;

$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\theta y} = \left(\frac{\partial w}{\partial r}\right)\left(\frac{y}{\sqrt{x^2+y^2}}\right) + \left(\frac{\partial w}{\partial \theta}\right)\left(\frac{x}{x^2+y^2}\right) = (\sin\theta)\frac{\partial w}{\partial r} + \left(\frac{\cos\theta}{r}\right)\frac{\partial w}{\partial \theta}$

92. $z_x = f_u \frac{\partial u}{\partial x} + f_v \frac{\partial v}{\partial x} = af_u + af_v$, and $z_y = f_u \frac{\partial u}{\partial y} + f_v \frac{\partial v}{\partial y} = bf_u - bf_v$

93. $\frac{\partial u}{\partial y} = b$ and $\frac{\partial u}{\partial x} = a \Rightarrow \frac{\partial w}{\partial x} = \frac{dw}{du}\frac{\partial u}{\partial x} = a\frac{dw}{du}$ and $\frac{\partial w}{\partial y} = \frac{dw}{du}\frac{\partial u}{\partial y} = b\frac{dw}{du} \Rightarrow \frac{1}{a}\frac{\partial w}{\partial x} = \frac{dw}{du}$ and $\frac{1}{b}\frac{\partial w}{\partial y} = \frac{dw}{du}$
$\Rightarrow \frac{1}{a}\frac{\partial w}{\partial x} = \frac{1}{b}\frac{\partial w}{\partial y} \Rightarrow b\frac{\partial w}{\partial x} = a\frac{\partial w}{\partial y}$

94. $\frac{\partial w}{\partial x} = \frac{2x}{x^2+y^2+2z} = \frac{2(r+s)}{(r+s)^2+(r-s)^2+4rs} = \frac{2(r+s)}{2(r^2+2rs+s^2)} = \frac{1}{r+s}$, $\frac{\partial w}{\partial y} = \frac{2y}{x^2+y^2+2z} = \frac{2(r-s)}{2(r+s)^2} = \frac{r-s}{(r+s)^2}$,
and $\frac{\partial w}{\partial z} = \frac{2}{x^2+y^2+2z} = \frac{1}{(r+s)^2} \Rightarrow \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = \frac{1}{r+s} + \frac{r-s}{(r+s)^2} + \left[\frac{1}{(r+s)^2}\right](2s) = \frac{2r+2s}{(r+s)^2}$
$= \frac{2}{r+s}$ and $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = \frac{1}{r+s} - \frac{r-s}{(r+s)^2} + \left[\frac{1}{(r+s)^2}\right](2r) = \frac{2}{r+s}$

95. $e^u \cos v - x = 0 \Rightarrow (e^u \cos v)\frac{\partial u}{\partial x} - (e^u \sin v)\frac{\partial v}{\partial x} = 1$; $e^u \sin v - y = 0 \Rightarrow (e^u \sin v)\frac{\partial u}{\partial x} + (e^u \cos v)\frac{\partial v}{\partial x} = 0$.
Solving this system yields $\frac{\partial u}{\partial x} = e^{-u}\cos v$ and $\frac{\partial v}{\partial x} = -e^{-u}\sin v$. Similarly, $e^u \cos v - x = 0$
$\Rightarrow (e^u \cos v)\frac{\partial u}{\partial y} - (e^u \sin v)\frac{\partial v}{\partial y} = 0$ and $e^u \sin v - y = 0 \Rightarrow (e^u \sin v)\frac{\partial u}{\partial y} + (e^u \cos v)\frac{\partial v}{\partial y} = 1$. Solving this second system yields $\frac{\partial u}{\partial y} = e^{-u}\sin v$ and $\frac{\partial v}{\partial y} = e^{-u}\cos v$. Therefore $\left(\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j}\right)\cdot\left(\frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j}\right)$
$= [(e^{-u}\cos v)\mathbf{i} + (e^{-u}\sin v)\mathbf{j}]\cdot[(-e^{-u}\sin v)\mathbf{i} + (e^{-u}\cos v)\mathbf{j}] = 0 \Rightarrow$ the vectors are orthogonal $\Rightarrow$ the angle between the vectors is the constant $\frac{\pi}{2}$.

96. $\frac{\partial g}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = (-r\sin\theta)\frac{\partial f}{\partial x} + (r\cos\theta)\frac{\partial f}{\partial y}$
$\Rightarrow \frac{\partial^2 g}{\partial \theta^2} = (-r\sin\theta)\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y\partial x}\frac{\partial y}{\partial \theta}\right) - (r\cos\theta)\frac{\partial f}{\partial x} + (r\cos\theta)\left(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial \theta}\right) - (r\sin\theta)\frac{\partial f}{\partial y}$
$= (-r\sin\theta)\left(\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta}\right) - (r\cos\theta) + (r\cos\theta)\left(\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta}\right) - (r\sin\theta)$
$= (-r\sin\theta + r\cos\theta)(-r\sin\theta + r\cos\theta) - (r\cos\theta + r\sin\theta) = (-2)(-2) - (0+2) = 4 - 2 = 2$ at $(r,\theta) = \left(2, \frac{\pi}{2}\right)$.

97. $(y+z)^2 + (z-x)^2 = 16 \Rightarrow \nabla f = -2(z-x)\mathbf{i} + 2(y+z)\mathbf{j} + 2(y+2z-x)\mathbf{k}$; if the normal line is parallel to the yz-plane, then x is constant $\Rightarrow \frac{\partial f}{\partial x} = 0 \Rightarrow -2(z-x) = 0 \Rightarrow z = x \Rightarrow (y+z)^2 + (z-z)^2 = 16 \Rightarrow y+z = \pm 4$.
Let $x = t \Rightarrow z = t \Rightarrow y = -t \pm 4$. Therefore the points are $(t, -t \pm 4, t)$, t a real number.

