# LESSON 8
## Mathematical Induction
Parts of §5.1, §5.2

| | | |
|---|---|---|
| IF ① $P(n_0)$ is true | PUSH OVER THE FIRST DOMINO | A... |
| AND ② $P(k)$ true $\Rightarrow P(k+1)$ true<br>↑<br>KEY | IF ONE FALLS THE NEXT ONE FALLS<br>↑<br>KEY | sequential unfoldment<br>word gap word<br>↑<br>KEY |
| THEN ③ $P(n)$ is true for all $n \geq n_0$ | THEY ALL FALL DOWN | all creation unfolds |

Examples

[1] The sum of the first $n$ positive integers is $\frac{n(n+1)}{2}$

$P(n)$: $1+2+3+\cdots+n = \frac{n(n+1)}{2}$

① $P(1)$ is true $\quad 1 = \frac{1(1+1)}{2}$ ✓

② If $P(k)$ is true, that is, if $1+2+3+\cdots+k = \frac{k(k+1)}{2}$ is true,

then $P(k+1)$ is true, that is, $1+2+3+\cdots+(k+1) = \frac{(k+1)(k+2)}{2}$

Why? $1+2+3+\cdots+(k+1) = \underbrace{1+2+3+\cdots+k}_{=\frac{k(k+1)}{2}}+(k+1)$

$$= \frac{k(k+1)}{2}+(k+1) = (k+1)\left(\frac{k}{2}+1\right) = (k+1)\left(\frac{k+2}{2}\right) = \frac{(k+1)(k+2)}{2}$$

③ Therefore, $P(n)$ is true for all $n \geq 1$.

That is, $1+2+3+\cdots+n = \frac{n(n+1)}{2}$.

$\boxed{\text{Exercise}}$: You do #4 on page 329.

[2] The number of subsets of a set A with $n$ elements is $2^n$

$P(n)$: Any set with $n$ elements has $2^n$ subsets.

① $P(0)$ is true. Why? $P(0)$: any set with no elements has $2^0 = 1$ element namely the empty set $\emptyset$. (By convention, $\emptyset$ is a subset of every set.

(2) Prove $P(k)$ true $\Rightarrow P(k+1)$ true.

Suppose $P(K)$ is true, that is, any set with $k$ elements has $2^k$ subsets.

Then $P(k+1)$ says any set with $k+1$ elements has $2^{k+1}$ subsets.

Let $S$ be a set with $k+1$ elements

$$S = \{a_1, a_2, \dots, a_{k+1}\} = \underbrace{\{a_1, a_2, \dots, a_k\}}_{\substack{\text{has } 2^k \text{ subsets} \\ \text{because } P(K) \text{ is true}}} \overset{\text{union}}{\cup} \{a_{k+1}\}.$$

Every subset of $S$ either contains $a_{k+1}$ or doesn't.

There are $2^k$ subsets that don't contain $a_{k+1}$ because $P(K)$ is true for $\{a_1, a_2, \dots, a_k\}$.

And for each of these $2^k$ subsets we can add $a_{k+1}$ and get a subset of $S$. And these will make all the subsets of $S$ containing $a_{k+1}$. There will therefore be $2^k$ subsets containing $a_{k+1}$.

Total # of subsets of $S = 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$.

Therefore $P(k+1)$ is true.

(3) Therefore $P(n)$ is true for all $n \geq 0$.

That is, any set with $n$ elements has $2^n$ subsets for $n \geq 0$.

$\boxed{\text{Exercise}}$ You prove #32 on p. 330.

3 divides $n^3 + 2n$ for all positive integers $n$.
(no remainder)

$\boxed{3}$ $2^n < n!$ for every $n \geq 4$. $P(n): 2^n < n!$

① $P(4): 2^4 < 4!$ True because LHS = 16, RHS = $4 \cdot 3 \cdot 2 \cdot 1 = 24$.

② If $P(k)$ is true, that is, if $2^k < k!$ is true

then $P(k+1)$ is true, that is, $2^{k+1} < (k+1)!$.

Why? $2^{k+1} = 2^k \cdot 2 \underset{\text{I.H.}}{<} k! \cdot 2$ since $P(k)$ is true.

Since $k \geq 4$, we have $2 < k+1$

Therefore $2^{k+1} < k! \cdot 2 < k!(k+1) = (k+1)!$

Therefore if $P(k)$ is true, then $P(k+1)$ is true.

③ Therefore $P(n)$ is true for all $n \geq 4$.

That is, $2^n < n!$ for all $n \geq 4$.

$\boxed{4}$ $\boxed{\text{Verifying algorithms}}$

Polynomial evaluation: $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$.

Given $n$, $a_0, a_1, \cdots, a_n$, and $x$

Step 1. Initialize: put $S = a_0$ and $k = 1$.

Step 2. While $k \leq n$, replace $S$ by $S + a_k x^k$ and $k$ by $k+1$.
Endwhile

Step 3. Print $p(x) = S$.

Verifying the algorithm means

$P(n)$: If the replacements in Step 2 are executed exactly $n$ times, at the end $S = a_0 + a_1 x + \cdots + a_n x^n$.

