LESSON 11

BASIC COUNTING TECHNIQUES

§6.1 and §6.2.
Notes

**Multiplication Principle**

If a procedure has two steps, with $n_1$ choices at the first step and, for each of these $n_1$ choices there are $n_2$ choices on the second step, then there are $n_1 \cdot n_2$ ways of completing the procedure.


$3 \cdot 3 = 9$ ways to complete the procedure.

This immediately generalizes to $n$ steps $n_1, n_2, \ldots, n_k$. In that case, there are $n_1 \cdot n_2 \cdot n_3 \cdots n_k$ ways of completing the procedure.

**Examples**

1. A bit is a binary digit 0 or 1 (on/off)

An $n$-bit string is an ordered sequence of $n$ bits.

How many 8 bit strings are there? $\underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2}$

There are $2^8$ possible 8 bit strings, two choices at each step from left to right.

There are $2^n$ possible $n$-bit strings

② How many 5 digit strings are there if we use decimal digits $\{0,1,2,\dots,9\}$?

$\underline{10}\ \underline{10}\ \underline{10}\cdot\underline{10}\ \underline{10} = 10^5 = 100{,}000$ strings

If we don't allow repetition, there are $\underline{10}\,\underline{9}\,\underline{8}\,\underline{7}\,\underline{6}$ strings.

$= 30240$ possible 5 digit strings.

How many of these start with a 7?

$\underline{1}\cdot\underline{9}\cdot\underline{8}\cdot\underline{7}\cdot\underline{6} = 3024$ ways.

**Sum Rule**

If a task can be done in one of $n_1$ ways or in one of $n_2$ ways, where none of the $n_1$ ways is the same as any of the $n_2$ ways, then the total number of ways in which the task can be done is $n_1 + n_2$.

**Example.** A new committee member needs to be chosen for an MCC committee. Eight people from the Business department qualify, and 4 people from the Math Dept qualify, and no one works for both the Business Dept and the Math Dept. Then there are $8+4=12$ ways to choose a person to serve on this committee.

If $S = A\cup B$ and $A\cap B \neq \phi$, then $|S| = |A| + |B|$

**Generalized Sum Rule**

If $S = A_1 \cup A_2 \cup \dots \cup A_k$ and $A_i \cap A_j = \phi$ for all $i \neq j$,

then $|S| = |A_1| + |A_2| + \dots + |A_k|$

number of elements in $A_k$.

**Example** How many 8 bit strings begin with 1011 or 01?

# of form $1011\,\underset{2}{\_}\,\underset{2}{\_}\,\underset{2}{\_}\,\underset{2}{\_}$ + # of form $\underline{0}\,\underline{1}\,\_\,\_\,\_\,\_\,\_\,\_$

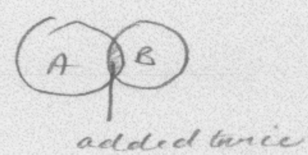
$= 2^4 + 2^6 = 16 + 64 = 80$

Example How many 8 bit strings begin with 1011 or end with 01?

1011 _ _ _ _ or _ _ _ _ _ _ 0 1

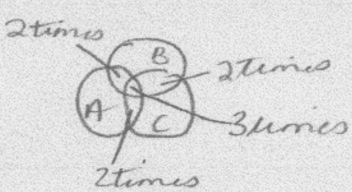
$2^4 + 2^6$ but we will have included those of the form 1011 _ _ 01 twice.

So the answer is $2^4 + 2^6 - 2^2 = 76$.




This is an example of the Inclusion - Exclusion Principle


$|A \cup B| = |A| + |B| - |A \cap B|$ ← number of elements in the set $A \cap B$




$|A \cup B \cup C| = |A| + |B| + |C|$ (too much)

$- |A \cap B| - |B \cap C| - |A \cap C|$ too little

$+ |A \cap B \cap C|$

and so on for 4 sets.

Example How many positive integers less than 2101 are divisible by at least one of the primes 2, 3, 5, 7?