98. Let $f(x,y,z) = xy + yz + zx - x - z^2 = 0$. If the tangent plane is to be parallel to the xy-plane, then ∇f is perpendicular to the xy-plane $\Rightarrow \nabla f \cdot \mathbf{i} = 0$ and $\nabla f \cdot \mathbf{j} = 0$. Now $\nabla f = (y+z-1)\mathbf{i} + (x+z)\mathbf{j} + (y+x-2z)\mathbf{k}$ so that $\nabla f \cdot \mathbf{i} = y+z-1 = 0 \Rightarrow y+z = 1 \Rightarrow y = 1-z$, and $\nabla f \cdot \mathbf{j} = x+z = 0 \Rightarrow x = -z$. Then $-z(1-z) + (1-z)z + z(-z) - (-z) - z^2 = 0 \Rightarrow z - 2z^2 = 0 \Rightarrow z = \frac{1}{2}$ or $z = 0$. Now $z = \frac{1}{2} \Rightarrow x = -\frac{1}{2}$ and $y = \frac{1}{2}$
$\Rightarrow \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is one desired point; $z = 0 \Rightarrow x = 0$ and $y = 1 \Rightarrow (0,1,0)$ is a second desired point.

99. $\nabla f = \lambda(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \Rightarrow \frac{\partial f}{\partial x} = \lambda x \Rightarrow f(x,y,z) = \frac{1}{2}\lambda x^2 + g(y,z)$ for some function g $\Rightarrow \lambda y = \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y}$
$\Rightarrow g(y,z) = \frac{1}{2}\lambda y^2 + h(z)$ for some function h $\Rightarrow \lambda z = \frac{\partial f}{\partial z} = \frac{\partial g}{\partial z} = h'(z) \Rightarrow h(z) = \frac{1}{2}\lambda z^2 + C$ for some arbitrary constant C $\Rightarrow g(y,z) = \frac{1}{2}\lambda y^2 + \left(\frac{1}{2}\lambda z^2 + C\right) \Rightarrow f(x,y,z) = \frac{1}{2}\lambda x^2 + \frac{1}{2}\lambda y^2 + \frac{1}{2}\lambda z^2 + C \Rightarrow f(0,0,a) = \frac{1}{2}\lambda a^2 + C$ and $f(0,0,-a) = \frac{1}{2}\lambda(-a)^2 + C \Rightarrow f(0,0,a) = f(0,0,-a)$ for any constant a, as claimed.

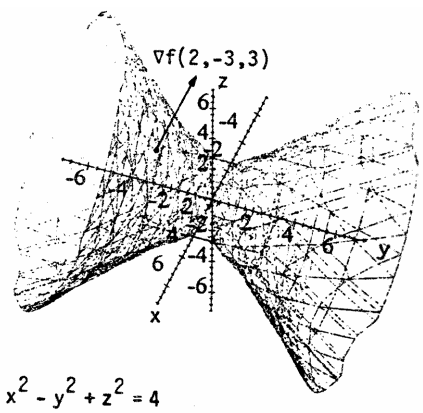
100. $\left(\frac{df}{ds}\right)_{\mathbf{u},(0,0,0)} = \lim_{s \to 0} \frac{f(0+su_1, 0+su_2, 0+su_3) - f(0,0,0)}{s}, s > 0$
$= \lim_{s \to 0} \frac{\sqrt{s^2u_1^2 + s^2u_2^2 + s^2u_3^2} - 0}{s}, s > 0$

$= \lim\limits_{s \to 0} \frac{s\sqrt{u_1^2 + u_2^2 + u_3^2}}{s} = \lim\limits_{s \to 0} |\mathbf{u}| = 1;$

however, $\nabla f = \frac{x}{\sqrt{x^2+y^2+z^2}}\,\mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\,\mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k}$ fails to exist at the origin $(0, 0, 0)$

101. Let $f(x, y, z) = xy + z - 2 \Rightarrow \nabla f = y\mathbf{i} + x\mathbf{j} + \mathbf{k}$. At $(1, 1, 1)$, we have $\nabla f = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow$ the normal line is $x = 1 + t$, $y = 1 + t$, $z = 1 + t$, so at $t = -1 \Rightarrow x = 0$, $y = 0$, $z = 0$ and the normal line passes through the origin.

102. (b) $f(x, y, z) = x^2 - y^2 + z^2 = 4$
$\Rightarrow \nabla f = 2x\mathbf{i} - 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow$ at $(2, -3, 3)$
the gradient is $\nabla f = 4\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}$ which is normal to the surface
(c) Tangent plane: $4x + 6y + 6z = 8$ or
$2x + 3y + 3z = 4$
Normal line: $x = 2 + 4t$, $y = -3 + 6t$, $z = 3 + 6t$



