LESSON 9
RECURRENCE RELATIONS
NOTES

$0! = 1$ ← initial condition

$\underbrace{n! = n(n-1)!}_{\text{recurrence relation}}$ for $n \geq 1$ $\Rightarrow$ $1! = 1(0!) = 1 \cdot 1 = 1$

$2! = 2(1!) = 2 \cdot 1 = 2$

$3! = 3(2!) = 3 \cdot 2 \cdot 1 = 6$

etc.

$F_1 = 1, F_2 = 1$ ← initial conditions

$\underbrace{F_n = F_{n-1} + F_{n-2}}$ for $n \geq 3$

recurrence relation

$F_1 = 3 \quad F_2 = 4$

$F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

HOW TO USE TO SOLVE A PROBLEM

Q. Knowing e_{n-1} how do we find e_n?

① Find the #edges in K_n: complete graph on n vertices.
e_n

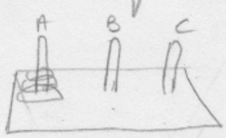
$K_{n-1} \rightarrow K_n$?

VIEWPOINT: 1 new vertex, $n-1$ new edges.

initial condition? $e_1 = 0 \quad K_1: \bullet$

$e_{n-1} + n - 1 = e_n$ for $n \geq 2$

② Towers of Hanoi.



Move all disks from spoke A to spoke C?

1 disk
2 disks
3 disks
4 disks
5 disks

- One disk at a time
- No larger disk can sit on top of a smaller disk.

n disks, min # of moves?

Experiment 1, 2, 3, 4 as above.

How can we solve this recursively?

VIEWPOINT m_n

$m_n \begin{cases} \text{① Get all top } n-1 \text{ disks from A to B somehow: } m_{n-1} \\ \text{② Move largest disk from A to C: } 1 \\ \text{③ Move all top } n-1 \text{ disks from B to C.: } m_{n-1} \end{cases}$

$m_n = m_{n-1} + 1 + m_{n-1} = 2m_{n-1} + 1.$

Initial condition? $m_1 = 1$

$m_1 = 1$
$m_2 = 3$
$m_3 = 7$
$m_4 = 15$
$m_5 = 31$
etc.

3. Grain elevator.

Harvest starts $g_0 = 600$ tons of corn already in elevator

200 tons rec'd during week

30% shipped out at beginning of each week.

How many tons of corn on hand at end of each week?

Week 0: $g_0 = 600$ tons at start

Week n: g_n = # tons at end of week n−1 − 0.3·(# tons at end of week n−1 + 200

$$g_n = g_{n-1} - 0.3\,g_{n-1} + 200$$

$$\boxed{\begin{array}{l} g_n = 0.7\,g_{n-1} + 200 \\ \text{with initial condition } g_0 = 600 \end{array}}$$

Recursion relation

4. Current status or projected status of financial accounts.

New house $200,000

- down payment $25,000
- 30 year mortgage.
- interest on unpaid balance 1% per month.
- monthly payments $1800.

How do we know how much is owed after n months?

b_n = balance owed at end of n^{th} month:

= balance owed at end of previous month + interest on balance owed at end of previous month − monthly payment

$$\boxed{\begin{array}{l} b_n = b_{n-1} + 0.01\,b_{n-1} - 1800 = 1.01\,b_{n-1} - 1800 \\ b_0 = 175{,}000. \end{array}}$$

Recursion relation

These are called discrete dynamical systems.

Can we find a formula for these that simply gives b_n as a function of n, rather than as a function of b_{n-1}?

Answer is YES.

LESSON 4B
RECURSION
METHOD OF ITERATION

STRATEGY

A recursion relation calculates s_n for each n sequentially starting from the initial condition. We may want an explicit formula for s_n that does not require all these steps.

We can:

1) make an educated guess at the formula by calculating the first few expressions for $s_1, s_2, s_3, \ldots$ and looking for a pattern.

2) Prove our guess is correct (or not) using math induction.

EXAMPLES

(5) Towers of Hanoi $\qquad m_n = 2m_{n-1} + 1$ for $n \geq 2$, $m_1 = 1$

$m_1 = 1$

$m_2 = 2(1) + 1 = 2 + 1$

$m_3 = 2(2+1) + 1 = 2^2 + 2 + 1$

$m_4 = 2(2^2+2+1) + 1 = 2^3 + 2^2 + 2 + 1$ (DON'T SIMPLIFY)

Guess: $\quad m_n = 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2 + 1 = \dfrac{2^n - 1}{2-1} = 2^n - 1$

(Recall the formula for the sum of a geometric series:

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x-1}.)$$

Prove: If $m_1, m_2, m_3, \ldots$ satisfy the recurrence relation $m_n = 2m_{n-1} + 1$ for $n \geq 2$ and $m_1 = 1$ then $m_n = 2^n - 1$ for every $n \geq 1$.

PROOF BY INDUCTION

Step 1) $S(1): m_1 = 1 \Rightarrow m_1 = 2^1 - 1$ because $2^1 - 1 = 2 - 1 = 1 = m_1$.

Step 2) Assume $S(k): m_k = 2^k - 1$ $\qquad$ Prove $S(k+1): m_{k+1} = 2^{k+1} - 1$

$m_{k+1} = 2m_k + 1 = 2(2^k - 1) + 1 = 2^{k+1} - 2 + 1 = 2^{k+1} - 1$

$\therefore S(k) \Rightarrow S(k+1)$.

