Modelling extreme ocean environments for structural design

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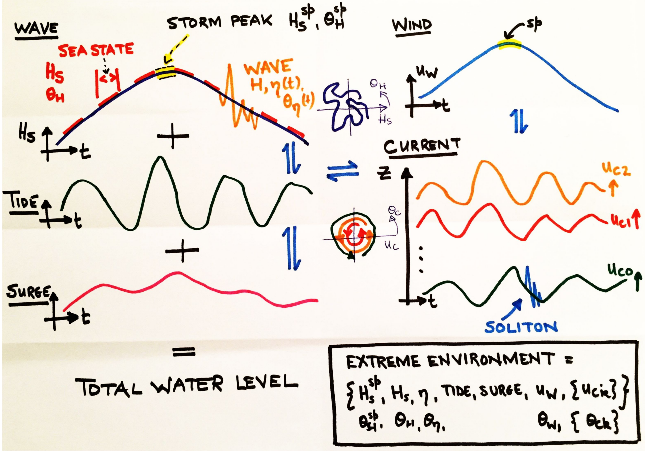
(Slides at *www.lancs.ac.uk/∼jonathan*)

... with thanks to countless colleagues!

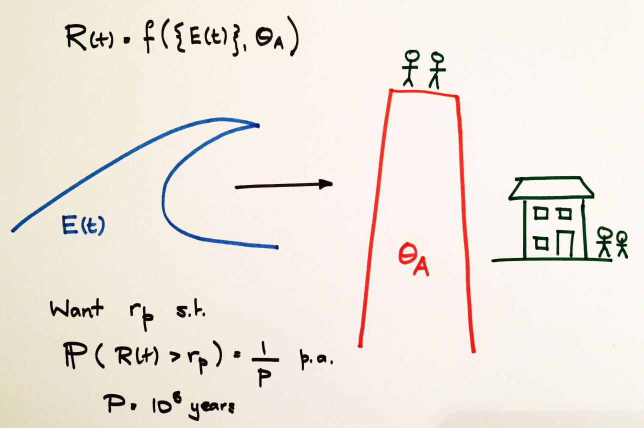
Lancaster University

Modelling ocean storm environment



- Multiple coupled physical processes
- Rare, extreme events

Modelling structural risk



- Ocean environment is harsh
- Marine structures at risk of failure
- Reliability standards must be met

Optimal design of marine structure

Set-up

- Storm storm peak events $\boldsymbol{X}^{\text{sp}}$ dependent on covariates $\boldsymbol{\Theta}^{\text{sp}}$
- An evolving within-storm environment $\{(\mathbf{X}_s, \boldsymbol{\Theta}_s)\}_{s \in S_T}$ for storm of length T
- Structural "loading" $\boldsymbol{Y}$
- Everything subject to sources of uncertainty $\boldsymbol{Z}$
- $\boldsymbol{Z}$, $\boldsymbol{\Theta}^{\text{sp}}$, $\boldsymbol{X}^{\text{sp}}$, $\{(\mathbf{X}_s, \boldsymbol{\Theta}_s)\}_{s \in S_T}$ and $\boldsymbol{Y}$ are multidimensional random variables

Unconditional distribution of loading for a random storm

$$
\begin{aligned}
F_{\boldsymbol{Y}}(\boldsymbol{y}) \quad = \quad & \int_{\zeta} \int_{(\{(\mathbf{x}_s, \boldsymbol{\theta}_s)\}_{s \in \mathcal{S}_\tau}, \tau)} \int_{\mathbf{x}^{\text{sp}}} \int_{\boldsymbol{\theta}^{\text{sp}}} \\
\times \quad & F_{\boldsymbol{Y} | \{(\mathbf{X}_s, \boldsymbol{\Theta}_s)\}_{s \in S_T}, \boldsymbol{Z}}(\boldsymbol{y} | \{(\mathbf{x}_s, \boldsymbol{\theta}_s)\}_{s \in S_\tau}, \boldsymbol{\zeta}) \\
\times \quad & f_{(\{(\mathbf{X}_s, \boldsymbol{\Theta}_s)\}_{s \in \mathcal{S}_T}, T) | \mathbf{X}^{\text{sp}}, \boldsymbol{\Theta}^{\text{sp}}, \boldsymbol{Z}} \left(\{(\mathbf{x}_s, \boldsymbol{\theta}_s)\}_{s \in \mathcal{S}_\tau}, \tau \mid \mathbf{x}^{\text{sp}}, \boldsymbol{\theta}^{\text{sp}}, \boldsymbol{\zeta} \right) \\
\times \quad & f_{\mathbf{X}^{\text{sp}} | \boldsymbol{\Theta}^{\text{sp}}, \boldsymbol{Z}}(\mathbf{x}^{\text{sp}} | \boldsymbol{\theta}^{\text{sp}}, \boldsymbol{\zeta}) \\
\times \quad & f_{\boldsymbol{\Theta}^{\text{sp}} | \boldsymbol{Z}}(\boldsymbol{\theta}^{\text{sp}} | \boldsymbol{\zeta}) \\
\times \quad & f_{\boldsymbol{Z}}(\boldsymbol{\zeta}) \\
\times \quad & \mathrm{d}\boldsymbol{\theta}^{\text{sp}} \, \mathrm{d}\mathbf{x}^{\text{sp}} \, \mathrm{d}(\{(\mathbf{x}_s, \boldsymbol{\theta}_s)\}_{s \in \mathcal{S}_\tau}, \tau) \, \mathrm{d}\boldsymbol{\zeta}
\end{aligned}
$$

Optimal design of marine structure

Typical

- Distribution of annual maximum loading (for univariate load here)

$$F_A(y) \quad = \quad \int_m [F_Y(y)]^m f_C(m)\, \mathrm{d}m$$

- Annual rate of occurrence f_C of storms
- Return value for return period P years given by $F_A^{-1}(1 - 1/P)$

More generally

- Expected annual utility for year with M random storms

$$\mathbb{E}(U_A|\boldsymbol{\mathcal{R}}) = \int_m \int_{\boldsymbol{y}_1} \ldots \int_{\boldsymbol{y}_m} U_A(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m|\boldsymbol{\mathcal{R}}) f_{\boldsymbol{Y}_1,\ldots,\boldsymbol{Y}_m,M}(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m, m)\, \mathrm{d}\boldsymbol{y}_1 \ldots \mathrm{d}\boldsymbol{y}_m\, \mathrm{d}m$$

- System annual utility $U_A(\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_m|\boldsymbol{\mathcal{R}})$ given system “strength” characteristics $\boldsymbol{\mathcal{R}}$
- $f_{\boldsymbol{Y}_1,\ldots,\boldsymbol{Y}_m,M}$ is the joint density of multivariate loading from M random storms
- Solve for $\boldsymbol{\mathcal{R}}$ to achieve required expected annual utility

Historical approach

Historical approach

Will discuss:

- Estimation for return values from small samples
  - This is still a major issue today (e.g. LOADS)

Generic historical issues:

- Weaker justification (?) for choice of distributional forms for extremes
- Neglect of covariate effects in extremes (direction, season, “climate change”)
- Neglect of spatial and temporal dependence in extremes
- Neglect of joint behaviour of extremes across multiple metocean variables (“associated values”)
- Neglect of uncertainty (“no UQ”)
- Dearth of data, data quality (measured, hindcast, ...) for extremes not clear
- Disconnect with risk (no direct connection with structural failure; “return values”, “design contours”)
- Missing interface between metocean specialists, structural engineers and “statistical modellers”
- “No full empirical model”

What is a return value?

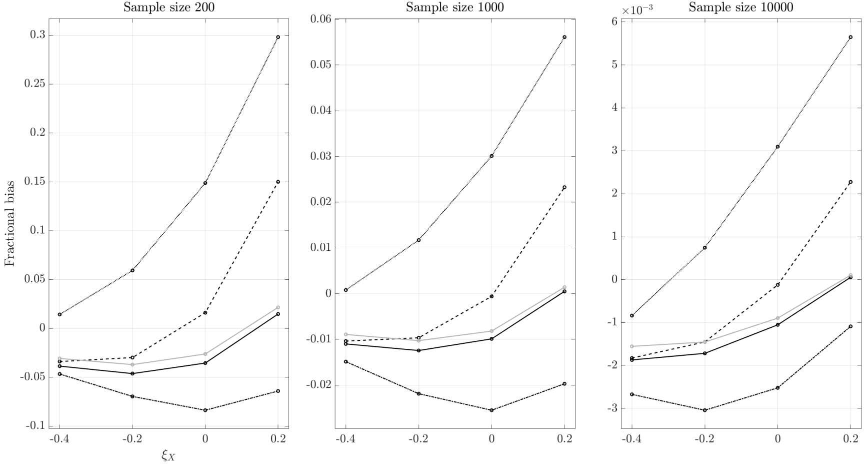
- $x_P = F_A^{-1}(1 - \frac{1}{P})$ for annual maximum event A
- $F_{A_P}(x_P) = \left(1 - \frac{1}{P}\right)^P \approx \exp(-1)$ for P-year maximum event A_P

- F_A or F_{A_P} estimated with uncertainty from a sample of data

- x_P can be estimated easily in the absence of uncertainty
- In the presence of uncertainty $\boldsymbol{Z}$, we can "integrate it out" using either
  - $\tilde{F}_Y(y) = \int_\zeta F_{Y|\boldsymbol{Z}}(x|\zeta) f_{\boldsymbol{Z}}(\zeta)\, d\zeta$, a predictive distribution from uncertain $F_{Y|\boldsymbol{Z}}$
  - $E[g(\boldsymbol{Z})] = \int_\zeta g(\zeta) f_{\boldsymbol{Z}}(\zeta)\, d\zeta$, a predictive mean from uncertain $g(\boldsymbol{Z})$

- Choices made lead to different estimates of return values and related quantities
- Bias effects can be proven theoretically (and demonstrated numerically)
- Effects are most dramatic for small sample sizes

Fractional bias of return value estimators



Fractional bias of return value estimates from different estimators using maximum likelihood, as a function of sample size and true GP shape ξ. LHS top to bottom: q_3, q_2, q_5, q_1, q_4.

- Knock-on effects for associated values of the form $\mathbb{E}_{\boldsymbol{Z}}(\boldsymbol{Y}|X = q, \boldsymbol{Z})$

Return value references and implications

References

- Return values: Serinaldi [2015], Jonathan et al. [2021]
- Associated values : Towe et al. [2023]
- Lots of other approaches for small samples (e.g empirical Bayes Zhang 2007, Zhang and Stephens 2009, Zhang 2010)

Implications for today

- Current EV models tend to have high effective dimensionality
- Effective number of degrees of freedom from sample for model fitting can be small $\Rightarrow$ we have small effective sample size
- Momentum in metocean community (e.g. AWARE, LOADS JIPs) to use Bayesian inference ... great in principle, but ...
- Characteristics of (posterior) predictive distributions highly dependent on prior specification. Yet not clear how to advise "diverse user community" regarding "rational prior specification".

