# Econ 210C Homework 4

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Due: 06/5/2024, 11:59PM PST. Submit pdf write-up and zipped code packet on Github.

## 1. Productivity Shocks in the Three Equation Model

The log-linearized NK model boils down to three equations:

$$\hat{y}_t = -\sigma[\hat{i}_t - E_t\{\hat{\pi}_{t+1}\}] + E_t\{\hat{y}_{t+1}\}$$
$$\hat{\pi}_t = \kappa(\hat{y}_t - \hat{y}_t^{flex}) + \beta E_t\{\hat{\pi}_{t+1}\}$$
$$\hat{i}_t = \phi_\pi \hat{\pi}_t + v_t$$

with $\hat{y}_t^{flex} = \frac{1+\varphi}{\gamma+\varphi}\hat{a}_t$.

For this part assume that $v_t = 0$ and that $\hat{a}_t = \rho_a \hat{a}_{t-1} + \epsilon_t$.

(a) Using the method of undetermined coefficients, solve for $\hat{y}_t$ and $\hat{\pi}_t$ as a function of $\hat{a}_t$.

With $v_t = 0$, we have:

$$\begin{aligned}
\hat{y}_t &= \eta_{ya}\hat{a}_t + \eta_{yv}\hat{v}_t = \eta_{ya}\hat{a}_t \\
\hat{\pi}_t &= \eta_{\pi a}\hat{a}_t + \eta_{\pi v}\hat{v}_t = \eta_{\pi a}\hat{a}_t \\
\hat{i}_t &= \eta_{ia}\hat{a}_t + \eta_{iv}\hat{v}_t = \eta_{ia}\hat{a}_t
\end{aligned}$$

Then for $\hat{y}_t$:

$$\begin{aligned}
\eta_{ya}\hat{a}_t &= -\sigma\left(\eta_{ia}\hat{a}_t - \mathbb{E}_t\left[\eta_{\pi a}\hat{a}_{t+1}\right]\right) + \mathbb{E}_t\left[\eta_{ya}\hat{a}_{t+1}\right] \\
&= -\sigma\left(\eta_{ia}\hat{a}_t - \eta_{\pi a}\left(\rho_a\hat{a}_t + \mathbb{E}_t\left[\epsilon_{t+1}\right]\right)\right) + \eta_{ya}\left(\rho_a\hat{a}_t + \mathbb{E}_t\left[\epsilon_{t+1}\right]\right) \\
&= -\sigma\left(\eta_{ia}\hat{a}_t - \eta_{\pi a}\rho_a\hat{a}_t\right) + \eta_{ya}\left(\rho_a\hat{a}_t\right) \\
\eta_{ya} &= -\sigma\eta_{ia} + \sigma\eta_{\pi a}\rho_a + \eta_{ya}\rho_a \\
\eta_{ya} &= \frac{\sigma\left(\eta_{\pi a}\rho_a - \eta_{ia}\right)}{1-\rho_a} \qquad (1)
\end{aligned}$$

Then for $\hat{\pi}_t$:

$$
\begin{aligned}
\eta_{\pi a}\hat{a}_t &= \kappa\hat{a}_t\left(\eta_{ya} - \frac{1+\varphi}{\gamma+\varphi}\right) + \beta\eta_{\pi a}\left(\rho_a\hat{a}_t + \mathbb{E}_t\left[\epsilon_{t+1}\right]\right) \\
&= \kappa\hat{a}_t\left(\eta_{ya} - \frac{1+\varphi}{\gamma\varphi}\right) + \beta\eta_{\pi a}\rho_a\hat{a}_t \\
\eta_{\pi a} &= \kappa\left(\eta_{ya} - \frac{1+\varphi}{\gamma+\varphi}\right) + \beta\eta_{\pi a}\rho_a \\
\eta_{\pi a} &= \frac{\kappa\left(\eta_{ya} - \frac{1+\varphi}{\gamma+}\right)}{1-\beta\rho_a} \qquad (2)
\end{aligned}
$$

Then for $\hat{i}_t$:

$$
\begin{aligned}
\eta_{ia}\hat{a}_t &= \phi_\pi\eta_{\pi a}\hat{a}_t \\
\eta_{ia} &= \phi_\pi\eta_{\pi a} \qquad (3)
\end{aligned}
$$

