MA2108S (AY24/25)
Mathematical Analysis I (S)
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Properties of Functions:
Let $f : A \to B$ be a function.
- f is *injective* if for any $x_1, x_2 \in A$, $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$.
- f is *surjective* if $f(A) = B$.
- f is *bijective* if it is both injective and surjective.

Inverse Function:
Given a bijective function $f : A \to B$, define the *inverse function* of f, denoted by $f^{-1} : B \to A$, as follows:
$$f^{-1}(b) = a \quad \text{if} \quad f(a) = b.$$

Composition Function:
Given $f : A \to B$ and $g : B \to C$, define the *composition function* $g \circ f : A \to C$ as follows:
$$(g \circ f)(a) = g(f(a)).$$

Well-ordering Principle of $\mathbb{N}$:
For any nonempty subset $S \subseteq \mathbb{N}$, S has a *minimal* element, that is, there exists $m \in S$ such that for any $s \in S$, $m \leq s$.

Principle of Induction:
Suppose that $S \subset \mathbb{N}$ satisfies the following:
1. $1 \in S$.
2. For any $k \in S$, $k+1 \in S$.

Then, $S = \mathbb{N}$.

Principle of Induction (in Practice):
Suppose we want to prove a statement $P(n)$ for any $n \in \mathbb{N}$. It suffices to prove the following:
1. $P(1)$ holds;
2. If $P(k)$ holds, then $P(k+1)$ holds.

Principle of Strong Induction:
Suppose we want to prove a statement $P(n)$ for any $n \in \mathbb{N}$. It suffices to prove the following:
1. $P(1)$ holds;
2. If for any $k' \leq k$, $P(k')$ holds, then $P(k+1)$ holds.

Finite and Infinite Sets:
$\emptyset$ has 0 elements and is *finite*.
A nonempty set S is finite if there exists a maximal $n \in \mathbb{N}$, called the *cardinality* of S, denoted by $|S|$, such that S has n elements. Otherwise, S is *infinite*.

Cardinality Inequality:
For $n \in \mathbb{N}$, suppose there exist n finite sets $S_1, S_2, \ldots, S_n$. Then the union $S = \bigcup_{i=1}^n S_i$ is also finite, and
$$|S| \leq \sum_{i=1}^n |S_i|.$$
Equality holds if and only if $S_i \cap S_j = \emptyset$ for any $1 \leq i, j \leq n$ with $i \neq j$.

Countability:
A set S is *countable* if there exists a bijective map $f : \mathbb{N} \to S$.
Any infinite subset of $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are countable. The set of real numbers $\mathbb{R}$ is uncountable.

Countable Union of Countable Sets:
For any $j \in \mathbb{N}$, suppose S_j is countable. Then
$$S = \bigcup_{j=1}^{\infty} S_j$$
is also countable.

Algebraic Properties of a Field:
1. $\forall a, b \in \mathbb{F}, a + b = b + a \in \mathbb{F}$;
2. $\forall a, b, c \in \mathbb{F}, (a + b) + c = a + (b + c) \in \mathbb{F}$;
3. $0 \in \mathbb{F}$ satisfies that for any $a \in \mathbb{F}, a + 0 = 0 + a = a$;
4. $\forall a \in \mathbb{F}, \exists -a \in \mathbb{F}$ such that $a + (-a) = (-a) + a = 0$;
5. $\forall a, b \in \mathbb{F}, a \times b = b \times a \in \mathbb{F}$;
6. $\forall a, b, c \in \mathbb{F}, (a \times b) \times c = a \times (b \times c) \in \mathbb{F}$;
7. $1 \in \mathbb{F}$ satisfies that for any $a \in \mathbb{F}, a \times 1 = 1 \times a = a$;
8. $\forall a \in \mathbb{F}\backslash\{0\}, \exists \frac{1}{a} \in \mathbb{F}\backslash\{0\}$ such that $a \times \frac{1}{a} = \frac{1}{a} \times a = 1$;
9. $\forall a, b, c \in \mathbb{F}, a \times (b + c) = a \times b + a \times c$.

any nonempty set $\mathbb{F}$ with operators $(+, \times)$ satisfying (1-9) is a *field*.
$\mathbb{Q}$ is a field, $\mathbb{N}$ and $\mathbb{Z}$ are not fields.

Order Property of $\mathbb{Q}$:
For $a, b \in \mathbb{Q}$,
- $a > b$ if $a - b$ is positive;
- $a < b$ if $a - b$ is negative;
- $a = b$ if $a - b = 0$.

The order in $\mathbb{Q}$ is a *total order*, that is, exactly one of the following holds:
$$(a > b); \quad (a < b); \quad (a = b).$$
- If $a >(\geq) b$, and $b >(\geq) c$, then $a >(\geq) c$.
- If $a \geq b$, and $c \geq d$, then $a + c \geq b + d$, with equality holding if and only if $a = b$ and $c = d$.
- If both a and b are positive (or negative), then $ab > 0$, $\frac{a}{b} > 0$.
- If one of a and b is positive and the other is negative, then $ab < 0$, $\frac{a}{b} < 0$.
- If $a >(\geq) b$, and $c > 0$, then $ac >(\geq) bc$.
- If $a >(\geq) b$, and $c < 0$, then $ac <(\leq) bc$.
- If $a \geq b > 0$, $c \geq d > 0$, then $ac \geq bd > 0$.
- If $a > b > 0$, then $\frac{1}{b} > \frac{1}{a} > 0$.

Supremum Property:
Given an ordered set S, a subset $E \subseteq S$ is *bounded from above* if there exists a unique $\beta \in S$ such that
$$e \leq \beta \quad \text{for all } e \in E.$$
The element β is an *upper bound* of E. If $\beta \in S$ satisfies:
1. β is an upper bound of E;
2. For any $\beta' < \beta$, β' is not an upper bound of E;

then β is the *supremum* of E, denoted by
$$\beta = \sup E.$$
An ordered set S has the *supremum property* if for any subset $E \subseteq S$, if E is bounded from above, then $\sup E$ exists.
$\mathbb{Q}$ does not have the supremum property.
The *infimum* of E, denoted by $\inf E$, is defined analogously.

Archimedean Property:
For any $x, y \in \mathbb{F}$ with $x, y > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$.
$\mathbb{Q}$ has the Archimedean property.

