Estimate Learning Parameters

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This document is going to set a framework of the expectation formation process for some empirical specifications and cross-sectional estimation.

Framework

This section develops a partial equilibrium framework. The model framework is inspired by Grossman and Stiglitz (1980). For tractability, the model is presented in a static, one-period environment. However, the core insights of the analysis naturally extend to a dynamic context. In the following, the subjective belief operator is distinguished from the realized term or the objective probability measure. For example, $\mathbb{E}$ means the subjective expectation, and $\mathbb{V}$ means the subjective variance. E means the objective expectation, and V means the objective variance.

Environment

There is a finite number of times. Without loss of generality, denote t as current and $t+1$ as one-period in the future. There are two securities. The price of the risk-free short-term bill at time t is $P_{S,t}$, and its gross return is $r_f = \frac{P_{S,t+1}}{P_{S,t}}$. The market-determined price of the risky long-term bond at time t is $P_{L,t}$. The total supply of the risky long-term bond is Z_t and it is exogenous determined.

One Representative Agent

The total wealth of the representative agent is W_t. The asset positions of this agent are denoted by $X_{j,t}$, where $j = S$ means the quantity of the short-term bill and $j = L$ means the quantity of the long-term bond. Therefore, the total wealth can be expressed as follows:

$$
\begin{aligned}
W_{t+1} &= X_{F,t}P_{S,t+1} + X_{L,t}P_{L,t+1} && (1) \\
&= \left(\frac{W_t - X_{L,t}P_{L,t}}{P_{S,t}}\right) P_{S,t+1} + X_{L,t}P_{L,t+1} && (2) \\
&= \left(W_t - X_{L,t}P_{L,t}\right) r_f + X_{L,t}P_{L,t+1} && (3)
\end{aligned}
$$

The initial wealth could be normalized to a constant, denoted by W_0. This agent has a CARA utility with the risk aversion α.

$$U\left(W_t\right) = -e^{-\alpha W_t}. \tag{4}$$

Given the maximization of wealth under the CARA utility:

$$\max_{X_{L,t}} \mathbb{E}_t[U(W_{t+1})] = \max_{X_{L,t}} -\mathbb{E}_t\left[e^{-\alpha W_t}\right] = \max_{X_{L,t}} -\left[e^{-a\mathbb{E}_t[W_{t+1}]+\frac{\alpha^2}{2}\mathbb{V}_t[W_{t+1}]}\right] \tag{5}$$

which is equivalent to

$$\max_{X_{L,t}} \ \mathbb{E}_t[W_{t+1}] - \frac{\alpha}{2}\mathbb{V}_t[W_{t+1}], \tag{6}$$

the optimal demand for the long-term bond is

$$X_{L,t} = \frac{\mathbb{E}_t[P_{L,t+1}] - r_f P_{L,t}}{\alpha \mathbb{V}_t[P_{L,t+1}]}. \tag{7}$$

Given the market clear condition $X_t = Z_t$, we have the long term bond equilibrium price:

$$P_{L,t} = \frac{1}{r_f}\mathbb{E}_t[P_{L,t+1}] - \frac{\alpha}{r_f}\mathbb{V}_t[P_{L,t+1}]Z_t \tag{8}$$

Notice that there are 4 components in the price function: risk-free rate r_f, risk aversion level α, subjective expectation on future price $\mathbb{P}_{L,t+1}$ and the subjective variance on future price

$\mathbb{V}_{L,t+1}$[1]. The current price is positively correlated with the expectation of the next period price and negatively correlated with the conditional variance of the next period price and the total supply Z_t.

Learning

Assume the representative agent or the synthetic agent, observes signals of the next periods of the long-term bond price, which could be the foundation of the economy or other leading indicators, and help to form the subjective expectation of the asset price. Suppose the signal s_t at time t is in the form:

$$s_t = \underbrace{f(P_t)}_{\text{Market Info}} + \underbrace{m_t}_{\text{Fundamental}} + \underbrace{\epsilon_t}_{\text{Noise}} \tag{9}$$

where f is some transform function of the price. It could incorporate some behavior bias patterns and suppose it is normal distributed. ϵ_t is the noise with the normal distribution and independent of other variables:

$$\epsilon_t \sim N(\mu_\epsilon, \sigma_\epsilon^2) \tag{10}$$

and m_t could reflect the economic conditions, such as GDP or CPI, which follows

$$m_t \sim N(\mu_m, \sigma_m^2). \tag{11}$$

Therefore the variance of the signal is $\sigma_s^2 = \sigma_{fp}^2 + \sigma_m^2 + \sigma_\epsilon^2$.

