CHAPTER 13 VECTOR-VALUED FUNCTIONS AND MOTION IN SPACE

13.1 VECTOR FUNCTIONS

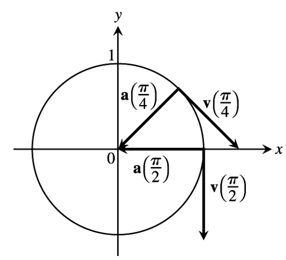
1. $x = t + 1$ and $y = t^2 - 1 \Rightarrow y = (x-1)^2 - 1 = x^2 - 2x$; $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{a} = \frac{d\mathbf{v}}{dt} = 2\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{a} = 2\mathbf{j}$ at $t = 1$

2. $x = t^2 + 1$ and $y = 2t - 1 \Rightarrow x = \left(\frac{y+1}{2}\right)^2 + 1 \Rightarrow x = \frac{1}{4}(y+1)^2 + 1$; $\mathbf{v} = \frac{d\mathbf{r}}{dt} = 2t\mathbf{i} + 2\mathbf{j} \Rightarrow \mathbf{a} = \frac{d\mathbf{v}}{dt} = 2\mathbf{i}$ $\Rightarrow \mathbf{v} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{a} = 2\mathbf{i}$ at $t = \frac{1}{2}$

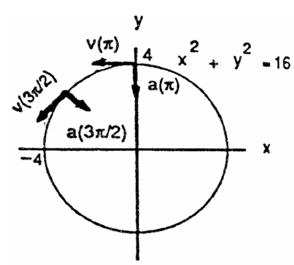
3. $x = e^t$ and $y = \frac{2}{9}e^{2t} \Rightarrow y = \frac{2}{9}x^2$; $\mathbf{v} = \frac{d\mathbf{r}}{dt} = e^t\mathbf{i} + \frac{4}{9}e^{2t}\mathbf{j} \Rightarrow \mathbf{a} = e^t\mathbf{i} + \frac{8}{9}e^{2t}\mathbf{j} \Rightarrow \mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{a} = 3\mathbf{i} + 8\mathbf{j}$ at $t = \ln 3$

4. $x = \cos 2t$ and $y = 3\sin 2t \Rightarrow x^2 + \frac{1}{9}y^2 = 1$; $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-2\sin 2t)\mathbf{i} + (6\cos 2t)\mathbf{j} \Rightarrow \mathbf{a} = \frac{d\mathbf{v}}{dt}$ $= (-4\cos 2t)\mathbf{i} + (-12\sin 2t)\mathbf{j} \Rightarrow \mathbf{v} = 6\mathbf{j}$ and $\mathbf{a} = -4\mathbf{i}$ at $t = 0$

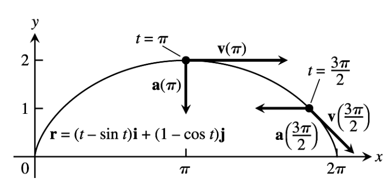
5. $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$ and $\mathbf{a} = \frac{d\mathbf{v}}{dt} = -(\sin t)\mathbf{i} - (\cos t)\mathbf{j}$ $\Rightarrow$ for $t = \frac{\pi}{4}$, $\mathbf{v}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$ and $\mathbf{a}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$; for $t = \frac{\pi}{2}$, $\mathbf{v}\left(\frac{\pi}{2}\right) = -\mathbf{j}$ and $\mathbf{a}\left(\frac{\pi}{2}\right) = -\mathbf{i}$



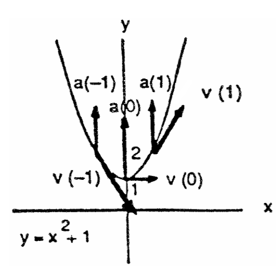
6. $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \left(-2\sin\frac{t}{2}\right)\mathbf{i} + \left(2\cos\frac{t}{2}\right)\mathbf{j}$ and $\mathbf{a} = \frac{d\mathbf{v}}{dt}$ $= \left(-\cos\frac{t}{2}\right)\mathbf{i} + \left(-\sin\frac{t}{2}\right)\mathbf{j} \Rightarrow$ for $t = \pi$, $\mathbf{v}(\pi) = -2\mathbf{i}$ and $\mathbf{a}(\pi) = -\mathbf{j}$; for $t = \frac{3\pi}{2}$, $\mathbf{v}\left(\frac{3\pi}{2}\right) = -\sqrt{2}\,\mathbf{i} - \sqrt{2}\,\mathbf{j}$ and $\mathbf{a}\left(\frac{3\pi}{2}\right) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$



7. $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (1 - \cos t)\mathbf{i} + (\sin t)\mathbf{j}$ and $\mathbf{a} = \frac{d\mathbf{v}}{dt}$ $= (\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow$ for $t = \pi$, $\mathbf{v}(\pi) = 2\mathbf{i}$ and $\mathbf{a}(\pi) = -\mathbf{j}$; for $t = \frac{3\pi}{2}$, $\mathbf{v}\left(\frac{3\pi}{2}\right) = \mathbf{i} - \mathbf{j}$ and $\mathbf{a}\left(\frac{3\pi}{2}\right) = -\mathbf{i}$



8. $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j}$ and $\mathbf{a} = \frac{d\mathbf{v}}{dt} = 2\mathbf{j} \Rightarrow$ for $t = -1$, $\mathbf{v}(-1) = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{a}(-1) = 2\mathbf{j}$; for $t = 0$, $\mathbf{v}(0) = \mathbf{i}$ and $\mathbf{a}(0) = 2\mathbf{j}$; for $t = 1$, $\mathbf{v}(1) = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{a}(1) = 2\mathbf{j}$



9. $\mathbf{r} = (t+1)\mathbf{i} + (t^2-1)\,\mathbf{j} + 2t\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = 2\mathbf{j}$; Speed: $|\mathbf{v}(1)| = \sqrt{1^2 + (2(1))^2 + 2^2} = 3$; Direction: $\frac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \frac{\mathbf{i}+2(1)\mathbf{j}+2\mathbf{k}}{3} = \frac{1}{3}\,\mathbf{i} + \frac{2}{3}\,\mathbf{j} + \frac{2}{3}\,\mathbf{k} \Rightarrow \mathbf{v}(1) = 3\left(\frac{1}{3}\,\mathbf{i} + \frac{2}{3}\,\mathbf{j} + \frac{2}{3}\,\mathbf{k}\right)$

10. $\mathbf{r} = (1+t)\mathbf{i} + \frac{t^2}{\sqrt{2}}\,\mathbf{j} + \frac{t^3}{3}\,\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + \frac{2t}{\sqrt{2}}\,\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \frac{2}{\sqrt{2}}\,\mathbf{j} + 2t\mathbf{k}$; Speed: $|\mathbf{v}(1)| = \sqrt{1^2 + \left(\frac{2(1)}{\sqrt{2}}\right)^2 + (1^2)^2} = 2$; Direction: $\frac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \frac{\mathbf{i}+\frac{2(1)}{\sqrt{2}}\,\mathbf{j}+(1^2)\mathbf{k}}{2} = \frac{1}{2}\,\mathbf{i} + \frac{1}{\sqrt{2}}\,\mathbf{j} + \frac{1}{2}\,\mathbf{k} \Rightarrow \mathbf{v}(1) = 2\left(\frac{1}{2}\,\mathbf{i} + \frac{1}{\sqrt{2}}\,\mathbf{j} + \frac{1}{2}\,\mathbf{k}\right)$

11. $\mathbf{r} = (2\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + 4t\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = (-2\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 4\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = (-2\cos t)\mathbf{i} - (3\sin t)\mathbf{j}$; Speed: $\left|\mathbf{v}\left(\frac{\pi}{2}\right)\right| = \sqrt{\left(-2\sin\frac{\pi}{2}\right)^2 + \left(3\cos\frac{\pi}{2}\right)^2 + 4^2} = 2\sqrt{5}$; Direction: $\frac{\mathbf{v}\left(\frac{\pi}{2}\right)}{\left|\mathbf{v}\left(\frac{\pi}{2}\right)\right|} = \left(-\frac{2}{2\sqrt{5}}\sin\frac{\pi}{2}\right)\mathbf{i} + \left(\frac{3}{2\sqrt{5}}\cos\frac{\pi}{2}\right)\mathbf{j} + \frac{4}{2\sqrt{5}}\,\mathbf{k} = -\frac{1}{\sqrt{5}}\,\mathbf{i} + \frac{2}{\sqrt{5}}\,\mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{2}\right) = 2\sqrt{5}\left(-\frac{1}{\sqrt{5}}\,\mathbf{i} + \frac{2}{\sqrt{5}}\,\mathbf{k}\right)$

12. $\mathbf{r} = (\sec t)\mathbf{i} + (\tan t)\mathbf{j} + \frac{4}{3}\,t\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = (\sec t\tan t)\mathbf{i} + (\sec^2 t)\,\mathbf{j} + \frac{4}{3}\,\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = (\sec t\tan^2 t + \sec^3 t)\,\mathbf{i} + (2\sec^2 t\tan t)\,\mathbf{j}$; Speed: $\left|\mathbf{v}\left(\frac{\pi}{6}\right)\right| = \sqrt{\left(\sec\frac{\pi}{6}\tan\frac{\pi}{6}\right)^2 + \left(\sec^2\frac{\pi}{6}\right)^2 + \left(\frac{4}{3}\right)^2} = 2$; Direction: $\frac{\mathbf{v}\left(\frac{\pi}{6}\right)}{\left|\mathbf{v}\left(\frac{\pi}{6}\right)\right|} = \frac{\left(\sec\frac{\pi}{6}\tan\frac{\pi}{6}\right)\mathbf{i} + \left(\sec^2\frac{\pi}{6}\right)\mathbf{j} + \frac{4}{3}\,\mathbf{k}}{2} = \frac{1}{3}\,\mathbf{i} + \frac{2}{3}\,\mathbf{j} + \frac{2}{3}\,\mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{6}\right) = 2\left(\frac{1}{3}\,\mathbf{i} + \frac{2}{3}\,\mathbf{j} + \frac{2}{3}\,\mathbf{k}\right)$

13. $\mathbf{r} = (2\ln(t+1))\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\,\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \left(\frac{2}{t+1}\right)\mathbf{i} + 2t\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \left[\frac{-2}{(t+1)^2}\right]\mathbf{i} + 2\mathbf{j} + \mathbf{k}$; Speed: $|\mathbf{v}(1)| = \sqrt{\left(\frac{2}{1+1}\right)^2 + (2(1))^2 + 1^2} = \sqrt{6}$; Direction: $\frac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \frac{\left(\frac{2}{1+1}\right)\mathbf{i} + 2(1)\mathbf{j} + (1)\mathbf{k}}{\sqrt{6}} = \frac{1}{\sqrt{6}}\,\mathbf{i} + \frac{2}{\sqrt{6}}\,\mathbf{j} + \frac{1}{\sqrt{6}}\,\mathbf{k} \Rightarrow \mathbf{v}(1) = \sqrt{6}\left(\frac{1}{\sqrt{6}}\,\mathbf{i} + \frac{2}{\sqrt{6}}\,\mathbf{j} + \frac{1}{\sqrt{6}}\,\mathbf{k}\right)$

14. $\mathbf{r} = (e^{-t})\,\mathbf{i} + (2\cos 3t)\mathbf{j} + (2\sin 3t)\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = (-e^{-t})\,\mathbf{i} - (6\sin 3t)\mathbf{j} + (6\cos 3t)\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = (e^{-t})\,\mathbf{i} - (18\cos 3t)\mathbf{j} - (18\sin 3t)\mathbf{k}$; Speed: $|\mathbf{v}(0)| = \sqrt{\left(-e^0\right)^2 + [-6\sin 3(0)]^2 + [6\cos 3(0)]^2} = \sqrt{37}$; Direction: $\frac{\mathbf{v}(0)}{|\mathbf{v}(0)|} = \frac{(-e^0)\,\mathbf{i} - 6\sin 3(0)\mathbf{j} + 6\cos 3(0)\mathbf{k}}{\sqrt{37}} = -\frac{1}{\sqrt{37}}\,\mathbf{i} + \frac{6}{\sqrt{37}}\,\mathbf{k} \Rightarrow \mathbf{v}(0) = \sqrt{37}\left(-\frac{1}{\sqrt{37}}\,\mathbf{i} + \frac{6}{\sqrt{37}}\,\mathbf{k}\right)$

15. $\mathbf{v} = 3\mathbf{i} + \sqrt{3}\,\mathbf{j} + 2t\mathbf{k}$ and $\mathbf{a} = 2\mathbf{k} \Rightarrow \mathbf{v}(0) = 3\mathbf{i} + \sqrt{3}\,\mathbf{j}$ and $\mathbf{a}(0) = 2\mathbf{k} \Rightarrow |\mathbf{v}(0)| = \sqrt{3^2 + \left(\sqrt{3}\right)^2 + 0^2} = \sqrt{12}$ and $|\mathbf{a}(0)| = \sqrt{2^2} = 2$; $\mathbf{v}(0)\cdot\mathbf{a}(0) = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

16. $\mathbf{v} = \frac{\sqrt{2}}{2}\,\mathbf{i} + \left(\frac{\sqrt{2}}{2} - 32t\right)\mathbf{j}$ and $\mathbf{a} = -32\mathbf{j} \Rightarrow \mathbf{v}(0) = \frac{\sqrt{2}}{2}\,\mathbf{i} + \frac{\sqrt{2}}{2}\,\mathbf{j}$ and $\mathbf{a}(0) = -32\mathbf{j} \Rightarrow |\mathbf{v}(0)| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = 1$ and $|\mathbf{a}(0)| = \sqrt{(-32)^2} = 32$; $\mathbf{v}(0)\cdot\mathbf{a}(0) = \left(\frac{\sqrt{2}}{2}\right)(-32) = -16\sqrt{2} \Rightarrow \cos\theta = \frac{-16\sqrt{2}}{1(32)} = -\frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{3\pi}{4}$

17. $\mathbf{v} = \left(\frac{2t}{t^2+1}\right)\mathbf{i} + \left(\frac{1}{t^2+1}\right)\mathbf{j} + t(t^2+1)^{-1/2}\,\mathbf{k}$ and $\mathbf{a} = \left[\frac{-2t^2+2}{(t^2+1)^2}\right]\mathbf{i} - \left[\frac{2t}{(t^2+1)^2}\right]\mathbf{j} + \left[\frac{1}{(t^2+1)^{3/2}}\right]\mathbf{k} \Rightarrow \mathbf{v}(0) = \mathbf{j}$ and $\mathbf{a}(0) = 2\mathbf{i} + \mathbf{k} \Rightarrow |\mathbf{v}(0)| = 1$ and $|\mathbf{a}(0)| = \sqrt{2^2 + 1^2} = \sqrt{5}$; $\mathbf{v}(0)\cdot\mathbf{a}(0) = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

18. $\mathbf{v} = \frac{2}{3}(1+t)^{1/2}\,\mathbf{i} - \frac{2}{3}(1-t)^{1/2}\,\mathbf{j} + \frac{1}{3}\,\mathbf{k}$ and $\mathbf{a} = \frac{1}{3}(1+t)^{-1/2}\,\mathbf{i} + \frac{1}{3}(1-t)^{-1/2}\,\mathbf{j} \Rightarrow \mathbf{v}(0) = \frac{2}{3}\,\mathbf{i} - \frac{2}{3}\,\mathbf{j} + \frac{1}{3}\,\mathbf{k}$ and $\mathbf{a}(0) = \frac{1}{3}\,\mathbf{i} + \frac{1}{3}\,\mathbf{j} \Rightarrow |\mathbf{v}(0)| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = 1$ and $|\mathbf{a}(0)| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{\sqrt{2}}{3}$; $\mathbf{v}(0)\cdot\mathbf{a}(0) = \frac{2}{9} - \frac{2}{9} = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

19. $\mathbf{v} = (1 - \cos t)\mathbf{i} + (\sin t)\mathbf{j}$ and $\mathbf{a} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{v}\cdot\mathbf{a} = (\sin t)(1 - \cos t) + (\sin t)(\cos t) = \sin t$. Thus, $\mathbf{v}\cdot\mathbf{a} = 0 \Rightarrow \sin t = 0 \Rightarrow t = 0, \pi$, or 2π

20. $\mathbf{v} = (\cos t)\mathbf{i} + \mathbf{j} - (\sin t)\mathbf{k}$ and $\mathbf{a} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{k} \Rightarrow \mathbf{v}\cdot\mathbf{a} = -\sin t\cos t + \sin t\cos t = 0$ for all $t \geq 0$

21. $\int_0^1 [t^3\mathbf{i} + 7\mathbf{j} + (t+1)\mathbf{k}]\,dt = \left[\frac{t^4}{4}\right]_0^1 \mathbf{i} + [7t]_0^1 \mathbf{j} + \left[\frac{t^2}{2} + t\right]_0^1 \mathbf{k} = \frac{1}{4}\mathbf{i} + 7\mathbf{j} + \frac{3}{2}\mathbf{k}$

22. $\int_1^2 \left[(6-6t)\mathbf{i} + 3\sqrt{t}\,\mathbf{j} + \left(\frac{4}{t^2}\right)\mathbf{k}\right] dt = [6t - 3t^2]_1^2\,\mathbf{i} + \left[2t^{3/2}\right]_1^2 \mathbf{j} + [-4t^{-1}]_1^2\,\mathbf{k} = -3\mathbf{i} + \left(4\sqrt{2} - 2\right)\mathbf{j} + 2\mathbf{k}$

23. $\int_{-\pi/4}^{\pi/4} [(\sin t)\mathbf{i} + (1 + \cos t)\mathbf{j} + (\sec^2 t)\,\mathbf{k}]\,dt = [-\cos t]_{-\pi/4}^{\pi/4}\,\mathbf{i} + [t + \sin t]_{-\pi/4}^{\pi/4}\,\mathbf{j} + [\tan t]_{-\pi/4}^{\pi/4}\,\mathbf{k}$
$= \left(\frac{\pi + 2\sqrt{2}}{2}\right)\mathbf{j} + 2\mathbf{k}$

24. $\int_0^{\pi/3} [(\sec t\tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2\sin t\cos t)\,\mathbf{k}]\,dt = \int_0^{\pi/3} [(\sec t\tan t)\mathbf{i} + (\tan t)\mathbf{j} + (\sin 2t)\mathbf{k}]\,dt$
$= [\sec t]_0^{\pi/3}\,\mathbf{i} + [-\ln(\cos t)]_0^{\pi/3}\,\mathbf{j} + \left[-\frac{1}{2}\cos 2t\right]_0^{\pi/3}\mathbf{k} = \mathbf{i} + (\ln 2)\mathbf{j} + \frac{3}{4}\mathbf{k}$

25. $\int_1^4 \left(\frac{1}{t}\,\mathbf{i} + \frac{1}{5-t}\,\mathbf{j} + \frac{1}{2t}\,\mathbf{k}\right) dt = \ = [\ln t]_1^4\,\mathbf{i} + [-\ln(5-t)]_1^4\,\mathbf{j} + \left[\frac{1}{2}\ln t\right]_1^4 \mathbf{k} = (\ln 4)\mathbf{i} + (\ln 4)\mathbf{j} + (\ln 2)\mathbf{k}$

26. $\int_0^1 \left(\frac{2}{\sqrt{1-t^2}}\,\mathbf{i} + \frac{\sqrt{3}}{1+t^2}\,\mathbf{k}\right) dt = [2\sin^{-1} t]_0^1\,\mathbf{i} + \left[\sqrt{3}\tan^{-1} t\right]_0^1 \mathbf{k} = \pi\mathbf{i} + \frac{\pi\sqrt{3}}{4}\,\mathbf{k}$

27. $\mathbf{r} = \int(-t\mathbf{i} - t\mathbf{j} - t\mathbf{k})\,dt = -\frac{t^2}{2}\,\mathbf{i} - \frac{t^2}{2}\,\mathbf{j} - \frac{t^2}{2}\,\mathbf{k} + \mathbf{C}$; $\mathbf{r}(0) = 0\mathbf{i} - 0\mathbf{j} - 0\mathbf{k} + \mathbf{C} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
$\Rightarrow \mathbf{r} = \left(-\frac{t^2}{2} + 1\right)\mathbf{i} + \left(-\frac{t^2}{2} + 2\right)\mathbf{j} + \left(-\frac{t^2}{2} + 3\right)\mathbf{k}$

28. $\mathbf{r} = \int [(180t)\mathbf{i} + (180t - 16t^2)\,\mathbf{j}]\,dt = 90t^2\mathbf{i} + \left(90t^2 - \frac{16}{3}t^3\right)\mathbf{j} + \mathbf{C}$; $\mathbf{r}(0) = 90(0)^2\mathbf{i} + \left[90(0)^2 - \frac{16}{3}(0)^3\right]\mathbf{j} + \mathbf{C}$
$= 100\mathbf{j} \Rightarrow \mathbf{C} = 100\mathbf{j} \Rightarrow \mathbf{r} = 90t^2\mathbf{i} + \left(90t^2 - \frac{16}{3}t^3 + 100\right)\mathbf{j}$

29. $\mathbf{r} = \int \left[\left(\frac{3}{2}(t+1)^{1/2}\right)\mathbf{i} + e^{-t}\mathbf{j} + \left(\frac{1}{t+1}\right)\mathbf{k}\right] dt = (t+1)^{3/2}\,\mathbf{i} - e^{-t}\,\mathbf{j} + \ln(t+1)\mathbf{k} + \mathbf{C}$;
$\mathbf{r}(0) = (0+1)^{3/2}\,\mathbf{i} - e^{-0}\,\mathbf{j} + \ln(0+1)\mathbf{k} + \mathbf{C} = \mathbf{k} \Rightarrow \mathbf{C} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$
$\Rightarrow \mathbf{r} = \left[(t+1)^{3/2} - 1\right]\mathbf{i} + (1 - e^{-t})\,\mathbf{j} + [1 + \ln(t+1)]\mathbf{k}$

30. $\mathbf{r} = \int [(t^3 + 4t)\,\mathbf{i} + t\mathbf{j} + 2t^2\mathbf{k}]\,dt = \left(\frac{t^4}{4} + 2t^2\right)\mathbf{i} + \frac{t^2}{2}\,\mathbf{j} + \frac{2t^3}{3}\,\mathbf{k} + \mathbf{C}$; $\mathbf{r}(0) = \left[\frac{0^4}{4} + 2(0)^2\right]\mathbf{i} + \frac{0^2}{2}\,\mathbf{j} + \frac{2(0)^3}{3}\,\mathbf{k} + \mathbf{C}$
$= \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{C} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{r} = \left(\frac{t^4}{4} + 2t^2 + 1\right)\mathbf{i} + \left(\frac{t^2}{2} + 1\right)\mathbf{j} + \frac{2t^3}{3}\mathbf{k}$

31. $\frac{d\mathbf{r}}{dt} = \int(-32\mathbf{k})\,dt = -32t\mathbf{k} + \mathbf{C}_1$; $\frac{d\mathbf{r}}{dt}(0) = 8\mathbf{i} + 8\mathbf{j} \Rightarrow -32(0)\mathbf{k} + \mathbf{C}_1 = 8\mathbf{i} + 8\mathbf{j} \Rightarrow \mathbf{C}_1 = 8\mathbf{i} + 8\mathbf{j}$
$\Rightarrow \frac{d\mathbf{r}}{dt} = 8\mathbf{i} + 8\mathbf{j} - 32t\mathbf{k}$; $\mathbf{r} = \int(8\mathbf{i} + 8\mathbf{j} - 32t\mathbf{k})\,dt = 8t\mathbf{i} + 8t\mathbf{j} - 16t^2\mathbf{k} + \mathbf{C}_2$; $\mathbf{r}(0) = 100\mathbf{k}$
$\Rightarrow 8(0)\mathbf{i} + 8(0)\mathbf{j} - 16(0)^2\mathbf{k} + \mathbf{C}_2 = 100\mathbf{k} \Rightarrow \mathbf{C}_2 = 100\mathbf{k} \Rightarrow \mathbf{r} = 8t\mathbf{i} + 8t\mathbf{j} + (100 - 16t^2)\,\mathbf{k}$

32. $\frac{d\mathbf{r}}{dt} = \int -(\mathbf{i} + \mathbf{j} + \mathbf{k})\,dt = -(t\mathbf{i} + t\mathbf{j} + t\mathbf{k}) + \mathbf{C}_1$; $\frac{d\mathbf{r}}{dt}(0) = \mathbf{0} \Rightarrow -(0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}) + \mathbf{C}_1 = \mathbf{0} \Rightarrow \mathbf{C}_1 = \mathbf{0}$
$\Rightarrow \frac{d\mathbf{r}}{dt} = -(t\mathbf{i} + t\mathbf{j} + t\mathbf{k})$; $\mathbf{r} = \int -(t\mathbf{i} + t\mathbf{j} + t\mathbf{k})\,dt = -\left(\frac{t^2}{2}\,\mathbf{i} + \frac{t^2}{2}\,\mathbf{j} + \frac{t^2}{2}\,\mathbf{k}\right) + \mathbf{C}_2$; $\mathbf{r}(0) = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k}$
$\Rightarrow -\left(\frac{0^2}{2}\,\mathbf{i} + \frac{0^2}{2}\,\mathbf{j} + \frac{0^2}{2}\,\mathbf{k}\right) + \mathbf{C}_2 = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k} \Rightarrow \mathbf{C}_2 = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k}$

$\Rightarrow \mathbf{r} = \left(-\frac{t^2}{2} + 10\right)\mathbf{i} + \left(-\frac{t^2}{2} + 10\right)\mathbf{j} + \left(-\frac{t^2}{2} + 10\right)\mathbf{k}$

33. $\mathbf{r}(t) = (\sin t)\mathbf{i} + (t^2 - \cos t)\,\mathbf{j} + e^t\mathbf{k} \Rightarrow \mathbf{v}(t) = (\cos t)\mathbf{i} + (2t + \sin t)\mathbf{j} + e^t\mathbf{k}$; $t_0 = 0 \Rightarrow \mathbf{v}(t_0) = \mathbf{i} + \mathbf{k}$ and $\mathbf{r}(t_0) = P_0 = (0, -1, 1) \Rightarrow x = 0 + t = t$, $y = -1$, and $z = 1 + t$ are parametric equations of the tangent line

34. $\mathbf{r}(t) = (2\sin t)\mathbf{i} + (2\cos t)\,\mathbf{j} + 5t\mathbf{k} \Rightarrow \mathbf{v}(t) = (2\cos t)\mathbf{i} - (2\sin t)\mathbf{j} + 5\mathbf{k}$; $t_0 = 4\pi \Rightarrow \mathbf{v}(t_0) = 2\mathbf{i} + 5\mathbf{k}$ and $\mathbf{r}(t_0) = P_0 = (0, 2, 20\pi) \Rightarrow x = 0 + 2t = 2t$, $y = 2$, and $z = 20\pi + 5t$ are parametric equations of the tangent line

35. $\mathbf{r}(t) = (a\sin t)\mathbf{i} + (a\cos t)\,\mathbf{j} + bt\mathbf{k} \Rightarrow \mathbf{v}(t) = (a\cos t)\mathbf{i} - (a\sin t)\mathbf{j} + b\mathbf{k}$; $t_0 = 2\pi \Rightarrow \mathbf{v}(t_0) = a\mathbf{i} + b\mathbf{k}$ and $\mathbf{r}(t_0) = P_0 = (0, a, 2b\pi) \Rightarrow x = 0 + at = at$, $y = a$, and $z = 2\pi b + bt$ are parametric equations of the tangent line

36. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\,\mathbf{j} + (\sin 2t)\mathbf{k} \Rightarrow \mathbf{v}(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (2\cos 2t)\mathbf{k}$; $t_0 = \frac{\pi}{2} \Rightarrow \mathbf{v}(t_0) = -\mathbf{i} - 2\mathbf{k}$ and $\mathbf{r}(t_0) = P_0 = (0, 1, 0) \Rightarrow x = 0 - t = -t$, $y = 1$, and $z = 0 - 2t = -2t$ are parametric equations of the tangent line

37. (a) $\mathbf{v}(t) = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}$;
   (i) $|\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow$ constant speed;
   (ii) $\mathbf{v} \cdot \mathbf{a} = (\sin t)(\cos t) - (\cos t)(\sin t) = 0 \Rightarrow$ yes, orthogonal;
   (iii) counterclockwise movement;
   (iv) yes, $\mathbf{r}(0) = \mathbf{i} + 0\mathbf{j}$

   (b) $\mathbf{v}(t) = -(2\sin 2t)\mathbf{i} + (2\cos 2t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(4\cos 2t)\mathbf{i} - (4\sin 2t)\mathbf{j}$;
   (i) $|\mathbf{v}(t)| = \sqrt{4\sin^2 2t + 4\cos^2 2t} = 2 \Rightarrow$ constant speed;
   (ii) $\mathbf{v} \cdot \mathbf{a} = 8\sin 2t\cos 2t - 8\cos 2t\sin 2t = 0 \Rightarrow$ yes, orthogonal;
   (iii) counterclockwise movement;
   (iv) yes, $\mathbf{r}(0) = \mathbf{i} + 0\mathbf{j}$

   (c) $\mathbf{v}(t) = -\sin\left(t - \frac{\pi}{2}\right)\mathbf{i} + \cos\left(t - \frac{\pi}{2}\right)\mathbf{j} \Rightarrow \mathbf{a}(t) = -\cos\left(t - \frac{\pi}{2}\right)\mathbf{i} - \sin\left(t - \frac{\pi}{2}\right)\mathbf{j}$;
   (i) $|\mathbf{v}(t)| = \sqrt{\sin^2\left(t - \frac{\pi}{2}\right) + \cos^2\left(t - \frac{\pi}{2}\right)} = 1 \Rightarrow$ constant speed;
   (ii) $\mathbf{v} \cdot \mathbf{a} = \sin\left(t - \frac{\pi}{2}\right)\cos\left(t - \frac{\pi}{2}\right) - \cos\left(t - \frac{\pi}{2}\right)\sin\left(t - \frac{\pi}{2}\right) = 0 \Rightarrow$ yes, orthogonal;
   (iii) counterclockwise movement;
   (iv) no, $\mathbf{r}(0) = 0\mathbf{i} - \mathbf{j}$ instead of $\mathbf{i} + 0\mathbf{j}$

   (d) $\mathbf{v}(t) = -(\sin t)\mathbf{i} - (\cos t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(\cos t)\mathbf{i} + (\sin t)\mathbf{j}$;
   (i) $|\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1 \Rightarrow$ constant speed;
   (ii) $\mathbf{v} \cdot \mathbf{a} = (\sin t)(\cos t) - (\cos t)(\sin t) = 0 \Rightarrow$ yes, orthogonal;
   (iii) clockwise movement;
   (iv) yes, $\mathbf{r}(0) = \mathbf{i} - 0\mathbf{j}$

   (e) $\mathbf{v}(t) = -(2t\sin t)\mathbf{i} + (2t\cos t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(2\sin t + 2t\cos t)\mathbf{i} + (2\cos t - 2t\sin t)\mathbf{j}$;
   (i) $|\mathbf{v}(t)| = \sqrt{[-(2t\sin t)]^2 + (2t\cos t)^2} = \sqrt{4t^2(\sin^2 t + \cos^2 t)} = 2|t| = 2t,\ t \geq 0$
   $\Rightarrow$ variable speed;
   (ii) $\mathbf{v} \cdot \mathbf{a} = 4\left(t\sin^2 t + t^2\sin t\cos t\right) + 4\left(t\cos^2 t - t^2\cos t\sin t\right) = 4t \neq 0$ in general
   $\Rightarrow$ not orthogonal in general;
   (iii) counterclockwise movement;
   (iv) yes, $\mathbf{r}(0) = \mathbf{i} + 0\mathbf{j}$

38. Let $\mathbf{p} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ denote the position vector of the point $(2, 2, 1)$ and let, $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ and $\mathbf{v} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$. Then $\mathbf{r}(t) = \mathbf{p} + (\cos t)\mathbf{u} + (\sin t)\mathbf{v}$. Note that $(2, 2, 1)$ is a point on the plane and $\mathbf{n} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ is normal to the plane. Moreover, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal unit vectors with $\mathbf{u} \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n} = 0 \Rightarrow \mathbf{u}$ and $\mathbf{v}$ are parallel to the plane. Therefore, $\mathbf{r}(t)$ identifies a point that lies in the plane for each t. Also, for each t, $(\cos t)\mathbf{u} + (\sin t)\mathbf{v}$

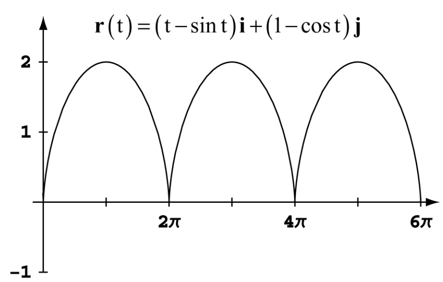
is a unit vector. Starting at the point $\left(2+\frac{1}{\sqrt{2}}, 2-\frac{1}{\sqrt{2}}, 1\right)$ the vector $\mathbf{r}(t)$ traces out a circle of radius 1 and center $(2, 2, 1)$ in the plane $x + y - 2z = 2$.