Dedekind Cuts:
A pair (A, A') is a *Dedekind cut* if it satisfies:
- $A, A' \neq \emptyset$;
- $A \cap A' = \emptyset$ and $\mathbb{Q} = A \cup A'$;
- For any $a \in A$ and $a' \in A'$, $a < a'$;
- For any $a \in A$, there exists $b \in A$ such that $b > a$.

Real Number System, $\mathbb{R}$:
$$\mathbb{R} := \{(A, A') \mid (A, A') \text{ is a Dedekind cut}\}.$$

Order Property of $\mathbb{R}$:
For $A, B \in \mathbb{R}$,
- $A < B$ if $A \subset B$;
- $A > B$ if $B \subset A$.

The order in $\mathbb{R}$ is a total order.
$\mathbb{R}$ has the supremum property.

Algebraic Operators in $\mathbb{R}$:
Let $m_1 = (M_1, M_1')$ and $m_2 = (M_2, M_2')$ be elements of $\mathbb{R}$. The sum of m_1 and m_2 is defined as
$$m_1 + m_2 = (M_3, M_3'),$$
where
$$M_3 := \{a_1 + a_2 \mid a_1 \in M_1, a_2 \in M_2\}.$$
For $m = (M, M')$, the additive inverse of m is defined as follows:
- If $m \in \mathbb{Q}$, then $-m$ is defined as in $\mathbb{Q}$;
- If $m \notin \mathbb{Q}$, then
$$-m = (-M', -M),$$
where
$$-M' := \{-m' \mid m' \in M'\}.$$
The product of m_1 and m_2 is defined as
$$m_1 \cdot m_2 = m_3 = (M_3, M_3'),$$
where the set M_3 is given by:
- If either $m_1 = 0$ or $m_2 = 0$, then $m_3 = 0$; - If both $m_1 > 0$ and $m_2 > 0$, then
$$M_3 = \{xy \mid x, y \geq 0, x \in M_1, y \in M_2\} \cup \{x \mid x < 0\}.$$
- If $m_1 > 0$ and $m_2 < 0$, then
$$m_3 = -(m_1 \cdot (-m_2));$$
- If $m_1 < 0$ and $m_2 > 0$, then
$$m_3 = -((-m_1) \cdot m_2);$$
- If $m_1 < 0$ and $m_2 < 0$, then
$$m_3 = (-m_1) \cdot (-m_2).$$
For $m = (M, M') \neq 0$, the reciprocal of m is defined as
$$\frac{1}{m} = (B, B'),$$
where the set B is given by:
- If $m \in \mathbb{Q}$, then $\frac{1}{m}$ is defined as in $\mathbb{Q}$;
- If $m \notin \mathbb{Q}$ and $m > 0$, then
$$B = \left\{ \frac{1}{x} \mid x > 0, x \in M' \right\} \cup \{x \mid x \leq 0\}.$$
- If $m < 0$, then
$$\frac{1}{m} = -\frac{1}{-m}.$$
$(\mathbb{R}, +, \cdot)$ forms a field.

Archimedean Property of $\mathbb{R}$:
$\mathbb{R}$ has the Archimedean property.
For any $c \in \mathbb{R}$, there exists a unique integer, denoted by $\lfloor c \rfloor \in \mathbb{Z}$, such that
$$\lfloor c \rfloor \leq c < \lfloor c \rfloor + 1.$$
For any $x > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.

Density of $\mathbb{Q}$ in $\mathbb{R}$:
For any $a < b$, there exists $r \in \mathbb{Q}$ such that
$$a < r < b.$$

Nested Interval Property:
Let $\{I_n = [a_n, b_n] : n \in \mathbb{N}\}$ be a sequence of closed intervals such that for any $n \in \mathbb{N}$, $I_{n+1} \subseteq I_n$, then there exists $x \in \mathbb{R}$ such that
$$x \in \bigcap_{n=1}^{\infty} I_n.$$

Bernoulli's Inequality:
For $x > -1$ and $n \in \mathbb{N}$,
$$(1+x)^n \geq 1 + nx.$$

QM-AM-GM-HM Inequality:
For $x_i > 0$ $(i = 1, \ldots, n)$, the quadratic mean (QM), arithmetic mean (AM), geometric mean (GM), and harmonic mean (HM) satisfy
$$\underbrace{\sqrt{\frac{1}{n}\sum_{i=1}^n x_i^2}}_{\text{QM}} \geq \underbrace{\frac{1}{n}\sum_{i=1}^n x_i}_{\text{AM}} \geq \underbrace{\Big(\prod_{i=1}^n x_i\Big)^{1/n}}_{\text{GM}} \geq \underbrace{\frac{n}{\sum_{i=1}^n \frac{1}{x_i}}}_{\text{HM}}.$$
Equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Triangle Inequality:
For $a, b \in \mathbb{R}$,
$$||a| - |b|| \leq |a + b| \leq |a| + |b|.$$
where
$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

Cauchy-Schwarz Inequality:
For $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in \mathbb{R}$,
$$\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right).$$
Equality holds if and only if there exists $k \in \mathbb{R}$ such that $a_i = kb_i$ for all $i = 1, \ldots, n$.

Sequences:
A *sequence* is a function $a : \mathbb{N} \to \mathbb{R}$.

Limits of Sequences:
A sequence $\{a_n : n \in \mathbb{N}\}$ *converges* to $A \in \mathbb{R}$, or that A is the *limit* of $\{a_n : n \in \mathbb{N}\}$ as $n \to \infty$, if
$$\forall \epsilon > 0, \ \exists N \in \mathbb{N} \ \text{ s.t. } \forall n \geq N, \ |a_n - A| < \epsilon.$$
If a sequence has a limit, the sequence is *convergent*. Otherwise, it is *divergent*.

Uniqueness of Limits:
If a sequence is convergent, then its limit is unique.

Boundedness of Limits:
A convergent sequence is *bounded*, that is, there exists $C > 0$ such that $|a_n| \leq C$ for all $n \in \mathbb{N}$.

Order Preserving Theorem:
Suppose $\{a_n\}$, $\{b_n\}$ are convergent sequences, and $a_n \geq b_n$ for all n. Then,
$$\lim_{n\to\infty} a_n \geq \lim_{n\to\infty} b_n.$$

Limit Laws:
Suppose $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then:
1. $\lim_{n\to\infty}(a_n \pm b_n) = a \pm b$;
2. $\lim_{n\to\infty}(a_n b_n) = ab$;
3. If $b_n \neq 0$ and $b \neq 0$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$.

