MA2104 (AY25/26 S1)
Multivariable Calculus
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Def. 1 (Distance Between Two Points)
For points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$:
$$|P_1P_2| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}.$$

Prop. 2 (Dot Product)
For vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle, \mathbf{v} = \langle v_1, v_2, v_3\rangle \in \mathbb{R}^3$:
$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3,$$
$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta,$$
$$|\mathbf{v}-\mathbf{u}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}|\,|\mathbf{v}|\cos\theta.$$
where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Def. 3 (Projection)
Projection of $\mathbf{u}$ onto $\mathbf{v}$:
$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}.$$

Prop. 4 (Cross Product Formula)
For vectors $\langle b_1, b_2, b_3\rangle$ and $\langle c_1, c_2, c_3\rangle$:
$$\langle b_1, b_2, b_3\rangle \times \langle c_1, c_2, c_3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Def. 5 (Cross Product)
For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$:
$$\mathbf{u}\times\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\sin\theta\,\mathbf{n},$$
where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, and $\mathbf{n}$ is the unit vector orthogonal to both, following the right–hand rule.

Def. 6 (Vector equation of a line)
The line L passing through $P_0(x_0, y_0, z_0)$ and parallel to a non-zero vector $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ is given by
$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \qquad -\infty < t < \infty,$$
where $\mathbf{r} = \overrightarrow{OP} = \langle x, y, z\rangle$ and $\mathbf{r}_0 = \overrightarrow{OP_0} = \langle x_0, y_0, z_0\rangle$.

Thm. 7 (Parametric equations of a line)
The line L passing through $P_0(x_0, y_0, z_0)$ and parallel to $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ can be parametrized as
$$x = x_0 + tv_1, \qquad y = y_0 + tv_2, \qquad z = z_0 + tv_3.$$

Prop. 8 (Distance from point to line)
The distance from a point S to a line passing through a point P and parallel to a vector $\mathbf{v}$ is
$$\operatorname{dist}(S, L) = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}.$$

Thm. 9 (Equation of a plane)
A plane containing the point $P_0(x_0, y_0, z_0)$ and normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ satisfies:
Vector equation:
$$\mathbf{n}\cdot\overrightarrow{PP_0} = 0.$$
Component equation:
$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0.$$
Simplified component equation:
$$Ax + By + Cz = D, \qquad D = Ax_0 + By_0 + Cz_0.$$

Def. 10 (Cylindrical Coordinates)
The relation between Cartesian (x, y, z) and cylindrical (r, θ, z) coordinates is:
$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$
with
$$r = \sqrt{x^2+y^2}, \quad \tan\theta = \frac{y}{x}.$$

Def. 11 (Spherical Coordinates)
The relation between Cartesian (x, y, z) and spherical (ρ, ϕ, θ) coordinates is:
$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi,$$
with
$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}, \quad r = \sqrt{x^2+y^2}.$$

Def. 12 (Cylindrical Surface)
A surface generated by moving a curve along a straight line in a fixed direction is called a *cylinder*.

Def. 13 (Quadric Surfaces)
$$\begin{aligned}
\text{Ellipsoid:} &\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,\\
\text{Elliptic Paraboloid:} &\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c},\\
\text{Elliptical Cone:} &\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2},\\
\text{Hyperboloid of One Sheet:} &\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,\\
\text{Hyperboloid of Two Sheets:} &\quad \frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,\\
\text{Hyperbolic Paraboloid:} &\quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, \quad c > 0.
\end{aligned}$$

Def. 14 (Limit of a Vector Function)
Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function, and let $\mathbf{L}$ be a vector. We say that $\mathbf{r}$ has limit $\mathbf{L}$ as $t \to t_0$ and write
$$\lim_{t\to t_0}\mathbf{r}(t) = \mathbf{L}$$
if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that
$$0 < |t - t_0| < \delta \implies |\mathbf{r}(t) - \mathbf{L}| < \varepsilon.$$

Def. 15 (Derivative of a Vector Function)
Let $\mathbf{r}(t)$ be a vector function. Its derivative is
$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t\to 0}\frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}.$$

Prop. 16
If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then
$$\mathbf{r}'(t) = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}.$$

Def. 17 (Tangent Line)
The tangent line of the curve at the point $\mathbf{r}(t)$ is the line spanned by $\mathbf{r}'(t)$.

Prop. 18
If $\mathbf{r}(t)$ is a differentiable vector function of constant length, then
$$\mathbf{r}(t)\cdot\frac{d\mathbf{r}}{dt} = 0.$$

Def. 19 (Indefinite Integral of a Vector Function)
The indefinite integral of a vector function $\mathbf{r}(t)$ with respect to t is the set of all anti-derivatives of $\mathbf{r}$, denoted by
$$\int \mathbf{r}(t)\,dt.$$
If $\mathbf{R}(t)$ is an anti-derivative of $\mathbf{r}(t)$ (i.e., $\mathbf{r}(t) = \frac{d\mathbf{R}}{dt}$), then
$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}.$$
Component-wise:
$$\int \mathbf{r}(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt\right\rangle.$$

Def. 20 (Definite Integral of a Vector Function)
If the components of a vector function
$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$
are integrable over $[a, b]$, then so is $\mathbf{r}$. The definite integral of $\mathbf{r}$ from a to b is
$$\int_a^b \mathbf{r}(t)\,dt = \Big(\int_a^b f(t)\,dt\Big)\mathbf{i} + \Big(\int_a^b g(t)\,dt\Big)\mathbf{j} + \Big(\int_a^b h(t)\,dt\Big)\mathbf{k}.$$

Def. 21 (Arc Length of a Smooth Curve)
Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ be a smooth curve for $a \le t \le b$. The length of the curve as t increases from $t = a$ to $t = b$ is
$$\begin{aligned} L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \\ &= \int_a^b |\mathbf{v}(t)|\,dt, \end{aligned}$$
where $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$ is the velocity vector. L is also called the arc length of the curve.

