# LESSON 10

## Sets, Functions, Relations

### WORKSHEET

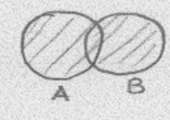
SET OPERATIONS

Space shuttle $S = \{1, 5, 6, 8\}$ experiments carried on the first trip $T = \{2, 4, 5, 8, 9\}$ experiments carried on the second trip

1) Which experiments were carried at least once? $\{ \qquad \}$

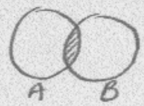
UNION $A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ or both}\}$

VENN DIAGRAM




2) Which experiments were carried on both trips? $\{ \qquad \}$

INTERSECTION $A \cap B = \{x : x \in A \text{ and } x \in B\}$




Example 2.1. If $A = \{1, 2, 4\}$ $B = \{2, 4, 6, 8\}$ $C = \{3, 6\}$, then

$A \cup B = \{ \qquad \}$

$A \cup C =$

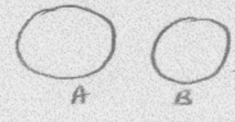
$B \cup C =$

$A \cap B =$

$A \cap C =$

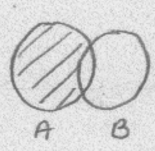
$B \cap C =$

DISJOINT A and B are disjoint if $A \cap B = \emptyset$.




3) which experiments were carried on the first trip but not the second trip? $\{$

DIFFERENCE $A - B = \{x : x \in A \text{ and } x \notin B\}$




Example 2.2. Using the sets A, B, C in example 2.1 above

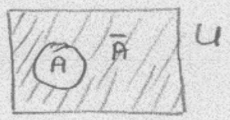
$A - B =$

$B - A =$

4) Which experiments were not taken on the first trip? $\{ \quad \}$

UNIVERSAL SET $\mathcal{U} = \{1, 2, 3, \cdots, 11, 12\}$

COMPLEMENT $\mathcal{U} - A = \bar{A}$




Example 2.3. Using the sets A, B, C in example 2.1 above

$\bar{B} =$ $\bar{C} =$

Draw a Venn diagram for $\overline{\bar{A} \cup B}$.

5) If $S = \{1, 5, 6, 8\}$ and $T = \{1, 2, 5, 6, 7, 8\}$ then S is a subset of T.

SUBSET $A \subseteq B$ if every element of A is also an element of B.

$A = B$ if A and B have the same elements
i.e. $A \subseteq B$ and $B \subseteq A$.

6) Convince yourself, using Venn diagrams, that if A, B, C are subsets of a universal set $\mathcal{U}$, then

(a) $A \cup B = B \cup A$ and $A \cap B = B \cap A$ commutative law

(b) $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$. associative rule

(c) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ distributive rule

(d) $\bar{\bar{A}} = A$

(e) $A \cup \bar{A} = \mathcal{U}$

(f) $A \cap \bar{A} = \phi$

(g) $A \subseteq A \cup B$ and $B \subseteq A \cup B$

(h) $A \cap B \subseteq A$ and $A \cap B \subseteq B$

(i) $A - B = A \cap \overline{B}$

De Morgan's Laws

(j) $\overline{A \cup B} = \overline{A} \cap \overline{B}$

(k) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

7) Use the above rules to simplify:

$\overline{\overline{A} \cup B}$ = ________ by deMorgan laws

= ________ by rule (d)

= ________ by rule (i)

Check your final answer by using a Venn diagram.

8) Use the above rules to prove that

$(A-B) \cup (B-A) = (A \cup B) - (A \cap B)$

$\Downarrow$ $\Downarrow$

9) Another way to prove two sets are equal: $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

Prove $\overline{A \cup B} = \overline{A} \cap \overline{B}$

FILL IN THE BLANK

Reason?

