Sharp: Short Relaxed Range Proofs

1 Motivation

Range proofs enable a prover to convince a verifier that a committed integer lies within an interval without revealing its value, serving as fundamental building blocks in privacy-preserving cryptosystems. Range proofs can be categorized into two main paradigms.

1.1 Binary Decomposition

The dominant approach, exemplified by Bulletproofs, decomposes the target value x into its binary representation and proves that each bit lies in $\{0, 1\}$ using inner-product arguments. Bulletproofs achieve asymptotically optimal proof sizes of $O(\lambda + \log N)$ group elements for N values, where λ is the security parameter, with logarithmic verification complexity. This approach benefits from transparent setup requirements and batching properties that make communication costs nearly independent of the number of proofs.

However, Bulletproofs incur substantial computational overhead. The protocol requires vector commitment schemes, multi-exponentiation techniques, and inner-product arguments that translate to significant prover and verifier computation.

1.2 Square Decomposition

An alternative approach leverages square decomposition techniques. These methods prove that a value x lies in a range by demonstrating that certain quadratic expressions can be decomposed into sums of squares. For instance, proving $x \in [0, B]$ reduces to showing that $1 + 4x(B - x)$ can be expressed as a sum of three squares, which guarantees non-negativity by leveraging number-theoretic properties.

Historically, square decomposition methods required integer commitment schemes instantiated over hidden-order groups such as RSA groups, leading to large element sizes (2048-3072 bits) and trusted setup requirements. These limitations rendered them uncompetitive with binary decomposition approaches for practical applications. The CKLR protocol (named after the four authors) revived this paradigm by introducing bounded integer commitments that operate over standard discrete logarithm groups. This eliminates the need for trusted setup while achieving proof sizes competitive with Bulletproofs.

1.3 CKLR Contributions and Limitations

CKLR demonstrates that square decomposition can be efficiently instantiated over discrete logarithm groups using a novel bounded integer commitment scheme. The protocol achieves this through relaxed soundness, where provers are bound to rational representatives m/d (with short numerator m and denominator d) rather than exact integers. For appropriate parameter choices, CKLR produces proofs approximately 15% shorter than Bulletproofs while requiring an order of magnitude fewer group operations.

Despite these theoretical advantages, CKLR faces several practical obstacles that limit its adoption. The protocol requires non-standard elliptic curves with 352-416 bit elements to achieve 128-bit security, whereas the cryptographic ecosystem has converged on highly optimized 256-bit implementations (such as libsecp256k1) that offer 10-20$\times$ performance improvements. Additionally, CKLR lacks efficient batching mechanisms. The bounded integer commitment scheme also provides only limited homomorphic properties compared to standard Pedersen commitments, restricting its applicability in certain protocols.

1.4 The Sharp Protocol Family

The Sharp family of protocols builds upon CKLR's foundation while addressing its practical limitations through a comprehensive set of optimizations. The key innovation lies in a modular decomposition of the range proof into two conceptually distinct components: the **Proof of Short Opening (PoSO)** and the **Proof of Decomposition (PoDec)**.

1.4.1 Technical Innovations

- Sharp employs a **polynomial test** to prove square decomposition. The prover computes a polynomial $f = z(\gamma B - z) - \sum_{i=1}^{3} z_i^2$ (where z, z_i are masked witnesses and γ is the challenge) and demonstrates that this polynomial has degree 1 in γ. The Schwartz-Zippel lemma ensures that the decomposition holds with overwhelming probability, while the polynomial structure enables compatibility with vector commitments.

- The polynomial technique allows Sharp to use **Pedersen multi-commitments**, committing to entire decompositions in single vector commitments rather than individual commitments per square.

- The modular PoSO/PoDec separation permits using **different cryptographic groups** optimized for each component. Typically, this involves 256-bit curves for commitments (leveraging existing optimized implementations) and larger curves for decomposition verification, connected through transparent parameter generation.

- Sharp accomplishes the same security guarantees as CKLR with just two scalars per repetition, independent of batch size.

2 Three Squares Decomposition Algorithm

2.1 Legendre's Three-Square Theorem

Legendre's three-square theorem states that a positive integer n can be expressed as a sum of three squares iff n is not of the form $4^a(8b+7)$ where $a, b \geq 0$ are integers.