Given (1), (2), (3), we can solve:

$$
\begin{aligned}
\eta_{ya} &= \frac{\sigma\frac{\kappa\left(\eta_{ya}-\frac{1+\varphi}{\gamma+\varphi}\right)}{1-\beta\rho_a}\left(\rho_a-\phi_\pi\right)}{1-\rho_a} = \frac{\left(\sigma\kappa\eta_{ya} - \sigma\kappa\frac{1+\varphi}{\gamma+\varphi}\right)\left(\rho_a-\phi_\pi\right)}{\left(1-\rho_a\right)\left(1-\beta\rho_a\right)} \\
\eta_{ya} &= -\frac{\sigma\kappa(1+\varphi)\left(\rho_a-\phi_\pi\right)}{\left(\left(1-\rho_a\right)\left(1-\beta\rho_a\right)-\sigma\kappa\left(\rho_a-\phi_\pi\right)\right)\left(\gamma+\varphi\right)} \qquad (4) \\
\eta_{\pi a} &= \frac{\kappa\left(-\frac{\sigma\kappa(1+\varphi)\left(\rho_a-\phi_\pi\right)}{\left(\left(1-\rho_a\right)\left(1-\beta\rho_a\right)-\sigma\kappa\left(\rho_a-\phi_\pi\right)\right)\left(\gamma+\varphi\right)} - \frac{1+\varphi}{\gamma+\varphi}\right)}{1-\beta\rho_a} \\
&= -\frac{\kappa(1+\varphi)\left(1-\rho_a\right)}{\left(\left(1-\rho_a\right)\left(1-\beta\rho_a\right)-\sigma\kappa\left(\rho_a-\phi_\pi\right)\right)\left(\gamma+\varphi\right)} \qquad (5) \\
\eta_{ia} &= \phi_\pi\eta_{\pi a} \qquad (6)
\end{aligned}
$$

Therefore,

$$
\hat{y}_t = -\frac{\sigma\kappa(1+\varphi)\left(\rho_a-\phi_\pi\right)}{\left(\left(1-\rho_a\right)\left(1-\beta\rho_a\right)-\sigma\kappa\left(\rho_a-\phi_\pi\right)\right)\left(\gamma+\varphi\right)}\hat{a}_t \qquad (7)
$$

$$
\hat{\pi}_t = -\frac{\kappa(1+\varphi)\left(1-\rho_a\right)}{\left(\left(1-\rho_a\right)\left(1-\beta\rho_a\right)-\sigma\kappa\left(\rho_a-\phi_\pi\right)\right)\left(\gamma+\varphi\right)}\hat{a}_t \qquad (8)
$$

$$
\hat{i}_t = \phi_\pi\left(-\frac{\kappa(1+\varphi)\left(1-\rho_a\right)}{\left(\left(1-\rho_a\right)\left(1-\beta\rho_a\right)-\sigma\kappa\left(\rho_a-\phi_\pi\right)\right)\left(\gamma+\varphi\right)}\right)\hat{a}_t \qquad (9)
$$

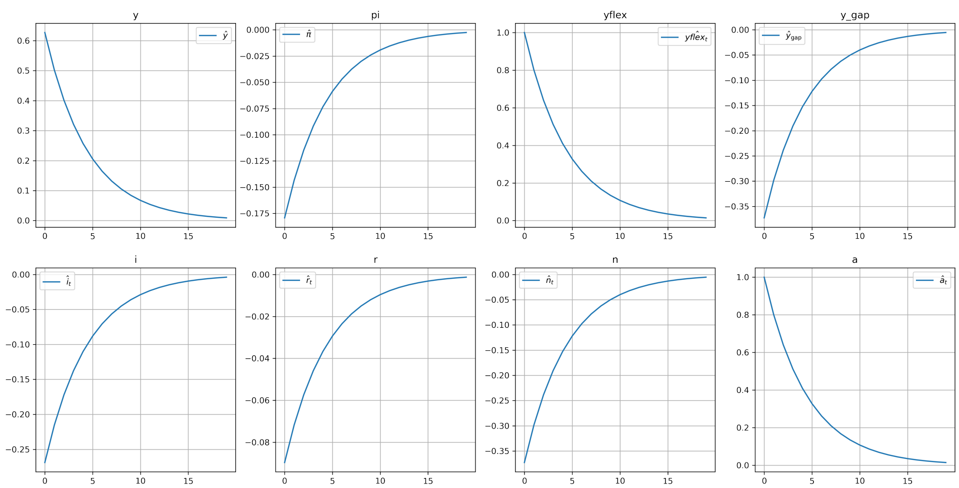
(b) Plot the impulse response function for $\hat{y}_t, \hat{\pi}_t, \hat{y}_t^{flex}, \hat{y}_t - \hat{y}_t^{flex}, \hat{i}_t, \mathbb{E}_t\hat{r}_{t+1}, \hat{n}_t, \hat{a}_t$ to a one unit shock to $\hat{a}_t$. Use the following parameter values:

$\beta = 0.99, \sigma = 1, \kappa = 0.1, \rho_a = 0.8, \phi_\pi = 1.5$

(c) Intuitively explain your results.

- The plot of $a_t$ reflects to AR(1) shock.
- If there is no sticky price, then a positive shock in productivity will lead to an equivalent increase in output $y^{\text{flex}}$.

(a) IRF From Analytical Solution




- Due to sticky prices, firms cannot freely adjust their cost, resulting in a smaller increase in output $y$.
- Both of these lead to the negative output gap $y^{\text{gap}}$.
- Employment $N$ decreases since productivity has increased more than output. The firms do not need as much employment as before.
- The increase in productivity allows firms to lower their prices and leads inflation $pi$ to be negative.
- According to the Taylor rule, the central bank will decrease the nominal interest rate $i$.
- Since $\phi_{pi} > 1$, the nominal interest rate decreases by more than inflation does. Then by the Fisher Equation, real interest rates $r$ decreases.

(d) Use the Jupyter notebook ”newkeynesianlinear.ipynb” to check that your plots in (b) are correct.

## 2. Non-linear NK model in Jupyter

Implement the standard new Keynesian model in Jupyter. We will write all conditions recursively and let the Sequence-Space Jacobian (SSJ) routines do the differentiation for us. Note that the first order conditions for firms and households are exactly as we have written in the lectures.

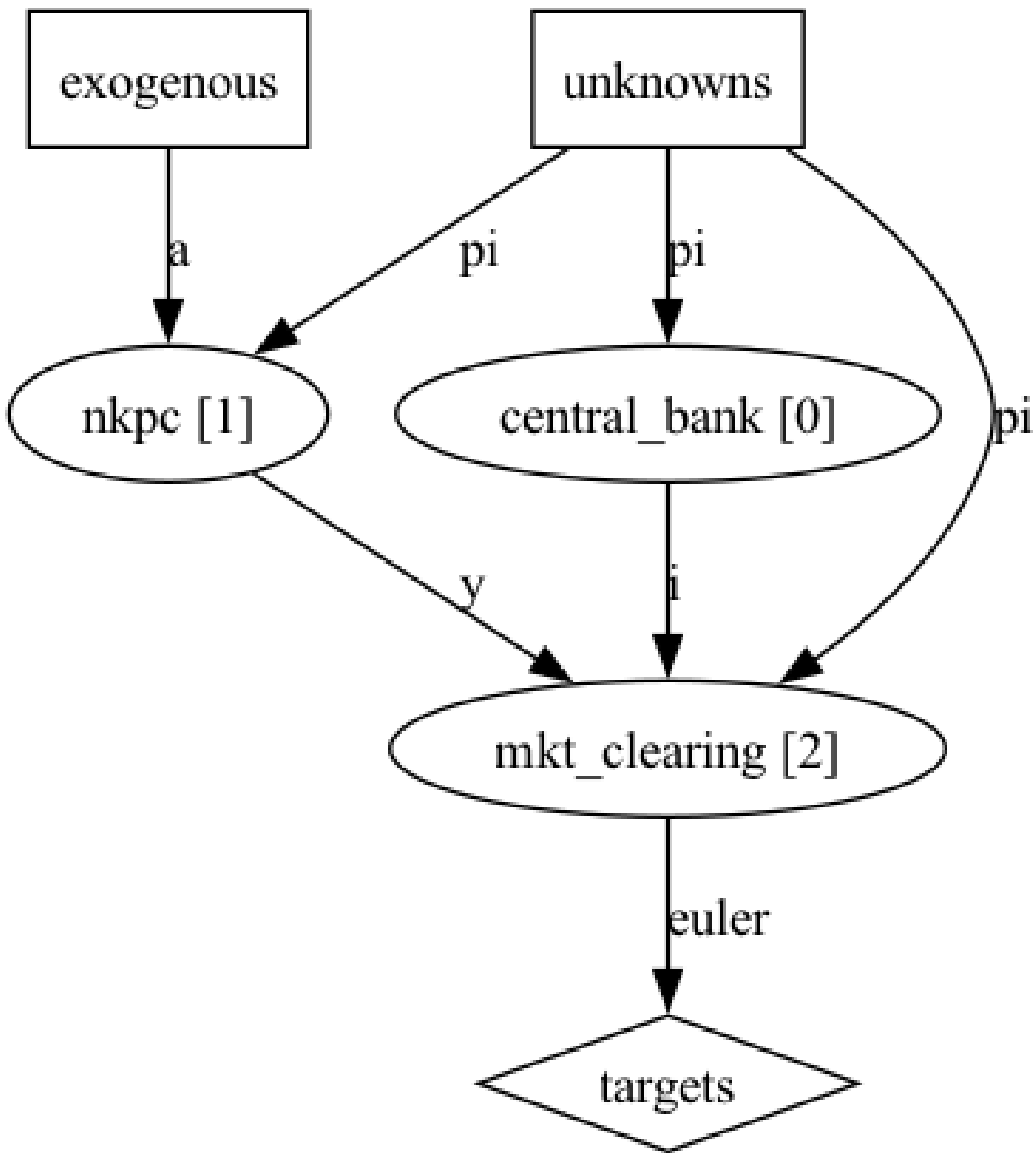
(a) The real reset price equation for the firm is,