$x^2 - y^2 + z^2 = 4$

CHAPTER 14 ADDITIONAL AND ADVANCED EXERCISES

1. By definition, $f_{xy}(0, 0) = \lim\limits_{h \to 0} \frac{f_x(0,h) - f_x(0,0)}{h}$ so we need to calculate the first partial derivatives in the numerator. For $(x, y) \neq (0, 0)$ we calculate $f_x(x, y)$ by applying the differentiation rules to the formula for $f(x, y)$: $f_x(x, y) = \frac{x^2y - y^3}{x^2+y^2} + (xy)\frac{(x^2+y^2)(2x) - (x^2-y^2)(2x)}{(x^2+y^2)^2} = \frac{x^2y - y^3}{x^2+y^2} + \frac{4x^2y^3}{(x^2+y^2)^2} \Rightarrow f_x(0, h) = -\frac{h^3}{h^2} = -h$.
For $(x, y) = (0, 0)$ we apply the definition: $f_x(0, 0) = \lim\limits_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim\limits_{h \to 0} \frac{0-0}{h} = 0$. Then by definition $f_{xy}(0, 0) = \lim\limits_{h \to 0} \frac{-h-0}{h} = -1$. Similarly, $f_{yx}(0, 0) = \lim\limits_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$, so for $(x, y) \neq (0, 0)$ we have $f_y(x, y) = \frac{x^3 - xy^2}{x^2+y^2} - \frac{4x^3y^2}{(x^2+y^2)^2} \Rightarrow f_y(h, 0) = \frac{h^3}{h^2} = h$; for $(x, y) = (0, 0)$ we obtain $f_y(0, 0) = \lim\limits_{h \to 0} \frac{f(0,h) - f(0,0)}{h}$ $= \lim\limits_{h \to 0} \frac{0-0}{h} = 0$. Then by definition $f_{yx}(0, 0) = \lim\limits_{h \to 0} \frac{h-0}{h} = 1$. Note that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$ in this case.

2. $\frac{\partial w}{\partial x} = 1 + e^x \cos y \Rightarrow w = x + e^x \cos y + g(y)$; $\frac{\partial w}{\partial y} = -e^x \sin y + g'(y) = 2y - e^x \sin y \Rightarrow g'(y) = 2y$ $\Rightarrow g(y) = y^2 + C$; $w = \ln 2$ when $x = \ln 2$ and $y = 0 \Rightarrow \ln 2 = \ln 2 + e^{\ln 2}\cos 0 + 0^2 + C \Rightarrow 0 = 2 + C$ $\Rightarrow C = -2$. Thus, $w = x + e^x \cos y + g(y) = x + e^x \cos y + y^2 - 2$.

3. Substitution of $u + u(x)$ and $v = v(x)$ in $g(u, v)$ gives $g(u(x), v(x))$ which is a function of the independent variable x. Then, $g(u, v) = \int_u^v f(t)\,dt \Rightarrow \frac{dg}{dx} = \frac{\partial g}{\partial u}\frac{du}{dx} + \frac{\partial g}{\partial v}\frac{dv}{dx} = \left(\frac{\partial}{\partial u}\int_u^v f(t)\,dt\right)\frac{du}{dx} + \left(\frac{\partial}{\partial v}\int_u^v f(t)\,dt\right)\frac{dv}{dx}$ $= \left(-\frac{\partial}{\partial u}\int_v^u f(t)\,dt\right)\frac{du}{dx} + \left(\frac{\partial}{\partial v}\int_u^v f(t)\,dt\right)\frac{dv}{dx} = -f(u(x))\frac{du}{dx} + f(v(x))\frac{dv}{dx} = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}$

4. Applying the chain rules, $f_x = \frac{df}{dr}\frac{\partial r}{\partial x} \Rightarrow f_{xx} = \left(\frac{d^2f}{dr^2}\right)\left(\frac{\partial r}{\partial x}\right)^2 + \frac{df}{dr}\frac{\partial^2 r}{\partial x^2}$. Similarly, $f_{yy} = \left(\frac{d^2f}{dr^2}\right)\left(\frac{\partial r}{\partial y}\right)^2 + \frac{df}{dr}\frac{\partial^2 r}{\partial y^2}$ and $f_{zz} = \left(\frac{d^2f}{dr^2}\right)\left(\frac{\partial r}{\partial z}\right)^2 + \frac{df}{dr}\frac{\partial^2 r}{\partial z^2}$. Moreover, $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} \Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{y^2+z^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$; $\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}$ $\Rightarrow \frac{\partial^2 r}{\partial y^2} = \frac{x^2+z^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$; and $\frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}} \Rightarrow \frac{\partial^2 r}{\partial z^2} = \frac{x^2+y^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$. Next, $f_{xx} + f_{yy} + f_{zz} = 0$
$\Rightarrow \left(\frac{d^2f}{dr^2}\right)\left(\frac{x^2}{x^2+y^2+z^2}\right) + \left(\frac{df}{dr}\right)\left(\frac{y^2+z^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right) + \left(\frac{d^2f}{dr^2}\right)\left(\frac{y^2}{x^2+y^2+z^2}\right) + \left(\frac{df}{dr}\right)\left(\frac{x^2+z^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right)$

$+\left(\frac{d^2f}{dr^2}\right)\left(\frac{z^2}{x^2+y^2+z^2}\right)+\left(\frac{df}{dr}\right)\left(\frac{x^2+y^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right)=0 \Rightarrow \frac{d^2f}{dr^2}+\left(\frac{2}{\sqrt{x^2+y^2+z^2}}\right)\frac{df}{dr}=0 \Rightarrow \frac{d^2f}{dr^2}+\frac{2}{r}\frac{df}{dr}=0$

$\Rightarrow \frac{d}{dr}(f')=\left(-\frac{2}{r}\right)f'$, where $f'=\frac{df}{dr} \Rightarrow \frac{df'}{f'}=-\frac{2\,dr}{r} \Rightarrow \ln f'=-2\ln r+\ln C \Rightarrow f'=Cr^{-2}$, or $\frac{df}{dr}=Cr^{-2} \Rightarrow f(r)=-\frac{C}{r}+b=\frac{a}{r}+b$ for some constants a and b (setting $a=-C$)

