Dr. Ardekani
EN.585.703.81.FA24 Applied Medical Image Processing
Module 2 Assignment
Johns Hopkins University
Student: Yves Greatti

Question 1

A convenient form of 2D Radon transform is to use the following equation:

$$\{\mathcal{R}\rho\}(t,\theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \rho(x,y)\delta(x\cos\theta + y\sin\theta - t)dxdy \tag{1}$$

using this definition:

1. calculate $\{\mathcal{R}\rho\}(t,0)$ for $\rho(x,y) = \Pi\left(\frac{x}{a}\right)\Pi\left(\frac{y}{b}\right)$ (10 points), Where:

$$\Pi(x) = \begin{cases} 1, & \text{if } |x| < 1/2 \\ 0, & \text{otherwise} \end{cases}$$

For $\theta = 0$:

$$\{\mathcal{R}\rho\}(t,0) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \rho(x,y)\delta(x - t)dxdy \tag{2}$$

Since $\int_{-\infty}^{+\infty} \rho(x,y)\delta(x-t)dx = \rho(t\ ,y)$; then

$$\begin{aligned}
\{\mathcal{R}\rho\}(t,0) &= \int_{-\infty}^{+\infty} \Pi(\frac{t}{a})\Pi(\frac{y}{b})\ dy \\
&= \Pi(\frac{t}{a}) \int_{-\infty}^{+\infty} \Pi(\frac{y}{b})\ dy \\
&= \Pi(\frac{t}{a}) \int_{-\frac{b}{2}}^{\frac{b}{2}}\ dy \\
&= \Pi(\frac{t}{a})[y]_{-\frac{b}{2}}^{\frac{b}{2}} \\
&= b\ \cdot \Pi(\frac{t}{a})
\end{aligned}$$

2. Calculate and compare the Fourier transform of $\{\mathcal{R}\rho\}(t, 45°)$ for a square object defined by:

$$\rho(x, y) = \Pi\left(\frac{x}{2}\right)\Pi\left(\frac{y}{2}\right)$$

using direct approach and projection-slice theorem (20 points).

In both [a] and [b] show your work.

According to the projection-slice theorem:

$$\mathcal{F}\left(\{\mathcal{R}\rho\}\right)(t, 45°) = \mathcal{F}\left(\mathcal{R}\rho\right)(t, 45°)$$

We make the change of variables from x and y to u and v. With

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\Rightarrow$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

With $\theta = 45°$

$$x = \frac{u - v}{\sqrt{2}}$$
$$y = \frac{u + v}{\sqrt{2}}$$

And

$$\text{Jacobian} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos\theta\cdot\cos\theta - (-\sin\theta)\cdot\sin\theta = \cos^2\theta + \sin^2\theta = 1$$

$$\begin{aligned}
\mathcal{R}f(t, 45°) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(x, y)\delta(x\cos(45°) + y\sin(45°) - t)\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Pi\left(\frac{x}{2}\right)\Pi\left(\frac{y}{2}\right)\delta\left(x\cos(45°) + y\sin(45°) - t\right)\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Pi\left(\frac{u - v}{2\sqrt{2}}\right)\Pi\left(\frac{u + v}{2\sqrt{2}}\right)\delta\left(t - u\right)\,du\,dv
\end{aligned}$$

$$
\begin{aligned}
\mathcal{R}f(t,45^\circ) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Pi\left(\frac{u-v}{2\sqrt{2}}\right)\Pi\left(\frac{u+v}{2\sqrt{2}}\right)\delta(t-u)\,du\,dv \\
&= \int_{-\infty}^{\infty} \Pi\left(\frac{t-v}{2\sqrt{2}}\right)\Pi\left(\frac{t+v}{2\sqrt{2}}\right)\,dv
\end{aligned}
$$

For $\Pi\left(\frac{t-v}{2\sqrt{2}}\right)$ to be 1:

$$\left|\frac{t-v}{2\sqrt{2}}\right| < \frac{1}{2}$$

$\Rightarrow$

$$
\begin{aligned}
|t-v| &< \sqrt{2} \\
-\sqrt{2} &< t-v < \sqrt{2} \\
t-\sqrt{2} &< v < t+\sqrt{2}
\end{aligned}
$$