Step 3) $\therefore S(n): m_n = 2^n - 1$ for every $n \geq 1$.

FILL IN THE BLANKS

⑥ $\boxed{\#\text{Edges in } K_n}$ $e_n = e_{n-1} + (n-1)$ for $n \geq 2$ and $e_1 = 0$.

$e_1 = 0$
$e_2 = e_1 + 1 = 1$
$e_3 = e_2 + 2 = 1+2$
$e_4 = e_3 + 3 = 1+2+3$
$e_5 = 1+2+3+4$

Guess: $e_n = 1+2+3+\cdots+(n-1)$
$= \dfrac{(n-1)n}{2}$

Prove: If $e_1, e_2, e_3, \ldots$ satisfy the recurrence relation
$e_n = e_{n-1} + (n-1)$ for $n \geq 2$ and $e_1 = 0$
then $e_n = \dfrac{(n-1)n}{2}$ for every $n \geq 1$.

$\boxed{\text{PROOF BY INDUCTION}}$

Step 1) $S(1)$: $e_1 = 0$ & $e_1 = \frac{0 \cdot 1}{2} = 0$

Step 2) Assume $S(k): e_k = \dfrac{(k-1)k}{2}$ Prove $S(k+1): e_{k+1} = \dfrac{k(k+1)}{2}$.

$S(k+1)$: $e_{k+1} = e_k + k = \dfrac{(k-1)k}{2} + k = \dfrac{k^2-k+2k}{2} = \dfrac{k^2+k}{2} = \dfrac{k(k+1)}{2}$

$\therefore S(k) \Rightarrow S(k+1)$. This completes the induction step.

Step 3) $\therefore S(n): e_n = \dfrac{(n-1)n}{2}$ for every $n \geq 1$.

⑦ $\boxed{\text{First order linear difference equations with constant coefficients}}$

$s_n = a s_{n-1} + b$, a, b constants; $a \neq 0$, s_0 given.

(e.g. Towers of Hanoi: $a = 2$, $b = 1$)

$s_0 = s_0$
$s_1 = a s_0 + b$
$s_2 = a s_1 + b = a(a s_0 + b) + b = a^2 s_0 + ab + b$
$s_3 = a s_2 + b = a(a^2 s_0 + ab + b) + b = a^3 s_0 + a^2 b + ab + b$
$\vdots$

Guess: $s_n = a^n s_0 + a^{n-1} b + a^{n-2} b + \cdots + a^2 b + ab + b$
$= a^n s_0 + b\,(\underbrace{a^{n-1} + a^{n-2} + \cdots + a^2 + a + 1})$

if $a \neq 1$ this is $\dfrac{a^n - 1}{a-1}$; if $a = 1$, this is n

$\boxed{s_n = a^n s_0 + \left(\frac{a^n-1}{a-1}\right) b \quad \text{if } a \neq 1}$ $\boxed{s_n = s_0 + bn \quad \text{if } a = 1}$

We can use this formula to solve recurrence relations.

$P(n)$:

Prove: If $s_1, s_2, s_3, \ldots$ satisfy $s_n = as_{n-1} + b$ with $a \neq 1$ and s_0 is given, then $s_n = a^n s_0 + \left(\frac{a^n - 1}{a-1}\right) b$ for all $n \geq 0$.

Step 1) $P(0)$: $s_0 = s_0 + \frac{1-1}{a-1} b = s_0$

Step 2) Assume $P(k)$: $s_k = a^k s_0 + \left(\frac{a^k - 1}{a-1}\right) b$ prove $P(k+1)$: $s_{k+1} = a^{k+1} s_0 + \left(\frac{a^{k+1} - 1}{a-1}\right) b$

$s_{k+1} = as_k + b$ by the recursive defn.

$$s_{k+1} = a\left(a^k s_0 + \left(\frac{a^k - 1}{a-1}\right) b\right) + b = a^{k+1} s_0 + a\left(\frac{a^k - 1}{a-1}\right) b + b$$

$$= a^{k+1} s_0 + b\left(a\frac{(a^k - 1)}{a-1} + 1\right)$$

$$= a^{k+1} s_0 + b\left(\frac{a(a^k - 1) + (a-1)}{a-1}\right)$$

$$= a^{k+1} s_0 + b\left(\frac{a^{k+1} \cancel{-a} + \cancel{a} - 1}{a-1}\right)$$

$\therefore P(k) \Rightarrow P(k+1)$.

Step 3. Therefore, this formula works for every $n \geq 1$.

Write out a proof by mathematical induction for the case $a = 1$ here now.

Example. Back to Example 4 with the mortgage on a house.

Amount owing after n months is

$$\begin{cases} b_n = 1.01 b_{n-1} - 1800 & \text{for } n \geq 1 \\ b_0 = 175{,}000 \end{cases}$$

This is of the form $ab_{n-1} + b$ where $a = 1.01$ and $b = -1800$

Using the formula we just derived, after 20 years they still owe

$$s_n = a^n s_0 + \left(\frac{a^n - 1}{a-1}\right) b \quad \text{with } n = 240 \text{ months}$$

$$s_{240} = (1.01)^{240}(175{,}000) + \frac{(1.01)^{240} - 1}{1.01 - 1}(-1800) = \$125{,}537.23$$