For range proofs, we use the expression:

$$4x(B-x)+1 = y_1^2 + y_2^2 + y_3^2 \tag{1}$$

When $x \in [0, B]$, we have $4x(B-x) \geq 0$, which means $4x(B-x)+1 \geq 1$.

2.2 Algorithm

Before attempting decomposition, we verify that the target number $n = 4x(B-x)+1$ satisfies Legendre's criterion:

$$\left\lfloor \frac{n}{4^{\mathrm{val}_4(n)}} \right\rfloor \bmod 8 \neq 7 \tag{2}$$

where $\mathrm{val}_4(n)$ is the largest power of 4 dividing n.

The main algorithm proceeds as follows. For each candidate $z \in [0, \lfloor\sqrt{n}\rfloor]$, we compute:

$$m = n - z^2 \tag{3}$$

In practice, empirical testing shows that for numbers up to 1024 bits, all required z values are found within the much smaller range $[0, 10^6]$, making the algorithm highly efficient even for large inputs.

We then determine if m can be written as $x^2 + y^2$ using the sum-of-two-squares theorem. An integer $m > 0$ can be expressed as a sum of two squares iff every prime $p \equiv 3 \pmod 4$ appears to an even power in the prime factorization of m.

When m satisfies this criterion, we find the actual decomposition using algebraic number theory over $\mathbb{Z}[i]$ (the Gaussian integers). Specifically, we find $\alpha = a + bi \in \mathbb{Z}[i]$ such that $N(\alpha) = a^2 + b^2 = m$.

2.3 Implementation via PARI/GP

The decomposition is computed using PARI/GP's built-in algebraic number theory functions.

A *Gaussian integer* is a complex number of the form $a + bi$ where $a, b \in \mathbb{Z}$ are integers and $i^2 = -1$. The set of all Gaussian integers forms a ring denoted $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$.

The *norm* of a Gaussian integer $\alpha = a + bi$ is defined as:

$$N(\alpha) = N(a+bi) = a^2 + b^2 \tag{4}$$

The key insight is that finding integers a, b such that $a^2+b^2 = m$ is equivalent to finding a Gaussian integer $\alpha = a+bi$ whose norm equals m. This transforms the sum-of-squares problem into a norm equation in the Gaussian integers. The PARI/GP function `bnfinit(x^2 + 1)` initializes the number field $\mathbb{Q}(i)$ and its ring of integers $\mathbb{Z}[i]$. The polynomial x^2+1 defines the minimal polynomial of i over $\mathbb{Q}$, establishing the algebraic structure where $i^2 = -1$. Subsequently, `bnfisintnorm(K, m)` efficiently finds all Gaussian integers $\alpha \in \mathbb{Z}[i]$ with $N(\alpha) = m$ by solving the norm equation in the ring. The algorithm returns these elements as polynomials in i, from which we extract the integer coefficients a and b to obtain the desired decomposition $m = a^2 + b^2$.

2.4 Connection to $\mathsf{Sharp}_{\mathsf{GS}}$

The three-square decomposition enables the range proof by establishing:

$$x \in [0, B] \iff 4x(B-x)+1 \text{ admits a three-square representation} \tag{5}$$

This reduces the range-checking problem to verifying polynomial relationships between the committed values and their square decompositions, which can be done efficiently using the polynomial test in $\mathsf{Sharp}_{\mathsf{GS}}$.

3 Setup and Parameters

Goal: Prove that N committed values $x_1, \ldots, x_N$ each belong to range $[0, B]$

3.1 Protocol Parameters

- N: number of values to prove
- R: number of repetitions for security (typically $R = \lceil \lambda / \log(\Gamma + 1) \rceil$)
- Γ: challenge space size $[0, \Gamma]$
- S: randomness space size $[0, S]$ (hiding parameter)
- L_x, L_r: masking overheads
- $\mathcal{R}_x, \mathcal{R}_r$: masking distributions
  These distributions sample masking randomness vectors of size L_x and L_r, respectively, drawn uniformly from their domains (x from $[0, p-1]$, r from $[0, q-1]$)
- $\mathsf{p}_x, \mathsf{p}_r$: masking abort probabilities
- Hash function: $\mathsf{Hash} : \{0,1\}^* \to \{0,1\}^{2\lambda}$

3.2 System Setup

The protocol requires a Common Reference String (CRS) that can be generated transparently as follows:

1. Generate two cryptographic groups:
   - $\mathbb{G}_{\mathsf{com}}$ (for main value commitments) with prime order $p > 2(B\Gamma^2 + 1)L_x$
   - $\mathbb{G}_{\mathsf{3sq}}$ (for decomposition proof) with prime order $q > 18((B\Gamma + 1)L_x)^2$

   Group switching optimizes efficiency by choosing appropriate group sizes for each purpose.