Def. 22 (Unit Tangent Vector)
Let $\mathbf{r}(t)$ be a smooth curve with $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} \neq 0$. The unit tangent vector of the curve at time t is
$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{d\mathbf{r}/dt}{ds/dt} = \frac{d\mathbf{r}}{ds},$$
where s is the arc length measured from a fixed point on the curve.

Def. 23 (Curvature of a Smooth Curve)
Let $\mathbf{r}(s)$ be a smooth curve parameterized by arc length s, with unit tangent vector $\mathbf{T}(s) = \frac{d\mathbf{r}}{ds}$. The curvature function κ of the curve is
$$\kappa = \left\|\frac{d\mathbf{T}}{ds}\right\| = \left\|\frac{d^2\mathbf{r}}{ds^2}\right\|.$$

Thm. 24 (Curvature for Arbitrary Parameterization)
Let $\mathbf{r}(t)$ be a smooth curve with velocity $\mathbf{v} = \frac{d\mathbf{r}}{dt} \neq 0$ and unit tangent vector $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$. The curvature of the curve is
$$\kappa = \frac{1}{|\mathbf{v}|}\left\|\frac{d\mathbf{T}}{dt}\right\|.$$

Prop. 25 (Curvature in Space)
Let $\mathbf{r}(t)$ be a smooth space curve. Then the curvature can also be expressed as
$$\kappa = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

Prop. 26 (Curvature in Plane)
For a plane curve $y = y(x)$, the curvature is
$$\kappa = \frac{|y''|}{(1+(y')^2)^{3/2}}.$$

Def. 27 (Interior and Boundary Points)
Let R be a domain in the xy-plane.
A point in R is called an *interior point* if it is the center of a small disk wholly contained in R.
A point is called a *boundary point* of R if every disk (no matter how small) centered at this point is partially in R and partially outside R.

Def. 28 (Level Curve)
Let $f(x, y)$ be a function of two variables with domain D. The set of points in the plane where $f(x, y)$ has a constant value $f(x, y) = c$ is called a *level curve* of f.

Def. 29 (Graph of a Function of Two Variables)
The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the *graph* of f. The graph is also called the surface $z = f(x, y)$.

Def. 30 (Level Surface)
Let $f(x, y, z)$ be a function of three independent variables. The set of points (x, y, z) in space where $f(x, y, z)$ has a constant value $f(x, y, z) = c$ is called a *level surface* of f.

Def. 31 (Limit of a Function of Two Variables)
Let $f(x, y)$ be a function with domain D. We say that f **approaches the limit** L as (x, y) approaches (x_0, y_0) and write
$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$
if for every $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that for all $(x, y) \in D$,
$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \implies |f(x,y) - L| < \epsilon.$$

Def. 32 (Continuity of a Function of Two Variables)
A function $f(x, y)$ is *continuous* at (x_0, y_0) if
1. (x_0, y_0) is in the domain of f,
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists, and
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

The function $f(x, y)$ is continuous if it is continuous at every point in its domain.

Def. 33 (Limit of a function of two variables)
Let $f(x, y)$ be defined on a neighborhood of (x_0, y_0) except possibly at (x_0, y_0). We say
$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$
if for every $\epsilon > 0$, there exists $\delta > 0$ such that
$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \implies |f(x,y) - L| < \epsilon.$$

Thm. 34 (Two-paths method)
If a function $f(x, y)$ has different limits along two different paths as (x, y) approaches (x_0, y_0), then
$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) \text{ does not exist.}$$

Def. 35 (Limit and continuity of functions of three or more variables)
The definitions of limit and continuity for functions of two variables extend to functions of three or more variables. Limits and continuity for sums, products, quotients, powers, and composites also extend in the same manner.

Def. 36 (Continuity at a point)
A function $f(x, y)$ is continuous at the point (x_0, y_0) if
1. f is defined at (x_0, y_0), i.e., (x_0, y_0) is in the domain of f,
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists,
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

Def. 37 (Continuous function)
A function $f(x, y)$ is continuous if it is continuous at every point in its domain.

Def. 38 (Partial derivative with respect to x)
The partial derivative of a function $f(x, y)$ with respect to x at a point (x_0, y_0) is
$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x, y_0) - f(x_0, y_0)}{\Delta x},$$
provided the limit exists. It is also denoted by $f_x(x_0, y_0)$.

Def. 39 (Partial derivative with respect to y)
The partial derivative of a function $f(x, y)$ with respect to y at a point (x_0, y_0) is
$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{\Delta y\to 0}\frac{f(x_0, y_0+\Delta y) - f(x_0, y_0)}{\Delta y},$$
provided the limit exists. It is also denoted by $f_y(x_0, y_0)$.

Def. 40 (Tangent plane to a surface)
Let $z = f(x, y)$ and (x_0, y_0, z_0) be a point on the surface, where $z_0 = f(x_0, y_0)$, $A = f_x(x_0, y_0)$, and $B = f_y(x_0, y_0)$. The tangent plane to the surface at (x_0, y_0, z_0) is given by
$$z = A(x-x_0) + B(y-y_0) + z_0.$$

Def. 41 (Linear approximation / Tangent plane approximation)
Let $z = f(x, y)$ and (x_0, y_0, z_0) be a point on the graph, with $A = f_x(x_0, y_0)$, $B = f_y(x_0, y_0)$, and $z_0 = f(x_0, y_0)$. The linear approximation (or tangent plane approximation) of f near (x_0, y_0) is
$$f(x, y) \approx A(x-x_0) + B(y-y_0) + z_0.$$

Thm. 42 (Increment Theorem for Functions of Two Variables)
Suppose that f_x and f_y are defined throughout an open region R containing (x_0, y_0) and are continuous at (x_0, y_0). Then for $(x_0+\Delta x, y_0+\Delta y) \in R$,
$$\begin{aligned} \Delta z &= f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0) \\ &= f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y, \end{aligned}$$
where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$.