Squeeze Theorem:
Let sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ satisfy
$$a_n \leq b_n \leq c_n.$$
Suppose
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = A.$$
Then,
$$\lim_{n\to\infty} b_n = A.$$

Subsequences:
Given a sequence $\{a_n : n \in \mathbb{N}\}$, $\{b_k : k \in \mathbb{N}\}$ is a *subsequence* of $\{a_n\}$ if for any $k \in \mathbb{N}$,
$$b_k = a_{n_k},$$
where $\{n_k : k \in \mathbb{N}\}$ satisfies that for any $k \in \mathbb{N}$, $n_k \in \mathbb{N}$ and $n_k < n_{k+1}$.

Subsequence Limit Theorem:
Let $\{b_k = a_{n_k}\}$ be a subsequence of $\{a_n\}$. Suppose $\lim_{n\to\infty} a_n = A$. Then $\lim_{k\to\infty} b_k = A$.

If $\{a_n\}$ has two subsequences with different limits, then $\{a_n\}$ is divergent. If $\{a_n\}$ is convergent, then $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$.

Monotonic Sequences:
A sequence $\{a_n : n \in \mathbb{N}\}$ is increasing (or decreasing) if for any $n \in \mathbb{N}$,
$$a_n \leq a_{n+1}, \quad (\text{or } a_n \geq a_{n+1}).$$

Monotone Convergence Theorem:
Suppose sequence $\{a_n\}$ is increasing (decreasing) and bounded from above (below). Then $\{a_n\}$ is convergent.

Monotone Subsequence Theorem:
Any sequence $\{a_n\}$ admits a monotone subsequence $\{a_{n_k}\}$.

Bolzano-Weierstrass Theorem:
Any bounded sequence $\{a_n\}$ admits a convergent subsequence $\{a_{n_k}\}$.

Cauchy Sequences:
$\{a_n\}$ is a *Cauchy sequence* if
$$\forall \epsilon > 0, \ \exists N \in \mathbb{N} \ \text{ s.t. } \forall m, n \geq N,$$
$$|a_m - a_n| < \epsilon.$$

Cauchy Convergence Criterion:
A sequence is convergent if and only if it is a Cauchy sequence.

Contractive Sequences:
A sequence $\{a_n\}$ is *contractive* if there exists a constant C, $0 < C < 1$, such that
$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n| \quad \forall n \in \mathbb{N}.$$
any contractive sequence is a Cauchy sequence.

Stolz–Cesàro Theorem:
$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = A \ \Rightarrow \ \lim_{n\to\infty} \frac{a_n}{b_n} = A.$$

Cesàro Means:
$$\lim_{n\to\infty} a_n = A \ \Rightarrow \ \lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = A.$$

Limits to Infinity:
A sequence $\{a_n\}$ tends to $+\infty$ if
$$\forall C \in \mathbb{R}, \ \exists N \in \mathbb{N} \ \text{ s.t. } \forall n \geq N, \ a_n > C.$$
A sequence $\{a_n\}$ tends to $-\infty$ if
$$\forall C \in \mathbb{R}, \ \exists N \in \mathbb{N} \ \text{ s.t. } \forall n \geq N, \ a_n < C.$$
A sequence $\{a_n\}$ tends to ∞ if $\{|a_n|\}$ tends to $+\infty$.

Squeeze Theorem:
Let sequences $\{a_n\}$, $\{b_n\}$, satisfy
$$a_n \leq b_n.$$
- Suppose $\lim_{n\to\infty} a_n = +\infty$, then $\lim_{n\to\infty} b_n = +\infty$;
- Suppose $\lim_{n\to\infty} b_n = -\infty$, then $\lim_{n\to\infty} a_n = -\infty$.

Limits of Reciprocals:
Suppose $a_n \neq 0$ and $\lim_{n\to\infty} a_n = \infty$. Then $\lim_{n\to\infty} \frac{1}{a_n} = 0$.
Suppose $a_n \neq 0$ and $\lim_{n\to\infty} a_n = 0$. Then $\lim_{n\to\infty} \frac{1}{a_n} = \infty$.

Limit Superior and Limit Inferior:
Suppose $\{a_n\}$ is bounded.
The *limit superior* of $\{a_n\}$ is defined as
$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} \sup\{a_k : k \geq n\}.$$
The *limit inferior* of $\{a_n\}$ is defined as
$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} \inf\{a_k : k \geq n\}.$$

Limit Point:
A number A is a limit point of a sequence $\{a_n\}$ if there exists a subsequence $\{b_k = a_{n_k}\}$ such that
$$\lim_{k\to\infty} b_k = A.$$
Let $\{a_n\}$ be a bounded sequence and let E denote the set of limit points of $\{a_n\}$. Then both the limit superior and the limit inferior of $\{a_n\}$ are contained in E, and
$$\limsup_{n\to\infty} a_n = \sup E, \quad \liminf_{n\to\infty} a_n = \inf E.$$
Also,
$$\inf_n a_n \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq \sup_n a_n.$$
Equality holds if and only if the sequence is convergent.

n^{th}-Term Test:
Suppose $\sum_{i=1}^{\infty} a_i$ is convergent. Then
$$\lim_{n\to\infty} a_n = 0.$$

Cauchy Criterion for Series:
$\sum_{i=1}^{\infty} a_i$ is convergent if
$$\forall \epsilon > 0, \ \exists N \in \mathbb{N} \ \text{ s.t. } \forall m > n \geq N,$$
$$\left| \sum_{k=n+1}^{m} a_k \right| < \epsilon.$$

Monotone Convergence Theorem:
Suppose $a_i \geq 0$ for any $i \in \mathbb{N}$, and
$$\exists C > 0, \ \sum_{i=1}^{n} a_i \leq C \ \forall n \in N.$$
Then $\sum_{i=1}^{\infty} a_i$ is convergent.
Suppose $a_i \geq 0$, and $\sum_{i=1}^{\infty} a_i$ is divergent. Then
$$\sum_{i=1}^{\infty} a_i = +\infty.$$

Absolute Convergence:
$\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if $\sum_{n=1}^{\infty} |a_n|$ is convergent.
If $\sum_{n=1}^{\infty} a_n$ is convergent but not absolutely convergent, it is *conditionally convergent.*

Comparison Test:
Suppose $\{a_n\}$ and $\{b_n\}$ satisfy that
$$\exists K \in \mathbb{N}, \lambda > 0, \ \text{ s.t. } \forall n \geq K,$$
$$0 \leq a_n \leq \lambda b_n.$$
Then,
- if $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
- if $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is divergent.