39. $\frac{d\mathbf{v}}{dt} = \mathbf{a} = 3\mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{v}(t) = 3t\mathbf{i} - t\mathbf{j} + t\mathbf{k} + \mathbf{C}_1$; the particle travels in the direction of the vector $(4-1)\mathbf{i} + (1-2)\mathbf{j} + (4-3)\mathbf{k} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$ (since it travels in a straight line), and at time $t = 0$ it has speed $2 \Rightarrow \mathbf{v}(0) = \frac{2}{\sqrt{9+1+1}}(3\mathbf{i} - \mathbf{j} + \mathbf{k}) = \mathbf{C}_1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{v}(t) = \left(3t + \frac{6}{\sqrt{11}}\right)\mathbf{i} - \left(t + \frac{2}{\sqrt{11}}\right)\mathbf{j} + \left(t + \frac{2}{\sqrt{11}}\right)\mathbf{k}$
$\Rightarrow \mathbf{r}(t) = \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t\right)\mathbf{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)\mathbf{k} + \mathbf{C}_2$; $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} = \mathbf{C}_2$
$\Rightarrow \mathbf{r}(t) = \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t + 1\right)\mathbf{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t - 2\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t + 3\right)\mathbf{k}$
$= \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)(3\mathbf{i} - \mathbf{j} + \mathbf{k}) + (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

40. $\frac{d\mathbf{v}}{dt} = \mathbf{a} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{v}(t) = 2t\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}_1$; the particle travels in the direction of the vector $(3-1)\mathbf{i} + (0-(-1))\mathbf{j} + (3-2)\mathbf{k} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ (since it travels in a straight line), and at time $t = 0$ it has speed 2
$\Rightarrow \mathbf{v}(0) = \frac{2}{\sqrt{4+1+1}}(2\mathbf{i} + \mathbf{j} + \mathbf{k}) = \mathbf{C}_1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{v}(t) = \left(2t + \frac{4}{\sqrt{6}}\right)\mathbf{i} + \left(t + \frac{2}{\sqrt{6}}\right)\mathbf{j} + \left(t + \frac{2}{\sqrt{6}}\right)\mathbf{k}$
$\Rightarrow \mathbf{r}(t) = \left(t^2 + \frac{4}{\sqrt{6}}t\right)\mathbf{i} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)\mathbf{k} + \mathbf{C}_2$; $\mathbf{r}(0) = \mathbf{i} - \mathbf{j} + 2\mathbf{k} = \mathbf{C}_2$
$\Rightarrow \mathbf{r}(t) = \left(t^2 + \frac{4}{\sqrt{6}}t + 1\right)\mathbf{i} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t - 1\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t + 2\right)\mathbf{k} = \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)(2\mathbf{i} + \mathbf{j} + \mathbf{k}) + (\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

41. The velocity vector is tangent to the graph of $y^2 = 2x$ at the point $(2, 2)$, has length 5, and a positive $\mathbf{i}$ component. Now, $y^2 = 2x \Rightarrow 2y\frac{dy}{dx} = 2 \Rightarrow \left.\frac{dy}{dx}\right|_{(2,2)} = \frac{2}{2\cdot 2} = \frac{1}{2} \Rightarrow$ the tangent vector lies in the direction of the vector $\mathbf{i} + \frac{1}{2}\mathbf{j} \Rightarrow$ the velocity vector is $\mathbf{v} = \frac{5}{\sqrt{1+\frac{1}{4}}}\left(\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = \frac{5}{\left(\frac{\sqrt{5}}{2}\right)}\left(\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = 2\sqrt{5}\mathbf{i} + \sqrt{5}\mathbf{j}$

42. (a)



(b) $\mathbf{v} = (1 - \cos t)\mathbf{i} + (\sin t)\mathbf{j}$ and $\mathbf{a} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}$; $|\mathbf{v}|^2 = (1 - \cos t)^2 + \sin^2 t = 2 - 2\cos t \Rightarrow |\mathbf{v}|^2$ is at a max when $\cos t = -1 \Rightarrow t = \pi, 3\pi, 5\pi$, etc., and at these values of t, $|\mathbf{v}|^2 = 4 \Rightarrow \max|\mathbf{v}| = \sqrt{4} = 2$; $|\mathbf{v}|^2$ is at a min when $\cos t = 1 \Rightarrow t = 0, 2\pi, 4\pi$, etc., and at these values of t, $|\mathbf{v}|^2 = 0 \Rightarrow \min|\mathbf{v}| = 0$; $|\mathbf{a}|^2 = \sin^2 t + \cos^2 t = 1$ for every $t \Rightarrow \max|\mathbf{a}| = \min|\mathbf{a}| = \sqrt{1} = 1$

43. $\mathbf{v} = (-3\sin t)\mathbf{j} + (2\cos t)\mathbf{k}$ and $\mathbf{a} = (-3\cos t)\mathbf{j} - (2\sin t)\mathbf{k}$; $|\mathbf{v}|^2 = 9\sin^2 t + 4\cos^2 t \Rightarrow \frac{d}{dt}\left(|\mathbf{v}|^2\right)$
$= 18\sin t\cos t - 8\cos t\sin t = 10\sin t\cos t$; $\frac{d}{dt}\left(|\mathbf{v}|^2\right) = 0 \Rightarrow 10\sin t\cos t = 0 \Rightarrow \sin t = 0$ or $\cos t = 0$
$\Rightarrow t = 0, \pi$ or $t = \frac{\pi}{2}, \frac{3\pi}{2}$. When $t = 0, \pi$, $|\mathbf{v}|^2 = 4 \Rightarrow |\mathbf{v}| = \sqrt{4} = 2$; when $t = \frac{\pi}{2}, \frac{3\pi}{2}$, $|\mathbf{v}| = \sqrt{9} = 3$.
Therefore max $|\mathbf{v}|$ is 3 when $t = \frac{\pi}{2}, \frac{3\pi}{2}$, and min $|\mathbf{v}| = 2$ when $t = 0, \pi$. Next, $|\mathbf{a}|^2 = 9\cos^2 t + 4\sin^2 t$
$\Rightarrow \frac{d}{dt}\left(|\mathbf{a}|^2\right) = -18\cos t\sin t + 8\sin t\cos t = -10\sin t\cos t$; $\frac{d}{dt}\left(|\mathbf{a}|^2\right) = 0 \Rightarrow -10\sin t\cos t = 0 \Rightarrow \sin t = 0$ or $\cos t = 0 \Rightarrow t = 0, \pi$ or $t = \frac{\pi}{2}, \frac{3\pi}{2}$. When $t = 0, \pi$, $|\mathbf{a}|^2 = 9 \Rightarrow |\mathbf{a}| = 3$; when $t = \frac{\pi}{2}, \frac{3\pi}{2}$, $|\mathbf{a}|^2 = 4 \Rightarrow |\mathbf{a}| = 2$.
Therefore, max $|\mathbf{a}| = 3$ when $t = 0, \pi$, and min $|\mathbf{a}| = 2$ when $t = \frac{\pi}{2}, \frac{3\pi}{2}$.

44. (a) $\mathbf{r}(t) = (r_0 \cos\theta)\mathbf{i} + (r_0 \sin\theta)\mathbf{j}$, and the distance traveled along the circle in time t is νt (rate times time) which equals the circular arc length $r_0\theta \Rightarrow \theta = \frac{\nu t}{r_0} \Rightarrow \mathbf{r}(t) = \left(r_0 \cos \frac{\nu t}{r_0}\right)\mathbf{i} + \left(r_0 \sin \frac{\nu t}{r_0}\right)\mathbf{j}$

(b) $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \left(-\nu \sin \frac{\nu t}{r_0}\right)\mathbf{i} + \left(\nu \cos \frac{\nu t}{r_0}\right)\mathbf{j} \Rightarrow \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \left(-\frac{\nu^2}{r_0} \cos \frac{\nu t}{r_0}\right)\mathbf{i} + \left(-\frac{\nu^2}{r_0} \sin \frac{\nu t}{r_0}\right)\mathbf{j}$
$= -\frac{\nu^2}{r_0^2}\left[\left(r_0 \cos \frac{\nu t}{r_0}\right)\mathbf{i} + \left(r_0 \sin \frac{\nu t}{r_0}\right)\mathbf{j}\right] = -\frac{\nu^2}{r_0^2}\,\mathbf{r}(t)$

(c) $\mathbf{F} = m\mathbf{a} \Rightarrow \left(-\frac{GmM}{r_0^2}\right)\frac{\mathbf{r}}{r_0} = m\left(-\frac{\nu^2}{r_0^2}\right)\mathbf{r} \Rightarrow -\frac{GmM}{r_0^3} = -\frac{m\nu^2}{r_0^2} \Rightarrow \nu^2 = \frac{GM}{r_0}$

(d) T is the time for the satellite to complete one full orbit $\Rightarrow \nu T =$ circumference of circle $\Rightarrow \nu T = 2\pi r_0$

(e) Substitute $\nu = \frac{2\pi r_0}{T}$ into $\nu^2 = \frac{GM}{r_0} \Rightarrow \frac{4\pi^2 r_0^2}{T^2} = \frac{GM}{r_0} \Rightarrow T^2 = \frac{4\pi^2 r_0^3}{GM} \Rightarrow T^2$ is proportional to r_0^3 since $\frac{4\pi^2}{GM}$ is a constant

45. $\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = \mathbf{v}\cdot\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{v}}{dt}\cdot\mathbf{v} = 2\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = 2\cdot 0 = 0 \Rightarrow \mathbf{v}\cdot\mathbf{v}$ is a constant $\Rightarrow |\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$ is constant

46. (a) $\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}) = \frac{d\mathbf{u}}{dt}\cdot(\mathbf{v}\times\mathbf{w}) + \mathbf{u}\cdot\frac{d}{dt}(\mathbf{v}\times\mathbf{w}) = \frac{d\mathbf{u}}{dt}\cdot(\mathbf{v}\times\mathbf{w}) + \mathbf{u}\cdot\left(\frac{d\mathbf{v}}{dt}\times\mathbf{w} + \mathbf{v}\times\frac{d\mathbf{w}}{dt}\right)$
$= \frac{d\mathbf{u}}{dt}\cdot(\mathbf{v}\times\mathbf{w}) + \mathbf{u}\cdot\frac{d\mathbf{v}}{dt}\times\mathbf{w} + \mathbf{u}\cdot\mathbf{v}\times\frac{d\mathbf{w}}{dt}$

(b) Each of the determinants is equivalent to each expression in Eq. 7 in part (a) because of the formual in Section 12.4 expressing the triple scalar product as a determinant.

47. $\frac{d}{dt}\left[\mathbf{r}\cdot\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)\right] = \frac{d\mathbf{r}}{dt}\cdot\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right) + \mathbf{r}\cdot\left(\frac{d^2\mathbf{r}}{dt^2}\times\frac{d^2\mathbf{r}}{dt^2}\right) + \mathbf{r}\cdot\left(\frac{d\mathbf{r}}{dt}\times\frac{d^3\mathbf{r}}{dt^3}\right) = \mathbf{r}\cdot\left(\frac{d\mathbf{r}}{dt}\times\frac{d^3\mathbf{r}}{dt^3}\right)$, since $\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B}) = 0$ and $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{B}) = 0$ for any vectors $\mathbf{A}$ and $\mathbf{B}$

48. $\mathbf{u} = \mathbf{C} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ with a, b, c real constants $\Rightarrow \frac{d\mathbf{u}}{dt} = \frac{da}{dt}\mathbf{i} + \frac{db}{dt}\mathbf{j} + \frac{dc}{dt}\mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

49. (a) $\mathbf{u} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \Rightarrow c\mathbf{u} = cf(t)\mathbf{i} + cg(t)\mathbf{j} + ch(t)\mathbf{k} \Rightarrow \frac{d}{dt}(c\mathbf{u}) = c\frac{df}{dt}\mathbf{i} + c\frac{dg}{dt}\mathbf{j} + c\frac{dh}{dt}\mathbf{k}$
$= c\left(\frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}\right) = c\frac{d\mathbf{u}}{dt}$

(b) $f\mathbf{u} = f f(t)\mathbf{i} + f g(t)\mathbf{j} + f h(t)\mathbf{k} \Rightarrow \frac{d}{dt}(f\mathbf{u}) = \left[\frac{df}{dt}f(t) + f\frac{df}{dt}\right]\mathbf{i} + \left[\frac{df}{dt}g(t) + f\frac{dg}{dt}\right]\mathbf{j} + \left[\frac{df}{dt}h(t) + f\frac{dh}{dt}\right]\mathbf{k}$
$= \frac{df}{dt}[f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}] + f\left[\frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}\right] = \frac{df}{dt}\mathbf{u} + f\frac{d\mathbf{u}}{dt}$

50. Let $\mathbf{u} = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$ and $\mathbf{v} = g_1(t)\mathbf{i} + g_2(t)\mathbf{j} + g_3(t)\mathbf{k}$. Then
$\mathbf{u} + \mathbf{v} = [f_1(t) + g_1(t)]\mathbf{i} + [f_2(t) + g_2(t)]\mathbf{j} + [f_3(t) + g_3(t)]\mathbf{k}$
$\Rightarrow \frac{d}{dt}(\mathbf{u} + \mathbf{v}) = [f_1'(t) + g_1'(t)]\mathbf{i} + [f_2'(t) + g_2'(t)]\mathbf{j} + [f_3'(t) + g_3'(t)]\mathbf{k}$
$= [f_1'(t)\mathbf{i} + f_2'(t)\mathbf{j} + f_3'(t)\mathbf{k}] + [g_1'(t)\mathbf{i} + g_2'(t)\mathbf{j} + g_3'(t)\mathbf{k}] = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}$;
$\mathbf{u} - \mathbf{v} = [f_1(t) - g_1(t)]\mathbf{i} + [f_2(t) - g_2(t)]\mathbf{j} + [f_3(t) - g_3(t)]\mathbf{k}$
$\Rightarrow \frac{d}{dt}(\mathbf{u} - \mathbf{v}) = [f_1'(t) - g_1'(t)]\mathbf{i} + [f_2'(t) - g_2'(t)]\mathbf{j} + [f_3'(t) - g_3'(t)]\mathbf{k}$
$= [f_1'(t)\mathbf{i} + f_2'(t)\mathbf{j} + f_3'(t)\mathbf{k}] - [g_1'(t)\mathbf{i} + g_2'(t)\mathbf{j} + g_3'(t)\mathbf{k}] = \frac{d\mathbf{u}}{dt} - \frac{d\mathbf{v}}{dt}$

51. Suppose $\mathbf{r}$ is continuous at $t = t_0$. Then $\lim_{t\to t_0}\mathbf{r}(t) = \mathbf{r}(t_0) \Leftrightarrow \lim_{t\to t_0}[f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}]$
$= f(t_0)\mathbf{i} + g(t_0)\mathbf{j} + h(t_0)\mathbf{k} \Leftrightarrow \lim_{t\to t_0} f(t) = f(t_0)$, $\lim_{t\to t_0} g(t) = g(t_0)$, and $\lim_{t\to t_0} h(t) = h(t_0) \Leftrightarrow$ f, g, and h are continuous at $t = t_0$.

52. $$\lim_{t\to t_0}[\mathbf{r}_1(t)\times\mathbf{r}_2(t)] = \lim_{t\to t_0}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1(t) & f_2(t) & f_3(t) \\ g_1(t) & g_2(t) & g_3(t) \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \lim_{t\to t_0} f_1(t) & \lim_{t\to t_0} f_2(t) & \lim_{t\to t_0} f_3(t) \\ \lim_{t\to t_0} g_1(t) & \lim_{t\to t_0} g_2(t) & \lim_{t\to t_0} g_3(t) \end{vmatrix}$$
$= \lim_{t\to t_0}\mathbf{r}_1(t)\times\lim_{t\to t_0}\mathbf{r}_2(t) = \mathbf{A}\times\mathbf{B}$

53. $\mathbf{r}'(t_0)$ exists $\Rightarrow$ $f'(t_0)\mathbf{i} + g'(t_0)\mathbf{j} + h'(t_0)\mathbf{k}$ exists $\Rightarrow$ $f'(t_0)$, $g'(t_0)$, $h'(t_0)$ all exist $\Rightarrow$ f, g, and h are continuous at $t = t_0$ $\Rightarrow$ $\mathbf{r}(t)$ is continuous at $t = t_0$

54. (a) $\int_a^b k\mathbf{r}(t)\,dt = \int_a^b [kf(t)\mathbf{i} + kg(t)\mathbf{j} + kh(t)\mathbf{k}]\,dt = \int_a^b [kf(t)]\,dt\,\mathbf{i} + \int_a^b [kg(t)]\,dt\,\mathbf{j} + \int_a^b [kh(t)]\,dt\,\mathbf{k}$
$= k\left(\int_a^b f(t)\,dt\,\mathbf{i} + \int_a^b g(t)\,dt\,\mathbf{j} + \int_a^b h(t)\,dt\,\mathbf{k}\right) = k\int_a^b \mathbf{r}(t)\,dt$

(b) $\int_a^b [\mathbf{r}_1(t) \pm \mathbf{r}_2(t)]\,dt = \int_a^b ([f_1(t)\mathbf{i} + g_1(t)\mathbf{j} + h_1(t)\mathbf{k}] \pm [f_2(t)\mathbf{i} + g_2(t)\mathbf{j} + h_2(t)\mathbf{k}])\,dt$
$= \int_a^b ([f_1(t) \pm f_2(t)]\,\mathbf{i} + [g_1(t) \pm g_2(t)]\,\mathbf{j} + [h_1(t) \pm h_2(t)]\,\mathbf{k})\,dt$
$= \int_a^b [f_1(t) \pm f_2(t)]\,dt\,\mathbf{i} + \int_a^b [g_1(t) \pm g_2(t)]\,dt\,\mathbf{j} + \int_a^b [h_1(t) \pm h_2(t)]\,dt\,\mathbf{k}$
$= \left[\int_a^b f_1(t)\,dt\,\mathbf{i} \pm \int_a^b f_2(t)\,dt\,\mathbf{i}\right] + \left[\int_a^b g_1(t)\,dt\,\mathbf{j} \pm \int_a^b g_2(t)\,dt\,\mathbf{j}\right] + \left[\int_a^b h_1(t)\,dt\,\mathbf{k} \pm \int_a^b h_2(t)\,dt\,\mathbf{k}\right]$
$= \int_a^b \mathbf{r}_1(t)\,dt \pm \int_a^b \mathbf{r}_2(t)\,dt$

(c) Let $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. Then $\int_a^b \mathbf{C}\cdot\mathbf{r}(t)\,dt = \int_a^b [c_1f(t) + c_2g(t) + c_3h(t)]\,dt$
$= c_1\int_a^b f(t)\,dt + c_2\int_a^b g(t)\,dt + c_3\int_a^b h(t)\,dt = \mathbf{C}\cdot\int_a^b \mathbf{r}(t)\,dt$;
$\int_a^b \mathbf{C}\times\mathbf{r}(t)\,dt = \int_a^b [c_2h(t) - c_3g(t)]\,\mathbf{i} + [c_3f(t) - c_1h(t)]\,\mathbf{j} + [c_1g(t) - c_2f(t)]\,\mathbf{k}\,dt$
$= \left[c_2\int_a^b h(t)\,dt - c_3\int_a^b g(t)\,dt\right]\mathbf{i} + \left[c_3\int_a^b f(t)\,dt - c_1\int_a^b h(t)\,dt\right]\mathbf{j} + \left[c_1\int_a^b g(t)\,dt - c_2\int_a^b f(t)\,dt\right]\mathbf{k}$
$= \mathbf{C}\times\int_a^b \mathbf{r}(t)\,dt$

55. (a) Let u and $\mathbf{r}$ be continuous on [a, b]. Then $\lim_{t\to t_0} u(t)\mathbf{r}(t) = \lim_{t\to t_0} [u(t)f(t)\mathbf{i} + u(t)g(t)\mathbf{j} + u(t)h(t)\mathbf{k}]$
$= u(t_0)f(t_0)\mathbf{i} + u(t_0)g(t_0)\mathbf{j} + u(t_0)h(t_0)\mathbf{k} = u(t_0)\mathbf{r}(t_0)$ $\Rightarrow$ $u\mathbf{r}$ is continuous for every t_0 in [a, b].

(b) Let u and $\mathbf{r}$ be differentiable. Then $\frac{d}{dt}(u\mathbf{r}) = \frac{d}{dt}[u(t)f(t)\mathbf{i} + u(t)g(t)\mathbf{j} + u(t)h(t)\mathbf{k}]$
$= \left(\frac{du}{dt}f(t) + u(t)\frac{df}{dt}\right)\mathbf{i} + \left(\frac{du}{dt}g(t) + u(t)\frac{dg}{dt}\right)\mathbf{j} + \left(\frac{du}{dt}h(t) + u(t)\frac{dh}{dt}\right)\mathbf{k}$
$= [f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}]\frac{du}{dt} + u(t)\left(\frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}\right) = \mathbf{r}\frac{du}{dt} + u\frac{d\mathbf{r}}{dt}$

56. (a) If $\mathbf{R}_1(t)$ and $\mathbf{R}_2(t)$ have identical derivatives on I, then $\frac{d\mathbf{R}_1}{dt} = \frac{df_1}{dt}\mathbf{i} + \frac{dg_1}{dt}\mathbf{j} + \frac{dh_1}{dt}\mathbf{k} = \frac{df_2}{dt}\mathbf{i} + \frac{dg_2}{dt}\mathbf{j} + \frac{dh_2}{dt}\mathbf{k}$
$= \frac{d\mathbf{R}_2}{dt}$ $\Rightarrow$ $\frac{df_1}{dt} = \frac{df_2}{dt}$, $\frac{dg_1}{dt} = \frac{dg_2}{dt}$, $\frac{dh_1}{dt} = \frac{dh_2}{dt}$ $\Rightarrow$ $f_1(t) = f_2(t) + c_1$, $g_1(t) = g_2(t) + c_2$, $h_1(t) = h_2(t) + c_3$
$\Rightarrow$ $f_1(t)\mathbf{i} + g_1(t)\mathbf{j} + h_1(t)\mathbf{k} = [f_2(t) + c_1]\mathbf{i} + [g_2(t) + c_2]\mathbf{j} + [h_2(t) + c_3]\mathbf{k}$ $\Rightarrow$ $\mathbf{R}_1(t) = \mathbf{R}_2(t) + \mathbf{C}$, where $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

(b) Let $\mathbf{R}(t)$ be an antiderivative of $\mathbf{r}(t)$ on I. Then $\mathbf{R}'(t) = \mathbf{r}(t)$. If $\mathbf{U}(t)$ is an antiderivative of $\mathbf{r}(t)$ on I, then $\mathbf{U}'(t) = \mathbf{r}(t)$. Thus $\mathbf{U}'(t) = \mathbf{R}'(t)$ on I $\Rightarrow$ $\mathbf{U}(t) = \mathbf{R}(t) + \mathbf{C}$.

57. $\frac{d}{dt}\int_a^t \mathbf{r}(\tau)\,d\tau = \frac{d}{dt}\int_a^t [f(\tau)\mathbf{i} + g(\tau)\mathbf{j} + h(\tau)\mathbf{k}]\,d\tau = \frac{d}{dt}\int_a^t f(\tau)\,d\tau\,\mathbf{i} + \frac{d}{dt}\int_a^t g(\tau)\,d\tau\,\mathbf{j} + \frac{d}{dt}\int_a^t h(\tau)\,d\tau\,\mathbf{k}$
$= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} = \mathbf{r}(t)$. Since $\frac{d}{dt}\int_a^t \mathbf{r}(\tau)\,d\tau = \mathbf{r}(t)$, we have that $\int_a^t \mathbf{r}(\tau)\,d\tau$ is an antiderivative of $\mathbf{r}$. If $\mathbf{R}$ is any antiderivative of $\mathbf{r}$, then $\mathbf{R}(t) = \int_a^t \mathbf{r}(\tau)\,d\tau + \mathbf{C}$ by Exercise 56(b). Then $\mathbf{R}(a) = \int_a^a \mathbf{r}(\tau)\,d\tau + \mathbf{C}$
$= \mathbf{0} + \mathbf{C}$ $\Rightarrow$ $\mathbf{C} = \mathbf{R}(a)$ $\Rightarrow$ $\int_a^t \mathbf{r}(\tau)\,d\tau = \mathbf{R}(t) - \mathbf{C} = \mathbf{R}(t) - \mathbf{R}(a)$ $\Rightarrow$ $\int_a^b \mathbf{r}(\tau)\,d\tau = \mathbf{R}(b) - \mathbf{R}(a)$.

58-61. Example CAS commands:

<u>Maple</u>:

```
> with( plots );
r := t -> [sin(t)-t*cos(t),cos(t)+t*sin(t),t^2];
```

```
t0 := 3*Pi/2;
lo := 0;
hi := 6*Pi;
P1 := spacecurve( r(t), t=lo..hi, axes=boxed, thickness=3 ):
display( P1, title="#58(a) (Section 13.1)" );
Dr := unapply( diff(r(t),t), t );                    # (b)
Dr(t0);                                              # (c)
q1 := expand( r(t0) + Dr(t0)*(t-t0) );
T := unapply( q1, t );
P2 := spacecurve( T(t), t=lo..hi, axes=boxed, thickness=3, color=black ):
display( [P1,P2], title="#58(d) (Section 13.1)" );
```

62-63. Example CAS commands:

Maple:

```
a := 'a'; b := 'b';
r := (a,b,t) -> [cos(a*t),sin(a*t),b*t];
Dr := unapply( diff(r(a,b,t),t), (a,b,t) );
t0 := 3*Pi/2;
q1 := expand( r(a,b,t0) + Dr(a,b,t0)*(t-t0) );
T := unapply( q1, (a,b,t) );
lo := 0;
hi := 4*Pi;
P := NULL:
for a in [ 1, 2, 4, 6 ] do
  P1 := spacecurve( r(a,1,t), t=lo..hi, thickness=3 ):
  P2 := spacecurve( T(a,1,t), t=lo..hi, thickness=3, color=black ):
  P := P, display( [P1,P2], axes=boxed, title=sprintf("#62 (Section 13.1)\n a=%a",a) );
end do:
display( [P], insequence=true );
```

58-63. Example CAS commands:

Mathematica: (assigned functions, parameters, and intervals will vary)

The x-y-z components for the curve are entered as a list of functions of t. The unit vectors **i**, **j**, **k** are not inserted.

If a graph is too small, highlight it and drag out a corner or side to make it larger.

Only the components of r[t] and values for t0, tmin, and tmax require alteration for each problem.

```
Clear[r, v, t, x, y, z]
r[t_]={ Sin[t] − t  Cos[t], Cos[t] + t  Sin[t], t2}
t0= 3π / 2;  tmin= 0;  tmax= 6π;
ParametricPlot3D[Evaluate[r[t]], {t, tmin, tmax}, AxesLabel → {x, y, z}];
v[t_]= r'[t]
tanline[t_]= v[t0] t + r[t0]
ParametricPlot3D[Evaluate[{r[t], tanline[t]}], {t, tmin, tmax}, AxesLabel → {x, y, z}];
```

For 62 and 63, the curve can be defined as a function of t, a, and b. Leave a space between a and t and b and t.

```
Clear[r, v, t, x, y, z, a, b]
r[t_,a_,b_]:={Cos[a t], Sin[a t], b t}
t0= 3π / 2;  tmin= 0;  tmax= 4π;
v[t_,a_,b_]= D[r[t, a, b], t]
tanline[t_,a_,b_]=v[t0, a, b] t + r[t0, a, b]
pa1=ParametricPlot3D[Evaluate[{r[t, 1, 1], tanline[t, 1, 1]}], {t,tmin, tmax}, AxesLabel → {x, y, z}];
```

```
pa2=ParametricPlot3D[Evaluate[{r[t, 2, 1], tanline[t, 2, 1]}], {t,tmin, tmax}, AxesLabel → {x, y, z}];
pa4=ParametricPlot3D[Evaluate[{r[t, 4, 1], tanline[t, 4, 1]}], {t,tmin, tmax}, AxesLabel → {x, y, z}];
pa6=ParametricPlot3D[Evaluate[{r[t, 6, 1], tanline[t, 6, 1]}], {t,tmin, tmax}, AxesLabel → {x, y, z}];
Show[GraphicsArray[{pa1, pa2, pa4, pa6}]]
```

13.2 MODELING PROJECTILE MOTION

1. $x = (v_0 \cos\alpha)t \Rightarrow (21\text{ km})\left(\frac{1000\text{ m}}{1\text{ km}}\right) = (840\text{ m/s})(\cos 60°)t \Rightarrow t = \frac{21{,}000\text{ m}}{(840\text{ m/s})(\cos 60°)} = 50$ seconds

2. $R = \frac{v_0^2}{g}\sin 2\alpha$ and maximum R occurs when $\alpha = 45° \Rightarrow 24.5\text{ km} = \left(\frac{v_0^2}{9.8\text{ m/s}^2}\right)(\sin 90°)$
$\Rightarrow v_0 = \sqrt{(9.8)(24{,}500)\text{ m}^2/\text{s}^2} = 490$ m/s

3. (a) $t = \frac{2v_0 \sin\alpha}{g} = \frac{2(500\text{ m/s})(\sin 45°)}{9.8\text{ m/s}^2} \approx 72.2$ seconds; $R = \frac{v_0^2}{g}\sin 2\alpha = \frac{(500\text{ m/s})^2}{9.8\text{ m/s}^2}(\sin 90°) \approx 25{,}510.2$ m
(b) $x = (v_0\cos\alpha)t \Rightarrow 5000\text{ m} = (500\text{ m/s})(\cos 45°)t \Rightarrow t = \frac{5000\text{ m}}{(500\text{ m/s})(\cos 45°)} \approx 14.14$ s; thus,
$y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \Rightarrow y \approx (500\text{ m/s})(\sin 45°)(14.14\text{ s}) - \frac{1}{2}(9.8\text{ m/s}^2)(14.14\text{ s})^2 \approx 4020$ m
(c) $y_{max} = \frac{(v_0\sin\alpha)^2}{2g} = \frac{((500\text{ m/s})(\sin 45°))^2}{2(9.8\text{ m/s}^2)} \approx 6378$ m

4. $y = y_0 + (v_0\sin\alpha)t - \frac{1}{2}gt^2 \Rightarrow y = 32\text{ ft} + (32\text{ ft/sec})(\sin 30°)t - \frac{1}{2}(32\text{ ft/sec}^2)t^2 \Rightarrow y = 32 + 16t - 16t^2$;
the ball hits the ground when $y = 0 \Rightarrow 0 = 32 + 16t - 16t^2 \Rightarrow t = -1$ or $t = 2 \Rightarrow t = 2$ sec since $t > 0$; thus,
$x = (v_0\cos\alpha)t \Rightarrow x = (32\text{ ft/sec})(\cos 30°)t = 32\left(\frac{\sqrt{3}}{2}\right)(2) \approx 55.4$ ft

5. $x = x_0 + (v_0\cos\alpha)t = 0 + (44\cos 45°)t = 22\sqrt{2}t$ and $y = y_0 + (v_0\sin\alpha)t - \frac{1}{2}gt^2 = 6.5 + (44\sin 45°)t - 16t^2$
$= 6.5 + 22\sqrt{2}t - 16t^2$; the shot lands when $y = 0 \Rightarrow t = \frac{22\sqrt{2}\pm\sqrt{968+416}}{32} \approx 2.135$ sec since $t > 0$; thus
$x = 22\sqrt{2}t \approx \left(22\sqrt{2}\right)(2.135) \approx 66.43$ ft

6. $x = 0 + (44\cos 40°)t \approx 33.706t$ and $y = 6.5 + (44\sin 40°)t - 16t^2 \approx 6.5 + 28.283t - 16t^2$; $y = 0$
$\Rightarrow t \approx \frac{28.283 + \sqrt{(28.283)^2 + 416}}{32} \approx 1.9735$ sec since $t > 0$; thus $x \approx (33.706)(1.9735) \approx 66.52$ ft $\Rightarrow$ the difference in distances is about $66.52 - 66.43 = 0.09$ ft or about 1 inch

7. $R = \frac{v_0^2}{g}\sin 2\alpha \Rightarrow 10\text{ m} = \left(\frac{v_0^2}{9.8\text{ m/s}^2}\right)(\sin 90°) \Rightarrow v_0^2 = 98\text{ m}^2\text{s}^2 \Rightarrow v_0 \approx 9.9$ m/s;
$6\text{m} \approx \frac{(9.9\text{ m/s})^2}{9.8\text{ m/s}^2}(\sin 2\alpha) \Rightarrow \sin 2\alpha \approx 0.59999 \Rightarrow 2\alpha \approx 36.87°$ or $143.12° \Rightarrow \alpha \approx 18.4°$ or $71.6°$

8. $v_0 = 5\times 10^6$ m/s and $x = 40$ cm $= 0.4$ m; thus $x = (v_0\cos\alpha)t \Rightarrow 0.4\text{m} = (5\times 10^6\text{ m/s})(\cos 0°)t$
$\Rightarrow t = 0.08\times 10^{-6}\text{ s} = 8\times 10^{-8}$ s; also, $y = y_0 + (v_0\sin\alpha)t - \frac{1}{2}gt^2$
$\Rightarrow y = (5\times 10^6\text{ m/s})(\sin 0°)(8\times 10^{-8}\text{ s}) - \frac{1}{2}(9.8\text{ m/s}^2)(8\times 10^{-8}\text{ s})^2 = -3.136\times 10^{-14}$ m or
-3.136×10^{-12} cm. Therefore, it drops 3.136×10^{-12} cm.