$$p_t^* \equiv \frac{P_t^*}{P_t} = (1+\mu)E_t\left\{\sum_{s=0}^{\infty} \frac{\theta^s \Lambda_{t,t+s} Y_{t+s}(P_{t+s}/P_t)^{\epsilon-1}}{\sum_{k=0}^{\infty}\theta^k \Lambda_{t,t+k} Y_{t+k}(P_{t+k}/P_t)^{\epsilon-1}} \frac{W_{t+s}/P_t}{A_{t+s}}\right\}$$

Explain why this expression is not recursive.

- The equation is not recursive because it does not contain $P_{t+1}^*$. When the firm selects its price to

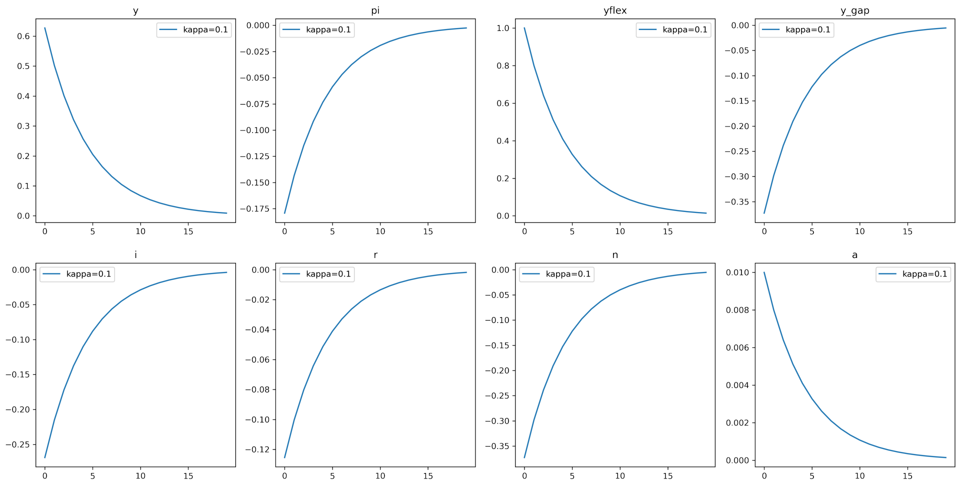
(a) DAG




optimize its profits, it does not need to consider any future $P^*$. This is due to the fact that the optimal price in the future is independent of the past prices.