1) Show $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$

Let $x \in \overline{A \cup B}$

then $x \notin A \cup B$ (by defn of complement)

$\therefore x \notin A$ and $x \notin B$ (by defn of union)

$\therefore x \in \overline{A}$ and $x \in \overline{B}$ (by defn of complement)

$\therefore x \in \overline{A} \cap \overline{B}$ (by defn of intersection)

2) Show $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$

Let $x \in \overline{A} \cap \overline{B}$

then $x \in$ ___ and $x \in$ ___ (

$\therefore x \notin$ ___ and $x \notin$ ___ (

$\therefore x \notin$ ________ (

$\therefore x \in \overline{A \cup B}$ (

10) <u>ORDERED PAIRS</u>

In a <u>set</u> order does not matter : $\{1,2,3\} = \{3,1,2\} = \{1,3,2\}$

If order <u>does</u> matter, we write $(1,2)$ instead of $\{1,2\}$.

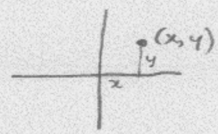
$(a,b) = (c,d)$ iff $a = c$ and $b = d$.

$(1,2) \neq (2,1)$.

11) <u>CARTESIAN PRODUCT</u>

$A \times B = \{(a,b) ; a \in A \text{ and } b \in B\}$

eg. Euclidean plane is $R \times R = \{(x,y) : x \in R \text{ and } y \in R\}$




Example 2.6. If $A = \{1,2,3\}$ and $B = \{3,4\}$, then

$A \times B = \{(1,3),(1,4),(2,3),(2,4),(3,3),(3,4)\}$

$B \times A = \{ \quad \}$

Note $A \times B \neq B \times A$

12) Write down the 3 ways we can show one set is equal to another

$\boxed{1}$

$\boxed{2}$

$\boxed{3}$

# Relations

"$\leq$" is a relation on the real numbers $\mathbb{R}$. ("less than or equal to")

Definition $r \leq s$ if $s - r \geq 0$. $3 \leq 5$ because $5-3=2 \geq 0$.

$\leq$ has three important properties:

① $r \leq r$ for all $r \in \mathbb{R}$ (reflexive). $r$ is related to itself $\because r-r=0 \geq 0$

② If $r \leq s$ and $s \leq r$ then $r = s$ (antisymmetric)
If $r$ is related to $s$ and $s$ is related to $r$ then $r=s$

③ If $r \leq s$ and $s \leq t$ then $r \leq t$ (transitive).
If $r$ is related to $s$ and $s$ is related to $t$ then $r$ is related to $t$.

These three properties make $\leq$ a "partial order".
But $\leq$ also has the property

④ For every pair of numbers $r, s \in \mathbb{R}$, either $r \leq s$ or $s \leq r$.
(any two numbers can be compared).

This makes $\leq$ into a "total order".

"$=$" is a relation on $\mathbb{R}$. Properties

① $r = r$ for all $r \in \mathbb{R}$ (reflexive)

②* $r = s \Rightarrow s = r$ for all $r, s \in \mathbb{R}$ (symmetric)

③ If $r = s$ and $s = t$ then $r = t$ (transitive).

This makes $=$ an <u>equivalence relation</u>.

We can view any relation $R$ on any set $S$ as a subset of $S \times S$, that is, as a set of ordered pairs in $S$.

Example: Let $S = \{1,2,3,4\}$. "$\leq$" $= \{(a,b) : a \leq b,\ a \in S, \text{and } b \in S\}$

then "$\leq$" is the set $\{(1,1),(1,2),(1,3),(1,4),(2,?)\ (2,2),(2,3),(2,4)\ \text{etc}\} \subseteq S \times S$.

We also write $1 \leq 1$, $1 \leq 2$, $1 \leq 3$, $1 \leq 4$, $2 \leq 2$, $2 \leq 3$, $2 \leq 4$, $3 \leq 3$, $3 \leq 4$

putting the relationship between $4 \leq 4$

$A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$ Cartesian product.

Any subset of $A \times B$ is called a <u>relation from A to B</u>.

We write $(a,b) \in R$ or $aRb$.

If $A = B$, $R$ is called a <u>relation on A</u>

<u>Example 1.</u> $A = \{$LA, Seattle, London, Frankfurt, Paris, Madrid, Dublin$\}$

$B = \{$Timmack, Jelinek, Rupp, Alfors, Tang, Washington, Ramirez

For $a \in A$ and $b \in B$, $(a,b) \in R$ if $a$ is a city requested by pilot $b$.