Suppose $f(P_t) = 0$, $\sigma_{fp}^2 = 0$. If the agent is a Bayesian learner and prior is a normal distribution, at each point in time, after the price is formed, the prior mean is set to be the previous price[2] P_{t-1} and the prior variance is the previous posterior $\sigma_p^2 = \mathbb{V}_{t-1}[P_{t+1}|s_{t-1}]$,

[1] Note that there is only one agent so the subjective expectation also becomes the consensus expectation.
[2] In this section, simplifies $P_{L,t}$ as P_t.

then

$$
\begin{aligned}
\mathbb{E}_t[P_{t+1}|s_t] &= \frac{\frac{1}{\sigma_p^2}P_{t-1} + \frac{1}{\sigma_s^2}s_t}{\frac{1}{\sigma_p^2} + \frac{1}{\sigma_s^2}} = (1-\kappa)P_{t-1} + \kappa s_t && (12)\\
\mathbb{V}_t[P_{t+1}|s_t] &= \frac{1}{\sigma_p^2} + \frac{1}{\sigma_s^2} && (13)
\end{aligned}
$$

Therefore, the learning scheme is

$$
\begin{aligned}
\mathbb{E}_t[P_{t+1}|s_t] &= \mathbb{E}_{t-1}[P_{t+1}|s_{t-1}] + \frac{\frac{1}{\sigma_s^2}}{\frac{1}{\sigma_p^2} + \frac{1}{\sigma_s^2}}\left(s_t - \mathbb{E}_{t-1}[P_{t+1}|s_{t-1}]\right) && (14)\\
\mathbb{V}_t[P_{t+1}|s_t] &= \frac{1}{\mathbb{V}_{t-1}[P_{t+1}|s_{t-1}]} + \frac{1}{\sigma_s^2} && (15)
\end{aligned}
$$

Here the signal s_t only includes fundamental information. We can assume that they are some functions of real GDP growth and CPI.

$$
s_t = m_t + \epsilon_t = h(g_t, \pi_t) + \epsilon_t \tag{16}
$$

Although real GDP growth and CPI are public information, the interpretation could be idiosyncratic. The revelation of these different interpretations can be measured by the forecast of fundamentals. Therefore, $h(g_t, \pi_t)$ can be expressed as some functions of expectation of the real GDP growth and CPI:

$$
s_t^i = h^i(g_t, \pi_t) + \epsilon_t \approx \alpha^i + \beta_1^i \mathbb{E}_t^i[g_{t+1}] + \beta_2^i \mathbb{E}_t^i[\pi_{t+1}] + \epsilon_t^i \tag{17}
$$

Therefore, the learning equation (12) and (14) becomes

$$
\mathbb{E}_t^i[P_{t+1}|s_t] = (1-\kappa)\mathbb{E}_{t-1}^i[P_{t+1}|s_{t-1}] + \kappa(\alpha^i + \beta_1^i \mathbb{E}_t^i[g_{t+1}] + \beta_2^i \mathbb{E}_t^i[\pi_{t+1}] + \epsilon_t^i) \tag{18}
$$

Then I can run the empirical specification to estimate parameters and test whether the financial sector analysts are following this learning rule with the blue-chip data:

$$
\mathbb{E}_t^i[r_{t+1}] = \alpha_t + \beta_0 \mathbb{E}_{t-1}^i[r_{t+1}] + \beta_1 \mathbb{E}_t^i[g_{t+1}] + \beta_2 \mathbb{E}_t^i[\pi_{t+1}] + \epsilon_t^i \tag{19}
$$

The above estimation could suffer from omitted variable bias. If we take the price function into the expectation equation:

$$
\begin{aligned}
\mathbb{E}_t^i[P_{t+1}|s_t] &= \frac{\frac{1}{\sigma_p^2}(\frac{1}{r_f}\widetilde{\mathbb{E}}_{t-1}[P_t] - \frac{\alpha}{r_f}\widetilde{\mathbb{V}}_{t-1}[P_t]Z_{t-1}) + \frac{1}{\sigma_s^2}s_t^i}{\frac{1}{\sigma_p^2} + \frac{1}{\sigma_s^2}} \\
&= \frac{\sigma_s^2}{\sigma_p^2+\sigma_s^2}\frac{1}{r_f}\widetilde{\mathbb{E}}_{t-1}[P_t] - \frac{\sigma_s^2}{\sigma_p^2+\sigma_s^2}\frac{\alpha}{r_f}\widetilde{\mathbb{V}}_{t-1}[P_t]Z_{t-1} + \frac{\sigma_p^2}{\sigma_p^2+\sigma_s^2}s_t \\
&= \frac{\sigma_s^2}{\sigma_p^2+\sigma_s^2}\frac{1}{r_f}\widetilde{\mathbb{E}}_{t-1}[P_t] - \frac{\sigma_s^2}{\sigma_p^2+\sigma_s^2}\frac{\alpha}{r_f}\widetilde{\mathbb{V}}_{t-1}[P_t]Z_{t-1} \\
&+ \frac{\sigma_p^2}{\sigma_p^2+\sigma_s^2}(\alpha^i + \beta_1^i\mathbb{E}_t^i[g_{t+1}] + \beta_2^i\mathbb{E}_t^i[\pi_{t+1}] + \epsilon_t^i) \qquad (20)
\end{aligned}
$$

