Ocean extremes: environmental risk, marginal and multivariate conditional extremes

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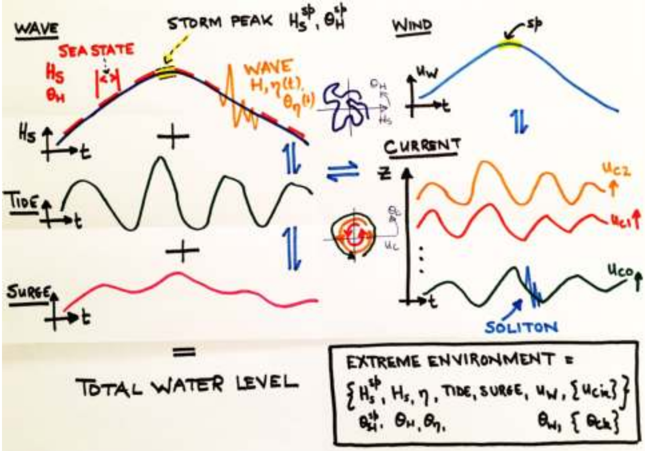
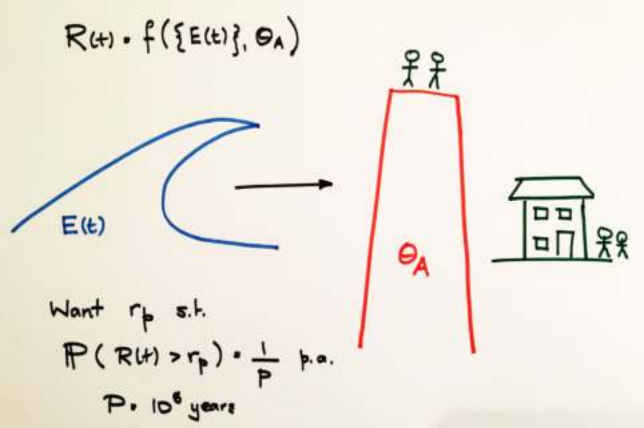
... with thanks to colleagues at Lancaster, Shell and elsewhere

Modelling ocean storm environment



- Multiple coupled physical processes
- Rare, extreme events

Modelling structural risk



- Ocean environment is harsh
- Marine structures at risk of failure
- Reliability standards must be met

Optimal design of marine structure

Set-up

- A marine system with "strength" specifications $\boldsymbol{\mathcal{S}}$
- An ocean environment $\boldsymbol{X}$ dependent on covariates $\boldsymbol{\Theta}$
- A structural "loading" $\boldsymbol{Y}$ as a result of environment X and covariates $\boldsymbol{\Theta}$
- System utility (or risk) $U(\boldsymbol{Y}|\boldsymbol{\mathcal{S}})$ for loading $\boldsymbol{Y}$ and specification $\boldsymbol{\mathcal{S}}$
- Desired U typically specified in terms of annual probability of failure
- $\boldsymbol{Y}|\boldsymbol{X}, \boldsymbol{\Theta}$ and $X|\boldsymbol{\Theta}$ (and U?) subject to uncertainty $\boldsymbol{Z}$
- $\boldsymbol{Z}, \boldsymbol{\Theta}, \boldsymbol{X}, \boldsymbol{Y}$ are multidimensional random variables

Optimal design

- A model $f_{X|\Theta,Z}$ for the environment
- A model $f_{Y|X,\Theta,Z}$ for environment-structure interaction
- A model $f_{\Theta|Z}$ for the covariates

$$\mathbb{E}[U|\boldsymbol{\mathcal{S}}] = \int_z \int_y \int_x \int_\theta U(\boldsymbol{y}|\boldsymbol{\mathcal{S}}, \boldsymbol{Z}) f_{Y|X,\Theta,Z}(\boldsymbol{y}|\boldsymbol{x}, \boldsymbol{\theta}, \boldsymbol{z}) f_{X|\Theta,Z}(\boldsymbol{x}|\boldsymbol{\theta}, \boldsymbol{z}) f_{\Theta|Z}(\boldsymbol{\theta}|\boldsymbol{z}) f_Z(\boldsymbol{z})\, d\boldsymbol{\theta}\, d\boldsymbol{x}\, d\boldsymbol{y}\, d\boldsymbol{z}$$

$\Rightarrow$ solve for $\boldsymbol{\mathcal{S}}$ to achieve required (safety) utility

Conventional engineering practice: environmental return values

- Estimating $\mathbb{E}[U|\boldsymbol{\mathcal{S}}]$ is difficult
- Ignore the structural response Y

- Design to extreme quantile of marginal annual distribution of single X instead

$$F_A(x) = \int_{\boldsymbol{Z}} \int_{\boldsymbol{\theta}} \int_k F^k_{X|\boldsymbol{\Theta},\boldsymbol{Z}}(x|\boldsymbol{\theta},\boldsymbol{Z}) f_{C|\boldsymbol{\Theta},\boldsymbol{Z}}(k|\boldsymbol{\theta},\boldsymbol{z}) f_{\boldsymbol{\Theta}|\boldsymbol{Z}}(\boldsymbol{\theta}|\boldsymbol{z}) f_{\boldsymbol{Z}}(\boldsymbol{z})\, dk\, d\boldsymbol{\theta}\, d\boldsymbol{z}$$

where $f_{C|\boldsymbol{\Theta},\boldsymbol{Z}}$ is the density of annual rate of events given covariate Θ.
- Set the return value x_T (for $T = 1000$ years say) such that

$$F_A(x_T) = 1 - \frac{1}{T}$$

- Specify conditional return values for other Xs given $X = x_T$
- Potentially as a function of covariates
- Ambiguous ordering of expectation operators ... more later

Better: model the non-stationary multivariate extreme environment

- Expected utility and return values are dominated by extreme environments
- Have to estimate tails of distributions well