Def. 43 (Differential)
The differential of f at (x_0, y_0) is
$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy,$$
where $dx = \Delta x$ and $dy = \Delta y$.

Thm. 44 (Clairaut's Theorem)
If f and its partial derivatives f_x, f_y, f_{xy}, f_{yx} are defined throughout an open region containing (a, b) and all are continuous at (a, b), then
$$f_{xy}(a, b) = f_{yx}(a, b).$$

Def. 45 (Differentiable function of two variables)
A function $z = f(x, y)$ is differentiable at (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and
$$\begin{aligned} \Delta z &= f(x, y) - f(x_0, y_0) \\ &= f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y, \end{aligned}$$
where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$, with $\Delta x = x - x_0$ and $\Delta y = y - y_0$. f is differentiable if it is differentiable at every point in its domain.

Thm. 46
If a function $f(x, y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

Def. 47 (Chain Rule for Two Independent Variables)
Let $w = f(x, y)$ be differentiable, and $x = x(t)$, $y = y(t)$ differentiable functions of t. Then $w(t) = f(x(t), y(t))$ is differentiable, and
$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Def. 48 (Chain Rule for Three Independent Variables)
Let $w = f(x, y, z)$ be differentiable, and $x = x(t)$, $y = y(t)$, $z = z(t)$ differentiable functions of t. Then $w(t) = f(x(t), y(t), z(t))$ is differentiable, and
$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Def. 49 (General Chain Rule)
Let $w = f(x_1, x_2, \ldots, x_n)$ be a differentiable function of n variables. Each $x_i = x_i(t_1, t_2, \ldots, t_m)$ is differentiable with respect to m variables. Then
$$w(t_1, \ldots, t_m) = f(x_1(t_1, \ldots, t_m), \ldots, x_n(t_1, \ldots, t_m))$$
is differentiable, and
$$\frac{\partial w}{\partial t_j} = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial t_j}.$$

Def. 50 (Directional Derivative)
Let $z = f(x, y)$ and $u = \langle u_1, u_2\rangle$ be a unit vector. The directional derivative of f at $P_0(x_0, y_0)$ in the direction of u is
$$\left(\frac{df}{ds}\right)_{u,P_0} = \lim_{s\to 0}\frac{f(x_0+su_1, y_0+su_2) - f(x_0, y_0)}{s},$$
provided the limit exists.

Thm. 51 (Directional Derivative via Gradient)
$$\begin{aligned} \left(\frac{df}{ds}\right)_{u,P_0} &= u_1f_x(x_0, y_0) + u_2f_y(x_0, y_0) \\ &= u\cdot\nabla f(x_0, y_0), \end{aligned}$$
where $\nabla f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle$.

Def. 52 (Gradient Vector)
The gradient of $f(x, y)$ at $P_0(x_0, y_0)$ is
$$\nabla f = f_xi + f_yj,$$
evaluated at P_0.

Prop. 53 (Gradient and Level Curves)
Let $f(x, y)$ be differentiable and D its domain. For a point $(x_0, y_0) \in D$, ∇f is perpendicular to the level curve passing through (x_0, y_0).

Def. 54 (Directional Derivative in Three Variables)
Let $f(x, y, z)$ and $u = \langle u_1, u_2, u_3\rangle$ be a unit vector. The directional derivative at (x_0, y_0, z_0) is
$$\left(\frac{df}{ds}\right)_{u,(x_0,y_0,z_0)} = \lim_{\Delta s\to 0}\cdots$$
$$\frac{f(x_0+u_1\Delta s, y_0+u_2\Delta s, z_0+u_3\Delta s) - f(x_0, y_0, z_0)}{\Delta s}.$$

Prop. 55 (Gradient and Level Surfaces)
Let $f(x, y, z)$ be differentiable and D its domain. For a point $P_0(x_0, y_0, z_0) \in D$, ∇f is perpendicular to the level surface passing through P_0.

Def. 56 (Tangent Plane and Normal Line)
The tangent plane at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$. The normal line of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$.

Rem. 57
The definition is meaningful only if $\nabla f|_{P_0} = \langle f_x(P_0), f_y(P_0), f_z(P_0)\rangle$ is a nonzero vector.

Def. 58 (Equation of Tangent Plane)
The tangent plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$ is
$$f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0) = 0.$$

Def. 59 (Parametric Equations of Normal Line)
The normal line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$ has parametric equations
$$\begin{aligned} x &= x_0 + f_x(P_0)t, \\ y &= y_0 + f_y(P_0)t, \\ z &= z_0 + f_z(P_0)t, \quad t \in \mathbb{R}. \end{aligned}$$

Def. 60 (Tangent Plane to a Graph $z = F(x, y)$)
Let $z_0 = F(x_0, y_0)$, $A = F_x(x_0, y_0)$, $B = F_y(x_0, y_0)$. The tangent plane at (x_0, y_0, z_0) is
$$z = A(x-x_0) + B(y-y_0) + z_0.$$

Def. 61 (Local Extreme Values)
Let $f(x, y)$ be defined on a region R containing the point (a, b).
- $f(a, b)$ is a *local maximum* if $f(a, b) \ge f(x, y)$ for all (x, y) in an open disk centered at (a, b).
- $f(a, b)$ is a *local minimum* if $f(a, b) \le f(x, y)$ for all (x, y) in an open disk centered at (a, b).
- (a, b) is a *local extreme point* if it is a local maximum or local minimum.

Def. 62 (Critical Point)
An interior point (a, b) of the domain of $f(x, y)$ is a *critical point* if either
$$f_x(a, b) = f_y(a, b) = 0$$
or one or both of f_x and f_y do not exist.