(b) We next show that we can write $B_t = E_t(F_{1t}/F_{2t})$, where both $F_{1t}, F_{2t}$ are recursive. First, show that

(a) IRF From Log Linearization




the denominator can be recursively written as,

$$
\begin{aligned}
F_{2t} &\equiv \sum_{k=0}^{\infty} \theta^k \Lambda_{t,t+k} Y_{t+k} (P_{t+k}/P_t)^{\epsilon-1} \\
&= Y_t + \theta \Pi_{t+1}^{\epsilon-1} \Lambda_{t,t+1} F_{2,t+1}
\end{aligned}
$$

noting that $\Lambda_{t,t+k} = \Lambda_{t,t+1}\Lambda_{t+1,t+k}$ for all $k \geqslant 1$. We have:

$$
\begin{aligned}
F_{2t} &= \sum_{k=0}^{\infty} \theta^k \Lambda_{t,t+k} Y_{t+k} \left(\frac{P_{t+k}}{P_t}\right)^{\epsilon-1} = Y_t + \sum_{k=1}^{\infty} \theta^k \Lambda_{t,t+k} Y_{t+k} \left(\frac{P_{t+k}}{P_t}\right)^{\epsilon-1} \\
&= Y_t + \sum_{k=0}^{\infty} \theta^{k+1} \Lambda_{t,t+k+1} Y_{t+k+1} \left(\frac{P_{t+k+1}}{P_t}\right)^{\epsilon-1}
\end{aligned}
$$

We also have

$$
\begin{aligned}
\Lambda_{t,t+k+1} &= \beta^{k+1} \left(\frac{C_{t+k+1}}{C_t}\right)^{-\gamma} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \beta^k \left(\frac{C_{t+1+k}}{C_{t+1}}\right) = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \Lambda_{t+1,t+1+k} \\
\left(\frac{P_{t+k+1}}{P_t}\right)^{\epsilon-1} &= \left(\frac{P_{t+k+1}P_{t+1}}{P_t P_{t+1}}\right)^{\epsilon-1} = \prod_{t+1}^{\epsilon-1} \left(\frac{p_{t+1+k}}{P_{t+1}}\right)^{\epsilon-1}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
F_{2t} &= Y_t + \sum_{k=0}^{\infty} \theta^{k+1} \Lambda_{t,t+k+1} Y_{t+k+1} \left( \frac{P_{t+k+1}}{P_t} \right)^{\epsilon-1} \\
&= Y_t + \theta\beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \Pi_{t+1}^{\epsilon-1} \sum_{k=0}^{\infty} \theta^k \Lambda_{t+1,t+1+k} Y_{t+k+1} \left( \frac{P_{t+1+k}}{P_{t+1}} \right)^{\epsilon-1} \\
&= Y_t + \theta\beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \Pi_{t+1}^{\epsilon-1} F_{2,t+1}
\end{aligned}
$$

(c) Second, show that the numerator can be recursively written as,

$$
\begin{aligned}
F_{1t} &\equiv (1+\mu) \sum_{s=0}^{\infty} \theta^s \Lambda_{t,t+s} Y_{t+s} (P_{t+s}/P_t)^{\epsilon-1} \frac{W_{t+s}/P_t}{A_{t+s}} \\
&= (1+\mu) Y_t \frac{W_t/P_t}{A_t} + \theta \Pi_{t+1}^{\epsilon} \Lambda_{t,t+k} F_{1,t+1}
\end{aligned}
$$

noting that $\Lambda_{t,t+k} \Lambda_{t,t+1} \Lambda_{t+1,t+k}$ for all $k \geqslant 1$.