Limit Comparison Test:
Suppose $\{a_n\}$ and $\{b_n\}$ are both positive sequences and
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lambda.$$
- If $\lambda > 0$, then $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\sum_{n=1}^{\infty} b_n$ is convergent.
- If $\lambda = 0$, then $\sum_{n=1}^{\infty} b_n$ convergent implies $\sum_{n=1}^{\infty} a_n$ is convergent. $\sum_{n=1}^{\infty} a_n$ divergent implies $\sum_{n=1}^{\infty} b_n$ is divergent.

p-series Test:
For $p > 1$,
$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is convergent.}$$

Cauchy Condensation Test:
Let $\{a_n\}$ be a positive decreasing sequence. Define $b_k := 2^k a_{2^k}$.
Then $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\sum_{k=1}^{\infty} b_k$ is convergent.

Leibniz Test:
Let $\{a_n\}$ be a positive decreasing sequence and $\lim_{n\to\infty} a_n = 0$. Then
$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \text{ is convergent.}$$

Root Test:
Let $\{a_n\}$ be a sequence.
- If $\exists \ 0 \leq r < 1, \ K \in \mathbb{N}$ s.t.
$$|a_n|^{1/n} \leq r \quad \forall n \geq K,$$
then
$$\sum_{n=1}^{\infty} a_n \text{ is absolutely convergent.}$$
- If $\exists K \in \mathbb{N}$ s.t.
$$|a_n|^{1/n} \geq 1 \quad \forall n \geq K,$$
then
$$\sum_{n=1}^{\infty} a_n \text{ is divergent.}$$

Ratio Test:
Let $\{a_n\}$ be a sequence with non-zero terms.
- If $\exists \ 0 \leq r < 1, \ K \in \mathbb{N}$ s.t.
$$\left| \frac{a_{n+1}}{a_n} \right| \leq r \quad \forall n \geq K,$$
then
$$\sum_{n=1}^{\infty} a_n \text{ is absolutely convergent.}$$
- If $\exists K \in \mathbb{N}$ s.t.
$$\left| \frac{a_{n+1}}{a_n} \right| \geq 1 \quad \forall n \geq K,$$
then
$$\sum_{n=1}^{\infty} a_n \text{ is divergent.}$$

Euler's Number, e:
$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}$$
$$\sum_{n=0}^{\infty} \frac{1}{n!} = e, \quad e \notin \mathbb{Q}$$

Abel's Summation Formula:
Let $A_n = \sum_{k=1}^{n} a_k$, $\quad r_k = \sum_{n=k}^{\infty} a_n$. For any $q > p$, $\sum_{n=p}^{q} a_n b_n$ can be expressed as
1. $A_q b_q - A_{p-1} b_p + \sum_{n=p}^{q-1} A_n (b_n - b_{n+1})$
2. $r_p b_{p-1} - r_{q+1} b_{q+1} + \sum_{k=p}^{q+1} r_k (b_k - b_{k-1})$

Dirichlet's Test:
Suppose $\sum_{i=1}^{\infty} a_i$ is convergent. Let $\{b_n\}$ be a positive decreasing sequence and $\lim_{n\to\infty} b_n = 0$, then
$$\sum_{n=1}^{\infty} a_n b_n \text{ is convergent.}$$

Abel's Test:
Suppose $\sum_{i=1}^{\infty} a_i$ is convergent. Let $\{b_n\}$ be a monotone and bounded sequence, then
$$\sum_{n=1}^{\infty} a_n b_n \text{ is convergent.}$$

Rearrangements:
Let $\{a_n\}$ be a sequence. A sequence $\{b_n\}$ is a *rearrangement* of $\{a_n\}$ if
$$b_n = a_{f(n)},$$
where $f : \mathbb{N} \to \mathbb{N}$ is a bijective map.

Absolute Convergence:
Suppose $\sum_{i=1}^{\infty} a_i$ is absolutely convergent. Let $\{b_n\}$ be a rearrangement of $\{a_n\}$. Then
$$\sum_{n=1}^{\infty} b_n \text{ is absolutely convergent, and}$$
$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

Riemann Rearrangement Theorem:
Suppose $\sum_{i=1}^{\infty} a_i$ is conditionally convergent. Then for any $A \in \mathbb{R}$, there exists a rearrangement $\{b_n\}$ of $\{a_n\}$ such that
$$\sum_{n=1}^{\infty} b_n = A.$$

Limit Points:
For $x \in \mathbb{R}$ and $\epsilon > 0$, denote
$$V_\epsilon(x) := (x - \epsilon, x + \epsilon).$$
Let $A \subset \mathbb{R}$ be a nonempty subset of $\mathbb{R}$. A point $x \in \mathbb{R}$ is a *limit point* of A if
$$\forall \epsilon > 0, \quad A \cap (V_\epsilon(x) \setminus \{x\}) \neq \emptyset.$$
$x \in \mathbb{R}$ is a limit point of A if and only if there exists a sequence $\{a_n\}$ such that $a_n \in A$, $a_n \neq x$, and
$$\lim_{n\to\infty} a_n = x.$$

Limit of Functions:
Let $f : A \to \mathbb{R}$ be a function and let c be a limit point of A. Suppose $L \in \mathbb{R}$ satisfies that
$$\forall \epsilon > 0, \ \exists \delta > 0 \ \text{ s.t.}$$
$$|f(x) - L| < \epsilon, \quad \forall x \in A \cap (V_\delta(c) \setminus \{c\}).$$
Then L is the limit of f at c, and denote
$$\lim_{x\to c} f(x) = L.$$
If f does not have a limit at a limit point c of A, then f is divergent at c.
The limits of functions are defined analogously to the limits of sequences. The following theorems apply:
1. Uniqueness of Limits;
2. Boundedness in a Neighbourhood;
3. Order Preserving Theorem;
4. Limit Laws;
5. Squeeze Theorem.

Sequential Criterion of Limits:
Let $f : A \to \mathbb{R}$ be a function and let c be a limit point of A. The following statements are equivalent:
1. $\lim_{x\to c} f(x) = L$;
2. For any sequence $\{a_n\} \subset A \setminus \{c\}$ such that $\lim_{n\to\infty} a_n = c$, then $\lim_{n\to\infty} f(a_n) = L$.