Thm. 63 (First Derivative Test)
If $f(x, y)$ has a local maximum or minimum at an interior point (a, b) of its domain and the first partial derivatives exist there, then
$$f_x(a, b) = 0 \quad\text{and}\quad f_y(a, b) = 0.$$

Def. 64 (Saddle Point)
A differentiable function $f(x, y)$ has a *saddle point* at a critical point (a, b) if in every open disk centered at (a, b) there are points (x, y) with $f(x, y) > f(a, b)$ and points (x, y) with $f(x, y) < f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a *saddle point of the surface.*

Thm. 65 (Second Derivative Test)
Suppose $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b), and (a, b) is a critical point $(f_x(a, b) = f_y(a, b) = 0)$. Let
$$D = f_{xx}f_{yy} - (f_{xy})^2.$$
- If $f_{xx} < 0$ and $D > 0$ at (a, b), then f has a local maximum at (a, b).
- If $f_{xx} > 0$ and $D > 0$ at (a, b), then f has a local minimum at (a, b).
- If $D < 0$ at (a, b), then f has a saddle point at (a, b).
- If $D = 0$ at (a, b), the test is inconclusive.

Prop. 66 (Extreme Values Location)
Extreme values of $f(x, y)$ can occur only at:
- boundary points of the domain,
- critical points (interior points where $f_x = f_y = 0$ or one of the partials does not exist).

Def. 67 (Lagrange Multiplier)
Let $f(x, y, z)$ be a function to be optimized subject to a constraint $g(x, y, z) = 0$. A scalar λ is called a *Lagrange multiplier* if at an extremum (x_0, y_0, z_0) of f subject to $g = 0$, we have
$$\nabla f(x_0, y_0, z_0) = \lambda\nabla g(x_0, y_0, z_0),$$
provided $\nabla g \neq \langle 0, 0, 0\rangle$.

Def. 68 (Method of Lagrange Multipliers)
To find the local maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$, solve simultaneously:
$$\nabla f(x, y, z) = \lambda\nabla g(x, y, z), \quad g(x, y, z) = 0$$
for unknowns x, y, z and λ.

Rem. 69
The method of Lagrange multipliers also applies to functions of two variables $f(x, y)$ with a constraint $g(x, y) = 0$ by removing the z-coordinate.

Def. 70 (Differential Operator Notation)
For a function $f(x, y)$, real numbers a, b, and a positive integer n,
$$(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y})^n f = \sum_{k=0}^n \binom{n}{k}a^{n-k}b^k\frac{\partial^n f}{\partial x^{n-k}\partial y^k}.$$

Thm. 71 (Taylor's Formula for Functions of Two Variables at the Origin)
Let $f(x, y)$ be a function of two variables. Then
$$\begin{aligned} f(x, y) &= f(0, 0) + xf_x + yf_y \\ &\quad + \frac{1}{2!}(x^2f_{xx} + 2xyf_{xy} + y^2f_{yy}) \\ &\quad + \cdots \\ &\quad + \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n f \\ &\quad + \frac{1}{(n+1)!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^{n+1} f\Big|_{(cx,cy)} \end{aligned}$$
for some $c \in (0, 1)$. The last term is the remainder (error term).

Def. 72 (Riemann Integral of a Function of Two Variables)
Let $f(x, y)$ be a function defined over a rectangular region $R \subset \mathbb{R}^2$. Subdivide R into n small rectangles $\Delta A_k = \Delta x_k\Delta y_k$, and let (x_k, y_k) be a point in ΔA_k. The Riemann sum is
$$S_n = \sum_{k=1}^n f(x_k, y_k)\Delta A_k = \sum_{k=1}^n f(x_k, y_k)\Delta x_k\Delta y_k.$$
If the limit exists as $n \to \infty$ and $\Delta A_k \to 0$, the Riemann integral of f over R is defined as
$$\iint_R f(x, y)\,dA = \lim_{n\to\infty} S_n.$$

Thm. 73 (Fubini's Theorem, First Form)
Let $f(x, y)$ be continuous on the rectangular region
$$R = \{(x, y) : a \le x \le b,\ c \le y \le d\}.$$
Then
$$\begin{aligned} \iint_R f(x, y)\,dA &= \int_c^d\int_a^b f(x, y)\,dx\,dy \\ &= \int_a^b\int_c^d f(x, y)\,dy\,dx. \end{aligned}$$

Thm. 74 (Properties of Double Integrals)
Let $f(x, y)$ and $g(x, y)$ be continuous on R.
1. For any $c \in \mathbb{R}$,
$$\iint_R cf(x, y)\,dA = c\iint_R f(x, y)\,dA.$$
2.
$$\begin{aligned} &\iint_R (f(x, y) \pm g(x, y))\,dA \\ &= \iint_R f(x, y)\,dA \pm \iint_R g(x, y)\,dA. \end{aligned}$$
3. (a) If $f(x, y) > 0$ on R, then $\iint_R f(x, y)\,dA > 0$.
   (b) If $f(x, y) > g(x, y)$ on R, then $\iint_R f(x, y)\,dA > \iint_R g(x, y)\,dA$.
4. If $R = R_1 \cup R_2$ with $R_1 \cap R_2 = \emptyset$, then
$$\iint_R f(x, y)\,dA = \iint_{R_1} f(x, y)\,dA + \iint_{R_2} f(x, y)\,dA.$$

Thm. 75 (Fubini's Theorem, Stronger Form)
Let $f(x, y)$ be continuous on a region R.
1. If $R = \{(x, y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}$, with g_1 and g_2 continuous on $[a, b]$, then
$$\iint_R f(x, y)\,dA = \int_a^b\int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx.$$
2. If $R = \{(x, y) : c \le y \le d, h_1(y) \le x \le h_2(y)\}$, with h_1 and h_2 continuous on $[c, d]$, then
$$\iint_R f(x, y)\,dA = \int_c^d\int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy.$$

Def. 76 (Area by Double Integration)
Let $R \subset \mathbb{R}^2$ be a closed bounded region. The area of R is defined as
$$A = \iint_R dA.$$