5. (a) Let $u=tx$, $v=ty$, and $w=f(u,v)=f(u(t,x),v(t,y))=f(tx,ty)=t^nf(x,y)$, where t, x, and y are independent variables. Then $nt^{n-1}f(x,y)=\frac{\partial w}{\partial t}=\frac{\partial w}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial t}=x\frac{\partial w}{\partial u}+y\frac{\partial w}{\partial v}$. Now, $\frac{\partial w}{\partial x}=\frac{\partial w}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial x}=\left(\frac{\partial w}{\partial u}\right)(t)+\left(\frac{\partial w}{\partial v}\right)(0)=t\frac{\partial w}{\partial u} \Rightarrow \frac{\partial w}{\partial u}=\left(\frac{1}{t}\right)\left(\frac{\partial w}{\partial x}\right)$. Likewise, $\frac{\partial w}{\partial y}=\frac{\partial w}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial y}=\left(\frac{\partial w}{\partial u}\right)(0)+\left(\frac{\partial w}{\partial v}\right)(t) \Rightarrow \frac{\partial w}{\partial v}=\left(\frac{1}{t}\right)\left(\frac{\partial w}{\partial y}\right)$. Therefore, $nt^{n-1}f(x,y)=x\frac{\partial w}{\partial u}+y\frac{\partial w}{\partial v}=\left(\frac{x}{t}\right)\left(\frac{\partial w}{\partial x}\right)+\left(\frac{y}{t}\right)\left(\frac{\partial w}{\partial y}\right)$. When $t=1$, $u=x$, $v=y$, and $w=f(x,y)$ $\Rightarrow \frac{\partial w}{\partial x}=\frac{\partial f}{\partial x}$ and $\frac{\partial w}{\partial y}=\frac{\partial f}{\partial x} \Rightarrow nf(x,y)=x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}$, as claimed.

(b) From part (a), $nt^{n-1}f(x,y)=x\frac{\partial w}{\partial u}+y\frac{\partial w}{\partial v}$. Differentiating with respect to t again we obtain $n(n-1)t^{n-2}f(x,y)=x\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial t}+x\frac{\partial^2 w}{\partial v\partial u}\frac{\partial v}{\partial t}+y\frac{\partial^2 w}{\partial u\partial v}\frac{\partial u}{\partial t}+y\frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial t}=x^2\frac{\partial^2 w}{\partial u^2}+2xy\frac{\partial^2 w}{\partial u\partial v}+y^2\frac{\partial^2 w}{\partial v^2}$.
Also from part (a), $\frac{\partial^2 w}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)=\frac{\partial}{\partial x}\left(t\frac{\partial w}{\partial u}\right)=t\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial x}+t\frac{\partial^2 w}{\partial v\partial u}\frac{\partial v}{\partial x}=t^2\frac{\partial^2 w}{\partial u^2}$, $\frac{\partial^2 w}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)$ $=\frac{\partial}{\partial y}\left(t\frac{\partial w}{\partial v}\right)=t\frac{\partial^2 w}{\partial u\partial v}\frac{\partial u}{\partial y}+t\frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial y}=t^2\frac{\partial^2 w}{\partial v^2}$, and $\frac{\partial^2 w}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)=\frac{\partial}{\partial y}\left(t\frac{\partial w}{\partial u}\right)=t\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial y}+t\frac{\partial^2 w}{\partial v\partial u}\frac{\partial v}{\partial y}$ $=t^2\frac{\partial^2 w}{\partial v\partial u} \Rightarrow \left(\frac{1}{t^2}\right)\frac{\partial^2 w}{\partial x^2}=\frac{\partial^2 w}{\partial u^2}$, $\left(\frac{1}{t^2}\right)\frac{\partial^2 w}{\partial y^2}=\frac{\partial^2 w}{\partial v^2}$, and $\left(\frac{1}{t^2}\right)\frac{\partial^2 w}{\partial y\partial x}=\frac{\partial^2 w}{\partial v\partial u}$
$\Rightarrow n(n-1)t^{n-2}f(x,y)=\left(\frac{x^2}{t^2}\right)\left(\frac{\partial^2 w}{\partial x^2}\right)+\left(\frac{2xy}{t^2}\right)\left(\frac{\partial^2 w}{\partial y\partial x}\right)+\left(\frac{y^2}{t^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right)$ for $t\neq 0$. When $t=1$, $w=f(x,y)$ and we have $n(n-1)f(x,y)=x^2\left(\frac{\partial^2 f}{\partial x^2}\right)+2xy\left(\frac{\partial^2 f}{\partial x\partial y}\right)+y^2\left(\frac{\partial^2 f}{\partial y^2}\right)$ as claimed.

6. (a) $\lim_{r\to 0}\frac{\sin 6r}{6r}=\lim_{t\to 0}\frac{\sin t}{t}=1$, where $t=6r$

(b) $f_r(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{\left(\frac{\sin 6h}{6h}\right)-1}{h}=\lim_{h\to 0}\frac{\sin 6h-6h}{6h^2}=\lim_{h\to 0}\frac{6\cos 6h-6}{12h}$
$=\lim_{h\to 0}\frac{-36\sin 6h}{12}=0$ (applying l'Hôpital's rule twice)

(c) $f_\theta(r,\theta)=\lim_{h\to 0}\frac{f(r,\theta+h)-f(r,\theta)}{h}=\lim_{h\to 0}\frac{\left(\frac{\sin 6r}{6r}\right)-\left(\frac{\sin 6r}{6r}\right)}{h}=\lim_{h\to 0}\frac{0}{h}=0$

7. (a) $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k} \Rightarrow r=|\mathbf{r}|=\sqrt{x^2+y^2+z^2}$ and $\nabla r=\frac{x}{\sqrt{x^2+y^2+z^2}}\mathbf{i}+\frac{y}{\sqrt{x^2+y^2+z^2}}\mathbf{j}+\frac{z}{\sqrt{x^2+y^2+z^2}}\mathbf{k}$ $=\frac{\mathbf{r}}{r}$

(b) $r^n=\left(\sqrt{x^2+y^2+z^2}\right)^n$
$\Rightarrow \nabla(r^n)=nx\left(x^2+y^2+z^2\right)^{(n/2)-1}\mathbf{i}+ny\left(x^2+y^2+z^2\right)^{(n/2)-1}\mathbf{j}+nz\left(x^2+y^2+z^2\right)^{(n/2)-1}\mathbf{k}$
$=nr^{n-2}\mathbf{r}$

(c) Let $n=2$ in part (b). Then $\frac{1}{2}\nabla(r^2)=\mathbf{r} \Rightarrow \nabla\left(\frac{1}{2}r^2\right)=\mathbf{r} \Rightarrow \frac{r^2}{2}=\frac{1}{2}(x^2+y^2+z^2)$ is the function.

