ASSIGNMENT 7
Algorithms
§3.1 & §3.3

§3.1

(4) Algorithm that takes as input a list of n integers and produces as output the largest difference obtained by subtracting an integer on the list from the one following it.

Subtract the first number on the list from the second.
Make this the temporary largest difference d.
Subtract the second number in the list from the third.
If it is larger than the temporary largest difference, make it the temporary largest difference d.
Repeat until all differences have been done.
stop
d will now be the largest difference.

20 Describe an algorithm for finding both the largest and the smallest integer in a finite sequence of integers.

Set the temporary minimum m equal to the first number and the temporary maximum M equal to the first number.
Compare the second number in the sequence with m. If it is smaller than m, set m equal to that.
Compare the second number with M. If it is larger set M equal to that.
Repeat these two steps with the third number in the sequence, then repeat over and over until there are no more numbers left.
Then m will be the smallest number in the sequence and M will be the largest number in the sequence

(10) Algorithm for x^n?

$(((x x) \cdot x) \cdot x) \cdots x$ $\quad n-1$ multiplications

Step 1. Set $P = x$ and $k = 1$

Step 2. While $k < n$ (i.e. $k = 1, 2, 3, \ldots, n-1$

(a) replace P by $P \cdot x$

(b) replace k by $k+1$.

Endwhile

Step 3. Print P.

How many operations?

In Step 2: For each k — Is $k = n$? (one comparison) — if no then perform step 2; if yes STOP. Do not perform step 2. Go straight to Step 3.

(a) one multiplication

(b) one addition

$\therefore$ for each k, there are 3 operations.

We do this step 2 $n-1$ times

So total $\quad n$ comparisons

$n-1$ multiplications

$n-1$ additions

Total $= 3n-2$ operations $= f(n)$

The complexity is $3n-2$ and this is $O(n)$.

SECTION 3.2 The Growth of Functions

2. Note that the choices of C and k witnesses are not unique.

a) Yes, since $17x + 11 \le 17x + x = 18x \le 18x^2$ for all $x > 11$. The witnesses are $C = 18$ and $k = 11$.

b) Yes, since $x^2 + 1000 \le x^2 + x^2 = 2x^2$ for all $x > \sqrt{1000}$. The witnesses are $C = 2$ and $k = \sqrt{1000}$.

c) Yes, since $x \log x \le x \cdot x = x^2$ for all x in the domain of the function. (The fact that $\log x < x$ for all x follows from the fact that $x < 2^x$ for all x, which can be seen by looking at the graphs of these two functions.) The witnesses are $C = 1$ and $k = 0$.

d) No. If there were a constant C such that $x^4/2 \le Cx^2$ for sufficiently large x, then we would have $C \ge x^2/2$. This is clearly impossible for a constant to satisfy.

e) No. If 2^x were $O(x^2)$, then the fraction $2^x/x^2$ would have to be bounded above by some constant C. It can be shown that in fact $2^x > x^3$ for all $x \ge 10$ (using mathematical induction—see Section 5.1—or calculus), so $2^x/x^2 \ge x^3/x^2 = x$ for large x, which is certainly not less than or equal to C.

f) Yes, since $\lfloor x \rfloor \lceil x \rceil \le x(x+1) \le x \cdot 2x = 2x^2$ for all $x > 1$. The witnesses are $C = 2$ and $k = 1$.

4. If $x > 5$, then $2^x + 17 \le 2^x + 2^x = 2 \cdot 2^x \le 2 \cdot 3^x$. This shows that $2^x + 17$ is $O(3^x)$ (the witnesses are $C = 2$ and $k = 5$).

6. We can use the following inequalities, valid for all $x > 1$ (note that making the denominator of a fraction smaller makes the fraction larger).

$$\frac{x^3 + 2x}{2x + 1} \le \frac{x^3 + 2x^3}{2x} = \frac{3}{2}x^2$$

This proves the desired statement, with witnesses $k = 1$ and $C = 3/2$.

10. Since $x^3 \le x^4$ for all $x > 1$, we know that x^3 is $O(x^4)$ (witnesses $C = 1$ and $k = 1$). On the other hand, if $x^4 \le Cx^3$, then (dividing by x^3) $x \le C$. Since this latter condition cannot hold for all large x, no matter what the value of the constant C, we conclude that x^4 is not $O(x^3)$.

14. a) No, by an argument similar to Exercise 10.

b) Yes, since $x^3 \le x^3$ for all x (witnesses $C = 1$, $k = 0$).

c) Yes, since $x^3 \le x^2 + x^3$ for all x (witnesses $C = 1$, $k = 0$).

d) Yes, since $x^3 \le x^2 + x^4$ for all x (witnesses $C = 1$, $k = 0$).

e) Yes, since $x^3 \le 2^x \le 3^x$ for all $x > 10$ (see Exercise 2e). Thus we have witnesses $C = 1$ and $k = 10$.

f) Yes, since $x^3 \le 2 \cdot (x^3/2)$ for all x (witnesses $C = 2$, $k = 0$).

16. The given information says that $|f(x)| \le C|x|$ for all $x > k$, where C and k are particular constants. Let k' be the larger of k and 1. Then since $|x| \le |x^2|$ for all $x > 1$, we have $|f(x)| \le C|x^2|$ for all $x > k'$, as desired.

Section 3.3

#5 $2n-1$

13 & 14 These algorithms are in the notes in plain language, not pseudocode. #13 has $\frac{3}{2}n^2 + \frac{7}{2}n + 1 = f(n)$ operations if you count the $k \to k+1$ additions. (See notes p 7-6) (of Lesson 7) for details

#14 is there, but students need to figure out the complexity.

$\underbrace{a_0 + a_1x + a_2x^2 + a_3x^3}$ $= a_0 + x(\underbrace{a_1 + a_2x + a_3x^2})$

$= a_0 + x(a_1 + x(a_2 + a_3x))$

start here S

Step 1. Set $S = a_n$ and $k = 1$

Step 2. While $k \le n$ (ie. $k = 1, 2, 3, \ldots, n$)

(a) Replace S with $xS + a_{n-k}$

(b) Replace k with $k+1$

Endwhile

Step 3. Print S

1 comparison: is $k \le n$?
1 subtraction: $n-k$
1 multiplication: xS
1 addition $xS + a_{n-k}$
1 addition $k+1$
Total 5 operations for each k.

We perform Step 2 n times, after which we perform one more comparison: is $k \le n$? To which the answer is now "no."

So total $5n+1$ operations. $= f(n)$ which is $O(n)$.

So Horner's algorithm is much more efficient than the ordinary algorithm, which is $O(n^2)$