Sign Preserving Theorem:
Let $f : A \to \mathbb{R}$ be a function and let c be a limit point of A.
- Suppose $\lim_{x\to c} f(x) = L > 0$. Then
$$\exists \delta > 0, \ C > 0 \ \text{ s.t.}$$
$$f(x) \geq C, \quad \forall x \in A \cap (V_\delta(c) \setminus \{c\}).$$
- Suppose $\lim_{x\to c} f(x) = L < 0$. Then
$$\exists \delta > 0, \ C < 0 \ \text{ s.t.}$$
$$f(x) \leq C, \quad \forall x \in A \cap (V_\delta(c) \setminus \{c\}).$$

One-sided Limtis:
Let $f : A \to \mathbb{R}$ be a function, and let c be a limit point of $A \cap (-\infty, c)$. If $L \in \mathbb{R}$ satisfies that
$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t.}$$
$$|f(x) - L| < \epsilon, \quad \forall x \in A \cap (c - \delta, c),$$
then L is the *left limit* of f at c, and denote
$$\lim_{x \to c^-} f(x) = L.$$
Let $f : A \to \mathbb{R}$ be a function, and let c be a limit point of $A \cap (c, +\infty)$. If $L \in \mathbb{R}$ satisfies that
$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t.}$$
$$|f(x) - L| < \epsilon, \quad \forall x \in A \cap (c, c + \delta),$$
then L is the *right limit* of f at c, and denote
$$\lim_{x \to c^+} f(x) = L.$$
The one-sided limits of functions are defined analogously to the limits of sequences. The following theorems apply:
1. Uniqueness of Limits;
2. Boundedness in a Neighbourhood;
3. Order Preserving Theorem;
4. Limit Laws;
5. Squeeze Theorem;
6. Sequential Criterion;
7. Sign-Preserving Theorem.

Two-sided Limits Theorem:
The following statements are equivalent:
1. $\lim_{x \to c} f(x) = L$;
2. $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$.

Infinite Limits:
Let $f : A \to \mathbb{R}$ and let c be a limit point of A.
- If $\forall L > 0,\ \exists \delta > 0$ s.t.
$$f(x) > L, \quad \forall x \in A \cap (V_\delta(c) \setminus \{c\}),$$
then
$$\lim_{x \to c} f(x) = +\infty$$
- If $\forall L > 0,\ \exists \delta > 0$ s.t.
$$f(x) < -L, \quad \forall x \in A \cap (V_\delta(c) \setminus \{c\}),$$
then
$$\lim_{x \to c} f(x) = -\infty$$
- $\forall L > 0,\ \exists \delta > 0$ s.t.
$$|f(x)| > L, \quad \forall x \in A \cap (V_\delta(c) \setminus \{c\}).$$
then
$$\lim_{x \to c} f(x) = \infty$$

The infinite limits (and one-sided infinite limits) are defined analogously to the limits of sequences. The following theorems apply:
1. Uniqueness of Limits;
2. Boundedness of Limits;
3. Order Preserving Theorem;
4. Limit Laws;
5. Squeeze Theorem;
6. Sequential Criterion;
7. Sign-Preserving Theorem.

Pointwise Continuity:
Let $f : A \to \mathbb{R}$ be a function and let $c \in A$. If
$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t.}$$
$$|f(x) - f(c)| < \epsilon, \quad \forall x \in A \cap V_\delta(c),$$
then f is *continuous* at c.
Otherwise, f is *discontinuous* at c.
Let $B \subseteq A$. If f is continuous at any point of B, f is continuous on B.
- If c is not a limit point of A, then f is always continuous at c.
- If c is a limit point of A, then f is continuous at c if and only if
$$\lim_{x \to c} f(x) = f(c).$$

Sequential Criterion of Continuity:
Let $f : A \to \mathbb{R}$ be a function and let $c \in A$. The following statements are equivalent:
- f is continuous at c;
- For any sequence $\{a_n\} \subset A \setminus \{c\}$ such that $\lim_{n\to\infty} a_n = c$, then $\lim_{n \to \infty} f(a_n) = f(c)$.

Compositions of Continuity:
Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be functions such that $f(A) \subseteq B$.
Suppose f is continuous at $c \in A$ and g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

Limit of Composite Functions:
Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be functions such that $f(A) \subseteq B$. Let c be a limit point of A and $\lim_{x\to c} f(x) = L$.
Suppose $L \in B$ and g is continuous at L, then
$$\lim_{x \to c} g(f(x)) = g(L).$$
Suppose further that $f(x) \neq L$ for $x \in A \cap (V_\delta(c) \setminus \{c\})$. If L is a limit point of B and $\lim_{y\to L} g(y) = M$, then
$$\lim_{x \to c} g(f(x)) = M.$$

Composition at Infinity:
Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be functions such that $f(A) \subseteq B$.
- Suppose there exists c such that $(c, +\infty) \subseteq A$. If $\lim_{x\to+\infty} f(x) = L$ and g is continuous at L, then
$$\lim_{x \to +\infty} g(f(x)) = g(L).$$
- Suppose there exists c such that $(-\infty, c) \subseteq A$. If $\lim_{x\to-\infty} f(x) = L$ and g is continuous at L, then
$$\lim_{x \to -\infty} g(f(x)) = g(L).$$
- Suppose there exists c such that $(c, +\infty) \subseteq B$. If $\lim_{x\to x_0} f(x) = +\infty$ and $\lim_{y\to+\infty} g(y) = L$, then
$$\lim_{x \to x_0} g(f(x)) = L.$$
- Suppose there exists c such that $(-\infty, c) \subseteq B$. If $\lim_{x\to x_0} f(x) = -\infty$ and $\lim_{y\to-\infty} g(y) = L$, then
$$\lim_{x \to x_0} g(f(x)) = L.$$

Boundedness Theorem:
A function $f : A \to \mathbb{R}$ is *bounded* if there exists a constant $C > 0$ such that
$$|f(x)| \le C, \quad \forall x \in A.$$
Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function on the closed interval $[a, b]$. Then f is bounded.