Def. 77 (Average Value of a Function over a Region)
Let $f(x, y)$ be defined on a region R with area $A = \iint_R dA$. The average value of f over R is

$$\text{Average value of } f \text{ over } R = \frac{1}{A}\iint_R f(x, y)\, dA.$$

Def. 78 (Area Element in Polar Coordinates)
For a small region in polar coordinates, the area element is

$$dA = r\, dr\, d\theta.$$

Def. 79 (Double Integral in Polar Coordinates)
Let $f(x, y)$ be defined over a region R that is more conveniently expressed in polar coordinates. Then the double integral of f over R is

$$\iint_R f(x, y)\, dA = \iint_R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Def. 80 (Area of a Region in Polar Coordinates)
Let R be a closed and bounded region in the polar coordinate plane. The area of R is

$$A = \iint_R r\, dr\, d\theta.$$

Def. 81 (Surface Area of a Graph)
Let S be the graph of $z = f(x, y)$ over a region $R \subset \mathbb{R}^2$. The surface area of S is

$$\begin{aligned}\text{Area}(S) &= \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA\\ &= \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy.\end{aligned}$$

Def. 82 (Normal Vector to a Surface)
For a surface defined as $F(x, y, z) = z - f(x, y) = 0$, a normal vector is

$$\nabla F = \langle F_x, F_y, F_z\rangle = \langle -f_x, -f_y, 1\rangle,$$

and the corresponding unit normal vector is

$$\mathbf{n} = \frac{\langle -f_x, -f_y, 1\rangle}{\sqrt{1 + f_x^2 + f_y^2}}.$$

Def. 83 (Riemann Integral of a Function of Three Variables)
Let D be a region in space and $F(x, y, z)$ be a continuous function. Divide D into n small rectangular boxes $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$. The mass of each small box is

$$F(x_k, y_k, z_k)\Delta V_k = F(x_k, y_k, z_k)\Delta x_k \Delta y_k \Delta z_k.$$

The triple integral is the limit of the Riemann sum:

$$\begin{aligned}S_n &= \sum_{k=1}^{n} F(x_k, y_k, z_k)\Delta V_k\\ &= \sum_{k=1}^{n} F(x_k, y_k, z_k)\Delta x_k \Delta y_k \Delta z_k,\\ \lim_{n\to\infty} S_n &= \iiint_D F(x, y, z)\, dV.\end{aligned}$$

Def. 84 (Volume of a Region)
The volume of a closed bounded region D in space is

$$V = \iiint_D dV.$$

Thm. 85 (Fubini's Theorem, First Form)
Let $D = \{(x, y, z) \in \mathbb{R}^3 : a_1 \le x \le a_2, b_1 \le y \le b_2, c_1 \le z \le c_2\}$. Then

$$\iiint_D F(x, y, z)\, dV = \int_{c_1}^{c_2}\int_{b_1}^{b_2}\int_{a_1}^{a_2} F(x, y, z)\, dx\, dy\, dz.$$

Thm. 86 (Fubini's Theorem, Stronger Form)
Let

$$D = \Big\{(x, y, z) \in \mathbb{R}^3 : a \le x \le b,\ g_1(x) \le y \le g_2(x),\ f_1(x, y) \le z \le f_2(x, y)\Big\}.$$

Then

$$\begin{aligned}&\iiint_D F(x, y, z)\, dV\\ &= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} F(x, y, z)\, dz\, dy\, dx.\end{aligned}$$

Prop. 87 (Properties of Triple Integrals)
Let $F(x, y, z)$ and $G(x, y, z)$ be continuous on D. Then

1. $\iiint_D kF\, dV = k\iiint_D F\, dV$ for any constant k.
2. $\iiint_D (F \pm G)\, dV = \iiint_D F\, dV \pm \iiint_D G\, dV$.
3. (a) If $F > 0$ on D, then $\iiint_D F\, dV > 0$.
   (b) If $F > G$ on D, then $\iiint_D F\, dV > \iiint_D G\, dV$.
4. If D is the union of two non-overlapping regions D_1 and D_2, then
$$\iiint_D F\, dV = \iiint_{D_1} F\, dV + \iiint_{D_2} F\, dV.$$

Def. 88 (Mass of a 2D Region)
Let D be a region in the xy-plane with density $\delta(x, y)$. The mass of D is

$$M = \iint_D \delta(x, y)\, dA.$$

Def. 89 (Mass of a 3D Region)
Let D be a region in space with density $\delta(x, y, z)$. The mass of D is

$$M = \iiint_D \delta(x, y, z)\, dV = \iiint_D \delta(x, y, z)\, dx\, dy\, dz.$$

Def. 90 (First Moments and Center of Mass in 2D)
For a thin plate occupying a region R in the xy-plane with density $\delta(x, y)$, the first moments are

$$M_x = \iint_R y\,\delta(x, y)\, dA, \quad M_y = \iint_R x\,\delta(x, y)\, dA.$$

The center of mass (centroid) is

$$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}.$$

Def. 91 (First Moments and Center of Mass in 3D)
For an object occupying a region D in three-dimensional space with density $\delta(x, y, z)$, the first moments are

$$\begin{aligned}M_{yz} &= \iiint_D x\,\delta(x, y, z)\, dV,\\ M_{xz} &= \iiint_D y\,\delta(x, y, z)\, dV,\\ M_{xy} &= \iiint_D z\,\delta(x, y, z)\, dV.\end{aligned}$$

The center of mass is

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}.$$

Def. 92 (Triple Integral in Cylindrical Coordinates)
The triple integral of a function $f(x, y, z)$ over a region D using cylindrical coordinates is

$$\begin{aligned}&\iiint_D f(x, y, z)\, dV\\ &= \iiint_D f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz.\end{aligned}$$