$$
\begin{aligned}
F_{1t} &= (1+\mu) \sum_{s=0}^{\infty} \theta^s \Lambda_{t,t+s} Y_{t+s} \left( \frac{P_{t+s}}{P_t} \right)^{\epsilon-1} \frac{W_{t+s}/P_t}{A_{t+s}} \\
&= (1+\mu) Y_t \frac{W_t}{P_t A_t} + \sum_{s=1}^{\infty} \theta^s \Lambda_{t,t+s} Y_{t+s} \left( \frac{P_{t+s}}{P_t} \right)^{\epsilon-1} \frac{W_{t+s}/P_t}{A_{t+s}} \\
&= (1+\mu) Y_t \frac{W_t}{P_t A_t} + \theta \sum_{s=0}^{\infty} \theta^s \Lambda_{t,t+s+1} Y_{t+1+s} \left( \frac{P_{t+1+s}}{P_t} \right)^{\epsilon-1} \frac{W_{t+1+s}/P_t}{A_{t+1+s}} \\
&= (1+\mu) Y_t \frac{W_t}{P_t A_t} + \theta\beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \Pi_{t+1}^{\epsilon-1} \frac{p_{t+1}}{p_t} \sum_{s=0}^{\infty} \theta^s \Lambda_{t+1,t+1+s} Y_{t+1+s} \left( \frac{P_{t+1+s}}{P_{t+1}} \right)^{\epsilon-1} \frac{W_{t+1+s}/P_{t+1}}{A_{t+1+s}} \\
&= (1+\mu) Y_t \frac{W_t}{P_t A_t} + \theta\beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \Pi_{t+1}^{\epsilon} F_{1,t+1}
\end{aligned}
$$

(d) Show that (gross) inflation can implicitly be written as

$$
1 = \theta \Pi_t^{\epsilon-1} + (1-\theta) p_t^{*1-\epsilon}
$$

The inflation is $\Pi_t = \frac{P_t}{P_{t-1}}$.

$$
\begin{aligned}
P_t &= \left( (1-\theta) \left(P_t^*\right)^{1-\epsilon} + \theta P_{t-1}^{1-\epsilon} \right)^{\frac{1}{1-\epsilon}} \\
P_t^{1-\epsilon} &= (1-\theta) \left(P_t^*\right)^{1-\epsilon} + \theta P_{t-1}^{1-\epsilon} \\
1 &= (1-\theta) \left( \frac{P_t^*}{P_t} \right)^{1-\epsilon} + \theta \left( \frac{P_{t-1}}{P_t} \right)^{1-\epsilon} \\
&= (1-\theta) B_t^{1-\epsilon} + \theta \Pi^{\epsilon-1}
\end{aligned}
$$

(e) Explain intuitively how when $p_t^* > 1$, then $\Pi_t > 1$.

- If $B_t$ is high that means that the price that firms that reset their price choose, $P_t^*$, is higher than the current price level $P_t$. The current price level is an average of $P_t^*$ and the previous price level, $P_{t-1}$. Therefore the $P_t^*$ is higher than $P_{t-1}$ and $P_t$ is higher than $P_{t-1}$. The higher $B_t$ (i.e. the higher $P_t^*$ relative to $P_t$ ) is the more the firms resetting their prices are going to pull the current price level higer than the previous price level causing higher inflation.

(f) Implement the non-linear NK using your recursive equations in Python using the Sequence Space Jacobian toolbox. For now, ignore the dispersion of labor in production and write the aggregate production function as $Y_t = A_t N_t$. Use the following parameters: $\beta = 0.99, \gamma = 1, \varphi = 1, \chi = 1, \epsilon = 10, \rho_a = 0.8, \phi_\pi = 1.5, \phi_y = 0$ where $A_t = (A_{t-1})^{\rho_a} e^{\epsilon_t^a}$. Productivity is the only shock. Price stickiness is specified below.