- Focus on a simple $\boldsymbol{Z}$-free 2-D environment with stationary dependence

$$F_{\boldsymbol{X}|\boldsymbol{\Theta},\boldsymbol{Z}}(\boldsymbol{x}|\boldsymbol{\theta},\boldsymbol{z}) = C\Big(F_{X_1|\Theta}(x_1|\theta), F_{X_2|\Theta}(x_2|\theta)\Big) \text{ for simplicity, so}$$

$$\begin{aligned} f_{\boldsymbol{X}|\boldsymbol{\Theta},\boldsymbol{Z}}(\boldsymbol{x}|\boldsymbol{\theta},\boldsymbol{z}) &= f_{X_1,X_2|\Theta}(x|\theta) \\ &= f_{X_1|\Theta}(x_1|\theta) f_{X_2|\Theta}(x_2|\theta) \times c\Big(F_{X_1|\Theta}(x_1|\theta), F_{X_2|\Theta}(x_2|\theta)\Big) \text{ typically} \end{aligned}$$

- Marginal models (non-stationary, extreme) $f_{X_1|\Theta}(x_1|\theta)$, $f_{X_2|\Theta}(x_2|\theta)$
- Multivariate model on standard marginal scale (stationary, "extreme") $c(u_1, u_2)$

Generalised Pareto distribution

- Suppose we have an exceedance X of high threshold $\psi \in \mathbb{R}$
- The Pickands-Balkema-De Haan theorem states

$$\begin{aligned}\lim_{\psi \to \infty} \mathbb{P}[X \leq x | X > \psi] &= \lim_{\psi \to \infty} \frac{F_X(x)}{1 - F_X(\psi)} \\ &= \mathrm{GP}(x|\xi, \sigma, \psi) \\ &= 1 - \left(1 + \frac{\xi}{\sigma}(x - \psi)\right)_+^{-1/\xi}, \quad \sigma > 0, \quad \xi \in \mathbb{R}\end{aligned}$$

Theory

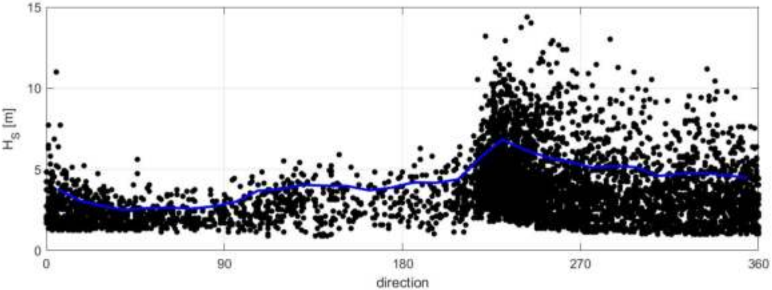
- Derived from max-stability of F_X
- Threshold-stability property
- "Poisson $\times$ GP $=$ GEV"

Practicalities

- How to isolate independent threshold exceedances from observed time-series?
- How to specify extreme threshold ψ?
- ξ, σ, ψ functions of covariates

Motivation

- Environmental extremes vary smoothly with multidimensional covariates
- Generic modelling framework for different covariate representations
- Statistical and computational efficiency for n-D covariates
- Full (Bayesian) uncertainty quantification



Typical data for northern North Sea. Storm peak H_S on direction, with $\tau = 0.8$ extreme value threshold.
Rate and size of occurrence varies with direction.

Model for size of occurrence

- Sample of storm peaks Y over threshold $\psi_\theta \in \mathbb{R}$, with 1-D covariate $\theta \in \mathcal{D}_\theta$
- Extreme value threshold ψ_θ assumed known
- Y assumed to follow generalised Pareto distribution with shape ξ_θ, (modified) scale ν_θ

$$f_{\mathrm{GP}}(y|\xi_\theta, \nu_\theta) = \frac{1}{\sigma_\theta}\left(1 + \frac{\xi_\theta}{\sigma_\theta}(y - \psi_\theta)\right)_+^{-1/\xi_\theta - 1} \text{ with } \nu_\theta = \sigma_\theta(1 + \xi_\theta)$$

- Shape parameter $\xi_\theta \in \mathbb{R}$ and scale parameter $\nu_\theta > 0$
- (Non-stationary Poisson model for rate of occurrence, with rate $\rho_\theta \geq 0$)

Covariate representations in 1-D

- Index set $\mathcal{I}_\theta = \{\theta_s\}_{s=1}^m$ on periodic covariate domain $\mathcal{D}_\theta$
- Each observation belongs to exactly one θ_s
- On $\mathcal{I}_\theta$, assume

$$\begin{aligned} \eta_s &= \sum_{k=1}^{n} B_{sk}\beta_k, s = 1, 2, ..., m, \text{ or} \\ \boldsymbol{\eta} &= \boldsymbol{B}\boldsymbol{\beta} \end{aligned}$$

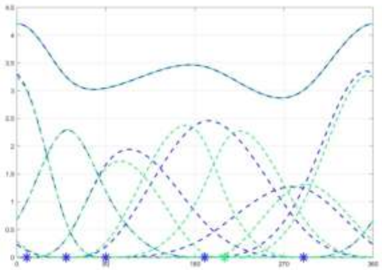
- $\eta \in (\xi, \nu)$ (and similar for ρ)
- $\boldsymbol{B} = \{B_{sk}\}_{s=1;k=1}^{m;n}$ basis for $\mathcal{D}_\theta$
- $\boldsymbol{\beta} = \{\beta_k\}_{k=1}^{n}$ basis coefficients
- Inference reduces to estimating $n_\xi, n_\nu, \boldsymbol{B}_\xi, \boldsymbol{B}_\nu, \boldsymbol{\beta}_\xi, \boldsymbol{\beta}_\nu$ (and roughnesses λ_ξ, λ_ν)

- P-splines, BARS and Voronoi are different forms of $\boldsymbol{B}$
- Tensor products and slick GLAM algorithms for n-D covariate representations

Basis representations ... BARS and others

Bayesian adaptive regression splines (BARS)

- n irregularly-spaced knots on $\mathcal{D}_\theta$
- $\boldsymbol{B}$ consists of n B-spline bases
- Order d
- Each using $d+1$ consecutive knot locations
- Local support
- Wrapped on $\mathcal{D}_\theta$
- Knot locations $\{r_k\}_{k=1}^n$ vary
- Number of basis functions n varies