$A_2 = \{$ even integers from 2 to 2100

$A_3 = \{$ multiples of 3 from 3 to 2100

$A_5 = \{$ multiples of 5 from 5 to 2100

$A_7 = \{$ multiples of 7 from 7 to 2100

$$\begin{aligned}
|A_2 \cup A_3 \cup A_5 \cup A_7| &= |A_2| + |A_3| + |A_4| + |A_7| \\
&- |A_2 \cap A_3| - |A_2 \cap A_5| - |A_2 \cap A_7| \\
&- |A_3 \cap A_5| - |A_3 \cap A_7| - |A_5 \cap A_7| \\
&+ |A_2 \cap A_3 \cap A_5| + |A_2 \cap A_3 \cap A_7| \\
&+ A_2 \cap A_5 \cap A_7| + |A_3 \cap A_5 \cap A_7| \\
&- |A_2 \cap A_3 \cap A_5 \cap A_7| \\
&= \frac{2100}{2} + \frac{2100}{3} + \frac{2100}{5} + \frac{2100}{7} \\
&- \frac{2100}{2.3} - \frac{2100}{2.5} - \frac{2100}{2.7} \\
&- \frac{2100}{3.5} - \frac{2100}{3.7} - \frac{2100}{5.7} \\
&+ \frac{2100}{2.3.5} + \frac{2100}{2.3.7} \\
&+ \frac{2100}{2.5.7} + \frac{2100}{3.5.7} \\
&- \frac{2100}{2.3.5.7} \\
&= 1620
\end{aligned}$$

How many are divisible by <u>none</u> of 2, 3, 5 or 7?

$2100 - 1620 = 480$ not that many.

$\boxed{\text{Division Rule}}$

How many ways are there to seat four people around a circular table?

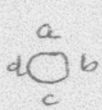
Select one particular seat and call it seat a.

Label the rest of the seats b, c, d proceeding clockwise around the table. Then there are $4.3.2.1$ ways of seating them.

But if everyone stands up and moves to their left one chair and sits down again – that is not really a different arrangement. Similarly if they move 2 seats left or 3 seats left.

So every arrangement in the original calculation has three others that are really the same.

these are same:
abcd
bcda
cdab
dabc




So there are really only

$$\frac{4\cdot 3\cdot 2\cdot 1}{4} = 6 \text{ different arrangements.}$$

**Pigeonhole Principle**

If the number of pigeons is more than the number of pigeonholes, then some pigeonhole must contain at least two pigeons.

**Example** In a group of at least 367 people there must be at least two with the same birthday.

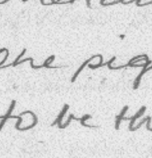
**In general,**

If the number of pigeons is more than $k$ times the number of pigeonholes, then some pigeonhole contains at least $k+1$ pigeons.

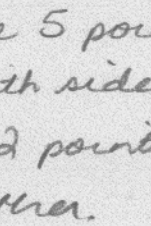
Proof. In the worst case, the pigeons will come home and fill up the pigeonholes so that there are $k$ pigeons in each hole. If there are more pigeons than $k$ times the number of pigeonholes, then there will be some pigeons still not housed. When these go into holes, those holes will contain at least $k+1$ pigeons.

Examples.

* How many distinct integers must be chosen to ensure there are at least 10 having the same remainder when divided by 7?
  The pigeons are integers and we take the pigeonholes to be the following sets:
  There are 7 possible remainders on division by 7, namely $0, 1, 2, 3, 4, 5, 6$.
  For $j = 0, 1, 2, 3, 4, 5, 6$, we define $A_j$ to be the set of all integers that have remainder $j$ on division by $j$. So here are 7 pigeonholes $A_0, A_1, A_2, A_3, A_4, A_5, A_6$.
  So to ensure at least 10 having the same remainder, we need to choose $9 \times 7 + 1 = 64$ integers.

* 


  If we choose 5 points inside an equilateral triangle with side 1 inch, then there are at least 2 points that lie within $\frac{1}{2}$ inch of each other.
  Why? Divide the triangle into 4 equal triangles.



5 points
4 pigeonholes
Points within a small triangle are at most $\frac{1}{2}$ inch from each other.

Caution

How many 5 digit strings of numbers from the set $S = \{1,2,3,4,5,6,7,8\}$ contain both a 1 and a 2?

We cannot use our usual procedure

_ _ _ _ _ moving left to right

because the number of choices at each step depends on the particular choices made at previous steps.

Here, if the first 3 digits chosen are 231, then there are 5 choices for the 4th digit: 4, 5, 6, 7, or 8.

But if the first 3 digits chosen are 567, then there are only 2 choices for the 4th digit: 1 or 2.

∴ Choosing how to fill the blanks from left to right does not work.

INSTEAD: Choose first which blanks 1 and 2 can go in. Then fill the remaining slots with digits 3, 4, 5, 6, 7, 8.

5 ways to choose the position for 1
4 ways left to choose the position for 2
6 digits to put in the first empty position
5 digits to " " " " second empty position
4 digits to put in the last empty position

∴ $5 \cdot 4 \cdot 6 \cdot 5 \cdot 4 = 2400$ choices that contain both 1 and 2.