Conditional Expected Shortfall

Nonparametric Estimation

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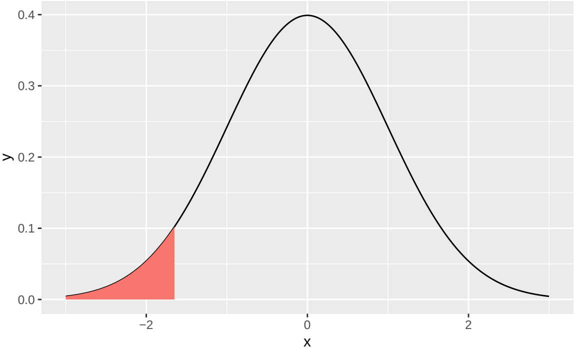
Expected Shortfall

Value at Risk



Tsay (2010) says that

Measure of loss under *normal* market conditions
Minimal loss under *extraordinary* market circumstances

Value at Risk

p: **Right** tail probability
l: Time horizon
$L(l)$: loss function of the asset
F: CDF of the loss

$$p = P\left[L(l) \geq VaR\right]$$

Subadditivity

Coherent risk measure

- Homogeneity
- Monotonicity
- Translation invariance (risk-free condition)
- Subadditivity

VaR

does not satisfy subadditivity

- When two portfolios are merged, the risk measure should not be greater than the sum of each.
- VaR *underestimates* the actual loss.

Conditional VaR

Stationary log-return $\{Y_t : t = 1, \dots n\}$
Exogenous variable $\{X_t : t = 1, \dots n\}$
Conditional VaR (CVaR) or Expected Shortfall (ES)

$$\nu_p(x) = S^{-1}(p \mid x)$$

where

$S(y \mid x) := 1 - F(y \mid x)$
F: conditional CDF of Y_t given $X_t = x$.

Conditional Expected Shortfall

We are interested in *Expected Shortfall given exogenous variable values*
Conditional Expected Shortfall (CES)

$$\mu_p(x) = E\left[Y_t \mid Y_t \geq \nu_p(x), X_t = x\right]$$

Formulating CES

Let $B \equiv \{\omega\colon Y_t \geq \nu_p(x)\} \in \mathcal{B}$. Then

$$
\begin{aligned}
\mu_p(x) &= E\left[Y_t \mid Y_t \geq \nu_p(x), X_t = x\right] \\
&= \frac{1}{P(B)} \int_B Y_t dP \\
&= \frac{1}{P\left(Y_t \geq \nu_p(x) \mid X_t = x\right)} \int_{\nu_p(x)}^{\infty} y f(y \mid x) dy \\
&= \frac{1}{p} \int_{\nu_p(x)}^{\infty} y f(y \mid x) dy
\end{aligned}
$$

Nonparametric Estimation

Workflow of Estimation

Plugging-in Method

$$\hat{\mu}_p(x) = \frac{1}{p} \int_{\hat{\nu}_p(x)}^{\infty} y \hat{f}(y \mid x) dy$$

What to estimate

Conditional PDF: $\hat{f}(y \mid x)$
CVaR: $\hat{\nu}_p(x) = \hat{S}^{-1}(p \mid x)$ by inverting the conditional CDF

Conditional Disribution

Taylor expansion

Consider any symmetric kernel $K_h(\cdot)$. Then

$$\begin{aligned} E[K_h(y - Y_t) \mid X_t = x] &= K_h * f_{y|x}(y) \\ &= f(y \mid x) + \frac{h^2}{2}\mu_2(K) f^{(2)}(y \mid x) + o(h^2) \end{aligned}$$

where $\mu_j(K) = \int_{\mathbb{R}} u^j K(u) du$.

Smoothing

$$f(y \mid x) \approx E\left[K_h(y - Y_t) \mid X_t = x\right]$$

Methods

Local Linear
Weighted Nadaraya Watson
WDKLL (Cai and Wang 2008)

Double Kernel Local Linear

Denote $Y_t^*(y) \equiv K_h(y - Y_t)$.

$$\hat{f}(y \mid x) = \underset{\alpha(x),\beta(x)}{\operatorname{argmin}} \sum_{t=1}^{n} W_\lambda(x - X_t) \left[Y_t^*(y) - \alpha(x) - \beta(x)(X_t - x) \right]^2$$

Since this is involved in the two kernel ($K_h(\cdot)$, $W_\lambda(\cdot)$), Cai and Wang (2008) names this as *double kernel*.