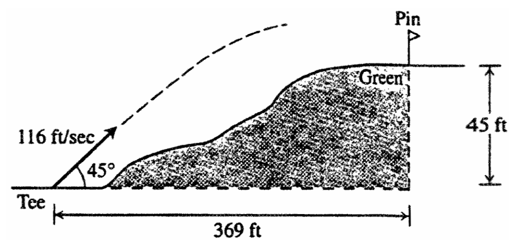
9. $R = \frac{v_0^2}{g}\sin 2\alpha \Rightarrow 3(248.8)\text{ ft} = \left(\frac{v_0^2}{32\text{ ft/sec}^2}\right)(\sin 18°) \Rightarrow v_0^2 \approx 77{,}292.84\text{ ft}^2/\text{sec}^2 \Rightarrow v_0 \approx 278.02$ ft/sec ≈ 190 mph

10. $v_0 = \frac{80\sqrt{10}}{3}$ ft/sec and $R = 200$ ft $\Rightarrow 200 = \frac{\left(\frac{80\sqrt{10}}{3}\right)^2}{32}(\sin 2\alpha) \Rightarrow \sin 2\alpha = 0.9 \Rightarrow 2\alpha \approx 64.2° \Rightarrow \alpha \approx 32.1°$; or
$2\alpha \approx 115.8° \Rightarrow \alpha \approx 57.9°$; If $\alpha \approx 32.1°$, $y_{max} = \frac{\left[\left(\frac{80\sqrt{10}}{3}\right)(\sin 32.1°)\right]^2}{2(32)} \approx 31.4$ ft. If $\alpha \approx 57.9°$, $y_{max} \approx 79.7$ ft > 75 ft. In order to reach the cushion, the angle of elevation will need to be about 32.1°. At this angle, the circus performer will go

31.4 ft into the air at maximum height and will not strike the 75 ft high ceiling.

11. $x = (v_0 \cos \alpha)t \Rightarrow 135 \text{ ft} = (90 \text{ ft/sec})(\cos 30°)t \Rightarrow t \approx 1.732 \text{ sec}; y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$
$\Rightarrow y \approx (90 \text{ ft/sec})(\sin 30°)(1.732 \text{ sec}) - \frac{1}{2}(32 \text{ ft/sec}^2)(1.732 \text{ sec})^2 \Rightarrow y \approx 29.94 \text{ ft} \Rightarrow$ the golf ball will clip the leaves at the top

12. $v_0 = 116$ ft/sec, $\alpha = 45°$, and $x = (v_0 \cos \alpha)t$
$\Rightarrow 369 = (116 \cos 45°)t \Rightarrow t \approx 4.50$ sec;
also $y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$
$\Rightarrow y = (116 \sin 45°)(4.50) - \frac{1}{2}(32)(4.50)^2$
≈ 45.11 ft. It will take the ball 4.50 sec to travel 369 ft. At that time the ball will be 45.11 ft in the air and will hit the green past the pin.



13. We do part b first.
(b) $x = (v_0 \cos \alpha)t \Rightarrow 315 \text{ ft} = (v_0 \cos 20°)t \Rightarrow v_0 = \frac{315}{t \cos 20°}$; also $y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$
$\Rightarrow 34 \text{ ft} = \left(\frac{315}{t \cos 20°}\right)(t \sin 20°) - \frac{1}{2}(32)t^2 \Rightarrow 34 = 315 \tan 20° - 16t^2 \Rightarrow t^2 \approx 5.04 \text{ sec}^2 \Rightarrow t \approx 2.25 \text{ sec}$
(a) $v_0 = \frac{315}{(2.25)(\cos 20°)} \approx 149$ ft/sec

14. $R = \frac{v_0^2}{g} \sin 2\alpha = \frac{v_0^2}{g}(2 \sin \alpha \cos \alpha) = \frac{v_0^2}{g}[2 \cos(90° - \alpha) \sin(90° - \alpha)] = \frac{v_0^2}{g}[\sin 2(90° - \alpha)]$

15. $R = \frac{v_0^2}{g} \sin 2\alpha \Rightarrow 16{,}000 \text{ m} = \frac{(400 \text{ m/s})^2}{9.8 \text{ m/s}^2} \sin 2\alpha \Rightarrow \sin 2\alpha = 0.98 \Rightarrow 2\alpha \approx 78.5°$ or $2\alpha \approx 101.5° \Rightarrow \alpha \approx 39.3°$ or $50.7°$

16. (a) $R = \frac{(2v_0)^2}{g} \sin 2\alpha = \frac{4v_0^2}{g} \sin 2\alpha = 4\left(\frac{v_0^2}{g} \sin \alpha\right)$ or 4 times the original range.
(b) Now, let the initial range be $R = \frac{v_0^2}{g} \sin 2\alpha$. Then we want the factor p so that pv_0 will double the range
$\Rightarrow \frac{(pv_0)^2}{g} \sin 2\alpha = 2\left(\frac{v_0^2}{g} \sin 2\alpha\right) \Rightarrow p^2 = 2 \Rightarrow p = \sqrt{2}$ or about 141%. The same percentage will approximately double the height: $\frac{(pv_0 \sin \alpha)^2}{2g} = \frac{2(v_0 \sin \alpha)^2}{2g} \Rightarrow p^2 = 2 \Rightarrow p = \sqrt{2}$.

17. $x = x_0 + (v_0 \cos \alpha)t = 0 + (v_0 \cos 40°)t \approx 0.766\, v_0 t$ and $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2 = 6.5 + (v_0 \sin 40°)t - 16t^2$
$\approx 6.5 + 0.643\, v_0 t - 16t^2$; now the shot went 73.833 ft $\Rightarrow 73.833 = 0.766\, v_0 t \Rightarrow t \approx \frac{96.383}{v_0}$ sec; the shot lands when $y = 0 \Rightarrow 0 = 6.5 + (0.643)(96.383) - 16\left(\frac{96.383}{v_0}\right)^2 \Rightarrow 0 \approx 68.474 - \frac{148{,}635}{v_0^2} \Rightarrow v_0 \approx \sqrt{\frac{148{,}635}{68.474}}$
≈ 46.6 ft/sec, the shot's initial speed

18. $y_{max} = \frac{(v_0 \sin \alpha)^2}{2g} \Rightarrow \frac{3}{4} y_{max} = \frac{3(v_0 \sin \alpha)^2}{8g}$ and $y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \Rightarrow \frac{3(v_0 \sin \alpha)^2}{8g} = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$
$\Rightarrow 3(v_0 \sin \alpha)^2 = (8gv_0 \sin \alpha)t - 4g^2t^2 \Rightarrow 4g^2t^2 - (8gv_0 \sin \alpha)t + 3(v_0 \sin \alpha)^2 = 0 \Rightarrow 2gt - 3v_0 \sin \alpha = 0$ or $2gt - v_0 \sin \alpha = 0 \Rightarrow t = \frac{3v_0 \sin \alpha}{2g}$ or $t = \frac{v_0 \sin \alpha}{2g}$. Since the time it takes to reach y_{max} is $t_{max} = \frac{v_0 \sin \alpha}{g}$, then the time it takes the projectile to reach $\frac{3}{4}$ of y_{max} is the shorter time $t = \frac{v_0 \sin \alpha}{2g}$ or half the time it takes to reach the maximum height.

19. $\frac{d\mathbf{r}}{dt} = \int(-g\mathbf{j})\, dt = -gt\mathbf{j} + \mathbf{C}_1$ and $\frac{d\mathbf{r}}{dt}(0) = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j} \Rightarrow -g(0)\mathbf{j} + \mathbf{C}_1 = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j}$
$\Rightarrow \mathbf{C}_1 = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha - gt)\mathbf{j}$; $\mathbf{r} = \int[(v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha - gt)\mathbf{j}]\, dt$
$= (v_0 t \cos \alpha)\mathbf{i} + \left(v_0 t \sin \alpha - \frac{1}{2}gt^2\right)\mathbf{j} + \mathbf{C}_2$ and $\mathbf{r}(0) = x_0\mathbf{i} + y_0\mathbf{j} \Rightarrow [v_0(0) \cos \alpha]\mathbf{i} + \left[v_0(0) \sin \alpha - \frac{1}{2}g(0)^2\right]\mathbf{j} + \mathbf{C}_2$
$= x_0\mathbf{i} + y_0\mathbf{j} \Rightarrow \mathbf{C}_2 = x_0\mathbf{i} + y_0\mathbf{j} \Rightarrow \mathbf{r} = (x_0 + v_0 t \cos \alpha)\mathbf{i} + \left(y_0 + v_0 t \sin \alpha - \frac{1}{2}gt^2\right)\mathbf{j} \Rightarrow x = x_0 + v_0 t \cos \alpha$ and

$y = y_0 + v_0 t \sin \alpha - \frac{1}{2} gt^2$

20. From Example 3(b) in the text, $v_0 \sin \alpha = \sqrt{(68)(64)} \Rightarrow v_0 \sin 56.5° \approx 65.97 \Rightarrow v_0 \approx 79$ ft/sec

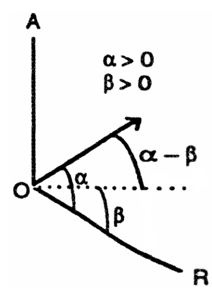
21. The horizontal distance from Rebollo to the center of the cauldron is 90 ft $\Rightarrow$ the horizontal distance to the nearest rim is $x = 90 - \frac{1}{2}(12) = 84 \Rightarrow 84 = x_0 + (v_0 \cos \alpha)t \approx 0 + \left(\frac{90g}{v_0 \sin \alpha}\right) t \Rightarrow 84 = \frac{(90)(32)}{\sqrt{(68)(64)}} t$
$\Rightarrow t = 1.92$ sec. The vertical distance at this time is $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2} gt^2$
$\approx 6 + \sqrt{(68)(64)}\,(1.92) - 16(1.92)^2 \approx 73.7$ ft $\Rightarrow$ the arrow clears the rim by 3.7 ft

22. The projectile rises straight up and then falls straight down, returning to the firing point.

23. Flight time = 1 sec and the measure of the angle of elevation is about 64° (using a protractor) so that $t = \frac{2v_0 \sin \alpha}{g} \Rightarrow 1 = \frac{2v_0 \sin 64°}{32} \Rightarrow v_0 \approx 17.80$ ft/sec. Then $y_{max} = \frac{(17.80 \sin 64°)^2}{2(32)} \approx 4.00$ ft and $R = \frac{v_0^2}{g} \sin 2\alpha \Rightarrow R = \frac{(17.80)^2}{32} \sin 128° \approx 7.80$ ft $\Rightarrow$ the engine traveled about 7.80 ft in 1 sec $\Rightarrow$ the engine velocity was about 7.80 ft/sec

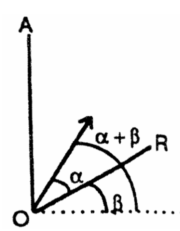
24. When marble A is located R units downrange, we have $x = (v_0 \cos \alpha)t \Rightarrow R = (v_0 \cos \alpha)t \Rightarrow t = \frac{R}{v_0 \cos \alpha}$. At that time the height of marble A is $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2} gt^2 = (v_0 \sin \alpha)\left(\frac{R}{v_0 \cos \alpha}\right) - \frac{1}{2} g \left(\frac{R}{v_0 \cos \alpha}\right)^2$
$\Rightarrow y = R \tan \alpha - \frac{1}{2} g \left(\frac{R^2}{v_0^2 \cos^2 \alpha}\right)$. The height of marble B at the same time $t = \frac{R}{v_0 \cos \alpha}$ seconds is $h = R \tan \alpha - \frac{1}{2} gt^2 = R \tan \alpha - \frac{1}{2} g \left(\frac{R^2}{v_0^2 \cos^2 \alpha}\right)$. Since the heights are the same, the marbles collide regardless of the initial velocity v_0.

25. (a) At the time t when the projectile hits the line OR we have $\tan \beta = \frac{y}{x}$; $x = [v_0 \cos(\alpha - \beta)]t$ and $y = [v_0 \sin(\alpha - \beta)]t - \frac{1}{2} gt^2 < 0$ since R is below level ground. Therefore let $|y| = \frac{1}{2} gt^2 - [v_0 \sin(\alpha - \beta)]t > 0$ so that $\tan \beta = \frac{\left[\frac{1}{2} gt^2 (v_0 \sin(\alpha - \beta))t\right]}{[v_0 \cos(\alpha - \beta)]t} = \frac{\left[\frac{1}{2} gt - v_0 \sin(\alpha - \beta)\right]}{v_0 \cos(\alpha - \beta)}$
$\Rightarrow v_0 \cos(\alpha - \beta) \tan \beta = \frac{1}{2} gt - v_0 \sin(\alpha - \beta)$
$\Rightarrow t = \frac{2v_0 \sin(\alpha - \beta) + 2v_0 \cos(\alpha - \beta) \tan \beta}{g}$, which is the time when the projectile hits the downhill slope. Therefore,



$x = [v_0 \cos(\alpha - \beta)]\left[\frac{2v_0 \sin(\alpha - \beta) + 2v_0 \cos(\alpha - \beta) \tan \beta}{g}\right] = \frac{2v_0^2}{g}\left[\cos^2(\alpha - \beta) \tan \beta + \sin(\alpha - \beta) \cos(\alpha - \beta)\right]$. If x is maximized, then OR is maximized: $\frac{dx}{d\alpha} = \frac{2v_0^2}{g}[-\sin 2(\alpha - \beta) \tan \beta + \cos 2(\alpha - \beta)] = 0$
$\Rightarrow -\sin 2(\alpha - \beta) \tan \beta + \cos 2(\alpha - \beta) = 0 \Rightarrow \tan \beta = \cot 2(\alpha - \beta) \Rightarrow 2(\alpha - \beta) = 90° - \beta$
$\Rightarrow \alpha - \beta = \frac{1}{2}(90° - \beta) \Rightarrow \alpha = \frac{1}{2}(90° + \beta) = \frac{1}{2}$ of $\angle AOR$.

(b) At the time t when the projectile hits OR we have $\tan \beta = \frac{y}{x}$; $x = [v_0 \cos(\alpha + \beta)]t$ and $y = [v_0 \sin(\alpha + \beta)]t - \frac{1}{2} gt^2$
$\Rightarrow \tan \beta = \frac{[v_0 \sin(\alpha + \beta)]t - \frac{1}{2} gt^2}{[v_0 \cos(\alpha + \beta)]t} = \frac{\left[v_0 \sin(\alpha + \beta) - \frac{1}{2} gt\right]}{v_0 \cos(\alpha + \beta)}$
$\Rightarrow v_0 \cos(\alpha + \beta) \tan \beta = v_0 \sin(\alpha + \beta) - \frac{1}{2} gt$
$\Rightarrow t = \frac{2v_0 \sin(\alpha + \beta) - 2v_0 \cos(\alpha + \beta) \tan \beta}{g}$, which is the time when the projectile hits the uphill slope. Therefore,



$x = [v_0 \cos(\alpha+\beta)]\left[\frac{2v_0 \sin(\alpha+\beta) - 2v_0 \cos(\alpha+\beta)\tan\beta}{g}\right] = \frac{2v_0^2}{g}\left[\sin(\alpha+\beta)\cos(\alpha+\beta) - \cos^2(\alpha+\beta)\tan\beta\right]$. If x is maximized, then OR is maximized: $\frac{dx}{d\alpha} = \frac{2v_0^2}{g}\left[\cos 2(\alpha+\beta) + \sin 2(\alpha+\beta)\tan\beta\right] = 0$
$\Rightarrow \cos 2(\alpha+\beta) + \sin 2(\alpha+\beta)\tan\beta = 0 \Rightarrow \cot 2(\alpha+\beta) + \tan\beta = 0 \Rightarrow \cot 2(\alpha+\beta) = -\tan\beta$
$= \tan(-\beta) \Rightarrow 2(\alpha+\beta) = 90° - (-\beta) = 90° + \beta \Rightarrow \alpha = \frac{1}{2}(90° - \beta) = \frac{1}{2}$ of $\angle AOR$. Therefore v_0 would bisect $\angle AOR$ for maximum range uphill.

26. (a) $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$; where $x(t) = (145 \cos 23° - 14)t$ and $y(t) = 2.5 + (145 \sin 23°)t - 16t^2$.
(b) $y_{max} = \frac{(v_0 \sin\alpha)^2}{2g} + 2.5 = \frac{(145 \sin 23°)^2}{64} + 2.5 \approx 52.655$ feet, which is reached at $t = \frac{v_0 \sin\alpha}{g} = \frac{145 \sin 23°}{32} \approx 1.771$ seconds.
(c) For the time, solve $y = 2.5 + (145 \sin 23°)t - 16t^2 = 0$ for t, using the quadratic formula
$t = \frac{145 \sin 23° + \sqrt{(145 \sin 23°)^2 + 160}}{32} \approx 3.585$ sec. Then the range at $t \approx 3.585$ is about $x = (145 \cos 23° - 14)(3.585)$
≈ 428.311 feet.
(d) For the time, solve $y = 2.5 + (145 \sin 23°)t - 16t^2 = 20$ for t, using the quadratic formula
$t = \frac{145 \sin 23° + \sqrt{(145 \sin 23°)^2 - 1120}}{32} \approx 0.342$ and 3.199 seconds. At those times the ball is about
$x(0.342) = (145 \cos 23° - 14)(0.342) \approx 40.860$ feet from home plate and $x(3.199) = (145 \cos 23° - 14)(3.199)$
≈ 382.195 feet from home plate.
(e) Yes. According to part (d), the ball is still 20 feet above the ground when it is 382 feet from home plate.

27. (a) (Assuming that "x" is zero at the point of impact:)
$\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$; where $x(t) = (35 \cos 27°)t$ and $y(t) = 4 + (35 \sin 27°)t - 16t^2$.
(b) $y_{max} = \frac{(v_0 \sin\alpha)^2}{2g} + 4 = \frac{(35 \sin 27°)^2}{64} + 4 \approx 7.945$ feet, which is reached at $t = \frac{v_0 \sin\alpha}{g} = \frac{35 \sin 27°}{32} \approx 0.497$ seconds.
(c) For the time, solve $y = 4 + (35 \sin 27°)t - 16t^2 = 0$ for t, using the quadratic formula
$t = \frac{35 \sin 27° + \sqrt{(-35 \sin 27°)^2 + 256}}{32} \approx 1.201$ sec. Then the range is about $x(1.201) = (35 \cos 27°)(1.201)$
≈ 37.453 feet.
(d) For the time, solve $y = 4 + (35 \sin 27°)t - 16t^2 = 7$ for t, using the quadratic formula
$t = \frac{35 \sin 27° + \sqrt{(-35 \sin 27°)^2 - 192}}{32} \approx 0.254$ and 0.740 seconds. At those times the ball is about
$x(0.254) = (35 \cos 27°)(0.254) \approx 7.921$ feet and $x(0.740) = (35 \cos 27°)(0.740) \approx 23.077$ feet the impact point, or about $37.453 - 7.921 \approx 29.532$ feet and $37.453 - 23.077 \approx 14.376$ feet from the landing spot.
(e) Yes. It changes things because the ball won't clear the net ($y_{max} \approx 7.945$).

28. The maximum height is $y = \frac{(v_0 \sin\alpha)^2}{2g}$ and this occurs for $x = \frac{v_0^2}{2g}\sin 2\alpha = \frac{v_0^2 \sin\alpha\cos\alpha}{g}$. These equations describe parametrically the points on a curve in the xy-plane associated with the maximum heights on the parabolic trajectories in terms of the parameter (launch angle) α. Eliminating the parameter α, we have $x^2 = \frac{v_0^4 \sin^2\alpha\cos^2\alpha}{g^2} = \frac{(v_0^4 \sin^2\alpha)(1-\sin^2\alpha)}{g^2}$
$= \frac{v_0^4 \sin^2\alpha}{g^2} - \frac{v_0^4 \sin^4\alpha}{g^2} = \frac{v_0^2}{g}(2y) - (2y)^2 \Rightarrow x^2 + 4y^2 - \left(\frac{2v_0^2}{g}\right)y = 0 \Rightarrow x^2 + 4\left[y^2 - \left(\frac{v_0^2}{2g}\right)y + \frac{v_0^4}{16g^2}\right] = \frac{v_0^4}{4g^2}$
$\Rightarrow x^2 + 4\left(y - \frac{v_0^2}{4g}\right)^2 = \frac{v_0^4}{4g^2}$, where $x \geq 0$.

29. $\frac{d^2\mathbf{r}}{dt^2} + k\frac{d\mathbf{r}}{dt} = -g\mathbf{j} \Rightarrow P(t) = k$ and $\mathbf{Q}(t) = -g\mathbf{j} \Rightarrow \int P(t)\,dt = kt \Rightarrow v(t) = e^{\int P(t)\,dt} = e^{kt} \Rightarrow \frac{d\mathbf{r}}{dt} = \frac{1}{v(t)}\int v(t)\,\mathbf{Q}(t)\,dt$
$= -ge^{-kt}\int e^{kt}\,\mathbf{j}\,dt = -ge^{-kt}\left[\frac{e^{kt}}{k}\mathbf{j} + \mathbf{C}_1\right] = -\frac{g}{k}\mathbf{j} + \mathbf{C}e^{-kt}$, where $\mathbf{C} = -g\mathbf{C}_1$; apply the initial condition:
$\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = (v_0 \cos\alpha)\mathbf{i} + (v_0 \sin\alpha)\mathbf{j} = -\frac{g}{k}\mathbf{j} + \mathbf{C} \Rightarrow \mathbf{C} = (v_0 \cos\alpha)\mathbf{i} + \left(\frac{g}{k} + v_0 \sin\alpha\right)\mathbf{j}$
$\Rightarrow \frac{d\mathbf{r}}{dt} = \left(v_0 e^{-kt}\cos\alpha\right)\mathbf{i} + \left(-\frac{g}{k} + e^{-kt}\left(\frac{g}{k} + v_0 \sin\alpha\right)\right)\mathbf{j}$, $\mathbf{r} = \int\left[\left(v_0 e^{-kt}\cos\alpha\right)\mathbf{i} + \left(-\frac{g}{k} + e^{-kt}\left(\frac{g}{k} + v_0 \sin\alpha\right)\right)\mathbf{j}\right]dt$
$= \left(-\frac{v_0}{k}e^{-kt}\cos\alpha\right)\mathbf{i} + \left(-\frac{gt}{k} - \frac{e^{-kt}}{k}\left(\frac{g}{k} + v_0 \sin\alpha\right)\right)\mathbf{j} + \mathbf{C}_2$; apply the initial condition:

$\mathbf{r}(0) = \mathbf{0} = \left(-\frac{v_0}{k}\cos\alpha\right)\mathbf{i} + \left(-\frac{g}{k^2} - \frac{v_0 \sin\alpha}{k}\right)\mathbf{j} + \mathbf{C}_2 \Rightarrow \mathbf{C}_2 = \left(\frac{v_0}{k}\cos\alpha\right)\mathbf{i} + \left(\frac{g}{k^2} + \frac{v_0 \sin\alpha}{k}\right)\mathbf{j}$
$\Rightarrow \mathbf{r}(t) = \left(\frac{v_0}{k}\left(1 - e^{-kt}\right)\cos\alpha\right)\mathbf{i} + \left(\frac{v_0}{k}\left(1 - e^{-kt}\right)\sin\alpha + \frac{g}{k^2}\left(1 - kt - e^{-kt}\right)\right)\mathbf{j}$

30. (a) $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$; where $x(t) = \left(\frac{152}{0.12}\right)(1 - e^{-0.12t})(\cos 20°)$ and $y(t) = 3 + \left(\frac{152}{0.12}\right)(1 - e^{-0.12t})(\sin 20°) + \left(\frac{32}{0.12^2}\right)(1 - 0.12t - e^{-0.12t})$
 (b) Solve graphically using a calculator or CAS: At $t \approx 1.484$ seconds the ball reaches a maximum height of about 40.435 feet.
 (c) Use a graphing calculator or CAS to find that $y = 0$ when the ball has traveled for ≈ 3.126 seconds. The range is about $x(3.126) = \left(\frac{152}{0.12}\right)\left(1 - e^{-0.12(3.126)}\right)(\cos 20°) \approx 372.311$ feet.
 (d) Use a graphing calculator or CAS to find that $y = 30$ for $t \approx 0.689$ and 2.305 seconds, at which times the ball is about $x(0.689) \approx 94.454$ feet and $x(2.305) \approx 287.621$ feet from home plate.
 (e) Yes, the batter has hit a home run since a graph of the trajectory shows that the ball is more than 14 feet above the ground when it passes over the fence.

31. (a) $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$; where $x(t) = \left(\frac{1}{0.08}\right)(1 - e^{-0.08t})(152\cos 20° - 17.6)$ and $y(t) = 3 + \left(\frac{152}{0.08}\right)(1 - e^{-0.08t})(\sin 20°) + \left(\frac{32}{0.08^2}\right)(1 - 0.08t - e^{-0.08t})$
 (b) Solve graphically using a calculator or CAS: At $t \approx 1.527$ seconds the ball reaches a maximum height of about 41.893 feet.
 (c) Use a graphing calculator or CAS to find that $y = 0$ when the ball has traveled for ≈ 3.181 seconds. The range is about $x(3.181) = \left(\frac{1}{0.08}\right)\left(1 - e^{-0.08(3.181)}\right)(152\cos 20° - 17.6) \approx 351.734$ feet.
 (d) Use a graphing calculator or CAS to find that $y = 35$ for $t \approx 0.877$ and 2.190 seconds, at which times the ball is about $x(0.877) \approx 106.028$ feet and $x(2.190) \approx 251.530$ feet from home plate.
 (e) No; the range is less than 380 feet. To find the wind needed for a home run, first use the method of part (d) to find that $y = 20$ at $t \approx 0.376$ and 2.716 seconds. Then define $x(w) = \left(\frac{1}{0.08}\right)\left(1 - e^{-0.08(2.716)}\right)(152\cos 20° + w)$, and solve $x(w) = 380$ to find $w \approx 12.846$ ft/sec.

13.3 ARC LENGTH AND THE UNIT TANGENT VECTOR T

1. $\mathbf{r} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + \sqrt{5}t\mathbf{k} \Rightarrow \mathbf{v} = (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + \sqrt{5}\mathbf{k}$
$\Rightarrow |\mathbf{v}| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + \left(\sqrt{5}\right)^2} = \sqrt{4\sin^2 t + 4\cos^2 t + 5} = 3$; $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= \left(-\frac{2}{3}\sin t\right)\mathbf{i} + \left(\frac{2}{3}\cos t\right)\mathbf{j} + \frac{\sqrt{5}}{3}\mathbf{k}$ and Length $= \int_0^\pi |\mathbf{v}|\,dt = \int_0^\pi 3\,dt = [3t]_0^\pi = 3\pi$

2. $\mathbf{r} = (6\sin 2t)\mathbf{i} + (6\cos 2t)\mathbf{j} + 5t\mathbf{k} \Rightarrow \mathbf{v} = (12\cos 2t)\mathbf{i} + (-12\sin 2t)\mathbf{j} + 5\mathbf{k}$
$\Rightarrow |\mathbf{v}| = \sqrt{(12\cos 2t)^2 + (-12\sin 2t)^2 + 5^2} = \sqrt{144\cos^2 2t + 144\sin^2 2t + 25} = 13$; $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= \left(\frac{12}{13}\cos 2t\right)\mathbf{i} - \left(\frac{12}{13}\sin 2t\right)\mathbf{j} + \frac{5}{13}\mathbf{k}$ and Length $= \int_0^\pi |\mathbf{v}|\,dt = \int_0^\pi 13\,dt = [13t]_0^\pi = 13\pi$

3. $\mathbf{r} = t\mathbf{i} + \frac{2}{3}t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + t^{1/2}\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + \left(t^{1/2}\right)^2} = \sqrt{1+t}$; $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{1+t}}\mathbf{i} + \frac{\sqrt{t}}{\sqrt{1+t}}\mathbf{k}$
and Length $= \int_0^8 \sqrt{1+t}\,dt = \left[\frac{2}{3}(1+t)^{3/2}\right]_0^8 = \frac{52}{3}$

4. $\mathbf{r} = (2+t)\mathbf{i} - (t+1)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$; $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$
and Length $= \int_0^3 \sqrt{3}\,dt = \left[\sqrt{3}t\right]_0^3 = 3\sqrt{3}$

5. $\mathbf{r} = (\cos^3 t)\,\mathbf{j} + (\sin^3 t)\,\mathbf{k} \Rightarrow \mathbf{v} = (-3\cos^2 t \sin t)\,\mathbf{j} + (3\sin^2 t \cos t)\,\mathbf{k} \Rightarrow |\mathbf{v}|$
$= \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2} = \sqrt{(9\cos^2 t \sin^2 t)(\cos^2 t + \sin^2 t)} = 3\,|\cos t \sin t|$;
$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-3\cos^2 t \sin t}{3\,|\cos t \sin t|}\,\mathbf{j} + \frac{3\sin^2 t \cos t}{3\,|\cos t \sin t|}\,\mathbf{k} = (-\cos t)\mathbf{j} + (\sin t)\mathbf{k}$, if $0 \le t \le \frac{\pi}{2}$, and
$\text{Length} = \int_0^{\pi/2} 3\,|\cos t \sin t|\, dt = \int_0^{\pi/2} 3\cos t \sin t\, dt = \int_0^{\pi/2} \frac{3}{2}\sin 2t\, dt = \left[-\frac{3}{4}\cos 2t\right]_0^{\pi/2} = \frac{3}{2}$

6. $\mathbf{r} = 6t^3\mathbf{i} - 2t^3\mathbf{j} - 3t^3\mathbf{k} \Rightarrow \mathbf{v} = 18t^2\mathbf{i} - 6t^2\mathbf{j} - 9t^2\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(18t^2)^2 + (-6t^2)^2 + (-9t^2)^2} = \sqrt{441t^4} = 21t^2$;
$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{18t^2}{21t^2}\,\mathbf{i} - \frac{6t^2}{21t^2}\,\mathbf{j} - \frac{9t^2}{21t^2}\,\mathbf{k} = \frac{6}{7}\,\mathbf{i} - \frac{2}{7}\,\mathbf{j} - \frac{3}{7}\,\mathbf{k}$ and $\text{Length} = \int_1^2 21t^2\, dt = \left[7t^3\right]_1^2 = 49$

7. $\mathbf{r} = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + \frac{2\sqrt{2}}{3}t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} + \left(\sqrt{2}\,t^{1/2}\right)\mathbf{k}$
$\Rightarrow |\mathbf{v}| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + \left(\sqrt{2}\,t\right)^2} = \sqrt{1 + t^2 + 2t} = \sqrt{(t+1)^2} = |t+1| = t+1$, if $t \ge 0$;
$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\cos t - t\sin t}{t+1}\right)\mathbf{i} + \left(\frac{\sin t + t\cos t}{t+1}\right)\mathbf{j} + \left(\frac{\sqrt{2}\,t^{1/2}}{t+1}\right)\mathbf{k}$ and $\text{Length} = \int_0^{\pi}(t+1)\, dt = \left[\frac{t^2}{2} + t\right]_0^{\pi} = \frac{\pi^2}{2} + \pi$

8. $\mathbf{r} = (t\sin t + \cos t)\mathbf{i} + (t\cos t - \sin t)\mathbf{j} \Rightarrow \mathbf{v} = (\sin t + t\cos t - \sin t)\mathbf{i} + (\cos t - t\sin t - \cos t)\mathbf{j}$
$= (t\cos t)\mathbf{i} - (t\sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t\cos t)^2 + (-t\sin t)^2} = \sqrt{t^2} = |t| = t$ if $\sqrt{2} \le t \le 2$; $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= \left(\frac{t\cos t}{t}\right)\mathbf{i} - \left(\frac{t\sin t}{t}\right)\mathbf{j} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$ and $\text{Length} = \int_{\sqrt{2}}^{2} t\, dt = \left[\frac{t^2}{2}\right]_{\sqrt{2}}^{2} = 1$

