Medical Imaging Modalities (sources) 1

1 Computed Tomography (CT) Reconstruction

Computed tomography is a projection based medical image modality where X-ray beam intensity is attenuated while passing through the body. Since this attenuation is a result of the beam passing thorough several layers of tissue, one single beam can not be used to decipher tissue complexity in the body. Many X-ray beams are needed to measure X-ray attenuation at different angles before a full image can be reconstructed. One of the techniques that has been proposed to reconstruct CT images is called **algebraic reconstruction**.

1.1 Algebraic reconstruction

It is an iterative technique to reconstruct CT images based on a series of angular projections. To illustrated the process, let's start with a simple example, where we have a simple tissue consist of four compartments with different attenuation coefficients: $u_1 \dots u_4$.

u_1	u_2
u_3	u_4

We have X-ray beams with constant initial intensity I_0 that are passing through this compartments at different angles, and are able to measure the intensity of X-ray exiting from the tissue (Fig. 1).

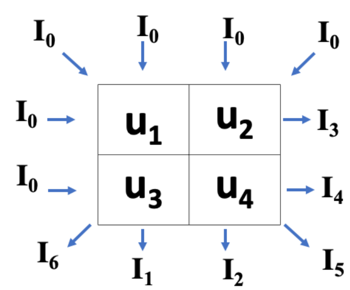

Figure 1: X-Ray beam projection example

Accordingly, we could write 6 equations describing the amount of attenuation of each X-ray beam:

$$
\begin{aligned}
x_1 &= u_1 + u_3 \\
x_2 &= u_2 + u_4 \\
x_3 &= u_1 + u_2 \\
x_4 &= u_3 + u_4 \\
x_5 &= u_1 + u_4 \\
x_6 &= u_2 + u_3
\end{aligned}
$$

These equations can be written in a Matrix form $AU = X$, where:

$$
\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}
$$

To solve this equation, we could proceed as follows:

$$
\begin{aligned}
AU &= X \\
A^T AU &= A^T X \\
(A^T A)^{-1}(A^T A)U &= (A^T A)^{-1} A^T X \\
\mathbb{I}U &= (A^T A)^{-1} A^T X \\
U &= (A^T A)^{-1} A^T X
\end{aligned}
\tag{1}
$$

Where $\mathbb{I}$ is the identity matrix, and A^T is the transpose of matrix A. In practice this approach is computationally very expensive and an iterative algorithms has been used instead. Before describing this iterative method, it would be beneficial to further explore the concept of X-Ray attenuation. Imaging we have an X-Ray beam entering a slab of body tissues with an initial intensity value of I_0 and passing through different compartments, each with an attenuation coefficient of μ_i. The path length of the X-ray through each voxel is represented by w_i (Fig. 2).

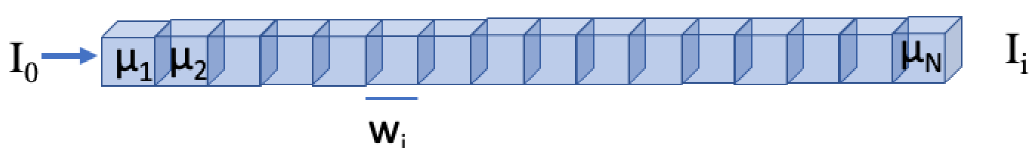

Figure 2: X-Ray path

Now, let's image that X-Ray with intensity I_0 has passed true the first voxel and has been attenuated to a new intensity I_1. I_1 will be further attenuated while passing through the second voxel to become I_2 and so forth: This can be described mathematically as:

$$
\begin{aligned}
&I_1 = I_0 e^{-(\omega_1 \mu_1)} \\
&I_2 = I_1 e^{-(\omega_2 \mu_2)} = I_0 e^{-(\omega_1 \mu_1)} e^{-(\omega_2 \mu_2)} \\
&I_3 = I_2 e^{-(\omega_3 \mu_3)} = I_0 e^{-(\omega_1 \mu_1)} e^{-(\omega_2 \mu_2)} e^{-(\omega_3 \mu_3)} \\
&I_i = I_0 e^{-(\omega_1 \mu_1 + \omega_2 \mu_2 + \omega_3 \mu_3 + \cdots + \omega_n \mu_n)} \\
&-\ln(\tfrac{I_i}{I_0}) = \omega_1 \mu_1 + \omega_2 \mu_2 + \omega_3 \mu_3 + \cdots + \omega_n \mu_n \\
&X = u_1 + u_2 + \ldots + u_n
\end{aligned}
$$

Which is the representation of X-ray intensity change due to attenuation sums while passing through a slab of tissue and exiting the body. Now we can discuss the algebraic reconstruction going back to our original example. Let's assume we have the following measurement recorded by a detector:

$$
\begin{aligned}
x_1 &= 10 \\
x_2 &= 10 \\
x_3 &= 14 \\
x_4 &= 6 \\
x_5 &= 12 \\
x_6 &= 8
\end{aligned}
$$

Since $x_1 = u_1 + u_3 = 10$, and we don't have any prior knowledge of the distribution of u_1 or u_3, we can assume that both are equally contributing to the attenuation, therefore they both equal to 5. Similarly, we could assume that both u_2 and u_4 are equal to 5. However, $x_3 = u_1 + u_2 = 14$, but based on the current estimation its value is $5 + 5 = 10$ which is 4 less than what it supposed to be. Similarly, the measured value for $x_3 = u_3 + u_4 = 6$, but based on the current estimation its value is $5 + 5 = 10$ which is 4 more than what it supposed to be. That means that $u_1 \ldots u_4$ are not homogenous and we need to make a correction.