(d) $d\mathbf{r}=dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k} \Rightarrow \mathbf{r}\cdot d\mathbf{r}=x\,dx+y\,dy+z\,dz$, and $dr=r_x\,dx+r_y\,dy+r_z\,dz=\frac{x}{r}dx+\frac{y}{r}dy+\frac{z}{r}dz$
$\Rightarrow r\,dr=x\,dx+y\,dy+z\,dz=\mathbf{r}\cdot d\mathbf{r}$

(e) $\mathbf{A}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k} \Rightarrow \mathbf{A}\cdot\mathbf{r}=ax+by+cz \Rightarrow \nabla(\mathbf{A}\cdot\mathbf{r})=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}=\mathbf{A}$

8. $f(g(t),h(t))=c \Rightarrow 0=\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}=\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}\right)\cdot\left(\frac{dx}{dt}\mathbf{i}+\frac{dy}{dt}\mathbf{j}\right)$, where $\frac{dx}{dt}\mathbf{i}+\frac{dy}{dt}\mathbf{j}$ is the tangent vector $\Rightarrow \nabla f$ is orthogonal to the tangent vector

9. $f(x,y,z)=xz^2-yz+\cos xy-1 \Rightarrow \nabla f=(z^2-y\sin xy)\mathbf{i}+(-z-x\sin xy)\mathbf{j}+(2xz-y)\mathbf{k} \Rightarrow \nabla f(0,0,1)=\mathbf{i}-\mathbf{j}$ $\Rightarrow$ the tangent plane is $x-y=0$; $\mathbf{r}=(\ln t)\mathbf{i}+(t\ln t)\mathbf{j}+t\mathbf{k} \Rightarrow \mathbf{r}'=\left(\frac{1}{t}\right)\mathbf{i}+(\ln t+1)\mathbf{j}+\mathbf{k}$; $x=y=0$, $z=1$ $\Rightarrow t=1 \Rightarrow \mathbf{r}'(1)=\mathbf{i}+\mathbf{j}+\mathbf{k}$. Since $(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}-\mathbf{j})=\mathbf{r}'(1)\cdot\nabla f=0$, $\mathbf{r}$ is parallel to the plane, and $\mathbf{r}(1)=0\mathbf{i}+0\mathbf{j}+\mathbf{k} \Rightarrow \mathbf{r}$ is contained in the plane.

10. Let $f(x, y, z) = x^3 + y^3 + z^3 - xyz \Rightarrow \nabla f = (3x^2 - yz)\,\mathbf{i} + (3y^2 - xz)\,\mathbf{j} + (3z^2 - xy)\,\mathbf{k} \Rightarrow \nabla f(0, -1, 1) = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ $\Rightarrow$ the tangent plane is $x + 3y + 3z = 0$; $\mathbf{r} = \left(\frac{t^3}{4} - 2\right)\mathbf{i} + \left(\frac{4}{t} - 3\right)\mathbf{j} + (\cos(t - 2))\,\mathbf{k}$ $\Rightarrow \mathbf{r}' = \left(\frac{3t^2}{4}\right)\mathbf{i} - \left(\frac{4}{t^2}\right)\mathbf{j} - (\sin(t - 2))\,\mathbf{k}$; $x = 0, y = -1, z = 1 \Rightarrow t = 2 \Rightarrow \mathbf{r}'(2) = 3\mathbf{i} - \mathbf{j}$. Since $\mathbf{r}'(2) \cdot \nabla f = 0 \Rightarrow \mathbf{r}$ is parallel to the plane, and $\mathbf{r}(2) = -\mathbf{i} + \mathbf{k} \Rightarrow \mathbf{r}$ is contained in the plane.

11. $\frac{\partial z}{\partial x} = 3x^2 - 9y = 0$ and $\frac{\partial z}{\partial y} = 3y^2 - 9x = 0 \Rightarrow y = \frac{1}{3}x^2$ and $3\left(\frac{1}{3}x^2\right)^2 - 9x = 0 \Rightarrow \frac{1}{3}x^4 - 9x = 0$ $\Rightarrow x(x^3 - 27) = 0 \Rightarrow x = 0$ or $x = 3$. Now $x = 0 \Rightarrow y = 0$ or $(0, 0)$ and $x = 3 \Rightarrow y = 3$ or $(3, 3)$. Next $\frac{\partial^2 z}{\partial x^2} = 6x$, $\frac{\partial^2 z}{\partial y^2} = 6y$, and $\frac{\partial^2 z}{\partial x \partial y} = -9$. For $(0, 0)$, $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = -81 \Rightarrow$ no extremum (a saddle point), and for $(3, 3)$, $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 243 > 0$ and $\frac{\partial^2 z}{\partial x^2} = 18 > 0 \Rightarrow$ a local minimum.