Similarly, for $\Pi\left(\frac{t+v}{2\sqrt{2}}\right)$ to be 1:

$$\left|\frac{t+v}{2\sqrt{2}}\right| < \frac{1}{2}$$

This implies:

$$
\begin{aligned}
|t+v| &< \sqrt{2} \\
-\sqrt{2} &< t+v < \sqrt{2} \\
-\left(t+\sqrt{2}\right) &< v < -\left(t-\sqrt{2}\right)
\end{aligned}
$$

We are interested in finding the values of v where both intervals; $\left[t-\sqrt{2}, t+\sqrt{2}\right]$ and $\left[-(t+\sqrt{2}), -(t-\sqrt{2})\right]$ overlap. For the intersection to be non-empty, v must satisfy:

$$
\begin{aligned}
v &> \max(t-\sqrt{2}, -(t+\sqrt{2})) \\
v &< \min(t+\sqrt{2}, -(t-\sqrt{2}))
\end{aligned}
$$

Case 1: $t-\sqrt{2} \geq -(t+\sqrt{2})$
In this case, $\max(t-\sqrt{2}, -(t+\sqrt{2})) = t-\sqrt{2}$.

We need:

$$t - \sqrt{2} \leq \min(t + \sqrt{2}, -(t - \sqrt{2}))$$

We have two sub-cases :

- If $t + \sqrt{2} \leq -(t - \sqrt{2})$:

  $$t - \sqrt{2} \leq t + \sqrt{2} \quad \text{(always verified)}$$

  We also need:

  $$t + \sqrt{2} \leq -(t - \sqrt{2})$$

  Simplifying this:

  $$t + \sqrt{2} \leq -t + \sqrt{2}$$

  $$2t \leq 0 \quad \Rightarrow \quad t \leq 0$$

- If $-(t - \sqrt{2}) \leq t + \sqrt{2}$:

  $$t - \sqrt{2} \leq -(t - \sqrt{2})$$

  Simplifying this:

  $$2t \leq 2\sqrt{2} \quad \Rightarrow \quad t \leq \sqrt{2}$$

So from Case 1, t must satisfy $0 \leq t \leq \sqrt{2}$.

Case 2: $t - \sqrt{2} < -(t + \sqrt{2})$

In this case, $\max(t - \sqrt{2}, -(t + \sqrt{2})) = -(t + \sqrt{2})$.

We need:

$$-(t + \sqrt{2}) \leq \min(t + \sqrt{2}, -(t - \sqrt{2}))$$

Again, two sub-cases:

- If $t + \sqrt{2} \leq -(t - \sqrt{2})$:

  $$-(t + \sqrt{2}) \leq t + \sqrt{2}$$

  Simplifying:

  $$-2t \leq 2\sqrt{2} \quad \Rightarrow \quad t \geq -\sqrt{2}$$

- If $-(t-\sqrt{2}) \leq t+\sqrt{2}$:

$$-(t+\sqrt{2}) \leq -(t-\sqrt{2})$$

  Simplifying:

$$-2\sqrt{2} \leq 0 \quad \text{(which is always true)}$$

So from Case 2, t must satisfy $t \geq -\sqrt{2}$.

The conditions from the cases give us that t must satisfy: $-\sqrt{2} \leq t \leq \sqrt{2}$. For $t \in \left[-\sqrt{2}, \sqrt{2}\right]$, the integrand is 1 over the interval where both Π functions are 1, which occurs when $-2\sqrt{2} \leq v \leq 2\sqrt{2}$. Therefore, the integral is:

$$\int_{-2\sqrt{2}}^{2\sqrt{2}} dv = 4\sqrt{2}$$

For $|t| > \sqrt{2}$, the integral is 0 because the intervals do not overlap.

The integral evaluates to

$$\mathcal{R}f(t,45^\circ) = \int_{-\infty}^{\infty} \Pi\left(\frac{t-v}{2\sqrt{2}}\right)\Pi\left(\frac{t+v}{2\sqrt{2}}\right)\,dv = \begin{cases} 4\sqrt{2}, & \text{if } |t| \leq \sqrt{2}, \\ 0 & \text{otherwise.} \end{cases}$$