LESSON 13
PROPositional Logic
§1.1

§1.2 Logic circuits only

§1.3

Definition

A proposition is a declarative sentence that is either true or false (even if we don't know which) but not both.

Examples:

(p. 12 #2)

Washington DC is the capital of the USA. T

Toronto is the capital of Canada. F

$1+1=2$ T

$2+2=3$ F

New propositions from old

We can combine propositions to form new propositions

(*) using connectives:

"not" $\neg p$ negation of p.

"and" $p \wedge q$ disjunction

"or" $p \vee q$ disjunction (either p or q or both)

"exclusive or" $p \oplus q$ exclusive or (either p or q but not both)

(*) using conditionals

"implies" $p \to q$ conditional (if p then q)

"iff" $p \leftrightarrow q$ bi-conditional (p implies q and q implies p).

$(p \to q) \wedge (q \to p)$

Examples.

p: My smartphone has at least 32GB of memory

$\neg p$: It is not the case that my smartphone has at least 32GB of memory

or My smartphone does not have at least 32GB of memory.

or My smartphone has less than 32GB of memory.

p: It is raining today
q: I will go for a walk.

$p \wedge q$ It is raining today and I will go for a walk.

(p.12 #4)

$p \vee q$ Either it is raining today or I will go for a walk (or both)

$p \oplus q$ Either it is raining today or I will go for a walk but not both. (p.14 #20)

$(\neg p) \rightarrow q$ If it is not raining today, I will go for a walk.

$\neg p \leftrightarrow q$ If it is not raining today, I will go for a walk and if I go for a walk, then it is not raining.

or I'm going for a walk today, but not if it's raining.

(p13 #8)

(p.13 #14)

$\boxed{\text{True or False}}$

Each connective and conditional has a truth table that tells us whether the proposition is true.
Read left to right.

p	¬p
T	F
F	T

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Need to explain. "If I go to town today, I will buy you some ice cream." $p \rightarrow q$

If I don't go to town, I haven't promised you anything. Whether I buy you ice cream or not, I have not broken my promise.

We use truth tables to determine T or F for compound propositions:

p	q	$p\to q$	$q\to p$	$p\leftrightarrow q=(p\to q)\wedge(q\to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(p15 #32.)

Application to Logic Circuits

We find microprocessors in many places these days. Cars, digital watches, electronic games, toasters, etc. These tiny electronic devices control the larger machines they are embedded in by responding to various inputs according to a preset pattern. They do this by means of circuits in them, using logic circuits, which consist of a combination of "logic gates".

Example (From Dossey, et al: Discrete Mathematics, 3rd edition, Addison-Wesley 1997, p. 474-475.

The sensitive electronic equipment in the control room of a recording studio needs to be protected from both high temperatures and excess humidity. An air conditioner is provided which must go on whenever either the temperature exceeds 80° or the humidity exceeds 50%. What is required is a control device that has two inputs, one coming from a thermostat and one from a humidistat, and one output going to the air conditioner. It must perform the function of turning on the air conditioner if it gets a yes signal from either of the input devices, as summarized in the following table.

Temperature > 80°?	*Humidity > 50%?*	*Air conditioner on?*
no	no	no
no	yes	yes
yes	no	yes
yes	yes	yes

We will follow the usual custom of using *x* and *y* to label our two inputs and 1 and 0 to stand for the input or output signals yes and no, respectively. Thus.

x and y can assume only the values 0 and 1; such variables are called **Boolean variables.** These conventions give our table a somewhat simpler form.

x	y	*Output*
0	0	0
0	1	1
1	0	1
1	1	1

The required device is an example of a **logical gate,** and the particular one whose working we have just described is called an **OR-gate,** since its output is 1 whenever either x *or* y is 1. We will denote the output of an OR-gate with inputs x and y by $x \vee y$, so that

$$x \vee y = \begin{cases} 1 & \text{if } x = 1 \text{ or } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We will not delve into the internal workings of the devices we call logical gates, but merely describe how they function. A logical gate is an electronic device that has either 1 or 2 inputs and a single output. These inputs and output are in one of two states, which we denote by 0 and 1. For example, the two states might be a low and high voltage.

Logical gates are represented graphically by standard symbols established by the Institute of Electrical and Electronics Engineers. The symbol for an OR-gate is shown in Figure 9.1.

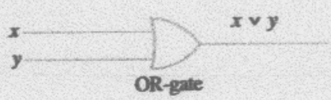


FIGURE 9.1

Here are the AND-gate and the NOT-gate.

AND-gate

NOT-gate

$\boxed{\text{Example}}$

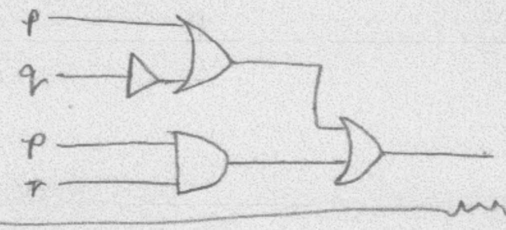
A printer attached to a PC prints only when the "on-line" button has been pressed and a paper sensor says there is paper in the printer

$\boxed{\text{Example}}$

A rental truck is equipped with a governor. If the speedometer exceeds 65 mph, the ignition in the truck shuts off.

More complicated "combinatorial" circuits are built up from these. Each circuit will have a corresponding truth table.

Exercise



Determine the expression of the output and its truth table.

Logical Equivalence

Tautology A proposition that is true for every possible assignment of truth values to its component propositions.

$p \vee [(p \wedge \neg q) \rightarrow r]$

$p \vee \neg p$

Construct a truth table and see that you get T for every possible values of T and F to p, q and r.

Contradiction. A proposition that is false for every possible assignment of truth values to its component propositions.

$(p \wedge \neg q) \wedge (\neg p \vee q)$

$p \wedge \neg p$

Two propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.

we write $p \equiv q$.

This will be true if the truth tables for p and q give the same values.

Example

De Morgan's laws

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

$p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

And finally,

Given $p \to q$: $q \to p$ is called the <u>converse</u> of $p \to q$.

$\neg q \to \neg p$ is called the <u>contrapositive</u> of $p \to q$.

$\neg p \to \neg q$ is called the <u>inverse</u> of $p \to q$.

If it isn't raining today, I am going to the beach.

Converse. If I go to the beach today, then it isn't raining.

Contrapositive. If I don't go to the beach today, it must be raining.

Inverse: If it is raining today, then I am not going to the beach.

p15 #28

Only the <u>contrapositive</u> is logically equivalent to proposition.

$$p \to q \equiv \neg q \to \neg p.$$

(You check this with a truth table.)