Periodic BARS knot birth and death

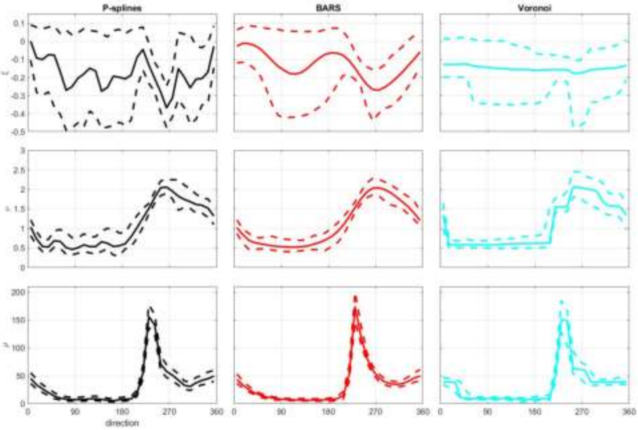
P-splines and Voronoi partition

- P-splines use fixed number of regularly-spaced knots
- Voronoi partition uses piecewise-constant representation, trivially extended to n-D

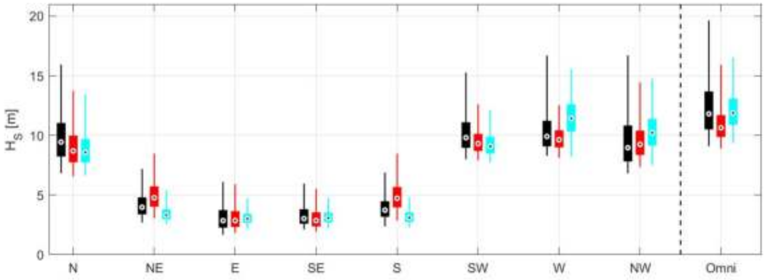
Posterior parameter estimates for ξ, ν and ρ for northern North Sea

- MCMC inference (Gibbs sampling, reversible jump, etc.)

- Note colour scheme
- Rate ρ and ν very similar
- Voronoi gives almost constant ξ
- Voronoi piecewise constant
- Land shadow effects
- General agreement
- ... for other parameters also



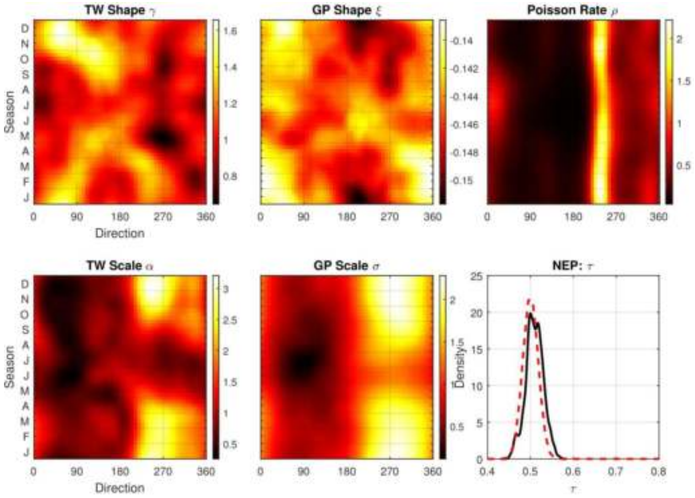
Directional posterior predictive distribution of $T = 1000$-year maximum



- Box-whiskers with 2.5%, 25%, 50%, 75% and 97.5% percentiles
- General agreement

- This is more-or-less what the engineer needs to design a "compliant" structure

Extension to 2D : directional-seasonal



- 2-D tensor product P-spline bases for same northern North Sea location
- Marginal posterior median estimates (plus posterior density for τ)

Recap: model the non-stationary multivariate extreme environment

- Expected utility dominated by extreme environments

$$\mathbb{E}[U|\boldsymbol{\mathcal{S}}] = \int_z \int_y \int_x \int_\theta U(\boldsymbol{y}|\boldsymbol{\mathcal{S}},\boldsymbol{Z}) f_{Y|X,\Theta,Z}(\boldsymbol{y}|\boldsymbol{x},\boldsymbol{\theta},\boldsymbol{z}) f_{X|\Theta,Z}(\boldsymbol{x}|\boldsymbol{\theta},\boldsymbol{z}) f_{\Theta|Z}(\boldsymbol{\theta}|\boldsymbol{z}) f_Z(\boldsymbol{z})\, d\boldsymbol{\theta}\, d\boldsymbol{x}\, d\boldsymbol{y}\, d\boldsymbol{z}$$

- Copulas (suppressing $\boldsymbol{Z}$ for clarity)

$$F_{\boldsymbol{X}|\boldsymbol{\Theta}}(\boldsymbol{x}|\boldsymbol{\theta}) = C\Big(F_{X_1|\boldsymbol{\Theta}}(x_1|\boldsymbol{\theta}), F_{X_2|\boldsymbol{\Theta}}(x_2|\boldsymbol{\theta}), ..., F_{X_p|\boldsymbol{\Theta}}(x_2|\boldsymbol{\theta})|\boldsymbol{\theta}\Big)$$

- We already have marginal models $F_{X_j|\boldsymbol{\Theta}}(x_j|\boldsymbol{\theta})$, $j = 1, 2, ..., p$
- Now we need a dependence model or copula $C = C(u_1, u_2, ..., u_p)$

Which dependence function?

