Applied Medical Image Processing Lecture Notes

Medical Imaging Modalities (sources) 2

A Review of Fourier Transform

Recall that the definition of Fourier transform in one dimension (1D) is:

$$[\mathcal{F}f](\xi) = \int_{-\infty}^{\infty} f\,(x)e^{-2\pi i x\xi}dx$$

or

$$[\mathcal{F}f](\omega) = \int_{-\infty}^{\infty} f\,(x)e^{-i\omega x}dx$$

From the linearity of Fourier transform, we could write:

$$\mathcal{F}(\alpha f + \beta g) = \alpha[\mathcal{F}f] + \beta[\mathcal{F}g]$$

The inverse Fourier transform in 1D is defined as:

$$\begin{aligned} [\mathcal{F}^{-1}f](x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f\,(\omega)e^{i\omega x}d\omega \\ \mathcal{F}^{-1}(\mathcal{F}f)x &= f(x) \end{aligned}$$

In two dimensions (2D) the Fourier and it's inverse transform take the following forms:

$$\begin{aligned} (\xi_1, \xi_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\,(x,y)e^{-i(x\xi_1 + y\xi_2)}dxdy \\ [\mathcal{F}_{2D}^{-1}g](x_1, x_2) &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G\,(\xi_1, \xi_2)e^{-i(x\xi_1 + y\xi_2)}d\xi_1 d\xi_2 \end{aligned}$$

Central Slice Theorem

Definition

One dimensional Fourier transform of a projected function such as Radon transform, is equal to the 2D Fourier transform of the original function taken on the slice through the origin parallel to the line, we projected our function on (see Fig. *1*).

The parallel 2-D projection in spatial domain and subsequent 1-D Fourier transform is identical to a slice through 2-D Fourier space. This image is taken from the text book "Maier, Andreas, et al., eds. Medical Imaging Systems: An Introductory Guide. Vol. 11111. Springer, 2018."

Mathematical derivation

Mathematically, this can be described as follows:

Taking a one dimensional Fourier transform of a Radon transformed function with respect to variable t

$$\mathcal{F}_t[Rf(t,\theta)](r,\theta) = \int_{-\infty}^{\infty} \underbrace{R(t,\theta)}_{\text{By definition: } \int_{-\infty}^{\infty} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\,ds} e^{-itr}dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\, e^{-itr}dsdt$$

Let's change variables from $t,\ s$ to $x,\ y$ using the following definition:

Performing the change of variables from t, s to x, y, requires calculating the determinant of the Jacobian of variable transformation as follows:

$$\begin{aligned} x(s) &= t\cos\theta - s\sin\theta \\ y(s) &= t\sin\theta + s\cos\theta \end{aligned} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} t \\ s \end{bmatrix}$$

$$\text{or } \vec{X} = A\vec{T}$$

$$\text{accordingly } \vec{T} = A^{-1}\vec{x}$$

$$A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \Rightarrow \begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$t = x\cos\theta + y\sin\theta$$

$$s = -x\sin\theta + y\cos\theta$$

Performing the change of variables from t, s to x, y requires calculating the determinant of the Jacobian of variable transformation as follows:

$$\begin{vmatrix} \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$$

$$= \cos^2\theta - (-\sin^2\theta) = \cos^2\theta + \sin^2\theta = 1$$

After replacing the variables in 1a, we get:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i(x\cos\theta + y\sin\theta)r}dxdy \qquad \text{(3a)}$$

$$= \mathcal{F}_{x,y}[f](r\cos\theta, r\sin\theta) \qquad \text{(3b)}$$

$$\mathcal{F}_{x,y}[f](r\cos\theta, r\sin\theta) = \mathcal{F}_t[\mathcal{R}f(t,\theta)](r,\theta) \qquad \text{(3c)}$$