2. Sample generators using transparent methods:

$$G_0, G_1, \ldots, G_N, G_{1,1}, G_{1,2}, G_{1,3}, \ldots, G_{N,1}, G_{N,2}, G_{N,3} \xleftarrow{\$} \mathbb{G}_{\mathsf{com}} \tag{6}$$

$$H_0, H_1, \ldots, H_N \xleftarrow{\$} \mathbb{G}_{\mathsf{3sq}} \tag{7}$$

3. Define commitment keys as:

$$\mathsf{ck}_{\mathbb{G}_{\mathsf{com}}} = \left(G_0, \{G_i\}_{i=1}^N, \{G_{i,j}\}_{i\in[1,N], j\in[1,3]}\right) \tag{8}$$

$$\mathsf{ck}_{\mathbb{G}_{\mathsf{3sq}}} = (H_0, \{H_i\}_{i=1}^N) \tag{9}$$

4. Define CRS as:

$$\mathsf{crs} = (\mathsf{ck}_{\mathbb{G}_{\mathsf{com}}}, \mathsf{ck}_{\mathbb{G}_{\mathsf{3sq}}}) \tag{10}$$

4 $\mathsf{Sharp}_{\mathsf{GS}}$ Algorithm

Input:

- **Both parties**: CRS $\mathsf{crs} = (\mathsf{ck}_{\mathbb{G}_{\mathsf{com}}}, \mathsf{ck}_{\mathbb{G}_{\mathsf{3sq}}})$, statement $C_x = r_x G_0 + \sum_{i=1}^{N} x_i G_i$ and range bound B
- **Prover**: Witnesses $(x_1, \ldots, x_N) \in [0, B]^N$ and randomness $r_x \in [0, S]$

4.1 Phase 1: Prover's First Message

4.1.1 Compute Three-Square Decomposition:

For each $i \in [1, N]$:

$$4x_i(B - x_i) + 1 = \sum_{j=1}^{3} y_{i,j}^2 \tag{11}$$

4.1.2 Commit to Decomposition:

$$C_y = r_y G_0 + \sum_{i=1}^{N} \sum_{j=1}^{3} y_{i,j} G_{i,j} \tag{12}$$

where $r_y \xleftarrow{\$} [0, S]$.

4.1.3 For each repetition $k \in [1, R]$:

a) Sample Random Masks:

- Opening masks: $\tilde{r}_{k,x}, \tilde{r}_{k,y} \xleftarrow{\$} \mathcal{R}_r$
- Value masks: $\tilde{x}_{k,i} \xleftarrow{\$} \mathcal{R}_x$ for $i \in [1, N]$
- Decomposition masks: $\tilde{y}_{k,i,j} \xleftarrow{\$} \mathcal{R}_x$ for $i \in [1, N], j \in [1, 3]$

b) Create Masked Commitments:

$$D_{k,x} = \tilde{r}_{k,x} G_0 + \sum_{i=1}^{N} \tilde{x}_{k,i} G_i \tag{13}$$

$$D_{k,y} = \tilde{r}_{k,y} G_0 + \sum_{i=1}^{N} \sum_{j=1}^{3} \tilde{y}_{k,i,j} G_{i,j} \tag{14}$$

c) Prepare Polynomial Coefficients:

For $i \in [1, N]$, the prover commits to coefficients $\alpha^*_{1,k,i}$ and $\alpha^*_{0,k,i}$ such that when the verifier later computes $f^*_{k,i} = 4z_{k,i}(\gamma_k B - z_{k,i}) + \gamma_k^2 - \sum_{j=1}^{3} z_{k,i,j}^2$, it will equal $\alpha^*_{1,k,i}\gamma_k + \alpha^*_{0,k,i}$ (degree 1 in γ_k) iff the three-square decomposition holds.

