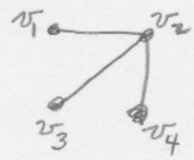
LESSON 2
GRAPHS
§2,6,10,3

WAYS TO REPRESENT A GRAPH

Dots and lines



$\mathcal{G}$

Adjacency list

$v_1 : v_2$

$v_2 : v_1, v_3, v_4$

$v_3 :$

$v_4 :$

Adjacency matrix

Square $n \times n$

$$\begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 0 & 0 \\ v_2 & & & & \\ v_3 & & & & \\ v_4 & & & & \end{array}$$

← Sum is $\deg v_1$

← Sum is $\deg v_2$

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge} \\ 0 & \text{if not} \end{cases}$$

If there are multiple edges, write the number of edges connecting v_i and v_j instead of 1.

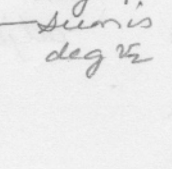
$\boxed{\text{Theorem}}$

Sum of row i = $\deg v_i$ = Sum of column i

Incidence matrix

$$\begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline v_1 & 1 & 0 & 0 \\ v_2 & & & \\ v_3 & & & \\ v_4 & & & \end{array}$$

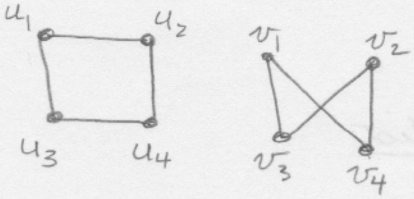
$a_{ij} = 1$ if v_i is incident to e_j.



ISOMORPHIC GRAPHS

When do two graphs have the same structure?

$$G_1 \cong G_2.$$



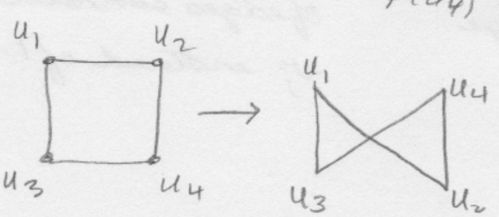
untwisting the second graph gives us the first graph. They are both just C_4.

Definition

Two simple graphs $G_1 = \{V_1, E_1\}$ and $G_2 = \{V_2, E_2\}$ are isomorphic if there is a one-to-one onto function $f: V_1 \to V_2$ such that vertices a and b are adjacent in G_1 iff $f(a)$ and $f(b)$ are adjacent in G_2.

Example above: $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_2$, $f(u_4) = v_3$

u_1 and u_2 adjacent iff $f(u_1) = v_1$ and $f(u_2) = v_4$ are adjacent.



label the vertices in G_2 with letters from the vertices in G_1. Check that all edges are preserved.

One way is to check the adjacency matrix of each and see if they are equal. Take care to write corresponding vertices in the order predicted by f.

$$G_1 \quad \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$G_2 \quad \begin{array}{c} \\ \\ v_1 \\ v_4 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{cccc} f(u_1) & f(u_2) & f(u_3) & f(u_4) \\ v_1 & v_4 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Same matrix

To summarize

To show 2 graphs G_1 and G_2 are isomorphic:

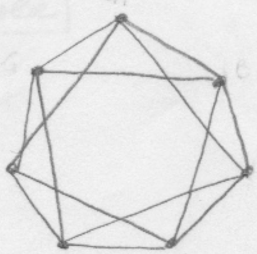
① Check there are the same number of vertices and the same number of edges.

② Find a 1-1 correspondence f from the vertices of G_1 to the vertices of G_2. (You may do this by labeling vertices of G_1, and then labeling the corresponding vertices in G_2 with the same letter.)

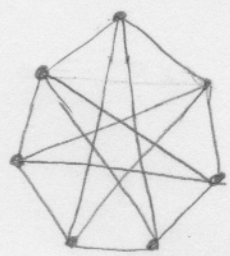
③ Ensure all edges correspond: if u and v are adjacent in G_1, then $f(u)$ and $f(v)$ are adjacent in G_2, and vice versa.

Example

G_1



G_2



Do G_1 and G_2 have the same number of vertices?

Label the vertices of G_1 with letters.

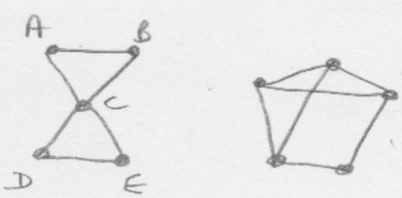
Try to label the vertices of G_2 with these same letters in such a way that adjacent vertices in G_1 are adjacent in G_2.

If two graphs are isomorphic, then

GRAPH INVARIANTS

- # vertices is the same
- # edges is the same
- corresponding edges have the same degree
- same number of vertices of degree n in both
- same number of triangles in both
- same number of quadrilaterals in both (4 sided figures)
- same number of circuits of length n in both.
- if 2 vertices are connected by a path of length n in one, there must be a path of length n connecting the two corresponding vertices in the other.
- one can be "unfolded" or redrawn to look like the other.
- etc.

If any of these are violated, the graphs are not isomorphic.



See also examples 9, 10, 11 in the textbook (pp. 672-673)

It is easier to show two graphs are not isomorphic than to show they are. Only one violation shows they are not isomorphic, but you need to find a 1-1 correspondence preserving edges in order to show they are isomorphic.

REVIEW OF MATRIX OPERATIONS

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

$m \times n$ matrix (size)
m rows
n columns

a_{ij} is the entry in row i and column j

A is <u>square</u> if $m = n$

A is <u>symmetric</u> if A is square and $a_{ij} = a_{ji}$

Example $A = \begin{bmatrix} 1 & 2 \\ 5 & 0 \\ 6 & 7 \end{bmatrix}$ size =

$a_{21} =$

$\boxed{A = B}$ if • same size

• $a_{ij} = b_{ij}$ for all i and j (corresponding elts are equal)

Example Are any of these matrices equal?

$C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix}$ $E = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

$\boxed{A + B}$ if • same size

• ij^{th} elt of $A+B$ is $a_{ij} + b_{ij}$ (add corresponding elts)

$$\begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 6 \\ 3 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} & \\ & \\ & \end{bmatrix}$$

$\boxed{AB}$ ij^{th} elt is dot product of i^{th} row of A with j^{th} column of B.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 & 8 \\ 0 & 1 & 6 & 9 \\ 2 & 4 & 7 & -1 \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \end{bmatrix}$$

size 2×3 size 3×4 size =

cols A = # rows B

$AB \neq BA$ usually.

Examples In the example above, does $CE = EC$?

Which of the following can be multiplied? CD, DC.