12. $f(x, y) = 6xye^{-(2x+3y)} \Rightarrow f_x(x, y) = 6y(1 - 2x)e^{-(2x+3y)} = 0$ and $f_y(x, y) = 6x(1 - 3y)e^{-(2x+3y)} = 0 \Rightarrow x = 0$ and $y = 0$, or $x = \frac{1}{2}$ and $y = \frac{1}{3}$. The value $f(0, 0) = 0$ is on the boundary, and $f\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{e^2}$. On the positive y-axis, $f(0, y) = 0$, and on the positive x-axis, $f(x, 0) = 0$. As $x \to \infty$ or $y \to \infty$ we see that $f(x, y) \to 0$. Thus the absolute maximum of f in the closed first quadrant is $\frac{1}{e^2}$ at the point $\left(\frac{1}{2}, \frac{1}{3}\right)$.

13. Let $f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \Rightarrow \nabla f = \frac{2x}{a^2}\mathbf{i} + \frac{2y}{b^2}\mathbf{j} + \frac{2z}{c^2}\mathbf{k} \Rightarrow$ an equation of the plane tangent at the point $P_0(x_0, y_0, y_0)$ is $\left(\frac{2x_0}{a^2}\right)x + \left(\frac{2y_0}{b^2}\right)y + \left(\frac{2z_0}{c^2}\right)z = \frac{2x_0^2}{a^2} + \frac{2y_0^2}{b^2} + \frac{2z_0^2}{c^2} = 2$ or $\left(\frac{x_0}{a^2}\right)x + \left(\frac{y_0}{b^2}\right)y + \left(\frac{z_0}{c^2}\right)z = 1$. The intercepts of the plane are $\left(\frac{a^2}{x_0}, 0, 0\right)$, $\left(0, \frac{b^2}{y_0}, 0\right)$ and $\left(0, 0, \frac{c^2}{z_0}\right)$. The volume of the tetrahedron formed by the plane and the coordinate planes is $V = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{a^2}{x_0}\right)\left(\frac{b^2}{y_0}\right)\left(\frac{c^2}{z_0}\right) \Rightarrow$ we need to maximize $V(x, y, z) = \frac{(abc)^2}{6}(xyz)^{-1}$ subject to the constraint $f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Thus, $\left[-\frac{(abc)^2}{6}\right]\left(\frac{1}{x^2yz}\right) = \frac{2x}{a^2}\lambda$, $\left[-\frac{(abc)^2}{6}\right]\left(\frac{1}{xy^2z}\right) = \frac{2y}{b^2}\lambda$, and $\left[-\frac{(abc)^2}{6}\right]\left(\frac{1}{xyz^2}\right) = \frac{2z}{c^2}\lambda$. Multiply the first equation by a^2yz, the second by b^2xz, and the third by c^2xy. Then equate the first and second $\Rightarrow a^2y^2 = b^2x^2$ $\Rightarrow y = \frac{b}{a}x$, $x > 0$; equate the first and third $\Rightarrow a^2z^2 = c^2x^2 \Rightarrow z = \frac{c}{a}x$, $x > 0$; substitute into $f(x, y, z) = 0$ $\Rightarrow x = \frac{a}{\sqrt{3}} \Rightarrow y = \frac{b}{\sqrt{3}} \Rightarrow z = \frac{c}{\sqrt{3}} \Rightarrow V = \frac{\sqrt{3}}{2}abc$.

14. $2(x - u) = -\lambda$, $2(y - v) = \lambda$, $-2(x - u) = \mu$, and $-2(y - v) = -2\mu v \Rightarrow x - u = v - y$, $x - u = -\frac{\mu}{2}$, and $y - v = \mu v \Rightarrow x - u = -\mu v = -\frac{\mu}{2} \Rightarrow v = \frac{1}{2}$ or $\mu = 0$.

CASE 1: $\mu = 0 \Rightarrow x = u$, $y = v$, and $\lambda = 0$; then $y = x + 1 \Rightarrow v = u + 1$ and $v^2 = u \Rightarrow v = v^2 + 1$ $\Rightarrow v^2 - v + 1 = 0 \Rightarrow v = \frac{1 \pm \sqrt{1-4}}{2} \Rightarrow$ no real solution.

CASE 2: $v = \frac{1}{2}$ and $u = v^2 \Rightarrow u = \frac{1}{4}$; $x - \frac{1}{4} = \frac{1}{2} - y$ and $y = x + 1 \Rightarrow x - \frac{1}{4} = -x - \frac{1}{2} \Rightarrow 2x = -\frac{1}{4}$ $\Rightarrow x = -\frac{1}{8} \Rightarrow y = \frac{7}{8}$. Then $f\left(-\frac{1}{8}, \frac{7}{8}, \frac{1}{4}, \frac{1}{2}\right) = \left(-\frac{1}{8} - \frac{1}{4}\right)^2 + \left(\frac{7}{8} - \frac{1}{2}\right)^2 = 2\left(\frac{3}{8}\right)^2 \Rightarrow$ the minimum distance is $\frac{3}{8}\sqrt{2}$. (Notice that f has no maximum value.)

15. Let (x_0, y_0) be any point in R. We must show $\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0)$ or, equivalently that $\lim_{(h, k) \to (0, 0)} |f(x_0 + h, y_0 + k) - f(x_0, y_0)| = 0$. Consider $f(x_0 + h, y_0 + k) - f(x_0, y_0)$ $= [f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)] + [f(x_0, y_0 + k) - f(x_0, y_0)]$. Let $F(x) = f(x, y_0 + k)$ and apply the Mean Value Theorem: there exists ξ with $x_0 < \xi < x_0 + h$ such that $F'(\xi)h = F(x_0 + h) - F(x_0) \Rightarrow hf_x(\xi, y_0 + k)$ $= f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)$. Similarly, $k\,f_y(x_0, \eta) = f(x_0, y_0 + k) - f(x_0, y_0)$ for some η with $y_0 < \eta < y_0 + k$. Then $|f(x_0 + h, y_0 + k) - f(x_0, y_0)| \le |hf_x(\xi, y_0 + k)| + |kf_y(x_0, \eta)|$. If M, N are positive real numbers such that $|f_x| \le M$ and $|f_y| \le N$ for all (x, y) in the xy-plane, then $|f(x_0 + h, y_0 + k) - f(x_0, y_0)|$ $\le M|h| + N|k|$. As $(h, k) \to 0$, $|f(x_0 + h, y_0 + k) - f(x_0, y_0)| \to 0 \Rightarrow \lim_{(h, k) \to (0, 0)} |f(x_0 + h, y_0 + k) - f(x_0, y_0)|$

$= 0 \Rightarrow$ f is continuous at (x_0, y_0).