Max-stability == multivariate extreme value distribution (MEVD)

- On uniform margins, extreme value copula: $C(\boldsymbol{u}) = C^k(\boldsymbol{u}^{1/k})$
- On Fréchet margins ($G_j(z) = \exp\left(-z^{-1}\right)$), $G(\boldsymbol{z}) = \exp\left(-V(\boldsymbol{z})\right)$, for exponent measure V such that $V(r\boldsymbol{z}) = r^{-1}V(\boldsymbol{z})$, homogeneity order -1
- Rich spatial extensions to max-stable processes, MSPs
- Multivariate generalised Pareto distribution, MGPD
- Max-stability involves a common but often unrealistic assumption ... component-wise maxima

AD and AI

- All MEVD distributions exhibit asymptotic dependence (AD)
- Many common distributions (e.g. the multivariate Gaussian) exhibit asymptotic independence (AI)
- So extreme value copulas are not general enough to describe extremal dependence in nature
- Other copula forms do exhibit AI
- The conditional extremes model admits AD (on the boundary) and AI

Conditional extremes ... moving beyond component-wise maxima

- $\boldsymbol{X} = (X_1, ..., X_j, ..., X_p)$
- Each X and Y have standard Laplace margins ($f(x) = \exp(-|x|)/2$, $x \in \mathbb{R}$)
- Seek a model for $\boldsymbol{X}|(Y = y)$ for $y > u$

- Assume we can find p-dimensional scaling $\boldsymbol{a}, \boldsymbol{b} > \boldsymbol{0}$ such that

$$\mathbb{P}(\boldsymbol{Z} \leq \boldsymbol{z}|Y = y) \quad \rightarrow \quad G(\boldsymbol{z}) \quad \text{as} \quad u \rightarrow \infty$$
$$\text{for} \quad \boldsymbol{Z} \quad = \quad \frac{\boldsymbol{X} - \boldsymbol{a}(y)}{\boldsymbol{b}(y)}$$

- Non-degenerate G is unknown, and estimated empirically

- Typical scaling is $\boldsymbol{a} = \boldsymbol{\alpha} y$ and $\boldsymbol{b} = y^{\boldsymbol{\beta}}$, $\boldsymbol{\alpha} \in [-1, 1]^p$, $\boldsymbol{\beta} \in (-\infty, 1]^p$
- So simply fit regression model

$$\boldsymbol{X}|(Y = y) = \boldsymbol{\alpha} y + y^{\boldsymbol{\beta}} \boldsymbol{Z}, \text{ for } y > u$$

- $\alpha = 1, \beta = 0$: perfect dependence and AD, and $\alpha \in (0, 1)$: AI

- Heffernan and Tawn [2004] find choices for α and β for popular bivariate cases
- Bivariate Gaussian : $\alpha = \rho^2$, $\beta = 1/2$

Developments of the conditional extremes model

Canonical extensions

- Basic: $X|(Y=y), y>u$
- Temporal: "heatwave model" $X_1, X_2, ..., X_\tau|(X_0=x_0), x_0>u$
- Spatial: "spatial conditional extremes" $X_1, X_2, ..., X_s|(X_0=x_0), x_0>u$

Idea

$$X_1, X_2, ..., X_p|(Y=y) = \boldsymbol{\alpha} y + y^{\boldsymbol{\beta}} \boldsymbol{Z}$$

- Impose appropriate structure on parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and distribution of $\boldsymbol{Z}$
  - e.g. $\boldsymbol{\alpha}$ evolves smoothly in space
  - e.g. $\boldsymbol{Z}$ follows a multivariate Gaussian or extension thereof with appropriate mean and covariance forms
- Make a simplifying assumption
  - e.g. apply a low-order model repetitively
    $X_{t+1}, X_{t+2}|(X_t=x) = [\alpha_1 \alpha_2]x + x^{[\beta_1 \beta_2]}[Z_1 Z_2]$

Further extensions

- Non-stationary and multivariate temporal and spatial models

Multivariate spatial conditional extremes (MSCE)

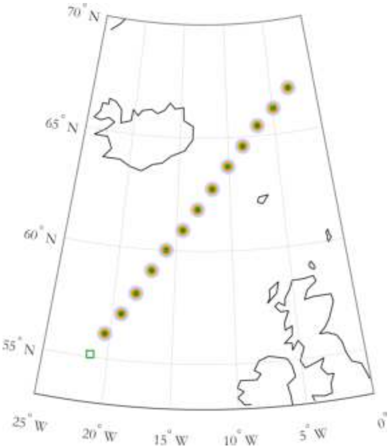
Motivation

- How useful are satellite observations of ocean waves and winds?
- Could they become the primary data source for decisions soon?
- What are the spatial characteristics of extremes from satellite observations?

Overview

- A look at the data : satellite wind, hindcast wind, hindcast wave
- Brief overview of methodology
- Results for joint spatial structure of extreme scatterometer wind speed, hindcast wind speed and hindcast significant wave height in the North Atlantic
- Implications for future practical applications

In a nut-shell



- Condition on large value x of first quantity X_{01} at one location $j = 0$ (green square)
- Estimate "conditional spatial profiles" for $m > 1$ quantities $\{X_{jk}\}_{j=1,k=1}^{p,m}$ at $p > 0$ other locations (green, orange and blue circles)

$$X_{jk} \sim \text{Lpl}$$

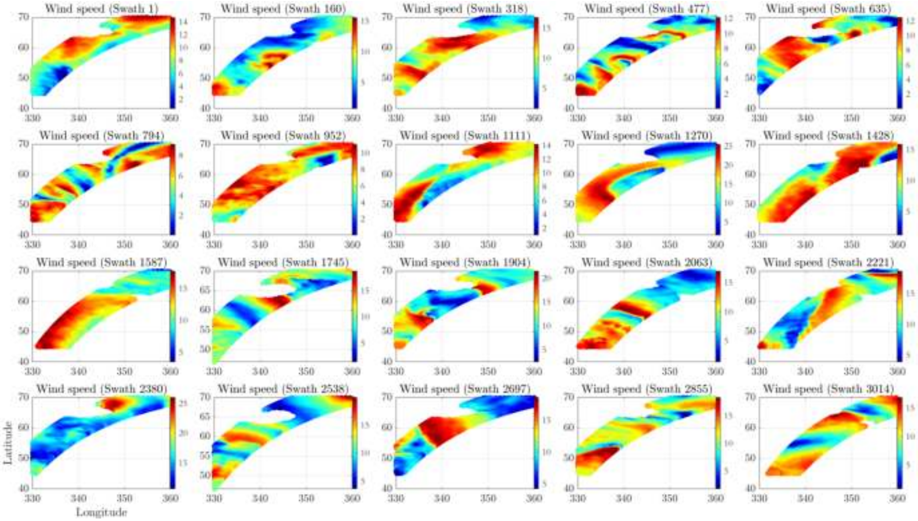
$$x > u$$

$$\boldsymbol{X}|\{X_{01} = x\} = \boldsymbol{\alpha}x + x^{\boldsymbol{\beta}}\boldsymbol{Z}$$

$$\boldsymbol{Z} \sim \text{DL}(\boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\delta}; \boldsymbol{\Sigma}(\boldsymbol{\lambda}, \boldsymbol{\rho}, \boldsymbol{\kappa}))$$

