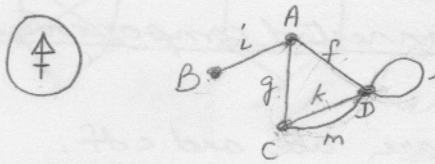
LESSON 3

Paths in Graphs

Notes

Undirected graph = $\left\{\begin{array}{l}\text{simple graph}\\ \text{multigraph}\\ \text{pseudograph}\end{array}\right\}$ multiple edges and loops allowed.



deg V = # edges incident with V including a loop as 2.
Here deg D = 5, deg C = 3, etc.

PATHS * $\boxed{\text{U-V path}}$ of length n

$U = V_1 e_1 V_2 e_2 \cdots V_n e_n V_{n+1} = V$

where e_i is an edge joining V_i and V_{i+1}.

Repetition is allowed

$U = V$ is allowed

In the graph above a CA path is CkDhDfA.

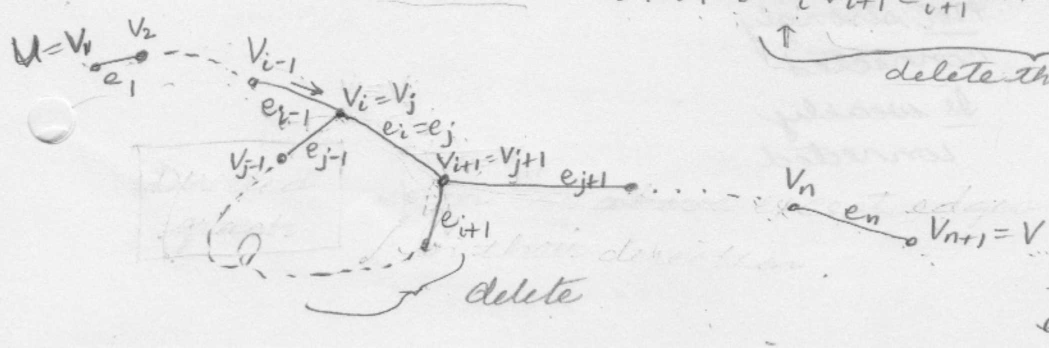
If there are no loops and no multiple edges, we can leave out the edges in this notation and just give the sequence of vertices.

* $\boxed{\text{Simple path}}$ – a path with <u>no edge</u> repeated (vertices are allowed to repeat)
* $\boxed{\text{Circuit}}$ – a path that begins and ends at the same vertex and has length at least 1.
* $\boxed{\text{Simple circuit}}$ A circuit with no repeated edges.
* $\boxed{\text{Every U-V path contains a simple U-V path.}}$

Why? Let $U = V_1 e_1 V_2 e_2 \cdots V_n e_n V_{n+1} = V$.

If $e_i = e_j$ for some $i \neq j$,

$$U = V_1 e_1 V_2 e_2 \cdots V_{i-1} e_{i-1} V_i \underbrace{e_i V_{i+1} e_{i+1} \cdots V_{j-1} e_{j-1} V_j e_j}_{\text{delete this string}} V_{j+1} e_{j+1} \cdots V_n e_n V_{n+1} = V$$

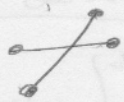


Is the remainder simple? Yes – done
No – repeat
etc.

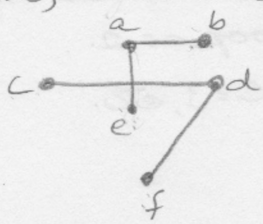
Since the number of edges is finite, the process must end in a simple graph.

Connectedness

Undirected graph: "Every pair of vertices is connected by some path." $\Leftrightarrow$ the graph is <u>connected</u>.

 is connected.  is not connected.

We can break up a graph that is not connected into disjoint, maximal, connected subgraphs called <u>connected components</u>.



connected components are abe and cdf.

Cut vertices and cut edges

Optional for now.

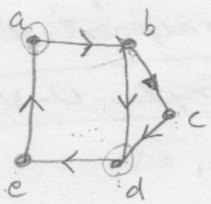
Directed graphs

All of the above is the same if we require paths to follow the directed edges according to their direction.

There are two types of connectedness for directed graphs:

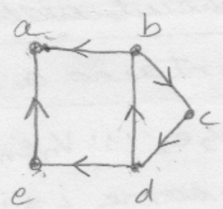
<u>Strongly connected</u>: For every pair of vertices a and b in the graph, there is an a-b path <u>and</u> a b-a path.

<u>Weakly connected</u>: The underlying <u>undirected</u> graph is connected. (Disregard the directions.)