Extreme Value Theorem:
A function $f : A \to \mathbb{R}$ has an *absolute maximum* (*absolute minimum*) on A if there exists $x^* \in A$ such that
$$f(x) \le (\ge)\ f(x^*), \quad \forall x \in A.$$
Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function on the closed interval $[a, b]$, then there exist points $c, d \in [a, b]$ such that
$$f(c) \ge f(x) \ge f(d), \quad \forall x \in [a, b].$$

Intermediate Value Theorem:
Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function on $[a, b]$. Let $m = \min\{f(a), f(b)\}$ and $M = \max\{f(a), f(b)\}$, then
$$\forall L \in [m, M],\ \exists c \in [a, b] \text{ s.t. } f(c) = L.$$
Also, the range $f([a, b])$ is a closed interval.
Suppose further that $m \le 0$ and $M \ge 0$, then there exists $c \in [a, b]$ such that
$$f(c) = 0.$$

Preservation of Intervals Theorem:
Let $S \subseteq \mathbb{R}$ contain at least two elements and satisfy the condition that for any $x < y \in S$, the interval $[x, y] \subseteq S$, then S is an interval, where the endpoints can be $\pm\infty$.
Let I be an interval. Suppose $f : I \to \mathbb{R}$ is a continuous function on I, then the range $f(I)$ is an interval.

Uniform Continuity Theorem:
A function $f : A \to \mathbb{R}$ is *uniformly continuous* on A if
$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } \forall x, x' \in I,$$
$$|x - x'| < \delta \implies |f(x) - f(x')| < \epsilon.$$
Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$, then f is uniformly continuous on $[a, b]$.

Lipschitz Continuity Theorem:
Let $f : A \to \mathbb{R}$ be a function. f is a *Lipschitz function* if there exists a constant $K > 0$ such that
$$|f(x) - f(y)| \le K|x - y|, \quad \forall x, y \in A.$$
K is the *Lipschitz constant* for f.
If $f : A \to \mathbb{R}$ is a Lipschitz function on A, then f is uniformly continuous on A.

Hölder Continuity:
Let $D \subseteq \mathbb{R}^n$ and let $0 < \alpha \le 1$. A function $f : D \to \mathbb{R}$ is *α-Hölder continuous* on D if there exists a constant $C > 0$ such that
$$\forall x, y \in D, \qquad |f(x) - f(y)| \le C\,\|x - y\|^\alpha.$$
α is the *Hölder exponent* and C is the *Hölder constant*.

Uniformly Continuous Functions:
Let $f : A \to \mathbb{R}$ be a uniformly continuous function, then for any Cauchy sequence $\{a_n\}$ from A, the sequence $\{f(a_n)\}$ is also Cauchy.

Continuous Extension Theorem:
Let $f : (a, b) \to \mathbb{R}$ be a function. Then, f is uniformly continuous on (a, b) if and only if f can be extended to a continuous function on $[a, b]$; that is, both $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist.

Properties of Monotone Functions:
Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a monotone function on I. Let $c \in I$ such that c is not an endpoint of I, then $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ both exist. The following statements are equivalent:
1. f is continuous at c;
2. $\lim_{x\to c^-} f(x) = f(c) = \lim_{x\to c^+} f(x)$;
3. $\sup\{f(x) : x < c\} = f(c) = \inf\{f(x) : x > c\}$.

The *jump* of f at c, denoted by $j_f(c)$, is defined as:
$$j_f(c) := \lim_{x \to c^+} f(x) - \lim_{x \to c^-} f(x).$$
f is continuous at c if and only if $j_f(c) = 0$.

Darboux-Froda Theorem:
Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a monotone function on I, then the set of discontinuities of f is countable.

Continuous Inverse Theorem:
Let $f : I \to \mathbb{R}$ be a strictly increasing (decreasing) and continuous function defined on I, then f^{-1} is also a strictly increasing (decreasing) and continuous function on $f(I)$.
Let $S \subseteq \mathbb{R}$ contain at least two elements and satisfy the condition that for any $x < y \in S$, the interval $[x, y] \subseteq S$. Then S is an interval, where the endpoints can be $\pm\infty$.

Topological Space:
A *topological space* is a nonempty set X with a collection of subsets $\mathcal{O} = \{O \subseteq X\}$, called *open sets*, with the following properties:
- $\emptyset, X \in \mathcal{O}$;
- For any finite subset $\mathcal{O}' \subseteq \mathcal{O}$,
$$\bigcap_{O \in \mathcal{O}'} O \in \mathcal{O};$$
- For any subset $\mathcal{O}' \subseteq \mathcal{O}$,
$$\bigcup_{O \in \mathcal{O}'} O \in \mathcal{O}.$$

In a topological space $(X, \mathcal{O})$, a subset $E \subseteq X$ is *closed* if $X \setminus E$ is open. Closed sets have the following properties:
- $\emptyset$ and X are closed;
- For any finite collection $\mathcal{E}$ of closed sets,
$$\bigcup_{E \in \mathcal{E}} E \text{ is closed;}$$
- For any collection $\mathcal{E}$ of closed sets,
$$\bigcap_{E \in \mathcal{E}} E \text{ is closed.}$$

Metric Space:
A *metric space* is a nonempty set X with a distance function
$$d : X \times X \to \mathbb{R}_{\ge 0},$$
with the following conditions:
- $d(x_1, x_2) = d(x_2, x_1)$
- $d(x, y) = 0$ if and only if $x = y$
- $d(x, z) \le d(x, y) + d(y, z)$

A metric d on X induces a topology on X where $O \subset X$ is open if and only if
$$\forall x \in O,\ \exists \epsilon > 0 \text{ s.t. } V_\epsilon(x) \subseteq O$$
where $V_\epsilon(x) := \{x' \in X : d(x', x) < \epsilon\}$.
Given a metric space (X, d) and open sets,
- $\emptyset$ and X are open.
- For any finite collection of open sets O', $\bigcap_{O \in O'} O$ is open.
- For any collection of open sets O', $\bigcup_{O \in O'} O$ is open.

Induced Topology:
Given a topological space X and a subset $Y \subset X$, a subset $U \subset Y$ is *open relative to* Y if there exists an open set $U_0 \subset X$ such that
$$U = U_0 \cap Y.$$
Let (X, d) be a metric space and $Y \subset X$. A subset $U \subset Y$ is open relative to Y if and only if for any $x \in U$ there exists $\delta > 0$ such that
$$V_\delta(x) \cap Y \subseteq U.$$

Limit Points in Metric Space:
Given a metric space (X, d), a sequence $\{x_n\} \subset X$ approaches $x \in X$ if
$$\lim_{n \to \infty} d(x_n, x) = 0.$$
Let $E \subset X$. A point $x \in X$ is a *limit point* of E if there exists a sequence $\{x_n\} \subset E$ such that $x_n \neq x$ for all n and
$$\lim_{n \to \infty} x_n = x.$$
The set of limit points of E is denoted by E'. A point x is a limit point of E if and only if
$$\forall \delta > 0,\ \exists x' \in E \text{ s.t. } x' \in V_\delta(x) \setminus \{x\}.$$

Characterization of Closed Sets:
Let (X, d) be a metric space and $E \subset X$. Then E is closed if and only if $E' \subseteq E$.
Define the *closure* of E as
$$\overline{E} := E \cup E',$$
then $\overline{E}$ is closed.