Def. 93 (Triple Integral in Spherical Coordinates)
The triple integral of a function $f(\rho, \phi, \theta)$ over a region D using spherical coordinates is

$$\iiint_D f(\rho, \phi, \theta)\, dV = \iiint_D f(\rho, \phi, \theta)\,\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

Rem. 94 (Differential in Triple Integrals)
The differential volume elements for triple integrals in different coordinate systems are:

$$dV = dx\, dy\, dz = r\, dr\, d\theta\, dz = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

Def. 95 (Jacobian Determinant)
Let $x = g(u, v)$ and $y = h(u, v)$ define a coordinate transformation $T : G \to R$. The Jacobian determinant of this transformation is

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Def. 96 (Change of Variables in Double Integrals)
Suppose $x = g(u, v)$ and $y = h(u, v)$ define a bijective coordinate transformation $T : G \to R$. Then for a function $f(x, y)$,

$$\begin{aligned}&\iint_R f(x, y)\, dx\, dy\\ &= \iint_G f(g(u, v), h(u, v))\, |J(u, v)|\, du\, dv,\end{aligned}$$

where $J(u, v)$ is the Jacobian determinant.

Def. 97 (Jacobian Determinant for Triple Integrals)
Let $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = k(u, v, w)$ define a coordinate transformation $T : G \to R$. The Jacobian determinant of this transformation is

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}.$$

Def. 98 (Change of Variables in Triple Integrals)
Suppose $x = g(u, v, w)$, $y = h(u, v, w)$, $z = k(u, v, w)$ define a bijective coordinate transformation $T : G \to R$. Then for a function $F(x, y, z)$,

$$\begin{aligned}&\iiint_R F(x, y, z)\, dx\, dy\, dz\\ &= \iiint_G H(u, v, w)\, |J(u, v, w)|\, du\, dv\, dw,\end{aligned}$$

where $H(u, v, w) = F(g(u, v, w), h(u, v, w), k(u, v, w))$ and $J(u, v, w)$ is the Jacobian determinant.

Def. 99 (Line Integral of a Scalar Function)
Let $f(x, y, z)$ be a function defined along a smooth curve C parametrized by

$$\mathbf{r}(t) = \langle g(t), h(t), l(t)\rangle, \quad a \le t \le b.$$

The line integral of f along C is

$$\begin{aligned}&\int_C f\, ds = \int_a^b f(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt\\ &= \int_a^b f(g(t), h(t), l(t))\sqrt{(g'(t))^2 + (h'(t))^2 + (l'(t))^2}\, dt.\end{aligned}$$

Rem. 100 (Path Independence of Line Integral)
The line integral $\int_C f\, ds$ depends only on the path C and not on the particular parametrization of the path. If $\mathbf{r}(t)$ and $\mathbf{s}(t)$ are two smooth parametrizations of the same curve C, then

$$\int_C f\, ds = \int_a^b f(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt = \int_c^d f(\mathbf{s}(t))\, |\mathbf{s}'(t)|\, dt.$$

Rem. 101 (Evaluation of Line Integrals)
To evaluate $\int_C f\, ds$ for a continuous function $f(x, y, z)$ over a curve C:

1. Find a smooth parametrization $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + l(t)\mathbf{k}$ of C.
2. Compute $|\mathbf{r}'(t)|$.
3. Evaluate $\int_a^b f(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt$.

The function f needs to be defined only along C.

Def. 102 (Vector Field on $\mathbb{R}^2$)
A vector field on $\mathbb{R}^2$ is a function

$$\mathbf{F}(x, y) = \langle M(x, y), N(x, y)\rangle$$

that assigns a vector to each point $(x, y) \in \mathbb{R}^2$. Equivalently, a vector field is a function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$.

Def. 103 (Vector Field on $\mathbb{R}^3$)
A vector field on $\mathbb{R}^3$ is a function

$$\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle$$

that assigns a vector to each point $(x, y, z) \in \mathbb{R}^3$. Equivalently, a vector field is a function $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$.

Def. 104 (Gradient Field)
Let $f(x, y, z)$ be a differentiable function. The gradient field of f is

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle.$$

The function f is called the potential function of the gradient field.

Def. 105 (Work Done by a Force)
Let $\mathbf{F} = \langle M, N, P\rangle$ be a force vector field and $r(t)$ a smooth curve from $t = a$ to $t = b$. The work done by $\mathbf{F}$ along the curve is

$$\begin{aligned}W &= \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt\\ &= \int_C \mathbf{F} \cdot d\mathbf{r}\\ &= \int_C \mathbf{F} \cdot \mathbf{T}\, ds\\ &= \int_a^b \Big(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt}\Big)dt\\ &= \int_C M\, dx + N\, dy + P\, dz\end{aligned}$$

The integral is called the work integral of $\mathbf{F}$ along the segment.

Rem. 106
The work integral depends on the path taken and the direction of travel along the curve. Moving backward along the curve reverses the sign of the work:

$$\int_{-C} \mathbf{F} \cdot \mathbf{T}\, ds = -\int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$

Def. 107 (Flow Along a Curve)
If $r(t)$ is a smooth curve in the domain of a continuous velocity field $\mathbf{F}$, the flow along the curve from $t = a$ to $t = b$ is

$$\text{Flow} = \int_a^b \mathbf{F} \cdot \mathbf{T}\, ds.$$

If the curve is closed, the integral is called the circulation about the curve. The integral in this case is called a flow integral.

Def. 108 (Conservative Fields)
Let F be a vector field defined on an open region D in space. Suppose for any two points A and B in D, the work done

$$\int_A^B F \cdot d\mathbf{r}$$

in moving from A to B is the same over all paths from A to B. Then the integral is path independent in D and the field F is *conservative* on D.

Def. 109 (Potential Functions)
If F is a vector field defined on D and

$$F = \nabla f$$

for some scalar function $f = f(x, y, z)$ on D, then f is called a *potential function* for F.

