Bayesian linear inspection planning for large-scale physical systems

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Abstract: Modelling of complex corroding industrial systems is critical to effective inspection and maintenance for assurance of system integrity. Wall thickness and corrosion rate are modelled for multiple dependent corroding components, given observations of minimum wall thickness per component. At each inspection, partial observations of the system are considered. A Bayes linear approach is adopted simplifying parameter estimation and avoiding often unrealistic distributional assumptions. Key system variances are modelled, making exchangeability assumptions to facilitate analysis for sparse inspection time series. A utility-based criterion is used to assess quality of inspection design and aid decision making. The model is applied to inspection data from pipework networks on a full-scale offshore platform.

Keywords: Bayes linear, exchangeability, inspection planning, corrosion, variance learning, dynamic linear model

1 INTRODUCTION AND MOTIVATION

Large industrial systems are susceptible to corrosion, leading to economic and environmental costs which can be mitigated by careful inspection and maintenance. Modelling these complex systems can improve the effectiveness of inspection and maintenance activities, providing a rational decision framework and preventing costly failures. Such systems can be thought of as collections of separate units or components. Most attempts to model corrosion concentrate on modelling wall thickness and corrosion rate for individual components. However, the corrosion behaviour of components is often interrelated, because, for example, of common usage, location, or age. This relation can be exploited to improve the quality of inspection information. Moreover, inspections are difficult and expensive, and rarely carried out system-wide; historical data for individual components is rather limited, but the number of components is often large.

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Consider a decision problem where an inspection design has to be specified. A particular inspection design, d, states which components are to be inspected. To quantify the value of a particular design, the benefit of reducing uncertainty about the current state of the system and thus reducing any potential loss incurred from component failures using that design is considered. The increased knowledge of the system must be balanced against the cost of the inspection. If the cost of gaining information about the system is greater than the benefit then it is not worth carrying out the inspection. The question which then arises is how to quantify the value of reducing the system integrity. Here the expected utility of any particular design is evaluated for this purpose.

Industry guidelines (e.g. [**1**] and [**2**]) treat the modelling of corrosion very generally, yet there is a vast body of engineering literature on this subject. Zhang and Mahadevan [**3**] outline mathematical expressions for initiation and evolution of different corrosion mechanism, including pitting and cracking. Kallen and van Noortwijk [**4**] discuss inspection and maintenance decisions based on imperfect inspection within a Bayesian framework using gamma processes.

Qin and Cui [**5**] provide a corrosion model describing the corrosion process on steel structures. Gasemyr and Natvig [**6**] present a Bayesian approach using partial inspections only. A number of authors discuss the inclusion of inspection data and expert judgement within a risk-based inspection framework. For example, Farber and Sorensen [**7**] present an approach to estimating system condition for inspection planning purposes using a combination of inspection observations and expert judgement, and Straub [**8**] describes generic approaches to risk-based inspection of steel structures.

Prior belief specification for large problems is usually very difficult. Even in small problems, with few sources of uncertainty, it can be difficult to estimate a satisfactory full joint prior probability specification over all of the possible outcomes. In practical problems, there may be hundreds of relevant sources of uncertainty about which prior judgements are made. In such problems, it is arguably impossible to carry out a full Bayesian analysis. If such a full prior specification were possible, it would often be the case that the specification was too time-consuming and too difficult to check. Furthermore, the resulting Bayes analysis would often be extremely computer intensive, particularly in areas such as experimental design. The Bayes linear approach is particularly appropriate whenever the full Bayes approach requires an unnecessarily exhaustive description and analysis of prior uncertainty. The Bayes linear approach can be viewed as either:

(a) offering a simple approximation to a full Bayes analysis, for problems where the full analysis would be too difficult or time-consuming;
(b) complementary to the full Bayes analysis, offering a variety of new interpretative and diagnostic tools which may be of value whatever ones viewpoint;
(c) a generalization of the full Bayes approach where the artificial constraint that requires a full probabilistic prior specification is lifted.

In the current work modelling a relatively low-dimensional system is considered, with straightforward forms for the observation and system evolution equations (2), for purposes of illustration only. In this case a full Bayesian analysis could have been conducted. In practice, however, systems are generally large and model structure can be complex (for example with non-linear or non-invertible functions in observation equations). In these cases, full prior specification would be difficult, and full Bayesian analysis would be computationally intractable. Bayes linear modelling, however, still provides a feasible approach. Goldstein and Wooff [**9**] provides a detailed reference for Bayes linear methodology, and Harrison and West [**10**] similarly gives comprehensive coverage of dynamic linear models (DLMs). Little *et al.* [**11**] uses a multivariate DLM to characterize the corrosion of large industrial storage tanks, using observations of component minima, and suggests approaches to optimal inspection planning. Little *et al.* [**12**] describes the application of a spatio-temporal DLM to modelling the corrosion of an industrial furnace using Bayes linear updating. Empirical distance-based estimates for covariances of DLM observation and system variances are used, and optimal inspection planning based on heuristic criteria is considered. Farrow and Goldstein [**13**] discusses Bayes linear methods for grouped multivariate repeated measurement studies with application to crossover trials. Wilkinson [**14**] discusses variance learning for a univariate linear growth DLM, and Wilkinson and Goldstein [**15**] describes Bayes linear covariance matrix adjustment for a multivariate constant DLM.

A simple dynamic linear model for corrosion is considered in section 2. In sections 3 and 4 it is shown how to update system levels and carry out variance learning using Bayes linear methods. A utility-based criterion for efficient inspection schemes is discussed in section 5. Providing an efficient method for evaluating the quality of inspection designs. Designs incorporating both expectation and variance learning are considered in section 5.5. An example based on corrosion assessment for an offshore platform is considered in section 6.

2 MODEL

Consider inspection of a collection of components over time. A linear growth DLM, is used for the system level, X_{ct}, and system slope α_{ct}, for component, c, at time t. Observations of the system state, Y_{ct}, are made subject to measurement error of the form, $\sigma_Y \epsilon_{Yct}$. The model equations are

$$\begin{aligned}
\text{Observation:}\quad & Y_{ct} = X_{ct} + \sigma_Y \epsilon_{Yct} \\
\text{System level:}\quad & X_{ct} = X_{c(t-1)} + \alpha_{ct} + \sigma_{Xc}\epsilon_{Xct} \\
\text{System slope:}\quad & \alpha_{ct} = \alpha_{c(t-1)} + \sigma_{\alpha c}\epsilon_{\alpha ct}
\end{aligned}$$

where σ_Y is the measurement error standard deviation and σ_{Xc} and $\sigma_{\alpha c}$ are the standard deviations for component-wise system evolution. Furthermore

$$\begin{aligned}
&\mathrm{E}(\epsilon_{Yct}) = 0, \quad \mathrm{E}(\epsilon_{Xct}) = 0, \quad \mathrm{E}(\epsilon_{\alpha ct}) = 0 \\
&\mathrm{var}(\epsilon_{Yct}) = 1, \quad \mathrm{var}(\epsilon_{Xct}) = 1, \quad \mathrm{var}(\epsilon_{\alpha ct}) = 1 \\
&\mathrm{cov}(\epsilon_{Yct}, \epsilon_{Yc't}) = \gamma_{Ycc'}, \quad \text{for } c \neq c', \\
&\quad \mathrm{cov}(\epsilon_{Yct}, \epsilon_{Yc't'}) = 0, \quad \text{for } t \neq t' \quad \forall c, c' \\
&\mathrm{cov}(\epsilon_{Xct}, \epsilon_{Xc't}) = \gamma_{Xcc'}, \quad \text{for } c \neq c', \\
&\quad \mathrm{cov}(\epsilon_{Xct}, \epsilon_{Xc't'}) = 0, \quad \text{for } t \neq t' \quad \forall c, c' \\
&\mathrm{cov}(\epsilon_{\alpha ct}, \epsilon_{\alpha c't}) = \gamma_{\alpha cc'}, \quad \text{for } c \neq c' \\
&\quad \mathrm{cov}(\epsilon_{\alpha ct}, \epsilon_{\alpha c't'}) = 0, \quad \text{for } t \neq t' \quad \forall c, c'
\end{aligned}$$

where ϵ_{Yct}, ϵ_{Xct}, and $\epsilon_{\alpha ct}$ are mutually uncorrelated random variables. System evolution is controlled by the system evolution residuals $\sigma_{Xc}\epsilon_{Xct}$ and $\sigma_{\alpha c}\epsilon_{\alpha ct}$. Other than this specification, no distributional assumptions are required for the model within the Bayes linear framework; partial specification of (prior) beliefs is sufficient. Nevertheless, even this specification is a challenge in general. As a basis for prior specification, incorporated in the current work are:

(a) estimates from analysis of similar corrosion circuits;
(b) expert judgements from corrosion engineers familiar with models of this form.

This model form is considered to be adequate to illustrate the general methodology for the current application, but note that it can be enhanced in various ways. For example, the system slope terms may be restricted to be non-positive, corresponding to non-increasing wall thickness. However, in practice there are situations (e.g. undocumented component replacement or repair) in which allowing unconstrained variation of system slope terms is advantageous. Transformations of variables may also be considered in cases where prior beliefs are consistent with these, or if preliminary modelling work suggested this approach.

3 BAYES LINEAR ANALYSIS

Full prior belief specification for modelling complex systems can be difficult or impractical. Bayes linear analysis provides a framework for modelling based around partial belief specification, similar in spirit to a full Bayes approach. Bayes linear analysis also provides a computationally efficient method for updating beliefs in applications where a full Bayes approach would be intractable. Bayes linear analysis can be viewed as a generalization of the full Bayes approach which relaxes the requirement for full probabilistic prior specifications.

In Bayes linear analysis, expectation rather than probability is treated as a primitive quantity [**16**]; prior beliefs are specified in terms of means, variances, and covariances. Beliefs about a vector $\boldsymbol{B}$, given observations on a vector $\boldsymbol{D}$ are updated via the adjusted expectation, $\mathrm{E}_{\boldsymbol{D}}(\boldsymbol{B})$

$$\mathrm{E}_{\boldsymbol{D}}(\boldsymbol{B}) = \mathrm{E}(\boldsymbol{B}) + \mathrm{cov}(\boldsymbol{B},\boldsymbol{D})[\mathrm{var}(\boldsymbol{D})]^{-1}[\boldsymbol{D} - \mathrm{E}(\boldsymbol{D})]$$

and the adjusted variance matrix is given by $\mathrm{var}_{\boldsymbol{D}}(\boldsymbol{B})$

$$\mathrm{var}_{\boldsymbol{D}}(\boldsymbol{B}) = \mathrm{var}(\boldsymbol{B}) - \mathrm{cov}(\boldsymbol{B},\boldsymbol{D})[\mathrm{var}(\boldsymbol{D})]^{-1}\mathrm{cov}(\boldsymbol{D},\boldsymbol{B})$$

For the model, in section 2, given inspection data, $\boldsymbol{Y}_{\mathrm{d}}$, from an inspection design d, updated beliefs about current system level and system slope, $\mathrm{E}_{Y_{\mathrm{d}}}(X_{ct})$ and $\mathrm{E}_{Y_{\mathrm{d}}}(\alpha_{ct})$ are computed

$$\begin{aligned}\mathrm{E}_{Y_{\mathrm{d}}}(X_{ct}) &= \mathrm{E}(X_{ct}) + \mathrm{cov}(X_{ct},\boldsymbol{Y}_{\mathrm{d}})\\ &\quad\times[\mathrm{var}(\boldsymbol{Y}_{\mathrm{d}})]^{-1}[Y_{\mathrm{d}} - \mathrm{E}(\boldsymbol{Y}_{\mathrm{d}})]\\ \mathrm{E}_{Y_{\mathrm{d}}}(\alpha_{ct}) &= \mathrm{E}(\alpha_{ct}) + \mathrm{cov}(\alpha_{ct},\boldsymbol{Y}_{\mathrm{d}})\\ &\quad\times[\mathrm{var}(\boldsymbol{Y}_{\mathrm{d}})]^{-1}[\boldsymbol{Y}_{\mathrm{d}} - \mathrm{E}(\boldsymbol{Y}_{\mathrm{d}})]\end{aligned}$$

4 BAYES LINEAR VARIANCE LEARNING

4.1 Squared linear combinations of observations

The Bayes linear approach can be used to learn about variance structures. Squared linear combinations of observations, involving only observation and system evolution residual terms can be specified, which facilitate variance learning.

Assuming observations equally spaced in time, consider, for component, c, the one-step time difference, given by

$$\begin{aligned}Y_{ct}^{(1)} &= Y_{ct} - Y_{c(t-1)}\\ &= X_{ct} - X_{c(t-1)} + \sigma_Y\epsilon_{Yct} - \sigma_Y\epsilon_{Yc(t-1)}\\ &= \alpha_{ct} + \sigma_{Xc}\epsilon_{Xct} + \sigma_Y\left(\epsilon_{Yct} - \epsilon_{Yc(t-1)}\right)\\ &= \alpha_{c(t-1)} + \sigma_{\alpha c}\epsilon_{Xct} + \sigma_{Xc}\epsilon_{Xct} + \sigma_Y\left(\epsilon_{Yct} - \epsilon_{Yc(t-1)}\right)\end{aligned}$$

and two-step difference, given by

$$\begin{aligned}Y_{ct}^{(2)} &= Y_{ct} - Y_{c(t-2)}\\ &= X_{ct} - X_{c(t-2)} + \sigma_Y\epsilon_{Yct} - \sigma_Y\epsilon_{Yc(t-2)}\\ &= \alpha_{ct} + X_{c(t-1)} - X_{c(t-2)} + \sigma_{Xc}\epsilon_{Xct}\\ &\quad + \sigma_Y\left(\epsilon_{Yct} - \epsilon_{Yc(t-2)}\right)\\ &= \alpha_{ct} + \alpha_{c(t-1)} + \sigma_{Xc}\left(\epsilon_{Xct} + \epsilon_{Xc(t-1)}\right)\\ &\quad + \sigma_Y\left(\epsilon_{Yct} - \epsilon_{Yc(t-2)}\right)\\ &= 2\alpha_{c(t-1)} + \sigma_{\alpha c}\epsilon_{Xct} + \sigma_{Xc}\left(\epsilon_{Xct} + \epsilon_{Xc(t-1)}\right)\\ &\quad + \sigma_Y\left(\epsilon_{Yct} - \epsilon_{Yc(t-2)}\right)\end{aligned}$$

and so the linear combination of differences

$$\begin{aligned}Y_{ct}^{(2)} - 2Y_{ct}^{(1)} &= -\sigma_{\alpha c}\epsilon_{Xct} + \sigma_{Xc}\left(\epsilon_{Xct} - \epsilon_{Xc(t-1)}\right)\\ &\quad + \sigma_Y\left(2\epsilon_{Yc(t-1)} - \epsilon_{Yct} - \epsilon_{Yc(t-2)}\right)\end{aligned}$$

gives an expression involving only the residuals and standard deviations, the square of which is informative for variance learning. Let

$$D_{ct} = (Y_{ct}^{(2)} - 2Y_{ct}^{(1)})^2$$

The expectation of which for all, t is

$$\mathrm{E}\,(D_{ct}) = \sigma_{\alpha c}^2 + 2\sigma_{Xc}^2 + 6\sigma_Y^2$$

To apply Bayes linear adjustment to squared residuals, judgements about fourth-order moments are required. In general, specification of high-order moments is difficult. Nevertheless, over time, as the body of evidence from analyses of corroding systems accumulates, improvements to this specification could be made. For simplicity, moments are assumed to be equal to those of a standard normal distribution; then $\mathrm{E}(\epsilon_{Xct}^4) = 3$, $\mathrm{E}(\epsilon_{\alpha ct}^4) = 3$, $\mathrm{E}(\epsilon_{Yct}^4) = 3$; it can then be shown that

$$\mathrm{var}\,(D_{ct}) = \sigma_{\alpha c}^4 + 8\sigma_{\alpha c}^2\sigma_{Xc}^2 + 24\sigma_{\alpha c}^2\sigma_Y^2 + 48\sigma_{Xc}^2\sigma_Y^2 + 8\sigma_{Xc}^4 + 72\sigma_Y^4 \qquad (1)$$

and the covariance between squared linear combinations at different times is given by

$$\mathrm{cov}\left(D_{ct}, D_{c(t-1)}\right) = 32\sigma_Y^4 + 16\sigma_{Xc}^2\sigma_Y^2 + 2\sigma_{Xc}^4 \qquad (2)$$

$$\mathrm{cov}\left(D_{ct}, D_{c(t-2)}\right) = 2\sigma_Y^4 \qquad (3)$$

$$\mathrm{cov}\left(D_{ct}, D_{c(t-k)}\right) = 0, \quad \text{where } k \geqslant 3 \qquad (4)$$

For a vector of observations, $\boldsymbol{Y}_c = (Y_{c1}, Y_{c2}, \ldots, Y_{cT})^{\mathrm{T}}$ the vector of squared differences, $\boldsymbol{D}_c$, is defined as $\boldsymbol{D}_c = (D_{c1}, D_{c2}, \ldots, D_{cT-2})^{\mathrm{T}}$.

More generally, for inspections that are incomplete and irregularly spaced in time, with arbitrary fourth-order moment specification, the analysis can be carried out as above, as illustrated in appendix 2.

4.2 Exchangeable error structures

Exchangeability is a central concept in the subjective theory of probability. In essence, exchangeability assumptions in a subjective analysis, (such as the current) provide a version of the mathematical framework corresponding to independence assumptions in classical inference [**16**]. For Bayes linear analysis, where only partial beliefs need to be specified, assumptions for error structures are restricted to exchangeability of first and second-order quantities.

Means, variances, and covariances of a second-order exchangeable sequence, $X = X_1, X_2, \ldots$, are invariant under permutation. Under the assumption of second-order exchangeability, the second-order exchangeability representation theorem, [**17**], can be used to express the quantities, X_i, in terms of the sum of two uncorrelated random quantities $\mathcal{M}(X)$ and $\mathcal{R}_i(X)$ which may be viewed as underlying population mean and discrepancies from the mean respectively.

Given a collection of quantities, $X = X_1, X_2, \ldots$, an infinite second-order exchangeable sequence with

$$\mathrm{E}(X_i) = \mu, \quad \mathrm{var}(X_i) = \Sigma, \quad \mathrm{cov}(X_i, X_j) = \Gamma, \quad \text{for } i \neq j$$

each X_i can be expressed as

$$X_i = \mathcal{M}(X) + \mathcal{R}_i(X)$$

where $\mathcal{M}(X)$ is a random vector known as the population mean with

$$\mathrm{E}[\mathcal{M}(X)] = \mu, \quad \mathrm{var}[\mathcal{M}(X)] = \Gamma$$

and the collection $\mathcal{R}_i(X)$ is also second-order exchangeable with:

$$\mathrm{E}[\mathcal{R}_i(X)] = 0, \quad \mathrm{var}[\mathcal{R}_i(X)] = \Sigma - \Gamma$$

Each pair $\mathcal{R}_i$ and $\mathcal{R}_j$ are uncorrelated $i \neq j$ and each $\mathcal{R}_i$ is uncorrelated with $\mathcal{M}(X)$.

4.3 Bayes linear variance learning

To update beliefs about the variances within the system, in the case of short time series, it is necessary to share information across components within the system. To do this, beliefs need to expressed, about the relationship between variances within the model, this is achieved by assuming exchangeability of the variances. Second-order exchangeability of system level evolution variance, σ_{Xc}^2, over components is assumed. This leads to representation statements for the variance of every component, $c = 1, 2, \ldots C$

$$\sigma_{Xc}^2 = V_{Xc} = \mathcal{M}(V_X) + \mathcal{R}_c(V_X)$$

where

$$\mathrm{E}(\sigma_{Xc}^2) = \mu_{V_X}, \quad \mathrm{var}(\sigma_{Xc}^2) = \Sigma_{V_X},$$

$$\mathrm{cov}(\sigma_{Xc}^2, \sigma_{Xc'}^2) = \Gamma_{V_X}, \quad \text{for } c \neq c'$$

The adjusted expectation $E_{\boldsymbol{D}}[\mathcal{M}(V_X)]$ gives an updated estimate of the system level evolution variance, σ^2_{Xc}. For the case described in section 4.1

$$\begin{aligned} E_{\boldsymbol{D}}[\mathcal{M}(V_X)] &= E[\mathcal{M}(V_X)] + \text{cov}\{[\mathcal{M}(V_X), \boldsymbol{D}] \\ &\quad \times [\text{var}(\boldsymbol{D})]^{-1}(\boldsymbol{D} - E(\boldsymbol{D}))\} \\ &= \mu_{V_X} + 2(\Gamma_{V_X} \ldots \Gamma_{V_X})[\text{var}(\boldsymbol{D})]^{-1} \\ &\quad \times [\boldsymbol{D} - 1_C(\sigma^2_{\alpha c} + 2\sigma^2_{Xc} + 6\sigma^2_Y)] \end{aligned}$$

where

$$\boldsymbol{D} = \begin{pmatrix} \boldsymbol{D}_1 \\ \boldsymbol{D}_2 \\ \vdots \\ \boldsymbol{D}_c \end{pmatrix}$$

and var($\boldsymbol{D}$) can be found using equations (1) to (4). The adjusted variance is then

$$\begin{aligned} \text{var}_{\boldsymbol{D}}[\mathcal{M}(V_X)] &= \text{var}[\mathcal{M}(V_X)] - \text{cov}\{[\mathcal{M}(V_X), \boldsymbol{D}] \\ &\quad \times [\text{var}(\boldsymbol{D})]^{-1}\text{cov}[\boldsymbol{D}, \mathcal{M}(V_X)]\} \\ &= \Gamma_{V_X} - 4(\Gamma_{V_X}, \ldots, \Gamma_{V_X}) \\ &\quad \times [\text{var}(\boldsymbol{D})]^{-1}(\Gamma_{V_X}, \ldots, \Gamma_{V_X})' \end{aligned}$$

and the amount of variation resolved by $\boldsymbol{D}$ is given by

$$\begin{aligned} \text{Rvar}_{\boldsymbol{D}}[\mathcal{M}(V_X)] &= \text{cov}\{[\mathcal{M}(V_X), \boldsymbol{D}][\text{var}(\boldsymbol{D})]^{-1} \\ &\quad \times \text{cov}[\boldsymbol{D}, \mathcal{M}(V_X)]\} \\ &= 4(\Gamma_{V_X}, \ldots, \Gamma_{V_X}) \\ &\quad \times [\text{var}(\boldsymbol{D})]^{-1}(\Gamma_{V_X}, \ldots, \Gamma_{V_X})' \end{aligned}$$

In the general case, appendix 2 gives the corresponding expressions to use.

5 EFFICIENT INSPECTION

5.1 The decision problem

Consider the problem of designing an efficient inspection scheme. Its value is assessed in terms of reducing uncertainty about system state, thus minimizing potential losses from component failure. In this section, mean updating for a single component using adjusted variance in considered. For collections of components, summation over components allows evaluation of complete designs. In section 5.5, a design approach also incorporating variance learning is presented.

To simplify the inspection problem, suppose that there are two possible outcomes, $o \in O$, namely failure, F, or survival, $\bar{F}$ per system component. System maintenance involves replacing a component, decision R, or leaving it alone, $\bar{R}$. Replacing a component incurs cost, L_R, whereas component failure costs L_F. Furthermore, when a component fails it also needs replacing, i.e. $L_R \leqslant L_F$. This cost structure can be summarized as:

	F	$\bar{F}$
R	L_R	L_R
$\bar{R}$	L_F	0

That is, the four possible decisions are as follows.

1. Replace component when it would have failed; cost L_R.
2. Replace component when it wouldn't have failed; cost L_R.
3. Don't replace component and it fails; cost L_F.
4. Don't replace component and it doesn't fail; cost zero.

5.2 Utility and expected loss

Utility quantifies preferences concerning different uncertain rewards. Loss is negative utility. In a space of possible decisions $\Delta = (R, \bar{R})$, the best decision procedure, δ^*, has maximum utility or minimum loss. For design d, yielding inspection data, $\boldsymbol{Y}_d$, the expected loss of decision $\delta(\boldsymbol{Y}_d)$ is

$$\begin{aligned} E\{L[O, \delta(\boldsymbol{Y}_d)]\} &= E\{E\{L[O, \delta(\boldsymbol{Y}_d)]\}|\boldsymbol{Y}_d\} \\ &= E\{L[F, \delta(\boldsymbol{Y}_d)]P(F|\boldsymbol{Y}_d) \\ &\quad + L[\bar{F}, \delta(\boldsymbol{Y}_d)]P(\bar{F}|\boldsymbol{Y}_d)\} \end{aligned} \tag{5}$$

The component is replaced, decision R, if $E[L(O, R)|\boldsymbol{Y}_d] < E[L(O, \bar{R})|\boldsymbol{Y}_d]$

$$\begin{aligned} E[L(O, R)|\boldsymbol{Y}_d] &= E[L(F, R)P(F|\boldsymbol{Y}_d) + L(\bar{F}, R)P(\bar{F}|\boldsymbol{Y}_d)] \\ &= L_R \end{aligned}$$

$$\begin{aligned} E[L(O, \bar{R})|\boldsymbol{Y}_d] &= E[L(F, \bar{R})P(F|\boldsymbol{Y}_d) + L(\bar{F}, \bar{R})P(\bar{F}|\boldsymbol{Y}_d)] \\ &= L_F P(F|\boldsymbol{Y}_d) \end{aligned}$$

Hence, the component is replaced if $L_R < L_F P(F|\boldsymbol{Y}_d)$, that is

$$\delta^*(\boldsymbol{Y}_d) = \begin{cases} R, & \text{if } p(F|\boldsymbol{Y}_d) \geqslant \rho \\ \bar{R}, & \text{if } p(F|\boldsymbol{Y}_d) < \rho \end{cases} \qquad \text{where } \rho = \frac{L_R}{L_F}$$

Let $q(\boldsymbol{Y}_d) = P(F|Y\boldsymbol{Y}_d)$, the probability of failure given current system observations. From equation (5) the expected loss of the optimal decision, $\delta^*(\boldsymbol{Y}_d)$, is

$$\begin{aligned}
\mathrm{E}\{L[O,\delta^*(\boldsymbol{Y}_d)]\} &= \mathrm{E}\{L[F,\delta^*(\boldsymbol{Y}_d)]P(F|\boldsymbol{Y}_d) \\
&\quad + L[\bar{F},\delta^*(\boldsymbol{Y}_d)]P(\bar{F}|\boldsymbol{Y}_d)\} \\
&= \mathrm{E}\{L[F,\delta^*(\boldsymbol{Y}_d)]q(\boldsymbol{Y}_d) \\
&\quad + L[\bar{F},\delta^*(\boldsymbol{Y}_d)][1-q(\boldsymbol{Y}_d)]\} \\
&= \mathrm{E}\big\{L[F,\delta^*(\boldsymbol{Y}_d)]q(\boldsymbol{Y}_d) + L[\bar{F},\delta^*(\boldsymbol{Y}_d)] \\
&\quad \times [1-q(\boldsymbol{Y}_d)]\,|\,q(\boldsymbol{Y}_d) \geqslant \rho\big\} \\
&\quad \times P[q(\boldsymbol{Y}_d) \geqslant \rho] \\
&\quad + \mathrm{E}\big\{L[F,\delta^*(\boldsymbol{Y}_d)]q(\boldsymbol{Y}_d) \\
&\quad + L[\bar{F},\delta^*(\boldsymbol{Y}_d)] \\
&\quad \times [1-q(\boldsymbol{Y}_d)]\,|\,q(\boldsymbol{Y}_d) < \rho\big\} \\
&\quad \times P[q(\boldsymbol{Y}_d) < \rho] \\
&= L_{\mathrm{R}}P[q(\boldsymbol{Y}_d) \geqslant \rho] \\
&\quad + L_{\mathrm{F}}\mathrm{E}[q(\boldsymbol{Y}_d)\,|\,q(\boldsymbol{Y}_d) < \rho] \\
&\quad \times P[q(\boldsymbol{Y}_d) < \rho] \\
&= L_{\mathrm{R}}\int_{\rho}^{1} p[q(\boldsymbol{Y}_d)]\mathrm{d}q(\boldsymbol{Y}_d) \\
&\quad + L_{\mathrm{F}}\int_{0}^{\rho} q(\boldsymbol{Y}_d)p[q(\boldsymbol{Y}_d)]\mathrm{d}q(\boldsymbol{Y}_d) \\
&= L_{\mathrm{R}}I_1 + L_{\mathrm{F}}I_2 \qquad (6)
\end{aligned}$$

Therefore, calculation of the expected loss of decision δ^*, requires evaluation of integrals I_1 and I_2 as explained in section 5.3, appendix 3.

5.3 Evaluating expected loss

A component is deemed to have failed if the system level falls below some critical value W_{c}. The probability of component failure before some future time $t+k$ is

$$q(\boldsymbol{Y}_d) = P(F|\boldsymbol{Y}_d) = P(X_{t+k} < W_{\mathrm{c}}|\boldsymbol{Y}_d)$$

where X_{t+k} is the unknown future system level at time $t+k$. To evaluate integrals I_1 and I_2 from equation (6) expressions for $q(\boldsymbol{Y}_d)$ and its probability distribution, $p[q(\boldsymbol{Y}_d)]$ are required, which can be evaluated for any proposed design. This is achieved using a combination of Bayes linear analysis and appropriate distributional assumptions.

For inspection data $\boldsymbol{Y}_d$, the adjusted mean and variance are

$$\begin{aligned}
\mathrm{E}_{Y_d}(X_{t+k}) &= \mathrm{E}(X_{t+k}) + \mathrm{cov}(X_{t+k},\boldsymbol{Y}_d) \\
&\quad \times \mathrm{var}(\boldsymbol{Y}_d)^{-1}\,[\boldsymbol{Y}_d - \mathrm{E}(\boldsymbol{Y}_d)] \\
\mathrm{var}_{Y_d}(X_{t+k}) &= \mathrm{var}(X_{t+k}) - \mathrm{cov}(X_{t+k},\boldsymbol{Y}_d) \\
&\quad \times \mathrm{var}(\boldsymbol{Y}_d)^{-1}\mathrm{cov}(\boldsymbol{Y}_d,X_{t+k})
\end{aligned}$$

Note that the adjusted variance, $\mathrm{var}_{Y_d}(X_{t+k})$ depends only on prior beliefs and the specific design, d. It does not depend on the observed inspection data, $\boldsymbol{Y}_d$. However, the adjusted expectation, $\mathrm{E}_{Y_d}(X_{t+k})$, depends directly on $\boldsymbol{Y}_d$. Bayes linear analysis is therefore also used to update beliefs about its mean, $\mathrm{E}[\mathrm{E}_{Y_d}(X_{t+k})]$, and variance, $\mathrm{var}[\mathrm{E}_{Y_d}(X_{t+k})]$. For the adjusted mean

$$\begin{aligned}
\mathrm{E}[\mathrm{E}_{Y_d}(X_{t+k})] &= \mathrm{E}\{\mathrm{E}(X_{t+k}) + \mathrm{cov}(X_{t+k},\boldsymbol{Y}_d) \\
&\quad \times \mathrm{var}(\boldsymbol{Y}_d)^{-1}[\boldsymbol{Y}_d - \mathrm{E}(\boldsymbol{Y}_d)]\} \\
&= \mathrm{E}(X_{t+k}) + \mathrm{cov}(X_{t+k},\boldsymbol{Y}_d) \\
&\quad \times \mathrm{var}(\boldsymbol{Y}_d)^{-1}[\mathrm{E}(\boldsymbol{Y}_d) - \mathrm{E}(\boldsymbol{Y}_d)] \\
&= \mathrm{E}(X_{t+k})
\end{aligned}$$

To find the adjusted variance $\mathrm{var}[\mathrm{E}_{Y_d}(X_{t+k})]$, use

$$\begin{aligned}
\mathrm{var}(X_{t+k}) &= \mathrm{var}\big[\mathrm{E}_{Y_d}(X_{t+k}) + X_{t+k} - \mathrm{E}_{Y_d}(X_{t+k})\big] \\
&= \mathrm{var}[\mathrm{E}_{Y_d}(X_{t+k})] + \mathrm{var}[X_{t+k} - \mathrm{E}_{Y_d}(X_{t+k})] \\
&= \mathrm{var}[\mathrm{E}_{Y_d}(X_{t+k})] + \mathrm{var}_{Y_d}(X_{t+k})
\end{aligned}$$

so that

$$\mathrm{var}[\mathrm{E}_{Y_d}(X_{t+k})] = \mathrm{var}(X_{t+k}) - \mathrm{var}_{Y_d}(X_{t+k})$$

These expressions for the first and second moments of X_{t+k}, and it adjusted expectation $\mathrm{E}_{Y_d}(X_{t+k})$, given specific distributional assumptions for these variables, permit calculation of I_1 and I_2. The forms of I_1 and I_2, under normal distribution assumptions, are given in appendix 3.

5.4 Design selection

Total inspection cost incorporates expected loss from above, along with other costs associated with the process of carrying out inspections. For example, any inspection will involve setup costs. Inspection of some components will be more costly. Different designs might involve inspection of different numbers of components. Optimal designs should be selected with respect to total inspection cost, not only expected loss. To calculate the total loss for an inspection design, expected loss for each component is summed component-wise and added to the associated inspection cost. It is possible therefore to quantify the value of any design, d prior to carrying it out, and to search for good designs.

A method of searching efficiently for good designs from the space of designs is required. For example, even in the current simple case, with a binary decision for each component, there are 2^n potential designs to choose from. Stepwise addition of components is one tractable search strategy; components are added sequentially to an empty starting design, such that at each step, the component added minimizes the incremental total inspection cost. Alternatively, a stepwise deletion, or any of a large number of possible search algorithms may be considered. In general, the authors have found that a combination of stepwise addition and deletion works relatively well in practice.

5.5 Designing for variance learning

Good inspection designs can be found which simultaneously learn about system expectation and variance. It is clear that designs for variance learning will have different characteristics to those discussed so far. When learning about system levels, good inspection designs favour inspecting components with a high risk of failure or components with high system level uncertainty. When learning about variances, however, observing a component several times to improve a variance estimates may be beneficial.

The following approach is used in the current work to select good designs for simultaneous expectation and variance learning:

Step 1: Observe the system until time t and update beliefs about system mean and variances.

Step 2: Choose a design d, update beliefs about system to time $t+k$ given design d, using expectation and variance learning. Find the adjusted and resolved variance for given design as in section 4.3.

Step 3: Specify a distribution for system level standard deviation given design d, $p_d(\sigma_X)$. Beliefs about the mean value of, σ_X, given the design has mean, μ_σ, and variance, $\mathrm{Rvar}_d[\mathcal{M}(V_X)]$, the resolved variance given design, d. A gamma distribution is fitting with this mean and variance

$$\sigma_X \sim \Gamma\left(\frac{\mu_\sigma^2}{\mathrm{Rvar}_d[\mathcal{M}(V_X)]}, \frac{\mathrm{Rvar}_d[\mathcal{M}(V_X)]}{\mu_\sigma}\right)$$

Step 4: Evaluate the expected loss of design d

$$\mathrm{E}\{L[O,\delta^*(\boldsymbol{Y}_d)]\} = \int \mathrm{E}\{L[O,\delta^*(\boldsymbol{Y}_d),\sigma_X]\}p_d(\sigma_X)\mathrm{d}\sigma_X$$

where $\mathrm{E}\{L[O,\delta^*(\boldsymbol{Y}_d),\sigma_X]\}$ is the evaluation of $\mathrm{E}\{L[O,\delta^*(\boldsymbol{Y}_d)]\}$ from equation (6) for a given σ_X. This is approximated by a discretized version of the Γ distribution using a range of values for $\sigma_X = (\sigma_{X_1}\cdots\sigma_{Xn})$

$$\mathrm{E}\{L[O,\delta^*(\boldsymbol{Y}_d)]\} = \sum_{i=1}^{n} \mathrm{E}\{L[(O,\delta^*(\boldsymbol{Y}_d),\sigma_X]\}p(\sigma_{Xi})$$

with centre of mass located at each of choices of σ_{Xi}, where

$$\begin{aligned}
p_d(\sigma_{X1}) &= P\left(\Gamma \leqslant \frac{\sigma_{X1}+\sigma_{X2}}{2}\right)\\
p_d(\sigma_{X2}) &= P\left(\Gamma \leqslant \frac{\sigma_{X2}+\sigma_{X3}}{2}\right)\\
&\quad - P\left(\Gamma \leqslant \frac{\sigma_{X1}+\sigma_{X2}}{2}\right)\\
p_d(\sigma_{Xi}) &= P\left(\Gamma \leqslant \frac{\sigma_{Xi}+\sigma_{X(i+1)}}{2}\right)\\
&\quad - P\left(\Gamma \leqslant \frac{\sigma_{X(i-1)}+\sigma_{Xi}}{2}\right)\\
p_d(\sigma_{Xn}) &= 1 - P\left(\Gamma \leqslant \frac{\sigma_{X(n-1)}+\sigma_{Xn}}{2}\right)
\end{aligned}$$

For each choice of σ_{Xi} the expected loss of the design is calculated using the method in section 5.

Step 5: Search across the space of designs, d, to choose design which minimizes total expected loss.

6 EXAMPLE

An application of the method to analysis of inspection data from a full-scale offshore platform is now considered. For inspection and maintenance purposes, the installation is considered as a set of corrosion circuits, (C_i), each consisting of multiple components for inspection. For the current application, a system of four corrosion circuits is modelled, consisting of a total of 64 pipework weld components.

Historical data for component minimum wall thickness, obtained during inspection campaigns using non-intrusive ultrasonic measurements for the period 1998–2005, are available. Based on the frequency of observations and the requirements for inspection planning, a monthly time-increment is used for modelling; the historical period therefore consists of 83 time points.

The actual historical inspection design is given in Fig. 1. From the figure it is clear that inspections are typically incomplete and irregularly spaced in time. A total of 174 observations of the system are available, corresponding to short time-series per component. A full set of prior system beliefs is given in Table 1. These represent a genuine attempt to put meaningful values on all of the uncertainties.

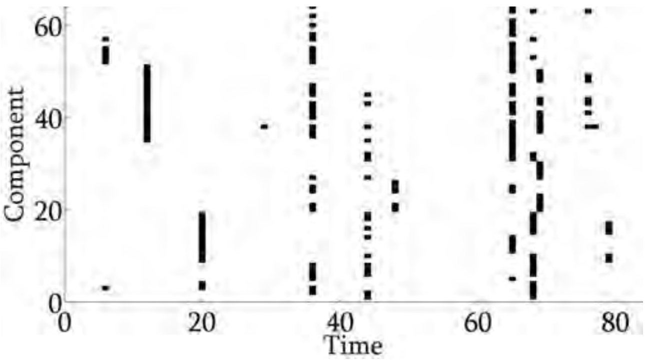

Fig. 1 Inspection design for the offshore application, consisting of 64 components over 83 time points. Black lines correspond to 174 observations of the system

Table 1 Prior values for offshore structure application

Uncertainty		Value
Number of components	N	64
Number of time points	T	83
Total number of inspections		174
Wall thickness variance	σ^2_{Xc}	0.1^2
Measurement error variance	σ^2_{Y}	0.16^2
Corrosion rate variance	σ^2_{ac}	0.1^2

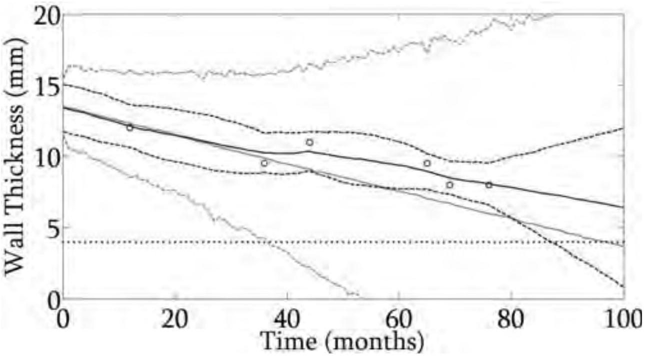

Fig. 2 Bayes linear updating for a single component of the system. Light grey lines show prior beliefs for the mean (solid) and uncertainties (dashed) of system state, dark grey lines show beliefs for the mean (solid) and uncertainties (dashed) of X_{ct} after updating our beliefs using the inspection data, $\boldsymbol{Y}_d$

Figure 2 illustrates Bayes linear updating for a single component of the system, ignoring the influence of other components. The critical wall thickness W_c, corresponding to component failure, is shown as a horizontal dotted line at 4 mm. Actual inspections of the component are shown as black circles. Light grey lines show prior beliefs for the mean (solid) and uncertainties (dashed) of system state, X_{ct}. Comparing prior beliefs with observation, it can be seen that corrosion rate is overestimated initially. Dark grey lines show

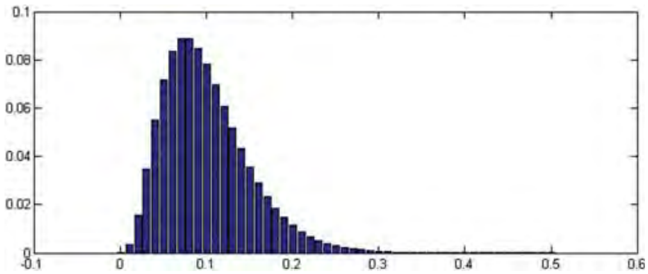

Fig. 3 Discretized gamma distribution for distribution of future variances

Table 2 Total expected loss values

Design	Total expected loss
No inspection	1.8774
Full inspection	1.8633
Every other component	1.9176

beliefs for the mean (solid) and uncertainties (dashed) of X_{ct} after updating our beliefs using the inspection data. Updating reduces the corrosion rate and our uncertainty about X_{ct}. As a result, the expected time of 'failure' for this component is further into the future than initially estimated.

In this example the cost of three different inspection designs are compared:

(a) no inspection;
(b) full inspection;
(c) inspection of half the system i.e. every other component.

In practice a large number of designs would be compared to try to find the optimal inspection scheme and this example is merely for illustration.

Figure 3 shows the discretized gamma used to generate probabilities to weight expected loss estimates.

Let

$$L_{\mathrm{R}} = 1, \quad L_{\mathrm{F}} = 100$$

so the cost of component failure is 100 times the cost of replacement, representing vital system components. The cost of setting up an inspection is 0.01 per component inspected, so for design 2 (full inspection) to be worthwhile the increased information about the system has to outweigh the increased cost.

In this case the total expected loss for each considered design is as in Table 2.

Thus, in this case a full system inspection is the best since the risk component failure is more costly.

Consider another case

$$L_{\mathrm{R}} = 1, \quad L_{\mathrm{F}} = 5$$

so the cost of component failure is only five times the cost of replacement, representing less important

Table 3 Total expected loss values for the second case

Design	Total expected loss
No inspection	1.0415
Full inspection	7.2471
Every other component	4.1348

system components. The cost of setting up an inspection is also more expensive at 0.1 per component inspected.

In this case the total expected loss for each considered design is as in Table 3.

Here, due to the less costly nature of the components, a design without any inspection, is best.

7 CONCLUSIONS

A method has been presented for learning about the mean and variances within a linear growth DLM using historical data. Using updated system estimates, a method was proposed to give a quick way of estimating the value of inspection designs, designing for both mean and variance learning using discretized gamma probabilities.

It is noted that variance learning can be achieved using different squared linear combinations of observations. In this work a single linear combination is used. In future it would be interesting to consider variance learning using multiple linear combinations from the same data. To do so reliably would require evaluation of variances and covariances between those linear combinations.

To estimate expected loss (section 5.3), integrals I_1 and I_2 are calculated (equation (6) and appendix 3) by making normality assumptions. In future, a direct Bayes linear updating the probability and expectation corresponding to I_1 and I_2 (equation (6)) will be considered. The utility function used here is also particularly simple, and would need to be generalized in practice to more adequately represent the complexity of decisions in reality. For example, the assumption that the only possible remedial action is component replacement is rather simplistic. In reality, components can be repaired to various extents. Furthermore, individual components are not usually replaced; rather, continuous sections (or spools) consisting of multiple components would be replaced. In addition, the costs associated with decisions are themselves subject to uncertainty. At present, expected loss in calculated per component, rather than across components. Any enhancement to the utility function used would need to retain the ability to estimate loss without recourse to full simulation, so that efficient search of the design space is possible.

ACKNOWLEDGEMENT

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APPENDIX 1

Notation

c	component in the system
C	total number of components in the system
d	observational inspection design
$\boldsymbol{D}_c$	data vector for component c
D_{ct}	data for component c at time t
$\mathrm{E}_{\boldsymbol{D}}[\boldsymbol{B}]$	adjusted expectation of beliefs $\boldsymbol{B}$ given data $\boldsymbol{D}$
F	failure of component
$\bar{F}$	survival of component
I_1	integral in expected loss calculation
I_2	integral in expected loss calculation
L	loss function (negative utility)
L_{F}	loss incurred through component failure
L_{R}	loss incurred through component replacement
$\mathcal{M}$	'population mean' vector in representation theorem
o	outcome
O	outcome space
q	probability of failure
R	decision to replace a component
$\bar{R}$	decision not to replace a component
$\mathcal{R}$	'population residual' vector in representation theorem
$\mathrm{Rvar}_{\boldsymbol{D}}[\boldsymbol{B}]$	variance resolved by updating of $\boldsymbol{B}$ given data $\boldsymbol{D}$
t	time point
T	total time points
X_{ct}	system level for component c at time t
V_X	exchangeability of across variances
$\mathrm{var}_{\boldsymbol{D}}[\boldsymbol{B}]$	adjusted variance of beliefs $\boldsymbol{B}$ given data $\boldsymbol{D}$
W_{c}	critical system state
$\boldsymbol{Y}_c$	vector of observations for component c
Y_{ct}	observation of system state for component c at time t
$\boldsymbol{Y}_d$	observed inspection data given a design d
$Y^{(i)}$	i-step difference of observations $Y_{ct} - Y_{c(t-i)}$
α_{ct}	system slope for component c at time t
$\gamma_{Xcc'}$	system level covariance between component c and c'
$\gamma_{Ycc'}$	observation covariance between component c and c'
$\gamma_{\alpha cc'}$	system slope covariance between component c and c'
Γ_{V_X}	$\mathrm{cov}(\sigma^2_{Xc}, \sigma^2_{Xc'})$ in second-order exchangeability representation
δ	decision
δ^*	optimal decision
Δ	space of decisions
ϵ_{Xct}	system level evolution residual for component c and time t
ϵ_{Yct}	measurement error residual for component c and time t
$\epsilon_{\alpha ct}$	system slope evolution residual for component c and time t
μ_{V_X}	$E(\sigma^2_{Xc})$ in second-order exchangeability representation
ρ	loss ratio $L_{\mathrm{R}}/L_{\mathrm{F}}$
σ_{Xc}	system level standard deviation for component c
σ_Y	measurement error standard deviation
$\sigma_{\alpha c}$	system slope standard deviation for component c
Σ_{V_X}	$\mathrm{var}(\sigma^2_{Xc})$ in second-order exchangeability representation

APPENDIX 2

Generalized variance updating

Similar types of linear combinations as discussion in section 4 can be found for the case of irregular and partial inspections. The system state equations of the DLM can be rewritten to give information about time steps longer than one step, thus

$$\begin{aligned} Y_{ct} &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ \alpha \end{pmatrix}_{ct} + \sigma_Y \epsilon_{Yct} \begin{pmatrix} X \\ \alpha \end{pmatrix}_{ct} \\ &= \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ \alpha \end{pmatrix}_{c(t-k)} \\ &\quad + \sum_{i=0}^{k-1} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_X \epsilon_X + \sigma_\alpha \epsilon_\alpha \\ \sigma_\alpha \epsilon_\alpha \end{pmatrix}_{ct-i} \end{aligned}$$

We have observations for component c

$$(Y_{ct_1}, Y_{ct_2}, \ldots, Y_{ct_{T_c}})$$

at times

$$(t_1, t_2, \ldots, t_{T_c})$$

then taking differences

$$
\begin{aligned}
Y_c^{(k_i)} &= Y_{ct_i} - Y_{c(t_i-k_i)} \\
&= \begin{pmatrix}1 & 0\end{pmatrix} X_{ct_i} - \begin{pmatrix}1 & 0\end{pmatrix} X_{c(t_i-k_i)} \\
&\quad + \sigma_Y \left(\epsilon_{Yct_i} - \epsilon_{Yc(t_i-k_i)}\right) \\
&= \begin{pmatrix}1 & k_i\end{pmatrix} X_{c(t_i-k_i)} - \begin{pmatrix}1 & 0\end{pmatrix} X_{c(t_i-k_i)} \\
&\quad + \begin{pmatrix}1 & 0\end{pmatrix} \xi_{c(t_i,t_i-k_i)} + \sigma_Y \left(\epsilon_{Yct_i} - \epsilon_{Yc(t_i-k_i)}\right) \\
&= \begin{pmatrix}0 & k_i\end{pmatrix} X_{c(t_i-k_i)} + \begin{pmatrix}1 & 0\end{pmatrix} \xi_{c(t_i,t_i-k_i)} \\
&\quad + \sigma_Y \left(\epsilon_{Yct_i} - \epsilon_{Yc(t_i-k_i)}\right) \\
&= \begin{pmatrix}0 & k_i\end{pmatrix} X_{c(t_i-l_i)} + \begin{pmatrix}0 & k_i\end{pmatrix} \xi_{c(t_i-k_i,t_i-l_i)} \\
&\quad + \begin{pmatrix}1 & 0\end{pmatrix} \xi_{c(t_i,t_i-k_i)} + \sigma_Y \left(\epsilon_{Yct_i} - \epsilon_{Yc(t_i-k_i)}\right)
\end{aligned}
$$

where

$$
\xi_{c(t_i,t_i-k_i)} = \sum_{i=0}^{k_i-1} \begin{pmatrix}1 & i \\ 0 & 1\end{pmatrix} \begin{pmatrix}\sigma_X\epsilon_X + \sigma_\alpha\epsilon_\alpha \\ \sigma_\alpha\epsilon_\alpha\end{pmatrix}_{ct_i-i}
$$

and

$$
\begin{aligned}
Y_c^{(l_i)} &= Y_{ct_i} - Y_{c(t_i-l_i)} \\
&= \begin{pmatrix}1 & 0\end{pmatrix} X_{ct_i} - \begin{pmatrix}1 & 0\end{pmatrix} X_{c(t_i-l_i)} \\
&\quad + \sigma_Y \left(\epsilon_{Yct_i} - \epsilon_{Yc(t_i-l_i)}\right) \\
&= \begin{pmatrix}1 & k_i\end{pmatrix} X_{c(t_i-k_i)} - \begin{pmatrix}1 & 0\end{pmatrix} X_{c(t_i-l_i)} \\
&\quad \times \begin{pmatrix}1 & 0\end{pmatrix} \xi_{c(t_i,t_i-k_i)} + \sigma_Y \left(\epsilon_{Yct_i} - \epsilon_{Yc(t_i-l_i)}\right) \\
&= \begin{pmatrix}0 & l_i\end{pmatrix} X_{c(t_i-l_i)} + \begin{pmatrix}1 & k_i\end{pmatrix} \xi_{c(t_i-k_i,t_i-l_i)} \\
&\quad + \begin{pmatrix}1 & 0\end{pmatrix} \xi_{c(t_i,t_i-k_i)} + \sigma_Y \left(\epsilon_{Yct_i} - \epsilon_{Yc(t_i-l_i)}\right)
\end{aligned}
$$

The following linear combination is considered to eliminate the effects of the wall thickness term

$$
\begin{aligned}
&k_i Y_c^{(l_i)} - l_i Y_c^{(k_i)} \\
&= k_i\Big[\begin{pmatrix}0 & l_i\end{pmatrix} X_{c(t_i-l_i)} + \begin{pmatrix}1 & k_i\end{pmatrix} \xi_{c(t_i-k_i,t_i-l_i)} \\
&\quad + \begin{pmatrix}1 & 0\end{pmatrix} \xi_{c(t_i,t_i-k_i)} + \sigma_Y \left(\epsilon_{Yct_i} - \epsilon_{Yc(t_i-l_i)}\right)\Big] \\
&\quad - l_i\Big[\begin{pmatrix}0 & k_i\end{pmatrix} X_{c(t_i-l_i)} + \begin{pmatrix}0 & k_i\end{pmatrix} \xi_{c(t_i-k_i,t_i-l_i)} \\
&\quad + \begin{pmatrix}1 & 0\end{pmatrix} \xi_{c(t_i,t_i-k_i)} + \sigma_Y \left(\epsilon_{Yct_i} - \epsilon_{Yc(t_i-l_i)}\right)\Big] \\
&= \begin{pmatrix}(k_i - l_i) & 0\end{pmatrix} \xi_{c(t_i,t_i-k_i)} \\
&\quad + \begin{pmatrix}k_i & k_i(k_i - l_i)\end{pmatrix} \xi_{c(t_i-k_i,t_i-l)} \\
&\quad + k_i\sigma_Y\left(\epsilon_{Yct_i} - \epsilon_{Yc(t_i-l_i)}\right) - l_i\sigma_Y\left(\epsilon_{Yct_i} - \epsilon_{Yc(t_i-k_i)}\right)
\end{aligned}
$$

It can be shown that

$$
\begin{aligned}
&\operatorname{var}(\xi_{c(t_i,t_i-k_i)}) \\
&= \sum_{i=0}^{k_i-1} \operatorname{var}\left[\begin{pmatrix}1 & i \\ 0 & 1\end{pmatrix} \begin{pmatrix}\sigma_X\epsilon_X + \sigma_\alpha\epsilon_\alpha \\ \sigma_\alpha\epsilon_\alpha\end{pmatrix}_{ct_i-i}\right] \\
&= \sum_{i=0}^{k_i-1} \begin{pmatrix}1 & i \\ 0 & 1\end{pmatrix} \begin{pmatrix}\sigma_X^2 + \sigma_\alpha^2 & \sigma_\alpha^2 \\ \sigma_\alpha^2 & \sigma_\alpha^2\end{pmatrix} \begin{pmatrix}1 & 0 \\ i & 1\end{pmatrix} \\
&= \begin{pmatrix} k_i\sigma_X^2 + \frac{1}{6}k_i(k_i+1)(2k_i+1)\sigma_\alpha^2 & \frac{1}{2}k_i(k_i+1)\sigma_\alpha^2 \\ \frac{1}{2}k_i(k_i+1)\sigma_\alpha^2 & k_i\sigma_\alpha^2 \end{pmatrix}
\end{aligned}
$$

Then

$$
\begin{aligned}
&\mathrm{E}[(k_i Y_c^{(l_i)} - l_i Y_c^{(k_i)})^2] \\
&= (k_i - 1)^2\left[k_i\sigma_X^2 + \frac{\sigma_\alpha^2 k_i(k_i+1)(2k_i+1)}{6}\right] \\
&\quad + k_i^2\Big[(l_i - k_i)\sigma_X^2 \\
&\quad + \frac{\sigma_\alpha^2(l_i - k_i)(l_i - k_i + 1)(2(l_i - k_i) + 1)}{6}\Big] \\
&\quad + 2k_i^2\left[(k_i - l_i)\sigma_X^2 + \frac{\sigma_\alpha^2(l_i - k_i)(l_i - k_i + 1)}{2}\right] \\
&\quad + k_i^2(k_i - l_i)^2(l_i - k_i)\sigma_\alpha^2 \\
&\quad + \left(2l_i^2 - 2k_il_i + 2k_i^2\right)\sigma_Y^2 \\
&= \frac{k_il_i(k_i - l_i)(2k_i^2 - 2l_i - 1)}{6}\sigma_\alpha^2 + k_il_i(l_i - k_i)\sigma_X^2 \\
&\quad + 2\left(l_i^2 - k_il_i + k_i^2\right)\sigma_Y^2 \qquad (7)
\end{aligned}
$$

To update ones beliefs about the variance it is necessary to compute $\mathrm{E}_{\boldsymbol{D}}[\mathcal{M}(V_X)]$ which is similar to the full inspections case, where

$$
D_{ct} = (k_i Y_c^{(l_i)} - l_i Y_c^{(k_i)})^2
$$

and

$$
\boldsymbol{D} = \begin{pmatrix} D_{11} \\ \vdots \\ D_{ct} \end{pmatrix}
$$

$$
\begin{aligned}
\mathrm{E}_{\boldsymbol{D}}[\mathcal{M}(V_X)] &= \mathrm{E}[\mathcal{M}(V_X)] + \operatorname{cov}[\mathcal{M}(V_X), \boldsymbol{D}] \\
&\quad \times [\operatorname{var}(\boldsymbol{D})]^{-1}[\boldsymbol{D} - \mathrm{E}(\boldsymbol{D})]
\end{aligned}
$$

where from equation (7)

$$\mathrm{E}(D_{ct}) = \frac{k_i l_i (k_i - l_i)(2k_i^2 - 2l_i - 1)}{6}\sigma_\alpha^2 + k_i l_i (l_i - k_i)\sigma_X^2 + 2\left(l_i^2 - k_i l_i + k_i^2\right)\sigma_Y^2$$

also

$$\mathrm{E}[\mathcal{M}(V_X)] = \sigma_X^2$$

and

$$\begin{aligned}
&\mathrm{cov}[\mathcal{M}(V_X), \boldsymbol{D}] \\
&\quad = \{\mathrm{cov}[\mathcal{M}(V_X), D_{11}], \mathrm{cov}[\mathcal{M}(V_X), D_{12}], \ldots, \\
&\qquad \mathrm{cov}[\mathcal{M}(V_X), D_{ct}]\} \\
&\mathrm{cov}[\mathcal{M}(V_X), \boldsymbol{D}_{ct}] \\
&\quad = \mathrm{cov}\big[\mathcal{M}(V_X), (k_i Y_c^{(l_i)} - l_i Y_c^{(k_i)})^2\big] \\
&\quad = \mathrm{cov}\big[\mathcal{M}(V_X), (((k_i - l_i) \quad 0)\xi_{c(t_i, t_i - k_i)} \\
&\qquad + (k_i \quad k_i(k_i - l_i))\xi_{c(t_i - k_i, t_i - l_i)} \\
&\qquad + k_i \sigma_Y (\epsilon_{Yct_i} - \epsilon_{Yc(t_i - l_i)}) \\
&\qquad - l_i \sigma_Y (\epsilon_{Yct_i} - \epsilon_{Yc(t_i - k_i)}))^2\big] \\
&\quad = k_i l_i (l_i - k_i)\Gamma_{V_X}
\end{aligned}$$

and

$$\mathrm{var}_{\boldsymbol{D}}[\mathcal{M}(V_X)] = \mathrm{var}[\mathcal{M}(V_X)] + \mathrm{cov}[\mathcal{M}(V_X), \boldsymbol{D}] \times [\mathrm{var}(\boldsymbol{D})]^{-1}\mathrm{cov}[\boldsymbol{D}, \mathcal{M}(V_X)]$$

where $\mathrm{var}(\boldsymbol{D})$ is evaluated as in section 4.1, but with irregular time steps and fourth-order moment specification and

$$\mathrm{var}[\mathcal{M}(V_X)] = \Sigma_{V_X}$$

APPENDIX 3

Evaluating Expected Loss under Normality

Expressions for the first and second moments of X_{t+k}, and its adjusted expectation $\mathrm{E}_{Y_d}(X_{t+k})$ are given in section 5.3. Henceforth, these quantities are assumed to be normally distributed

$$X_{t+k}(Y_d) \sim N[\mu_{t+k}(Y_d), \sigma_{t+k}^2]$$

$$\mu_{t+k}(Y_d) \sim N[\mathrm{E}(X_{t+k}), \mathrm{var}(X_{t+k}) - \sigma_{t+k}^2]$$

where

$$\mu_{t+k}(Y_d) = \mathrm{E}_{Y_d}(X_{t+k}), \quad \sigma_{t+k}^2 = \mathrm{var}_{Y_d}(X_{t+k})$$

From equation (6) the expected loss for a given design, d, is given by

$$\begin{aligned}
\mathrm{E}\{L[O, \delta^*(Y_d)]\} &= L_{\mathrm{R}} \int_\rho^1 p(q(Y_d))\mathrm{d}q(Y_d) \\
&\quad + L_{\mathrm{F}} \int_0^\rho q(Y_d)p[q(Y_d)]\mathrm{d}q(Y_d) \\
&= L_{\mathrm{R}} I_1 + L_{\mathrm{F}} I_2
\end{aligned}$$

3.1 Evaluating I_1

The probability of component failure is given by

$$q(Y_d) = P(F|Y_d) = P(X_{t+k} < W_{\mathrm{c}}|Y_d)$$

Therefore, using the normality and standardizing

$$\begin{aligned}
&P(X_{t+k} < W_{\mathrm{c}}|Y_d) \\
&\quad = P\left[\frac{X_{t+k} - (\mu_{t+k}|Y_d)}{\sigma_{t+k}|Y_d} < \frac{W_{\mathrm{c}} - (\mu_{t+k}|Y_d)}{\sigma_{t+k}|Y_d}\middle| Y_d\right]
\end{aligned}$$

$$q(Y_d) = \Phi\left[\frac{W_{\mathrm{c}} - (\mu_{t+k}|Y_d)}{\sigma_{t+k}|Y_d}\right] \tag{8}$$

Let

$$z = \frac{W_{\mathrm{c}} - (\mu_{t+k}|Y_d)}{\sigma_{t+k}|Y_d} \tag{9}$$

Then

$$\begin{aligned}
\mathrm{E}(z) &= \mathrm{E}\left[\frac{W_{\mathrm{c}} - (\mu_{t+k}|Y_d)}{\sigma_{t+k}}\right] \\
&= \frac{W_{\mathrm{c}} - \mathrm{E}(\mu_{t+k}|Y_d)}{\sigma_{t+k}} \\
&= \frac{W_{\mathrm{c}} - \mathrm{E}(X_{t+k})}{\sigma_{t+k}} \\
&= \mu_z
\end{aligned}$$

$$\begin{aligned}
\mathrm{var}(z) &= \mathrm{var}\left[\frac{W_{\mathrm{c}} - (\mu_{t+k}|Y_d)}{\sigma_{t+k}}\right] \\
&= \frac{\mathrm{var}(\mu_{t+k}|Y_d)}{\sigma_{t+k}^2} \\
&= \frac{\mathrm{var}(X_{t+k}) - \mathrm{var}_{Y_d}(X_{t+k})}{\sigma_{t+k}^2} \\
&= \sigma_{t+k}^2
\end{aligned}$$

Thus, to calculate I_1

$$I_1 = \int_\rho^1 p[q(Y_d)]\mathrm{d}q(Y_d) = P[q(Y_d) \geqslant \rho]$$

Then, from equations (8) and (9)

$$P\left[q(Y_d) \geqslant \rho\right] = P\left\{\Phi\left[\frac{W_c - (\mu_{t+k}|Y_d)}{\sigma_{t+k}|Y_d}\right] \geqslant \rho\right\}$$

$$= P[\Phi(z) \geqslant \rho]$$

$$= P[z \geqslant \Phi^{-1}(\rho)]$$

$$= P\left[\frac{z-\mu_z}{\sigma_z} \geqslant \frac{\Phi^{-1}(\rho)-\mu_z}{\sigma_z}\right]$$

$$= 1 - \Phi\left[\frac{\Phi^{-1}(\rho)-\mu_z}{\sigma_z}\right]$$

$$= \Phi\left[\frac{\mu_z - \Phi^{-1}(\rho)}{\sigma_z}\right]$$

3.2 Evaluating I_2

Continuing from equations (8) and (9), the expression for I_2 becomes

$$I_2 = \int_0^{\rho} q(Y_d)p\left[q(Y_d)\right]\mathrm{d}q(Y_d)$$

$$= \int_{-\infty}^{\Phi^{-1}(\rho)} \Phi(z)f_q[\Phi(z)]\phi(z)\mathrm{d}z$$

where $f_q[\Phi(z)]$ is given by the derivative of $F_q = P[q(Y_d) < x]$ and $\phi(z)$ is the standard normal density

$$P[q(Y_d) < x] = P\left\{\Phi\left[\frac{W_c - (\mu_{t+k}|Y_d)}{\sigma_{t+k}|Y_d}\right] < x\right\}$$

$$= P[\Phi(z) < x]$$

$$= P[z < \Phi^{-1}(x)]$$

$$= P\left[\frac{z-\mu_z}{\sigma_z} < \frac{\Phi^{-1}(x)-\mu_z}{\sigma_z}\right]$$

$$= \Phi\left[\frac{\Phi^{-1}(x)-\mu_z}{\sigma_z}\right]$$

Therefore

$$f_q = \frac{\mathrm{d}F_q}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left\{\Phi\left[\frac{\Phi^{-1}(x)-\mu_z}{\sigma_z}\right]\right\}$$

$$= \frac{1}{\sigma_z}\phi\left[\frac{\Phi^{-1}(x)-\mu_z}{\sigma_z}\right] \times \frac{1}{\phi(\Phi^{-1}(x))}$$

Then

$$I_2 = \int_{-\infty}^{\Phi^{-1}(\rho)} \Phi(z)f_q[\Phi(z)]\phi(z)\mathrm{d}z$$

$$= \int_{-\infty}^{\Phi^{-1}(\rho)} \Phi(z)\frac{1}{\sigma_z}\phi\left\{\frac{\Phi^{-1}[\Phi(z)]-\mu_z}{\sigma_z}\right\}$$

$$\times \frac{1}{\phi\{\Phi^{-1}[\Phi(z)]\}}\phi(z)\mathrm{d}z$$

$$= \int_{-\infty}^{\Phi^{-1}(\rho)} \Phi(z)\frac{1}{\sigma_z}\phi\left(\frac{z-\mu_z}{\sigma_z}\right) \times \frac{1}{\phi(z)}\phi(z)\mathrm{d}z$$

$$= \int_{-\infty}^{\Phi^{-1}(\rho)} \Phi(z)\phi\left(\frac{z-\mu_z}{\sigma_z}\right)\frac{\mathrm{d}z}{\sigma_z}$$