9. Let $P(t_0)$ denote the point. Then $\mathbf{v} = (5\cos t)\mathbf{i} - (5\sin t)\mathbf{j} + 12\mathbf{k}$ and $26\pi = \int_0^{t_0}\sqrt{25\cos^2 t + 25\sin^2 t + 144}\, dt$
$= \int_0^{t_0} 13\, dt = 13t_0 \Rightarrow t_0 = 2\pi$, and the point is $P(2\pi) = (5\sin 2\pi, 5\cos 2\pi, 24\pi) = (0, 5, 24\pi)$

10. Let $P(t_0)$ denote the point. Then $\mathbf{v} = (12\cos t)\mathbf{i} + (12\sin t)\mathbf{j} + 5\mathbf{k}$ and
$-13\pi = \int_0^{t_0}\sqrt{144\cos^2 t + 144\sin^2 t + 25}\, dt = \int_0^{t_0} 13\, dt = 13t_0 \Rightarrow t_0 = -\pi$, and the point is
$P(-\pi) = (12\sin(-\pi), -12\cos(-\pi), -5\pi) = (0, 12, -5\pi)$

11. $\mathbf{r} = (4\cos t)\mathbf{i} + (4\sin t)\mathbf{j} + 3t\mathbf{k} \Rightarrow \mathbf{v} = (-4\sin t)\mathbf{i} + (4\cos t)\mathbf{j} + 3\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2}$
$= \sqrt{25} = 5 \Rightarrow s(t) = \int_0^t 5\, d\tau = 5t \Rightarrow \text{Length} = s\left(\frac{\pi}{2}\right) = \frac{5\pi}{2}$

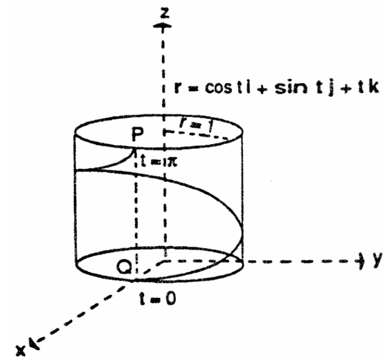
12. $\mathbf{r} = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} \Rightarrow \mathbf{v} = (-\sin t + \sin t + t\cos t)\mathbf{i} + (\cos t - \cos t + t\sin t)\mathbf{j}$
$= (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t\cos t)^2 + (t\cos t)^2} = \, = \sqrt{t^2} = t$, since $\frac{\pi}{2} \le t \le \pi \Rightarrow s(t) = \int_0^t \tau\, d\tau = \frac{t^2}{2}$
$\Rightarrow \text{Length} = s(\pi) - s\left(\frac{\pi}{2}\right) = \frac{\pi^2}{2} - \frac{\left(\frac{\pi}{2}\right)^2}{2} = \frac{3\pi^2}{8}$

13. $\mathbf{r} = (e^t\cos t)\,\mathbf{i} + (e^t\sin t)\,\mathbf{j} + e^t\mathbf{k} \Rightarrow \mathbf{v} = (e^t\cos t - e^t\sin t)\,\mathbf{i} + (e^t\sin t + e^t\cos t)\,\mathbf{j} + e^t\mathbf{k}$
$\Rightarrow |\mathbf{v}| = \sqrt{(e^t\cos t - e^t\sin t)^2 + (e^t\sin t + e^t\cos t)^2 + (e^t)^2} = \, = \sqrt{3e^{2t}} = \sqrt{3}\,e^t \Rightarrow s(t) = \int_0^t \sqrt{3}\,e^{\tau}\, d\tau$
$= \sqrt{3}\,e^t - \sqrt{3} \Rightarrow \text{Length} = s(0) - s(-\ln 4) = 0 - \left(\sqrt{3}\,e^{-\ln 4} - \sqrt{3}\right) = \frac{3\sqrt{3}}{4}$

14. $\mathbf{r} = (1+2t)\mathbf{i} + (1+3t)\mathbf{j} + (6-6t)\mathbf{k} \Rightarrow \mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{2^2 + 3^2 + (-6)^2} = 7 \Rightarrow s(t) = \int_0^t 7\, d\tau = 7t$
$\Rightarrow \text{Length} = s(0) - s(-1) = 0 - (-7) = 7$

15. $\mathbf{r} = \left(\sqrt{2}t\right)\mathbf{i} + \left(\sqrt{2}t\right)\mathbf{j} + (1-t^2)\mathbf{k} \Rightarrow \mathbf{v} = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} - 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2 + (-2t)^2} = \sqrt{4+4t^2}$ $= 2\sqrt{1+t^2} \Rightarrow \text{Length} = \int_0^1 2\sqrt{1+t^2}\,dt = \left[2\left(\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln\left(t+\sqrt{1+t^2}\right)\right)\right]_0^1 = \sqrt{2} + \ln\left(1+\sqrt{2}\right)$

16. Let the helix make one complete turn from $t = 0$ to $t = 2\pi$. Note that the radius of the cylinder is $1 \Rightarrow$ the circumference of the base is 2π. When $t = 2\pi$, the point P is $(\cos 2\pi, \sin 2\pi, 2\pi) = (1, 0, 2\pi) \Rightarrow$ the cylinder is 2π units high. Cut the cylinder along PQ and flatten. The resulting rectangle has a width equal to the circumference of the cylinder $= 2\pi$ and a height equal to 2π, the height of the cylinder. Therefore, the rectangle is a square and the portion of the helix from $t = 0$ to $t = 2\pi$ is its diagonal.



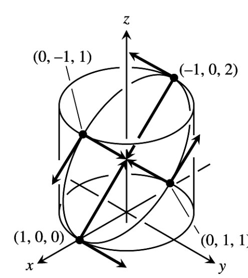
17. (a) $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (1-\cos t)\mathbf{k},\ 0 \le t \le 2\pi \Rightarrow x = \cos t,\ y = \sin t,\ z = 1 - \cos t \Rightarrow x^2 + y^2$ $= \cos^2 t + \sin^2 t = 1$, a right circular cylinder with the z-axis as the axis and radius $= 1$. Therefore $P(\cos t, \sin t, 1-\cos t)$ lies on the cylinder $x^2 + y^2 = 1$; $t = 0 \Rightarrow P(1,0,0)$ is on the curve; $t = \frac{\pi}{2} \Rightarrow Q(0,1,1)$ is on the curve; $t = \pi \Rightarrow R(-1,0,2)$ is on the curve. Then $\overrightarrow{PQ} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\overrightarrow{PR} = -2\mathbf{i} + 2\mathbf{k}$

$$\Rightarrow \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ -2 & 0 & 2 \end{bmatrix} = 2\mathbf{i} + 2\mathbf{k} \text{ is a vector normal to the plane of P, Q, and R. Then the}$$

plane containing P, Q, and R has an equation $2x + 2z = 2(1) + 2(0)$ or $x + z = 1$. Any point on the curve will satisfy this equation since $x + z = \cos t + (1 - \cos t) = 1$. Therefore, any point on the curve lies on the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + z = 1 \Rightarrow$ the curve is an ellipse.

(b) $\mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (\sin t)\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\sin^2 t + \cos^2 t + \sin^2 t} = \sqrt{1+\sin^2 t} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$ $= \frac{(-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (\sin t)\mathbf{k}}{\sqrt{1+\sin^2 t}} \Rightarrow \mathbf{T}(0) = \mathbf{j},\ \mathbf{T}\left(\frac{\pi}{2}\right) = \frac{-\mathbf{i}+\mathbf{k}}{\sqrt{2}},\ \mathbf{T}(\pi) = -\mathbf{j},\ \mathbf{T}\left(\frac{3\pi}{2}\right) = \frac{\mathbf{i}-\mathbf{k}}{\sqrt{2}}$

(c) $\mathbf{a} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j} + (\cos t)\mathbf{k}$; $\mathbf{n} = \mathbf{i} + \mathbf{k}$ is normal to the plane $x + z = 1 \Rightarrow \mathbf{n} \cdot \mathbf{a} = -\cos t + \cos t$ $= 0 \Rightarrow \mathbf{a}$ is orthogonal to $\mathbf{n} \Rightarrow \mathbf{a}$ is parallel to the plane; $\mathbf{a}(0) = -\mathbf{i} + \mathbf{k},\ \mathbf{a}\left(\frac{\pi}{2}\right) = -\mathbf{j},\ \mathbf{a}(\pi) = \mathbf{i} - \mathbf{k},$ $\mathbf{a}\left(\frac{3\pi}{2}\right) = \mathbf{j}$



(d) $|\mathbf{v}| = \sqrt{1+\sin^2 t}$ (See part (b) $\Rightarrow L = \int_0^{2\pi} \sqrt{1+\sin^2 t}\,dt$

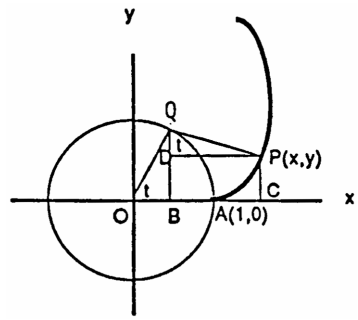
(e) $L \approx 7.64$ (by *Mathematica*)

18. (a) $\mathbf{r} = (\cos 4t)\mathbf{i} + (\sin 4t)\mathbf{j} + 4t\mathbf{k} \Rightarrow \mathbf{v} = (-4\sin 4t)\mathbf{i} + (4\cos 4t)\mathbf{j} + 4\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-4\sin 4t)^2 + (4\cos 4t)^2 + 4^2}$ $= \sqrt{32} = 4\sqrt{2} \Rightarrow \text{Length} = \int_0^{\pi/2} 4\sqrt{2}\,dt = \left[4\sqrt{2}\,t\right]_0^{\pi/2} = 2\pi\sqrt{2}$

(b) $\mathbf{r} = \left(\cos\frac{t}{2}\right)\mathbf{i} + \left(\sin\frac{t}{2}\right)\mathbf{j} + \frac{t}{2}\mathbf{k} \Rightarrow \mathbf{v} = \left(-\frac{1}{2}\sin\frac{t}{2}\right)\mathbf{i} + \left(\frac{1}{2}\cos\frac{t}{2}\right)\mathbf{j} + \frac{1}{2}\mathbf{k}$ $\Rightarrow |\mathbf{v}| = \sqrt{\left(-\frac{1}{2}\sin\frac{t}{2}\right)^2 + \left(\frac{1}{2}\cos\frac{t}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{\sqrt{2}}{2} \Rightarrow \text{Length} = \int_0^{4\pi} \frac{\sqrt{2}}{2}\,dt = \left[\frac{\sqrt{2}}{2}\,t\right]_0^{4\pi} = 2\pi\sqrt{2}$

(c) $\mathbf{r} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} - t\mathbf{k} \Rightarrow \mathbf{v} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j} - \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-\sin t)^2 + (-\cos t)^2 + (-1)^2} = \sqrt{1+1}$ $= \sqrt{2} \Rightarrow \text{Length} = \int_{-2\pi}^{0} \sqrt{2}\,dt = \left[\sqrt{2}\,t\right]_{-2\pi}^{0} = 2\pi\sqrt{2}$

19. $\angle PQB = \angle QOB = t$ and $PQ = \text{arc}(AQ) = t$ since
PQ = length of the unwound string = length of arc (AQ);
thus $x = OB + BC = OB + DP = \cos t + t\sin t$, and
$y = PC = QB - QD = \sin t - t\cos t$



20. $\mathbf{r} = (\cos t + t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} \Rightarrow \mathbf{v} = (-\sin t + t\cos t + \sin t)\mathbf{i} + (\cos t - (t(-\sin t) + \cos t))\mathbf{j}$
$= (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t\cos t)^2 + (t\sin t)^2} = \sqrt{t^2} = |t| = t,\ t \ge 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{t\cos t}{t}\mathbf{i} + \frac{t\sin t}{t}\mathbf{j}$
$= \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$

13.4 CURVATURE AND THE UNIT NORMAL VECTOR N

1. $\mathbf{r} = t\mathbf{i} + \ln(\cos t)\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + \left(\frac{-\sin t}{\cos t}\right)\mathbf{j} = \mathbf{i} - (\tan t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (-\tan t)^2} = \sqrt{\sec^2 t} = |\sec t| = \sec t$, since $-\frac{\pi}{2} < t < \frac{\pi}{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sec t}\right)\mathbf{i} - \left(\frac{\tan t}{\sec t}\right)\mathbf{j} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$; $\frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}$
$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}$;
$\kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sec t} \cdot 1 = \cos t.$

2. $\mathbf{r} = \ln(\sec t)\mathbf{i} + t\mathbf{j} \Rightarrow \mathbf{v} = \left(\frac{\sec t \tan t}{\sec t}\right)\mathbf{i} + \mathbf{j} = (\tan t)\mathbf{i} + \mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(\tan t)^2 + 1^2} = \sqrt{\sec^2 t} = |\sec t| = \sec t$,
since $-\frac{\pi}{2} < t < \frac{\pi}{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\tan t}{\sec t}\right)\mathbf{i} - \left(\frac{1}{\sec t}\right)\mathbf{j} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}$; $\frac{d\mathbf{T}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$
$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(\cos t)^2 + (-\sin t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$;
$\kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sec t} \cdot 1 = \cos t.$

3. $\mathbf{r} = (2t+3)\mathbf{i} + (5-t^2)\mathbf{j} \Rightarrow \mathbf{v} = 2\mathbf{i} - 2t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{2^2 + (-2t)^2} = 2\sqrt{1+t^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{2\sqrt{1+t^2}}\mathbf{i} + \frac{-2t}{2\sqrt{1+t^2}}\mathbf{j}$
$= \frac{1}{\sqrt{1+t^2}}\mathbf{i} - \frac{t}{\sqrt{1+t^2}}\mathbf{j}$; $\frac{d\mathbf{T}}{dt} = \frac{-t}{\left(\sqrt{1+t^2}\right)^3}\mathbf{i} - \frac{1}{\left(\sqrt{1+t^2}\right)^3}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(\frac{-t}{\left(\sqrt{1+t^2}\right)^3}\right)^2 + \left(-\frac{1}{\left(\sqrt{1+t^2}\right)^3}\right)^2}$
$= \sqrt{\frac{1}{(1+t^2)^2}} = \frac{1}{1+t^2} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \frac{-t}{\sqrt{1+t^2}}\mathbf{i} - \frac{1}{\sqrt{1+t^2}}\mathbf{j}$;
$\kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{2\sqrt{1+t^2}} \cdot \frac{1}{1+t^2} = \frac{1}{2(1+t^2)^{3/2}}$

4. $\mathbf{r} = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} \Rightarrow \mathbf{v} = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t\cos t)^2 + (t\sin t)^2} = \sqrt{t^2} = |t|$
$= t$, since $t > 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(t\cos t)\mathbf{i} + (t\sin t)\mathbf{j}}{t} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$; $\frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$
$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$; $\kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{t} \cdot 1 = \frac{1}{t}$

5. (a) $\kappa(x) = \frac{1}{|\mathbf{v}(x)|} \cdot \left|\frac{d\mathbf{T}(x)}{dt}\right|$. Now, $\mathbf{v} = \mathbf{i} + f'(x)\mathbf{j} \Rightarrow |\mathbf{v}(x)| = \sqrt{1 + [f'(x)]^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= \left(1 + [f'(x)]^2\right)^{-1/2}\mathbf{i} + f'(x)\left(1 + [f'(x)]^2\right)^{-1/2}\mathbf{j}$. Thus $\frac{d\mathbf{T}}{dt}(x) = \frac{-f'(x)f''(x)}{\left(1+[f'(x)]^2\right)^{3/2}}\mathbf{i} + \frac{f''(x)}{\left(1+[f'(x)]^2\right)^{3/2}}\mathbf{j}$
$\Rightarrow \left|\frac{d\mathbf{T}(x)}{dt}\right| = \sqrt{\left[\frac{-f'(x)f''(x)}{\left(1+[f'(x)]^2\right)^{3/2}}\right]^2 + \left(\frac{f''(x)}{\left(1+[f'(x)]^2\right)^{3/2}}\right)^2} = \sqrt{\frac{[f''(x)]^2\left(1+[f'(x)]^2\right)}{\left(1+[f'(x)]^2\right)^3}} = \frac{|f''(x)|}{\left|1+[f'(x)]^2\right|}$

Thus $\kappa(x) = \frac{1}{(1+[f'(x)]^2)^{1/2}} \cdot \frac{|f''(x)|}{|1+[f'(x)]^2|} = \frac{|f''(x)|}{\left(1+[f'(x)]^2\right)^{3/2}}$

(b) $y = \ln(\cos x) \Rightarrow \frac{dy}{dx} = \left(\frac{1}{\cos x}\right)(-\sin x) = -\tan x \Rightarrow \frac{d^2y}{dx^2} = -\sec^2 x \Rightarrow \kappa = \frac{|-\sec^2 x|}{[1+(-\tan x)^2]^{3/2}} = \frac{\sec^2 x}{|\sec^3 x|}$
$= \frac{1}{\sec x} = \cos x$, since $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(c) Note that $f''(x) = 0$ at an inflection point.

6. (a) $\mathbf{r} = f(t)\mathbf{i} + g(t)\mathbf{j} = x\mathbf{i} + y\mathbf{j} \Rightarrow \mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{\dot{x}^2+\dot{y}^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\mathbf{i} + \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\mathbf{j}$

$\frac{d\mathbf{T}}{dt} = \frac{\dot{y}(\dot{y}\ddot{x}-\dot{x}\ddot{y})}{(\dot{x}^2+\dot{y}^2)^{3/2}}\mathbf{i} + \frac{\dot{x}(\dot{x}\ddot{y}-\dot{y}\ddot{x})}{(\dot{x}^2+\dot{y}^2)^{3/2}}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left[\frac{\dot{y}(\dot{y}\ddot{x}-\dot{x}\ddot{y})}{(\dot{x}^2+\dot{y}^2)^{3/2}}\right]^2 + \left[\frac{\dot{x}(\dot{x}\ddot{y}-\dot{y}\ddot{x})}{(\dot{x}^2+\dot{y}^2)^{3/2}}\right]^2} = \sqrt{\frac{(\dot{y}^2+\dot{x}^2)(\dot{y}\ddot{x}-\dot{x}\ddot{y})^2}{(\dot{x}^2+\dot{y}^2)^3}}$
$= \frac{|\dot{y}\ddot{x}-\dot{x}\ddot{y}|}{|\dot{x}^2+\dot{y}^2|}$; $\kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}} \cdot \frac{|\dot{y}\ddot{x}-\dot{x}\ddot{y}|}{|\dot{x}^2+\dot{y}^2|} = \frac{|\dot{y}\ddot{x}-\dot{x}\ddot{y}|}{(\dot{x}^2+\dot{y}^2)^{3/2}}$.

(b) $\mathbf{r}(t) = t\mathbf{i} + \ln(\sin t)\mathbf{j}$, $0 < t < \pi \Rightarrow x = t$ and $y = \ln(\sin t) \Rightarrow \dot{x} = 1, \ddot{x} = 0; \dot{y} = \frac{\cos t}{\sin t} = \cot t, \ddot{y} = -\csc^2 t$
$\Rightarrow \kappa = \frac{|-\csc^2 t - 0|}{(1+\cot^2 t))^{3/2}} = \frac{\csc^2 t}{\csc^3 t} = \sin t$

(c) $\mathbf{r}(t) = \tan^{-1}(\sinh t)\mathbf{i} + \ln(\cosh t)\mathbf{j} \Rightarrow x = \tan^{-1}(\sinh t)$ and $y = \ln(\cosh t) \Rightarrow \dot{x} = \frac{\cosh t}{1+\sinh^2 t} = \frac{1}{\cosh t}$
$= \operatorname{sech} t, \ddot{x} = -\operatorname{sech} t \tanh t; \dot{y} = \frac{\sinh t}{\cosh t} = \tanh t, \ddot{y} = \operatorname{sech}^2 t \Rightarrow \kappa = \frac{|\operatorname{sech}^3 t + \operatorname{sech} t \tanh^2 t|}{(\operatorname{sech}^2 t + \tanh^2 t)} = |\operatorname{sech} t|$
$= \operatorname{sech} t$

7. (a) $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \Rightarrow \mathbf{v} = f'(t)\mathbf{i} + g'(t)\mathbf{j}$ is tangent to the curve at the point $(f(t), g(t))$;
$\mathbf{n} \cdot \mathbf{v} = [-g'(t)\mathbf{i} + f'(t)\mathbf{j}] \cdot [f'(t)\mathbf{i} + g'(t)\mathbf{j}] = -g'(t)f'(t) + f'(t)g'(t) = 0$; $-\mathbf{n} \cdot \mathbf{v} = -(\mathbf{n} \cdot \mathbf{v}) = 0$; thus, $\mathbf{n}$ and $-\mathbf{n}$ are both normal to the curve at the point

(b) $\mathbf{r}(t) = t\mathbf{i} + e^{2t}\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + 2e^{2t}\mathbf{j} \Rightarrow \mathbf{n} = -2e^{2t}\mathbf{i} + \mathbf{j}$ points toward the concave side of the curve; $\mathbf{N} = \frac{\mathbf{n}}{|\mathbf{n}|}$ and $|\mathbf{n}| = \sqrt{4e^{4t}+1} \Rightarrow \mathbf{N} = \frac{-2e^{2t}}{\sqrt{1+4e^{4t}}}\mathbf{i} + \frac{1}{\sqrt{1+4e^{4t}}}\mathbf{j}$

(c) $\mathbf{r}(t) = \sqrt{4-t^2}\,\mathbf{i} + t\mathbf{j} \Rightarrow \mathbf{v} = \frac{-t}{\sqrt{4-t^2}}\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{n} = -\mathbf{i} - \frac{t}{\sqrt{4-t^2}}\mathbf{j}$ points toward the concave side of the curve;
$\mathbf{N} = \frac{\mathbf{n}}{|\mathbf{n}|}$ and $|\mathbf{n}| = \sqrt{1+\frac{t^2}{4-t^2}} = \frac{2}{\sqrt{4-t^2}} \Rightarrow \mathbf{N} = -\frac{1}{2}\left(\sqrt{4-t^2}\,\mathbf{i} + t\mathbf{j}\right)$

8. (a) $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{3}t^3\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + t^2\mathbf{j} \Rightarrow \mathbf{n} = t^2\mathbf{i} - \mathbf{j}$ points toward the concave side of the curve when $t < 0$ and $-\mathbf{n} = -t^2\mathbf{i} + \mathbf{j}$ points toward the concave side when $t > 0 \Rightarrow \mathbf{N} = \frac{1}{\sqrt{1+t^4}}(t^2\mathbf{i} - \mathbf{j})$ for $t < 0$ and
$\mathbf{N} = \frac{1}{\sqrt{1+t^4}}(-t^2\mathbf{i} + \mathbf{j})$ for $t > 0$

(b) From part (a), $|\mathbf{v}| = \sqrt{1+t^4} \Rightarrow \mathbf{T} = \frac{1}{\sqrt{1+t^4}}\mathbf{i} + \frac{t^2}{\sqrt{1+t^4}}\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \frac{-2t^3}{(1+t^4)^{3/2}}\mathbf{i} + \frac{2t}{(1+t^4)^{3/2}}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{4t^6+4t^2}{(1+t^4)^3}}$
$= \frac{2|t|}{1+t^4}$; $\mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \frac{1+t^4}{2|t|}\left(\frac{-2t^3}{(1+t^4)^{3/2}}\mathbf{i} + \frac{2t}{(1+t^4)^{3/2}}\mathbf{j}\right) = \frac{-t^3}{|t|\sqrt{1+t^4}}\mathbf{i} + \frac{t}{|t|\sqrt{1+t^4}}\mathbf{j}; t \neq 0$
$\mathbf{N}$ does not exist at $t = 0$, where the curve has a point of inflection; $\left.\frac{d\mathbf{T}}{dt}\right|_{t=0} = 0$ so the curvature $\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$
$= \left|\frac{d\mathbf{T}}{dt} \cdot \frac{dt}{ds}\right| = 0$ at $t = 0 \Rightarrow \mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}$ is undefined. Since $x = t$ and $y = \frac{1}{3}t^3 \Rightarrow y = \frac{1}{3}x^3$, the curve is the cubic power curve which is concave down for $x = t < 0$ and concave up for $x = t > 0$.

9. $\mathbf{r} = (3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 4t\mathbf{k} \Rightarrow \mathbf{v} = (3\cos t)\mathbf{i} + (-3\sin t)\mathbf{j} + 4\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(3\cos t)^2 + (-3\sin t)^2 + 4^2}$
$= \sqrt{25} = 5 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{3}{5}\cos t\right)\mathbf{i} - \left(\frac{3}{5}\sin t\right)\mathbf{j} + \frac{4}{5}\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{3}{5}\sin t\right)\mathbf{i} - \left(\frac{3}{5}\cos t\right)\mathbf{j}$
$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(-\frac{3}{5}\sin t\right)^2 + \left(-\frac{3}{5}\cos t\right)^2} = \frac{3}{5} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}$; $\kappa = \frac{1}{5} \cdot \frac{3}{5} = \frac{3}{25}$

10. $\mathbf{r} = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} + 3\mathbf{k} \Rightarrow \mathbf{v} = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t\cos t)^2 + (t\sin t)^2} = \sqrt{t^2}$
$= |t| = t$, if $t > 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$, $t > 0 \Rightarrow \frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$
$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$; $\kappa = \frac{1}{t} \cdot 1 = \frac{1}{t}$

11. $\mathbf{r} = (e^t \cos t)\,\mathbf{i} + (e^t \sin t)\,\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{v} = (e^t \cos t - e^t \sin t)\,\mathbf{i} + (e^t \sin t + e^t \cos t)\,\mathbf{j} \Rightarrow$
$|\mathbf{v}| = \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2} = \sqrt{2e^{2t}} = e^t\sqrt{2}\,;$
$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{\sin t + \cos t}{\sqrt{2}}\right)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(\frac{-\sin t - \cos t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)\mathbf{j}$
$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(\frac{-\sin t - \cos t}{\sqrt{2}}\right)^2 + \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \left(\frac{-\cos t - \sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{-\sin t + \cos t}{\sqrt{2}}\right)\mathbf{j}\,;$
$\kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{e^t\sqrt{2}} \cdot 1 = \frac{1}{e^t\sqrt{2}}$

12. $\mathbf{r} = (6 \sin 2t)\mathbf{i} + (6 \cos 2t)\mathbf{j} + 5t\mathbf{k} \Rightarrow \mathbf{v} = (12 \cos 2t)\mathbf{i} - (12 \sin 2t)\mathbf{j} + 5\mathbf{k}$
$\Rightarrow |\mathbf{v}| = \sqrt{(12 \cos 2t)^2 + (-12 \sin 2t)^2 + 5^2} = \sqrt{169} = 13 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= \left(\frac{12}{13} \cos 2t\right)\mathbf{i} - \left(\frac{12}{13} \sin 2t\right)\mathbf{j} + \frac{5}{13}\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{24}{13} \sin 2t\right)\mathbf{i} - \left(\frac{24}{13} \cos 2t\right)\mathbf{j}$
$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(-\frac{24}{13} \sin 2t\right)^2 + \left(-\frac{24}{13} \cos 2t\right)^2} = \frac{24}{13} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin 2t)\mathbf{i} - (\cos 2t)\mathbf{j}\,;$
$\kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{13} \cdot \frac{24}{13} = \frac{24}{169}.$

13. $\mathbf{r} = \left(\frac{t^3}{3}\right)\mathbf{i} + \left(\frac{t^2}{2}\right)\mathbf{j},\, t > 0 \Rightarrow \mathbf{v} = t^2\mathbf{i} + t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{t^4 + t^2} = t\sqrt{t^2 + 1}$, since $t > 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= \frac{t}{\sqrt{t^2 + t}}\mathbf{i} + \frac{1}{\sqrt{t^2 + 1}}\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \frac{1}{(t^2+1)^{3/2}}\,\mathbf{i} - \frac{t}{(t^2+1)^{3/2}}\,\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(\frac{1}{(t^2+1)^{3/2}}\right)^2 + \left(\frac{-t}{(t^2+1)^{3/2}}\right)^2}$
$= \sqrt{\frac{1 + t^2}{(t^2+1)^3}} = \frac{1}{t^2 + 1} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \frac{1}{\sqrt{t^2+1}}\,\mathbf{i} - \frac{t}{\sqrt{t^2+1}}\,\mathbf{j}\,; \kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{t\sqrt{t^2+1}} \cdot \frac{1}{t^2+1} = \frac{1}{t\,(t^2+1)^{3/2}}.$

14. $\mathbf{r} = (\cos^3 t)\,\mathbf{i} + (\sin^3 t)\,\mathbf{j},\, 0 < t < \frac{\pi}{2} \Rightarrow \mathbf{v} = (-3 \cos^2 t \sin t)\,\mathbf{i} + (3 \sin^2 t \cos t)\,\mathbf{j}$
$\Rightarrow |\mathbf{v}| = \sqrt{(-3 \cos^2 t \sin t)^2 + (3 \sin^2 t \cos t)^2} = \sqrt{9 \cos^4 t \sin^2 t + 9 \sin^4 t \cos^2 t} = 3 \cos t \sin t$, since $0 < t < \frac{\pi}{2}$
$\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = (-\cos t)\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|}$
$= (\sin t)\mathbf{i} + (\cos t)\mathbf{j}; \kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{3 \cos t \sin t} \cdot 1 = \frac{1}{3 \cos t \sin t}.$

15. $\mathbf{r} = t\mathbf{i} + \left(a \cosh \frac{t}{a}\right)\mathbf{j},\, a > 0 \Rightarrow \mathbf{v} = \mathbf{i} + \left(\sinh \frac{t}{a}\right)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1 + \sinh^2\left(\frac{t}{a}\right)} = \sqrt{\cosh^2\left(\frac{t}{a}\right)} = \cosh \frac{t}{a}$
$\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\operatorname{sech} \frac{t}{a}\right)\mathbf{i} + \left(\tanh \frac{t}{a}\right)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{1}{a} \operatorname{sech} \frac{t}{a} \tanh \frac{t}{a}\right)\mathbf{i} + \left(\frac{1}{a} \operatorname{sech}^2 \frac{t}{a}\right)\mathbf{j}$
$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{1}{a^2} \operatorname{sech}^2\left(\frac{t}{a}\right) \tanh^2\left(\frac{t}{a}\right) + \frac{1}{a^2} \operatorname{sech}^4\left(\frac{t}{a}\right)} = \frac{1}{a} \operatorname{sech}\left(\frac{t}{a}\right) \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \left(-\tanh \frac{t}{a}\right)\mathbf{i} + \left(\operatorname{sech} \frac{t}{a}\right)\mathbf{j}\,;$
$\kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\cosh \frac{t}{a}} \cdot \frac{1}{a} \operatorname{sech}\left(\frac{t}{a}\right) = \frac{1}{a} \operatorname{sech}^2\left(\frac{t}{a}\right).$

16. $\mathbf{r} = (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = (\sinh t)\mathbf{i} - (\cosh t)\mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\sinh^2 t + (-\cosh t)^2 + 1} = \sqrt{2} \cosh t$
$\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{2}} \tanh t\right)\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j} + \left(\frac{1}{\sqrt{2}} \operatorname{sech} t\right)\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(\frac{1}{\sqrt{2}} \operatorname{sech}^2 t\right)\mathbf{i} - \left(\frac{1}{\sqrt{2}} \operatorname{sech} t \tanh t\right)\mathbf{k}$
$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{1}{2} \operatorname{sech}^4 t + \frac{1}{2} \operatorname{sech}^2 t \tanh^2 t} = \frac{1}{\sqrt{2}} \operatorname{sech} t \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (\operatorname{sech} t)\mathbf{i} - (\tanh t)\mathbf{k}\,;$
$\kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sqrt{2} \cosh t} \cdot \frac{1}{\sqrt{2}} \operatorname{sech} t = \frac{1}{2} \operatorname{sech}^2 t.$

17. $y = ax^2 \Rightarrow y' = 2ax \Rightarrow y'' = 2a$; from Exercise 5(a), $\kappa(x) = \frac{|2a|}{(1 + 4a^2x^2)^{3/2}} = |2a|\,(1 + 4a^2x^2)^{-3/2}$
$\Rightarrow \kappa'(x) = -\frac{3}{2}\,|2a|\,(1 + 4a^2x^2)^{-5/2}\,(8a^2x)$; thus, $\kappa'(x) = 0 \Rightarrow x = 0$. Now, $\kappa'(x) > 0$ for $x < 0$ and $\kappa'(x) < 0$ for $x > 0$ so that $\kappa(x)$ has an absolute maximum at $x = 0$ which is the vertex of the parabola. Since $x = 0$ is the only critical point for $\kappa(x)$, the curvature has no minimum value.