Full probabilistic modelling

Full probabilistic modelling

- Model components of “full empirical model”
  - Storm peaks
  - Within-storm evolution
  - Fluid loading
- Marginal modelling
- Dependence modelling

The full "forward" model

Unconditional distribution of loading from a random storm

$$
\begin{aligned}
F_{\boldsymbol{Y}}(\boldsymbol{y}) \quad = \quad & \int_{\zeta} \int_{(\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in\mathcal{S}_\tau},\tau)} \int_{\mathbf{x}^{\text{sp}}} \int_{\boldsymbol{\theta}^{\text{sp}}} \\
\times \quad & F_{\boldsymbol{Y}|\{(\mathbf{X}_s,\boldsymbol{\Theta}_s)\}_{s\in\mathcal{S}_T},\boldsymbol{Z}}(\boldsymbol{y}|\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in\mathcal{S}_\tau},\boldsymbol{\zeta}) \\
\times \quad & f_{(\{(\mathbf{X}_s,\boldsymbol{\Theta}_s)\}_{s\in\mathcal{S}_T},T)|\mathbf{X}^{\text{sp}},\boldsymbol{\Theta}^{\text{sp}},\boldsymbol{Z}}\left(\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in\mathcal{S}_\tau},\tau \mid \mathbf{x}^{\text{sp}},\boldsymbol{\theta}^{\text{sp}},\boldsymbol{\zeta}\right) \\
\times \quad & f_{\mathbf{X}^{\text{sp}}|\boldsymbol{\Theta}^{\text{sp}},\boldsymbol{Z}}(\mathbf{x}^{\text{sp}}|\boldsymbol{\theta}^{\text{sp}},\boldsymbol{\zeta}) \\
\times \quad & f_{\boldsymbol{\Theta}^{\text{sp}}|\boldsymbol{Z}}(\boldsymbol{\theta}^{\text{sp}}|\boldsymbol{\zeta}) \\
\times \quad & f_{\boldsymbol{Z}}(\boldsymbol{\zeta}) \\
\times \quad & \mathrm{d}\boldsymbol{\theta}^{\text{sp}}\,\mathrm{d}\mathbf{x}^{\text{sp}}\,\mathrm{d}(\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in\mathcal{S}_\tau},\tau)\,\mathrm{d}\boldsymbol{\zeta}
\end{aligned}
$$

Issues

- Temporal "inter-storm" effects (clustering, climate change)
  - "Random storm" model invalid; even conditional independence assumption invalid (?)
- Spatial dependence of extremes
  - Spatial risk: e.g. de-manning multiple structures
- Estimating each model component is challenging!

Full model for fluid loading

General approach

- Linear wave spectrum model
  - e.g. JONSWAP
  - Multivariate extreme value model for all spectral model parameters
  - $\Rightarrow$ Simulation of arbitrary sea state spectra
- Linear wave theory (potential theory)
  - Linearised boundary conditions
  - Linear surface elevation and kinematics
  - $\Rightarrow$ Simulation of linear time-series given linear spectrum
- Non-linear transformation (Swan 2020, Gibson 2020)
  - Non-linear surface elevation
  - "Stretched" kinematics
  - $\Rightarrow$ Simulation of non-linear time-series given linear spectrum
- Conditional simulation of Gaussian time-series (Taylor et al. 1997)
  - Embed extreme excursions in surface elevation and associated kinematics
  - $\Rightarrow$ Efficient simulation of extreme time-series
- Estimate marginal distribution of structural response from random storm
  - Efficient integration using importance sampling and conditional simulation
  - Optimal design in environmental space (Gramstad et al. 2020, Speers et al. 2024)

Marginal extremes

Model for size of occurrence

- Sample of storm peaks X over threshold $\psi_\theta \in \mathbb{R}$, with 1-D covariate $\theta \in \mathcal{D}_\theta$
- Extreme value threshold ψ_θ assumed known
- X assumed to follow generalised Pareto distribution with shape ξ_θ, (modified) scale ν_θ

$$f_{\text{GP}}(x|\xi_\theta, \nu_\theta) = \frac{1}{\sigma_\theta}\left(1 + \frac{\xi_\theta}{\sigma_\theta}(x - \psi_\theta)\right)_+^{-1/\xi_\theta - 1} \text{ with } \nu_\theta = \sigma_\theta(1 + \xi_\theta)$$

- Shape parameter $\xi_\theta \in \mathbb{R}$ and scale parameter $\nu_\theta > 0$
- (Non-stationary Poisson model for rate of occurrence, with rate $\rho_\theta \geq 0$)

Covariate representations in 1-D

- Index set $\mathcal{I}_\theta = \{\theta_s\}_{s=1}^m$ on periodic covariate domain $\mathcal{D}_\theta$
- Each observation belongs to exactly one θ_s
- On $\mathcal{I}_\theta$, assume

$$\begin{aligned} \eta_s &= \sum_{k=1}^{n} B_{sk}\beta_k, s = 1, 2, ..., m, \text{ or} \\ \boldsymbol{\eta} &= \boldsymbol{B}\boldsymbol{\beta} \end{aligned}$$

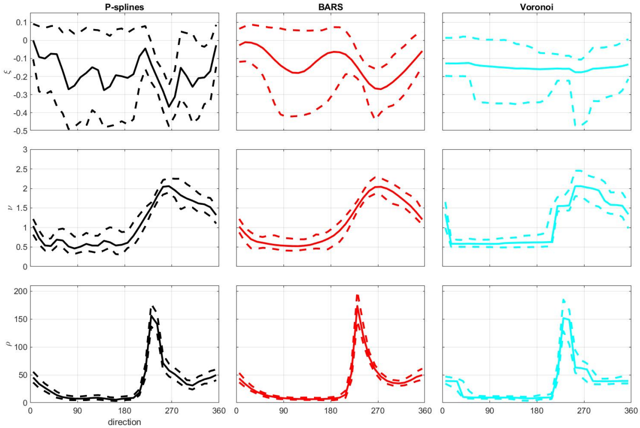
- $\eta \in (\xi, \nu)$ (and similar for ρ)
- $\boldsymbol{B} = \{B_{sk}\}_{s=1;k=1}^{m;n}$ basis for $\mathcal{D}_\theta$
- $\boldsymbol{\beta} = \{\beta_k\}_{k=1}^{n}$ basis coefficients
- Inference reduces to estimating $n_\xi, n_\nu, \boldsymbol{B}_\xi, \boldsymbol{B}_\nu, \boldsymbol{\beta}_\xi, \boldsymbol{\beta}_\nu$ (and roughnesses λ_ξ, λ_ν)

- P-splines, BARS and Voronoi are different forms of $\boldsymbol{B}$
- Tensor products and slick GLAM algorithms for n-D covariate representations

Posterior parameter estimates for ξ, ν and ρ for northern North Sea

- MCMC inference (Gibbs sampling, reversible jump, etc.)

- Note colour scheme
- Rate ρ and ν very similar
- Voronoi gives almost constant ξ
- Voronoi piecewise constant
- Land shadow effects
- General agreement
- ... for other parameters also



- Covariate effects are everywhere, margins and dependence ...

Practical implications of modelling choices

Practical implications of modelling choices

- How do “arbitrary choices” in the modelling procedure effect output?
- Case studies (like a southern North Sea location)

Effects of

- Generalised Pareto (GP) model parameterisation
  - Orthogonal
  - “Mean-max”
- Relative penality for GP shape and scale
  - Relatively high
  - Very high
- Cross-validation strategy
  - 10-fold
  - Repeated random 2-fold
- Choice of estimator for return value
  - Mean quantile
  - Quantile mean

Findings

- Material impact on estimates of return values

Issues and opportunities

Issues

- EV threshold modelling and UQ
- Many tuning parameters which should be optimised, but rarely are, and UQ w.r.t. these
- Model misspecification
  - Measurement scale, sub-asymptotic models
  - Missing covariates
- Prior specification (or equivalent frequentist choices)
- UQ generally

Opportunities

- Incorporate new data sources
  - Satellite (e.g. scatterometry)
  - GCM output (but CMIP6 inconsistency)
  - Large simulations (over 10^3s of years; so just “interpolate”)
- Overly-complex models
  - Standard Norge [2022] “immature methodologies”
  - Diagnostics
- “Black box” AI/ML (e.g. KAUST, Saudi A.)
  - “ExaGeoStat” (Genton)
  - Sensible extremes (e.g. GP tail, “interpretable” plus “uninterpretable” covariate effects; Hüser, Richards)
- Just “do the whole planet” and be done with it!

Marginal extremes references

- Theory : Embrechts et al. [2003], Beirlant et al. [2004]
- Method : Coles [2001], Dey and Yan [2016]
- Motivation : Davison and Smith [1990]
- Covariate effects : Wood [2003], Chavez-Demoulin and Davison [2005], Brezger and Lang [2006], Youngman [2022]

- Metocean : Jonathan and Ewans [2013], Feld et al. [2019], Vanem et al. [2022]
- Metocean applications : Randell et al. [2016], Zanini et al. [2020]
- Machine learning: Abdulah et al. [2018], Richards and Huser [2024]

- Uncertainties: Tendijck et al [2024]

Multivariate extremes

Multivariate extremes

- Max-stability, AD and AI
- Conditional extremes basics
- Time-series conditional extremes
- Multivariate spatial conditional extremes
- SPAR
- covXtreme

Modelling margins and dependence

Context

$$F_{\mathbf{X}^{\text{sp}}|\mathbf{\Theta}^{\text{sp}},\boldsymbol{Z}}(\mathbf{x}^{\text{sp}}|\boldsymbol{\theta}^{\text{sp}},\boldsymbol{\zeta}) \quad = \quad C(F_{X_1^{\text{sp}}|\mathbf{\Theta}^{\text{sp}},\boldsymbol{Z}}(x_1^{\text{sp}}|\boldsymbol{\theta}^{\text{sp}},\boldsymbol{\zeta}),...,F_{X_p^{\text{sp}}|\mathbf{\Theta}^{\text{sp}},\boldsymbol{Z}}(x_p^{\text{sp}}|\boldsymbol{\theta}^{\text{sp}},\boldsymbol{\zeta})|\boldsymbol{\theta}^{\text{sp}},\boldsymbol{\zeta})$$

- We already have marginal models $F_{X_j^{\text{sp}}|\mathbf{\Theta}^{\text{sp}}}(x_j^{\text{sp}}|\boldsymbol{\theta}^{\text{sp}},\boldsymbol{Z})$, $j = 1, 2, ..., p$
- Now we need a dependence model or copula $C = C(u_1, u_2, ..., u_p|\boldsymbol{\theta}^{\text{sp}},\boldsymbol{\zeta})$

Which dependence function?

Max-stability == multivariate extreme value distribution, MEVD

- The copula is not unique
- Max-stability is one popular assumption, which itself involves a common but often unrealistic assumption of component-wise maxima
- On uniform margins, extreme value copula: $C(\boldsymbol{u}) = C^k(\boldsymbol{u}^{1/k})$
- On Fréchet margins ($G_j(z) = \exp\left(-z^{-1}\right)$), $G(\boldsymbol{z}) = \exp\left(-V(\boldsymbol{z})\right)$, for exponent measure V such that $V(r\boldsymbol{z}) = r^{-1}V(\boldsymbol{z})$, homogeneity order -1
- Rich spatial extensions to max-stable processes, MSPs
- Multivariate generalised Pareto distribution, MGPD

AD and AI

- All MEVD distributions exhibit asymptotic dependence (AD)
- Many common distributions (e.g. the multivariate Gaussian) exhibit asymptotic independence (AI)
- So extreme value copulas are not general enough to describe extremal dependence in nature
- Other (e.g. inverted) copula forms do exhibit AI
- The conditional extremes model admits AD (on the boundary) and AI
- SPAR admits AD and AI

Conditional extremes

Conditional extremes ... moving beyond component-wise maxima

- Random variables $\boldsymbol{X} = (X_1, ..., X_j, ..., X_p)$ and Y
- Each X and Y have standard Laplace margins ($f(x) = \exp(-|x|)/2, x \in \mathbb{R}$)
- Seek a model for $\boldsymbol{X}|(Y = y)$ for $y > u$

- Assume we can find p-dimensional scaling $\boldsymbol{a}, \boldsymbol{b} > \boldsymbol{0}$ such that

$$\mathbb{P}(\boldsymbol{Z} \leq \boldsymbol{z}|Y = y) \quad \rightarrow \quad G(\boldsymbol{z}) \quad \text{as} \quad u \rightarrow \infty$$
$$\text{for} \quad \boldsymbol{Z} \quad = \quad \frac{\boldsymbol{X} - \boldsymbol{a}(y)}{\boldsymbol{b}(y)}$$

- Non-degenerate G is unknown, and estimated empirically

- Typical scaling is $\boldsymbol{a} = \boldsymbol{\alpha} y$ and $\boldsymbol{b} = y^{\boldsymbol{\beta}}$, $\boldsymbol{\alpha} \in [-1, 1]^p$, $\boldsymbol{\beta} \in (-\infty, 1]^p$
- So simply fit regression model

$$\boldsymbol{X}|(Y = y) = \boldsymbol{\alpha} y + y^{\boldsymbol{\beta}} \boldsymbol{Z}, \text{ for } y > u$$

- $\alpha = 1, \beta = 0$: perfect dependence and AD, and $\alpha \in (0, 1)$: AI

- Heffernan and Tawn [2004] find choices for α and β for popular bivariate cases
- Bivariate Gaussian : $\alpha = \rho^2, \beta = 1/2$

Developments of the conditional extremes model

Canonical extensions

- Basic: $X|(Y=y), y>u$
- Temporal: "heatwave model" $X_1, X_2, ..., X_\tau|(X_0=x_0), x_0>u$
- Spatial: "spatial conditional extremes" $X_1, X_2, ..., X_s|(X_0=x_0), x_0>u$

Idea

$$X_1, X_2, ..., X_p|(Y=y) = \boldsymbol{\alpha} y + y^{\boldsymbol{\beta}} \boldsymbol{Z}$$

- Impose appropriate structure on parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and distribution of $\boldsymbol{Z}$
  - e.g. $\boldsymbol{\alpha}$ evolves smoothly in space
  - e.g. $\boldsymbol{Z}$ follows a multivariate Gaussian or extension thereof with appropriate mean and covariance forms
- Make a simplifying assumption
  - e.g. apply a low-order model repeatedly $X_{t+1}, X_{t+2}|(X_t=x) = [\alpha_1 \alpha_2]x + x^{[\beta_1 \beta_2]}[Z_1 Z_2]$

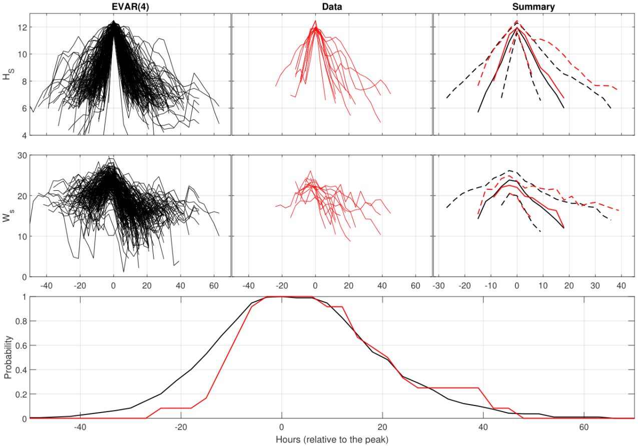
Further extensions

- Non-stationary and multivariate temporal and spatial models

Extremal vector auto-regression (EVAR) for within-storm evolution

On Laplace margins, with component-wise operations and $\boldsymbol{X}_t \in \mathbb{R}^d$:

$$\boldsymbol{X}_{t+k} \mid (\boldsymbol{X}_t, \ldots, \boldsymbol{X}_{t+k-1}, X_{t,1} = y) = \sum_{\ell=1}^{k} \boldsymbol{A}_\ell \boldsymbol{X}_{t+k-\ell} + y^{\boldsymbol{b}} \boldsymbol{Z}, \quad y > u \gg 0$$



Excursions of H_S (top) and W_S (middle) from EVAR(4) model (left; black), observed (middle; red) on original margins with storm peak $H_S \in [11.5, 12.5]$; right-hand plots summarise the observed (red) and EVAR(4) (black) excursions, using median (solid), 10% and 90% quantiles (dashed). In the bottom panel, we plot survival probabilities for observed (red) and EVAR(4) (black) excursions relative to the time of the excursion maximum.

Multivariate spatial conditional extremes (MSCE)

MSCE

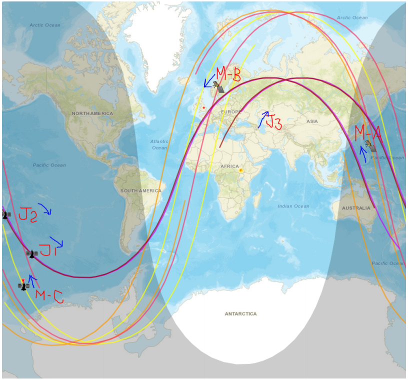
Motivation

- How useful are satellite observations of ocean waves and winds?
- Could they become the primary data source for decisions soon?
- What are the spatial characteristics of extremes from satellite observations?

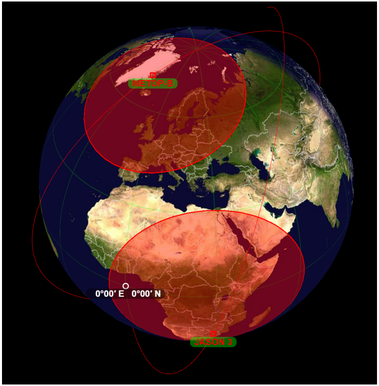
Overview

- A look at the data : satellite wind, hindcast wind, hindcast wave
- Brief overview of methodology
- Results for joint spatial structure of extreme scatterometer wind speed, hindcast wind speed and hindcast significant wave height in the North Atlantic
- Implications for future practical applications

JASON and METOP



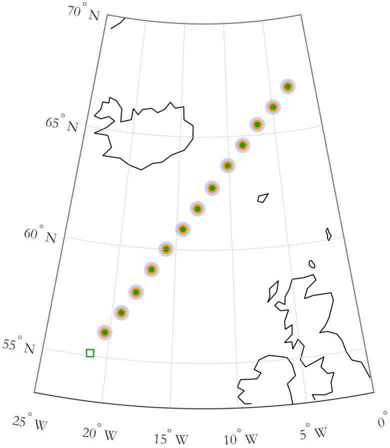
[n2yo.com, accessed 06.09.21 at around 1100UK]



[stltracker.github.io, accessed 27.08.2021 at around 1235UK]

- JASON and METOP similar polar orbits
- JASON all ascending, METOP all descending over North Atlantic
- Joint occurrence of JASON and METOP over North Atlantic rare

Methodology in a nut-shell



- Transform to standard margins using independent non-stationary GP models
- Condition on large value x of first quantity X_{01} at one location $j = 0$ (green square)
- Estimate "conditional spatial profiles" for $m > 1$ quantities $\{X_{jk}\}_{j=1,k=1}^{p,m}$ at $p > 0$ other locations (green, orange and blue circles)

$$X_{jk} \sim \text{Lpl}$$

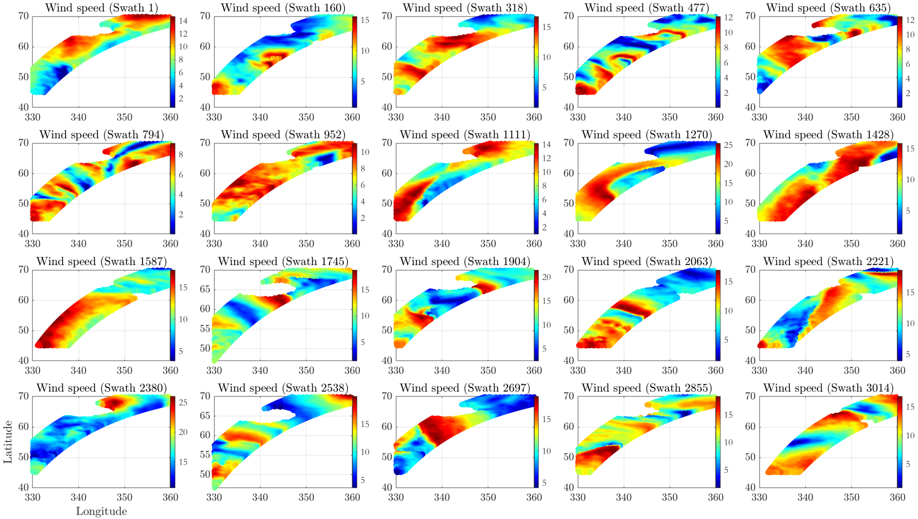
$$x > u$$

$$\boldsymbol{X} | \{X_{01} = x\} = \boldsymbol{\alpha} x + x^{\boldsymbol{\beta}} \boldsymbol{Z}$$

$$\boldsymbol{Z} \sim \text{DL}(\boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\delta}; \boldsymbol{\Sigma}(\boldsymbol{\lambda}, \boldsymbol{\rho}, \boldsymbol{\kappa}))$$

- MCMC to estimate $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\mu}$, $\boldsymbol{\sigma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\rho}$, $\boldsymbol{\kappa}$, $\boldsymbol{\lambda}$
- $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\mu}$, $\boldsymbol{\sigma}$, $\boldsymbol{\delta}$ spatially smooth for each quantity
- DL = delta-Laplace = generalised Gaussian
- Residual correlation $\boldsymbol{\Sigma}$ for conditional Gaussian field, powered-exponential decay with distance

Swath wind speeds



Daily descending METOP swaths. Satellite swath location changes over time. Spatial structure evident

Inference

$$X_{jk} \sim \text{Lpl}, \quad x > u, \quad \boldsymbol{X} | \{X_{01} = x\} = \boldsymbol{\alpha} x + x^{\boldsymbol{\beta}} \boldsymbol{Z}, \quad \boldsymbol{Z} \sim \text{DL}(\boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\delta}; \boldsymbol{\Sigma}(\boldsymbol{\lambda}, \boldsymbol{\rho}, \boldsymbol{\kappa}))$$

- Delta-Laplace residual margins

$$f_{Z_{j,k}}(z_{j,k}) = \frac{\delta_{j,k}}{2\kappa_{j,k}\sigma_{j,k}\Gamma\left(\frac{1}{\delta_{j,k}}\right)} \exp\left\{-\left|\frac{z - \mu_{j,k}}{\kappa_{j,k}\sigma_{j,k}}\right|^{\delta_{j,k}}\right\}, \quad \kappa_{j,k}^2 = \Gamma\left(1/\delta_{j,k}\right)/\Gamma\left(3/\delta_{j,k}\right)$$

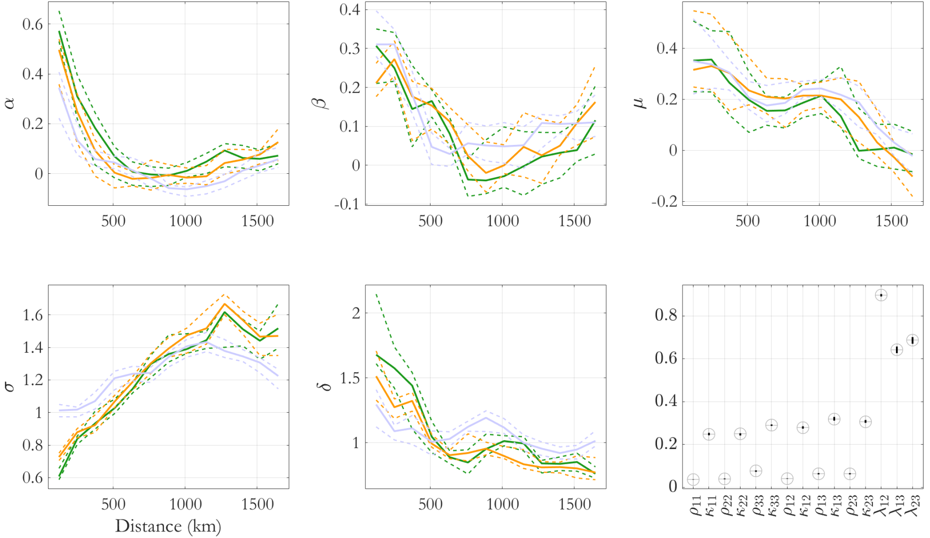
- Gaussian residual dependence

$$\boldsymbol{\Sigma}^*_{\mathcal{A}^*(j,k)\mathcal{A}^*(j',k')} = \lambda_{k,k'}^{|k-k'|} \exp\left(-\left(\frac{\text{dist}(r_j, r_{j'})}{\rho_{k,k'}}\right)^{\kappa_{k,k'}}\right)$$

- Piecewise linear forms for $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\mu}$, $\boldsymbol{\sigma}$, $\boldsymbol{\delta}$ with distance using n_{Nod} spatial nodes

- Adaptive MCMC, Roberts and Rosenthal [2009]
- Total of $m(5n_{\text{Nod}} + (3m+1)/2)$ parameters
- Rapid convergence, 10k iterations sufficient

Parameter estimates



Estimates for α, β, μ, σ and δ with distance, and residual process estimates ρ, κ and λ. Model fitted with $\tau = 0.75$
StlWnd (green), HndWnd (orange), HndWav(blue)
Residual Gaussian field : ρ=scale (need to $\times$100km), κ=exponent (need to $\times$5), λ=cross-correlation

Applied conditional extremes references

- Non-stationary : Jonathan et al. [2014]
- Time-series : Winter and Tawn [2016], Tendijck et al. [2019], Tendijck et al. [2024]
- Mixture model : Tendijck et al. [2023]
- Spatial : Shooter et al. [2021b], Shooter et al. [2021a], Shooter et al. [2022]
- Lots more

Semi-parametric angular-radial representations (SPAR)

SPAR

Basics

- Radial R and angular Q components. Then joint density factorised as

$$f_{R,Q}(r,q) = f_Q(q) f_{R|Q}(r|q)$$

- Assume GP conditional tail for $R|(Q=q)$, with parameters varying smoothly with angle q above some threshold $\psi(q)$ with non-exceedance probability $\tau(q)$

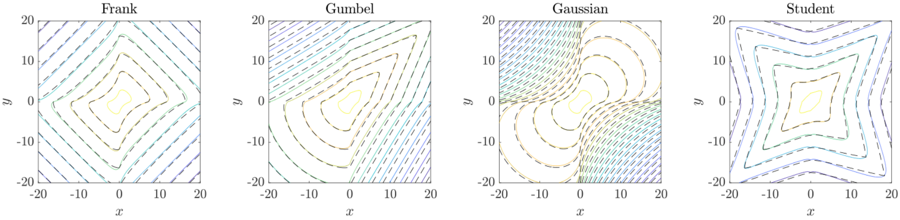
$$f_{R,Q}(r,q) = f_Q(q) \times \tau(q) f_{GP}(r - \psi(q)|\xi(q), \sigma(q)), \quad r > \psi(q)$$

  with smoothly varying $\psi(q)$, $\tau(q)$, $\xi(q)$ and $\sigma(q)$. Also assume angular density $f_Q(q)$ varies smoothly with q

- SPAR representation shown to provide good approximations to a large set of copula functions on standard margins
- Is transformation to standard margins necessary?
- Different possible angular-radial decompositions using "generalised co-ordinates"

- $\Rightarrow$ multivariate extremes is just "non-stationary univariate" extremes!

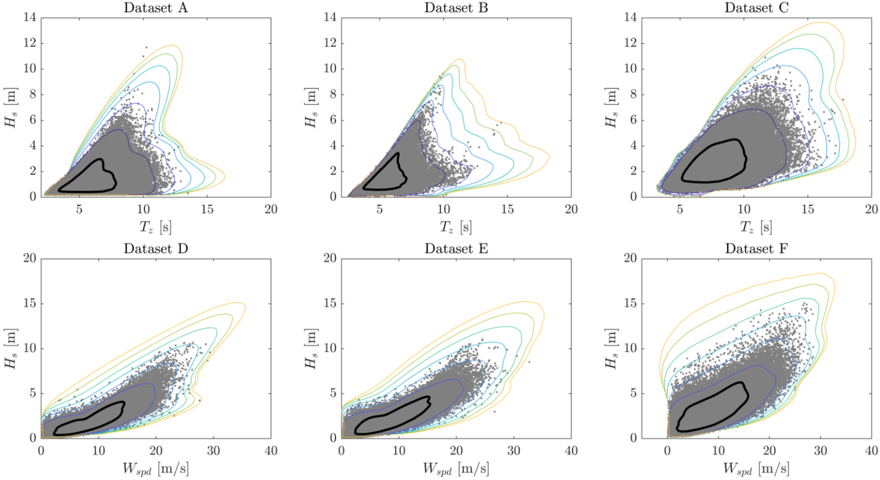
SPAR fits to extreme value copulas



Density contours of various copulas on Laplace margins. All copulas have Pearson correlation coefficient 0.6. Student-t copula has two degrees of freedom. Solid lines: true contours at logarithmic increments. Dashed lines: SPAR-estimated contours.

- SPAR admits asymptotic independence (e.g. upper tails of Frank and Gaussian) and asymptotic dependence (e.g. upper tails of Gumbel and Student-t)
- SPAR handles all directions (not just “first quadrant”)
- Link to limit sets

Density contours from SPAR fits to data



Density contours from SPAR model for 6 samples.

covXtreme

Pragmatic non-stationary multivariate extremes with UQ

- Fit generalised Pareto marginal models for peaks over threshold data $\dot{\boldsymbol{X}}$ and $\dot{Y}$
  - Physics-based identification of peaks from time-series
  - Multiple thresholds, simple piecewise constant model for covariates $\boldsymbol{\Theta}$
  - Diagnostics: threshold stability
- Transform to standard Laplace scale $\boldsymbol{X}$ and Y
  - Transform full sample
- Fit conditional extremes model $\boldsymbol{X}|(Y = y)$ for $y > u$
  - Multiple thresholds, simple piecewise constant covariate model for α
  - Diagnostics: threshold stability, residual structure
- Calculate probabilities of extreme sets
  - MC simulation, importance sampling
  - Estimate environmental contours

- Free covXtreme software for MATLAB does all of above
  - UQ: incorporates epistemic uncertainty using bootstrapping cradle to grave
  - Model averaging: incorporates multiple models for different threshold combinations
  - Multidimensional X and covariates
  - Cross-validation for optimal parameter roughness in marginal and dependence models
  - Careful return value and associated value definitions
  - `https://lfenergy.org/projects/covXtreme/`, Towe et al. [2024]

Multivariate extremes references

- Theory : Beirlant et al. [2004]
- Copulas : Joe [2014]
- Method : Dey and Yan [2016]
- Key ideas in AI : Ledford and Tawn [1996], Ledford and Tawn [1997], Coles et al. [1999], Heffernan and Tawn [2004]
- Modelling across dependence classes : Wadsworth et al. [2017], Huser and Wadsworth [2022]
- “Geometric extremes”, limit sets and SPAR : Nolde and Wadsworth [2022], Mackay and Jonathan [2023], Huser et al. [2024], Murphy-Barltrop et al. [2024], Papastathopoulos et al. [2024], Simpson and Tawn [2024], Wadsworth and Campbell [2024], Mackay et al. [2025]
- Metocean : Parametric conditional models (e.g. Haver 1987, Bitner-Gregersen and Haver 1991), design contours (e.g. Huseby et al. 2013, Haselsteiner et al. 2021).
- covXtreme: Towe et al. [2024]

Summary

Summary

Why?

- Careful quantification of “rare-event” risk
- Characterise tails of (multivariate) distributions
- Limited observations
- Combine solid theory and pragmatic application, UQ
- Immediate real-world consequences

The next 10 years?

- Univariate : fuller covariate descriptions, exploit measurement scale / sub-asymptotics, UQ, provide real-world decision-support
- Multivariate : theoretical development, computational tractability, expansion in scope (time-series, spatial), serious real-world applications
- More demanding regulatory framework

Tusen takk! / Diolch yn fawr!

References I

S. Abdulah, H. Ltaief, Y. Sun, M. G. Genton, and D. E. Keyes. ExaGeoStat: A high performance unified software for geostatistics on manycore systems. *IEEE Trans. Parallel Distrib. Syst.*, 29:2771–2784, 2018.

J. Beirlant, Y. Goegebeur, J. Segers, and J. Teugels. *Statistics of extremes: theory and applications*. Wiley, Chichester, UK, 2004.

E. M. Bitner-Gregersen and S. Haver. Joint environmental model for reliability calculations. In *Proc. ISOPE 91, pp. 246-253*, 1991.

A. Brezger and S. Lang. Generalized structured additive regression based on Bayesian P-splines. *Comput. Statist. Data Anal.*, 50:967–991, 2006.

V. Chavez-Demoulin and A. C. Davison. Generalized additive modelling of sample extremes. *J. Roy. Statist. Soc. Series C: Appl. Stat.*, 54:207–222, 2005.

S. Coles. *An introduction to statistical modelling of extreme values*. Springer, 2001.

S Coles, J Heffernan, and J Tawn. Dependence measures for extreme value analyses. *Extremes*, 2:339–365, 1999.

A.C. Davison and R. L. Smith. Models for exceedances over high thresholds. *J. Roy. Statist. Soc. B*, 52:393, 1990.

D. Dey and J. Yan, editors. *Extreme value modeling and risk analysis: methods and applications*. CRC Press, Boca Raton, USA, 2016.

P. Embrechts, C. Klueppelberg, and T. Mikosch. *Modelling extremal events for insurance and finance*. Springer-Verlag, 2003.

G. Feld, D. Randell, E. Ross, and P. Jonathan. Design conditions for waves and water levels using extreme value analysis with covariates. *Ocean Eng.*, 173: 851–866, 2019.

R. Gibson. Extreme Environmental Loading of Fixed Offshore Structures: Summary Report, Component 2. `https://www.hse.gov.uk/offshore/assets/docs/summary-report-component2.pdf`, 2020.

O. Gramstad, C. Agrell, E. Bitner-Gregersen, B. Guo, E. Ruth, and E. Vanem. Sequential sampling method using gaussian process regression for estimating extreme structural response. *Marine Structures*, 72:102780, 2020.

H. F. Hansen, D. Randell, A. R. Zeeberg, and P. Jonathan. Directional-seasonal extreme value analysis of North Sea storm conditions. *Ocean Eng.*, 195:106665, 2020.

Andreas F. Haselsteiner, Ryan G. Coe, Lance Manuel, Wei Chai, Bernt Leira, Guilherme Clarindo, C. Guedes Soares, Ásta Hannesdóttir, Nikolay Dimitrov, Aljoscha Sander, Jan Hendrik Ohlendorf, Klaus Dieter Thoben, Guillaume de Hauteclocque, Ed Mackay, Philip Jonathan, Chi Qiao, Andrew Myers, Anna Rode, Arndt Hildebrandt, Boso Schmidt, Erik Vanem, and Arne Bang Huseby. A benchmarking exercise for environmental contours. *Ocean Engineering*, 236:109504, 2021.

S. Haver. On the joint distribution of heights and periods of sea waves. *Ocean Eng.*, 14:359–376, 1987.

J. E. Heffernan and J. A. Tawn. A conditional approach for multivariate extreme values. *J. Roy. Statist. Soc. B*, 66:497–546, 2004.

A. B. Huseby, E. Vanem, and B. Natvig. A new approach to environmental contours for ocean engineering applications based on direct monte carlo simulations. *Ocean Eng.*, 60:125–135, 2013.

References II

R. Huser and J. L. Wadsworth. Advances in statistical modeling of spatial extremes. *WIREs Computational Statistics*, 14:e1537, 2022.

R. Huser, T. Opitz, and J. Wadsworth. Modeling of spatial extremes in environmental data science: Time to move away from max-stable processes. *arXiv preprint arxiv:2401.17430*, 2024.

H. Joe. *Dependence modelling with copulas*. CRC Press, 2014.

P. Jonathan and K. Ewans. Statistical modelling of extreme ocean environments with implications for marine design : a review. *Ocean Eng.*, 62:91–109, 2013.

P. Jonathan, K. C. Ewans, and D. Randell. Non-stationary conditional extremes of northern North Sea storm characteristics. *Environmetrics*, 25:172–188, 2014.

P. Jonathan, D. Randell, J. Wadsworth, and J.A. Tawn. Uncertainties in return values from extreme value analysis of peaks over threshold using the generalised Pareto distribution. *Ocean Eng.*, 220:107725, 2021.

A. W. Ledford and J. A. Tawn. Statistics for near independence in multivariate extreme values. *Biometrika*, 83:169–187, 1996.

A. W. Ledford and J. A. Tawn. Modelling dependence within joint tail regions. *J. R. Statist. Soc. B*, 59:475–499, 1997.

E Mackay and P Jonathan. Modelling multivariate extremes through angular-radial decomposition of the density function. *(arXiv preprint arXiv:2310.12711)*, 2023.

E. Mackay, C. J. R. Murphy-Barltrop, and P. Jonathan. The SPAR model: a new paradigm for multivariate extremes. Application to joint distributions of metocean variables. *J. Offshore Mech. Arct. Eng.*, 147:011205, 2025.

C. J. R. Murphy-Barltrop, E. Mackay, and P. Jonathan. Inference for bivariate extremes via a semi-parametric angular-radial model. *Extremes*, 2024. URL `https://doi.org/10.1007/s10687-024-00492-2`.

N. Nolde and J. L. Wadsworth. Linking representations for multivariate extremes via a limit set. *Adv. Appl. Probab.*, 54:688–717, 2022.

I. Papastathopoulos, L. de Monte, R. Campbell, and H. Rue. Statistical inference for radially-stable generalized pareto distributions and return level-sets in geometric extremes, 2024. URL `https://arxiv.org/abs/2310.06130`.

D. Randell, K. Turnbull, K. Ewans, and P. Jonathan. Bayesian inference for non-stationary marginal extremes. *Environmetrics*, 27:439–450, 2016.

A. Ribal and I. R. Young. 33 years of globally calibrated wave height and wind speed data based on altimeter observations. *Sci. Data*, 6:77, 2019.

A. Ribal and I. R. Young. Global calibration and error estimation of altimeter, scatterometer, and radiometer wind speed using triple collocation. *Remote Sens.*, 12:1997, 2020.

J. Richards and R. Huser. Regression modelling of spatiotemporal extreme U.S. wildfires via partially-interpretable neural networks. *arXiv preprint arxiv:2208.07581*, 2024.

G. O. Roberts and J. S. Rosenthal. Examples of adaptive MCMC. *J. Comp. Graph. Stat.*, 18:349–367, 2009.

F. Serinaldi. Dismissing return periods! *Stoch. Env. Res. Risk A.*, 29:1179–1189, 2015.

References III

R. Shooter, E Ross, A. Ribal, I. R. Young, and P. Jonathan. Spatial conditional extremes for significant wave height from satellite altimetry. *Environmetrics*, 32: e2674, 2021a.

R Shooter, J A Tawn, E Ross, and P Jonathan. Basin-wide spatial conditional extremes for severe ocean storms. *Extremes*, 24:241–265, 2021b.

R. Shooter, E Ross, A. Ribal, I. R. Young, and P. Jonathan. Multivariate spatial conditional extremes for extreme ocean environments. *Ocean Eng.*, 247:110647, 2022.

E. S. Simpson and J. A. Tawn. Inference for new environmental contours using extreme value analysis. *J. Agric. Biol. Environ. Stat.*, 2024. URL `https://doi.org/10.1007/s13253-024-00612-2`.

M. Speers, D. Randell, J. A. Tawn, and P. Jonathan. Estimating metocean environments associated with extreme structural response. *Ocean Eng.*, 311:118754, 2024.

Standard Norge. Shall NORSOK N-0031 and NORSOK N-0062 be updated as a result of findings in LOADS JIP? Conclusions from the evaluation committee. `https://standard.no/globalassets/fagomrader-sektorer/petroleum/loads-jip-and-norsok-n_003.pdf`, 2022.

C. Swan. Extreme Environmental Loading of Fixed Offshore Structures: Summary Report, Component 1. `https://www.hse.gov.uk/offshore/assets/docs/summary-report-component1.pdf`, 2020.

P. H. Taylor, P. Jonathan, and L. A. Harland. Time domain simulation of jack-up dynamics with the extremes of a Gaussian process. *J. Vib. Acoust.*, 119: 624–628, 1997.

S. Tendijck, E. Ross, D. Randell, and P. Jonathan. A non-stationary statistical model for the evolution of extreme storm events. *Environmetrics*, 30:e2541, 2019.

S. Tendijck, E. Eastoe, J. Tawn, D. Randell, and P. Jonathan. Modeling the extremes of bivariate mixture distributions with application to oceanographic data. *J. Am. Statist. Soc.*, 118:1373–1384, 2023.

S. Tendijck, P. Jonathan, D. Randell, and J. A. Tawn. Temporal evolution of the extreme excursions of multivariate kth order Markov processes with application to oceanographic data. *Environmetrics*, 35:e2834, 2024.

R Towe, D Randell, J Kensler, G Feld, and P Jonathan. Estimation of associated values from conditional extreme value models. *Ocean Eng.*, 272:113808, 2023.

R. Towe, E. Ross, D. Randell, and P. Jonathan. covXtreme: MATLAB software for non-stationary penalised piecewise constant marginal and conditional extreme value models. *Environ. Model. Softw.*, 177:106035, 2024.

E Vanem, T Zhu, and A Babanin. Statistical modelling of the ocean environment: a review of recent developments in theory and applications. *Marine Structures*, 86:103297, 2022.

J L Wadsworth and R Campbell. Statistical inference for multivariate extremes via a geometric approach. *J. Roy. Statist. Soc. B*, 2024. URL `https://doi.org/10.1093/jrsssb/qkae030`.

J. L. Wadsworth, J. A. Tawn, A. C. Davison, and D. M. Elton. Modelling across extremal dependence classes. *J. Roy. Statist. Soc. C*, 79:149–175, 2017.

References IV

H. C. Winter and J. A. Tawn. Modelling heatwaves in central France: a case-study in extremal dependence. *J. Roy. Statist. Soc. C*, 65:345–365, 2016.

S. N. Wood. Thin plate regression splines. *J. Roy. Statist. Soc. B*, 65:95–114, 2003.

B. Youngman. evgam: generalised additive extreme value models. `https://cran.r-project.org/package=evgam`, 2022.

E. Zanini, E. Eastoe, M. J. Jones, D. Randell, and P. Jonathan. Flexible covariate representations for extremes. *Environmetrics*, 31:e2624, 2020.

J. Zhang. Improving on estimation for the generalized Pareto distribution. *Technometrics*, 52:335–339, 2010.

J. Zhang and M. A. Stephens. A new and efficient estimation method for the generalized Pareto distribution. *Technometrics*, 51:316–325, 2009.

Jin Zhang. Likelihood moment estimation for the generalized Pareto distribution. *Australian & New Zealand Journal of Statistics*, 49:69–77, 2007.

Backup

What is a return value?

- Random variable A represents the maximum value of some physical quantity X per annum
- Forget about all complicating issues like serial dependence, covariates and other sources of dependence and uncertainty
- The P-year return value x_P of X is then defined by the equation

$$F_A(x_P) = \Pr(A \leq x_P) = 1 - \frac{1}{P}$$

- Or

$$x_P = F_A^{-1}(1 - \frac{1}{P})$$

- Typically $P \in [10^2, 10^8]$ years

An alternative definition

- Random variable A_P represents the P-year maximum value of X
- The P-year return value x'_P of X can be found from F_{A_P} for large P, assuming independent annual maxima since

$$\begin{aligned} F_A(x_P) &= 1 - \frac{1}{P} \\ \Rightarrow F_{A_P}(x_P) &= \left(1 - \frac{1}{P}\right)^P \approx \exp(-1) \end{aligned}$$

- Use $F_{A_P}(x'_P) = \exp(-1)$ to define an alternative return value x'_P

Estimating a return value

- To estimate x_P, we need knowledge of the distribution function F_A of the annual maximum
- We might estimate F_A using extreme value analysis on a sample of independent observations of A
- Typically more efficient to estimate the distribution $F_{X|X>\psi}$ of threshold exceedances of X above some high threshold ψ using a sample of independent observations of X, and use this in turn to estimate F_A and x_P
- How is this done?

Estimating a return value

- Asymptotic theory suggests for high threshold $\psi \in (-\infty, \infty)$ that

$$F_{X|X>\psi}(x|\psi, \sigma, \xi) = 1 - \left(1 + \frac{\xi}{\sigma}(x - \psi)\right)_+^{-1/\xi}$$

  for $x > \psi$, shape $\xi \in (-\infty, \infty)$ and scale $\sigma \in (0, \infty)$
- The full distribution of X is $F_X(x) = \tau + (1-\tau)F_{X|X>\psi}(x)$ where $\tau = \Pr(X \leq \psi)$
- Thus

$$F_A(x) = \Pr(A \leq x) = \sum_{k=0}^{\infty} f_C(k) F_X^k(x)$$

  where C is the number of occurrences of X per annum, with probability mass function f_C to be estimated (say with a Poisson model with parameter λ)

- So what's the problem?

Parameter uncertainty

- x_p can be estimated easily in the absence of uncertainty

- In reality, we estimate parameters λ, ψ, σ and ξ from a sample of data, and we cannot know their values exactly
- How does this epistemic uncertainty affect return value estimates?

- A number of different plausible estimators for return values under uncertainty
- Different estimators perform differently (bias and variance)
- Which estimators are likely to perform reasonably in fairly general circumstances?

- Is it even **sensible or desirable** to estimate return values?

Incorporating uncertainty

- If a distribution $F_{Y|\boldsymbol{Z}}$ of random variable Y is known conditional on random variables $\boldsymbol{Z}$, and the joint density $f_{\boldsymbol{Z}}$ of $\boldsymbol{Z}$ is also known, the unconditional predictive distribution $\tilde{F}_Y$ can be evaluated using

$$\tilde{F}_Y(y) = \int_\zeta F_{Y|\boldsymbol{Z}}(x|\zeta) f_{\boldsymbol{Z}}(\zeta)\, d\zeta$$

- Th expected value of deterministic function g of parameters $\boldsymbol{Z}$ given joint density $f_{\boldsymbol{Z}}$ is

$$E[g(\boldsymbol{Z})] = \int_\zeta g(\zeta) f_{\boldsymbol{Z}}(\zeta)\, d\zeta$$

- $\zeta = (\lambda, \psi, \sigma, \xi)$, $Y = A$ (or $Y = A_P$)

Different estimators of return value

- Uncertain estimates of GP model parameters from fit to sample represented by random variables $\boldsymbol{Z}$
- Estimate distribution $F_{A|\boldsymbol{Z}}$ of annual maximum event using $\boldsymbol{Z}$
- Estimate P-year return value by finding the $1-1/P$ quantile of $F_{A|\boldsymbol{Z}}$

- Various options available, including:

$$
\begin{aligned}
q_1 &= F_{A|\boldsymbol{Z}}^{-1}(1-1/P \mid \mathbb{E}_{\boldsymbol{Z}}[\boldsymbol{Z}]) = F_{A|\boldsymbol{Z}}^{-1}(1-1/P \mid \int_\zeta \zeta f_{\boldsymbol{Z}}(\zeta) d\zeta) \\
q_2 &= \mathbb{E}_{\boldsymbol{Z}}[F_{A|\boldsymbol{Z}}^{-1}(1-1/P \mid \boldsymbol{Z})] = \int_\zeta F_{A|\boldsymbol{Z}}^{-1}(1-1/P \mid \zeta) f_{\boldsymbol{Z}}(\zeta) d\zeta \\
q_3 &= \tilde{F}_A^{-1}(1-1/P) \text{ where } \tilde{F}_A(x) = \int_\zeta F_{A|\boldsymbol{Z}}(x \mid \zeta) f_{\boldsymbol{Z}}(\zeta) d\zeta \\
q_4 &= \tilde{F}_{A_P}^{-1}(\exp(-1)) \text{ where } \tilde{F}_{A_P}(x) = \tilde{F}_A^P(x) \\
q_5 &= \text{med}_{\boldsymbol{Z}}[F_{A|\boldsymbol{Z}}^{-1}(1-1/P \mid \boldsymbol{Z})]
\end{aligned}
$$

- For small samples, these have very different properties

Storm peaks

Context: unconditional distribution of loading

$$
\begin{aligned}
F_Y(y) \quad = \quad & \int_\zeta \int_{(\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in\mathcal{S}_\tau},\tau)} \int_{\mathbf{x}^{\text{sp}}} \int_{\boldsymbol{\theta}^{\text{sp}}} \\
\times \quad & F_{Y|\{(\mathbf{X}_s,\boldsymbol{\Theta}_s)\}_{s\in\mathcal{S}_T},Z}(y|\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in\mathcal{S}_\tau},\zeta) \\
\times \quad & f_{(\{(\mathbf{X}_s,\boldsymbol{\Theta}_s)\}_{s\in\mathcal{S}_T},T)|\mathbf{X}^{\text{sp}},\boldsymbol{\Theta}^{\text{sp}},Z}\left(\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in\mathcal{S}_\tau},\tau \mid \mathbf{x}^{\text{sp}},\boldsymbol{\theta}^{\text{sp}},\zeta\right) \\
\times \quad & f_{\mathbf{X}^{\text{sp}}|\boldsymbol{\Theta}^{\text{sp}},Z}(\mathbf{x}^{\text{sp}}|\boldsymbol{\theta}^{\text{sp}},\zeta) \\
\times \quad & f_{\boldsymbol{\Theta}^{\text{sp}}|Z}(\boldsymbol{\theta}^{\text{sp}}|\zeta) \\
\times \quad & f_Z(\zeta) \\
\times \quad & \mathrm{d}\boldsymbol{\theta}^{\text{sp}}\,\mathrm{d}\mathbf{x}^{\text{sp}}\,\mathrm{d}(\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in\mathcal{S}_\tau},\tau)\,\mathrm{d}\zeta
\end{aligned}
$$

Storm peaks: modelling margins and dependence

$$
\begin{aligned}
f_{\mathbf{X}^{\text{sp}}|\boldsymbol{\Theta}^{\text{sp}},Z}(\mathbf{x}^{\text{sp}}|\boldsymbol{\theta}^{\text{sp}},\zeta) \quad = \quad & \left[\prod_{j=1}^{p} f_{X_j^{\text{sp}}|\boldsymbol{\Theta}^{\text{sp}},Z}(x_j^{\text{sp}}|\boldsymbol{\theta}^{\text{sp}},\zeta)\right] \\
\times \quad & c(F_{X_1^{\text{sp}}|\boldsymbol{\Theta}^{\text{sp}},Z}(x_1^{\text{sp}}|\boldsymbol{\theta}^{\text{sp}},\zeta),\ldots,F_{X_p^{\text{sp}}|\boldsymbol{\Theta}^{\text{sp}},Z}(x_p^{\text{sp}}|\boldsymbol{\theta}^{\text{sp}},\zeta)|\boldsymbol{\theta}^{\text{sp}},\zeta)
\end{aligned}
$$

More to come in a minute!

Within-storm evolution

Context: unconditional distribution of loading

$$
\begin{aligned}
F_Y(y) \quad = \quad & \int_\zeta \int_{(\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in\mathcal{S}_\tau},\tau)} \int_{\mathbf{x}^{\mathrm{sp}}} \int_{\boldsymbol{\theta}^{\mathrm{sp}}} \\
\times \quad & F_{Y|\{(\mathbf{X}_s,\boldsymbol{\Theta}_s)\}_{s\in S_T},Z}(y|\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in S_\tau},\zeta) \\
\times \quad & f_{(\{(\mathbf{X}_s,\boldsymbol{\Theta}_s)\}_{s\in\mathcal{S}_T},T)|\mathbf{X}^{\mathrm{sp}},\boldsymbol{\Theta}^{\mathrm{sp}},Z}\left(\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in\mathcal{S}_\tau},\tau \mid \mathbf{x}^{\mathrm{sp}},\boldsymbol{\theta}^{\mathrm{sp}},\zeta\right) \\
\times \quad & f_{\mathbf{X}^{\mathrm{sp}}|\boldsymbol{\Theta}^{\mathrm{sp}},Z}(\mathbf{x}^{\mathrm{sp}}|\boldsymbol{\theta}^{\mathrm{sp}},\zeta) \\
\times \quad & f_{\boldsymbol{\Theta}^{\mathrm{sp}}|Z}(\boldsymbol{\theta}^{\mathrm{sp}}|\zeta) \\
\times \quad & f_Z(\zeta) \\
\times \quad & \mathrm{d}\boldsymbol{\theta}^{\mathrm{sp}}\,\mathrm{d}\mathbf{x}^{\mathrm{sp}}\,\mathrm{d}(\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in\mathcal{S}_\tau},\tau)\,\mathrm{d}\zeta
\end{aligned}
$$

Models for within-storm evolution

- History matching: Feld et al. [2019], Hansen et al. [2020]
- Extreme value time-series model (an extension of conditional extremes): Tendijck et al. [2019], Tendijck et al. [2024]

Fluid loading

Context: unconditional distribution of loading

$$
\begin{aligned}
F_{\boldsymbol{Y}}(\boldsymbol{y}) \quad = \quad & \int_{\zeta} \int_{(\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in\mathcal{S}_\tau},\tau)} \int_{\mathbf{x}^{\mathrm{sp}}} \int_{\boldsymbol{\theta}^{\mathrm{sp}}} \\
\times \quad & F_{\boldsymbol{Y}|\{(\mathbf{X}_s,\boldsymbol{\Theta}_s)\}_{s\in\mathcal{S}_T},\boldsymbol{Z}}(\boldsymbol{y}|\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in\mathcal{S}_\tau},\zeta) \\
\times \quad & f_{(\{(\mathbf{X}_s,\boldsymbol{\Theta}_s)\}_{s\in\mathcal{S}_T},T)|\mathbf{X}^{\mathrm{sp}},\boldsymbol{\Theta}^{\mathrm{sp}},\boldsymbol{Z}}\left(\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in\mathcal{S}_\tau},\tau \mid \mathbf{x}^{\mathrm{sp}},\boldsymbol{\theta}^{\mathrm{sp}},\zeta\right) \\
\times \quad & f_{\mathbf{X}^{\mathrm{sp}}|\boldsymbol{\Theta}^{\mathrm{sp}},\boldsymbol{Z}}(\mathbf{x}^{\mathrm{sp}}|\boldsymbol{\theta}^{\mathrm{sp}},\zeta) \\
\times \quad & f_{\boldsymbol{\Theta}^{\mathrm{sp}}|\boldsymbol{Z}}(\boldsymbol{\theta}^{\mathrm{sp}}|\zeta) \\
\times \quad & f_{\boldsymbol{Z}}(\zeta) \\
\times \quad & \mathrm{d}\boldsymbol{\theta}^{\mathrm{sp}}\,\mathrm{d}\mathbf{x}^{\mathrm{sp}}\,\mathrm{d}(\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in\mathcal{S}_\tau},\tau)\,\mathrm{d}\zeta
\end{aligned}
$$

Models for fluid loading

- Incorporate kinematics, estimate Morison loads (e.g. LOADS, AWARE JIPs): Swan [2020], Gibson [2020]
- Interface environment and fluid loading software for full “forward model”
- Fundamentals paper: Speers et al. [2024]

Motivating marginal extremes

Storm peaks: modelling margins and dependence

$$
\begin{aligned}
f_{\mathbf{X}^{sp}|\boldsymbol{\Theta}^{sp},\boldsymbol{Z}}(\mathbf{x}^{sp}|\theta^{sp},\zeta) &= \left[\prod_{j=1}^{p} f_{X_j^{sp}|\boldsymbol{\Theta}^{sp},\boldsymbol{Z}}(x_j^{sp}|\theta^{sp},\zeta)\right] \\
&\times \quad c(F_{X_1^{sp}|\boldsymbol{\Theta}^{sp},\boldsymbol{Z}}(x_1^{sp}|\theta^{sp},\zeta),\ldots,F_{X_p^{sp}|\boldsymbol{\Theta}^{sp},\boldsymbol{Z}}(x_p^{sp}|\theta^{sp},\zeta)|\theta^{sp},\zeta)
\end{aligned}
$$

Generalised Pareto distribution

- Suppose we have an exceedance X of high threshold $\psi \in \mathbb{R}$
- The Pickands-Balkema-De Haan theorem states

$$\begin{aligned}
\lim_{\psi \to \infty} \mathbb{P}[X \leq x | X > \psi] &= \lim_{\psi \to \infty} \frac{F_X(x)}{1 - F_X(\psi)} \\
&= \mathrm{GP}(x|\xi, \sigma, \psi) \\
&= 1 - \left(1 + \frac{\xi}{\sigma}(x - \psi)\right)_+^{-1/\xi}, \quad \sigma > 0, \quad \xi \in \mathbb{R}
\end{aligned}$$

Theory

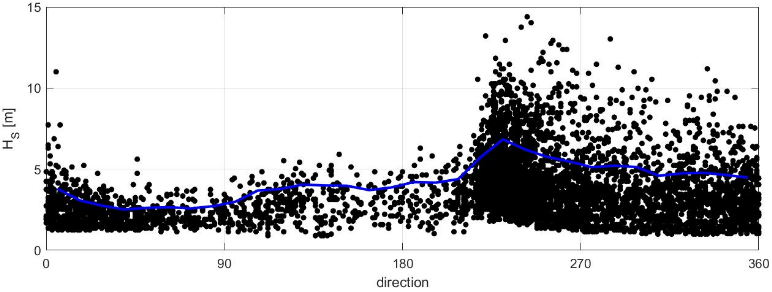
- Derived from max-stability of F_X
- Threshold-stability property
- "Poisson $\times$ GP $=$ GEV"

Practicalities

- How to isolate independent threshold exceedances from observed time-series?
- How to specify extreme threshold ψ?
- ξ, σ, ψ functions of covariates

Motivation

- Environmental extremes vary smoothly with multidimensional covariates
- Generic modelling framework for different covariate representations
- Statistical and computational efficiency for n-D covariates
- Thorough Bayesian uncertainty quantification

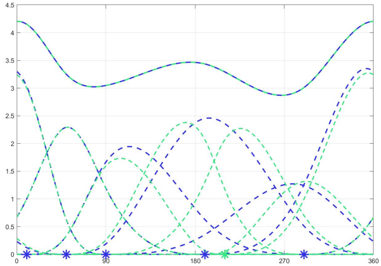


Typical data for northern North Sea. Storm peak H_S on direction, with $\tau = 0.8$ extreme value threshold.
Rate and size of occurrence varies with direction.

Basis representations ... BARS and others

Bayesian adaptive regression splines (BARS)

- n irregularly-spaced knots on $\mathcal{D}_\theta$
- $\boldsymbol{B}$ consists of n B-spline bases
- Order d
- Each using $d+1$ consecutive knot locations
- Local support
- Wrapped on $\mathcal{D}_\theta$
- Knot locations $\{r_k\}_{k=1}^{n}$ vary
- Number of basis functions n varies

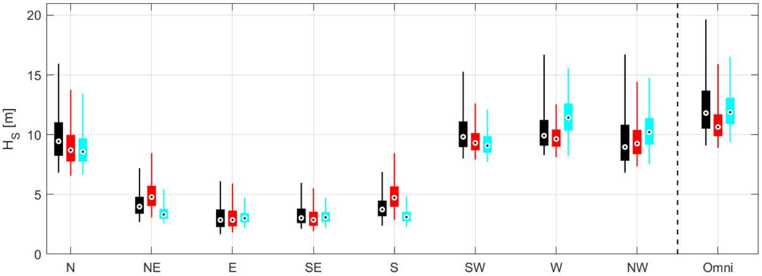


Periodic BARS knot birth and death

P-splines and Voronoi partition

- P-splines use fixed number of regularly-spaced knots
- Voronoi partition uses piecewise-constant representation, trivially extended to n-D

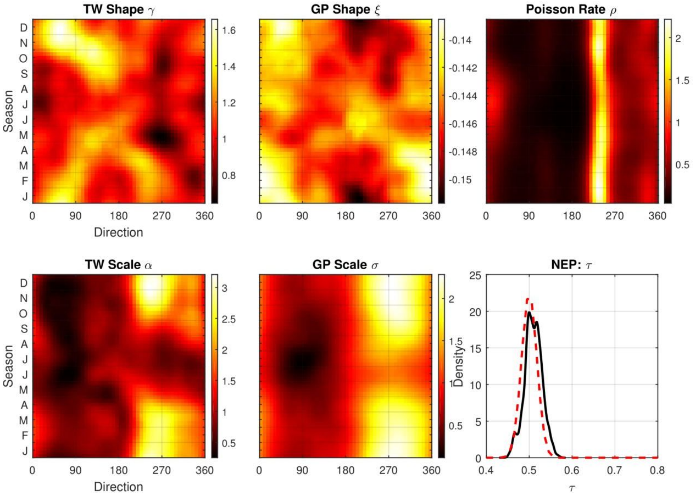
Directional posterior predictive distribution of $P = 1000$-year maximum



- Box-whiskers with 2.5%, 25%, 50%, 75% and 97.5% percentiles
- General agreement

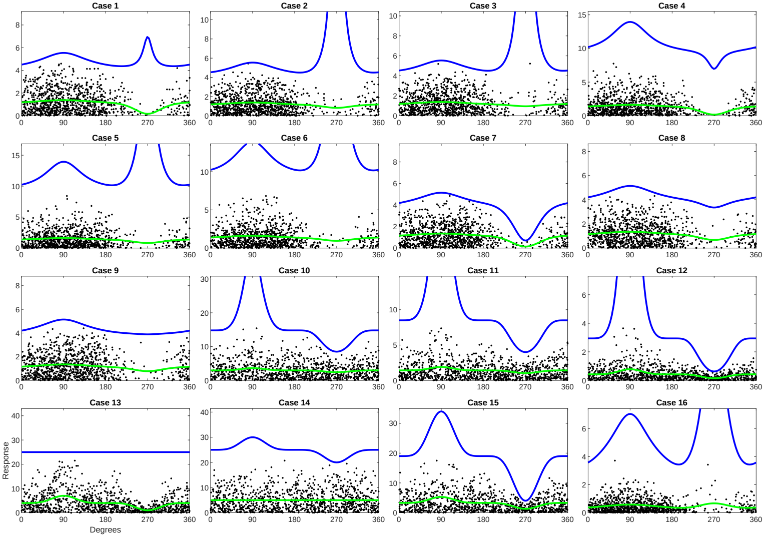
- This is more-or-less what the engineer currently uses to design a “compliant” structure

Extension to 2D : directional-seasonal



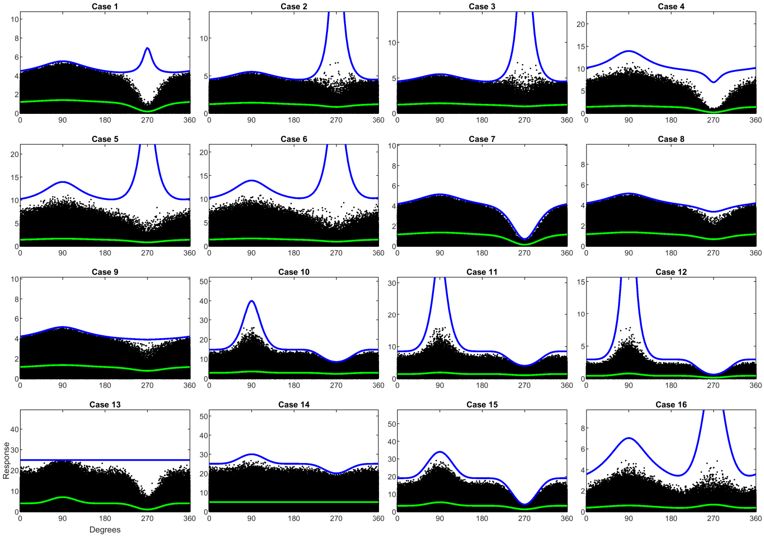
- 2-D tensor product P-spline bases for same northern North Sea location
- Marginal posterior median estimates (plus posterior density for τ)

Case studies



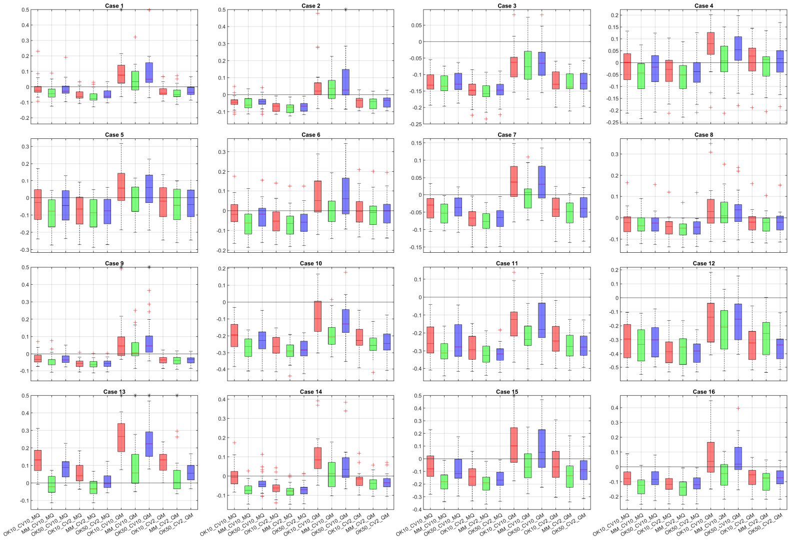
- Small samples

Case studies



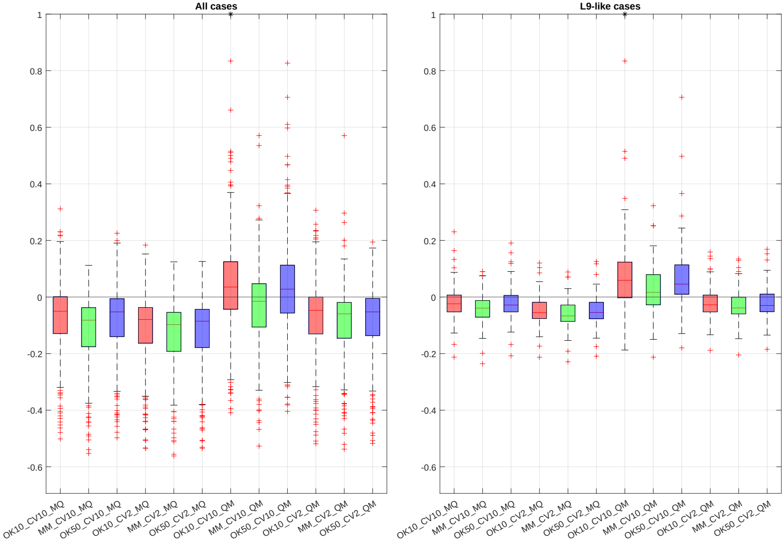
- Large samples

Case studies



- Performance by case

Case studies



- Aggregate performance

Extremal vector auto-regression (EVAR) for within-storm evolution

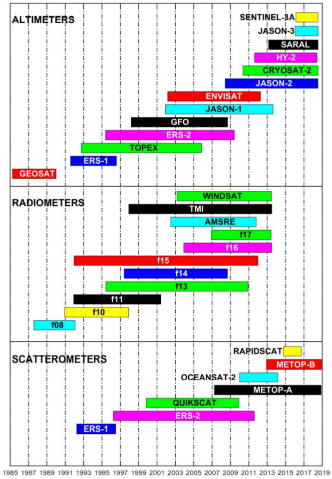
Context: unconditional distribution of loading

$$
\begin{aligned}
F_Y(y) \quad = \quad & \int_\zeta \int_{(\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in\mathcal{S}_\tau},\tau)} \int_{\mathbf{x}^{\mathrm{sp}}} \int_{\boldsymbol{\theta}^{\mathrm{sp}}} \\
\times \quad & F_{Y|\{(\mathbf{X}_s,\boldsymbol{\Theta}_s)\}_{s\in S_T},\mathbf{Z}}(y|\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in S_\tau},\zeta) \\
\times \quad & f_{(\{(\mathbf{X}_s,\boldsymbol{\Theta}_s)\}_{s\in\mathcal{S}_T},T)|\mathbf{X}^{\mathrm{sp}},\boldsymbol{\Theta}^{\mathrm{sp}},\mathbf{Z}}\left(\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in\mathcal{S}_\tau},\tau \mid \mathbf{x}^{\mathrm{sp}},\boldsymbol{\theta}^{\mathrm{sp}},\zeta\right) \\
\times \quad & f_{\mathbf{X}^{\mathrm{sp}}|\boldsymbol{\Theta}^{\mathrm{sp}},\mathbf{Z}}(\mathbf{x}^{\mathrm{sp}}|\boldsymbol{\theta}^{\mathrm{sp}},\zeta) \\
\times \quad & f_{\boldsymbol{\Theta}^{\mathrm{sp}}|\mathbf{Z}}(\boldsymbol{\theta}^{\mathrm{sp}}|\zeta) \\
\times \quad & f_{\mathbf{Z}}(\zeta) \\
\times \quad & \mathrm{d}\boldsymbol{\theta}^{\mathrm{sp}}\,\mathrm{d}\mathbf{x}^{\mathrm{sp}}\,\mathrm{d}(\{(\mathbf{x}_s,\boldsymbol{\theta}_s)\}_{s\in\mathcal{S}_\tau},\tau)\,\mathrm{d}\zeta
\end{aligned}
$$

Models for within-storm evolution

- History matching: Feld et al. [2019], Hansen et al. [2020]
- Extreme value time-series model (an extension of conditional extremes): Tendijck et al. [2019], Tendijck et al. [2024]

Satellite observation



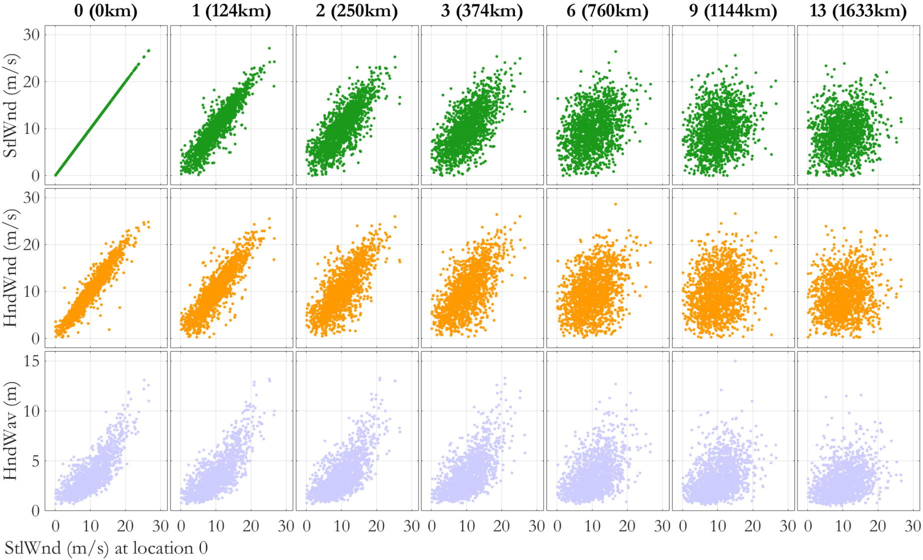
[Ribal and Young 2019]

Features

- Altimetry: H_S and U_{10}
- Scatterometry: best for U_{10} and direction
- > 30 years of observations
- Spatial coverage is by no means complete: one observation daily if all well
- Calibration necessary (to buoys and reanalysis datasets, Ribal and Young 2020)
- METOP(-A,-B,-C) since 2007

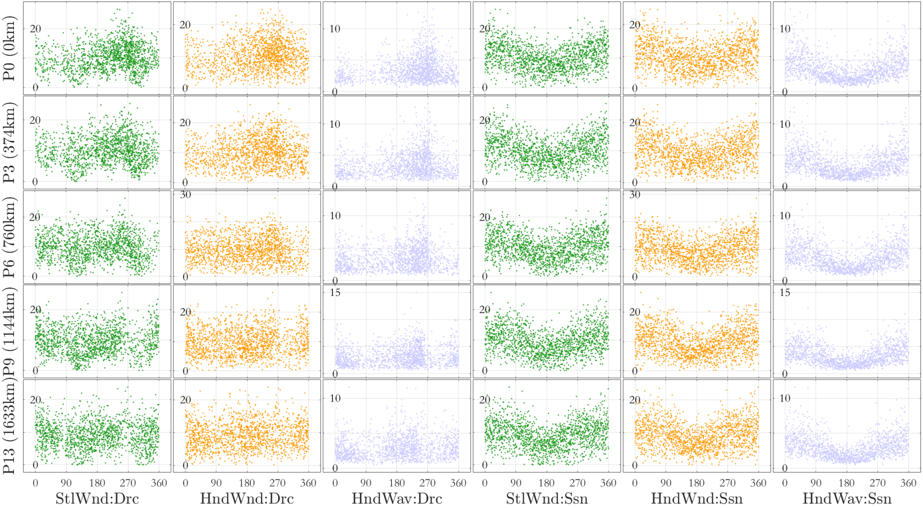
H_S: significant wave height (m)
U_{10}: wind speed (ms^{-1}) at 10m (calibrated to 10-minute average wind speed)

Scatter plots on physical scale



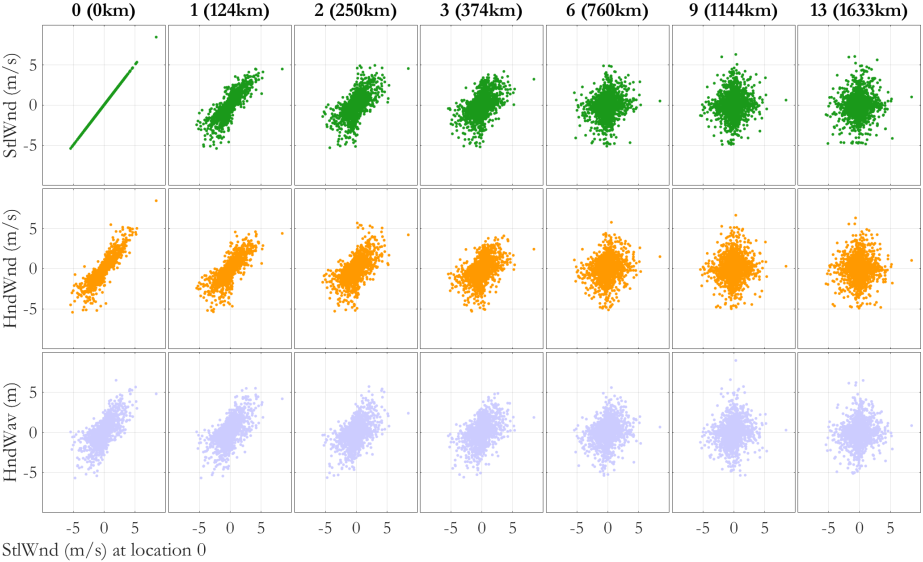
Scatter plots of registered data : StlWnd (green), HndWnd (orange), HndWav(blue)

Covariate dependence on physical scale



Directional and seasonal dependence. “Direction” is that from which fluid flows measured clockwise from North
StlWnd (green), HndWnd (orange), HndWav(blue)

Scatter plots on Laplace scale



Registered data on Laplace scale: StlWnd (green), HndWnd (orange), HndWav(blue)

Generalised extreme value distribution

- F_X^n is the distribution of the maximum of n independent draws of X
- If F_X^n "looks like" $F_X^{n'}$, we say F_X is max-stable

- More formally, F_X is max-stable if there exist sequences of constants $a_n > 0$, b_n, and non-degenerate G_ξ such that

$$\lim_{n \to \infty} F_X^n (a_n x + b_n) = G_\xi(x)$$

- We say $F_X \in D(G_\xi)$ or that F_X lies in the max-domain of attraction of G_ξ
- The Fisher–Tippett–Gnedenko theorem states that G_ξ is the generalised extreme value distribution with parameter ξ

$$G_\xi(y) = \exp\left(-(1+\xi y)^{-1/\xi}\right), \quad \xi \in \mathbb{R}$$

- For large n, makes sense to model block maxima of n iid draws using G_ξ (with $(x-\mu)/\sigma$ in place of y above)

Multivariate extreme value distribution (MEVD)

- $\boldsymbol{X}_i = (X_{i1}, \ldots, X_{ij}, \ldots, X_{ip}), i = 1, \ldots, n$ iid p-vectors, distribution F
- $M_{n,j} = \max_i X_{ij}$, component-wise maximum
- The component-wise maximum is not "observed" (especially as $n \to \infty$)

- Then for $Z_{n,j} = (M_{n,j} - b_{n,j})/a_{n,j}$, normalised with scaling constants:

$$\mathbb{P}(\boldsymbol{Z} \leq \boldsymbol{z}) = F^n\left(\boldsymbol{a}_n \boldsymbol{z} + \boldsymbol{b}_n\right) \to G(\boldsymbol{z}) \quad \text{as} \quad n \to \infty$$

- Non-degenerate $G(\boldsymbol{z})$ must be max-stable, so $\forall k \in \mathbb{N}$, $\exists\ \boldsymbol{\alpha_k} > \boldsymbol{0}, \boldsymbol{\beta_k}$ s.t.

$$G^k(\boldsymbol{\alpha}_k \boldsymbol{z} + \boldsymbol{\beta}_k) = G(\boldsymbol{z})$$

- We say $F \in D(G)$
- Margins $G_1, \ldots, G_p$ are unique GEV, but $G(\boldsymbol{z})$ is not unique

MEVD on common margins

- On standard Fréchet margins with pseudo-polars $(r, \boldsymbol{w})$

$$
\begin{aligned}
G(\boldsymbol{z}) &= \exp\left(-V(\boldsymbol{z})\right) \\
\text{with } V(\boldsymbol{z}) &= \int_{\Delta} \max_{j}\{\frac{w_j}{z_j}\}\, S(d\boldsymbol{w}), \quad \text{on } \Delta = \{\boldsymbol{w} \in \mathbb{R}^p : ||\boldsymbol{w}|| = 1\} \\
\text{and } 1 &= \int_{\Delta} w_j\, S(d\boldsymbol{w}), \quad \forall j, \text{ for angular measure } S
\end{aligned}
$$

- Condition of multivariate regular variation, MRV

$$
\frac{1 - F(t\boldsymbol{x})}{1 - F(t\mathbf{1})} \to \lambda(\boldsymbol{x}) \text{ as } t \to \infty, \boldsymbol{x} \in \mathbb{R}^p
$$

  useful to prove that $F \in D(G)$ for some MEVD G

- Lots more

Asymptotic dependence ... admitted by MEVD

- On uniform margins

$$\chi(u) = \frac{\mathbb{P}(U > u, V > u)}{\mathbb{P}(U > u)} = \frac{\bar{C}(u,u)}{1-u} \quad \to \chi \text{ as } u \to 1$$

- $\chi = 1$ perfect dependence
- $\chi \in (0,1)$ asymptotic dependence, AD
- $\chi = 0$ perfect independence

$$\theta(u) = \frac{\log \mathbb{P}(U \leq u, V \leq u)}{\log \mathbb{P}(U \leq u)} = \frac{\log C(u,u)}{\log u} \quad \to \theta \text{ as } u \to 1$$

- $\theta = 2 - \chi$

- χ and θ describe AD
- MEVD admits AD

Asymptotic independence ... not admitted by MEVD

- On uniform margins

$$\bar{\chi}(u) = 2\frac{\log \mathbb{P}(U > u)}{\log \mathbb{P}(U > u, V > u)} - 1 = 2\frac{\log(1-u)}{\log \bar{C}(u,u)} - 1 \quad \to \bar{\chi} \text{ as } u \to 1$$

- $\bar{\chi} = 1$ perfect dependence and AD
- $\bar{\chi} \in (0,1)$ asymptotic independence, AI
- $\bar{\chi} = 0$ perfect independence
- On Fréchet margins ($F(z) = \exp\left(-z^{-1}\right)$), assume

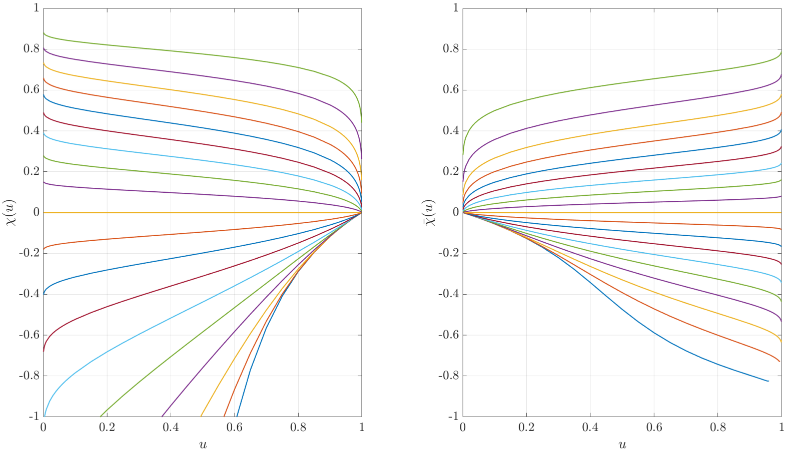
$$\frac{\mathbb{P}(Z_1 > z, Z_2 > z)}{\left(\mathbb{P}(Z_1 > z)\right)^{1/\eta}} = \mathcal{L}(z)$$

where $\mathcal{L}$ is slowly varying : $\mathcal{L}(xz)/\mathcal{L}(z) \to 1$ as $z \to \infty$

- $\bar{\chi} = 2\eta - 1$
- Idea : use non-extreme value copulas or inverted EV copulas
- Also $\mathbb{P}(Z_2 > z | Z_1 > z) \approx Cz^{1-1/\eta}$ from above
- Idea : assume a max-stable-like normalisation for conditional extremes

Extremal dependence (bivariate Gaussian)

- Many (almost all?) environmental extremes problems involve asymptotic independence, at least in part ... bivariate Gaussian is one example!

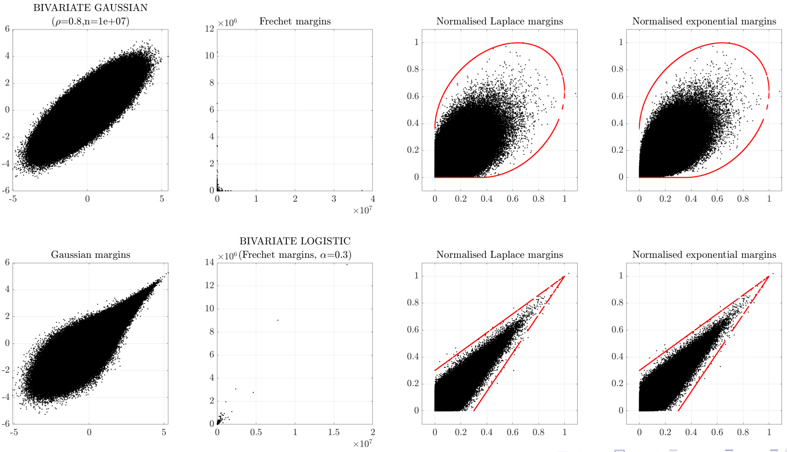


$\chi(u)$ and $\bar{\chi}(u)$ for bivariate Gaussian ($\Rightarrow \chi = 0, \bar{\chi} = \rho$)
Colours are correlations ρ on -0.9, -0.8, ..., 0.9 (Recreated from Coles et al. 1999)

Limit sets

Intuition

- Transform your sample $\boldsymbol{X}$ (empirically) to certain standard margins $\boldsymbol{X}_S$ (e.g. Laplace or exponential)
- Divide each value of $\boldsymbol{X}_S$ by a simple known function of n (like $\log(n/2)$ for Laplace) appropriate for that marginal scale
- The normalised values must be contained within a limit set in red below (which you can work out from theory)
- The cloud shape reveals dependence structure (e.g. AI (top) or AD (bottom))
- Value of HT α where red curve touches $y = 1$ or $x = 1$



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