Applied Medical Image Processing Lecture Notes

1 Automatic thresholding:

A set of all pixels is partitioned into two disjoint clusters, C_0 and C_1, using a threshold value q:

$$(u, v) \in \begin{cases} C_0 \text{ if } I(u, v) \leq q & \text{(background)} \\ C_1 \text{ if } I(u, v) > q & \text{(foreground)} \end{cases}$$

Question: How can the optimum threshold be identified to binarize the image?

1.1 Iterative Threshold Selection (ISODATA Algorithm):

This algorithm is similar to k-means clustering with the following assumption: The image histogram consists of two separate distributions representing the foreground and the background. Each distribution is assumed to be Gaussian with equal variance.

Let $h(g)$ represent the histogram of the image I, with pixels having K possible intensity values:

$$0 \leq g < K$$

The algorithm follows these steps (see Algorithm 1 for details):

1. Make an initial guess for the threshold (e.g., the mean or median of the intensity values).
2. Split the pixels into two groups based on the initial threshold.
3. Calculate the mean of each group to update the threshold value (i.e., the mean of the sets).
4. Repeat the process until the threshold value does not change significantly.

Algorithm 1 Isodata Threshold Algorithm

Input: $h : [0, K-1] \rightarrow N$ ▷ gray scale histogram
$K \leftarrow \mathrm{size}(h)$ ▷ number of intensity values
$q \leftarrow \mathrm{mean}(h, 0, K-1)$ ▷ set the initial threshold to overall mean
  $\mu_I \leftarrow \frac{1}{N} \sum_{g=0}^{K-1} g \cdot h(g)$ ▷ overall mean
repeat
  $n_0 \leftarrow \mathrm{count}(h, 0, q)$ ▷ background population
  $n_1 \leftarrow \mathrm{count}(h, q+1, K-1)$ ▷ foreground population
  if $n_0 = 0$ **or** $n_1 = 0$ **then**
    return -1 ▷ invalid threshold
  end if
  $\mu_0 \leftarrow \mathrm{Mean}(h, 0, q)$ ▷ background mean
  $\mu_1 \leftarrow \mathrm{Mean}(h, q+1, K-1)$ ▷ foreground mean
  $q' \leftarrow q$ ▷ previous threshold
  $q \leftarrow \left\lfloor \frac{\mu_0 + \mu_1}{2} \right\rfloor$
until $q = q'$
return q

$$\mathrm{count}(h, a, b) := \sum_{g=a}^{b} h(g)$$

$$\mathrm{Mean}(h, a, b) := \frac{\sum_{g=a}^{b} g \cdot h(g)}{\sum_{g=a}^{b} h(g)}$$

1.2 Otsu's Method

This method assumes two classes with unknown intensity distributions. The goal is to find a threshold q such that the two distributions are maximally separated. This separation is achieved by meeting the following criteria:

1. The variances within each class are minimal.
2. The centers (means) of the two classes are as distant as possible.

The within-class variance for the background class at intensity threshold q, $\sigma_0^2(q)$, is given by:

$$\sigma_0^2(q) = \frac{1}{n_0(q)} \sum_{g=0}^{q} (g - \mu_0(q))^2 \cdot h(g)$$

Here, $n_0(q)$, the number of pixels in the background class is defined as:

$$n_0(q) = |C_0| = \sum_{g=0}^{q} h(g)$$

Similarly, the within-class variance for the foreground class, $\sigma_1^2(q)$, is given by:

$$\sigma_1^2(q) = \frac{1}{n_1(q)} \sum_{g=q+1}^{K-1} (g - \mu_1(q))^2 \cdot h(g)$$

The number of pixels in the foreground class, $n_1(q)$, is defined as:

$$n_1(q) = |C_1| = \sum_{g=q+1}^{K-1} h(g)$$

The within-class variance $\sigma_w^2(q)$ is a weighted sum of the variances of the two classes:

$$\sigma_w^2(q) = \mathbb{P}_0(q) \cdot \sigma_0^2(q) + \mathbb{P}_1(q) \cdot \sigma_1^2(q)$$