18. $\mathbf{r} = (a\cos t)\mathbf{i} + (b\sin t)\mathbf{j} \Rightarrow \mathbf{v} = (-a\sin t)\mathbf{i} + (b\cos t)\mathbf{j} \Rightarrow \mathbf{a} = (-a\cos t)\mathbf{i} - (b\sin t)\mathbf{j} \Rightarrow \mathbf{v}\times\mathbf{a}$
$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & b\cos t & 0 \\ -a\cos t & -b\sin t & 0 \end{vmatrix} = ab\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}| = |ab| = ab$, since $a > b > 0$; $\kappa(t) = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$
$= ab\left(a^2\sin^2 t + b^2\cos^2 t\right)^{-3/2}$; $\kappa'(t) = -\frac{3}{2}(ab)\left(a^2\sin^2 t + b^2\cos^2 t\right)^{-5/2}\left(2a^2\sin t\cos t - 2b^2\sin t\cos t\right)$
$= -\frac{3}{2}(ab)\left(a^2 - b^2\right)(\sin 2t)\left(a^2\sin^2 t + b^2\cos^2 t\right)^{-5/2}$; thus, $\kappa'(t) = 0 \Rightarrow \sin 2t = 0 \Rightarrow t = 0, \pi$ identifying points on the major axis, or $t = \frac{\pi}{2}, \frac{3\pi}{2}$ identifying points on the minor axis. Furthermore, $\kappa'(t) < 0$ for $0 < t < \frac{\pi}{2}$ and for $\pi < t < \frac{3\pi}{2}$; $\kappa'(t) > 0$ for $\frac{\pi}{2} < t < \pi$ and $\frac{3\pi}{2} < t < 2\pi$. Therefore, the points associated with $t = 0$ and $t = \pi$ on the major axis give absolute maximum curvature and the points associated with $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ on the minor axis give absolute minimum curvature.

19. $\kappa = \frac{a}{a^2+b^2} \Rightarrow \frac{d\kappa}{da} = \frac{-a^2+b^2}{\left(a^2+b^2\right)^2}$; $\frac{d\kappa}{da} = 0 \Rightarrow -a^2+b^2 = 0 \Rightarrow a = \pm b \Rightarrow a = b$ since $a, b \ge 0$. Now, $\frac{d\kappa}{da} > 0$ if $a < b$ and $\frac{d\kappa}{da} < 0$ if $a > b \Rightarrow \kappa$ is at a maximum for $a = b$ and $\kappa(b) = \frac{b}{b^2+b^2} = \frac{1}{2b}$ is the maximum value of κ.

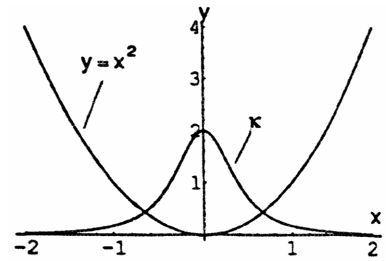
20. (a) From Example 5, the curvature of the helix $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\mathbf{k}$, $a, b \ge 0$ is $\kappa = \frac{a}{a^2+b^2}$; also $|\mathbf{v}| = \sqrt{a^2+b^2}$. For the helix $\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t\mathbf{k}$, $0 \le t \le 4\pi$, $a = 3$ and $b = 1 \Rightarrow \kappa = \frac{3}{3^2+1^2} = \frac{3}{10}$ and $|\mathbf{v}| = \sqrt{10} \Rightarrow K = \int_0^{4\pi} \frac{3}{10}\sqrt{10}\,dt = \left[\frac{3}{\sqrt{10}}t\right]_0^{4\pi} = \frac{12\pi}{\sqrt{10}}$

(b) $y = x^2 \Rightarrow x = t$ and $y = t^2$, $-\infty < t < \infty \Rightarrow \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + 2t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1+4t^2}$;
$\mathbf{T} = \frac{1}{\sqrt{1+4t^2}}\mathbf{i} + \frac{2t}{\sqrt{1+4t^2}}\mathbf{j}$; $\frac{d\mathbf{T}}{dt} = \frac{-4t}{\left(1+4t^2\right)^{3/2}}\mathbf{i} + \frac{2}{\left(1+4t^2\right)^{3/2}}\mathbf{j}$; $\left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{16t^2+4}{\left(1+4t^2\right)^3}} = \frac{2}{1+4t^2}$. Thus
$\kappa = \frac{1}{\sqrt{1+4t^2}} \cdot \frac{2}{1+4t^2} = \frac{2}{\left(\sqrt{1+4t^2}\right)^3}$. Then $K = \int_{-\infty}^{\infty} \frac{2}{\left(\sqrt{1+4t^2}\right)^3}\left(\sqrt{1+4t^2}\right)dt = \int_{-\infty}^{\infty} \frac{2}{1+4t^2}\,dt$
$= \lim_{a\to-\infty}\int_a^0 \frac{2}{1+4t^2}\,dt + \lim_{b\to\infty}\int_0^b \frac{2}{1+4t^2}\,dt = \lim_{a\to-\infty}\left[\tan^{-1} 2t\right]_a^0 + \lim_{b\to\infty}\left[\tan^{-1} 2t\right]_0^b$
$= \lim_{a\to-\infty}\left(-\tan^{-1} 2a\right) + \lim_{b\to\infty}\left(\tan^{-1} 2b\right) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

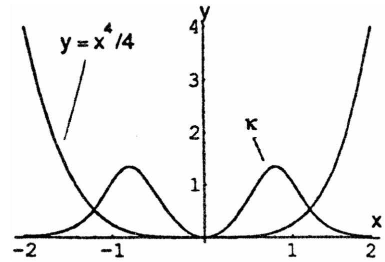
21. $\mathbf{r} = t\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + (\cos t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (\cos t)^2} = \sqrt{1+\cos^2 t} \Rightarrow \left|\mathbf{v}\left(\frac{\pi}{2}\right)\right| = \sqrt{1+\cos^2\left(\frac{\pi}{2}\right)} = 1$; $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= \frac{\mathbf{i}+\cos t\,\mathbf{j}}{\sqrt{1+\cos^2 t}} \Rightarrow \frac{d\mathbf{T}}{dt} = \frac{\sin t\cos t}{\left(1+\cos^2 t\right)^{3/2}}\mathbf{i} + \frac{-\sin t}{\left(1+\cos^2 t\right)^{3/2}}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\sin t|}{1+\cos^2 t}$; $\left|\frac{d\mathbf{T}}{dt}\right|_{t=\frac{\pi}{2}} = \frac{\left|\sin\frac{\pi}{2}\right|}{1+\cos^2\left(\frac{\pi}{2}\right)} = \frac{1}{1} = 1$. Thus $\kappa\left(\frac{\pi}{2}\right) = \frac{1}{1}\cdot 1 = 1$
$\Rightarrow \rho = \frac{1}{1} = 1$ and the center is $\left(\frac{\pi}{2}, 0\right) \Rightarrow \left(x - \frac{\pi}{2}\right)^2 + y^2 = 1$

22. $\mathbf{r} = (2\ln t)\mathbf{i} - \left(t + \frac{1}{t}\right)\mathbf{j} \Rightarrow \mathbf{v} = \left(\frac{2}{t}\right)\mathbf{i} - \left(1 - \frac{1}{t^2}\right)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{\frac{4}{t^2} + \left(1 - \frac{1}{t^2}\right)^2} = \frac{t^2+1}{t^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2t}{t^2+1}\mathbf{i} - \frac{t^2-1}{t^2+1}\mathbf{j}$;
$\frac{d\mathbf{T}}{dt} = \frac{-2\left(t^2-1\right)}{\left(t^2+1\right)^2}\mathbf{i} - \frac{4t}{\left(t^2+1\right)^2}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{4\left(t^2-1\right)^2 + 16t^2}{\left(t^2+1\right)^4}} = \frac{2}{t^2+1}$. Thus $\kappa = \frac{1}{|\mathbf{v}|}\cdot\left|\frac{d\mathbf{T}}{dt}\right| = \frac{t^2}{t^2+1}\cdot\frac{2}{t^2+1} = \frac{2t^2}{\left(t^2+1\right)^2} \Rightarrow \kappa(1) = \frac{2}{2^2}$
$= \frac{1}{2} \Rightarrow \rho = \frac{1}{\kappa} = 2$. The circle of curvature is tangent to the curve at $P(0, -2) \Rightarrow$ circle has same tangent as the curve $\Rightarrow \mathbf{v}(1) = 2\mathbf{i}$ is tangent to the circle $\Rightarrow$ the center lies on the y-axis. If $t \ne 1$ $(t > 0)$, then $(t-1)^2 > 0$
$\Rightarrow t^2 - 2t + 1 > 0 \Rightarrow t^2 + 1 > 2t \Rightarrow \frac{t^2+1}{t} > 2$ since $t > 0 \Rightarrow t + \frac{1}{t} > 2 \Rightarrow -\left(t + \frac{1}{t}\right) < -2 \Rightarrow y < -2$ on both sides of $(0, -2) \Rightarrow$ the curve is concave down $\Rightarrow$ center of circle of curvature is $(0, -4) \Rightarrow x^2 + (y+4)^2 = 4$ is an equation of the circle of curvature

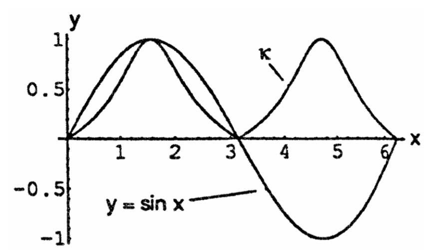
23. $y = x^2 \Rightarrow f'(x) = 2x$ and $f''(x) = 2$
$\Rightarrow \kappa = \frac{|2|}{(1+(2x)^2)^{3/2}} = \frac{2}{(1+4x^2)^{3/2}}$



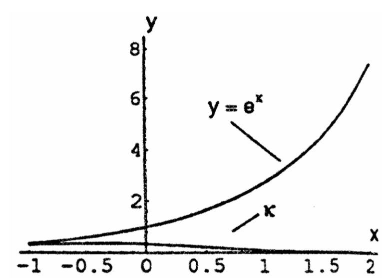
24. $y = \frac{x^4}{4} \Rightarrow f'(x) = x^3$ and $f''(x) = 3x^2$
$\Rightarrow \kappa = \frac{|3x^2|}{\left(1+(x^3)^2\right)^{3/2}} = \frac{3x^2}{(1+x^6)^{3/2}}$



25. $y = \sin x \Rightarrow f'(x) = \cos x$ and $f''(x) = -\sin x$
$\Rightarrow \kappa = \frac{|-\sin x|}{(1+\cos^2 x)^{3/2}} = \frac{|\sin x|}{(1+\cos^2 x)^{3/2}}$



26. $y = e^x \Rightarrow f'(x) = e^x$ and $f''(x) = e^x$
$\Rightarrow \kappa = \frac{|e^x|}{\left(1+(e^x)^2\right)^{3/2}} = \frac{e^x}{(1+e^{2x})^{3/2}}$



27-34. Example CAS commands:

Maple:

```
with( plots );
r := t -> [3*cos(t),5*sin(t)];
lo := 0;
hi := 2*Pi;
t0 := Pi/4;
P1 := plot( [r(t)[], t=lo..hi] ):
display( P1, scaling=constrained, title="#27(a) (Section 13.4)" );
CURVATURE := (x,y,t) ->simplify(abs(diff(x,t)*diff(y,t,t)-diff(y,t)*diff(x,t,t))/(diff(x,t)^2+diff(y,t)^2)^(3/2));
kappa := eval(CURVATURE(r(t)[],t),t=t0);
UnitNormal := (x,y,t) ->expand( [-diff(y,t),diff(x,t)]/sqrt(diff(x,t)^2+diff(y,t)^2) );
N := eval( UnitNormal(r(t)[],t), t=t0 );
C := expand( r(t0) + N/kappa );
OscCircle := (x-C[1])^2+(y-C[2])^2 = 1/kappa^2;
evalf( OscCircle );
P2 := implicitplot( (x-C[1])^2+(y-C[2])^2 = 1/kappa^2, x=-7..4, y=-4..6, color=blue ):
```

```
display( [P1,P2], scaling=constrained, title="#27(e) (Section 13.4)" );
```

Mathematica: (assigned functions and parameters may vary)

In Mathematica, the dot product can be applied either with a period "." or with the word, "Dot".

Similarly, the cross product can be applied either with a very small "x" (in the palette next to the arrow) or with the word, "Cross". However, the Cross command assumes the vectors are in three dimensions

For the purposes of applying the cross product command, we will define the position vector r as a three dimensional vector with zero for its z-component. For graphing, we will use only the first two components.

```
Clear[r, t, x, y]
r[t_]={3 Cos[t], 5 Sin[t] }
t0= π /4;  tmin= 0;  tmax= 2π;
r2[t_]= {r[t][[1]], r[t][[2]]}
pp=ParametricPlot[r2[t], {t, tmin, tmax}];
mag[v_]=Sqrt[v.v]
vel[t_]= r'[t]
speed[t_]=mag[vel[t]]
acc[t_]= vel'[t]
curv[t_]= mag[Cross[vel[t],acc[t]]]/speed[t]^3//Simplify
unittan[t_]= vel[t]/speed[t]//Simplify
unitnorm[t_]= unittan'[t] / mag[unittan'[t]]
ctr= r[t0] + (1 / curv[t0]) unitnorm[t0] //Simplify
{a,b}= {ctr[[1]], ctr[[2]]}
```

To plot the osculating circle, load a graphics package and then plot it, and show it together with the original curve.

```
<<Graphics`ImplicitPlot`
pc=ImplicitPlot[(x − a)2 + (y − b)2 == 1/curv[t0]^2 , {x, −8, 8},{y, −8, 8}]
radius=Graphics[Line[{{a, b}, r2[t0]}]]
Show[pp, pc, radius, AspectRatio → 1]
```

13.5 TORSION AND THE UNIT BINORMAL VECTOR B

1. By Exercise 9 in Section 13.4, $\mathbf{T} = \left(\frac{3}{5}\cos t\right)\mathbf{i} + \left(-\frac{3}{5}\sin t\right)\mathbf{j} + \frac{4}{5}\mathbf{k}$ and $\mathbf{N} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}$ so that $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ $= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3}{5}\cos t & -\frac{3}{5}\sin t & \frac{4}{5} \\ -\sin t & -\cos t & 0 \end{vmatrix} = \left(\frac{4}{5}\cos t\right)\mathbf{i} - \left(\frac{4}{5}\sin t\right)\mathbf{j} - \frac{3}{5}\mathbf{k}$. Also $\mathbf{v} = (3\cos t)\mathbf{i} + (-3\sin t)\mathbf{j} + 4\mathbf{k}$ $\Rightarrow \mathbf{a} = (-3\sin t)\mathbf{i} + (-3\cos t)\mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = (-3\cos t)\mathbf{i} + (3\sin t)\mathbf{j}$ and $\mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \end{vmatrix}$ $= (12\cos t)\mathbf{i} - (12\sin t)\mathbf{j} - 9\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}|^2 = (12\cos t)^2 + (-12\sin t)^2 + (-9)^2 = 225$. Thus $\tau = \frac{\begin{vmatrix} 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \\ -3\cos t & 3\sin t & 0 \end{vmatrix}}{225} = \frac{4\cdot(-9\sin^2 t - 9\cos^2 t)}{225} = \frac{-36}{225} = -\frac{4}{25}$

2. By Exercise 10 in Section 13.4, $\mathbf{T} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ and $\mathbf{N} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$; thus $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ $= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \end{vmatrix} = (\cos^2 t + \sin^2 t)\,\mathbf{k} = \mathbf{k}$. Also $\mathbf{v} = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j}$ $\Rightarrow \mathbf{a} = (t(-\sin t) + \cos t)\mathbf{i} + (t\cos t + \sin t)\mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = (-t\cos t - \sin t - \sin t)\mathbf{i} + (-t\sin t + \cos t + \cos t)\mathbf{j}$ $= (-t\cos t - 2\sin t)\mathbf{i} + (2\cos t - t\sin t)\mathbf{j}$. Thus $\mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t\cos t & t\sin t & 0 \\ (-t\sin t + \cos t) & (t\cos t + \sin t) & 0 \end{vmatrix}$ $= [(t\cos t)(t\cos t + \sin t) - (t\sin t)(-t\sin t + \cos t)]\mathbf{k} = t^2\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}|^2 = (t^2)^2 = t^4$. Thus

$\tau = \frac{\begin{vmatrix} t\cos t & t\sin t & 0 \\ \cos t - t\sin t & \sin t + t\cos t & 0 \\ -2\sin t - t\cos t & 2\cos t - t\sin t & 0 \end{vmatrix}}{t^4} = \frac{0}{t^4} = 0$

3. By Exercise 11 in Section 13.4, $\mathbf{T} = \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{\sin t + \cos t}{\sqrt{2}}\right)\mathbf{j}$ and $\mathbf{N} = \left(\frac{-\cos t - \sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{-\sin t + \cos t}{\sqrt{2}}\right)\mathbf{j}$; Thus $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\cos t - \sin t}{\sqrt{2}} & \frac{\sin t + \cos t}{\sqrt{2}} & 0 \\ \frac{-\cos t - \sin t}{\sqrt{2}} & \frac{-\sin t + \cos t}{\sqrt{2}} & 0 \end{vmatrix} = \left[\left(\frac{\cos^2 t - 2\cos t \sin t + \sin^2 t}{2}\right) + \left(\frac{\sin^2 t + 2\sin t \cos t + \cos^2 t}{2}\right)\right]\mathbf{k}$
$= \left[\left(\frac{1 - \sin(2t)}{2}\right) + \left(\frac{1 + \sin(2t)}{2}\right)\right]\mathbf{k} = \mathbf{k}$. Also, $\mathbf{v} = (e^t\cos t - e^t\sin t)\,\mathbf{i} + (e^t\sin t + e^t\cos t)\,\mathbf{j}$
$\Rightarrow \mathbf{a} = [\,e^t(-\sin t - \cos t) + e^t(\cos t - \sin t)\,]\,\mathbf{i} + [\,e^t(\cos t - \sin t) + e^t(\sin t + \cos t)\,]\,\mathbf{j} = (-2e^t\sin t)\,\mathbf{i} + (2e^t\cos t)\,\mathbf{j}$
$\Rightarrow \frac{d\mathbf{a}}{dt} = -2e^t(\cos t + \sin t)\,\mathbf{i} + 2e^t(-\sin t + \cos t)\mathbf{j}$. Thus $\mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t(\cos t - \sin t) & e^t(\sin t + \cos t) & 0 \\ -2e^t\sin t & 2e^t\cos t & 0 \end{vmatrix} = 2e^{2t}\mathbf{k}$
$\Rightarrow |\mathbf{v} \times \mathbf{a}|^2 = (2e^{2t})^2 = 4e^{4t}$. Thus $\tau = \frac{\begin{vmatrix} e^t(\cos t - \sin t) & e^t(\sin t + \cos t) & 0 \\ -2e^t\sin t & 2e^t\cos t & 0 \\ -2e^t(\cos t + \sin t) & 2e^t(-\sin t + \cos t) & 0 \end{vmatrix}}{4e^{4t}} = 0$

4. By Exercise 12 in Section 13.4, $\mathbf{T} = \left(\frac{12}{13}\cos 2t\right)\mathbf{i} - \left(\frac{12}{13}\sin 2t\right)\mathbf{j} + \frac{5}{13}\mathbf{k}$ and $\mathbf{N} = (-\sin 2t)\mathbf{i} - (\cos 2t)\mathbf{j}$ so $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \left(\frac{12}{13}\cos 2t\right) & \left(-\frac{12}{13}\sin 2t\right) & \frac{5}{13} \\ (-\sin 2t) & (-\cos 2t) & 0 \end{vmatrix} = \left(\frac{5}{13}\cos 2t\right)\mathbf{i} - \left(\frac{5}{13}\sin 2t\right)\mathbf{j} - \frac{12}{13}\mathbf{k}$. Also,
$\mathbf{v} = (12\cos 2t)\mathbf{i} - (12\sin 2t)\mathbf{j} + 5\mathbf{k} \Rightarrow \mathbf{a} = (-24\sin 2t)\mathbf{i} - (24\cos 2t)\mathbf{j}$ and $\frac{d\mathbf{a}}{dt} = (-48\cos 2t)\mathbf{i} + (48\sin 2t)\mathbf{j}$
$\mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12\cos 2t & -12\sin 2t & 5 \\ -24\sin 2t & -24\cos 2t & 0 \end{vmatrix} = (120\cos 2t)\mathbf{i} - (120\sin 2t)\mathbf{j} - 288\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}|^2$
$= (120\cos 2t)^2 + (-120\sin 2t)^2 + (-288)^2 = 120^2(\cos^2 2t + \sin^2 2t) + 288^2 = 97344$. Thus
$\tau = \frac{\begin{vmatrix} 12\cos 2t & -12\sin 2t & 5 \\ -24\sin 2t & -24\cos 2t & 0 \\ -48\cos 2t & 48\sin 2t & 0 \end{vmatrix}}{97344} = \frac{5\cdot(-24\cdot 48)}{97344} = -\frac{10}{169}$

5. By Exercise 13 in Section 13.4, $\mathbf{T} = \frac{t}{(t^2+1)^{1/2}}\mathbf{i} + \frac{1}{(t^2+1)^{1/2}}\mathbf{j}$ and $\mathbf{N} = \frac{1}{\sqrt{t^2+1}}\mathbf{i} - \frac{t}{\sqrt{t^2+1}}\mathbf{j}$ so that $\mathbf{B} = \mathbf{T} \times \mathbf{N}$
$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{t}{\sqrt{t^2+1}} & \frac{1}{\sqrt{t^2+1}} & 0 \\ \frac{1}{\sqrt{t^2+1}} & \frac{-t}{\sqrt{t^2+1}} & 0 \end{vmatrix} = -\mathbf{k}$. Also, $\mathbf{v} = t^2\mathbf{i} + t\mathbf{j} \Rightarrow \mathbf{a} = 2t\mathbf{i} + \mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = 2\mathbf{i}$ so that $\begin{vmatrix} t^2 & t & 0 \\ 2t & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix} = 0 \Rightarrow \tau = 0$

6. By Exercise 14 in Section 13.4, $\mathbf{T} = (-\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ and $\mathbf{N} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}$ so that $\mathbf{B} = \mathbf{T} \times \mathbf{N}$
$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\cos t & \sin t & 0 \\ \sin t & \cos t & 0 \end{vmatrix} = -\mathbf{k}$. Also, $\mathbf{v} = (-3\cos^2 t\sin t)\,\mathbf{i} + (3\sin^2 t\cos t)\,\mathbf{j}$
$\Rightarrow \mathbf{a} = \frac{d}{dt}(-3\cos^2 t\sin t)\,\mathbf{i} + \frac{d}{dt}(3\sin^2 t\cos t)\,\mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = \frac{d}{dt}\left(\frac{d}{dt}(-3\cos^2 t\sin t)\right)\mathbf{i} + \frac{d}{dt}\left(\frac{d}{dt}(3\sin^2 t\cos t)\right)\mathbf{j}$
$\Rightarrow \begin{vmatrix} -3\cos^2 t\sin t & 3\sin^2 t\cos t & 0 \\ \frac{d}{dt}(-3\cos^2 t\sin t) & \frac{d}{dt}(3\sin^2 t\cos t) & 0 \\ \frac{d}{dt}\left(\frac{d}{dt}(-3\cos^2 t\sin t)\right) & \frac{d}{dt}\left(\frac{d}{dt}(3\sin^2 t\cos t)\right) & 0 \end{vmatrix} = 0 \Rightarrow \tau = 0$

7. By Exercise 15 in Section 13.4, $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\operatorname{sech}\frac{t}{a}\right)\mathbf{i} + \left(\tanh\frac{t}{a}\right)\mathbf{j}$ and $\mathbf{N} = \left(-\tanh\frac{t}{a}\right)\mathbf{i} + \left(\operatorname{sech}\frac{t}{a}\right)\mathbf{j}$ so that $\mathbf{B} = \mathbf{T} \times \mathbf{N}$
$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \operatorname{sech}\left(\frac{t}{a}\right) & \tanh\left(\frac{t}{a}\right) & 0 \\ -\tanh\left(\frac{t}{a}\right) & \operatorname{sech}\left(\frac{t}{a}\right) & 0 \end{vmatrix} = \mathbf{k}$. Also, $\mathbf{v} = \mathbf{i} + \left(\sinh\frac{t}{a}\right)\mathbf{j} \Rightarrow \mathbf{a} = \left(\frac{1}{a}\cosh\frac{t}{a}\right)\mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = \frac{1}{a^2}\sinh\left(\frac{t}{a}\right)\mathbf{j}$ so that

$$\begin{vmatrix} 1 & \sinh\left(\frac{t}{a}\right) & 0 \\ 0 & \frac{1}{a}\cosh\left(\frac{t}{a}\right) & 0 \\ 0 & \frac{1}{a^2}\sinh\left(\frac{t}{a}\right) & 0 \end{vmatrix} = 0 \Rightarrow \tau = 0$$

8. By Exercise 16 in Section 13.4, $\mathbf{T} = \left(\frac{1}{\sqrt{2}}\tanh t\right)\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j} + \left(\frac{1}{\sqrt{2}}\operatorname{sech} t\right)\mathbf{k}$ and $\mathbf{N} = (\operatorname{sech} t)\mathbf{i} - (\tanh t)\mathbf{k}$ so that $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{2}}\tanh t & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\operatorname{sech} t \\ \operatorname{sech} t & 0 & -\tanh t \end{vmatrix} = \left(\frac{1}{\sqrt{2}}\tanh t\right)\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \left(\frac{1}{\sqrt{2}}\operatorname{sech} t\right)\mathbf{k}$. Also, $\mathbf{v} = (\sinh t)\mathbf{i} - (\cosh t)\mathbf{j} + \mathbf{k}$
$\mathbf{a} = (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = (\sinh t)\mathbf{i} - (\cosh t)\mathbf{j}$ and $\mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sinh t & -\cosh t & 1 \\ \cosh t & -\sinh t & 0 \end{vmatrix}$
$= (\sinh t)\mathbf{i} + (\cosh t)\mathbf{j} + (\cosh^2 t - \sinh^2 t)\mathbf{k} = (\sinh t)\mathbf{i} + (\cosh t)\mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}|^2 = \sinh^2 t + \cosh^2 t + 1$. Thus
$\tau = \dfrac{\begin{vmatrix} \sinh t & -\cosh t & 1 \\ \cosh t & -\sinh t & 0 \\ \sinh t & -\cosh t & 0 \end{vmatrix}}{\sinh^2 t + \cosh^2 t + 1} = \frac{-1}{\sinh^2 t + \cosh^2 t + 1} = \frac{-1}{2\cosh^2 t}$.

9. $\mathbf{r} = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\mathbf{k} \Rightarrow \mathbf{v} = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2}$
$= \sqrt{a^2 + b^2} \Rightarrow a_T = \frac{d}{dt}|\mathbf{v}| = 0$; $\mathbf{a} = (-a\cos t)\mathbf{i} + (-a\sin t)\mathbf{j} \Rightarrow |\mathbf{a}| = \sqrt{(-a\cos t)^2 + (-a\sin t)^2} = \sqrt{a^2} = |a|$
$\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{|\mathbf{a}|^2 - 0^2} = |\mathbf{a}| = |a| \Rightarrow \mathbf{a} = (0)\mathbf{T} + |a|\,\mathbf{N} = |a|\,\mathbf{N}$

10. $\mathbf{r} = (1 + 3t)\mathbf{i} + (t - 2)\mathbf{j} - 3t\mathbf{k} \Rightarrow \mathbf{v} = 3\mathbf{i} + \mathbf{j} - 3\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{3^2 + 1^2 + (-3)^2} = \sqrt{19} \Rightarrow a_T = \frac{d}{dt}|\mathbf{v}| = 0$; $\mathbf{a} = \mathbf{0}$
$\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = 0 \Rightarrow \mathbf{a} = (0)\mathbf{T} + (0)\mathbf{N} = \mathbf{0}$

11. $\mathbf{r} = (t + 1)\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + 2\mathbf{j} + 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + 2^2 + (2t)^2} = \sqrt{5 + 4t^2} \Rightarrow a_T = \frac{1}{2}(5 + 4t^2)^{-1/2}(8t)$
$= 4t(5 + 4t^2)^{-1/2} \Rightarrow a_T(1) = \frac{4}{\sqrt{9}} = \frac{4}{3}$; $\mathbf{a} = 2\mathbf{k} \Rightarrow \mathbf{a}(1) = 2\mathbf{k} \Rightarrow |\mathbf{a}(1)| = 2 \Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{2^2 - \left(\frac{4}{3}\right)^2}$
$= \sqrt{\frac{20}{9}} = \frac{2\sqrt{5}}{3} \Rightarrow \mathbf{a}(1) = \frac{4}{3}\mathbf{T} + \frac{2\sqrt{5}}{3}\mathbf{N}$

12. $\mathbf{r} = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{v} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} + 2t\mathbf{k}$
$\Rightarrow |\mathbf{v}| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + (2t)^2} = \sqrt{5t^2 + 1} \Rightarrow a_T = \frac{1}{2}(5t^2 + 1)^{-1/2}(10t)$
$= \frac{5t}{\sqrt{5t^2 + 1}} \Rightarrow a_T(0) = 0$; $\mathbf{a} = (-2\sin t - t\cos t)\mathbf{i} + (2\cos t - t\sin t)\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{a}(0) = 2\mathbf{j} + 2\mathbf{k} \Rightarrow |\mathbf{a}(0)|$
$= \sqrt{2^2 + 2^2} = 2\sqrt{2} \Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{\left(2\sqrt{2}\right)^2 - 0^2} = 2\sqrt{2} \Rightarrow \mathbf{a}(0) = (0)\mathbf{T} + 2\sqrt{2}\mathbf{N} = 2\sqrt{2}\mathbf{N}$

13. $\mathbf{r} = t^2\mathbf{i} + \left(t + \frac{1}{3}t^3\right)\mathbf{j} + \left(t - \frac{1}{3}t^3\right)\mathbf{k} \Rightarrow \mathbf{v} = 2t\mathbf{i} + (1 + t^2)\mathbf{j} + (1 - t^2)\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(2t)^2 + (1 + t^2)^2 + (1 - t^2)^2}$
$= \sqrt{2(t^4 + 2t^2 + 1)} = \sqrt{2}(1 + t^2) \Rightarrow a_T = 2t\sqrt{2} \Rightarrow a_T(0) = 0$; $\mathbf{a} = 2\mathbf{i} + 2t\mathbf{j} - 2t\mathbf{k} \Rightarrow \mathbf{a}(0) = 2\mathbf{i} \Rightarrow |\mathbf{a}(0)| = 2$
$\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{2^2 - 0^2} = 2 \Rightarrow \mathbf{a}(0) = (0)\mathbf{T} + 2\mathbf{N} = 2\mathbf{N}$

14. $\mathbf{r} = (e^t\cos t)\mathbf{i} + (e^t\sin t)\mathbf{j} + \sqrt{2}e^t\mathbf{k} \Rightarrow \mathbf{v} = (e^t\cos t - e^t\sin t)\mathbf{i} + (e^t\sin t + e^t\cos t)\mathbf{j} + \sqrt{2}e^t\mathbf{k}$
$\Rightarrow |\mathbf{v}| = \sqrt{(e^t\cos t - e^t\sin t)^2 + (e^t\sin t + e^t\cos t)^2 + \left(\sqrt{2}e^t\right)^2} = \sqrt{4e^{2t}} = 2e^t \Rightarrow a_T = 2e^t \Rightarrow a_T(0) = 2$;
$\mathbf{a} = (e^t\cos t - e^t\sin t - e^t\sin t - e^t\cos t)\mathbf{i} + (e^t\sin t + e^t\cos t + e^t\cos t - e^t\sin t)\mathbf{j} + \sqrt{2}e^t\mathbf{k}$
$= (-2e^t\sin t)\mathbf{i} + (2e^t\cos t)\mathbf{j} + \sqrt{2}e^t\mathbf{k} \Rightarrow \mathbf{a}(0) = 2\mathbf{j} + \sqrt{2}\mathbf{k} \Rightarrow |\mathbf{a}(0)| = \sqrt{2^2 + \left(\sqrt{2}\right)^2} = \sqrt{6}$

$\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{\left(\sqrt{6}\right)^2 - 2^2} = \sqrt{2} \Rightarrow \mathbf{a}(0) = 2\mathbf{T} + \sqrt{2}\mathbf{N}$

15. $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - \mathbf{k} \Rightarrow \mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{T}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$; $\frac{d\mathbf{T}}{dt} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\cos t)^2 + (-\sin t)^2}$
$= 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j} \Rightarrow \mathbf{N}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$; $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \mathbf{k}$
$\Rightarrow \mathbf{B}\left(\frac{\pi}{4}\right) = \mathbf{k}$, the normal to the osculating plane; $\mathbf{r}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} - \mathbf{k} \Rightarrow P = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -1\right)$ lies on the osculating plane $\Rightarrow 0\left(x - \frac{\sqrt{2}}{2}\right) + 0\left(y - \frac{\sqrt{2}}{2}\right) + (z - (-1)) = 0 \Rightarrow z = -1$ is the osculating plane; $\mathbf{T}$ is normal to the normal plane $\Rightarrow \left(-\frac{\sqrt{2}}{2}\right)\left(x - \frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(y - \frac{\sqrt{2}}{2}\right) + 0(z - (-1)) = 0 \Rightarrow -\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y = 0$
$\Rightarrow -x + y = 0$ is the normal plane; $\mathbf{N}$ is normal to the rectifying plane
$\Rightarrow \left(-\frac{\sqrt{2}}{2}\right)\left(x - \frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{2}}{2}\right)\left(y - \frac{\sqrt{2}}{2}\right) + 0(z - (-1)) = 0 \Rightarrow -\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y = -1 \Rightarrow x + y = \sqrt{2}$ is the rectifying plane

16. $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= \left(-\frac{1}{\sqrt{2}}\sin t\right)\mathbf{i} + \left(\frac{1}{\sqrt{2}}\cos t\right)\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{1}{\sqrt{2}}\cos t\right)\mathbf{i} + \left(-\frac{1}{\sqrt{2}}\sin t\right)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right|$
$= \sqrt{\frac{1}{2}\cos^2 t + \frac{1}{2}\sin^2 t} = \frac{1}{\sqrt{2}} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j}$; thus $\mathbf{T}(0) = \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$ and $\mathbf{N}(0) = -\mathbf{i}$
$\Rightarrow \mathbf{B}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1 & 0 & 0 \end{vmatrix} = -\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$, the normal to the osculating plane; $\mathbf{r}(0) = \mathbf{i} \Rightarrow P(1,0,0)$ lies on the osculating plane $\Rightarrow 0(x-1) - \frac{1}{\sqrt{2}}(y-0) + \frac{1}{\sqrt{2}}(z-0) = 0 \Rightarrow y - z = 0$ is the osculating plane; $\mathbf{T}$ is normal to the normal plane $\Rightarrow 0(x-1) + \frac{1}{\sqrt{2}}(y-0) + \frac{1}{\sqrt{2}}(z-0) = 0 \Rightarrow y + z = 0$ is the normal plane; $\mathbf{N}$ is normal to the rectifying plane $\Rightarrow -1(x-1) + 0(y-0) + 0(z-0) = 0 \Rightarrow x = 1$ is the rectifying plane

17. Yes. If the car is moving along a curved path, then $\kappa \neq 0$ and $a_N = \kappa\,|\mathbf{v}|^2 \neq 0 \Rightarrow \mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} \neq \mathbf{0}$.