16. At extreme values, ∇f and $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ are orthogonal because $\frac{df}{dt} = \nabla f \cdot \frac{d\mathbf{r}}{dt} = 0$ by the First Derivative Theorem for Local Extreme Values.

17. $\frac{\partial f}{\partial x} = 0 \Rightarrow f(x,y) = h(y)$ is a function of y only. Also, $\frac{\partial g}{\partial y} = \frac{\partial f}{\partial x} = 0 \Rightarrow g(x,y) = k(x)$ is a function of x only. Moreover, $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \Rightarrow h'(y) = k'(x)$ for all x and y. This can happen only if $h'(y) = k'(x) = c$ is a constant. Integration gives $h(y) = cy + c_1$ and $k(x) = cx + c_2$, where c_1 and c_2 are constants. Therefore $f(x,y) = cy + c_1$ and $g(x,y) = cx + c_2$. Then $f(1,2) = g(1,2) = 5 \Rightarrow 5 = 2c + c_1 = c + c_2$, and $f(0,0) = 4 \Rightarrow c_1 = 4 \Rightarrow c = \frac{1}{2}$ $\Rightarrow c_2 = \frac{9}{2}$. Thus, $f(x,y) = \frac{1}{2}y + 4$ and $g(x,y) = \frac{1}{2}x + \frac{9}{2}$.

18. Let $g(x,y) = D_{\mathbf{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b$. Then $D_{\mathbf{u}}g(x,y) = g_x(x,y)a + g_y(x,y)b$ $= f_{xx}(x,y)a^2 + f_{yx}(x,y)ab + f_{xy}(x,y)ba + f_{yy}(x,y)b^2 = f_{xx}(x,y)a^2 + 2f_{xy}(x,y)ab + f_{yy}(x,y)b^2$.

19. Since the particle is heat-seeking, at each point (x, y) it moves in the direction of maximal temperature increase, that is in the direction of $\nabla T(x,y) = (e^{-2y}\sin x)\,\mathbf{i} + (2e^{-2y}\cos x)\,\mathbf{j}$. Since $\nabla T(x,y)$ is parallel to the particle's velocity vector, it is tangent to the path $y = f(x)$ of the particle $\Rightarrow f'(x) = \frac{2e^{-2y}\cos x}{e^{-2y}\sin x} = 2\cot x$. Integration gives $f(x) = 2\ln|\sin x| + C$ and $f\left(\frac{\pi}{4}\right) = 0 \Rightarrow 0 = 2\ln\left|\sin\frac{\pi}{4}\right| + C \Rightarrow C = -2\ln\frac{\sqrt{2}}{2} = \ln\left(\frac{2}{\sqrt{2}}\right)^2$ $= \ln 2$. Therefore, the path of the particle is the graph of $y = 2\ln|\sin x| + \ln 2$.

20. The line of travel is $x = t$, $y = t$, $z = 30 - 5t$, and the bullet hits the surface $z = 2x^2 + 3y^2$ when $30 - 5t = 2t^2 + 3t^2 \Rightarrow t^2 + t - 6 = 0 \Rightarrow (t+3)(t-2) = 0 \Rightarrow t = 2$ (since $t > 0$). Thus the bullet hits the surface at the point $(2, 2, 20)$. Now, the vector $4x\mathbf{i} + 6y\mathbf{j} - \mathbf{k}$ is normal to the surface at any (x, y, z), so that $\mathbf{n} = 8\mathbf{i} + 12\mathbf{j} - \mathbf{k}$ is normal to the surface at $(2, 2, 20)$. If $\mathbf{v} = \mathbf{i} + \mathbf{j} - 5\mathbf{k}$, then the velocity of the particle after the ricochet is $\mathbf{w} = \mathbf{v} - 2\,\mathrm{proj}_{\mathbf{n}}\,\mathbf{v} = \mathbf{v} - \left(\frac{2\mathbf{v}\cdot\mathbf{n}}{|\mathbf{n}|^2}\right)\mathbf{n} = \mathbf{v} - \left(\frac{2\cdot 25}{209}\right)\mathbf{n} = (\mathbf{i} + \mathbf{j} - 5\mathbf{k}) - \left(\frac{400}{209}\mathbf{i} + \frac{600}{209}\mathbf{j} - \frac{50}{209}\mathbf{k}\right)$ $= -\frac{191}{209}\mathbf{i} - \frac{391}{209}\mathbf{j} - \frac{995}{209}\mathbf{k}$.

21. (a) $\mathbf{k}$ is a vector normal to $z = 10 - x^2 - y^2$ at the point $(0, 0, 10)$. So directions tangential to S at $(0, 0, 10)$ will be unit vectors $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$. Also, $\nabla T(x,y,z) = (2xy + 4)\,\mathbf{i} + (x^2 + 2yz + 14)\,\mathbf{j} + (y^2 + 1)\,\mathbf{k}$ $\Rightarrow \nabla T(0,0,10) = 4\mathbf{i} + 14\mathbf{j} + \mathbf{k}$. We seek the unit vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ such that $D_{\mathbf{u}}T(0,0,10)$ $= (4\mathbf{i} + 14\mathbf{j} + \mathbf{k})\cdot(a\mathbf{i} + b\mathbf{j}) = (4\mathbf{i} + 14\mathbf{j})\cdot(a\mathbf{i} + b\mathbf{j})$ is a maximum. The maximum will occur when $a\mathbf{i} + b\mathbf{j}$ has the same direction as $4\mathbf{i} + 14\mathbf{j}$, or $\mathbf{u} = \frac{1}{\sqrt{53}}(2\mathbf{i} + 7\mathbf{j})$.

