Lesson 15
Boolean functions
§ 12.1, 12.2

Boolean algebra for $\{0,1\}$

The set $B=\{0,1\}$ together with the operations $+$, $\cdot$, $\overline{\ }$ (Boolean sum / OR, product / AND, complement)
satisfying the properties

$+$	0	1
0	0	1
1	1	1

$\cdot$	0	1
0	0	0
1	0	1

$\overline{0}=1$
$\overline{1}=0$

Order of operations in expressions: ① complement ② product ③ sum

Example 1. $1\cdot 0+\overline{(0+1)} = 0+\overline{1} = 0+0=0$
Evaluate expressions in this simple system using its rules.

Boolean Algebra for $\{0,1\}$ is parallel to how T and F work in propositional logic with $\neg, \wedge, \vee$.

$1 \leftrightarrow T$ $\quad \neg \leftrightarrow \overline{\ }$ $\quad + \leftrightarrow \vee$
$0 \leftrightarrow F$ $\quad \cdot \leftrightarrow \wedge$

Example 2. $1\cdot 0+\overline{(0+1)} = 0$

$(T\wedge F) \vee \neg(F\vee T) \equiv F$
and, or, or, logical equivalence

Example 3. Vice versa $(T\wedge T)\vee \neg F \equiv T$

$(1\cdot 1)+\overline{0} = 1$

Definition: B^n is the set of all possible n-tuples of 0 and 1.

$B^n = \{(x_1,x_2,\dots,x_n) \mid x_i\in B \text{ for } 1\le i\le n\}$.
$B = \{0,1\}$

Boolean variable x only takes on values 0 and 1.
Boolean function from B^n to B, degree n.

Boolean expression.

Example. $F(x,y) = x\bar{y}$

$F: B^2 \to B$.

F	0	1
0	0	0
1	1	0

We build up all the Boolean expressions defining Boolean function recursively.

- $0, 1, x_1, x_2, \dots, x_n$ are Boolean expressions
- If E_1 and E_2 are Boolean expressions then

  $\bar{E}_1$, $E_1 + E_2$, E_1E_2 are Boolean expressions.

Values of Boolean functions are obtained by substituting 0 and 1 for the variables in the expression. The values of a Boolean function are put into a table as in Example 5 on p 813. (Like a truth table for logic.)

(*) We now put all Boolean functions of a given degree n together into a big set and define operations on the functions.

Boolean functions are equivalent if

$F(b_1, b_2, \dots, b_n) = G(b_1, b_2, \dots, b_n)$

for all n-tuples of elements of $B = \{0,1\}$. (Like logical equivalence

Complement: $\bar{F}$:

$\bar{F}(x_1, \dots, x_n) = \overline{F(x_1, \dots, x_n)}$

Boolean Sum: $F+G$

$(F+G)(x_1, \dots, x_n) = F(x_1, \dots, x_n) + G(x_1, \dots, x_n)$

Boolean Product FG

$(FG)(x_1, \dots, x_n) = F(x_1, \dots, x_n)G(x_1, \dots, x_n)$.

Example. There are 16 different Boolean functions of degree 2.

See Table 3 on p. 814.

Properties (Identities) satisfied by Boolean functions:

Listed on page 815 in Table 5.

These are the same as the properties of logical equivalences in Logic (See section 1.3)

These are the same as the properties of set identities in Set Theory (See Section 2.2).

You should look back at these two sections and see how they are parallel. These are natural laws operating in the same way in different contexts.

These laws are all verifiable using Tables just as we do for logical equivalence.

These rules for Boolean functions, for logical equivalence, for equality of sets can be made into one abstract definition of a "Boolean Algebra"

See definition 1 on page 817.

Every theorem that can be proved from these axioms then can be applied to <u>every</u> system satisfying these axioms.

A Boolean algebra is a set B with two binary operations, $\vee$ and $\wedge$, elements 0 and 1, and a unary operation $\bar{\ }$ such that for all $x, y, z \in B$:

$$x \vee 0 = x$$
$$x \wedge 1 = x$$
Identity laws

$$x \vee \bar{x} = 1$$
$$x \wedge \bar{x} = 0$$
Complement laws

$$(x \vee y) \vee z = x \vee (y \vee z)$$
$$(x \wedge y) \wedge z = x \wedge (y \wedge z)$$
Associative laws

$$x \vee y = y \vee x$$
$$x \wedge y = y \wedge x$$
Commutative laws

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$
Distributive laws.

We now see how to find a Boolean expression for a given specific table of values. This is done by using "minterms" – Boolean products of the variables $x_1, x_2, \ldots, x_n$, but with $\bar{x}_i$ or x_i appearing in the ith place.

$x_1 x_2 \bar{x}_3 x_4$ for example.

Ex2 on p. 820.

Specify that the expression evaluates to 1 if $x_1 = x_3 = 0$ and $x_2 = x_4 = x_5 = 1$. and otherwise evaluates to 0.

$$\underbrace{\bar{x}_1 x_2 \bar{x}_3 x_4 x_5}_{\text{minterm}}$$ will do this.

This is because a minterm $y_1 y_2 \cdots y_n$ has value 1 iff each y_i is 1.

A sum of minterms will give us a way of representing any Boolean expression.

See example 3 on p. 820.

$\therefore$ Every Boolean function can be expressed as a Boolean sum of minterms.

This means every Boolean function can be expressed in terms of $\cdot$, $+$, and $^{-}$.

We call the set $\{\cdot, +, ^{-}\}$ <u>functionally complete</u>.