Here are the requests

Los Angeles: Timmack, Jelinek, Rupp
Seattle: Alfors, Timmack, Tang, Washington
London: Timmack, Tang, Washington
Frankfort: Alfors, Tang, Rupp, Ramirez
Paris: Jelinek, Washington, Rupp
Madrid: Jelinek, Ramirez
Dublin: Timmack, Rupp, Ramirez.

(LA, Timmack) is one element of this relation. Write down six more elements of the relation:

Notice that we can depict a relation by a graph. In the case of the airline pilots, it will be a bipartite graph with pilots on one side and destinations on the other side. An edge is drawn if a pilot requests that destination.

<u>Example 2</u> $S$ is the set of all positive integers $= \{1, 2, 3, \ldots\}$

$x R y$ if $x$ divides into $y$ with no remainder.

Write down 10 elements of $R$.

$R = \{ (1,2), (3,6), \qquad \}$

<u>Example 3</u> $S = \mathcal{P}(X)$, the set of all subsets of the set $X$.

$A R B$ if $A \subseteq B$.

For $X = \{1,2,3\}$, write down all the elements of $S$.

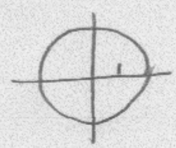
$\{ \phi, \{1\}, \qquad \}$

Then write down all the pairs in $R$.

$\{ (\phi, \{1\}), \qquad \}$.

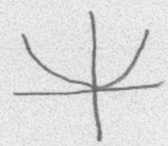
<u>Example 4</u> $\{(x,y) : x^2 + y^2 = 1\}$ is a relation on $\mathbb{R}$.

$x R y$ if $x^2 + y^2 = 1$




<u>Example 5</u> $\{(x,y) : y = x^2\}$ is a relation on $\mathbb{R}$

$x R y$ if $y = x^2$

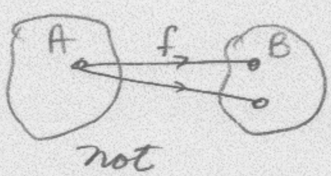


<u>Example 6</u>

Functions are a particular type of relation.

Given two sets $A$ and $B$, then $f$ is a function from $A$ to $B$ if $f$ is a subset of $A \times B$ such that if $(a,b) \in f$ and $(a,c) \in f$, then $b = c$.

In our usual treatment of functions, this corresponds to the idea that $f$ is a function from $A$ to $B$ if it maps every element $a$ of $A$ to exactly one element of $B$.

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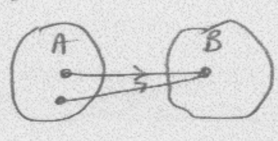
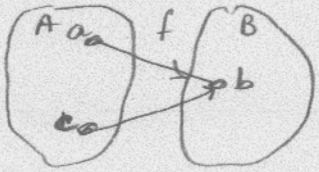



<u>is</u> allowed.

A function $f: A \to B$ is <u>one-to-one</u> if and only if $(a,b) \in f$ and $(c,b) \in f$ then $a = c$.

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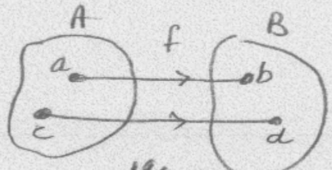
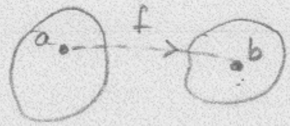



<u>allowed</u>

A function maps $A$ <u>onto</u> $B$ if for every element $b$ in $B$, there is an element $a$ in $A$ such that $f(a) = b$.




## PROPERTIES OF RELATIONS ON A SET S

These properties may or may not be true for a particular relation.

Reflexive: $xRx$ for all $x \in S$

Symmetric: $xRy$ implies $yRx$ for all $x, y \in S$.

Transitive: $xRy$ and $yRz$ implies $xRz$ for all $x, y, z \in S$.

Check which of these properties hold for Examples 2-5 above.

2)

3)

4)

5)

## EQUIVALENCE RELATION

is a relation satisfying all three properties:

* Reflexive
* Symmetric
* Transitive

For example, equality $=$ is an equivalence relation.