Where the probabilities $\mathbb{P}_0(q)$ and $\mathbb{P}_1(q)$ are given by:

$$\mathbb{P}_0(q) = \sum_{i=0}^{q} p(i) = \frac{1}{N} \sum_{i=0}^{q} h(i) = \frac{n_0(q)}{N}$$

$$\mathbb{P}_1(q) = \sum_{i=q+1}^{K-1} p(i) = \frac{1}{N} \sum_{i=q+1}^{K-1} h(i) = \frac{n_1(q)}{N}$$

Accordingly, $\sigma_w^2(q)$ is given by:

$$\sigma_w^2(q) = \frac{1}{N} \left[n_0(q) \cdot \sigma_0^2(q) + n_1(q) \cdot \sigma_1^2(q) \right]$$

The between-class variance $\sigma_b^2(q)$ measures the distance between the cluster means μ_0 and μ_1 relative to the overall mean μ_I:

$$
\begin{aligned}
\sigma_b^2(q) &= \mathbb{P}_0(q)\left(\mu_0(q)-\mu_I\right)^2+\mathbb{P}_1(q)\left(\mu_1(q)-\mu_I\right)^2 \\
&= \frac{1}{N}\left[n_0(q)\left(\mu_0(q)-\mu_I\right)^2+n_1(q)\left(\mu_1(q)-\mu_I\right)^2\right]
\end{aligned}
$$

The total variance can be expressed as:

$$
\sigma^2=\sigma_w^2(q)+\sigma_b^2(q)
$$

To minimize the total variance, we can either minimize $\sigma_w^2(q)$ or maximize $\sigma_b^2(q)$. Maximizing $\sigma_b^2(q)$ is preferred because it only relies on first-order statistics.

$$
\begin{aligned}
\sigma_b^2(q) &= \mathbb{P}_0(q)\left(\mu_0(q)-\mu_I\right)^2+\mathbb{P}_1(q)\left(\mu_1(q)-\mu_I\right)^2 \\
&= \mathbb{P}_0(q)\left[\mu_0^2(q)+\mu_I^2-2\mu_0(q)\mu_I\right]+\mathbb{P}_1(q)\left[\mu_1^2(q)+\mu_I^2-2\mu_1(q)\mu_I\right] \\
&= \mathbb{P}_0(q)\mu_0^2(q)+\mathbb{P}_0(q)\mu_I^2-\mathbb{P}_0(q)2\mu_0(q)\mu_I+\mathbb{P}_1(q)\mu_1^2(q)+\mathbb{P}_1(q)\mu_I^2-\mathbb{P}_1(q)2\mu_1(q)\mu_I \\
&\text{Note: } (\mathbb{P}_0(q)+\mathbb{P}_1(q))\mu_I^2=\mu_I^2 \\
&\text{Also since: } \mu_I=\mathbb{P}_0\mu_0(q)+\mathbb{P}_1\mu_1(q) \\
&2\mu_I\left[\mathbb{P}_0(q)\mu_0(q)+\mathbb{P}_1(q)\mu_1(q)\right]=2\mu_I^2 \\
&= \mathbb{P}_0(q)\mu_0^2(q)+\mathbb{P}_1(q)\mu_1^2(q)+\mu_I^2-2\mu_I^2 \\
&= \mathbb{P}_0(q)\mu_0^2(q)+\mathbb{P}_1(q)\mu_1^2(q)-\mu_I^2 \\
&= \mathbb{P}_0(q)\mu_0^2(q)+\mathbb{P}_1(q)\mu_1^2(q)-\left[\mathbb{P}_0(q)\mu_0(q)+\mathbb{P}_1(q)\mu_1(q)\right]^2 \\
&= \left[\mathbb{P}_0(q)\mu_0^2(q)+\mathbb{P}_1(q)\mu_1^2(q)\right]-\left[\mathbb{P}_0^2(q)\mu_0^2(q)+\mathbb{P}_1^2(q)\mu_1^2(q)+2\mathbb{P}_0(q)\mathbb{P}_1(q)\mu_0(q)\mu_1(q)\right] \\
&= \mathbb{P}_0(q)\mu_0^2(q)-\mathbb{P}_0^2(q)\mu_0^2(q)+\mathbb{P}_1(q)\mu_1^2(q)-\mathbb{P}_1^2(q)\mu_1^2(q)-2\mathbb{P}_0(q)\mathbb{P}_1(q)\mu_0(q)\mu_1(q) \\
&= \mathbb{P}_0(q)\mu_0^2(q)\left[1-\mathbb{P}_0(q)\right]+\mathbb{P}_1(q)\mu_1^2(q)\left[1-\mathbb{P}_1(q)\right]-2\mathbb{P}_0(q)\mathbb{P}_1(q)\mu_0(q)\mu_1(q) \\
&= \mathbb{P}_0(q)\mathbb{P}_1(q)\mu_0^2(q)+\mathbb{P}_1(q)\mathbb{P}_0(q)\mu_1^2(q)-2\mathbb{P}_0(q)\mathbb{P}_1(q)\mu_0(q)\mu_1(q) \\
&= \mathbb{P}_0(q)\mathbb{P}_1(q)\left[\mu_0^2(q)+\mu_1^2(q)-2\mu_0(q)\mu_1(q)\right] \\
&= \mathbb{P}_0(q)\mathbb{P}_1(q)\left[\left(\mu_0(q)-\mu_1(q)\right)^2\right] \\
\sigma_b^2(q) &= \frac{1}{N^2}n_0(q)n_1(q)\left(\mu_0(q)-\mu_1(q)\right)^2
\end{aligned}
$$