$$\alpha^*_{1,k,i} = 4\tilde{x}_{k,i} B - 8x_i \tilde{x}_{k,i} - 2\sum_{j=1}^{3} y_{i,j} \tilde{y}_{k,i,j} \quad \text{(coefficient of } \gamma_k\text{)} \tag{15}$$

$$\alpha^*_{0,k,i} = -\left(4\tilde{x}_{k,i}^2 + \sum_{j=1}^{3} \tilde{y}_{k,i,j}^2\right) \quad \text{(constant term)} \tag{16}$$

Commit to these:

$$C_{k,*} = r_k^* H_0 + \sum_{i=1}^{N} \alpha_{1,k,i}^* H_i \tag{17}$$

$$D_{k,*} = \tilde{r}_k^* H_0 + \sum_{i=1}^{N} \alpha_{0,k,i}^* H_i \tag{18}$$

where $r_k^* \xleftarrow{\$} [0, S]$ and $\tilde{r}_k^* \xleftarrow{\$} \mathcal{R}_r$.

Send $C_y, \{C_{k,*}, D_{k,x}, D_{k,y}, D_{k,*}\}_{k=1}^{R}$ to verifier.

4.2 Phase 2: Verifier's Challenge

Samples challenges $\gamma_k \xleftarrow{\$} [0, \Gamma]$ for each repetition $k \in [1, R]$. **Send** $\{\gamma_k\}_{k=1}^{R}$ to prover.

4.3 Phase 3: Prover's Response

For each repetition $k \in [1, R]$:

4.3.1 Mask the Witnesses

$$z_{k,i} = \mathsf{mask}_x(\gamma_k \cdot x_i, \tilde{x}_{k,i}) \quad \text{(masked value)} \tag{19}$$

$$z_{k,i,j} = \mathsf{mask}_x(\gamma_k \cdot y_{i,j}, \tilde{y}_{k,i,j}) \quad \text{(masked decomposition)} \tag{20}$$

where $\mathsf{mask}_x(v, r)$ outputs $v + r$ if $v + r \in [0, (B\Gamma + 1)L_x]$, else $\perp$.

4.3.2 Mask the Randomness

$$t_{k,x} = \mathsf{mask}_r(\gamma_k r_x, \tilde{r}_{k,x}) \tag{21}$$

$$t_{k,y} = \mathsf{mask}_r(\gamma_k \cdot r_y, \tilde{r}_{k,y}) \tag{22}$$

$$t_k^* = \mathsf{mask}_r(\gamma_k \cdot r_k^*, \tilde{r}_k^*) \tag{23}$$

4.3.3 Abort Handling

If any mask ($z_{k,i}$, $z_{k,i,j}$, $t_{k,x}$, $t_{k,y}$, t_k^*) returns $\perp$, the prover must restart the masking for that repetition k:

1. Discard all masks $\{z_{k,i},\ z_{k,i,j},\ t_{k,x},\ t_{k,y},\ t_k^*\}$.
2. Resample new randomness $\{\tilde{x}_{k,i}, \tilde{y}_{k,i,j}, \tilde{r}_{k,x}, \tilde{r}_{k,y}, \tilde{r}_k^*\}$.
3. Recompute masked values via mask_x and mask_r.
4. Repeat the abort check.

This rejection-sampling ensures both the hiding property and the numeric bounds.

Send $\{z_{k,i,j}, z_{k,i}, t_{k,x}, t_{k,y}, t_k^*\}_{k\in[1,R], i\in[1,N], j\in[1,3]}$ to verifier.

4.4 Phase 4: Verifier's Verification

For each repetition $k \in [1, R]$:

4.4.1 Check 1: Commitment Consistency

Verify:

$$D_{k,x} + \gamma_k C_x \stackrel{?}{=} t_{k,x} G_0 + \sum_{i=1}^{N} z_{k,i} G_i \tag{24}$$

$$D_{k,y} + \gamma_k C_y \stackrel{?}{=} t_{k,y} G_0 + \sum_{i=1}^{N} \sum_{j=1}^{3} z_{k,i,j} G_{i,j} \tag{25}$$