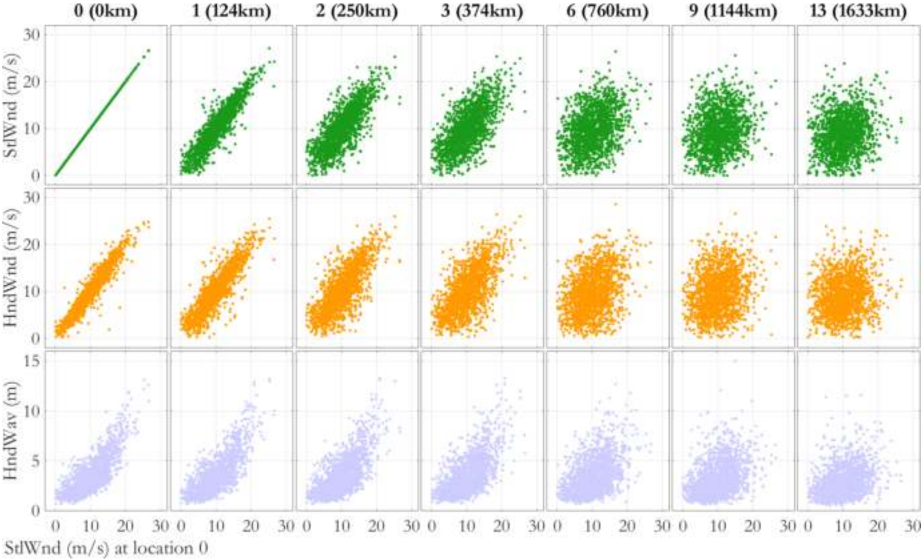
- MCMC to estimate $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\mu}$, $\boldsymbol{\sigma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\rho}$, $\boldsymbol{\kappa}$, $\boldsymbol{\lambda}$
- $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\mu}$, $\boldsymbol{\sigma}$, $\boldsymbol{\delta}$ spatially smooth for each quantity
- DL = delta-Laplace = generalised Gaussian
- Residual correlation $\boldsymbol{\Sigma}$ for conditional Gaussian field, powered-exponential decay with distance

Swath wind speeds



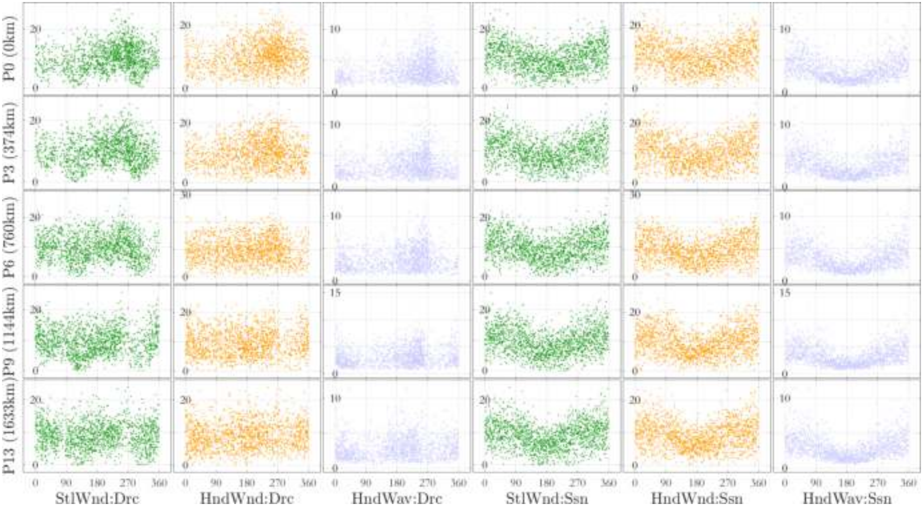
Daily descending METOP swaths. Satellite swath location changes over time. Spatial structure evident

Scatter plots on physical scale



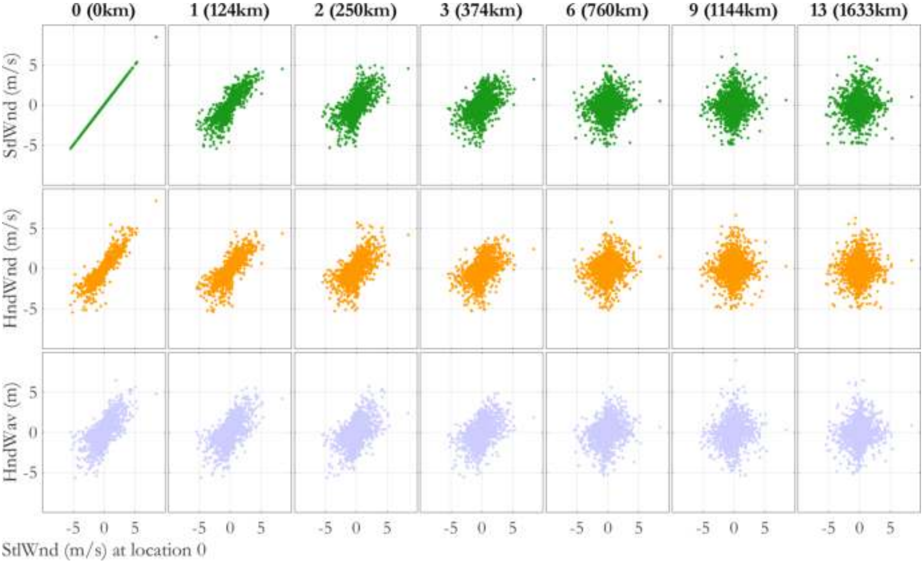
Scatter plots of registered data : StlWnd (green), HndWnd (orange), HndWav(blue)