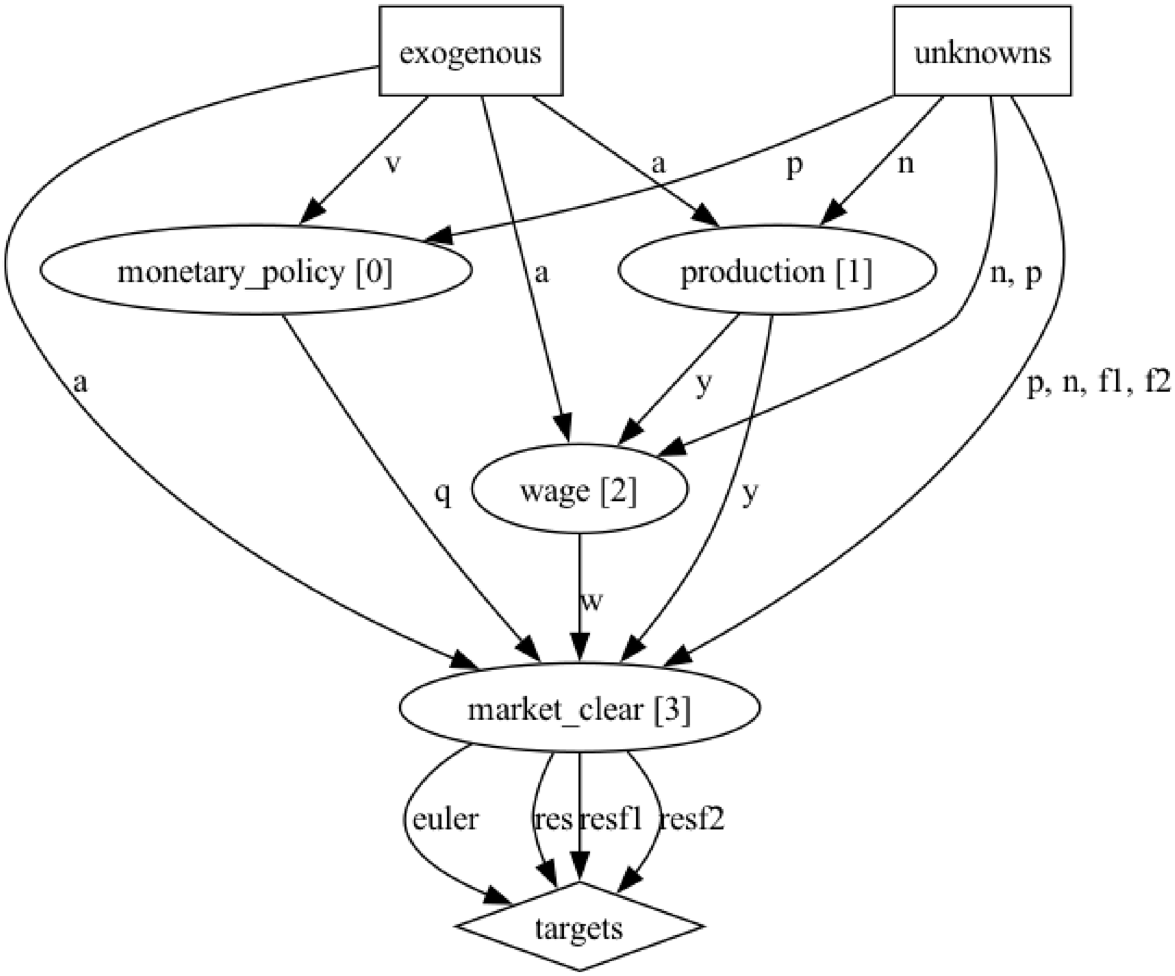
The raw non-linear equations are:

$$
\begin{aligned}
\frac{W_t}{P_t} &= \frac{\chi N_t^{\varphi}}{C_t^{-\gamma}} \\
1 &= \beta E_t \left\{ Q_t \frac{P_t}{P_{t+1}} \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \right\} \\
P_t &= \left[\theta P_{t-1}^{1-\varepsilon} + (1-\theta) P_t^{*1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}} \\
P_t^* &= (1+\mu) E_t \left\{ \sum_{s=0}^{\infty} \frac{\theta^s \Lambda_{t,t+s} Y_{t+s} P_{t+s}^{\varepsilon-1}}{\sum_{k=0}^{\infty} \theta^k \Lambda_{t,t+k} Y_{t+k} P_{t+k}^{\varepsilon-1}} \frac{W_{t+s}}{A_{t+s}} \right\} \\
Y_t &= C_t \\
Y_t &= A_t N_t \left[ \int_0^1 \left( \frac{N_t(i)}{N_t} \right)^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} \\
Q_t &= \beta^{-1} \left( \frac{P_t}{P_{t-1}} \right)^{\phi_\pi} e^{V_t}
\end{aligned}
$$