4.4.2 Check 2: Polynomial Degree

Compute:

$$f^*_{k,i} = 4 z_{k,i} (\gamma_k B - z_{k,i}) + \gamma_k^2 - \sum_{j=1}^{3} z^2_{k,i,j} \tag{26}$$

Verify:

$$D_{k,*} + \gamma_k C_{k,*} \stackrel{?}{=} t^*_k H_0 + \sum_{i=1}^{N} f^*_{k,i} H_i \tag{27}$$

If the three-square decomposition holds, then the polynomials $f^*_{k,i}$ should have degree exactly 1 in γ_k. Suppose the difference polynomial is non-zero of degree d, it can only vanish on at most d points of S. By the Schwartz-Zippel lemma,

$$\Pr_{\gamma \leftarrow S}\big[g(\gamma) = 0\big] \;\le\; \frac{d}{|S|} = \frac{d}{\Gamma + 1}$$

which ensures soundness by detecting non-zero difference polynomials with high probability.

For all $i \in [1, N], j \in [1, 3], k \in [1, R]$:

4.4.3 Check 3: Range Verification

Verify:

$$z_{k,i}, \;\; z_{k,i,j} \stackrel{?}{\in} [0, \;(B\Gamma + 1) L_x] \tag{28}$$

Accept iff all checks succeed for all repetitions $k \in [1, R]$.

5 Hash Function Optimizations

The hash function $\mathsf{Hash} : \{0,1\}^* \to \{0,1\}^{2\lambda}$ enables two optimizations.

5.1 Communication Optimization

In the basic protocol, Phase 1 requires the prover to send $4R$ group elements

$$\mathcal{D} = \{C_{k,*}, D_{k,x}, D_{k,y}, D_{k,*}\}_{k=1}^{R} \tag{29}$$

The prover can replace the commitment transmission with a compact hash through the following process:

1. Compute hash $\Delta \leftarrow \mathsf{Hash}(\mathcal{D})$
2. Send compressed message (C_y, Δ) instead of $(C_y, \mathcal{D})$
3. The verifier recomputes $\mathcal{D}$ during verification and checks $\mathsf{Hash}(\mathcal{D}) \stackrel{?}{=} \Delta$

Communication savings:

$$\text{Original size} \quad 4R \times 32\text{-}48 \text{ bytes} \tag{30}$$

$$\text{Optimized size} \quad \frac{2\lambda}{8} \text{ bytes} = 32 \text{ bytes (for } \lambda = 128) \tag{31}$$

The hash optimization preserves security through collision resistance. It would be impossible for any adversary attempting to forge different commitments $\mathcal{D}' \neq \mathcal{D}$ with the same hash $\mathsf{Hash}(\mathcal{D}') = \mathsf{Hash}(\mathcal{D}) = \Delta$.

5.2 Fiat-Shamir Transformation

The hash function enables conversion to a non-interactive zero-knowledge proof via the Fiat-Shamir transformation.

Instead of receiving challenges from the verifier, the prover computes them deterministically. The hash function is applied to generate a seed, which is then used to derive individual challenges:

$$\text{seed} \leftarrow \mathsf{Hash}(\text{statement} \| C_y \| \Delta \| \text{context}) \tag{32}$$

$$\gamma_k \leftarrow \mathsf{Hash}(\text{seed} \| k) \bmod (\Gamma + 1) \quad \text{for } k \in [1, R] \tag{33}$$

where context includes any additional protocol parameters or public inputs, and $\Delta = \mathsf{Hash}(\mathcal{D})$ when using hash optimization.

The modified protocol proceeds as follows:

1. Prover computes first message $(C_y, \mathcal{D})$
2. Prover generates challenges $\{\gamma_k\}_{k=1}^{R}$ using Hash
3. Prover computes response $\{z_{k,i,j}, z_{k,i}, t_{k,x}, t_{k,y}, t_k^*\}$
4. Prover outputs proof $\pi = (C_y, \Delta, \text{response})$
5. Verifier recomputes challenges using Hash and verifies.

The Fiat-Shamir transformation is provably secure in the Random Oracle Model (ROM), where Hash is modeled as a truly random function. The transformation preserves the soundness and zero-knowledge properties of the original interactive protocol.