Backprojection Algorithm

To get the original signal at position (x, y), we need to calculate the $f(x, y)$ as follow:

$$f(x,y) = \mathcal{F}_{x,y}^{-1}\left[\mathcal{F}_{x,y}f\right](x,y)$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\mathcal{F}_{x,y}f\right](\xi_1,\xi_2)e^{i(x\xi_1+y\xi_2)}d\xi_1 d_1\xi_2$$

$$\xi_1 = s\cos\theta$$

$$\xi_2 = s\sin\theta$$

$$\begin{vmatrix} \frac{\partial\xi_1}{\partial s} & \frac{\partial\xi_1}{\partial\theta} \\ \frac{\partial\xi_2}{\partial s} & \frac{\partial\xi_2}{\partial\theta} \end{vmatrix} =$$

$$\begin{vmatrix} \cos\theta & -s\sin\theta \\ \sin\theta & s\cos\theta \end{vmatrix} \quad = |s\cos^2\theta + s\sin^2\theta|$$

$$= |s|\underbrace{|\cos^2\theta + \sin^2\theta|}_{1} = |s|$$

$$d\xi_1 d\xi_2 = |s|dsd\theta$$

$$\Rightarrow \frac{1}{4\pi^2}\int_0^{\pi}\int_{-\infty}^{\infty}\mathcal{F}_{2D}\, f(s\cos\theta, s\sin\theta)e^{is(\cos\theta+y\sin)}|s|dsd\theta$$

$$= \frac{1}{4\pi^2}\int_0^{\pi}\int_{-\infty}^{\infty}\mathcal{F}\left[\mathcal{R}f(s,\theta)e^{is(\cos\theta+y\sin)}|s|dsd\theta\right.$$

$$\int_{-\infty}^{\infty}\mathcal{F}\left[\mathcal{R}f(s,\theta)\right]e^{is(x\cos\theta+y\sin\theta)}|s|ds$$

$$= 2\pi\left(\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{F}\left[\mathcal{R}f(s,\theta)\right]e^{is(x\cos\theta+y\sin\theta)}|s|ds}_{\mathcal{F}^{-1}[|s|\mathcal{F}(\mathcal{R}f(s,\theta))](x\cos\theta+y\sin\theta,\theta)}\right)$$

$$f(x,y) = \frac{1}{4\pi^2}2\pi\int_0^{\pi}\mathcal{F}^{-1}\left[|s|\mathcal{F}(\mathcal{R}f)(s,\theta)\right](x\cos\theta + y\sin\theta, \theta)d\theta$$

$$f(x,y) = 1/2 \quad \underbrace{\mathcal{B}}_{\text{Back projection}} \quad \{\mathcal{F}^{-1}[|s|\mathcal{F}(\mathcal{R}f)(s,\theta)]\}$$

Practice Example for the Radon Transform:

Using the definition of Radon transform find:

$$\mathcal{R}\left[e^{-\pi\left(x^2+y^2\right)}\right](t,\theta)$$

Solution:

$$[Rf](t,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\,(x,y)\delta(x\cos\theta + y\sin\theta - t)dxdy$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$u = x\cos\theta + y\sin\theta$$

$$v = -x\sin\theta + y\cos\theta$$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

$$[Rf](t,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(u^2+v^2)}\,\delta(t-u)dvdv$$

since $x^2 + y^2 = u^2 + v^2$ as shown here

$$u^2 + v^2 = x^2\cos^2\theta + y^2\sin^2\theta + 2xy\cos\theta\sin\theta$$
$$+x^2\sin^2\theta + y^2\cos^2\theta - 2xy\cos\theta\sin\theta$$

$$[Rf](t,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)}\,\delta(x\cos\theta + y\sin\theta - t)dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(u^2+v^2)}\,\delta(u-t)dudv$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} e^{-\pi u^2 - \pi v^2}\,\delta(u-t)dudv$$

Note that $\int_{-\infty}^{\infty} e^{-\pi u^2}\,\delta(u-t)du = e^{-\pi t^2}$

Therefore $[Rf](t,\theta) = e^{-\pi t^2}\underbrace{\int_{-\sigma}^{\infty} e^{-\pi v^2}\,dv}_{1} = e^{-\pi t^2}$

Note that: (Gaussian distribution)

$$\int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x^2}{\sigma^2}\right)}dx = 1$$

Assuming $\sigma = \sqrt{\frac{1}{2\pi}}$

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{\frac{1}{2\pi}}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x^2}{\left(\sqrt{\frac{1}{2\pi}}\right)^2}\right)}$$

$$= \int_{-\infty}^{\infty} e^{-\pi x^2} = 1$$