Thm. 110 (Fundamental Theorem of Line Integrals)
Let $F(x, y, z)$ be a vector field such that

$$F(x, y, z) = \nabla f(x, y, z) = \langle f_x, f_y, f_z\rangle$$

for some differentiable function $f(x, y, z)$. Let $r(t)$ be a curve in the domain of f, with $A = r(a)$ and $B = r(b)$ being the start and end points of the curve respectively. Then

$$f(B) - f(A) = \int_a^b F(r(t)) \cdot r'(t)\, dt.$$

The path integral on the right depends only on the start point A and end point B, and not on the actual path $r(t)$.

Thm. 111
Let $F = \langle M, N, P\rangle$ be a vector field whose components are continuous on an open connected region D in space. Then F is conservative if and only if F is the gradient field ∇f of a differentiable function f.

Thm. 112
The following statements are equivalent for a vector field F defined on an open connected region D:

1. $\oint_C F \cdot d\mathbf{r} = 0$ around every oriented closed curve C in D.
2. The field F is conservative on D.

Def. 113 (Open Domains)
An open domain $D \subset \mathbb{R}^2$ is one such that for every point x in D, there exists a small disk wholly contained in D with center x. An open domain $D \subset \mathbb{R}^3$ is one such that for every point x in D, there exists a small ball wholly contained in D with center x. Essentially, an open domain means that the domain does not contain its boundary.

Def. 114 (Simply-connected Domains)
A domain D is *simply-connected* if every closed curve within D can be continuously shrunk within D to a point.

Prop. 115 (Component Test for Conservative Fields)
Let D be a simply connected open domain in $\mathbb{R}^3$. Let

$$F(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle$$

be a vector field defined on D, where M, N, and P are differentiable functions with continuous first partial derivatives. Then F is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Def. 116 (Closed Curve)
A *closed curve* is a curve whose starting point is the same as the end point. A *simple closed curve* is a closed curve which does not cross itself.

Def. 117 (Directed Closed Curve)
A closed curve with a direction is called a *directed* or *oriented closed curve*. An oriented simple closed curve moving in an anti-clockwise direction is said to have a *positive orientation.* An oriented simple closed curve moving in a clockwise direction is said to have a *negative orientation.*

Def. 118 (Path Integral)
Let $F(x, y) = \langle M(x, y), N(x, y)\rangle$ be a vector field on the xy-plane. Let $r(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$, be a positively oriented closed curve C. The *path integral* of F along C is

$$\oint_C F \cdot d\mathbf{r} = \int_a^b F(r(t)) \cdot r'(t)\, dt = \int_C M\, dx + N\, dy.$$

Thm. 119 (Green's Theorem, Circulation-Curl / Tangential Form)
Let C be a positively oriented closed curve and let R be the region enclosed by C. Let $F(x, y) = \langle M(x, y), N(x, y)\rangle$ be a differentiable vector field on the xy-plane. Then

$$\oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy.$$

Thm. 120 (Green's Theorem, Flux-Divergence / Normal Form)
Let C be a positively oriented closed curve and let R be the region enclosed by C. Let $F(x, y) = \langle M(x, y), N(x, y)\rangle$ be a differentiable vector field on the xy-plane. Then

$$\oint_C M\, dy - N\, dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy.$$

Prop. 121 (Green's Theorem Area Formula)
Let C be a positively oriented simple closed curve enclosing region R. Then the area of R is

$$\text{Area}(R) = \frac{1}{2}\oint_C x\, dy - y\, dx.$$

Def. 122 (Parameterized Surfaces)
A *parametrization* of a surface S is a continuous and injective vector function $r : R \to \mathbb{R}^3$ where R is an open subset in the uv-plane. The variables u and v are called the *parameters*, R is the *parameter domain*, and the range of r is the surface S defined by r.

Rem. 123
In general, a surface S can be divided into pieces

$$S = S_1 \cup S_2 \cup \cdots \cup S_k$$

where each piece S_j has a parametrization

$$r_j : R_j \to S_j$$

with R_j a subset of the xy-plane, yz-plane, or xz-plane, and S_j is a graph.

Def. 124 (Surface Area)
The *surface area* of a surface S defined by $f(x, y, z) = c$ over a closed and bounded plane region R is

$$\text{Surface area} = \iint_R \frac{|\nabla f|}{|\nabla f \cdot p|}\, dA,$$

where p is a unit vector normal to R and $\nabla f \cdot p \ne 0$.

Prop. 125
For a surface given as a graph $z = g(x, y)$, we can set $f(x, y, z) = z - g(x, y) = 0$. Then the surface area is

$$\text{Area} = \iint_R \sqrt{1 + g_x^2 + g_y^2}\, dx\, dy.$$

Def. 126 (Surface Integrals)
Let S be a surface defined by $f(x, y, z) = c$, and let g be a continuous function defined on S. If R is the shadow region of S, then the *surface integral* of g over S is

$$\iint_S g\, d\sigma = \iint_R g(x, y, z)\frac{|\nabla f|}{|\nabla f \cdot p|}\, dA,$$

where p is a unit vector normal to R and $\nabla f \cdot p \ne 0$. The differential

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot p|}\, dA$$

is called the *surface area differential*, and the integral $\iint_S g\, d\sigma$ is called the *differential formula for surface integrals.*

Rem. 127
If a surface S is the union of two surfaces, $S = S_1 \cup S_2$, then

$$\iint_S g\, d\sigma = \iint_{S_1} g\, d\sigma + \iint_{S_2} g\, d\sigma.$$

Prop. 128
For a surface S given explicitly as the graph $z = f(x, y)$ over a region R in the xy-plane, the surface integral of a continuous function G over S is

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(x, y, f(x, y))\sqrt{1 + f_x^2 + f_y^2}\, dx\, dy.$$

Def. 129 (Mass of a Surface)
If a surface S has density $\delta(x, y, z)$, then the *mass of the surface* is given by the surface integral

$$\text{Mass} = \iint_S \delta\, d\sigma.$$

Def. 130 (Oriented Surface)
An *orientation* of a surface is an assignment of a unit normal vector $n(p)$ at every point p on the surface, such that $n(p)$ changes continuously as p moves on the surface. A surface with an orientation is called *orientable*, and a surface with a prescribed orientation is called an *oriented surface.*

Rem. 131
If $n(p)$ is an orientation, then $-n(p)$ is also an orientation. For a connected surface S, there are either 0 or 2 orientations. Closed surfaces are orientable, and the outward pointing vector is usually chosen as the positive direction.