Covariate dependence on physical scale



Directional and seasonal dependence. “Direction” is that from which fluid flows measured clockwise from North
StlWnd (green), HndWnd (orange), HndWav(blue)

Scatter plots **on Laplace scale**



Registered data on Laplace scale: StlWnd (green), HndWnd (orange), HndWav(blue)

Inference

$$X_{jk} \sim \text{Lpl}, \quad x > u, \quad \boldsymbol{X} | \{X_{01} = x\} = \boldsymbol{\alpha} x + x^{\boldsymbol{\beta}} \boldsymbol{Z}, \quad \boldsymbol{Z} \sim \text{DL}(\boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\delta}; \boldsymbol{\Sigma}(\boldsymbol{\lambda}, \boldsymbol{\rho}, \boldsymbol{\kappa}))$$

- Delta-Laplace residual margins

$$f_{Z_{j,k}}(z_{j,k}) = \frac{\delta_{j,k}}{2\kappa_{j,k}\sigma_{j,k}\Gamma\left(\frac{1}{\delta_{j,k}}\right)} \exp\left\{ - \left| \frac{z - \mu_{j,k}}{\kappa_{j,k}\sigma_{j,k}} \right|^{\delta_{j,k}} \right\}, \quad \kappa_{j,k}^2 = \Gamma\left(1/\delta_{j,k}\right) / \Gamma\left(3/\delta_{j,k}\right)$$

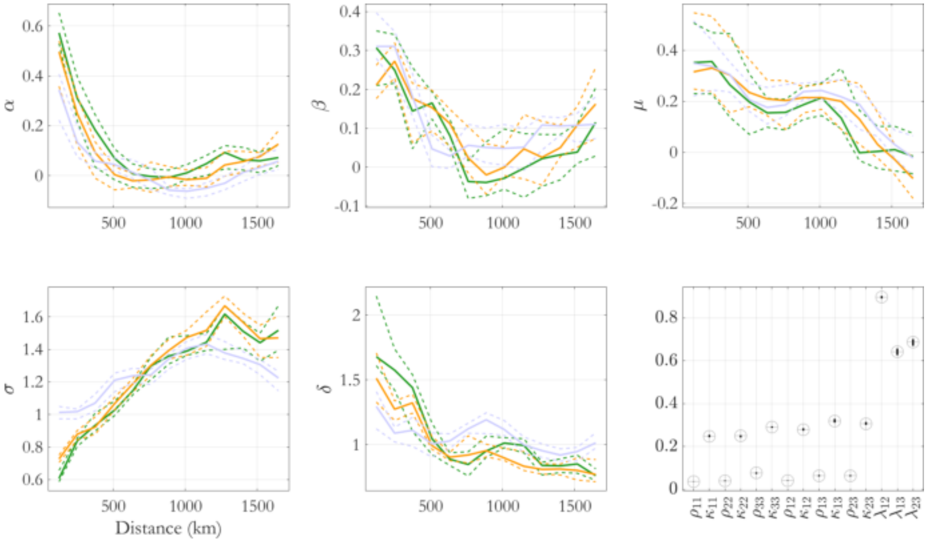
- Gaussian residual dependence

$$\boldsymbol{\Sigma}^*_{\mathcal{A}^*(j,k)\mathcal{A}^*(j',k')} = \lambda_{k,k'}^{|k-k'|} \exp\left(- \left(\frac{\text{dist}(r_j, r_{j'})}{\rho_{k,k'}} \right)^{\kappa_{k,k'}} \right)$$

- Piecewise linear forms for $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\mu}$, $\boldsymbol{\sigma}$, $\boldsymbol{\delta}$ with distance using n_{Nod} spatial nodes

- Adaptive MCMC, Roberts and Rosenthal [2009]
- Total of $m(5n_{\text{Nod}} + (3m+1)/2)$ parameters
- Rapid convergence, 10k iterations sufficient

Parameter estimates



Estimates for α, β, μ, σ and δ with distance, and residual process estimates ρ, κ and λ. Model fitted with $\tau = 0.75$
StlWnd (green), HndWnd (orange), HndWav(blue)
Residual Gaussian field : ρ=scale (need to $\times$100km), κ=exponent (need to $\times$5), λ=cross-correlation

Estimating return values and associated values

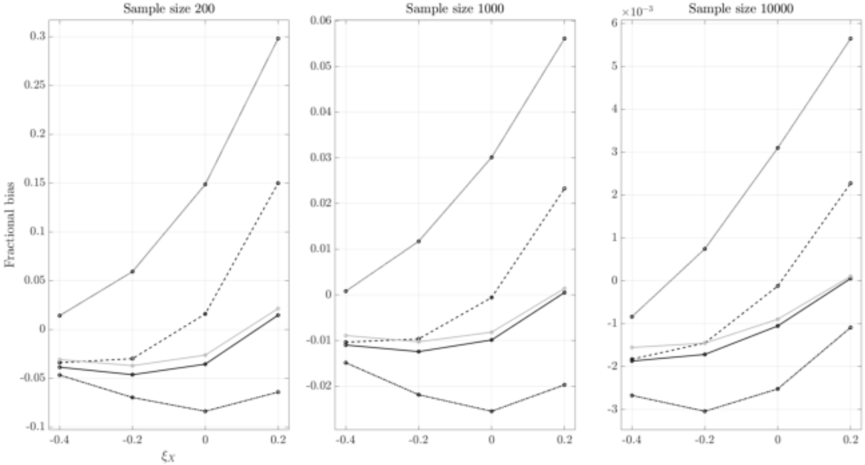
- Uncertain estimates of GP model parameters from fit to sample represented by random variables $\boldsymbol{Z}$
- Estimate distribution $F_{A|\boldsymbol{Z}}$ of annual maximum event using $\boldsymbol{Z}$
- Estimate N-year return value by finding the $1 - 1/N$ quantile of $F_{A|\boldsymbol{Z}}$
- What possibly could go wrong?