5	5	$\underset{\rightarrow}{10}$ 14
5	5	$\underset{\rightarrow}{10}$ 6

To make a correction for u_1 and u_2, we need to add 4 to their sum. Again, since we don't have any knowledge about u_1 and u_2, we equally add 2 to each of them to make them $u_1 = 7$ and $u_2 = 7$. With regards to u_3 and u_4, we need to subtract 4 from their sum or 2 from each to get $u_1 = 3$ and $u_2 = 3$.

7	7
3	3

Following the same logic, $x_5 = u_1 + u_4 = 12$, but based on the current estimation their sum is 10, so we add one to each of u_1 and u_4 to get $u_1 = 8$ and $u_4 = 4$.

8	7
3	4

Finally $x_6 = u_2 + u_3 = 8$, but based on the current estimation their sum is 10, so we subtract one from each of u_2 and u_3 to get $u_2 = 6$ and $u_3 = 2$.

8	6
2	4

Now all the sums are consistent with our measurements. In practice, due to noise and imperfect measurements, we never get the exact values, however, we could define a threshold at which we can accept the estimation of attenuation coefficients and stop the iterative process.

1.2 Radon Transform

A more general approach to solve for the X-ray attenuation in tissue is called Radon transform. To describe this mathematical tool, we need to write the X-ray attenuation equations in a general form described as:

$$\frac{dI}{dx} = -A(x)I(x) \tag{2}$$

Let the initial conditions are defined as follows:

$$\begin{cases} I(x_0) = I_0 \\ I(x_1) = I_1 \end{cases} \tag{3}$$

We can solve for the X-ray beam intensity I change passing through small tissue dx by getting the integrals of both sides in equation 2 to have:

$$\begin{aligned} \int_{x_0}^{x_1} \frac{dI}{dx} &= \int_{x_0}^{x_1} -A(x)I(x) \\ \int_{x_0}^{x_1} \frac{dI}{I(x)} &= \int_{x_0}^{x_1} -A(x)d(x) \text{ , } I(x) \text{ and } dx \text{ are switched.} \\ \ln I(x)\Big|_{x_0}^{x_1} &= -\int_{x_0}^{x_1} A(x)dx \\ \ln I_1 - \ln I_0 = \ln\frac{I_1}{I_0} &= -\int_{x_0}^{x_1} A(x)I(x) \\ \ln\frac{I_0}{I_1} &= \int_{x_0}^{x_1} A(x)I(x) \end{aligned} \tag{4}$$

Here $A(x)$ is referred to as attenuation coefficient which is proportional to the density of tissue. Before describing the Radon transform we need to define a mathematical way, equation of lines in space, that allows us dealing with parallel lines (X-ray beams) passing through a tissue. We could define a line " L " in $\mathbb{R}^2$ as : $ax + by = c$ where $a, b, c \in \mathbb{R}$ and $a^2 + b^2 \neq 0$. Furthermore, since $\sqrt{a^2 + b^2} \neq 0$ then we could also write:

$$\frac{a}{\sqrt{a^2+b^2}}x + \frac{b}{\sqrt{a^2+b^2}}y = \frac{c}{\sqrt{a^2+b^2}}$$

Defining

$$\omega = (\omega_1, \omega_2) = \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right)$$

and noticing that

$$\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2 = 1$$

We can conclude that ω is a point on the unit circle (using polar coordinates): $\omega = (\cos\theta, \sin\theta)$ $\theta \in [0, 2\pi)$. Also let's define $t = \frac{c}{\sqrt{a^2+b^2}}$ and $z : (x, y)$ being a point in 2D space,then:

$$\begin{aligned}\langle z, \omega\rangle &= x\omega_1 + y\omega_2 \\ &= \frac{a}{\sqrt{a^2+b^2}}x + \frac{b}{\sqrt{a^2+\omega^2}}y = \frac{c}{\sqrt{a^2+b^2}} \\ &= \boxed{x\cos\theta + y\sin\theta = t}\end{aligned}$$

