LESSON 12

Permutations and Combinations

§6.3 and §6.4.

NOTES

Set S with n elements

$n!$ permutations (arrangements)

$$P(n,r) = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

\# ways of choosing an <u>ordered list</u> of r elements from S <u>without</u> repetition

"PERMUTATIONS"

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

\# subsets of S of size r

\# ways of choosing r elements from S without regard to order

"COMBINATIONS"

Example How many ways can we choose a President and a Secretary from a set of 6 people?

{Jeff, Seth, Naha, Redda, Kassaw, Derartu}

Order matters: Naha President, Derartu Secretary $\neq$ Derartu President, Naha Secretary

$$\therefore\ P(n,r) = P(6,2) = \frac{6!}{4!} = 6.5 = 30$$

How many ways can we choose a 2-person committee from a set of 6 people?

Order doesn't matter {Seth, Kassaw} = {Kassaw, Seth}

$$C(n,r) = C(6,2) = \binom{6}{2} = \frac{6!}{2!4!} = \frac{6.5}{2} = 15.$$

Note

$\binom{n}{r}$ is pronounced "n choose r"

Where do these formulas come from?

$P(n,r)$ = # ways of choosing an ordered list of r elements from n elements

$$= \underbrace{n \cdot n-1 \cdot n-2 \cdot \ldots \cdot n-r+1}_{r \text{ elements}} = \frac{n!}{(n-r)!}$$ multiplication rule from the previous lesson

$C(n,r)$ = # ways of choosing a subset of r elements from n elements.

If we go ahead and choose an ordered list of r elements as above, there are $P(n,r) = \frac{n!}{(n-r)!}$ ways of doing that. But if the order in each list doesn't matter then there is a lot of duplication here. For the list $a_1 a_2 a_3 \ldots a_r$

there are $r!$ ways of rearranging that list. All of those rearrangements give the same set $\{a_1, a_2, a_3, \ldots, a_r\}$. So dividing $P(n,r)$ by $r!$ will give us the number of subsets of S with r elements.

Therefore, $C(n,r) = \dfrac{n!}{r!(n-r)!}$

$\boxed{\text{Identities}}$

① $C(n,r) = C(n,n-r)$ Why? $C(n,r) = \dfrac{n!}{r!(n-r)!} = \dfrac{n!}{(n-r)!\,r!} = C(n,n-r)$.

② $C(n,r) = C(n-1,r-1) + C(n-1,r)$ for $n = 1,2,3,\ldots$ and $r = 0,1,2,\ldots,n$

Proof. $S = \{a_1, a_2, \ldots, a_{n-1}, a_n\} = \{a_1, a_2, \ldots, a_{n-1}\} \cup \{a_n\}$

subsets of S with r elements = # subsets of S of r elements including a_n + # subsets of S of r elements not containing a_n.

= # of subsets of $\{a_1, a_2, \ldots, a_{n-1}\}$ of $r-1$ elements + # subsets of $\{a_1, a_2, \ldots, a_{n-1}\}$ with r elements

$= C(n-1,r-1) + C(n-1,r)$

This is the principle behind $\boxed{\text{Pascal's Triangle}}$.

$$\begin{array}{ccccccccc}
 & & & & C(0,0) & & & & \\
 & & & C(1,0) & & C(1,1) & & & \\
 & & C(2,0) & & C(2,1) & & C(2,2) & & \\
 & C(3,0) & & C(3,1) & & \overset{n-1,r-1}{C(3,2)} & & \overset{n-1,r}{C(3,3)} & \\
C(4,0) & & C(4,1) & & C(4,2) & & \overset{n\ r}{C(4,3)} & & C(4,4)
\end{array}$$

and so on.

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

and so on.

Pascal's Triangle also gives the coefficients of successive binomial Theorems. Coefficients of $(x+y)^n$ appear in the n^{th} row

$$1 = C(0,0)$$

$$x+y = 1\cdot x + 1\cdot y = C(1,0)x + C(0,1)y$$

$$(x+y)^2 = x^2+2xy+y^2 = C(2,0)x^2 + C(2,1)xy + C(2,2)y^2$$

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3 = C(3,0)x^3 + C(3,1)x^2y + C(3,2)xy^2 + C(3,3)y^3.$$

$$\vdots$$

$$(x+y)^n = C(n,0)x^n + C(n,1)x^{n-1}y + C(n,2)x^{n-2}y^2 + \cdots + C(n,n)y^n.$$

<u>Proof</u> Look for the pattern in $(x+y)^3$.

$$(x+y)^3 = (x+y)(x+y)(x+y) = \overset{x^3}{xxx} + \overbrace{xxy + xyx + yxx}^{x^2y \text{ terms}} + \underbrace{xyy + yxy + yyx}_{xy^2 \text{ terms}} + \underset{y^3}{yyy}$$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

- coefficient of x^3 is 1, which is $C(3,0) = \binom{3}{0}$
- coefficient of x^2y is 3, which is the number of ways of choosing which of the $x+y$ factors the y could come from $= C(3,1) = \binom{3}{1}$
- coefficient of xy^2 is 3, which is the number of ways of choosing 2 of the three $x+y$ factors to draw the two y's from. $= C(3,2) = \binom{3}{2}$

- coefficient of y^3 is 1, which is the number of ways of choosing all three $x+y$ factors to draw the three y's from. $= C(3,3)$

This generalizes easily to $(x+y)^n = (x+y)(x+y)\cdots(x+y)$

The coefficient of $x^{n-r}y^r$ is the number of ways of choosing r of the n $x+y$ factors to draw the r y's from.
$= C(n,r) = \binom{n}{r}$

$\boxed{\text{Binomial Theorem}}$

If x and y are variables and n is a non-negative integer then

$$\begin{aligned}(x+y)^n &= C(n,0)x^n + C(n,1)x^{n-1}y + C(n,2)x^{n-2}y^2 + \cdots + C(n,n)y^n \\ &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n \\ &= \sum_{i=0}^{n} \binom{n}{i}x^{n-i}y^i\end{aligned}$$