- Various options available, including:

$$
\begin{aligned}
q_1 &= F^{-1}_{A|\boldsymbol{Z}}(1 - 1/N \mid \mathbb{E}_{\boldsymbol{Z}}[\boldsymbol{Z}]) = F^{-1}_{A|\boldsymbol{Z}}(1 - 1/N \mid \int_{\boldsymbol{z}} \boldsymbol{z} f_{\boldsymbol{Z}}(\boldsymbol{z}) d\boldsymbol{z}) \\
q_2 &= \mathbb{E}_{\boldsymbol{Z}}[F^{-1}_{A|\boldsymbol{Z}}(1 - 1/N \mid \boldsymbol{Z})] = \int_{\boldsymbol{z}} F^{-1}_{A|\boldsymbol{Z}}(1 - 1/N \mid \boldsymbol{z}) f_{\boldsymbol{Z}}(\boldsymbol{z}) d\boldsymbol{z} \\
q_3 &= \tilde{F}^{-1}_{A}(1 - 1/N) \text{ where } \tilde{F}_A(x) = \int_{\boldsymbol{z}} F_{A|\boldsymbol{Z}}(x \mid \boldsymbol{z}) f_{\boldsymbol{Z}}(\boldsymbol{z}) d\boldsymbol{z} \\
q_4 &= \tilde{F}^{-1}_{A_N}(\exp(-1)) \text{ where } \tilde{F}_{A_N}(x) = \tilde{F}^{N}_{A}(x) \\
q_5 &= \text{med}_{\boldsymbol{Z}}[F^{-1}_{A|\boldsymbol{Z}}(1 - 1/N \mid \boldsymbol{Z})]
\end{aligned}
$$

- For small samples, these have very different properties

Fractional bias of return value estimators



Fractional bias of return value estimates from different estimators using maximum likelihood, as a function of sample size and true GP shape ξ. LHS top to bottom: q_3, q_2, q_5, q_1, q_4.

- Knock-on effects for associated values of the form $\mathbb{E}_{\boldsymbol{Z}}(Y|X = q, \boldsymbol{Z})$

Pragmatic non-stationary multivariate extremes with UQ

- Fit generalised Pareto marginal models for peaks over threshold data $\dot{X}$ and $\dot{Y}$
  - Multiple thresholds, simple covariate model (e.g. piecewise constant)
  - Diagnostics: threshold stability
- Transform to standard Laplace scale X and Y
  - Transform full sample
- Fit conditional extremes model $X|(Y = y)$ for $y > u$
  - Multiple thresholds, simple covariate model (e.g. piecewise constant) for α
  - Diagnostics: threshold stability, residual structure
- Calculate probabilities of extreme sets
  - MC simulation, importance sampling
  - Estimate environmental contours

- Free PPC software for MATLAB does all of above
  - UQ: incorporates epistemic uncertainty using bootstrapping cradle to grave
  - Model averaging: incorporates multiple models for different threshold combinations
  - Multidimensional X and covariates
  - Cross-validation for optimal parameter roughness in marginal and dependence models
  - Careful return value and associated value definitions
  - `https://github.com/ECSADES/ecsades-matlab`

Summary

Why?

- Careful quantification of “rare-event” risk
- Characterise tails of (multivariate) distributions
- Limited observations
- Combine solid theory and pragmatic application, UQ
- Immediate real-world consequences

The next 10 years?

- Univariate : fuller covariate descriptions, exploit measurement scale / sub-asymptotics, UQ, provide real-world decision-support
- Multivariate : theoretical development, computational tractability, expansion in scope (time-series, spatial), serious real-world applications
- More demanding regulatory framework

Thanks for listening / Diolch am wrando!

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Backup

Marginal extremes

- Theory : Embrechts et al. [2003], Beirlant et al. [2004]
- Method : Coles [2001], Dey and Yan [2016]
- Motivation : Davison and Smith [1990], Chavez-Demoulin and Davison [2005]

- Practicalities : Jonathan and Ewans [2013], Feld et al. [2019]
- Semi-parametric : Randell et al. [2016], Zanini et al. [2020]
- Lots more : Wood [2003]

Generalised extreme value distribution

- F_X^n is the distribution of the maximum of n independent draws of X
- If F_X^n "looks like" $F_X^{n'}$, we say F_X is max-stable

- More formally, F_X is max-stable if there exist sequences of constants $a_n > 0$, b_n, and non-degenerate G_ξ such that

$$\lim_{n\to\infty} F_X^n\left(a_n x + b_n\right) = G_\xi(x)$$

- We say $F_X \in D(G_\xi)$ or that F_X lies in the max-domain of attraction of G_ξ
- The Fisher–Tippett–Gnedenko theorem states that G_ξ is the generalised extreme value distribution with parameter ξ

$$G_\xi(y) = \exp\left(-\left(1+\xi y\right)^{-1/\xi}\right), \quad \xi \in \mathbb{R}$$

- For large n, makes sense to model block maxima of n iid draws using G_ξ (with $(x-\mu)/\sigma$ in place of y above)

Multivariate extremes

- Theory : Beirlant et al. [2004]
- Copulas : Joe [2014]
- Method : Dey and Yan [2016]
- Key ideas in AI : Ledford and Tawn [1996], Ledford and Tawn [1997], Coles et al. [1999], Heffernan and Tawn [2004]
- Modelling across dependence classes : Wadsworth et al. [2017], Huser and Wadsworth [2020]

Multivariate extreme value distribution, MEVD

- $\boldsymbol{X}_i = (X_{i1}, ..., X_{ij}, ..., X_{ip}), i = 1, ..., n$ iid p-vectors, distribution F
- $M_{n,j} = \max_i X_{ij}$, component-wise maximum
- The component-wise maximum is not "observed" (especially as $n \to \infty$)

- Then for $Z_{n,j} = (M_{n,j} - b_{n,j})/a_{n,j}$, normalised with scaling constants:

$$\mathbb{P}(\boldsymbol{Z} \leq \boldsymbol{z}) = F^n(\boldsymbol{a}_n \boldsymbol{z} + \boldsymbol{b}_n) \to G(\boldsymbol{z}) \quad \text{as} \quad n \to \infty$$