6 Cost Analysis

The computational complexity of $\mathsf{Sharp_{GS}}$ can be expressed as:

$$\mathcal{O}(R \cdot N)_{\mathbb{G}_{\mathsf{com}}} + \mathcal{O}(R \cdot N)_{\mathbb{G}_{\mathsf{3sq}}} + \mathcal{O}(R \cdot N)_{\mathbb{F}_p} + \mathcal{O}(R \cdot N)_{\mathbb{F}_q}$$

where:

- $\mathbb{G}_{\mathsf{com}}$ represents group operations in the commitment group (256-333 bits)
- $\mathbb{G}_{\mathsf{3sq}}$ represents group operations in the three-squares group (256-411 bits)
- $\mathbb{F}_p$ and $\mathbb{F}_q$ represent field operations in their respective finite fields

7 Security Guarantees

Let p be the prime order of our group and let N, D be positive integers satisfying

$$N \cdot D < \frac{p}{2}.$$

We define

$$\mathbb{Q}_{N,D} \;=\; \left\{ \tfrac{n}{d} \in \mathbb{Q} \;\middle|\; |n| \leq N,\ 1 \leq d \leq D \right\}.$$

Then for any $x \in \mathbb{Z}_p$ which admits a representative in $\mathbb{Q}_{N,D}$, that representative is unique. This unique fraction is denoted by

$$[x]_{\mathbb{Q}} \;=\; \frac{n}{d} \;\in\; \mathbb{Q}_{N,D}, \quad \text{characterized by} \quad n \;\equiv\; x \cdot d \pmod{p}.$$

$\mathsf{Sharp_{GS}}$ proves that each x_i has a rational representative $[x_i]_{\mathbb{Q}}$ in $\left[-\frac{1}{4}B, B + \frac{1}{4}B\right]_{\mathbb{Q}}$ with numerator bounded by $K = (B\Gamma + 1)L_x$ and denominator bounded by Γ. This relaxed binding enables several optimizations that would be impossible with integer binding.

7.1 Group Switching Optimization

By accepting rational representatives, $\mathsf{Sharp_{GS}}$ can work over finite field groups $\mathbb{G}_{\mathsf{com}}$ and $\mathbb{G}_{\mathsf{3sq}}$ instead of hidden order groups. This allows independent optimization of group sizes for commitments (256-333 bits) versus decomposition proofs (256-411 bits).

7.2 Polynomial Verification Technique

The three-square decomposition verification in Phase 4 uses polynomial relationships over finite fields rather than exact arithmetic in hidden order groups, which works efficiently over $\mathbb{Z}_p$ when only rational binding is required.

7.3 Efficient Batching

Pedersen multi-commitments $C_y = r_y G_0 + \sum_{i,j} y_{i,j} G_{i,j}$ work efficiently with rational representatives, enabling batch proofs for N values simultaneously. Integer binding would require separate commitments.

7.4 Simplified Masking Protocol

The masking scheme in Phase 3 operates over finite fields with bounds $z_{k,i} \in [0, (B\Gamma + 1)L_x]$ that correspond directly to rational representative numerator bounds. This enables uniform rejection sampling instead of Gaussian sampling required for integer binding schemes.

8 Other Sharp Algorithm Variants

8.1 $\mathsf{Sharp}_{\mathsf{SO}}^{\mathsf{Po}}$

$\mathsf{Sharp}_{\mathsf{SO}}^{\mathsf{Po}}$ optimizes runtime performance through a fractional shortness test, enabling single-scalar repetitions regardless of batch size. It is designed specifically for standard 256-bit elliptic curves.

8.2 $\mathsf{Sharp}_{\mathsf{RSA}}$

$\mathsf{Sharp}_{\mathsf{RSA}}$ augments the basic construction with RSA groups to achieve standard soundness, binding provers to integer values rather than rationals. Requires trusted RSA parameter generation.

8.3 $\mathsf{Sharp}_{\mathsf{CL}}$

$\mathsf{Sharp}_{\mathsf{CL}}$ uses class groups of imaginary quadratic fields to bind provers to dyadic rationals of form $m/2^k$. Maintains transparent setup.

8.4 $\mathsf{Sharp}_{\mathsf{HO}}$

$\mathsf{Sharp}_{\mathsf{HO}}$ provides a general framework for augmenting any Sharp variant with hidden order groups. Supports RSA groups, class groups, and other hidden order instantiations.

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