    (b) A vector normal to S at $(1, 1, 8)$ is $\mathbf{n} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Now, $\nabla T(1,1,8) = 6\mathbf{i} + 31\mathbf{j} + 2\mathbf{k}$ and we seek the unit vector $\mathbf{u}$ such that $D_{\mathbf{u}}T(1,1,8) = \nabla T\cdot\mathbf{u}$ has its largest value. Now write $\nabla T = \mathbf{v} + \mathbf{w}$, where $\mathbf{v}$ is parallel to ∇T and $\mathbf{w}$ is orthogonal to ∇T. Then $D_{\mathbf{u}}T = \nabla T\cdot\mathbf{u} = (\mathbf{v} + \mathbf{w})\cdot\mathbf{u} = \mathbf{v}\cdot\mathbf{u} + \mathbf{w}\cdot\mathbf{u} = \mathbf{w}\cdot\mathbf{u}$. Thus $D_{\mathbf{u}}T(1,1,8)$ is a maximum when $\mathbf{u}$ has the same direction as $\mathbf{w}$. Now, $\mathbf{w} = \nabla T - \left(\frac{\nabla T\cdot\mathbf{n}}{|\mathbf{n}|^2}\right)\mathbf{n}$ $= (6\mathbf{i} + 31\mathbf{j} + 2\mathbf{k}) - \left(\frac{12 + 62 + 2}{4 + 4 + 1}\right)(2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = \left(6 - \frac{152}{9}\right)\mathbf{i} + \left(31 - \frac{152}{9}\right)\mathbf{j} + \left(2 - \frac{76}{9}\right)\mathbf{k}$ $= -\frac{98}{9}\mathbf{i} + \frac{127}{9}\mathbf{j} - \frac{58}{9}\mathbf{k} \Rightarrow \mathbf{u} = \frac{\mathbf{w}}{|\mathbf{w}|} = -\frac{1}{\sqrt{29{,}097}}(98\mathbf{i} - 127\mathbf{j} + 58\mathbf{k})$.

22. Suppose the surface (boundary) of the mineral deposit is the graph of $z = f(x,y)$ (where the z-axis points up into the air). Then $-\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$ is an outer normal to the mineral deposit at (x, y) and $\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$ points in the direction of steepest ascent of the mineral deposit. This is in the direction of the vector $\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$ at $(0, 0)$ (the location of the 1st borehole) that the geologists should drill their fourth borehole. To approximate this vector we use the fact that $(0, 0, -1000)$, $(0, 100, -950)$, and $(100, 0, -1025)$ lie on the graph of $z = f(x,y)$. The plane containing these three points is a good approximation to the tangent plane to $z = f(x,y)$ at the point

$(0,0,0)$. A normal to this plane is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 100 & 50 \\ 100 & 0 & -25 \end{vmatrix} = -2500\mathbf{i} + 5000\mathbf{j} - 10{,}000\mathbf{k}$, or $-\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$. So at $(0,0)$ the vector $\frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}$ is approximately $-\mathbf{i} + 2\mathbf{j}$. Thus the geologists should drill their fourth borehole in the direction of $\frac{1}{\sqrt{5}}\,(-\mathbf{i} + 2\mathbf{j})$ from the first borehole.

23. $w = e^{rt} \sin \pi x \Rightarrow w_t = re^{rt} \sin \pi x$ and $w_x = \pi e^{rt} \cos \pi x \Rightarrow w_{xx} = -\pi^2 e^{rt} \sin \pi x$; $w_{xx} = \frac{1}{c^2}\, w_t$, where c^2 is the positive constant determined by the material of the rod $\Rightarrow -\pi^2 e^{rt} \sin \pi x = \frac{1}{c^2}\,(re^{rt} \sin \pi x)$
$\Rightarrow (r + c^2\pi^2)\, e^{rt} \sin \pi x = 0 \Rightarrow r = -c^2\pi^2 \Rightarrow w = e^{-c^2\pi^2 t} \sin \pi x$

24. $w = e^{rt} \sin kx \Rightarrow w_t = re^{rt} \sin kx$ and $w_x = ke^{rt} \cos kx \Rightarrow w_{xx} = -k^2 e^{rt} \sin kx$; $w_{xx} = \frac{1}{c^2}\, w_t$
$\Rightarrow -k^2 e^{rt} \sin kx = \frac{1}{c^2}\,(re^{rt} \sin kx) \Rightarrow (r + c^2k^2)\, e^{rt} \sin kx = 0 \Rightarrow r = -c^2k^2 \Rightarrow w = e^{-c^2k^2 t} \sin kx$.
Now, $w(L,t) = 0 \Rightarrow e^{-c^2k^2 t} \sin kL = 0 \Rightarrow kL = n\pi$ for n an integer $\Rightarrow k = \frac{n\pi}{L} \Rightarrow w = e^{-c^2n^2\pi^2 t/L^2} \sin\left(\frac{n\pi}{L}\, x\right)$.
As $t \to \infty$, $w \to 0$ since $\left|\sin\left(\frac{n\pi}{L}\, x\right)\right| \le 1$ and $e^{-c^2n^2\pi^2 t/L^2} \to 0$.