LESSON 14
Predicates and Quantifiers
§1.4 and §1.5

We cannot use propositional logic to handle all statements we would like to make.

We handle statements involving variables by predicates and quantifiers.

"$x > 3$" x is the variable
"is greater than 3" is the predicate P
$P(x)$ is the value of the propositional function P at x.
When we assign a value to x, the statement $P(x)$ becomes a proposition and has a truth value.
$P(4)$ is the proposition $4 > 3$ which is true.
$P(2)$ is the proposition $2 > 3$ which is false.

$x = y + 3$ $Q(x, y)$ where x and y are variables and Q is the predicate.
$Q(1,2)$ is the statement $1 = 2 + 3$ which is false
$Q(3,0)$ is the statement $3 = 0 + 3$ which is true.

In general,

$$\underset{\substack{\uparrow \\ \text{propositional} \\ \text{function } P}}{P}(\underbrace{x_1, x_2, \ldots, x_n}_{n \text{ variables}})$$

Example: "If $x > 0$ then $x := x + 1$" in a computer program.
The current value of x is inserted into $P(x)$ "$x > 0$".
If $P(x)$ is true for this value of x, $x := x + 1$ is carried out and the value of x is increased by 1.
If $P(x)$ is false, $x := x + 1$ is not carried out and x is not changed.

Quantification of $P(x)$

Defn. "$P(x)$ for all values in the domain"

is called the universal quantification of the propositional function P. and we write this statement

$\forall x P(x)$ "for all x, $P(x)$"

An element for which $P(x)$ is false is called a counterexample of $\forall x P(x)$.

$\forall x P(x)$ is true if $P(x)$ is true for every x.

false if there is an x for which $P(x)$ is false.

example: $P(x)$: $x+1>x$. $\forall x P(x)$ is true.

$P(x)$: $x<2$ $\forall x P(x)$ is false

counterexample $x=3$.

If the domain is finite: $x_1, x_2, \cdots, x_n$ then $\forall x P(x)$ is just

$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$.

Example. $P(x)$ is "$x^2<10$" and the domain is $\{1,2,3,4\}$

$\forall x P(x)$ is $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$.

$\downarrow$

$4^2=16 \nless 10$ so $\forall x P(x)$ is false.

Example $N(x)$: "Computer x is connected to the campus network"

Domain: all computers on campus

$\forall x N(x)$: every computer on campus is connected to the network.

__Defn.__ "There exists an element x in the __domain__ such that $P(x)$" is called the __existential__ quantification of the propositional function $P(x)$. We write $\exists x P(x)$.

"there exists x s.t. $P(x)$."

$\exists x P(x)$ is true if there is an x such that $P(x)$ is true.
false if $P(x)$ is false for every x.

__Example.__ $P(x)$: "$x>3$" $\exists x P(x)$ is true because $4>3$ for example.

$Q(x)$: "$x=x+1$" is false for every x in the real numbers.
So $\exists x P(x)$ is false.

Convention: all domains must be nonempty.

If the domain is finite: $x_1, x_2, \ldots, x_n$ then $\exists x P(x)$ is just

$$P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$$

Again look at $P(x)$: "$x^2>10$" with domain $\{1,2,3,4\}$.

$\exists x P(x)$ is $P(1) \vee P(2) \vee P(3) \vee P(4)$.

Since $4^2>10$ ($P(4)$) is true, therefore $\exists x P(x)$ is true.

__Logical equivalence__

Two statements involving quantifiers are __logically equivalent__ if they have the same truth value no matter which predicates are substituted & no matter which domain is used. $S \equiv T$.

__Example__ $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$.

$$\exists x (P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$$

Proof of the first:

Show if $\forall x(P(x) \wedge Q(x))$ is true then $\forall x P(x) \wedge \forall x Q(x)$ is true and vice versa.

Let $P(x)$ and $Q(x)$ satisfy $\forall x(P(x) \wedge Q(x))$ is true.

then if a is in the domain

$P(a) \wedge Q(a)$ is true.

$\therefore$ $P(a)$ is true and $Q(a)$ is true

Since $P(a)$ is true for every elt in the domain, $\forall x P(x)$ is true.

Similarly $\forall x Q(x)$ is true.

$\therefore$ $\forall x P(x) \wedge \forall x Q(x)$ is true.

The converse works similarly.

IMPORTANT

How to negate quantified expressions

DeMorgan's Laws for Quantifiers

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

It is not the case that $P(x)$ is true for all x iff There exists an x such that $P(x)$ is not true.

eg. Every student in this class has passed college algebra.

Negation: Not every student has passed college algebra.

There is at least one student who has not passed college algebra.

$$\neg \exists x P(x) \equiv \forall x \neg P(x).$$

Please peruse Examples 20, 21, 23, 24 in the textbook. pp 47-49

Nested quantifiers

$\forall x \exists y (x+y=0)$ is $\forall x Q(x)$ where $Q(x)$ is $\exists y P(x,y)$
and $P(x,y)$ is $x+y=0$.

Examples

$\forall x \forall y (x+y=y+x)$ says $x+y=y+x$ for all real numbers x and y -
Commutative rule for addition of numbers.

$\forall x \exists y (x+y)=0$ says for every real number x there is y
such that $x+y=0$. Every real number has an additive inverse.

Translate into English:

$$\forall x \forall y ((x>0) \wedge (y<0) \rightarrow (xy<0))$$

Table 1. p. 60 When true when false for 2 quantifiers

Practice translating math statements into statements involving nested quantifiers, and vice versa.

Negation of nested quantifiers

Negate $\forall x \exists y (xy=1)$ so that no negation precedes a quantifier.

$$\neg \forall x \exists y (xy=1) \equiv \exists x \neg \exists y (xy=1) \equiv \exists x \forall y \neg (xy=1) \equiv \exists x \forall y (xy \neq 1)$$

using De Morgan's laws,