Quaternions Lecture Notes

Applied Medical Image Processing

Quaternions

Rotation in 2D plane using complex numbers

The root of quaternions goes back to the concept of imaginary numbers.

assume a complex number as defined below:

$$z \in \mathrm{C} \text{ and } Z = a + ib$$

$$a, b \in \mathrm{R}, i^2 = -1$$

For a complex number, simple mathematical operations are performed element wise:

1) $(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$
2) $\lambda(a_1 + ib_1) = \lambda a_1 + i\lambda$
3) $(a_1 + ib_1)(a_2 + ib_2) = a_1a_2 + ia_1b_2 + ia_2b_1 + (i^2)b_1b_2 = a_1a_2 - b_1b_2 + i(a_1b_2 + a_2b_1)$

Interestingly, complex numbers can be used to represent rotations in 2D space. Let's use an example:

$$\begin{aligned}
&i^0 = 1 \\
&i^1 = i \\
&i^2 = i \cdot i = -1 \\
&i^3 = i \cdot i \cdot i = -i \\
&i^{-4} = i \cdot i \cdot i \cdot i = 1
\end{aligned}$$

Note that starting with $1 + 0i$, multiplying by i, we get a new vector that is rotated by $90\mathrm{deg}$. Another example is to define a vector in the 2D imaginary plane as follows:

$$\vec{p} = 2 + i$$

$$\begin{aligned}
pi &= 2i - 1 \\
pi^2 &= -2 - i \\
pi &= 1 - 2i \\
pi^3 &= 2 + i
\end{aligned}$$

if we call $\vec{q} = \vec{p}i$, then these two vectors should be perpendicular to each other based on previous two examples. To test that we could compute the inner product of these two vectors to check if it will be zero. Note that we are working with complex numbers hence dot product is implemented using complex conjugate:

$$p \cdot q^* = \langle 2, i \rangle \cdot \langle -1, -i2 \rangle$$
$$-2 + 2 = 0$$

In general, we can rotate a point in complex 2D plane by angle theta by multiplying it by $\cos\theta + i\sin\theta = e^{i\theta}$. So to rotate $\vec{p} = a + ib$ by angle θ counterclockwise, we could use the following steps:

$$p = a + ib$$
$$q = \cos\theta + i\sin\theta$$
$$pq = (a + ib)(\cos\theta + i\sin\theta)$$
$$= (a\cos\theta - b\sin\theta) + i(b\cos\theta + a\sin\theta)$$

Alternatively, we could write a rotation matrix and perform the following matrix multiplication:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \end{bmatrix}$$

Rotation in 3D plane using quaternions

The concept of quaternions was originally introduced by Hamilton, a mathematician, as an effort to extend the concept of 2D imaginary plane rotation to higher dimensions. Their general form is:

$$q = q_0 + q_1 i + q_2 j + q_3 k$$

where

$$q_0, q_1, q_2, q_3 \in \mathbb{R}$$

$$i^2 = j^2 = k^2 = ijk = -1$$

Using these relationships, we can drive the following additional relationships:

$$ij = k, jk = i, ki = j$$

$$ji = -k, kj = -i, ik = -j$$

Example derivation:

$$\begin{cases} ijk = -1 \\ kk = -1 \end{cases} \Rightarrow ij = k$$

Another example: What is ji ?

$$jj = -1 \overset{*i}{\Rightarrow} jji = -i \Rightarrow$$
$$ijji = \underbrace{ij}_{k}ji = kji = -ii = 1$$
$$\overset{*k}{\Rightarrow} \underbrace{kk}_{-1}ji = k \Rightarrow -ji = k \Rightarrow$$
$$ij = -k$$

Quaternions can also be defined as:

$$q = [q_0, \vec{v}] \quad q_0 \in R \quad \vec{v} \in R^3$$

$$= \left[q_0, q_1\vec{\imath} + q_2\vec{\jmath} + q_3\vec{k}\right]$$

common operations between two quartenions are defined as follows:

$$q_a = [q_a, \vec{a}]$$

$$q_b = \left[q_1 b, \vec{b}\right]$$

$$q_a + q_b = \left[q_{0a} + q_{0b}, \vec{a} + \vec{b}\right]$$

$$q_a - q_b = \left[q_{0a} - q_{0b}, \vec{a} - \vec{b}\right]$$

Accordingly, what is $\mathbb{pq}$?

$$\mathbb{pq} = \left(p_0 + p_1\vec{\imath} + p_2\vec{\jmath} + p_3\vec{k}\right)\left(q_0 + q_1\vec{\imath} + q_2\vec{\jmath} + q_3\vec{k}\right)$$

$$= \ \& p\_\{0\} q\_\{0\} - \vdash(p\_\{1\} q\_\{1\} + p\_\{2\} q\_\{2\} + p\_\{3\} q\_\{3\}\dashv) +$$

$$p_0\left(q_1\vec{\imath} + q_2\vec{\jmath} + q_3\vec{k}\right) +$$

$$q_0\left(p_1\vec{\imath} + p_2\vec{\jmath} + p_3\vec{k}\right) +$$

$$(p_2q_3 - p_3q_2)\vec{\imath} + (p_3q_1 - p_1q_3)\vec{\jmath} +$$

$$(p_1q_2 - p_2q_1)\vec{k}$$

It can be simplified if we use the following steps:

$$\vec{p} = (p_1, p_2, p_3)$$

$$\vec{q} = (q_1, q_2, q_3)$$

$$p_1q_1 + p_2q_2 + p_3q_3 = \vec{p} \cdot \vec{q}$$

$$p_0\left(q_1\vec{\imath} + q_2\vec{\jmath} + q_3\vec{k}\right) = p_0\vec{q}$$

$$\vec{p} \times \vec{q} = (p_2 q_3 - p_3 q_2)\vec{\imath} + (p_3 q_1 - p_1 q_3)\vec{\jmath} +$$

$$(p_1 q_2 - p_2 q_1)\vec{k}$$

Using these simplifications, we could write:

$$\mathrm{pq} = [p_0 q_0 - \vec{p} \cdot \vec{q}] + p_0 \vec{q} + q_0 \vec{p} + \vec{p} \times \vec{q}$$

Additional properties: if $q = [q_0, \vec{q}]$ then

1. $\lambda q_k = [\lambda q_0, \lambda \vec{q}] \quad \lambda \in R$
2. other definitions:

$$q = [0, \vec{q}] = q_1 \vec{\imath} + q_2 \vec{\jmath} + q_3 \vec{k}$$

$$q = [q_0, \vec{q}] = \underbrace{[q_0, \vec{0}]}_{real} + \underbrace{[0, \vec{q}]}_{pure}$$

3. conjugate: $q^* = [q_0, -\vec{q}]$
4. $qq^* = q_0^2 + \underbrace{\vec{q} \cdot \vec{q}}_{|\vec{q}|^2} + \underbrace{q_0 \cdot \vec{q} - q_0 \cdot \vec{q}}_{0} + \underbrace{\vec{q} \times \overline{-\vec{q}}}_{0} = q_0^2 + |\vec{q}|^2 = |q|^2$
5. $\mathbb{q}_1^{-1} = \frac{\mathbb{q}^*}{\|\mathbb{q}\|^2} \quad \mathbb{q}\mathbb{q}^{-1} = \frac{\mathbb{q}\mathbb{q}^*}{\|\mathbb{q}\|^2} = \frac{\|\mathbb{q}\|^2}{\|\mathbb{q}\|^2} = 1$

To explore 3D rotation using quaternions let's use $q = [\cos\theta, \sin\theta\vec{q}] = [q_0, \lambda\vec{q}] s.t. \quad |q| = 1.$ let's also define a vector in 3D as:

$$P = [0, \vec{p}] = [0, p, \vec{\imath} + p_2 \vec{\jmath} + p_3 \vec{\jmath}]$$

Accordingly using the example from 2D complex numbers we define P_{rot} as:

$$\mathrm{qP} = -\lambda \vec{\mathrm{q}} \cdot \vec{\mathrm{p}} + \mathrm{q}_0 \vec{\mathrm{p}} + \lambda \vec{\mathrm{q}} \times \vec{\mathrm{p}}$$

$$= [-\lambda \vec{q} \cdot \vec{p}, q_0 \vec{p} + \lambda \vec{q} \times \vec{p}]$$

To get a better insight let's use the following example:

$$\left\{\begin{array}{ll} \vec{q} = \frac{\sqrt{2}}{2}\vec{\imath} + \frac{\sqrt{2}}{2}\vec{k} & q = [co\,s\,\theta, \sin\theta\vec{\mathrm{q}} \\ \vec{p} = 2\vec{\imath} & p = [0, 2\vec{\imath}] \end{array}\right]$$

$$P_{\mathrm{rt}} = qP = [-\sin\theta\vec{q} \cdot \vec{p}, \cos\theta\vec{p} + \sin\theta \cdot \vec{q} \times \vec{p}]$$

$$\text{For } \theta = 45^\circ q = \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\vec{\imath} + \frac{\sqrt{2}}{2}\vec{k}\right)\right]$$

$$P_{rot} = qP = \left(\frac{\sqrt{2}}{2} + \frac{1}{2}\vec{\imath} + 1/2\vec{k}\right)(2\vec{\imath})$$

$$P_{r_ot} = \left[-\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\vec{\imath} + \frac{\sqrt{2}}{2}\vec{k}\right) \cdot 2\vec{\imath}, \frac{\sqrt{2}}{2}2\vec{\imath} + \frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\vec{\imath} + \frac{\sqrt{2}}{2}\vec{k}\right) \times 2\vec{\imath}\right]$$

$$\left(-1/2\vec{\imath} - 1/2\vec{k}\right) \cdot 2\vec{\imath} = -1$$

$$\frac{\sqrt{2}}{2}2\vec{\imath} = \sqrt{2}\vec{\imath}$$

$$\left(\frac{1}{2}\vec{\imath} + 1/2\vec{k}\right) \times (2\vec{\imath}) = \vec{\jmath}$$

$\text{Prot} = \left[-1, \sqrt{2}\vec{\imath} + \vec{\jmath}\right]$This is not a pure vector.

We started with pure vector and needed to end with pure as well. Now let's try:

$$qpq^{-\mathbb{1}} \quad q = \left[\cos\theta, \sin\theta\left(\frac{\sqrt{2}}{2}\vec{\imath} + \frac{\sqrt{2}}{2}\vec{k}\right)\right]$$

since $|q| = 1 \Rightarrow q^{-\mathbb{1}} = q^*$

$$q^{-\mathbb{1}} = \left[\cos\theta, -\sin\theta\left(\frac{\sqrt{2}}{2}\vec{\imath} + \frac{\sqrt{2}}{2}\vec{k}\right)\right]$$

For $\theta = 45^\circ \quad q^{-\mathbb{1}} = \left[\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\vec{\imath} + \frac{\sqrt{2}}{2}\vec{k}\right)\right]$

$$= 1/2\left[\sqrt{2}, -\left(\vec{\imath} + \vec{k}\right)\right]$$

$$qPq^{-1} = \left[-1, \sqrt{2}\vec{\imath} + \vec{\jmath}\right]1/2\left[\sqrt{2}, -\left(\vec{\imath} + \vec{k}\right)\right]$$

$$= \frac{1}{2}[-\sqrt{2} - (\sqrt{2}\,i + j) \cdot (-\vec{\imath} - \vec{k}) + (\vec{\imath} + \vec{k}) +$$

$$\sqrt{2}(\sqrt{2}\,\vec{\imath} + \vec{\jmath}) + (\sqrt{2}\vec{\imath} + \vec{\jmath}) \times (-\vec{\imath} - \vec{k})]$$

note: $\left(\sqrt{2}\vec{\imath} + \vec{\jmath}\right) \times \left(-\vec{\imath} - \vec{k}\right) = 0 + \sqrt{2}\vec{\jmath} + \vec{k} - \vec{\imath}$

$$qPq^{-1} = \frac{1}{2}[-\sqrt{2} - (\sqrt{2}\,\vec{\imath} + \vec{\jmath}) \cdot (-\vec{\imath} - \vec{k}) +$$

$$\vec{\imath} + \vec{k} + \sqrt{2}(\sqrt{2}\,\vec{\imath} + \vec{\jmath}) + \sqrt{2}\,\vec{\jmath} + \vec{k} - \vec{\imath})$$

$$= 1/2\left[-\underbrace{\sqrt{2}+\sqrt{2}}_{0} + 2\vec{k} + 2\sqrt{2}\vec{j} + 2\vec{\imath}\right]$$

$$= 1/2\left[0 + 2\left(\vec{\imath} + \sqrt{2}\vec{j} + \vec{k}\right)\right]$$

$$= \left[0, \vec{\imath} + \sqrt{2}\vec{j} + \vec{k}\right] \text{ pure}$$

Original and rotated vectors should have the same magnitude:

$$\left.\begin{array}{l} \|\mathbb{P}_{\text{rot}}\| = \sqrt{1+2+1} = \sqrt{4} = 2 \\ \|\ \mathbb{P}\ \| = \sqrt{2^2} = 2 \end{array}\right\} \quad \text{no change}$$

However, the rotation is 90° not 45°

To rotate by 45° we can define the following quaternion:

$$q = \left[\cos\frac{\theta}{2}, \quad \sin\frac{\theta}{2} \underbrace{\vec{q}}_{\text{rotation axis}}\right]$$

Theorem:

For any unit quaternion:

$$q = q_0 + \vec{q} = \cos\frac{\theta}{2}, \quad \sin\frac{\theta}{2}\vec{u}$$

and any vector $\vec{p} \in R^3$, $L_q(p) = \mathbb{q}\mathbb{P}\mathbb{p}^*$ is equivalent to a rotation of a vector through an angle θ about vector $\vec{u}$ as the axis of rotation.

$$L_q(p) = \mathbb{q}\mathbb{P}\mathbb{p}^*$$

$$= (q_0^2 - |\vec{q}|^2\vec{p} + 2(\vec{q}\cdot\vec{p})q + 2q_0(\vec{q}\times\vec{p})$$

$$L_q(p) = \left((q_0^2 - |\vec{q}|^2) \underbrace{I_3}_{3\times 3 \text{ identity matrix}} + 2(q\ \cdot)q_+ 2q_0\vec{q}x\right)\vec{p}$$

$$\vec{q}\text{x} = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}$$

$$|q_1| = 1$$

Therefore

$$L_q(p) = R\vec{p}$$

$$R = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 - 2q_0q_3 & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 + q_2^2 - q_1^2 - q_3^2 & 2(q_2q_3 - q_1q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 - q_0q_1) & q_0^2 + q_3^2 - q_2^2 - q_1^2 \end{bmatrix}$$

NIFTI method 3 coordinate system

In method 3, there is a variable called *sform* which is greater than 0. In this method coordinate transformation is defined as follows:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} Srow_x[0] & Srow_x[1] & Srow_x[2] \\ Srow_y[0] & Srow_y[1] & Srow_y[2] \\ Srow_z[0] & Srow_z[1] & Srow_z[2] \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix} + \begin{bmatrix} Srow_x[3] \\ Srow_y[3] \\ Srow_z[3] \end{bmatrix}$$

To summarize, method one is just for backward compatibility for Analyze 7.5. Method two is used to represent scanner coordinate system. If further processing is done (Ex. registration), then method three can be incorporated to show coordinate in the standard space (atlas space).