Local Linear Solution

Note that the local linear estimate is equivalent to WLS.

$$\mathbf{Y}_y^* = (Y_1(y), \ldots, Y_n(y))^T \in \mathbb{R}^n$$
$$\mathbf{b}_x(x_t) := (1, x_t - x)^T \in \mathbb{R}^2 \text{ and } \mathbf{b}_x(x) = \mathbf{e}_1 := (1, 0)^T$$
$$X_x := \left(\mathbf{b}_x(x_i)^T\right) \in \mathbb{R}^{n\times 2}$$
$$W_x := diag(W_\lambda(x - X_j)) \in \mathbb{R}^{n\times n}$$

Then $\hat{f}_{ll} = \hat{\alpha}$:

$$\begin{aligned}\hat{f}_{ll}(y \mid x) &= \mathbf{e}_1^T (X_x^T W_x X_x)^{-1} X_x^T W_x \mathbf{Y}_y^* \\ &= \mathbf{l}(x)^T \mathbf{Y}_y^* \\ &\equiv \sum_{t=1}^{n} l_t(x) Y_t^*(y)\end{aligned}$$

Linear Smoother

$$\mathbf{l}(x)^T = \mathbf{e}_1^T (X_x^T W_x X_x)^{-1} X_x^T W_x$$

By annoying arithmetic,

$$l_t(x) = \frac{S_2(x) - (X_t - x) S_1(x)}{S_0(x) S_2(x) - [S_1(x)]^2} W_\lambda(x - X_t)$$

where $S_j(x) := \sum_{t=1}^{n} W_\lambda(x - X_t)(X_t - x)^j$.

Matrix computations

Let $w_t \equiv W_\lambda(x - X_t)$

$$(X_x^T W_x X_x) = \begin{bmatrix} \sum_t w_t & \sum_t w_t(x_t - x) \\ \sum_t w_t(x_t - x) & \sum_t w_t(x_t - x)^2 \end{bmatrix} \equiv \begin{bmatrix} S_0 & S_1 \\ S_1 & S_2 \end{bmatrix}$$

$$X_x^T W_x = \begin{bmatrix} w_1 & \cdots & w_n \\ w_1(x_1 - x) & \cdots & w_n(x_n - x) \end{bmatrix}$$

Thus,

$$\mathbf{l}(x)^T = \frac{1}{S_0 S_2 - S_1^2} \begin{bmatrix} S_2 w_1 - S_1 w_1(x_1 - x) & \cdots & S_2 w_n - S_1 w_n(x_n - x) \end{bmatrix}$$

Discrete Moments Conditions

$$S_j(x) := \sum_{t=1}^{n} W_\lambda(x - X_t)(X_t - x)^j = \delta_{0,j} = \begin{cases} 1 & j = 0 \\ 0 & \text{o/w} \end{cases}$$

will be used when showing the asymptotic properties

CVaR

Invert $\hat{F}_{ll}(y \mid x)$

Conditional CDF

$$\begin{aligned}\hat{F}_{ll}(y \mid x) &= \int_{\infty}^{y} \hat{f}_{ll}(y \mid x) dy \\ &= \sum_{t=1}^{n} l_t(x) G_h(y - Y_t)\end{aligned}$$

where $G(\cdot)$ is the cdf of $K(\cdot)$.

Problem

It must be $\hat{F}_{ll} \in [0, 1]$ and monotone increasing
However, LL does not guarantee these properties.

Weighted Nadaraya Watson

To get the right shape of CDF

$$\hat{F}_{NW}(y \mid x) = \sum_{t=1}^{n} H_t(x, \lambda) I(Y_t \leq y)$$

where

$$H_t(x, \lambda) = \frac{p_t(x) W_\lambda(x - X_t)}{\sum_{i=1}^{n} p_i(x) W_\lambda(x - X_i)}$$

$p_t(x)$ is *weighted* for each NW weight.
Cai (2001) finds the best weights $\{p_t\}_1^n$ by maximizing the *empirical likelihood*.