- Non-degenerate $G(\boldsymbol{z})$ must be max-stable, so $\forall k \in \mathbb{N}$, $\exists\ \boldsymbol{\alpha_k} > \boldsymbol{0}, \boldsymbol{\beta_k}$ s.t.

$$G^k(\boldsymbol{\alpha}_k \boldsymbol{z} + \boldsymbol{\beta}_k) = G(\boldsymbol{z})$$

- We say $F \in D(G)$
- Margins $G_1, ..., G_p$ are unique GEV, but $G(\boldsymbol{z})$ is not unique

MEVD on common margins

- On standard Fréchet margins with pseudo-polars $(r, \boldsymbol{w})$

$$\begin{aligned} G(\boldsymbol{z}) &= \exp\left(-V(\boldsymbol{z})\right) \\ \text{with } V(\boldsymbol{z}) &= \int_\Delta \max_j \{\frac{w_j}{z_j}\}\, S(d\boldsymbol{w}), \quad \text{on } \Delta = \{\boldsymbol{w} \in \mathbb{R}^p : ||\boldsymbol{w}|| = 1\} \\ \text{and } 1 &= \int_\Delta w_j\, S(d\boldsymbol{w}), \quad \forall j, \text{ for angular measure } S \end{aligned}$$

- Condition of multivariate regular variation, MRV

$$\frac{1 - F(t\boldsymbol{x})}{1 - F(t\mathbf{1})} \to \lambda(\boldsymbol{x}) \text{ as } t \to \infty, \boldsymbol{x} \in \mathbb{R}^p$$

useful to prove that $F \in D(G)$ for some MEVD G

- Lots more

Asymptotic dependence ... admitted by MEVD

- On uniform margins

$$\chi(u) = \frac{\mathbb{P}(U > u, V > u)}{\mathbb{P}(U > u)} = \frac{\bar{C}(u,u)}{1-u} \quad \to \chi \text{ as } u \to 1$$

- $\chi = 1$ perfect dependence
- $\chi \in (0,1)$ asymptotic dependence, AD
- $\chi = 0$ perfect independence

$$\theta(u) = \frac{\log \mathbb{P}(U \leq u, V \leq u)}{\log \mathbb{P}(U \leq u)} = \frac{\log C(u,u)}{\log u} \quad \to \theta \text{ as } u \to 1$$

- $\theta = 2 - \chi$

- χ and θ describe AD
- MEVD admits AD

Asymptotic independence ... not admitted by MEVD

- On uniform margins

$$\bar{\chi}(u) = 2\frac{\log \mathbb{P}(U > u)}{\log \mathbb{P}(U > u, V > u)} - 1 = 2\frac{\log(1-u)}{\log \bar{C}(u,u)} - 1 \quad \to \bar{\chi} \text{ as } u \to 1$$

- $\bar{\chi} = 1$ perfect dependence and AD
- $\bar{\chi} \in (0,1)$ asymptotic independence, AI
- $\bar{\chi} = 0$ perfect independence
- On Fréchet margins ($F(z) = \exp\left(-z^{-1}\right)$), assume

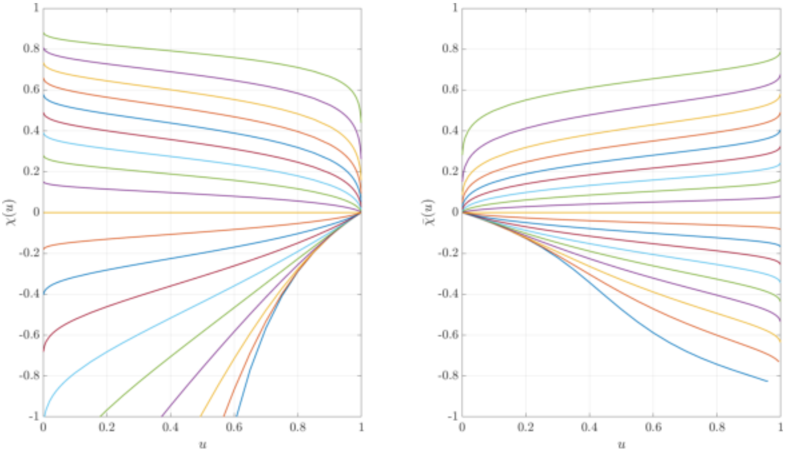
$$\frac{\mathbb{P}(Z_1 > z, Z_2 > z)}{\left(\mathbb{P}(Z_1 > z)\right)^{1/\eta}} = \mathcal{L}(z)$$

where $\mathcal{L}$ is slowly varying : $\mathcal{L}(xz)/\mathcal{L}(z) \to 1$ as $z \to \infty$

- $\bar{\chi} = 2\eta - 1$
- Idea : use non-extreme value copulas or inverted EV copulas
- Also $\mathbb{P}(Z_2 > z | Z_1 > z) \approx C z^{1-1/\eta}$ from above
- Idea : assume a max-stable-like normalisation for conditional extremes

Extremal dependence (bivariate Gaussian)

- Many (almost all?) environmental extremes problems involve asymptotic independence, at least in part ... bivariate Gaussian is one example!



$\chi(u)$ and $\bar{\chi}(u)$ for bivariate Gaussian ($\Rightarrow \chi = 0, \bar{\chi} = \rho$)
Colours are correlations ρ on -0.9, -0.8, ..., 0.9 (Recreated from Coles et al. 1999)

Conditional extremes in practice

- Non-stationary : Jonathan et al. [2014]
- Time-series : Winter and Tawn [2016], Tendijck et al. [2019]
- Mixture model : Tendijck et al. [2021]
- Spatial : Shooter et al. [2021b], Shooter et al. [2021a]
- Lots more

- Multivariate spatial : Shooter et al. [2022]

Estimating return values and associated values

- Return values: Serinaldi [2015], Jonathan et al. [2021]
- Associated values : Towe et al. [2022]