According to (b), (c), (d), we have

$$
\begin{aligned}
\frac{W_t}{P_t} &= \frac{\chi N_t^{\varphi}}{C_t^{-\gamma}} \\
1 &= \beta E_t \left\{ Q_t \frac{P_t}{P_{t+1}} \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \right\} \\
B_t &= \frac{F_{1t}}{F_{2t}} \\
F_{1,t} &= (1+\mu) Y_t \frac{W_t}{P_t A_t} + \theta\beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \Pi_{t+1}^{\epsilon} F_{1,t+1} \\
F_{2,t} &= Y_t + \theta\beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \Pi_{t+1}^{\epsilon-1} F_{2,t+1} \\
1 &= (1-\theta) B_t^{1-\epsilon} + \theta \Pi^{\epsilon-1} \\
Y_t &= C_t \\
Y_t &= A_t N_t \\
Q_t &= \beta^{-1} \left( \frac{P_t}{P_{t-1}} \right)^{\phi_\pi} e^{V_t}
\end{aligned}
$$

(a) DAG



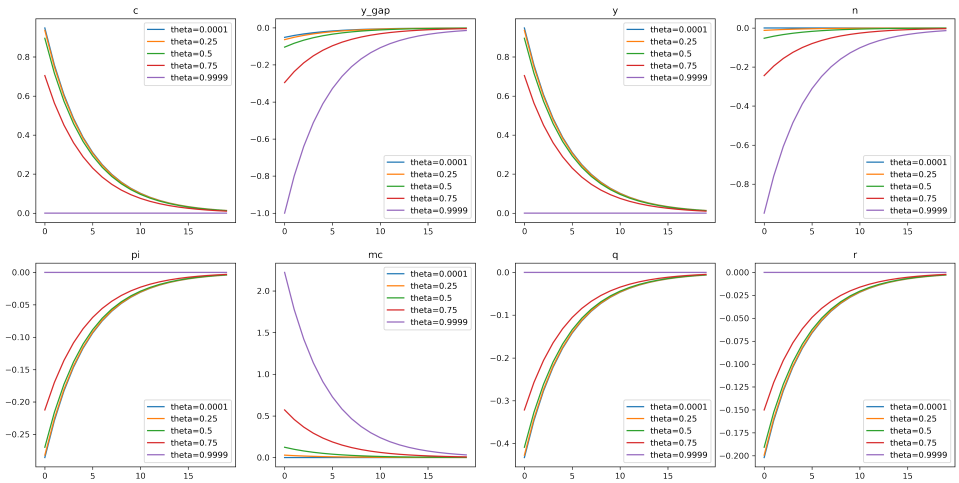

(g) Compute IRFs for $\theta \in \{0.0001, 0.25, 0.5, 0.75, 0.9999\}$ using a first order approximation to your non-linear equations.

Report the IRFs for consumption, the output gap, the level of output, employment, inflation, the mark-up, the nominal interest rate, and the ex-ante real interest rate. Your graph for each variable should contain all cases for $\theta$, appropriately labelled.

(h) Intuitively explain how the impulse response functions depend on the value of $\theta$.

- $\theta$ is the proportion of firms that can flexibly adjust their price.
  - When $\theta$ is close to 0, almost all firms can freely adjust prices in response to a productivity shock.
  - When $\theta$ is close to 1 almost all firms cannot adjust the prices.
- When firms can adjust their prices,
  - an increase in productivity would lead to an increase in output from firms and hence also increase consumption, and therefore the output gap is small.
  - Employment doesn't change because households' trade-offs between leisure and consumption are unchanged. Then the markup is unchanged.

(a) IRF from Nonlinear




  - An increase in productivity leads to lower inflation and interest rates as firms' average cost decreases leading to all firms lowering their prices. By the Taylor rule, the impact on nominal interest rates is greater than the impact on inflation.

- When firms cannot freely adjust their prices:
  - The inflation, nominal interest rates, and real interest rates do not respond to the productivity shock.
  - Since the economy is demand-driven, and productivity does impact the demand, output, and consumption do not change. Therefore, this leads to a large output gap.
  - Because the output is unchanged and firms are more productive, employment decreases.

(i) What would you expect to see from the same shock in an RBC model without capital? (No derivation should be necessary.)

- The same shock in an RBC model should be similar to the case that $\theta$ is close to 0 because in the RBC case, firms can freely adjust the price and therefore the output and consumption will be sensitive to the productivity shock.
- However, in RBC, there is no inflation and nominal interest rate, therefore we cannot observe the response of these two variables.