Characterization of Open Sets:
A subset $U \subset \mathbb{R}$ is open if and only if it is a disjoint union of countably many open intervals.

Compactness:
Let X be a topological space and $K \subseteq X$. If $K \subseteq \bigcup_{\alpha\in\mathcal{A}} U_\alpha$ and U_α is open for any $\alpha \in \mathcal{A}$, then $C = \{U_\alpha\}_{\alpha\in\mathcal{A}}$ is an *open cover* of K.
K is *compact* if for any open cover of K, there exists a *finite subcover* $\{U_{\alpha_i}\}_{i=1}^n$ such that $K \subseteq \bigcup_{i=1}^n U_{\alpha_i}$.

Theorems of Compact Set:
Let X be a topological space.
- Let $K \subset Y \subset X$, then K is compact in Y (w.r.t. the induced topology) if and only if K is compact in X.
- Let $K \subset X$ be compact and $Y \subset X$ be a closed subset, then $Y \cap K$ is compact.

Let (X, d) be a metric space, then any compact subset of X is closed.

Heine-Borel Theorem:
$K \subset \mathbb{R}^n$ is compact if and only if K is closed and bounded.

n-dimensional NIP:
Let $K \subset \mathbb{R}^n$ be an *n-cube* if
$$K = [a_1, b_1] \times \cdots \times [a_n, b_n].$$
Define the *diameter* of K as
$$\operatorname{diam}(K) = \max\{|b_\ell - a_\ell| : \ell = 1, \ldots, n\}.$$
Let $\{K_i\}$ be a sequence of n-cubes such that $K_{i+1} \subset K_i$ for all $i \in \mathbb{N}$, then
$$\bigcap_{i=1}^{\infty} K_i \neq \emptyset.$$
If $\operatorname{diam}(K_i) \to 0$, then the intersection is a singleton.

n-dimensional Weierstrass Theorem:
any bounded sequence $\{x_n\}$ in $\mathbb{R}^n$ admits a convergent subsequence.

Finite Intersection Theorem:
Let $\{K_\alpha : \alpha \in \mathcal{A}\}$ be a collection of compact subsets of a metric space X such that the intersection of any finite subcollection is nonempty, then
$$\bigcap_{\alpha \in \mathcal{A}} K_\alpha \neq \emptyset.$$
Let $\{K_n : n \in \mathbb{N}\}$ be a sequence of nonempty compact subsets of a metric space X satisfying $K_{n+1} \subseteq K_n$ for any $n \in \mathbb{N}$, then
$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

Sequential Compactness:
Let (X, d) be a metric space. A subset $K \subseteq X$ is *sequentially compact* if any infinite subset $E \subseteq K$ has a limit point in K.
$K \subseteq X$ is sequentially compact if and only if K is compact.

Separable Space:
A topological space X is *separable* if it admits a countable, dense subset.

Bases:
Let X be a topological space. A collection of open subsets of X, $\{U_\alpha : \alpha \in A\}$, is a *base* if for any $x \in X$ and any open subset U containing x, there exists $\alpha \in A$ such that
$$x \in U_\alpha \subseteq U.$$

Second-Countable Space:
A topological space X is *second countable* if it admits a countable base.
If X is second countable with base $\{U_n : n \in \mathbb{N}\}$, then any open set U can be written as
$$U = \bigcup_{k=1}^{\infty} U_{n_k}.$$

δ-Separated Sets:
A subset $M \subseteq X$ is *δ-separated* if for any two distinct points $x_1, x_2 \in M$,
$$d(x_1, x_2) \ge \delta.$$

Sequentially Compact Metric Space:
Let (X, d) be a sequentially compact metric space. The following statements hold:
- For any $\delta > 0$ and any δ-separated subset $E \subseteq X$, E is finite;
- X is second countable;
- Any open cover of X admits a countable subcover.

Connectedness:
Let X be a topological space. A subset $S \subseteq X$ is *disconnected* if there exist open sets A, B in X such that:
- $S \subseteq A \cup B$;
- $S \cap A \neq \emptyset$ and $S \cap B \neq \emptyset$;
- $S \cap A \cap B = \emptyset$.

Otherwise, S is *connected*.
$S \subseteq \mathbb{R}$ is connected if and only if S is an interval.

Topological Continuity:
Let X and Y be topological spaces. A map $f : X \to Y$ is *continuous* if for any open set $U \subseteq Y$, the preimage
$$f^{-1}(U) := \{x \in X : f(x) \in U\}$$
is open in X.
Let $f : A \to \mathbb{R}$ be a function. The following statements are equivalent:
1. f is continuous at any $c \in A$;
2. For any open (closed) set $U \subseteq \mathbb{R}$, $f^{-1}(U)$ is open (closed) in A (w.r.t. the induced topology).

Let $f : X \to Y$ be a continuous map and $U \subseteq X$ be open (closed), then $f(U)$ is not necessarily open (closed).

Theorems of Continuous Map:
Let $f : X \to Y$ be a continuous map.
- For any compact subset $K \subseteq X$, $f(K)$ is compact. This theorem applied to $f : [a, b] \to \mathbb{R}$ implies Boundedness Theorem and Extreme Value Theorem.
- For any connected subset $C \subseteq X$, $f(C)$ is connected. This theorem applied to $f : [a, b] \to \mathbb{R}$ implies Intermediate Value Theorem.