Choosing weights

Constraints

$p_t(x) \geq 0$
$\sum_t p_t(x) = 1$
Discrete moments conditions $\sum_{t=1}^{n} H_t(x, \lambda)(X_t - x)^j = \delta_{0,j},\ 0 \leq j \leq 1$

Empirical likelihood

Maximize $\sum_t \ln p_t(x)$. Lagrangian multiplier gives that

$$p_t(x) = \frac{1}{n\left[1 + \gamma(X_t - x)W_\lambda(x - X_i)\right]} \geq 0$$

and γ uniquely maximizing the log of the empirical likelihood

$$L_n(\gamma) = -\sum^{n} \ln\left[1 + \gamma(X_t - x)W_\lambda(x - X_i)\right]$$

Weighted Double Kernel Local Linear

In a local linear scheme,
replace linear smoother with WNW weight

$$\hat{f}_{cai}(y \mid x) = \sum_{t=1}^{n} H_t(x, \lambda) Y_t^*(y)$$

and hence,

$$\begin{aligned}\hat{F}_{cai}(y \mid x) &= \int_{\infty}^{y} \hat{f}_{cai}(y \mid x) dy \\ &= \sum_{t=1}^{n} H_t(x, \lambda) G_h(y - Y_t)\end{aligned}$$

Inverting and Plugging-in

CVaR

$$\hat{\nu}_p^{(cai)}(x) = \hat{S}_{cai}^{-1}(p \mid x)$$

where $\hat{S}_{cai}(y \mid x) = 1 - \hat{F}_{cai}(y \mid x)$

CES

$$\hat{\mu}_p(x) = \frac{1}{p} \sum_{t=1}^{n} H_t(x, \lambda) \left[Y_t \bar{G}_h(\hat{\nu}_p(x) - Y_t) + h G_{1,h}(\hat{\nu}_p(x) - Y_t) \right]$$

where $\bar{G}(u) = 1 - G(u)$ and $G_1(u) = \int_u^\infty vK(v)dv$.

Statistical Properties

Asymptotic Normality

Investigate

$\hat{f}_{cai}(y \mid x)$
$\hat{S}_{cai}(y \mid x) = 1 - \hat{F}_{cai}(y \mid x)$
$\hat{\nu}_p(x)$
$\hat{\mu}_p(x)$

at both

Interior	Boundary
x	$x = c\lambda$

Notations

$$
\begin{aligned}
&\alpha(K) = \int_{-\infty}^{\infty} uK(u)\bar{G}(u)du \\
&\mu(W) = \int_{-\infty}^{\infty} u^j W(u)du \\
&l_j(u \mid v) = E\left[Y_t^j I(Y_t \geq u) \mid X_t = v \right] \\
&l_j^{a,b}(u \mid v) = \frac{\partial^{ab}}{\partial u^a \partial v^b} l_j(u \mid v)
\end{aligned}
$$

Interior

$$\sqrt{n\lambda}\left[\hat{\mu}_p(x) - \mu(x) - B_\mu(x)\right] \xrightarrow{\mathcal{D}} N\left(0, \sigma_\mu^2(x)\right)$$

Boundary

W.L.O.G. the left boundary point $x = c\lambda$ s.t.

$$sptK = [-1, 1]$$
$$c \in (0, 1)$$

$$\sqrt{n\lambda}\left[\hat{\mu}_p(c\lambda) - \mu(c\lambda) - B_{\mu,c}\right] \xrightarrow{\mathcal{D}} N\left(0, \sigma^2_{\mu,c}\right)$$

Simulation for Asymptotic Normality

Main Packages

```r
# devtools::install_github("ygeunkim/ceshat")
library(ceshat)
# devtools::install_github("ygeunkim/youngtool")
library(youngtool)
# GARCH
library(fGarch)
```

For details, see my Github package repositories[1]

[1] github.com/ygeunkim/ceshat and github.com/ygeunkim/youngtool

Models

AR(1)-GARCH(1, 0)

$$\begin{cases} X_t = Y_{t-1} \\ Y_t = 0.01 + 0.62X_t + \sigma_t \epsilon_t \\ \sigma_t^2 = -0.15 + 0.65\sigma_{t-1}^2 \\ \epsilon_t \sim N(0,1) \end{cases}$$

Random number generation

Monte Carlo Samples:

For fixed x_t
Generate GARCH(1, 0): (σ_t, ϵ_t)
$X_t = Y_{t-1}$
AR(1): $Y_t = 0.01 + 0.62 Y_{t-1} + \sigma_t \epsilon_t$

```r
garch_sim <- function(n, cond, ar_mu = .01, ar = .62) {
  garch_spec <-
    garchSpec(
      cond.dist = "norm",
      model = list(
        omega = .15, alpha = 0, beta = .65
      )
    )
  tibble(garch = garchSim(garch_spec, n = n) %>% as.numeric
    mutate(
      x = cond,
      y = ar_mu + ar * x + garch
    ) %>%
    select(y) %>% # to use youngtool (experimental stage)
    pull()
}
```

Monte Carlo Simulation

```
cond_sim <- function(n, m, xcond) {
  mc_data(garch_sim, N = n, M = m, cond = xcond) %>%
    .[,
      xcond := xcond] %>%
    .[]
}
```

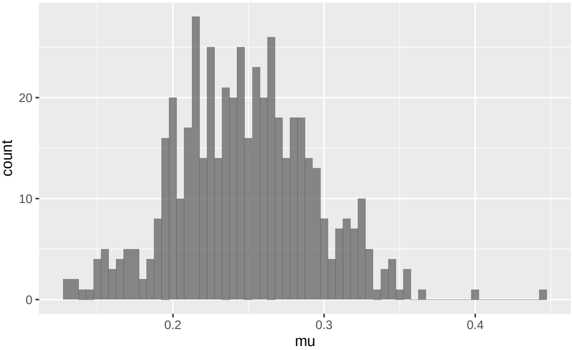
Empirical Distribution

```
x <- runif(1)
mc <- cond_sim(200, 500, x)
```

Interior

At $x = 0.266$,

```
CES <-
  mc[,
     .(mu =
         wdkll_ces(x ~ xcond, .SD) %>%
         predict(newx = unique(xcond))),
     by = mc]
```



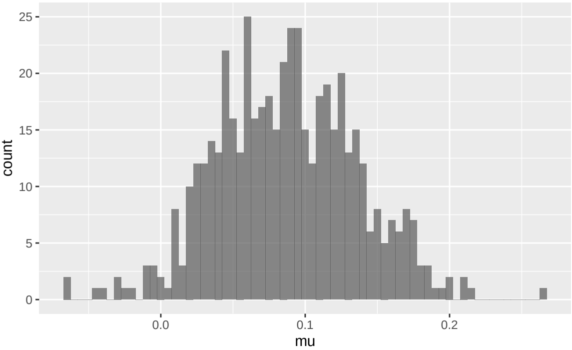
Empirical distribution of $\hat{\mu}_p(0.266)$
Shape of Normal distribution

Boundary

```
bound_c <- runif(1) * 200^(-4/5)
mc2 <- cond_sim(200, 500, bound_c)
```

At $x = 0.009$,

```
CES2 <-
  mc2[,
      .(mu =
          wdkll_ces(x ~ xcond, .SD) %>%
          predict(newx = unique(xcond))),
      by = mc]
```



Empricial distribution of $\hat{\mu}_p(x)$ at the left boundary point
Shape of Normal distribution

Future Study

Bandwidth Selection

Two bandwidths

Initial bandwidth h: insensitive to the final estimation

WNW bandwith λ

Strategy

Use linear estimators

- WNW estimator: select one $\tilde{h}$
  - $h \leq 0.1\tilde{h}$: take small initial bandwidth
- Given h
  - Use $\hat{F}_{cai}$

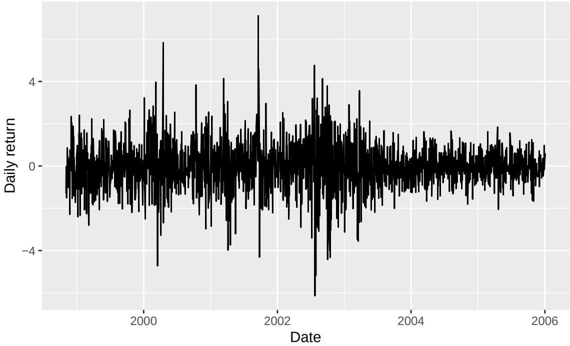
Criterion

Nonparametric AIC (Cai and Tiwari 2000)

GCV?

Real Data

Cai and Wang (2008) used Dow Jones index with daily return defined by $y_t := -100 \ln \frac{P_t}{P_{t-1}}$



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