Def. 132 (Flux)
Let $\mathbf{F}(x, y, z)$ be a vector field and $n(x, y, z)$ an orientation on a surface S. The *flux* of $\mathbf{F}$ across S in the direction of n is defined as

$$\begin{aligned}\text{Flux} &= \iint_S \mathbf{F} \cdot n\, d\sigma\\ &= \iint_S \mathbf{F} \cdot d\mathbf{S}.\end{aligned}$$

Rem. 133
Flux can be interpreted as the amount of a vector field (e.g., water flow) passing through a surface. If $\mathbf{F}$ is constant and S is perpendicular to $\mathbf{F}$, the flux equals $|\mathbf{F}| \times \text{Area}(S)$. If the flow is at an angle θ, the flux equals $|\mathbf{F}| \times \text{Area}(S) \times \cos\theta = (\mathbf{F} \cdot n)\text{Area}(S)$.

Prop. 134
For a surface S defined by $g(x, y, z) = 0$, with orientation $n = \pm\frac{\nabla g}{|\nabla g|}$ and $d\sigma = \frac{|\nabla g|}{|\nabla g \cdot k|}\, dx\, dy$ where $k = \langle 0, 0, 1\rangle$, the flux of a vector field $\mathbf{F}$ across S is

$$\text{Flux} = \iint_S \mathbf{F} \cdot n\, d\sigma = \pm\iint_R \frac{\mathbf{F} \cdot \nabla g}{|\nabla g \cdot k|}\, dx\, dy.$$

Prop. 135
If S is the graph of $z = f(x, y)$, set $g(x, y, z) = z - f(x, y)$. Then the flux of a vector field $\mathbf{F}$ across S is

$$\text{Flux} = \iint_R \mathbf{F} \cdot \langle -f_x, -f_y, 1\rangle\, dx\, dy.$$

Def. 136 (Curl in 3D)
Let $\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle$ be a vector field. The *curl* of $\mathbf{F}$ is defined by

$$\text{curl}\,\mathbf{F} = \langle P_y - N_z, M_z - P_x, N_x - M_y\rangle = \nabla \times \mathbf{F}.$$

Def. 137 (Curl in 2D)
If $\mathbf{F}(x, y) = \langle M(x, y), N(x, y)\rangle$ is a vector field on the xy-plane, we set $P = 0$ and define

$$\text{curl}\,\mathbf{F} = \langle 0, 0, N_x - M_y\rangle = (N_x - M_y)\mathbf{k}.$$

Thm. 138 (Green's Theorem)
Let D be an open connected and simply connected region on the xy-plane with boundary $C = \partial D$. Let $\mathbf{F} = \langle M, N\rangle$ be a vector field on D. Then *Green's theorem* in terms of curl states:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \text{curl}\,\mathbf{F} \cdot \mathbf{k}\, dA.$$

Thm. 139 (Stokes' Theorem)
Let $\mathbf{F}(x, y, z) = \langle M, N, P\rangle$ be a vector field, and S a connected and simply connected oriented surface in $\mathbb{R}^3$ with boundary $C = \partial S$ oriented such that the right-hand rule gives the surface's normal vector $\mathbf{n}$. Then *Stokes' Theorem* states:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma.$$

Rem. 140
If the surface S contains holes, Stokes' Theorem still holds. Let C be the outer boundary oriented counterclockwise with respect to $\mathbf{n}$, and $C_1, C_2, \ldots, C_m$ the boundaries of the holes oriented clockwise with respect to $\mathbf{n}$. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} + \sum_{j=1}^{m}\oint_{C_j} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma.$$

Thm. 141
Let f be a smooth function defined on an open domain in $\mathbb{R}^3$. Then

$$\nabla \times (\nabla f) = \mathbf{0}.$$

Thm. 142
Let D be a simply connected open region in space and $\mathbf{F}$ a vector field on D with $\nabla \times \mathbf{F} = \mathbf{0}$. Then for any oriented piecewise-smooth closed path C in D,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Thm. 143
Let D be a connected, simply connected, and open domain in $\mathbb{R}^3$. Let $\mathbf{F}$ be a vector field on D. The following statements are equivalent:

1. $\mathbf{F}$ is a gradient field, i.e. $\mathbf{F} = \nabla f$ for some function $f(x, y, z)$ on D.
2. $\mathbf{F}$ is conservative, i.e. path-independent.
3. $\mathbf{F}$ satisfies
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$
for every closed curve C in D.
4. $\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \langle 0, 0, 0\rangle$.

Def. 144 (Divergence)
Let $\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle$ be a vector field. The *divergence* of $\mathbf{F}$ is

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

Def. 145 (Incompressible Vector Field)
A vector field $\mathbf{F}(x, y, z)$ is *incompressible* if

$$\text{div}\,\mathbf{F} = 0$$

at all points (x, y, z).

Thm. 146 (Divergence Theorem)
Let D be a region in $\mathbb{R}^3$ with boundary $S = \partial D$, oriented with outward unit normal $\mathbf{n}$. Let $\mathbf{F}$ be a vector field defined on D and its boundary. Then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \text{div}\,\mathbf{F}\, dV.$$

Rem. 147
The Divergence Theorem requires $\mathbf{F}$ to be defined at all points of D.