① $P(0)$: Step 2 is executed 0 times. Therefore $S = a_0$, so $P(0)$ is true.

② Prove $P(k)$ true implies $P(k+1)$ true.

If $P(k)$: If step 2 is executed $k$ times, then $S = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k$

then $P(k+1)$: If step 2 is executed $k+1$ times, then

$$S = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k + a_{k+1}x^{k+1}.$$

Why? Because on the $k+1$ step, $a_0 + a_1x + \cdots + a_kx^k$ is replaced by $S + a_{k+1}x^{k+1} = (a_0 + a_1x + \cdots + a_kx^k) + a_{k+1}x^{k+1}$.

③ Therefore, $P(n)$ is true for all $n \geq 0$.

This example may appear quite trivial, but the loops in a more complex program can be very complicated.

Example 5

Theorem. Given a number $x$ and a list of $2^n$ numbers in non-decreasing order, then at most $n+1$ comparisons are necessary to determine if $x$ is in the list.

Proof by induction on $n$

① $P(0)$, given a number $x$ and a list of $2^0 = 1$ number only, 1 comparison will determine if this is the number or not.

② Prove: if $P(k)$ is true then $P(k+1)$ is true.

Suppose $P(k)$: given a number $x$ and a list of $2^k$ numbers, $k+1$ comparisons are enough to determine if $x$ is in the list.

Then $P(k+1)$ is true: given a number $x$ and a list of $2^{k+1}$ numbers, $k+2$ comparisons are enough to determine if $x$ is in the list.

Why?

Because: Divide a list of $2^{k+1}$ members in the middle into 2 parts.

$$\underbrace{a_1, a_2, \ldots, a_{2^k}}_{2^k \text{ members}}, \underbrace{a_{2^k+1}, \ldots, a_{2^{k+1}}}_{2^k \text{ members}}$$

Compare $x$ with $a_{2^k}$

if $x \le a_{2^k}$

then $P(k)$ tells us that $k+1$ comparisons are enough to determine if $x$ is in this list

if $x > a_{2^k}$

then $P(k)$ tells us that $k+1$ comparisons are enough to determine if $x$ is in this list.

In either case, a total of $1 + (k+1) = k+2$ comparisons are enough to determine if $x$ is in the list.

Therefore $P(k+1)$ is true and this completes the inductive step.

3 Therefore, $P(n)$ is true for all $n \ge 0$.

$n+1$ comparisons will determine if $x$ is there.

**Strong Induction**

1 Show $P(n_0)$ is true.

2 Show: If $P(n_0), P(n_0+1), P(n_0+2), \ldots, P(k)$ are all true, then $P(k+1)$ is true.

3 Then $P(n)$ is true for all $n \ge n_0$.

$\boxed{\text{Example}}$ (Example 4 on p 337).

For $n \geq 12$; a postage of $n$ cents can be formed using just 4-cent stamps and 5-cent stamps. $P(n)$.

(A) <u>Proof using plain mathematical induction.</u>

① $P(12)$ is true because $12¢ = 4¢ + 4¢ + 4¢$ (3 4¢ stamps)

② Suppose $P(k)$ is true: a postage of $k¢$ can be formed using just 4¢ and 5¢ stamps.

Show $P(k+1)$ is true: a postage of $k+1$ ¢ can be formed using just 4¢ and 5¢ stamps.

Note that $k \geq 12$.

There are two cases:

ⓐ At least one 4¢ stamp was used in forming $k¢$.
Then replace it by a 5¢ stamp. This forms $k+1$ ¢ using just 4¢ and 5¢ stamps.

or ⓑ If no 4¢ stamps were used in forming $k¢$,
Then only 5¢ stamps were used to form $k¢$.
Since $k \geq 12$, there must be at least three 5¢ stamps in the formation of $k¢$.
Replace these three 5¢ stamps with four 4¢ stamps.
This forms $k+1$ ¢ with just 4¢ and 5¢ stamps.

This completes the induction step.

③ Therefore $P(n)$ is true for all $n \geq 12$: a postage of $n$ cents can be formed using just 4-cent and 5-cent stamps.

(B) <u>Proof using strong induction.</u>

① Basis step: Show $P(12)$, $P(13)$, $P(14)$, $P(15)$ are all true.

$12 = 4+4+4$ $\quad$ $14 = 4+5+5$

$13 = 4+4+5$ $\quad$ $15 = 5+5+5$

② Show. If $P(j)$ is true for all $j \leq k$, then $P(k+1)$ is true.

$P(k+1)$: postage of $k+1$ cents can be formed with only 4¢ and 5¢ stamps.

By the induction hypothesis, $P(k-3)$ is true: postage of $k-3$ cents can be formed with only 4¢ and 5¢ stamps.

Add one 4¢ stamp and we have formed postage of $k+1$ cents.

Therefore $P(k+1)$ is true.

Therefore $P(k)$ true $\Rightarrow$ $P(k+1)$ is true.

③ Therefore $P(n)$ is true for all $n \geq 12$:

postage of $n$ cents can be formed using only 4¢ and 5¢ stamps.

$\boxed{\text{Exercise}}$ You do problem 4 on page 341.