18. $|\mathbf{v}|$ constant $\Rightarrow a_T = \frac{d}{dt}|\mathbf{v}| = 0 \Rightarrow \mathbf{a} = a_N\mathbf{N}$ is orthogonal to $\mathbf{T} \Rightarrow$ the acceleration is normal to the path

19. $\mathbf{a} \perp \mathbf{v} \Rightarrow \mathbf{a} \perp \mathbf{T} \Rightarrow a_T = 0 \Rightarrow \frac{d}{dt}|\mathbf{v}| = 0 \Rightarrow |\mathbf{v}|$ is constant

20. $\mathbf{a}(t) = a_T\mathbf{T} + a_N\mathbf{N}$, where $a_T = \frac{d}{dt}|\mathbf{v}| = \frac{d}{dt}(10) = 0$ and $a_N = \kappa\,|\mathbf{v}|^2 = 100\kappa \Rightarrow \mathbf{a} = 0\mathbf{T} + 100\kappa\mathbf{N}$. Now, from Exercise 5(a) Section 13.4, we find for $y = f(x) = x^2$ that $\kappa = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}} = \frac{2}{\left[1 + (2x)^2\right]^{3/2}} = \frac{2}{\left(1 + 4x^2\right)^{3/2}}$; also, $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$ is the position vector of the moving mass $\Rightarrow \mathbf{v} = \mathbf{i} + 2t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1 + 4t^2}$
$\Rightarrow \mathbf{T} = \frac{1}{\sqrt{1+4t^2}}(\mathbf{i} + 2t\mathbf{j})$. At $(0,0)$: $\mathbf{T}(0) = \mathbf{i}$, $\mathbf{N}(0) = \mathbf{j}$ and $\kappa(0) = 2 \Rightarrow \mathbf{F} = m\mathbf{a} = m(100\kappa)\mathbf{N} = 200m\,\mathbf{j}$;
At $\left(\sqrt{2}, 2\right)$: $\mathbf{T}\left(\sqrt{2}\right) = \frac{1}{3}\left(\mathbf{i} + 2\sqrt{2}\mathbf{j}\right) = \frac{1}{3}\mathbf{i} + \frac{2\sqrt{2}}{3}\mathbf{j}$, $\mathbf{N}\left(\sqrt{2}\right) = -\frac{2\sqrt{2}}{3}\mathbf{i} + \frac{1}{3}\mathbf{j}$, and $\kappa\left(\sqrt{2}\right) = \frac{2}{27} \Rightarrow \mathbf{F} = m\mathbf{a}$
$= m(100\kappa)\mathbf{N} = \left(\frac{200}{27}m\right)\left(-\frac{2\sqrt{2}}{3}\mathbf{i} + \frac{1}{3}\mathbf{j}\right) = -\frac{400\sqrt{2}}{81}m\,\mathbf{i} + \frac{200}{81}m\,\mathbf{j}$

21. $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$, where $a_T = \frac{d}{dt}|\mathbf{v}| = \frac{d}{dt}(\text{constant}) = 0$ and $a_N = \kappa\,|\mathbf{v}|^2 \Rightarrow \mathbf{F} = m\mathbf{a} = m\kappa\,|\mathbf{v}|^2\mathbf{N} \Rightarrow |\mathbf{F}| = m\kappa\,|\mathbf{v}|^2$
$= \left(m\,|\mathbf{v}|^2\right)\kappa$, a constant multiple of the curvature κ of the trajectory

22. $a_N = 0 \Rightarrow \kappa|\mathbf{v}|^2 = 0 \Rightarrow \kappa = 0$ (since the particle is moving, we cannot have zero speed) $\Rightarrow$ the curvature is zero so the particle is moving along a straight line

23. From Example 1, $|\mathbf{v}| = t$ and $a_N = t$ so that $a_N = \kappa|\mathbf{v}|^2 \Rightarrow \kappa = \frac{a_N}{|\mathbf{v}|^2} = \frac{t}{t^2} = \frac{1}{t}$, $t \neq 0 \Rightarrow \rho = \frac{1}{\kappa} = t$

24. $\mathbf{r} = (x_0 + At)\mathbf{i} + (y_0 + Bt)\mathbf{j} + (z_0 + Ct)\mathbf{k} \Rightarrow \mathbf{v} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k} \Rightarrow \mathbf{a} = \mathbf{0} \Rightarrow \mathbf{v} \times \mathbf{a} = \mathbf{0} \Rightarrow \kappa = 0$. Since the curve is a plane curve, $\tau = 0$.

25. If a plane curve is sufficiently differentiable the torsion is zero as the following argument shows:
$\mathbf{r} = f(t)\mathbf{i} + g(t)\mathbf{j} \Rightarrow \mathbf{v} = f'(t)\mathbf{i} + g'(t)\mathbf{j} \Rightarrow \mathbf{a} = f''(t)\mathbf{i} + g''(t)\mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = f'''(t)\mathbf{i} + g'''(t)\mathbf{j}$

$$\Rightarrow \tau = \frac{\begin{vmatrix} f'(t) & g'(t) & 0 \\ f''(t) & g''(t) & 0 \\ f'''(t) & g'''(t) & 0 \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = 0$$

26. From Example 2, $\tau = \frac{b}{a^2+b^2} \Rightarrow \tau'(b) = \frac{a^2-b^2}{(a^2+b^2)^2}$; $\tau'(b) = 0 \Rightarrow \frac{a^2-b^2}{(a^2+b^2)^2} = 0 \Rightarrow a^2 - b^2 = 0 \Rightarrow b = \pm a$
$\Rightarrow b = a$ since $a, b > 0$. Also $b < a \Rightarrow \tau' > 0$ and $b > a \Rightarrow \tau' < 0$ so τ_{max} occurs when $b = a \Rightarrow \tau_{max} = \frac{a}{a^2+a^2}$
$= \frac{1}{2a}$

27. $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \Rightarrow \mathbf{v} = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$; $\mathbf{v}\cdot\mathbf{k} = 0 \Rightarrow h'(t) = 0 \Rightarrow h(t) = C$
$\Rightarrow \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + C\mathbf{k}$ and $\mathbf{r}(a) = f(a)\mathbf{i} + g(a)\mathbf{j} + C\mathbf{k} = \mathbf{0} \Rightarrow f(a) = 0, g(a) = 0$ and $C = 0 \Rightarrow h(t) = 0$.

28. From Example 2, $\mathbf{v} = -(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{a^2+b^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= \frac{1}{\sqrt{a^2+b^2}}[-(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}]$; $\frac{d\mathbf{T}}{dt} = \frac{1}{\sqrt{a^2+b^2}}[-(a\cos t)\mathbf{i} - (a\sin t)\mathbf{j}] \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|}$

$$= -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}\,;\ \mathbf{B} = \mathbf{T}\times\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{a\sin t}{\sqrt{a^2+b^2}} & \frac{a\cos t}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$= \frac{b\sin t}{\sqrt{a^2+b^2}}\mathbf{i} - \frac{b\cos t}{\sqrt{a^2+b^2}}\mathbf{j} + \frac{a}{\sqrt{a^2+b^2}}\mathbf{k} \Rightarrow \frac{d\mathbf{B}}{dt} = \frac{1}{\sqrt{a^2+b^2}}[(b\cos t)\mathbf{i} + (b\sin t)\mathbf{j}] \Rightarrow \frac{d\mathbf{B}}{dt}\cdot\mathbf{N} = -\frac{b}{\sqrt{a^2+b^2}}$
$\Rightarrow \tau = -\frac{1}{|\mathbf{v}|}\left(\frac{d\mathbf{B}}{dt}\cdot\mathbf{N}\right) = \left(-\frac{1}{\sqrt{a^2+b^2}}\right)\left(-\frac{b}{\sqrt{a^2+b^2}}\right) = \frac{b}{a^2+b^2}$, which is consistent with the result in Example 2.

29-32. Example CAS commands:

<u>Maple</u>:

```
with( LinearAlgebra );
r := < t*cos(t) | t*sin(t) | t >;
t0 := sqrt(3);
rr := eval( r, t=t0 );
v := map( diff, r, t );
vv := eval( v, t=t0 );
a := map( diff, v, t );
aa := eval( a, t=t0 );
s := simplify(Norm( v, 2 )) assuming t::real;
ss := eval( s, t=t0 );
T := v/s;
TT := vv/ss ;
q1 := map( diff, simplify(T), t ):
NN := simplify(eval( q1/Norm(q1,2), t=t0 ));
```

```
BB := CrossProduct( TT, NN );
kappa := Norm(CrossProduct(vv,aa),2)/ss^3;
tau := simplify( Determinant(< vv, aa, eval(map(diff,a,t),t=t0) >)/Norm(CrossProduct(vv,aa),2)^3 );
a_t := eval( diff( s, t ), t=t0 );
a_n := evalf[4]( kappa*ss^2 );
```

Mathematica: (assigned functions and value for t0 will vary)

```
Clear[t, v, a, t]
mag[vector_]:=Sqrt[vector.vector]
Print["The position vector is ", r[t_]={t Cos[t], t Sin[t], t}]
Print["The velocity vector is ", v[t_]= r'[t]]
Print["The acceleration vector is ", a[t_]= v'[t]]
Print["The speed is ", speed[t_]= mag[v[t]]//Simplify]
Print["The unit tangent vector is ", utan[t_]= v[t]/speed[t] //Simplify]
Print["The curvature is ", curv[t_]= mag[Cross[v[t],a[t]]] / speed[t]^3 //Simplify]
Print["The torsion is ", torsion[t_]= Det[{v[t], a[t], a'[t]}] / mag[Cross[v[t],a[t]]]^2 //Simplify]
Print["The unit normal vector is ", unorm[t_]= utan'[t] / mag[utan'[t]] //Simplify]
Print["The unit binormal vector is ", ubinorm[t_]= Cross[utan[t],unorm[t]] //Simplify]
Print["The tangential component of the acceleration is ", at[t_]=a[t].utan[t] //Simplify]
Print["The normal component of the acceleration is ", an[t_]=a[t].unorm[t] //Simplify]
```

You can evaluate any of these functions at a specified value of t.

```
t0= Sqrt[3]
{utan[t0], unorm[t0], ubinorm[t0]}
N[{utan[t0], unorm[t0], ubinorm[t0]}]
{curv[t0], torsion[t0]}
N[{curv[t0], torsion[t0]}]
{at[t0], an[t0]}
N[{at[t0], an[t0]}]
```

To verify that the tangential and normal components of the acceleration agree with the formulas in the book:

```
at[t]== speed'[t] //Simplify
an[t]==curv [t] speed[t]^2 //Simplify
```

13.6 PLANETARY MOTION AND SATELLITES

1. $\frac{T^2}{a^3} = \frac{4\pi^2}{GM} \Rightarrow T^2 = \frac{4\pi^2}{GM} a^3 \Rightarrow T^2 = \frac{4\pi^2}{(6.6726\times 10^{-11}\ Nm^2kg^{-2})\,(5.975\times 10^{24}\ kg)} (6{,}808{,}000\ m)^3$
$\approx 3.125 \times 10^7\ sec^2 \Rightarrow T \approx \sqrt{3125 \times 10^4\ sec^2} \approx 55.90 \times 10^2\ sec \approx 93.2\ min$

2. $e = 0.0167$ and perihelion distance $= 149{,}577{,}000$ km and $e = \frac{r_0 v_0^2}{GM} - 1$
$\Rightarrow 0.0167 = \frac{(149{,}577{,}000{,}000\ m)v_0^2}{(6.6726\times 10^{-11}\ Nm^2kg^{-2})\,(1.99\times 10^{30}\ kg)} - 1 \Rightarrow v_0^2 \approx 9.03 \times 10^8\ m^2/sec^2$
$\Rightarrow v_0 \approx \sqrt{9.03 \times 10^8\ m^2/sec^2} \approx 3.00 \times 10^4\ m/sec$

3. $92.25\ min = 5535\ sec$ and $\frac{T^2}{a^3} = \frac{4\pi^2}{GM} \Rightarrow a^3 = \frac{GM}{4\pi^2} T^2$
$\Rightarrow a^3 = \frac{(6.6726\times 10^{-11}\ Nm^2kg^{-2})\,(5.975\times 10^{24}\ kg)}{4\pi^2} (5535\ sec)^2 = 3.094 \times 10^{20}\ m^3 \Rightarrow a \approx \sqrt[3]{3.094 \times 10^{20}\ m^3}$
$= 6.764 \times 10^6\ m \approx 6764\ km$. Note that $6764\ km \approx \frac{1}{2}(12{,}757\ km + 183\ km + 589\ km)$.

4. $T = 1639\ min = 98{,}340\ sec$ and mass of Mars $= 6.418 \times 10^{23}\ kg \Rightarrow a^3 = \frac{GM}{4\pi^2} T^2$
$= \frac{(6.6726\times 10^{-11}\ Nm^2kg^{-2})\,(6.418\times 10^{23}\ kg)\,(98{,}340\ sec)^2}{4\pi^2} \approx 1.049 \times 10^{22}\ m^3 \Rightarrow a \approx \sqrt[3]{1.049 \times 10^{22}\ m^3}$
$= 2.19 \times 10^7\ m = 21{,}900\ km$

5. $2a = \text{diameter of Mars} + \text{perigee height} + \text{apogee height} = D + 1499\text{ km} + 35{,}800\text{ km}$
$\Rightarrow\ 2(21{,}900)\text{ km} = D + 37{,}299\text{ km} \ \Rightarrow\ D = 6501\text{ km}$

6. $a = 22{,}030\text{ km} = 2.203 \times 10^7\text{ m}$ and $T^2 = \frac{4\pi^2}{GM}\, a^3$
$\Rightarrow\ T^2 = \frac{4\pi^2}{(6.6720\times 10^{-11}\text{ Nm}^2\text{kg}^{-2})\,(6.418\times 10^{23}\text{ kg})}\,(2.203 \times 10^7\text{ m})^3 \approx 9.856 \times 10^9\text{ sec}^2$
$\Rightarrow\ T \approx \sqrt{9.856 \times 10^8\text{ sec}^2} \approx 9.928 \times 10^4\text{ sec} \approx 1655\text{ min}$

7. (a) Period of the satellite $=$ rotational period of the Earth $\Rightarrow$ period of the satellite $= 1436.1$ min
$= 86{,}166\text{ sec};\ a^3 = \frac{GMT^2}{4\pi^2} \ \Rightarrow\ a^3 = \frac{(6.6726\times 10^{-11}\text{ Nm}^2\text{kg}^{-2})\left(5.975\times 10^{24}\text{ kg}\right)(86{,}166\text{ sec})^2}{4\pi^2}$
$\approx\ 7.4980 \times 10^{22}\text{ m}^3 \ \Rightarrow\ a \approx \sqrt[3]{74.980 \times 10^{21}\text{ m}^3} \approx 4.2168 \times 10^7\text{ m} = 42{,}168\text{ km}$
(b) The radius of the Earth is approximately 6379 km $\Rightarrow$ the height of the orbit is $42{,}168 - 6379 = 35{,}789$ km
(c) Symcom 3, GOES 4, and Intelsat 5

8. $T = 1477.4\text{ min} = 88{,}644\text{ sec} \ \Rightarrow\ a^3 = \frac{GMT^2}{4\pi^2}$
$= \frac{(6.6726\times 10^{-11}\text{ Nm}^2\text{kg}^{-2})\,(6.418\times 10^{23}\text{ kg})\,(88{,}644\text{ sec})^2}{4\pi^2} = 8.524 \times 10^{21}\text{ m}^3 \ \Rightarrow\ a \approx \sqrt[3]{8.524 \times 10^{21}\text{ m}^3}$
$\approx 2.043 \times 10^7\text{ m} = 20{,}430\text{ km}$

9. Period of the Moon $= 2.36055 \times 10^6\text{ sec} \ \Rightarrow\ a^3 = \frac{GMT^2}{4\pi^2}$
$= \frac{(6.6726\times 10^{-11}\text{ Nm}^2\text{kg}^{-2})\left(5.975\times 10^{24}\text{ kg}\right)(2.36055\times 10^6\text{ sec})^2}{4\pi^2} \approx 5.627 \times 10^{25}\text{ m}^3 \ \Rightarrow\ a \approx \sqrt[3]{5.627 \times 10^{25}\text{ m}^3}$
$\approx 3.832 \times 10^8$ m $= 383{,}200$ km from the center of the Earth.

10. $r = \frac{GM}{v^2} \ \Rightarrow\ v^2 = \frac{GM}{r} \ \Rightarrow\ |v| = \sqrt{\frac{GM}{r}} = \sqrt{\frac{(6.6726\times 10^{-11}\text{ Nm}^2\text{kg}^{-2})\,(5.975\times 10^{24}\text{ kg})}{r}} \approx 1.9967 \times 10^7 r^{-1/2}\text{ m/sec}$

11. Solar System: $\frac{T^2}{a^3} = \frac{4\pi^2}{(6.6726\times 10^{-11}\text{ Nm}^2\text{kg}^{-2})\,(1.99\times 10^{30}\text{ kg})} \approx 2.97 \times 10^{-19}\text{ sec}^2/\text{m}^3;$
Earth: $\frac{T^2}{a^3} = \frac{4\pi^2}{(6.6726\times 10^{-11}\text{ Nm}^2\text{kg}^{-2})\,(5.975\times 10^{24}\text{ kg})} \approx 9.902 \times 10^{-14}\text{ sec}^2/\text{m}^3;$
Moon: $\frac{T^2}{a^3} = \frac{4\pi^2}{(6.6726\times 10^{-11}\text{ Nm}^2\text{kg}^{-2})\,(7.354\times 10^{22}\text{ kg})} \approx 8.045 \times 10^{-12}\text{ sec}^2/\text{m}^3;$

12. $e = \frac{r_0 v_0^2}{GM} - 1 \ \Rightarrow\ v_0^2 = \frac{GM(e+1)}{r_0} \ \Rightarrow\ v_0 = \sqrt{\frac{GM(e+1)}{r_0}}\,;$
Circle: $e = 0 \ \Rightarrow\ v_0 = \sqrt{\frac{GM}{r_0}}$
Ellipse: $0 < e < 1 \ \Rightarrow\ \sqrt{\frac{GM}{r_0}} < v_0 < \sqrt{\frac{2GM}{r_0}}$
Parabola: $e = 1 \ \Rightarrow\ v_0 = \sqrt{\frac{2GM}{r_0}}$
Hyperbola: $e > 1 \ \Rightarrow\ v_0 > \sqrt{\frac{2GM}{r_0}}$

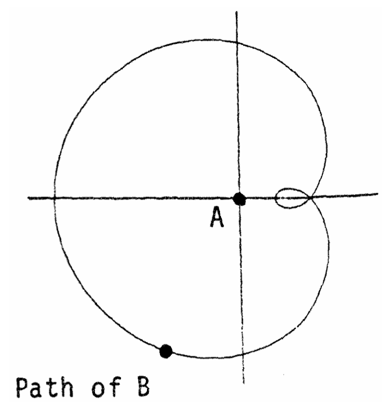
13. $r = \frac{GM}{v^2} \ \Rightarrow\ v^2 = \frac{GM}{r} \ \Rightarrow\ v = \sqrt{\frac{GM}{r}}$ which is constant since G, M, and r (the radius of orbit) are constant

14. $\Delta A = \frac{1}{2}\,|\mathbf{r}(t+\Delta t) \times \mathbf{r}(t)| \ \Rightarrow\ \frac{\Delta A}{\Delta t} = \frac{1}{2}\left|\frac{\mathbf{r}(t+\Delta t)}{\Delta t} \times \mathbf{r}(t)\right| = \frac{1}{2}\left|\frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t) + \mathbf{r}(t)}{\Delta t} \times \mathbf{r}(t)\right|$
$= \frac{1}{2}\left|\frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \times \mathbf{r}(t) + \frac{1}{\Delta t}\,\mathbf{r}(t) \times \mathbf{r}(t)\right| = \frac{1}{2}\left|\frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \times \mathbf{r}(t)\right| \ \Rightarrow\ \frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{1}{2}\left|\frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \times \mathbf{r}(t)\right|$
$= \frac{1}{2}\left|\frac{d\mathbf{r}}{dt} \times \mathbf{r}(t)\right| = \frac{1}{2}\left|\mathbf{r}(t) \times \frac{d\mathbf{r}}{dt}\right| = \frac{1}{2}\,|\mathbf{r} \times \dot{\mathbf{r}}|$

15. $T = \left(\frac{2\pi a^2}{r_0 v_0}\right)\sqrt{1-e^2} \Rightarrow T^2 = \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2}\right)(1-e^2) = \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2}\right)\left[1-\left(\frac{r_0 v_0^2}{GM}-1\right)^2\right]$ (from Equation 32)
$= \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2}\right)\left[-\frac{r_0^2 v_0^4}{G^2M^2}+2\left(\frac{r_0 v_0^2}{GM}\right)\right] = \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2}\right)\left[\frac{2GMr_0 v_0^2 - r_0^2 v_0^4}{G^2M^2}\right] = \frac{(4\pi^2 a^4)(2GM - r_0 v_0^2)}{r_0 G^2M^2}$
$= (4\pi^2 a^4)\left(\frac{2GM - r_0 v_0^2}{2r_0 GM}\right)\left(\frac{2}{GM}\right) = (4\pi^2 a^4)\left(\frac{1}{2a}\right)\left(\frac{2}{GM}\right)$ (from Equation 35) $\Rightarrow T^2 = \frac{4\pi^2 a^3}{GM} \Rightarrow \frac{T^2}{a^3} = \frac{4\pi^2}{GM}$

16. Let $\mathbf{r}_{AB}(t)$ denote the vector from planet A to planet B at time t. Then $\mathbf{r}_{AB}(t) = \mathbf{r}_B(t) - \mathbf{r}_A(t)$
$= [3\cos(\pi t) - 2\cos(2\pi t)]\mathbf{i} + [3\sin(\pi t) - 2\sin(2\pi t)]\mathbf{j}$
$= [3\cos(\pi t) - 2(\cos^2(\pi t) - \sin^2(\pi t))]\,\mathbf{i} + [3\sin(\pi t) - 4\sin(\pi t)\cos(\pi t)]\mathbf{j}$
$= [3\cos(\pi t) - 4\cos^2(\pi t) + 2]\,\mathbf{i} + [(3 - 4\cos(\pi t))\sin(\pi t)]\mathbf{j} \Rightarrow$ parametric equations for the path are
$x(t) = 2 + [3 - 4\cos(\pi t)]\cos(\pi t)$ and $y(t) = [3 - 4\cos(\pi t)]\sin(\pi t)$

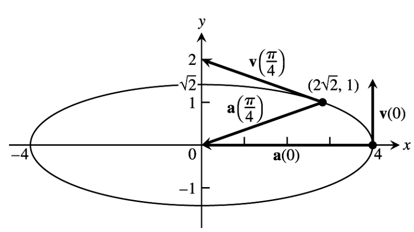
17. The graph of the path of planet B is the limaçon at the right.



18. (i) Perihelion is the time t such that $|\mathbf{r}(t)|$ is a minimum.
(ii) Aphelion is the time t such that $|\mathbf{r}(t)|$ is a maximum.
(iii) Equinox is the time t such that $\mathbf{r}(t) \cdot \mathbf{w} = 0$.
(iv) Summer solstice is the time t such that the angle between $\mathbf{r}(t)$ and $\mathbf{w}$ is a maximum.
(v) Winter solstice is the time t such that the angle between $\mathbf{r}(t)$ and $\mathbf{w}$ is a minimum.

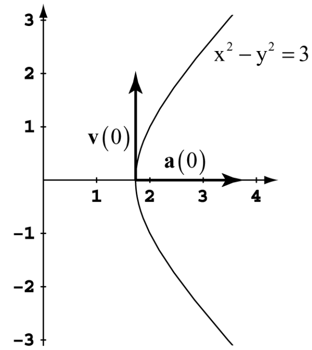
CHAPTER 13 PRACTICE EXERCISES

1. $\mathbf{r}(t) = (4\cos t)\mathbf{i} + \left(\sqrt{2}\sin t\right)\mathbf{j} \Rightarrow x = 4\cos t$ and $y = \sqrt{2}\sin t \Rightarrow \frac{x^2}{16} + \frac{y^2}{2} = 1$;
$\mathbf{v} = (-4\sin t)\mathbf{i} + \left(\sqrt{2}\cos t\right)\mathbf{j}$ and
$\mathbf{a} = (-4\cos t)\mathbf{i} - \left(\sqrt{2}\sin t\right)\mathbf{j}$; $\mathbf{r}(0) = 4\,\mathbf{i}$, $\mathbf{v}(0) = \sqrt{2}\mathbf{j}$,
$\mathbf{a}(0) = -4\mathbf{i}$; $\mathbf{r}\left(\frac{\pi}{4}\right) = 2\sqrt{2}\mathbf{i} + \mathbf{j}$, $\mathbf{v}\left(\frac{\pi}{4}\right) = -2\sqrt{2}\mathbf{i} + \mathbf{j}$,
$\mathbf{a}\left(\frac{\pi}{4}\right) = -2\sqrt{2}\mathbf{i} - \mathbf{j}$; $|\mathbf{v}| = \sqrt{16\sin^2 t + 2\cos^2 t}$



$\Rightarrow a_T = \frac{d}{dt}|\mathbf{v}| = \frac{14\sin t\cos t}{\sqrt{16\sin^2 t + 2\cos^2 t}}$; at $t = 0$: $a_T = 0$, $a_N = \sqrt{|\mathbf{a}|^2 - 0} = 4$, $\kappa = \frac{a_N}{|\mathbf{v}|^2} = \frac{4}{2} = 2$;
at $t = \frac{\pi}{4}$: $a_T = \frac{7}{\sqrt{8+1}} = \frac{7}{3}$, $a_N = \sqrt{9 - \frac{49}{9}} = \frac{4\sqrt{2}}{3}$, $\kappa = \frac{a_N}{|\mathbf{v}|^2} = \frac{4\sqrt{2}}{27}$

2. $\mathbf{r}(t) = \left(\sqrt{3}\sec t\right)\mathbf{i} + \left(\sqrt{3}\tan t\right)\mathbf{j} \Rightarrow x = \sqrt{3}\sec t$ and $y = \sqrt{3}\tan t \Rightarrow \frac{x^2}{3} - \frac{y^2}{3} = \sec^2 t - \tan^2 t = 1;$
$\Rightarrow x^2 - y^2 = 3;\ \mathbf{v} = \left(\sqrt{3}\sec t\tan t\right)\mathbf{i} + \left(\sqrt{3}\sec^2 t\right)\mathbf{j}$
and
$\mathbf{a} = \left(\sqrt{3}\sec t\tan^2 t + \sqrt{3}\sec^3 t\right)\mathbf{i} - \left(2\sqrt{3}\sec^2 t\tan t\right)\mathbf{j};$
$\mathbf{r}(0) = \sqrt{3}\mathbf{i},\ \mathbf{v}(0) = \sqrt{3}\mathbf{j},\ \mathbf{a}(0) = \sqrt{3}\mathbf{i};$
$|\mathbf{v}| = \sqrt{3\sec^2 t\tan^2 t + 3\sec^4 t}$
$\Rightarrow a_T = \frac{d}{dt}|\mathbf{v}| = \frac{6\sec^2 t\tan^3 t + 18\sec^4 t\tan t}{2\sqrt{3\sec^2 t\tan^2 t + 3\sec^4 t}};$
at $t = 0$: $a_T = 0,\ a_N = \sqrt{|\mathbf{a}|^2 - 0} = \sqrt{3},$
$\kappa = \frac{a_N}{|\mathbf{v}|^2} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$



3. $\mathbf{r} = \frac{1}{\sqrt{1+t^2}}\mathbf{i} + \frac{t}{\sqrt{1+t^2}}\mathbf{j} \Rightarrow \mathbf{v} = -t\left(1+t^2\right)^{-3/2}\mathbf{i} + \left(1+t^2\right)^{-3/2}\mathbf{j}$
$\Rightarrow |\mathbf{v}| = \sqrt{\left[-t\left(1+t^2\right)^{-3/2}\right]^2 + \left[\left(1+t^2\right)^{-3/2}\right]^2} = \frac{1}{1+t^2}$. We want to maximize $|\mathbf{v}|$: $\frac{d|\mathbf{v}|}{dt} = \frac{-2t}{\left(1+t^2\right)^2}$ and
$\frac{d|\mathbf{v}|}{dt} = 0 \Rightarrow \frac{-2t}{\left(1+t^2\right)^2} = 0 \Rightarrow t = 0$. For $t < 0$, $\frac{-2t}{\left(1+t^2\right)^2} > 0$; for $t > 0$, $\frac{-2t}{\left(1+t^2\right)^2} < 0 \Rightarrow |\mathbf{v}|_{\max}$ occurs when
$t = 0 \Rightarrow |\mathbf{v}|_{\max} = 1$

4. $\mathbf{r} = (e^t\cos t)\mathbf{i} + (e^t\sin t)\mathbf{j} \Rightarrow \mathbf{v} = (e^t\cos t - e^t\sin t)\mathbf{i} + (e^t\sin t + e^t\cos t)\mathbf{j}$
$\Rightarrow \mathbf{a} = (e^t\cos t - e^t\sin t - e^t\sin t - e^t\cos t)\mathbf{i} + (e^t\sin t + e^t\cos t + e^t\cos t - e^t\sin t)\mathbf{j}$
$= (-2e^t\sin t)\mathbf{i} + (2e^t\cos t)\mathbf{j}$. Let θ be the angle between $\mathbf{r}$ and $\mathbf{a}$. Then $\theta = \cos^{-1}\left(\frac{\mathbf{r}\cdot\mathbf{a}}{|\mathbf{r}|\,|\mathbf{a}|}\right)$
$= \cos^{-1}\left(\frac{-2e^{2t}\sin t\cos t + 2e^{2t}\sin t\cos t}{\sqrt{(e^t\cos t)^2 + (e^t\sin t)^2}\sqrt{(-2e^t\sin t)^2 + (2e^t\cos t)^2}}\right) = \cos^{-1}\left(\frac{0}{2e^{2t}}\right) = \cos^{-1} 0 = \frac{\pi}{2}$ for all t

5. $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{a} = 5\mathbf{i} + 15\mathbf{j} \Rightarrow \mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 0 \\ 5 & 15 & 0 \end{vmatrix} = 25\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}| = 25;\ |\mathbf{v}| = \sqrt{3^2 + 4^2} = 5$
$\Rightarrow \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{25}{5^3} = \frac{1}{5}$

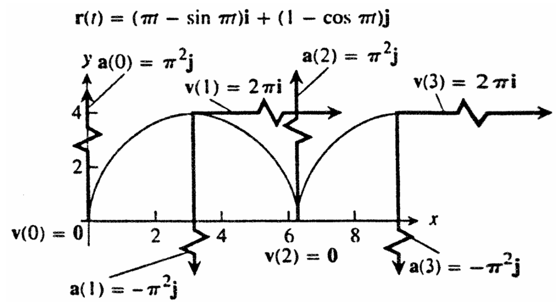
6. $\kappa = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = e^x\left(1 + e^{2x}\right)^{-3/2} \Rightarrow \frac{d\kappa}{dx} = e^x\left(1 + e^{2x}\right)^{-3/2} + e^x\left[-\frac{3}{2}\left(1 + e^{2x}\right)^{-5/2}\left(2e^{2x}\right)\right]$
$= e^x\left(1 + e^{2x}\right)^{-3/2} - 3e^{3x}\left(1 + e^{2x}\right)^{-5/2} = e^x\left(1 + e^{2x}\right)^{-5/2}\left[\left(1 + e^{2x}\right) - 3e^{2x}\right] = e^x\left(1 + e^{2x}\right)^{-5/2}\left(1 - 2e^{2x}\right);$
$\frac{d\kappa}{dx} = 0 \Rightarrow \left(1 - 2e^{2x}\right) = 0 \Rightarrow e^{2x} = \frac{1}{2} \Rightarrow 2x = -\ln 2 \Rightarrow x = -\frac{1}{2}\ln 2 = -\ln\sqrt{2} \Rightarrow y = \frac{1}{\sqrt{2}}$; therefore κ is at a maximum at the point $\left(-\ln\sqrt{2}, \frac{1}{\sqrt{2}}\right)$

7. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} \Rightarrow \mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$ and $\mathbf{v}\cdot\mathbf{i} = y \Rightarrow \frac{dx}{dt} = y$. Since the particle moves around the unit circle
$x^2 + y^2 = 1,\ 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = -\frac{x}{y}(y) = -x$. Since $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = -x$, we have
$\mathbf{v} = y\mathbf{i} - x\mathbf{j} \Rightarrow$ at $(1, 0)$, $\mathbf{v} = -\mathbf{j}$ and the motion is clockwise.