Cauchy Property:
Let (X, d) be a metric space. A sequence in X, $\{x_n : n \in \mathbb{N}\}$, is *Cauchy* if
$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } \forall m, n \ge N,$$
$$d(x_m, x_n) < \epsilon.$$

Diameter:
Let (X, d) be a metric space and E be a nonempty subset of X. Define $S := \{\, d(p, q) : p \in E,\ q \in E \,\}$.
Define the *diameter* of E as
$$\operatorname{diam}(E) = \sup S.$$

Completeness:
Let (X, d) be a metric space. (X, d) is *complete* if for any Cauchy sequence $\{x_n : n \in \mathbb{N}\}$ in X, there exists $x \in X$ such that
$$\lim_{n \to \infty} d(x, x_n) = 0.$$

Completion of (X, d):
- $X^* := \{P = \{p_n\} : \{p_n\} \subseteq X\}$, where $\{p_n\}$ is Cauchy in (X, d);
- $D(P, Q) := \lim_{n\to\infty} d(p_n, q_n)$;
- $P = Q$ if and only if $D(P, Q) = 0$.

(X^*, D) is a complete metric space.
The completion of $(\mathbb{Q}, |\cdot|)$ is $(\mathbb{R}, |\cdot|)$.

Space of Continuous Functions:
$X = C[a, b] := \{f : [a, b] \to \mathbb{R}\}$, where f is continuous on $[a, b]$.

Pointwise Topology:
A sequence of functions $\{f_n(x) : n \in \mathbb{N}\}$ *converges pointwise* to $f(x)$ if for any $x \in [a, b]$,
$$\lim_{n \to \infty} f_n(x) = f(x).$$
$C[a, b]$ is not closed in the pointwise topology (not closed under pointwise convergence).

Product Topology:
Given an index set A and a collection of topological spaces $\{X_\alpha : \alpha \in \mathcal{A}\}$, the *product topology* on
$$Y = \prod_{\alpha \in A} X_\alpha$$
is defined by the following bases:
$$\left\{ \prod_{\alpha \in A} U_\alpha \right\}$$
where $U_\alpha \subseteq X_\alpha$ is open and $U_\alpha = X_\alpha$ for all but finitely many α's.
The product topology is the coarsest topology ensuring the continuity of every projection $\pi_\alpha : Y \to X_\alpha$, which maps a point in Y to its α-coordinate.
A map $f : X \to Y$ can be considered as an element in $\prod_{x\in X} Y$.

Uniform Topology:
Define the *uniform norm* of a function f as
$$\|f\|_{\sup} := \sup\{|f(x)| : x \in [a, b]\}.$$
A sequence of functions $\{f_n(x) : n \in \mathbb{N}\}$ *uniformly converges* to $f(x)$ if
$$\lim_{n \to \infty} \|f_n - f\|_{\sup} = 0.$$
The *uniform topology* is induced from the metric defined as follows:
$$d(f, g) := \|f - g\|_{\sup}.$$
$C[a, b]$ is closed in the uniform topology (closed under uniform convergence).
$(X, \|\cdot\|_{\sup})$ is complete.

Dini's Theorem:
Let K be a compact metric space, and let $f : K \to \mathbb{R}$ be a continuous function. Suppose that $\{f_n : n \in \mathbb{N}\}$ is a sequence of continuous functions on K which converges pointwise to f and $\{f_n(u)\}$ is increasing, then $\{f_n\}$ converges uniformly to f on K.

Baire Category Theorem:
Let X be a non-empty complete metric space.
Let $\{G_n\}$ be a countable collection of dense open subsets of X. Then, $\bigcap_{n=1}^\infty G_n$ is non-empty.
Let $\{F_n\}$ be a countable collection of closed subsets of X such that $X = \bigcup_{n=1}^\infty F_n$. Then, there exists some open set $U \subseteq X$ with $\overline{U} \subseteq F_n$ for some $n \in \mathbb{N}$.

Weierstrass M-test:
If the series $\sum_{n=1}^\infty \|f_n\|_{\sup}$ is convergent, then the partial sum
$$S_k(x) := \sum_{n=1}^{k} f_n(x)$$
uniformly converges to
$$S(x) = \sum_{n=1}^{\infty} f_n(x).$$

Radius of Convergence:
Let $L = \limsup_{n\to\infty} |a_n|^{1/n}$.
- If $L = +\infty$, $\sum_{n=1}^\infty a_n x^n$ diverges for any $x \in \mathbb{R}$.
- If $L < +\infty$, $\sum_{n=1}^\infty a_n x^n$ absolutely converges for $|x| < \frac{1}{L}$, and diverges for $|x| > \frac{1}{L}$.

$R = \frac{1}{L}$ is the *radius of convergence* of the *power series* $\sum_{n=1}^\infty a_n x^n$.
Let $R > 0$ be the radius of convergence of the power series $S(x) := \sum_{n=1}^\infty a_n x^n$, then $S(x)$ is continuous on $(-R, R)$.

Abel's Theorem:
If $S(R) = \sum_{n=1}^\infty a_n R^n$ converges, then
$$\lim_{x \to R^-} S(x) = S(R).$$

Infinite Product:
Given a positive sequence $\{a_n\}$, let $P_n = \prod_{k=1}^n a_k$. If $\lim_{n\to\infty} P_n = P$ exists and $P \neq 0$, the *infinite product*
$$\prod_{k=1}^{\infty} a_k$$
is convergent and denote $\prod_{k=1}^\infty a_k = P$.

Cauchy Criterion for Infinite Product:
Let $\{a_n\}$ be a positive sequence. $\prod_{k=1}^\infty a_k$ is convergent if and only if
$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } \forall m > n \ge N,$$
$$\left| \prod_{k=n+1}^{m} a_k - 1 \right| < \varepsilon.$$

Divergence Test:
Let $\{a_n\}$ be a positive sequence. If $\prod_{k=1}^\infty a_k$ is convergent, then
$$\lim_{n \to \infty} a_n = 1.$$

Convergence Criterion:
Let $\{x_n\}$ be a positive sequence, then the following statements are equivalent:
1. $\prod_{k=1}^\infty (1 + x_k)$ is convergent;
2. $\sum_{k=1}^\infty x_k$ is convergent.

Euler's Product:
Let $\mathfrak{P} = \{p_1 < p_2 < \cdots < p_k < \cdots\}$ be the set of prime numbers, then for $s > 1$,
$$\prod_{k=1}^{\infty} \frac{1}{1 - p_k^{-s}} = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$
Also,
$$\sum_{k=1}^{\infty} \frac{1}{p_k} \text{ is divergent.}$$

Important Limits:
$$\lim_{x \to \infty} \left(1 + \tfrac{1}{x}\right)^x = e, \qquad \lim_{x \to 0} \frac{\log(1 + x)}{x} = 1,$$
$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1, \qquad \lim_{x \to 0} \frac{(1 + x)^a - 1}{x} = a,$$
$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}.$$