1.2.1 Implementation:

Algorithm 2 Otsu Threshold

Input: $h : [0 \rightarrow K-1] \rightarrow N$ ▷ gray scale histogram
Output: Optimal threshold value or -1 if not found
$K \leftarrow \text{size}(h)$ ▷ number of intensity levels
$(\mu_0, \mu_1, N) \leftarrow \text{MakeMeanTable}(h, K)$ ▷ table of means
$\sigma^2_{\text{max}} \leftarrow 0$
$q_{\text{max}} \leftarrow -1$
$n_0 \leftarrow 0$
for $q = 0$ to $K-2$ **do**
    $n_0 \leftarrow n_0 + h(q)$
    $n_1 \leftarrow N - n_0$
    if $n_0 > 0 \wedge n_1 > 0$ **then**
        $\sigma^2_b \leftarrow \frac{1}{N^2} n_0 n_1 (\mu_0(q) - \mu_1(q))^2$
        if $\sigma^2_b > \sigma^2_{\text{max}}$ **then**
            $\sigma^2_{\text{max}} \leftarrow \sigma^2_b$
            $q_{\text{max}} \leftarrow q$
        end if
    end if
end for
return q_{max}

Algorithm 3 Make Mean Tables

Input: $h : [0 \rightarrow K-1] \rightarrow N, K$ ▷ gray scale histogram
Output: μ_0, μ_1, N ▷ tables of means and total count
$n_0 \leftarrow 0, S_0 \leftarrow 0$
for $q = 0$ to $K-1$ **do**
    $n_0 \leftarrow n_0 + h(q)$
    $S_0 \leftarrow S_0 + q \cdot h(q)$
    $\mu_0(q) \leftarrow \begin{cases} S_0/n_0 & \text{if } n_0 > 0 \\ -1 & \text{otherwise} \end{cases}$
end for
$N \leftarrow n_0$
$n_1 \leftarrow 0, S_1 \leftarrow 0$
$\mu_1(K-1) \leftarrow 0$
for $q = K-2$ to 0 **do**
    $n_1 \leftarrow n_1 + h(q+1)$
    $S_1 \leftarrow S_1 + (q+1) \cdot h(q+1)$
    $\mu_1(q) \leftarrow \begin{cases} S_1/n_1 & \text{if } n_1 > 0 \\ -1 & \text{otherwise} \end{cases}$
end for
return (μ_0, μ_1, N)

Bibliography

[1] Wilhelm Burger, Mark J. Burge , Principles of Digital Image Processing: Advanced Methods, Chapter 2 - Automatic Thresholding, 2013