In the one representative agent setting, $\widetilde{\mathbb{E}}$ and $\widetilde{\mathbb{V}}$ are essentially just $\mathbb{E}$ and $\mathbb{V}$. In the empirical setting, I can estimate parameters and test it by regression:

$$
\mathbb{E}_t^i[r_{t+1}] = \alpha_t + \gamma_1 r_{t-1} + \gamma_2 Z_t + \beta_1\mathbb{E}_t^i[g_{t+1}] + \beta_2\mathbb{E}_t^i[\pi_{t+1}] + \epsilon_t^i \qquad (21)
$$

Here β_1 and β_2 are the interest of estimation. They reflect one perspective of the expectation formation process: how does the expectation of interest rate respond to the expectation of economic growth and inflation?

References

Grossman, Sanford J., and Joseph E. Stiglitz, 1980, On the Impossibility of Informationally Efficient Markets, *The American Economic Review* 70, 393–408, Publisher: American Economic Association.

APPENDIX

Data and Institution Background

This project mainly relies on the subjective expectation forecast data. The subjective expectation data is the Blue Chip financial forecast (BCFF) data from January 1988 and January 2021. BCFF provides monthly surveys of professional financial economists' expectations about all maturities of the U.S. yield curve and economic fundamentals, such as GDP and inflation. I mainly focus on the one-year forecast horizon, and therefore keep the forecast for the 4 quarters ahead and the first month of each quarter. To get the sector-level expectation, I assign each BCFF participant by the institution type based on the firms' names that are commonly used[3], and take the cross-sectional average as the forecast. The table 1 shows the category and each institution.

[3]For institutions with different businesses, I report its primary type.

Table 1: Institution Details

Institution	Candidates
bank	Regions Financial Corporation, Barnett Banks, Transohio Savings Bank, Lasalle National Bank, First National Bank of Chicago, Society National Bank, Wachovia Bank of Atlanta, First National Bank of Boston, Keycorp, Security Pacific Bank Washington, Banc One corp, First National Bank of Atlanta, PNC Bank corp., RBC, National Westminster Bank, Wells Fargo, Wachovia, Bank of America, Bank of America Merrill Lynch, NationsBanc Capital Markets, Suntrust Banks, BNP Paribas, Federal Home Loan Bank of San Francisco, Corestates Financial corp. Bank of Tokyo- Mitsubishi, Huntington National Bank, First City Bancorporation, Chemical Bank, First Jersey National Bank, National City Bank, Societe Generale, Bank One, Scotiabank, Comerica Bank, Valley National Bank, First Interstate Bank, Rainier National Bank, Chase Manhattan Bank, Fleet/Norstar Financial Group, Mellon Bank, Texas Commerce Bancshares, First Union corp., Manufacturers National Bank of Detroit, First Fidelity corp., Wells Fargo Bank, Marine Midland Bank, Citibank, UFJ Bank, Manufacturers Hanover Trust, Tokai Bank, Fleet Financial Group, MUFG Union Bank
broker	Prudential securities, Nomura, The Chicago Corporation, BMO Capital Markets, Barclays, Sanwa Securities (USA), Dean Witter Reynolds, Goldman Sachs & co., MF Global, Smith Barney, Lehman Brothers, Prudential Equity Group, Deutsche Bank, Daiwa Capital Markets America, RBS Securities, CRT Government Securities, Bear Stearns, Drexel Burnham Lambert inc., J.P. Morgan, Indosuez Carr Futures , Amherst Pierpont Securities, Merrill Lynch, Daiwa Securities America, Tucker Anthony inc., Harris Trust & Savings Bank, Aubrey G. Lanston & co., NatWest Markets, County NatWest Securities, Credit Suisse, UBS
insurance	Conning & Company, ING Financial Markets, Prudential Insurance, Equitable Life Assurance Society, Investor Briefing, New York Life Insurance comp., AIG, Swiss Re Group, Metropolitan Insurance comp., Kemper Financial Services inc.
fund	Bankers Trust, J.W. Coons Advisors llc, Mesirow Financial, C.J. Lawrence, I.B.J. Schroders inc., The Northern Trust comp., Via Nova Investment Management, ASB Capital Management, Ridgeworth Investments, Russell Investments, Stone Harbor Investment Partners, Trusco Capital Management, MCM, ING investment mgt., Wintrust Wealth Management, U.S. Trust Company, Wayne Hummer & Company, J.P Morgan privare wealth mgt. Chase Wealth Management, Crestar Investment Bank, Sanford c. Bernstein & co., Chicago Capital, Loomis, Sayles & Company, ACIMA Private Wealth, ClearBridge Investments, Aeltus Investment, Beacon Investment Company, Wells Capital Management