8. $9y = x^3 \Rightarrow 9\frac{dy}{dt} = 3x^2\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{1}{3}x^2\frac{dx}{dt}$. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, where x and y are differentiable functions of t,
then $\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$. Hence $\mathbf{v}\cdot\mathbf{i} = 4 \Rightarrow \frac{dx}{dt} = 4$ and $\mathbf{v}\cdot\mathbf{j} = \frac{dy}{dt} = \frac{1}{3}x^2\frac{dx}{dt} = \frac{1}{3}(3)^2(4) = 12$ at $(3, 3)$. Also,
$\mathbf{a} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}$ and $\frac{d^2y}{dt^2} = \left(\frac{2}{3}x\right)\left(\frac{dx}{dt}\right)^2 + \left(\frac{1}{3}x^2\right)\frac{d^2x}{dt^2}$. Hence $\mathbf{a}\cdot\mathbf{i} = -2 \Rightarrow \frac{d^2x}{dt^2} = -2$ and
$\mathbf{a}\cdot\mathbf{j} = \frac{d^2y}{dt^2} = \frac{2}{3}(3)(4)^2 + \frac{1}{3}(3)^2(-2) = 26$ at the point $(x, y) = (3, 3)$.

9. $\frac{d\mathbf{r}}{dt}$ orthogonal to $\mathbf{r}$ $\Rightarrow$ $0 = \frac{d\mathbf{r}}{dt} \cdot \mathbf{r} = \frac{1}{2}\frac{d\mathbf{r}}{dt} \cdot \mathbf{r} + \frac{1}{2}\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = \frac{1}{2}\frac{d}{dt}(\mathbf{r} \cdot \mathbf{r})$ $\Rightarrow$ $\mathbf{r} \cdot \mathbf{r} = K$, a constant. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, where x and y are differentiable functions of t, then $\mathbf{r} \cdot \mathbf{r} = x^2 + y^2$ $\Rightarrow$ $x^2 + y^2 = K$, which is the equation of a circle centered at the origin.

10. (b) $\mathbf{v} = (\pi - \pi \cos \pi t)\mathbf{i} + (\pi \sin \pi t)\mathbf{j}$
$\Rightarrow$ $\mathbf{a} = (\pi^2 \sin \pi t)\,\mathbf{i} + (\pi^2 \cos \pi t)\,\mathbf{j}$;
$\mathbf{v}(0) = \mathbf{0}$ and $\mathbf{a}(0) = \pi^2\mathbf{j}$;
$\mathbf{v}(1) = 2\pi\mathbf{i}$ and $\mathbf{a}(1) = -\pi^2\mathbf{j}$;
$\mathbf{v}(2) = \mathbf{0}$ and $\mathbf{a}(2) = \pi^2\mathbf{j}$;
$\mathbf{v}(3) = 2\pi\mathbf{i}$ and $\mathbf{a}(3) = -\pi^2\mathbf{j}$

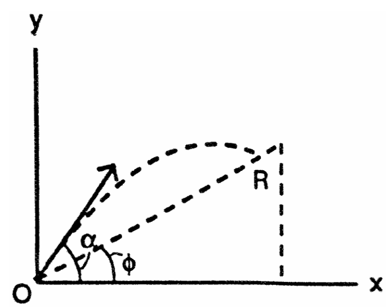


(c) Forward speed at the topmost point is $|\mathbf{v}(1)| = |\mathbf{v}(3)| = 2\pi$ ft/sec; since the circle makes $\frac{1}{2}$ revolution per second, the center moves π ft parallel to the x-axis each second $\Rightarrow$ the forward speed of C is π ft/sec.

11. $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2$ $\Rightarrow$ $y = 6.5 + (44 \text{ ft/sec})(\sin 45°)(3 \text{ sec}) - \frac{1}{2}(32 \text{ ft/sec}^2)(3 \text{ sec})^2 = 6.5 + 66\sqrt{2} - 144$ ≈ -44.16 ft $\Rightarrow$ the shot put is on the ground. Now, $y = 0$ $\Rightarrow$ $6.5 + 22\sqrt{2}t - 16t^2 = 0$ $\Rightarrow$ $t \approx 2.13$ sec (the positive root) $\Rightarrow$ $x \approx (44 \text{ ft/sec})(\cos 45°)(2.13 \text{ sec}) \approx 66.27$ ft or about 66 ft, 3 in. from the stopboard

12. $y_{max} = y_0 + \frac{(v_0 \sin \alpha)^2}{2g} = 7 \text{ ft} + \frac{[(80 \text{ ft/sec})(\sin 45°)]^2}{(2)(32 \text{ ft/sec}^2)} \approx 57$ ft

13. $x = (v_0 \cos \alpha)t$ and $y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$ $\Rightarrow$ $\tan \phi = \frac{y}{x} = \frac{(v_0 \sin \alpha)t - \frac{1}{2}gt^2}{(v_0 \cos \alpha)t} = \frac{(v_0 \sin \alpha) - \frac{1}{2}gt}{v_0 \cos \alpha}$
$\Rightarrow$ $v_0 \cos \alpha \tan \phi = v_0 \sin \alpha - \frac{1}{2}gt$ $\Rightarrow$ $t = \frac{2v_0 \sin \alpha - 2v_0 \cos \alpha \tan \phi}{g}$, which is the time when the golf ball hits the upward slope. At this time
$x = (v_0 \cos \alpha)\left(\frac{2v_0 \sin \alpha - 2v_0 \cos \alpha \tan \phi}{g}\right)$
$= \left(\frac{2}{g}\right)(v_0^2 \sin \alpha \cos \alpha - v_0^2 \cos^2 \alpha \tan \phi)$. Now
$OR = \frac{x}{\cos \phi}$ $\Rightarrow$ $OR = \left(\frac{2}{g}\right)\left(\frac{v_0^2 \sin \alpha \cos \alpha - v_0^2 \cos^2 \alpha \tan \phi}{\cos \phi}\right)$
$= \left(\frac{2v_0^2 \cos \alpha}{g}\right)\left(\frac{\sin \alpha}{\cos \phi} - \frac{\cos \alpha \tan \phi}{\cos \phi}\right)$
$= \left(\frac{2v_0^2 \cos \alpha}{g}\right)\left(\frac{\sin \alpha \cos \phi - \cos \alpha \sin \phi}{\cos^2 \phi}\right)$
$= \left(\frac{2v_0^2 \cos \alpha}{g \cos^2 \phi}\right)[\sin(\alpha - \phi)]$. The distance OR is maximized when x is maximized: $\frac{dx}{d\alpha} = \left(\frac{2v_0^2}{g}\right)(\cos 2\alpha + \sin 2\alpha \tan \phi) = 0 \Rightarrow (\cos 2\alpha + \sin 2\alpha \tan \phi) = 0$ $\Rightarrow$ $\cot 2\alpha + \tan \phi = 0$
$\Rightarrow$ $\cot 2\alpha = \tan(-\phi)$ $\Rightarrow$ $2\alpha = \frac{\pi}{2} + \phi$ $\Rightarrow$ $\alpha = \frac{\phi}{2} + \frac{\pi}{4}$



14. $R = \frac{v_0^2}{g}\sin 2\alpha$ $\Rightarrow$ $v_0 = \sqrt{\frac{Rg}{\sin 2\alpha}}$; for 4325 yards: 4325 yards = 12,975 ft $\Rightarrow$ $v_0 = \sqrt{\frac{(12{,}975 \text{ ft})(32 \text{ ft/sec}^2)}{(\sin 90°)}}$
≈ 644 ft/sec; for 4752 yards: 4752 yards = 14,256 ft $\Rightarrow$ $v_0 = \sqrt{\frac{(14{,}256 \text{ ft})(32 \text{ ft/sec}^2)}{(\sin 90°)}} \approx 675$ ft/sec

15. (a) $R = \frac{v_0^2}{g}\sin 2\alpha$ $\Rightarrow$ $109.5 \text{ ft} = \left(\frac{v_0^2}{32 \text{ ft/sec}^2}\right)(\sin 90°)$ $\Rightarrow$ $v_0^2 = 3504 \text{ ft}^2/\text{sec}^2$ $\Rightarrow$ $v_0 = \sqrt{3504 \text{ ft}^2/\text{sec}^2}$
≈ 59.19 ft/sec

(b) $x = (v_0 \cos \alpha)t$ and $y = 4 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2$; when the cork hits the ground, $x = 177.75$ ft and $y = 0$
$\Rightarrow$ $177.75 = \left(v_0 \frac{1}{\sqrt{2}}\right)t$ and $0 = 4 + \left(v_0 \frac{1}{\sqrt{2}}\right)t - 16t^2$ $\Rightarrow$ $16t^2 = 4 + 177.75$ $\Rightarrow$ $t = \frac{\sqrt{181.75}}{4}$
$\Rightarrow$ $v_0 = \frac{(177.75)\sqrt{2}}{t} = \frac{4(177.75)\sqrt{2}}{\sqrt{181.75}} \approx 74.58$ ft/sec

16. (a) $x = v_0(\cos 40°)t$ and $y = 6.5 + v_0(\sin 40°)t - \frac{1}{2}gt^2 = 6.5 + v_0(\sin 40°)t - 16t^2$; $x = 262\frac{5}{12}$ ft and $y = 0$ ft
$\Rightarrow 262\frac{5}{12} = v_0(\cos 40°)t$ or $v_0 = \frac{262.4167}{(\cos 40°)t}$ and $0 = 6.5 + \left[\frac{262.4167}{(\cos 40°)t}\right](\sin 40°)t - 16t^2 \Rightarrow t^2 = 14.1684$
$\Rightarrow t \approx 3.764$ sec. Therefore, $262.4167 \approx v_0(\cos 40°)(3.764 \text{ sec}) \Rightarrow v_0 \approx \frac{262.4167}{(\cos 40°)(3.764 \text{ sec})} \Rightarrow v_0 \approx 91$ ft/sec

(b) $y_{max} = y_0 + \frac{(v_0 \sin \alpha)^2}{2g} \approx 6.5 + \frac{((91)(\sin 40°))^2}{(2)(32)} \approx 60$ ft

17. $x^2 = (v_0^2 \cos^2 \alpha)\, t^2$ and $\left(y + \frac{1}{2}gt^2\right)^2 = (v_0^2 \sin^2 \alpha)\, t^2 \Rightarrow x^2 + \left(y + \frac{1}{2}gt^2\right)^2 = v_0^2 t^2$

18. $\ddot{s} = \frac{d}{dt}\sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\dot{x}\ddot{x} + \dot{y}\ddot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Rightarrow \ddot{x}^2 + \ddot{y}^2 - \ddot{s}^2 = \ddot{x}^2 + \ddot{y}^2 - \frac{(\dot{x}\ddot{x} + \dot{y}\ddot{y})^2}{\dot{x}^2 + \dot{y}^2}$
$= \frac{(\ddot{x}^2 + \ddot{y}^2)(\dot{x}^2 + \dot{y}^2) - (\dot{x}^2\ddot{x}^2 + 2\dot{x}\ddot{x}\dot{y}\ddot{y} + \dot{y}^2\ddot{y}^2)}{\dot{x}^2 + \dot{y}^2} = \frac{\dot{x}^2\ddot{y}^2 + \dot{y}^2\ddot{x}^2 - 2\dot{x}\ddot{x}\dot{y}\ddot{y}}{\dot{x}^2 + \dot{y}^2} = \frac{(\dot{x}\ddot{y} - \dot{y}\ddot{x})^2}{\dot{x}^2 + \dot{y}^2}$
$\Rightarrow \sqrt{\ddot{x}^2 + \ddot{y}^2 - \ddot{s}^2} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Rightarrow \frac{\dot{x}^2 + \dot{y}^2}{\sqrt{\ddot{x}^2 + \ddot{y}^2 - \ddot{s}^2}} = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|} = \frac{1}{\kappa} = \rho$

19. $\mathbf{r}(t) = \left[\int_0^t \cos\left(\frac{1}{2}\pi\theta^2\right) d\theta\right]\mathbf{i} + \left[\int_0^t \sin\left(\frac{1}{2}\pi\theta^2\right) d\theta\right]\mathbf{j} \Rightarrow \mathbf{v}(t) = \cos\left(\frac{\pi t^2}{2}\right)\mathbf{i} + \sin\left(\frac{\pi t^2}{2}\right)\mathbf{j} \Rightarrow |\mathbf{v}| = 1;$

$\mathbf{a}(t) = -\pi t \sin\left(\frac{\pi t^2}{2}\right)\mathbf{i} + \pi t \cos\left(\frac{\pi t^2}{2}\right)\mathbf{j} \Rightarrow \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\left(\frac{\pi t^2}{2}\right) & \sin\left(\frac{\pi t^2}{2}\right) & 0 \\ -\pi t \sin\left(\frac{\pi t^2}{2}\right) & \pi t \cos\left(\frac{\pi t^2}{2}\right) & 0 \end{vmatrix}$

$= \pi t\mathbf{k} \Rightarrow \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \pi t$; $|\mathbf{v}(t)| = \frac{ds}{dt} = 1 \Rightarrow s = t + C$; $\mathbf{r}(0) = \mathbf{0} \Rightarrow s(0) = 0 \Rightarrow C = 0 \Rightarrow \kappa = \pi s$

20. $s = a\theta \Rightarrow \theta = \frac{s}{a} \Rightarrow \phi = \frac{s}{a} + \frac{\pi}{2} \Rightarrow \frac{d\phi}{ds} = \frac{1}{a} \Rightarrow \kappa = \left|\frac{1}{a}\right| = \frac{1}{a}$ since $a > 0$

21. $\mathbf{r} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{v} = (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (2t)^2}$
$= 2\sqrt{1 + t^2} \Rightarrow \text{Length} = \int_0^{\pi/4} 2\sqrt{1 + t^2}\, dt = \left[t\sqrt{1 + t^2} + \ln\left|t + \sqrt{1 + t^2}\right|\right]_0^{\pi/4} = \frac{\pi}{4}\sqrt{1 + \frac{\pi^2}{16}} + \ln\left(\frac{\pi}{4} + \sqrt{1 + \frac{\pi^2}{16}}\right)$

22. $\mathbf{r} = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + 2t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = (-3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 3t^{1/2}\mathbf{k}$
$\Rightarrow |\mathbf{v}| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + \left(3t^{1/2}\right)^2} = \sqrt{9 + 9t} = 3\sqrt{1 + t} \Rightarrow \text{Length} = \int_0^3 3\sqrt{1 + t}\, dt = \left[2(1 + t)^{3/2}\right]_0^3$
$= 14$

23. $\mathbf{r} = \frac{4}{9}(1 + t)^{3/2}\mathbf{i} + \frac{4}{9}(1 - t)^{3/2}\mathbf{j} + \frac{1}{3}t\mathbf{k} \Rightarrow \mathbf{v} = \frac{2}{3}(1 + t)^{1/2}\mathbf{i} - \frac{2}{3}(1 - t)^{1/2}\mathbf{j} + \frac{1}{3}\mathbf{k}$
$\Rightarrow |\mathbf{v}| = \sqrt{\left[\frac{2}{3}(1 + t)^{1/2}\right]^2 + \left[-\frac{2}{3}(1 - t)^{1/2}\right]^2 + \left(\frac{1}{3}\right)^2} = 1 \Rightarrow \mathbf{T} = \frac{2}{3}(1 + t)^{1/2}\mathbf{i} - \frac{2}{3}(1 - t)^{1/2}\mathbf{j} + \frac{1}{3}\mathbf{k}$
$\Rightarrow \mathbf{T}(0) = \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$; $\frac{d\mathbf{T}}{dt} = \frac{1}{3}(1 + t)^{-1/2}\mathbf{i} + \frac{1}{3}(1 - t)^{-1/2}\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}(0) = \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}(0)\right| = \frac{\sqrt{2}}{3}$
$\Rightarrow \mathbf{N}(0) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$; $\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{vmatrix} = -\frac{1}{3\sqrt{2}}\mathbf{i} + \frac{1}{3\sqrt{2}}\mathbf{j} + \frac{4}{3\sqrt{2}}\mathbf{k}$;

$\mathbf{a} = \frac{1}{3}(1 + t)^{-1/2}\mathbf{i} + \frac{1}{3}(1 - t)^{-1/2}\mathbf{j} \Rightarrow \mathbf{a}(0) = \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j}$ and $\mathbf{v}(0) = \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k} \Rightarrow \mathbf{v}(0) \times \mathbf{a}(0)$
$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{vmatrix} = -\frac{1}{9}\mathbf{i} + \frac{1}{9}\mathbf{j} + \frac{4}{9}\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = \frac{\sqrt{2}}{3} \Rightarrow \kappa(0) = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{\left(\frac{\sqrt{2}}{3}\right)}{1^3} = \frac{\sqrt{2}}{3}$;

$\dot{\mathbf{a}} = -\frac{1}{6}(1 + t)^{-3/2}\mathbf{i} + \frac{1}{6}(1 - t)^{-3/2}\mathbf{j} \Rightarrow \dot{\mathbf{a}}(0) = -\frac{1}{6}\mathbf{i} + \frac{1}{6}\mathbf{j} \Rightarrow \tau(0) = \frac{\begin{vmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{6} & 0 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{\left(\frac{1}{3}\right)\left(\frac{2}{18}\right)}{\left(\frac{\sqrt{2}}{3}\right)^2} = \frac{1}{6}$;

$t = 0 \Rightarrow \left(\frac{4}{9}, \frac{4}{9}, 0\right)$ is the point on the curve

24. $\mathbf{r} = (e^t \sin 2t)\,\mathbf{i} + (e^t \cos 2t)\,\mathbf{j} + 2e^t\mathbf{k} \Rightarrow \mathbf{v} = (e^t \sin 2t + 2e^t \cos 2t)\,\mathbf{i} + (e^t \cos 2t - 2e^t \sin 2t)\,\mathbf{j} + 2e^t\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{(e^t \sin 2t + 2e^t \cos 2t)^2 + (e^t \cos 2t - 2e^t \sin 2t)^2 + (2e^t)^2} = 3e^t \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$

$= \left(\frac{1}{3} \sin 2t + \frac{2}{3} \cos 2t\right)\mathbf{i} + \left(\frac{1}{3} \cos 2t - \frac{2}{3} \sin 2t\right)\mathbf{j} + \frac{2}{3}\,\mathbf{k} \Rightarrow \mathbf{T}(0) = \frac{2}{3}\,\mathbf{i} + \frac{1}{3}\,\mathbf{j} + \frac{2}{3}\,\mathbf{k}$;

$\frac{d\mathbf{T}}{dt} = \left(\frac{2}{3} \cos 2t - \frac{4}{3} \sin 2t\right)\mathbf{i} + \left(-\frac{2}{3} \sin 2t - \frac{4}{3} \cos 2t\right)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}(0) = \frac{2}{3}\,\mathbf{i} - \frac{4}{3}\,\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}(0)\right| = \frac{2}{3}\sqrt{5}$

$\Rightarrow \mathbf{N}(0) = \frac{\left(\frac{2}{3}\mathbf{i} - \frac{4}{3}\mathbf{j}\right)}{\left(\frac{2\sqrt{5}}{3}\right)} = \frac{1}{\sqrt{5}}\,\mathbf{i} - \frac{2}{\sqrt{5}}\,\mathbf{j}$; $\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \end{vmatrix} = \frac{4}{3\sqrt{5}}\,\mathbf{i} + \frac{2}{3\sqrt{5}}\,\mathbf{j} - \frac{5}{3\sqrt{5}}\,\mathbf{k}$;

$\mathbf{a} = (4e^t \cos 2t - 3e^t \sin 2t)\,\mathbf{i} + (-3e^t \cos 2t - 4e^t \sin 2t)\,\mathbf{j} + 2e^t\mathbf{k} \Rightarrow \mathbf{a}(0) = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v}(0) = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$\Rightarrow \mathbf{v}(0) \times \mathbf{a}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 4 & -3 & 2 \end{vmatrix} = 8\mathbf{i} + 4\mathbf{j} - 10\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = \sqrt{64 + 16 + 100} = 6\sqrt{5}$ and $|\mathbf{v}(0)| = 3$

$\Rightarrow \kappa(0) = \frac{6\sqrt{5}}{3^3} = \frac{2\sqrt{5}}{9}$;

$\dot{\mathbf{a}} = (4e^t \cos 2t - 8e^t \sin 2t - 3e^t \sin 2t - 6e^t \cos 2t)\,\mathbf{i} + (-3e^t \cos 2t + 6e^t \sin 2t - 4e^t \sin 2t - 8e^t \cos 2t)\,\mathbf{j} + 2e^t\mathbf{k}$

$= (-2e^t \cos 2t - 11e^t \sin 2t)\,\mathbf{i} + (-11\, e^t \cos 2t + 2e^t \sin 2t)\,\mathbf{j} + 2e^t\mathbf{k} \Rightarrow \dot{\mathbf{a}}(0) = -2\mathbf{i} - 11\mathbf{j} + 2\mathbf{k}$

$\Rightarrow \tau(0) = \frac{\begin{vmatrix} 2 & 1 & 2 \\ 4 & -3 & 2 \\ -2 & -11 & 2 \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{-80}{180} = -\frac{4}{9}$; $t = 0 \Rightarrow (0, 1, 2)$ is on the curve

25. $\mathbf{r} = t\mathbf{i} + \frac{1}{2}\, e^{2t}\,\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + e^{2t}\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1 + e^{4t}} \Rightarrow \mathbf{T} = \frac{1}{\sqrt{1+e^{4t}}}\,\mathbf{i} + \frac{e^{2t}}{\sqrt{1+e^{4t}}}\,\mathbf{j} \Rightarrow \mathbf{T}(\ln 2) = \frac{1}{\sqrt{17}}\,\mathbf{i} + \frac{4}{\sqrt{17}}\,\mathbf{j}$;

$\frac{d\mathbf{T}}{dt} = \frac{-2e^{4t}}{(1+e^{4t})^{3/2}}\,\mathbf{i} + \frac{2e^{2t}}{(1+e^{4t})^{3/2}}\,\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}(\ln 2) = \frac{-32}{17\sqrt{17}}\,\mathbf{i} + \frac{8}{17\sqrt{17}}\,\mathbf{j} \Rightarrow \mathbf{N}(\ln 2) = -\frac{4}{\sqrt{17}}\,\mathbf{i} + \frac{1}{\sqrt{17}}\,\mathbf{j}$;

$\mathbf{B}(\ln 2) = \mathbf{T}(\ln 2) \times \mathbf{N}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} & 0 \\ -\frac{4}{\sqrt{17}} & \frac{1}{\sqrt{17}} & 0 \end{vmatrix} = \mathbf{k}$; $\mathbf{a} = 2e^{2t}\,\mathbf{j} \Rightarrow \mathbf{a}(\ln 2) = 8\mathbf{j}$ and $\mathbf{v}(\ln 2) = \mathbf{i} + 4\mathbf{j}$

$\Rightarrow \mathbf{v}(\ln 2) \times \mathbf{a}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 0 \\ 0 & 8 & 0 \end{vmatrix} = 8\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = 8$ and $|\mathbf{v}(\ln 2)| = \sqrt{17} \Rightarrow \kappa(\ln 2) = \frac{8}{17\sqrt{17}}$; $\dot{\mathbf{a}} = 4e^{2t}\,\mathbf{j}$

$\Rightarrow \dot{\mathbf{a}}(\ln 2) = 16\mathbf{j} \Rightarrow \tau(\ln 2) = \frac{\begin{vmatrix} 1 & 4 & 0 \\ 0 & 8 & 0 \\ 0 & 16 & 0 \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = 0$; $t = \ln 2 \Rightarrow (\ln 2, 2, 0)$ is on the curve

26. $\mathbf{r} = (3 \cosh 2t)\mathbf{i} + (3 \sinh 2t)\mathbf{j} + 6t\mathbf{k} \Rightarrow \mathbf{v} = (6 \sinh 2t)\mathbf{i} + (6 \cosh 2t)\mathbf{j} + 6\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{36 \sinh^2 2t + 36 \cosh^2 2t + 36} = 6\sqrt{2} \cosh 2t \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{2}} \tanh 2t\right)\mathbf{i} + \frac{1}{\sqrt{2}}\,\mathbf{j} + \left(\frac{1}{\sqrt{2}} \operatorname{sech} 2t\right)\mathbf{k}$

$\Rightarrow \mathbf{T}(\ln 2) = \frac{15}{17\sqrt{2}}\,\mathbf{i} + \frac{1}{\sqrt{2}}\,\mathbf{j} + \frac{8}{17\sqrt{2}}\,\mathbf{k}$; $\frac{d\mathbf{T}}{dt} = \left(\frac{2}{\sqrt{2}} \operatorname{sech}^2 2t\right)\mathbf{i} - \left(\frac{2}{\sqrt{2}} \operatorname{sech} 2t \tanh 2t\right)\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt}(\ln 2)$

$= \left(\frac{2}{\sqrt{2}}\right)\left(\frac{8}{17}\right)^2 \mathbf{i} - \left(\frac{2}{\sqrt{2}}\right)\left(\frac{8}{17}\right)\left(\frac{15}{17}\right)\mathbf{k} = \frac{128}{289\sqrt{2}}\,\mathbf{i} - \frac{240}{289\sqrt{2}}\,\mathbf{k} \Rightarrow \left|\frac{d\mathbf{T}}{dt}(\ln 2)\right| = \sqrt{\left(\frac{128}{289\sqrt{2}}\right)^2 + \left(-\frac{240}{289\sqrt{2}}\right)^2} = \frac{8\sqrt{2}}{17}$

$\Rightarrow \mathbf{N}(\ln 2) = \frac{8}{17}\,\mathbf{i} - \frac{15}{17}\,\mathbf{k}$; $\mathbf{B}(\ln 2) = \mathbf{T}(\ln 2) \times \mathbf{N}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{15}{17\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{8}{17\sqrt{2}} \\ \frac{8}{17} & 0 & -\frac{15}{17} \end{vmatrix} = -\frac{15}{17\sqrt{2}}\,\mathbf{i} + \frac{1}{\sqrt{2}}\,\mathbf{j} - \frac{8}{17\sqrt{2}}\,\mathbf{k}$;

$\mathbf{a} = (12 \cosh 2t)\mathbf{i} + (12 \sinh 2t)\mathbf{j} \Rightarrow \mathbf{a}(\ln 2) = 12\left(\frac{17}{8}\right)\mathbf{i} + 12\left(\frac{15}{8}\right)\mathbf{j} = \frac{51}{2}\,\mathbf{i} + \frac{45}{2}\,\mathbf{j}$ and

$\mathbf{v}(\ln 2) = 6\left(\frac{15}{8}\right)\mathbf{i} + 6\left(\frac{17}{8}\right)\mathbf{j} + 6\mathbf{k} = \frac{45}{4}\,\mathbf{i} + \frac{51}{4}\,\mathbf{j} + 6\mathbf{k} \Rightarrow \mathbf{v}(\ln 2) \times \mathbf{a}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{45}{4} & \frac{51}{4} & 6 \\ \frac{51}{2} & \frac{45}{2} & 0 \end{vmatrix}$

$= -135\mathbf{i} + 153\mathbf{j} - 72\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = 153\sqrt{2}$ and $|\mathbf{v}(\ln 2)| = \frac{51}{4}\sqrt{2} \Rightarrow \kappa(\ln 2) = \frac{153\sqrt{2}}{\left(\frac{51}{4}\sqrt{2}\right)^3} = \frac{32}{867}$;

$\dot{\mathbf{a}} = (24 \sinh 2t)\mathbf{i} + (24 \cosh 2t)\mathbf{j} \Rightarrow \dot{\mathbf{a}}(\ln 2) = 45\mathbf{i} + 51\mathbf{j} \Rightarrow \tau(\ln 2) = \dfrac{\begin{vmatrix} \frac{45}{4} & \frac{51}{4} & 6 \\ \frac{51}{2} & \frac{45}{2} & 0 \\ 45 & 51 & 0 \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{32}{867}$;

$t = \ln 2 \Rightarrow \left(\frac{51}{8}, \frac{45}{8}, 6 \ln 2\right)$ is on the curve

27. $\mathbf{r} = (2 + 3t + 3t^2)\,\mathbf{i} + (4t + 4t^2)\,\mathbf{j} - (6 \cos t)\mathbf{k} \Rightarrow \mathbf{v} = (3 + 6t)\mathbf{i} + (4 + 8t)\mathbf{j} + (6 \sin t)\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{(3 + 6t)^2 + (4 + 8t)^2 + (6 \sin t)^2} = \sqrt{25 + 100t + 100t^2 + 36 \sin^2 t}$

$\Rightarrow \frac{d|\mathbf{v}|}{dt} = \frac{1}{2}\left(25 + 100t + 100t^2 + 36 \sin^2 t\right)^{-1/2}(100 + 200t + 72 \sin t \cos t) \Rightarrow a_T(0) = \frac{d|\mathbf{v}|}{dt}(0) = 10$;

$\mathbf{a} = 6\mathbf{i} + 8\mathbf{j} + (6 \cos t)\mathbf{k} \Rightarrow |\mathbf{a}| = \sqrt{6^2 + 8^2 + (6 \cos t)^2} = \sqrt{100 + 36 \cos^2 t} \Rightarrow |\mathbf{a}(0)| = \sqrt{136}$

$\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{136 - 10^2} = \sqrt{36} = 6 \Rightarrow \mathbf{a}(0) = 10\mathbf{T} + 6\mathbf{N}$

28. $\mathbf{r} = (2 + t)\mathbf{i} + (t + 2t^2)\,\mathbf{j} + (1 + t^2)\,\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + (1 + 4t)\mathbf{j} + 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (1 + 4t)^2 + (2t)^2}$

$= \sqrt{2 + 8t + 20t^2} \Rightarrow \frac{d|\mathbf{v}|}{dt} = \frac{1}{2}\left(2 + 8t + 20t^2\right)^{-1/2}(8 + 40t) \Rightarrow a_T = \frac{d|\mathbf{v}|}{dt}(0) = 2\sqrt{2}$; $\mathbf{a} = 4\mathbf{j} + 2\mathbf{k}$

$\Rightarrow |\mathbf{a}| = \sqrt{4^2 + 2^2} = \sqrt{20} \Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{20 - \left(2\sqrt{2}\right)^2} = \sqrt{12} = 2\sqrt{3} \Rightarrow \mathbf{a}(0) = 2\sqrt{2}\mathbf{T} + 2\sqrt{3}\mathbf{N}$