Note that $x\cos\theta + y\sin\theta = t$ is the equation of a line for a fixed θ and t values. So we could write:

$$l_{t,\theta} = \{z \in \mathbb{R}^2 :< z, \omega >= t\}$$

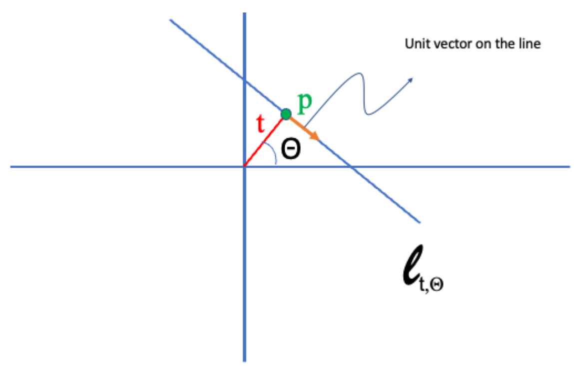

Figure 3: Line in 2D space

Referring to Fig. 3, t is the length of the line starting at the origin and perpendicular to line $l_{t,\theta}$ at point p. Point p has the following coordinates:

$$p = \begin{cases} x &= t\cos\theta \\ y &= t\sin\theta \end{cases}$$

placing the coordinates of this point into the line equation, we will have:

$$\begin{aligned}&x\cos\theta + y\sin\theta = t \\ &(t\cos\theta)\cos\theta + (t\sin\theta)\sin\theta = t \\ &t\cos^2\theta + t\sin^2\theta = t \\ &t(\cos^2\theta + \sin^2\theta) = t\end{aligned}$$

Another way to define points on a line is to start at point p and move along the unit vector on the line which is perpendicular to line t (Fig 3). Using this definition, we will have:

$$(x(s), y(s)) :< t\cos\theta, t\sin\theta > + < -s\sin\theta, s\cos\theta > \text{ for } s \in \mathbb{R}$$

Now using the line equations, we could define the Radon transform as:

$$\mathcal{R}\{f\}(t,\theta) = \int_{-\infty}^{\infty} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)ds$$

which is a line integral of a function f in 2D space parametrized by s (Fig. 4).

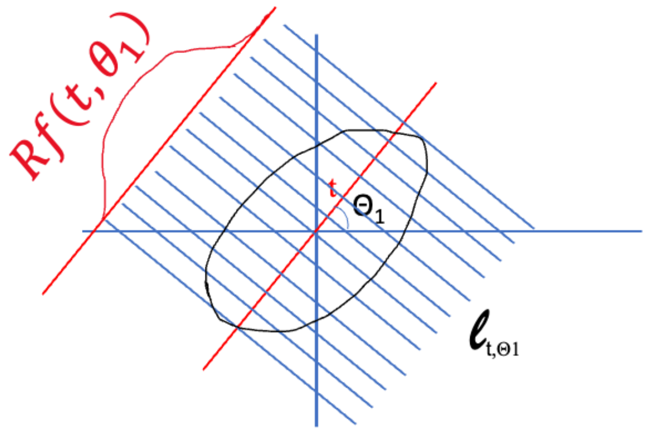

Figure 4: Radon Transform

To see the correlation between the Radon transform and the X-ray attenuation in tissue, we will refer to equation 4.

$$\ln\left(\frac{I_0}{I_1}\right) = \int_{x_0}^{x_1} A(x)dx \tag{5a}$$

$$\mathcal{R}\{f\}(t,\theta) = \int_{-\infty}^{\infty} f(x(s), y(s))ds \tag{5b}$$

The goal is to find an inverse function that can recover $f(.,.)$ or $A(x)$. Before proceeding with describing a method that allows us to achieve this task, lets list few properties of the Radon transform. Here, we assume that α and β are constant and f and g are continuous functions:

- Linear Property

  $\mathcal{R}(\alpha f + \beta g) = \alpha\mathcal{R}\{f\} + \beta\mathcal{R}\{g\}$

- Scaling property

  $\mathcal{R}f(at, a\theta) = ?$

  We would like to drive the scaling property of the Radon transform:

$$\vec{\omega} : (\cos\theta, \sin\theta) \tag{6a}$$

$$\vec{z} : (x, y) \tag{6b}$$

$$x\cos\theta + y\sin\theta = t \tag{6c}$$

$$\vec{\omega} \cdot \vec{z} = t \tag{6d}$$

$$\mathcal{R}f : \int_{-\infty}^{\infty} f(\vec{z})\delta(t - \vec{\omega} \cdot \vec{z})d\vec{z} \tag{6e}$$

$$\mathcal{R}f \approx \tilde{f}(t, \vec{\omega}) \tag{6f}$$

$$\tilde{f}(at, a\vec{\omega}) = ? \tag{6g}$$

$$\tilde{f}(at, a\vec{\omega}) = \mathcal{R}f = \int_{-\infty}^{\infty} f(\vec{z})\delta(at - a\vec{\omega} \cdot \vec{z})d\vec{z} \tag{6h}$$

$$\delta(ax) = \frac{1}{|a|}\delta(x) \tag{6i}$$

$$\mathcal{R}f = \int_{-\infty}^{\infty} f(\vec{z})\frac{\delta(t - \vec{\omega} \cdot \vec{z})}{|a|}d\vec{z} \tag{6j}$$

$$\mathcal{R}f = |a|^{-1}\int_{-\infty}^{\infty} f(\vec{z})\delta(t - \vec{\omega} \cdot \vec{z})d\vec{z} \tag{6k}$$

$$\tilde{f}(at, a\vec{\omega}) = |a|^{-1}\tilde{f}(t, \vec{\omega}) \tag{6l}$$