G

Strongly connected



H

<u>Not</u> strongly connected

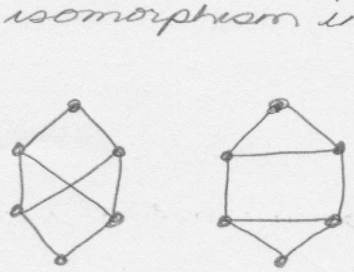
<u>Is</u> weakly connected

Strongly connected $\Rightarrow$ weakly connected

but not the reverse.

Relationship to Isomorphism

Existence of a simple circuit of a particular length is an isomorphism invariant.



are not isomorphic.

There are simple circuits of length 3 in the graph on the right, but not in the graph on the left.

Find an isomorphism between two graph we suspect are isomorphic can be helped by trying to preserve paths and circuits according to degree of vertices. See example 14 on page 687.

Counting paths between vertices

If A is the adjacency matrix of a graph (directed or undirected, with loops and multiple edges allowed) then the ij-element of A^r gives us the number of paths of length r from vertex V_i to vertex V_j.

[Leave to the next lesson]

Euler Path
- includes every edge exactly once.
- first and last vertices are different.

(4) does not contain an Euler path.

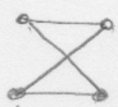
but remove edge k and it does have an Euler path.

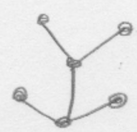
- mailman.
- FedEx delivery.
- Trash collection.

Euler circuit
- includes every edge exactly once
- first and last vertices are the same

(not necessarily a cycle, vertices are allowed to repeat)

(4) has no Euler circuit


E.C


no EC
no EP


EP
no EC

How do we know if there is an Euler circuit?

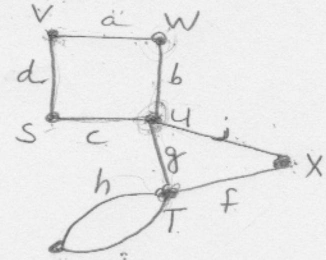
In a connected graph G with at least 2 vertices:

YES iff every vertex has even degree.

Proof ① YES ⇒ you must enter and exit every vertex along a different edge ⇒ every vertex has even degree.

② Every vertex has even degree ⇒

Start anywhere, say at vertex V.
There must be at least one adjacent vertex, since G is connected and has at least 2 vertices.
Travel from V to W, say VaW.
W has even degree so you can exit it along a different edge. VaWbU



G

Keep going until you get back to V.

You *must* get back to V because *every* vertex has even degree — if you enter a vertex you can leave by a different edge. Say VaWbUcSdV.

Have you traveled *every* edge of G.

Yes — done

No — There must be a vertex in the path that is incident with an edge not yet traveled, because G is connected. Say U.

Go to U and keep going, always leaving a vertex by an edge not yet traveled. Even degree lets you do this. You must eventually come back to U, say UjXfTgU.

Insert this circuit in the previous one.

VaWbU cSdV

jXfTgU

iYhT insert in previous one.

Repeat this loop until you run out of edges. You must run out of edges, since there are only a finite number of edges.

Result: an Euler circuit VaWbUjXfTiYhTgUcSdV.

$\boxed{\text{How do we know if there is an Euler path?}}$

Yes $\Leftrightarrow$ exactly 2 vertices have odd degree.

Proof. Add an edge to the graph between the two odd vertices. Find an Euler circuit as above. Remove the added edge. Result — an Euler path.

Hamiltonian Path

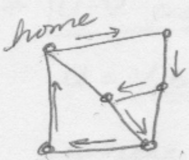
A simple path in G that passes through every vertex exactly once. (Edges cannot repeat, edges may be left out)

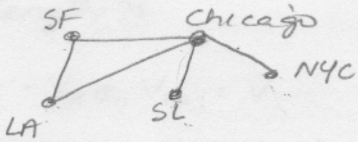
Hamiltonian Circuit

A simple circuit in G that passes through every vertex exactly once.

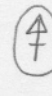
Traveling salesperson

Airline routes



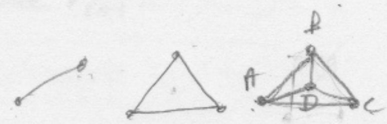


None



HC no
HP yes

K_n: Yes there is HC if $n \geq 3$.



K_2 K_3

How do we know there is a Hamiltonian circuit?

There are no necessary and sufficient conditions.

This is a <u>much</u> harder problem than existence of Euler circuits.

Dirac's theorem and Ore's theorem give sufficient conditions. (optional). p.701