29. $\mathbf{r} = (\sin t)\mathbf{i} + \left(\sqrt{2} \cos t\right)\mathbf{j} + (\sin t)\mathbf{k} \Rightarrow \mathbf{v} = (\cos t)\mathbf{i} - \left(\sqrt{2} \sin t\right)\mathbf{j} + (\cos t)\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{(\cos t)^2 + \left(-\sqrt{2} \sin t\right)^2 + (\cos t)^2} = \sqrt{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{2}} \cos t\right)\mathbf{i} - (\sin t)\mathbf{j} + \left(\frac{1}{\sqrt{2}} \cos t\right)\mathbf{k}$;

$\frac{d\mathbf{T}}{dt} = \left(-\frac{1}{\sqrt{2}} \sin t\right)\mathbf{i} - (\cos t)\mathbf{j} - \left(\frac{1}{\sqrt{2}} \sin t\right)\mathbf{k} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(-\frac{1}{\sqrt{2}} \sin t\right)^2 + (-\cos t)^2 + \left(-\frac{1}{\sqrt{2}} \sin t\right)^2} = 1$

$\Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \left(-\frac{1}{\sqrt{2}} \sin t\right)\mathbf{i} - (\cos t)\mathbf{j} - \left(\frac{1}{\sqrt{2}} \sin t\right)\mathbf{k}$; $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{2}} \cos t & -\sin t & \frac{1}{\sqrt{2}} \cos t \\ -\frac{1}{\sqrt{2}} \sin t & -\cos t & -\frac{1}{\sqrt{2}} \sin t \end{vmatrix}$

$= \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{k}$; $\mathbf{a} = (-\sin t)\mathbf{i} - \left(\sqrt{2} \cos t\right)\mathbf{j} - (\sin t)\mathbf{k} \Rightarrow \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & -\sqrt{2} \sin t & \cos t \\ -\sin t & -\sqrt{2} \cos t & -\sin t \end{vmatrix}$

$= \sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = \sqrt{4} = 2 \Rightarrow \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{2}{\left(\sqrt{2}\right)^3} = \frac{1}{\sqrt{2}}$; $\dot{\mathbf{a}} = (-\cos t)\mathbf{i} + \left(\sqrt{2} \sin t\right)\mathbf{j} - (\cos t)\mathbf{k}$

$\Rightarrow \tau = \dfrac{\begin{vmatrix} \cos t & -\sqrt{2} \sin t & \cos t \\ -\sin t & -\sqrt{2} \cos t & -\sin t \\ -\cos t & \sqrt{2} \sin t & -\cos t \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{(\cos t)\left(\sqrt{2}\right) - \left(\sqrt{2} \sin t\right)(0) + (\cos t)\left(-\sqrt{2}\right)}{4} = 0$

30. $\mathbf{r} = \mathbf{i} + (5 \cos t)\mathbf{j} + (3 \sin t)\mathbf{k} \Rightarrow \mathbf{v} = (-5 \sin t)\mathbf{j} + (3 \cos t)\mathbf{k} \Rightarrow \mathbf{a} = (-5 \cos t)\mathbf{j} - (3 \sin t)\mathbf{k}$

$\Rightarrow \mathbf{v} \cdot \mathbf{a} = 25 \sin t \cos t - 9 \sin t \cos t = 16 \sin t \cos t$; $\mathbf{v} \cdot \mathbf{a} = 0 \Rightarrow 16 \sin t \cos t = 0 \Rightarrow \sin t = 0$ or $\cos t = 0$

$\Rightarrow t = 0, \frac{\pi}{2}$ or π

31. $\mathbf{r} = 2\mathbf{i} + \left(4 \sin \frac{t}{2}\right)\mathbf{j} + \left(3 - \frac{t}{\pi}\right)\mathbf{k} \Rightarrow 0 = \mathbf{r} \cdot (\mathbf{i} - \mathbf{j}) = 2(1) + \left(4 \sin \frac{t}{2}\right)(-1) \Rightarrow 0 = 2 - 4 \sin \frac{t}{2} \Rightarrow \sin \frac{t}{2} = \frac{1}{2} \Rightarrow \frac{t}{2} = \frac{\pi}{6}$

$\Rightarrow t = \frac{\pi}{3}$ (for the first time)

32. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1 + 4t^2 + 9t^4} \Rightarrow |\mathbf{v}(1)| = \sqrt{14}$

$\Rightarrow \mathbf{T}(1) = \frac{1}{\sqrt{14}}\mathbf{i} + \frac{2}{\sqrt{14}}\mathbf{j} + \frac{3}{\sqrt{14}}\mathbf{k}$, which is normal to the normal plane

$\Rightarrow \frac{1}{\sqrt{14}}(x - 1) + \frac{2}{\sqrt{14}}(y - 1) + \frac{3}{\sqrt{14}}(z - 1) = 0$ or $x + 2y + 3z = 6$ is an equation of the normal plane. Next we calculate $\mathbf{N}(1)$ which is normal to the rectifying plane. Now, $\mathbf{a} = 2\mathbf{j} + 6t\mathbf{k} \Rightarrow \mathbf{a}(1) = 2\mathbf{j} + 6\mathbf{k} \Rightarrow \mathbf{v}(1) \times \mathbf{a}(1)$

$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} = 6\mathbf{i} - 6\mathbf{j} + 2\mathbf{k} \Rightarrow |\mathbf{v}(1) \times \mathbf{a}(1)| = \sqrt{76} \Rightarrow \kappa(1) = \frac{\sqrt{76}}{\left(\sqrt{14}\right)^3} = \frac{\sqrt{19}}{7\sqrt{14}}$; $\frac{ds}{dt} = |\mathbf{v}(t)| \Rightarrow \left.\frac{d^2s}{dt^2}\right|_{t=1}$

$= \frac{1}{2}\left(1 + 4t^2 + 9t^4\right)^{-1/2}\left(8t + 36t^3\right)\Big|_{t=1} = \frac{22}{\sqrt{14}}$, so $\mathbf{a} = \frac{d^2s}{dt^2}\,\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N} \Rightarrow 2\mathbf{j} + 6\mathbf{k}$

$= \frac{22}{\sqrt{14}}\left(\frac{\mathbf{i}+2\mathbf{j}+3\mathbf{k}}{\sqrt{14}}\right) + \frac{\sqrt{19}}{7\sqrt{14}}\left(\sqrt{14}\right)^2\mathbf{N} \Rightarrow \mathbf{N} = \frac{\sqrt{14}}{2\sqrt{19}}\left(-\frac{11}{7}\,\mathbf{i} - \frac{8}{7}\,\mathbf{j} + \frac{9}{7}\,\mathbf{k}\right) \Rightarrow -\frac{11}{7}(x-1) - \frac{8}{7}(y-1) + \frac{9}{7}(z-1)$

$= 0$ or $11x + 8y - 9z = 10$ is an equation of the rectifying plane. Finally, $\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1)$

$= \left(\frac{\sqrt{14}}{2\sqrt{19}}\right)\left(\frac{1}{\sqrt{14}}\right)\left(\frac{1}{7}\right)\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -11 & -8 & 9 \end{vmatrix} = \frac{1}{\sqrt{19}}(3\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \Rightarrow 3(x-1) - 3(y-1) + (z-1) = 0$ or $3x - 3y + z$

$= 1$ is an equation of the osculating plane.

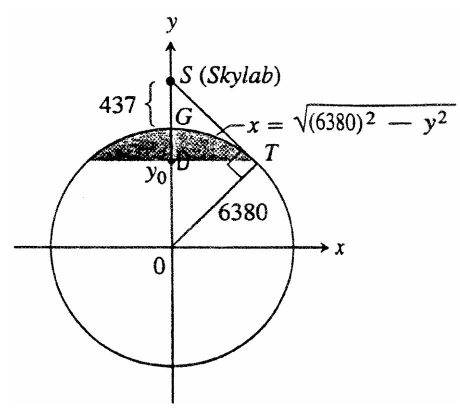
33. $\mathbf{r} = e^t\mathbf{i} + (\sin t)\mathbf{j} + \ln(1-t)\mathbf{k} \Rightarrow \mathbf{v} = e^t\mathbf{i} + (\cos t)\mathbf{j} - \left(\frac{1}{1-t}\right)\mathbf{k} \Rightarrow \mathbf{v}(0) = \mathbf{i} + \mathbf{j} - \mathbf{k}$; $\mathbf{r}(0) = \mathbf{i} \Rightarrow (1, 0, 0)$ is on the line
$\Rightarrow x = 1 + t$, $y = t$, and $z = -t$ are parametric equations of the line

34. $\mathbf{r} = \left(\sqrt{2}\cos t\right)\mathbf{i} + \left(\sqrt{2}\sin t\right)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = \left(-\sqrt{2}\sin t\right)\mathbf{i} + \left(\sqrt{2}\cos t\right)\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{4}\right)$
$= \left(-\sqrt{2}\sin\frac{\pi}{4}\right)\mathbf{i} + \left(\sqrt{2}\cos\frac{\pi}{4}\right)\mathbf{j} + \mathbf{k} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ is a vector tangent to the helix when $t = \frac{\pi}{4} \Rightarrow$ the tangent line is parallel to $\mathbf{v}\left(\frac{\pi}{4}\right)$; also $\mathbf{r}\left(\frac{\pi}{4}\right) = \left(\sqrt{2}\cos\frac{\pi}{4}\right)\mathbf{i} + \left(\sqrt{2}\sin\frac{\pi}{4}\right)\mathbf{j} + \frac{\pi}{4}\mathbf{k} \Rightarrow$ the point $\left(1, 1, \frac{\pi}{4}\right)$ is on the line
$\Rightarrow x = 1 - t$, $y = 1 + t$, and $z = \frac{\pi}{4} + t$ are parametric equations of the line

35. (a) $\Delta SOT \approx \Delta TOD \Rightarrow \frac{DO}{OT} = \frac{OT}{SO} \Rightarrow \frac{y_0}{6380} = \frac{6380}{6380+437}$
$\Rightarrow y_0 = \frac{6380^2}{6817} \Rightarrow y_0 \approx 5971$ km;

(b) $VA = \int_{5971}^{6380} 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$
$= 2\pi\int_{5971}^{6817}\sqrt{6380^2 - y^2}\left(\frac{6380}{\sqrt{6380^2 - y^2}}\right)dy$
$= 2\pi\int_{5971}^{6817} 6380\,dy = 2\pi\left[6380y\right]_{5971}^{6817}$
$= 16{,}395{,}469\ \text{km}^2 \approx 1.639 \times 10^7\ \text{km}^2$;

(c) percentage visible $\approx \frac{16{,}395{,}469\ \text{km}^2}{4\pi(6380\ \text{km})^2} \approx 3.21\%$



CHAPTER 13 ADDITIONAL AND ADVANCED EXERCISES

1. (a) The velocity of the boat at (x, y) relative to land is the sum of the velocity due to the rower and the velocity of the river, or $\mathbf{v} = \left[-\frac{1}{250}(y-50)^2 + 10\right]\mathbf{i} - 20\mathbf{j}$. Now, $\frac{dy}{dt} = -20 \Rightarrow y = -20t + c$; $y(0) = 100$
$\Rightarrow c = 100 \Rightarrow y = -20t + 100 \Rightarrow \mathbf{v} = \left[-\frac{1}{250}(-20t+50)^2 + 10\right]\mathbf{i} - 20\mathbf{j} = \left(-\frac{8}{5}t^2 + 8t\right)\mathbf{i} - 20\mathbf{j}$
$\Rightarrow \mathbf{r}(t) = \left(-\frac{8}{15}t^3 + 4t^2\right)\mathbf{i} - 20t\mathbf{j} + \mathbf{C}_1$; $\mathbf{r}(0) = 0\mathbf{i} + 100\mathbf{j} \Rightarrow 100\mathbf{j} = \mathbf{C}_1 \Rightarrow \mathbf{r}(t)$
$= \left(-\frac{8}{15}t^3 + 4t^2\right)\mathbf{i} + (100 - 20t)\mathbf{j}$

(b) The boat reaches the shore when $y = 0 \Rightarrow 0 = -20t + 100$ from part (a) $\Rightarrow t = 5$
$\Rightarrow \mathbf{r}(5) = \left(-\frac{8}{15}\cdot 125 + 4\cdot 25\right)\mathbf{i} + (100 - 20\cdot 5)\mathbf{j} = \left(-\frac{200}{3} + 100\right)\mathbf{i} = \frac{100}{3}\,\mathbf{i}$; the distance downstream is therefore $\frac{100}{3}$ m

2. (a) Let $a\mathbf{i} + b\mathbf{j}$ be the velocity of the boat. The velocity of the boat relative to an observer on the bank of the river is $\mathbf{v} = a\mathbf{i} + \left[b - \frac{3x(20-x)}{100}\right]\mathbf{j}$. The distance x of the boat as it crosses the river is related to time by
$x = at \Rightarrow \mathbf{v} = a\mathbf{i} + \left[b - \frac{3at(20-at)}{100}\right]\mathbf{j} = a\mathbf{i} + \left(b + \frac{3a^2t^2 - 60at}{100}\right)\mathbf{j} \Rightarrow \mathbf{r}(t) = at\mathbf{i} + \left(bt + \frac{a^2t^3}{100} - \frac{30at^2}{100}\right)\mathbf{j} + \mathbf{C}$;

$\mathbf{r}(0) = 0\mathbf{i} + 0\mathbf{j} \Rightarrow \mathbf{C} = 0 \Rightarrow \mathbf{r}(t) = at\mathbf{i} + \left(bt + \frac{a^2t^3 - 30at^2}{100}\right)\mathbf{j}$. The boat reaches the shore when $x = 20$

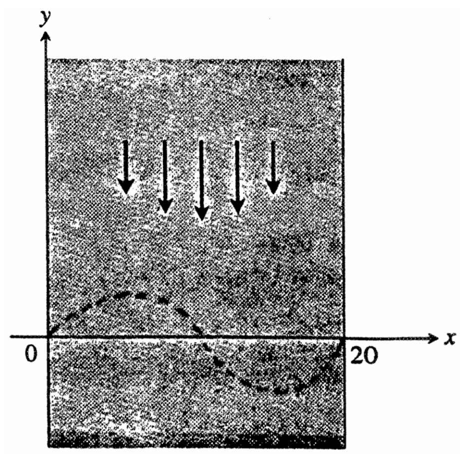
$\Rightarrow 20 = at \Rightarrow t = \frac{20}{a}$ and $y = 0 \Rightarrow 0 = b\left(\frac{20}{a}\right) + \frac{a^2\left(\frac{20}{a}\right)^3 - 30a\left(\frac{20}{a}\right)^2}{100} = \frac{20b}{a} + \frac{(20)^3 - 30(20)^2}{100a}$

$= \frac{2000b + 8000 - 12{,}000}{100a} \Rightarrow b = 2$; the speed of the boat is $\sqrt{20} = |\mathbf{v}| = \sqrt{a^2 + b^2} = \sqrt{a^2 + 4} \Rightarrow a^2 = 16$

$\Rightarrow a = 4$; thus, $\mathbf{v} = 4\mathbf{i} + 2\mathbf{j}$ is the velocity of the boat

(b) $\mathbf{r}(t) = at\mathbf{i} + \left(bt + \frac{a^2t^3 - 30at^2}{100}\right)\mathbf{j} = 4t\mathbf{i} + \left(2t + \frac{16t^3}{100} - \frac{120t^2}{100}\right)\mathbf{j}$ by part (a), where $0 \le t \le 5$

(c) $x = 4t$ and $y = 2t + \frac{16t^3}{100} - \frac{120t^2}{100}$
$= \frac{4}{25}t^3 - \frac{6}{5}t^2 + 2t = \frac{2}{25}t\,(2t^2 - 15t + 25)$
$= \frac{2}{25}t(2t - 5)(t - 5)$, which is the graph of the cubic displayed here



3. (a) $\mathbf{r}(\theta) = (a\cos\theta)\mathbf{i} + (a\sin\theta)\mathbf{j} + b\theta\mathbf{k} \Rightarrow \frac{d\mathbf{r}}{dt} = [(-a\sin\theta)\mathbf{i} + (a\cos\theta)\mathbf{j} + b\mathbf{k}]\frac{d\theta}{dt}$; $|\mathbf{v}| = \sqrt{2gz} = \left|\frac{d\mathbf{r}}{dt}\right|$

$= \sqrt{a^2 + b^2}\,\frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \sqrt{\frac{2gz}{a^2 + b^2}} = \sqrt{\frac{2gb\theta}{a^2 + b^2}} \Rightarrow \left.\frac{d\theta}{dt}\right|_{\theta = 2\pi} = \sqrt{\frac{4\pi gb}{a^2 + b^2}} = 2\sqrt{\frac{\pi gb}{a^2 + b^2}}$

(b) $\frac{d\theta}{dt} = \sqrt{\frac{2gb\theta}{a^2 + b^2}} \Rightarrow \frac{d\theta}{\sqrt{\theta}} = \sqrt{\frac{2gb}{a^2 + b^2}}\,dt \Rightarrow 2\theta^{1/2} = \sqrt{\frac{2gb}{a^2 + b^2}}\,t + C$; $t = 0 \Rightarrow \theta = 0 \Rightarrow C = 0$

$\Rightarrow 2\theta^{1/2} = \sqrt{\frac{2gb}{a^2 + b^2}}\,t \Rightarrow \theta = \frac{gbt^2}{2(a^2 + b^2)}$; $z = b\theta \Rightarrow z = \frac{gb^2t^2}{2(a^2 + b^2)}$

(c) $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = [(-a\sin\theta)\mathbf{i} + (a\cos\theta)\mathbf{j} + b\mathbf{k}]\frac{d\theta}{dt} = [(-a\sin\theta)\mathbf{i} + (a\cos\theta)\mathbf{j} + b\mathbf{k}]\left(\frac{gbt}{a^2 + b^2}\right)$, from part (b)

$\Rightarrow \mathbf{v}(t) = \left[\frac{(-a\sin\theta)\mathbf{i} + (a\cos\theta)\mathbf{j} + b\mathbf{k}}{\sqrt{a^2 + b^2}}\right]\left(\frac{gbt}{\sqrt{a^2 + b^2}}\right) = \frac{gbt}{\sqrt{a^2 + b^2}}\,\mathbf{T}$;

$\frac{d^2\mathbf{r}}{dt^2} = [(-a\cos\theta)\mathbf{i} - (a\sin\theta)\mathbf{j}]\left(\frac{d\theta}{dt}\right)^2 + [(-a\sin\theta)\mathbf{i} + (a\cos\theta)\mathbf{j} + b\mathbf{k}]\frac{d^2\theta}{dt^2}$

$= \left(\frac{gbt}{a^2 + b^2}\right)^2[(-a\cos\theta)\mathbf{i} - (a\sin\theta)\mathbf{j}] + [(-a\sin\theta)\mathbf{i} + (a\cos\theta)\mathbf{j} + b\mathbf{k}]\left(\frac{gb}{a^2 + b^2}\right)$

$= \left[\frac{(-a\sin\theta)\mathbf{i} + (a\cos\theta)\mathbf{j} + b\mathbf{k}}{\sqrt{a^2 + b^2}}\right]\left(\frac{gb}{\sqrt{a^2 + b^2}}\right) + a\left(\frac{gbt}{a^2 + b^2}\right)^2[(-\cos\theta)\mathbf{i} - (\sin\theta)\mathbf{j}]$

$= \frac{gb}{\sqrt{a^2 + b^2}}\,\mathbf{T} + a\left(\frac{gbt}{a^2 + b^2}\right)^2\mathbf{N}$ (there is no component in the direction of $\mathbf{B}$).

4. (a) $\mathbf{r}(\theta) = (a\theta\cos\theta)\mathbf{i} + (a\theta\sin\theta)\mathbf{j} + b\theta\mathbf{k} \Rightarrow \frac{d\mathbf{r}}{dt} = [(a\cos\theta - a\theta\sin\theta)\mathbf{i} + (a\sin\theta + a\theta\cos\theta)\mathbf{j} + b\mathbf{k}]\frac{d\theta}{dt}$;

$|\mathbf{v}| = \sqrt{2gz} = \left|\frac{d\mathbf{r}}{dt}\right| = (a^2 + a^2\theta^2 + b^2)^{1/2}\left(\frac{d\theta}{dt}\right) \Rightarrow \frac{d\theta}{dt} = \frac{\sqrt{2gb\theta}}{\sqrt{a^2 + a^2\theta^2 + b^2}}$

(b) $s = \int_0^t |\mathbf{v}|\,dt = \int_0^t (a^2 + a^2\theta^2 + b^2)^{1/2}\frac{d\theta}{dt}\,dt = \int_0^t (a^2 + a^2\theta^2 + b^2)^{1/2}\,d\theta = \int_0^\theta (a^2 + a^2u^2 + b^2)^{1/2}\,du$

$= \int_0^\theta a\sqrt{\frac{a^2 + b^2}{a^2} + u^2}\,du = a\int_0^\theta \sqrt{c^2 + u^2}\,du$, where $c = \frac{\sqrt{a^2 + b^2}}{|a|}$

$\Rightarrow s = a\left[\frac{u}{2}\sqrt{c^2 + u^2} + \frac{c^2}{2}\ln\left|u + \sqrt{c^2 + u^2}\right|\right]_0^\theta = \frac{a}{2}\left(\theta\sqrt{c^2 + \theta^2} + c^2\ln\left|\theta + \sqrt{c^2 + \theta^2}\right| - c^2\ln c\right)$

5. $r = \frac{(1 + e)r_0}{1 + e\cos\theta} \Rightarrow \frac{dr}{d\theta} = \frac{(1 + e)r_0(e\sin\theta)}{(1 + e\cos\theta)^2}$; $\frac{dr}{d\theta} = 0 \Rightarrow \frac{(1 + e)r_0(e\sin\theta)}{(1 + e\cos\theta)^2} = 0 \Rightarrow (1 + e)r_0(e\sin\theta) = 0$

$\Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$ or π. Note that $\frac{dr}{d\theta} > 0$ when $\sin\theta > 0$ and $\frac{dr}{d\theta} < 0$ when $\sin\theta < 0$. Since $\sin\theta < 0$ on $-\pi < \theta < 0$ and $\sin\theta > 0$ on $0 < \theta < \pi$, r is a minimum when $\theta = 0$ and $r(0) = \frac{(1 + e)r_0}{1 + e\cos 0} = r_0$

6. (a) $f(x) = x - 1 - \frac{1}{2}\sin x = 0 \Rightarrow f(0) = -1$ and $f(2) = 2 - 1 - \frac{1}{2}\sin 2 \geq \frac{1}{2}$ since $|\sin 2| \leq 1$; since f is continuous on $[0, 2]$, the Intermediate Value Theorem implies there is a root between 0 and 2

(b) Root ≈ 1.4987011335179

7. (a) $\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$ and $\mathbf{v} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta = \left(\frac{dr}{dt}\right)[(\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}] + \left(r\frac{d\theta}{dt}\right)[(-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}] \Rightarrow \mathbf{v}\cdot\mathbf{i} = \frac{dx}{dt}$ and $\mathbf{v}\cdot\mathbf{i} = \frac{dr}{dt}\cos\theta - r\frac{d\theta}{dt}\sin\theta \Rightarrow \frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\frac{d\theta}{dt}\sin\theta$; $\mathbf{v}\cdot\mathbf{j} = \frac{dy}{dt}$ and $\mathbf{v}\cdot\mathbf{j} = \frac{dr}{dt}\sin\theta + r\frac{d\theta}{dt}\cos\theta$ $\Rightarrow \frac{dy}{dt} = \frac{dr}{dt}\sin\theta + r\frac{d\theta}{dt}\cos\theta$

(b) $\mathbf{u}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} \Rightarrow \mathbf{v}\cdot\mathbf{u}_r = \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta$
$= \left(\frac{dr}{dt}\cos\theta - r\frac{d\theta}{dt}\sin\theta\right)(\cos\theta) + \left(\frac{dr}{dt}\sin\theta + r\frac{d\theta}{dt}\cos\theta\right)(\sin\theta)$ by part (a),
$\Rightarrow \mathbf{v}\cdot\mathbf{u}_r = \frac{dr}{dt}$; therefore, $\frac{dr}{dt} = \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta$;
$\mathbf{u}_\theta = -(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j} \Rightarrow \mathbf{v}\cdot\mathbf{u}_\theta = -\frac{dx}{dt}\sin\theta + \frac{dy}{dt}\cos\theta$
$= \left(\frac{dr}{dt}\cos\theta - r\frac{d\theta}{dt}\sin\theta\right)(-\sin\theta) + \left(\frac{dr}{dt}\sin\theta + r\frac{d\theta}{dt}\cos\theta\right)(\cos\theta)$ by part (a) $\Rightarrow \mathbf{v}\cdot\mathbf{u}_\theta = r\frac{d\theta}{dt}$;
therefore, $r\frac{d\theta}{dt} = -\frac{dx}{dt}\sin\theta + \frac{dy}{dt}\cos\theta$

8. $\mathbf{r} = f(\theta) \Rightarrow \frac{dr}{dt} = f'(\theta)\frac{d\theta}{dt} \Rightarrow \frac{d^2r}{dt^2} = f''(\theta)\left(\frac{d\theta}{dt}\right)^2 + f'(\theta)\frac{d^2\theta}{dt^2}$; $\mathbf{v} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta$
$= \left(\cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt}\right)\mathbf{i} + \left(\sin\theta\frac{dr}{dt} + r\cos\theta\frac{d\theta}{dt}\right)\mathbf{j} \Rightarrow |\mathbf{v}| = \left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right]^{1/2} = \left[(f')^2 + f^2\right]^{1/2}\left(\frac{d\theta}{dt}\right)$;
$|\mathbf{v}\times\mathbf{a}| = |\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}|$, where $x = r\cos\theta$ and $y = r\sin\theta$. Then $\frac{dx}{dt} = (-r\sin\theta)\frac{d\theta}{dt} + (\cos\theta)\frac{dr}{dt}$
$\Rightarrow \frac{d^2x}{dt^2} = (-2\sin\theta)\frac{d\theta}{dt}\frac{dr}{dt} - (r\cos\theta)\left(\frac{d\theta}{dt}\right)^2 - (r\sin\theta)\frac{d^2\theta}{dt^2} + (\cos\theta)\frac{d^2r}{dt^2}$; $\frac{dy}{dt} = (r\cos\theta)\frac{d\theta}{dt} + (\sin\theta)\frac{dr}{dt}$
$\Rightarrow \frac{d^2y}{dt^2} = (2\cos\theta)\frac{d\theta}{dt}\frac{dr}{dt} - (r\sin\theta)\left(\frac{d\theta}{dt}\right)^2 + (r\cos\theta)\frac{d^2\theta}{dt^2} + (\sin\theta)\frac{d^2r}{dt^2}$. Then $|\mathbf{v}\times\mathbf{a}|$
$=$ (after <u>much</u> algebra) $r^2\left(\frac{d\theta}{dt}\right)^3 + r\frac{d^2\theta}{dt^2}\frac{dr}{dt} - r\frac{d\theta}{dt}\frac{d^2r}{dt^2} + 2\frac{d\theta}{dt}\left(\frac{dr}{dt}\right)^2 = \left(\frac{d\theta}{dt}\right)^3\left(f^2 - f\cdot f'' + 2(f')^2\right)$
$\Rightarrow \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|} = \frac{f^2 - f\cdot f'' + 2(f')^2}{\left[(f')^2 + f^2\right]^{3/2}}$

9. (a) Let $r = 2 - t$ and $\theta = 3t \Rightarrow \frac{dr}{dt} = -1$ and $\frac{d\theta}{dt} = 3 \Rightarrow \frac{d^2r}{dt^2} = \frac{d^2\theta}{dt^2} = 0$. The halfway point is $(1, 3) \Rightarrow t = 1$;
$\mathbf{v} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta \Rightarrow \mathbf{v}(1) = -\mathbf{u}_r + 3\mathbf{u}_\theta$; $\mathbf{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\mathbf{u}_\theta \Rightarrow \mathbf{a}(1) = -9\mathbf{u}_r - 6\mathbf{u}_\theta$

(b) It takes the beetle 2 min to crawl to the origin $\Rightarrow$ the rod has revolved 6 radians
$\Rightarrow L = \int_0^6 \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta = \int_0^6 \sqrt{\left(2 - \frac{\theta}{3}\right)^2 + \left(-\frac{1}{3}\right)^2}\,d\theta = \int_0^6 \sqrt{4 - \frac{4\theta}{3} + \frac{\theta^2}{9} + \frac{1}{9}}\,d\theta$
$= \int_0^6 \sqrt{\frac{37 - 12\theta + \theta^2}{9}}\,d\theta = \frac{1}{3}\int_0^6 \sqrt{(\theta - 6)^2 + 1}\,d\theta = \frac{1}{3}\left[\frac{(\theta - 6)}{2}\sqrt{(\theta - 6)^2 + 1} + \frac{1}{2}\ln\left|\theta - 6 + \sqrt{(\theta - 6)^2 + 1}\right|\right]_0^6$
$= \sqrt{37} - \frac{1}{6}\ln\left(\sqrt{37} - 6\right) \approx 6.5$ in.

10. $\mathbf{L}(t) = \mathbf{r}(t)\times m\mathbf{v}(t) \Rightarrow \frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{r}}{dt}\times m\mathbf{v}\right) + \left(\mathbf{r}\times m\frac{d^2\mathbf{r}}{dt^2}\right) \Rightarrow \frac{d\mathbf{L}}{dt} = (\mathbf{v}\times m\mathbf{v}) + (\mathbf{r}\times m\mathbf{a}) = \mathbf{r}\times m\mathbf{a}$; $\mathbf{F} = m\mathbf{a} \Rightarrow -\frac{c}{|\mathbf{r}|^3}\mathbf{r}$ $= m\mathbf{a} \Rightarrow \frac{d\mathbf{L}}{dt} = \mathbf{r}\times m\mathbf{a} = \mathbf{r}\times\left(-\frac{c}{|\mathbf{r}|^3}\mathbf{r}\right) = -\frac{c}{|\mathbf{r}|^3}(\mathbf{r}\times\mathbf{r}) = \mathbf{0} \Rightarrow \mathbf{L} =$ constant vector

11. (a) $\mathbf{u}_r\times\mathbf{u}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} = \mathbf{k} \Rightarrow$ a right-handed frame of unit vectors

(b) $\frac{d\mathbf{u}_r}{d\theta} = (-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j} = \mathbf{u}_\theta$ and $\frac{d\mathbf{u}_\theta}{d\theta} = (-\cos\theta)\mathbf{i} - (\sin\theta)\mathbf{j} = -\mathbf{u}_r$

(c) From Eq. (7), $\mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta + \dot{z}\mathbf{k} \Rightarrow \mathbf{a} = \dot{\mathbf{v}} = (\ddot{r}\mathbf{u}_r + \dot{r}\,\dot{\mathbf{u}}_r) + \left(\dot{r}\,\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta + r\dot{\theta}\,\dot{\mathbf{u}}_\theta\right) + \ddot{z}\,\mathbf{k}$
$= \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{u}_r + \left(r\ddot{\theta} + 2\dot{r}\,\dot{\theta}\right)\mathbf{u}_\theta + \ddot{z}\,\mathbf{k}$

12. (a) $x = r\cos\theta \Rightarrow dx = \cos\theta\,dr - r\sin\theta\,d\theta$; $y = r\sin\theta \Rightarrow dy = \sin\theta\,dr + r\cos\theta\,d\theta$; thus
$dx^2 = \cos^2\theta\,dr^2 - 2r\sin\theta\cos\theta\,dr\,d\theta + r^2\sin^2\theta\,d\theta^2$ and

$dy^2 = \sin^2\theta\, dr^2 + 2r\sin\theta\cos\theta\, dr\, d\theta + r^2\cos^2\theta\, d\theta^2 \Rightarrow dx^2 + dy^2 + dz^2 = dr^2 + r^2\, d\theta^2 + dz^2$

(c) $r = e^\theta \Rightarrow dr = e^\theta\, d\theta$

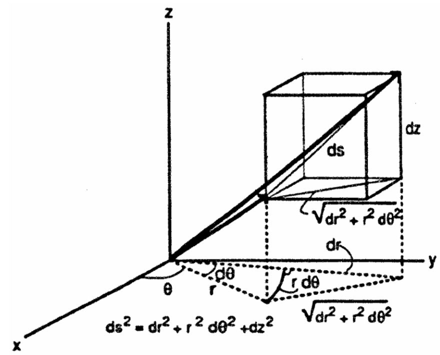
$\Rightarrow L = \int_0^{\ln 8} \sqrt{dr^2 + r^2\, d\theta^2 + dz^2}$

$= \int_0^{\ln 8} \sqrt{e^{2\theta} + e^{2\theta} + e^{2\theta}}\, d\theta$

$= \int_0^{\ln 8} \sqrt{3}e^\theta\, d\theta = \left[\sqrt{3}\, e^\theta\right]_0^{\ln 8}$

$= 8\sqrt{3} - \sqrt{3} = 7\sqrt{3}$

(b)



NOTES: