Applied Medical Image Processing Lecture Notes

1 Unsupervised Tissue Classification:

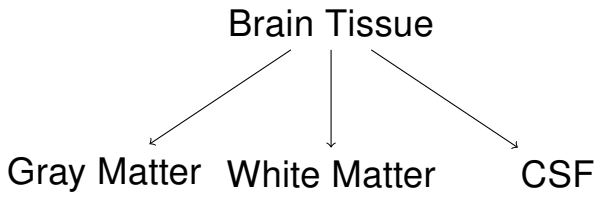
1.1 Introduction

Tissue classification is a crucial task in medical image processing, particularly in MRI scans where distinguishing between different types of brain tissue is necessary for diagnosis and treatment planning. This process can be performed in a supervised or unsupervised manner. Here, we discuss an unsupervised approach to tissue classification. Previously, we discussed binary classification in which the goal is to separate the image into foreground and background.

$$\text{Binary Classification} \rightarrow \text{Foreground}, \text{Background}$$

In MRI neuroimaging, when focusing on foreground (brain tissue), the objective is to classify it into different tissue types such as gray matter, white matter, and cerebrospinal fluid (CSF).

Brain Tissue $\rightarrow$ Classify $\rightarrow$ Gray Matter, White Matter, CSF (Cerebrospinal Fluid)



1.2 Assumptions

1. The image is noisy.
2. There is intensity inhomogeneity in the MR image which indicates the same tissue class, such as white matter, can have different intensity profiles.

3. The image is piecewise constant.

1.3 Mathematical Model

The true image is composed of K tissue classes, each with a distinct but unknown class mean, γ_k.

$$\gamma_k : \text{Class mean for } k = 1, \ldots, K$$

The class mean $\gamma_k(\vec{x})$ varies from one location to another due to intensity inhomogeneity. Here $\vec{x}$ is pixel location.

$$\vec{x} \in \Omega : \text{Image domain}$$

The observed image intensity can be modeled as follows:

$$y_j = g_j \sum_{k=1}^{K} z_{jk}\gamma_k + \eta_j$$

where:

- y_j is the observed intensity at pixel j.
- g_j is multiplicative gain factor for field inhomogeneity.
- z_{jk} is the indicator function for class membership.
- γ_k is the class mean.
- η_j is Gaussian noise with unknown variance.

$$\sum_{k} z_{jk} = 1 \text{ (No overlapping classes)}$$

The goal is to find z_{jk} such that:

$$z_{jk} = \begin{cases} 1 & \text{if pixel } j \text{ belongs to tissue class } k \\ 0 & \text{otherwise} \end{cases}$$

This approach to unsupervised tissue classification involves modeling the observed image intensity with a mixture of Gaussian distributions for each tissue class. The primary goal is to accurately classify each pixel into the correct tissue class based on the observed intensity and underlying statistical properties.

2 Bayesian Framework

In a Bayesian framework, given the observed data y, the objective is to determine the class labels (posterior probabilities), the class means, and associated parameters. The Bayesian framework combines the observed data likelihood with a prior that incorporates spatial regularization. Posterior probability can be described as follows:

$$P(z|y) \approx P(y|z;\theta)P(z)$$
$$f(z|y) \approx f(y|z;\theta)f(z) \text{ Different notation}$$

This approach helps in segmenting the image into meaningful regions by considering both the intensity values and their spatial relationships.

Here, we assume that the likelihood function takes the following form:

$$f(y|z;\theta) = \prod_{i\in\Omega} \left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y_j - g_j \sum_k z_{jk}\gamma_k)^2}{2\sigma^2} \right) \right)$$

Incorporating spatial information into labeling can be done using Markov random fields for the prior function, which discourage the formation of regions consisting of disconnected pixels.

$$P(z) = \frac{1}{Z_z} \prod_{j\in\Omega} e^{\left(-\beta \sum_{(i\in N_j} Z_j^T V Z_i\right)}$$

where Z_z is the normalizing factor, β controls the strength of the prior, and N_j defines the neighborhood structure. For a 2D image, the neighbors are typically the 4 nearest pixels, and for a 3D image, the neighbors are the 6 nearest pixels.

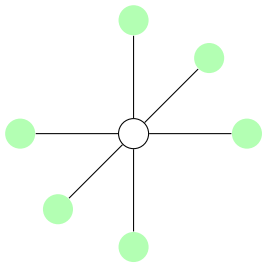


Figure 1: An example of 6 nearest pixel in a 3D image.

The matrix V is a $K \times K$ matrix defined as:

$$V = [1, \ldots, 1]^T [1, \ldots, 1] - \mathbb{I}$$

Where $\mathbb{I}$ is identity matrix. Matrix V ensures that a pixel belonging to the same class as its neighbors is favored over other configurations.

2.1 Posterior Probability

Combining the likelihood and prior probability terms, we will have:

$$\begin{aligned} P(z|y) &\approx P(y|z;\theta)P(z) \\ &= \prod_{j\in\Omega} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y_j - g_j \sum_{k=1}^{K} z_{jk}\gamma_k)^2}{2\sigma^2}\right) \cdot \frac{1}{Z_z} \prod_{j\in\Omega} \exp\left(-\beta \sum_{i\in N_j} z_j^T V z_i\right) \end{aligned}$$

We could ignore the normalizing factors which are $\frac{1}{\sqrt{2\pi\sigma^2}}$ and $\frac{1}{Z_z}$. Accordingly, the log-likelihood of the posterior probability is given by:

$$\ln P(z|y) = \ln P(y|z;\theta) + \ln P(z)$$
$$\ln P(z|y) \approx \frac{1}{2\sigma^2}\sum_j -(y_j - g_j\sum_k z_{jk}\gamma_k)^2 - \beta\sum_{i\in N_j} z_j^T V z_i$$
$$\ln P(z|y) \approx \frac{1}{2\sigma^2}\sum_j (y_j - g_j\sum_k z_{jk}\gamma_k)^2 + \beta\sum_{i\in N_j} z_j^T V z_i$$

Expanding the sum of squares, we get:

$$\sum_j (y_j - g_j\sum_k z_{jk}\gamma_k)^2$$

$$= \sum_j \left(y_j^2 + g_j^2\sum_k z_{jk}^2\gamma_k^2 - 2y_j g_j\sum_k z_{jk}\gamma_k \right)$$

Given that $\sum_k z_{jk} = 1$ and $z_{jk}^2 = \underbrace{z_{jk}z_{jk}}_{\text{either 0 or 1}} = z_{jk}$, the above equation can be rewritten as:

$$= \sum_j \left(y_j^2\sum_k z_{jk} + g_j^2\sum_k z_{jk}^2\gamma_k^2 - 2\sum_k y_j g_j z_{jk}\gamma_k \right)$$
$$= \sum_j\sum_k z_{jk}\left[y_j^2 + g_j^2\gamma_k^2 - 2y_j g_j\gamma_k\right]$$
$$= \sum_j\sum_k z_{jk}\left(y_j - g_j\gamma_k\right)^2$$

2.1.1 Energy Function Components

Now if we refer to $\ln p(z|y)$ as energy function E, this function can be decomposed into two components:

$$E = \frac{1}{2\sigma^2} E_1 + \beta E_2$$

where

$$E_1 = \sum_{j \in \Omega} \sum_{k=1}^{K} z_{jk} (y_j - g_j \gamma_k)^2$$

$$E_2 = \sum_{j \in \Omega} \sum_{i \in N_j} z_j^T V z_i$$

2.1.2 Solution

Estimate means γ_k by solving:

$$\partial E / \partial \gamma_k = \frac{1}{2\sigma^2} \partial E_1 / \partial \gamma_k + \beta \partial E_2 / \partial \gamma_k = 0$$

Given that $\partial E_2 / \partial \gamma_k = 0$ because it is not a function of γ_k:

$$\partial E / \partial \gamma_k = \partial E_1 / \partial \gamma_k = \frac{\partial}{\partial \gamma_k} \sum_j \sum_k z_{jk} (y_j - g_j \gamma_k)^2$$

Differentiating:

$$= 2 \sum_j z_{jk} (y_j - g_j \gamma_k)(-g_j)$$

$$= -2 \sum_j z_{jk} g_j y_j + 2 \sum_j z_{jk} g_j^2 \gamma_k = 0$$

Solving for γ_k:

$$\sum_j z_{jk} g_j^2 \gamma_k = \sum_j z_{jk} g_j y_j$$

$$\boxed{\gamma_k = \frac{\sum_j z_{jk} g_j y_j}{\sum_j z_{jk} g_j^2}}$$

To estimate g_j, we need a smooth function to model g_j as a low-degree polynomial function.

$$g_j = \sum_{n=1}^{N} f_n P_n(j)$$

where $P_n(j)$ are polynomial basis functions and f_n are coefficients. A common choice for $P_n(j)$ is the Chebyshev polynomial basis function.

2.2 Algorithm

The algorithm follows these steps:

1. Obtain initial estimates of centroids γ_k and assume $g_j = 1$.
2. Solve for the indicator function. Note that:

$$z_j^* = \arg\min_{z_j}((y_j - g_j \sum_k z_{jk}\gamma_k)^2 + \beta \sum_{i \in N_j} z_j^T V z_i)$$

3. Solve for the means γ_k.
4. Solve for the gain field by solving the following equation for coefficients f_n:

$$[z_{jk}\gamma_k P_n(j)]\,[f_n] = [z_{jk} y_j]$$

Matrix Formulation for gain field

We have:

$$Y = Pf$$

$$P^T Y = P^T P f$$

$$(P^T P)^{-1} P^T Y = (P^T P)^{-1} (P^T P) f$$

$$f = (P^T P)^{-1} P^T Y \quad \text{(assuming } P^T P \text{ is invertible)}$$

where the dimensions are:

$$Y : (Q \times R \times K) \times 1 \quad \text{lattice}$$

$$P : (Q \times R \times K) \times N$$

$$f : N \times 1$$

For example, $P_n(j)$ can be represented as:

$$P_n(j) = \left[x_j, y_j, z_j, x_j^2, y_j^2, z_j^2, x_j y_j, x_j z_j, y_j z_j, 1\right]$$

This representation uses the coordinates x_j, y_j, z_j and their polynomial combinations to form the basis functions for the polynomial model instead of Chebyshev polynomial.

Bibliography

[1] Unsupervised tissue classification, DL Pham, PL Bazin, Handbook of Medical Image Processing and Analysis, 209-222

[2] Robust unsupervised tissue classification in MR image (2004), Dzung L. Pham etal. in Proceedings of the IEEE International Symposium on Biomedical Imaging