Recitation 7

DS-GA 1013 Mathematical Tools for Data Science

1. Early Stopping

2. Which of the following cosine functions all have a period of 2π?

   A. $cos(t), cos(t/2), cost(t/3), \ldots$

   B. $cos(\pi t), cos(2\pi t), cos(3\pi t), \ldots$

   C. $cos(t), cost(2t), cos(3t) \ldots$

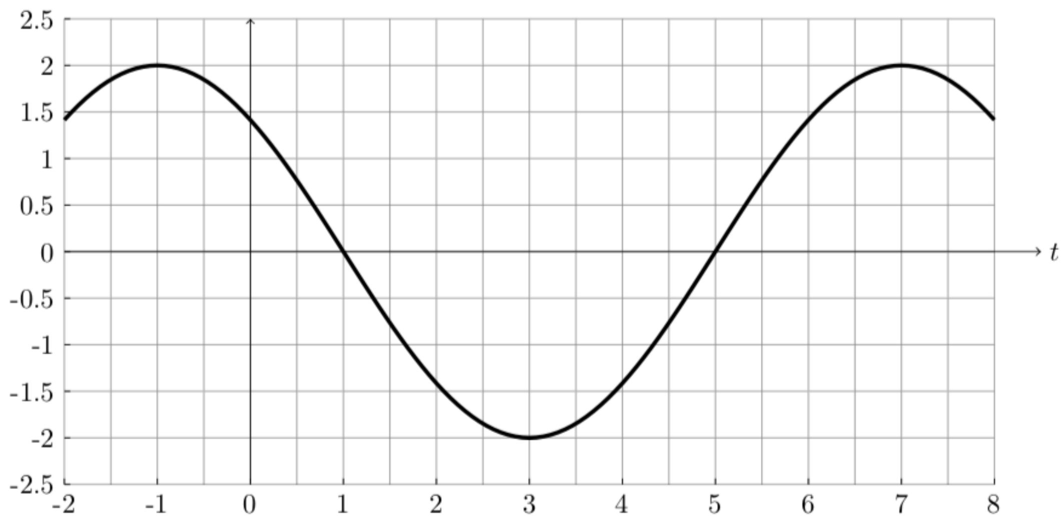
3. What is the fundamental period of

   1. $sin(\pi t/3)$
   2. $|sin(t)|$
   3. $cos^2(3t)$
   4. $f(t) = cos(t) + cos(2t) + cos(3t)$?

Solution:

1. 6
2. π
3. $\pi/3$
4. 2π

4. Express the following sinusoidal function in the form $Acos(\omega t - \varphi)$ where $A \in \mathbb{R}^+ \cup \{0\}$ and $\omega, \varphi \in \mathbb{R}$



Solution: $2cos(\frac{\pi}{4}t + \pi/4)$

5. Now express the sinusoid above as $\sum_j r_j e^{i\varphi_j} e^{i\omega_j t}$ where $r_j \in \mathbb{R}^+ \cup \{0\}$ and $\omega_j, \varphi_j \in \mathbb{R}$

Solution:

$$\begin{aligned} Acos(\omega t - \varphi) &= A\frac{e^{i(\omega t-\varphi)} + e^{-i(\omega t-\varphi)}}{2} \\ &= \frac{A}{2}e^{-i\varphi}e^{i(\omega t)} + \frac{A}{2}e^{i\varphi}e^{i(-\omega t)} \end{aligned}$$

6. What's the fundamental period of $e^{j\omega t}$? What is the projection of $e^{j\omega t}$ to both the axes on complex plane? Animation. Negative frequency.

Here we list some useful facts about complex numbers. Below $z \in \mathbb{C}$ and $a, b \in \mathbb{R}$.

- $z = a + bi = \mathrm{Re}(z) + i\,\mathrm{Im}(z)$
- $(a + bi)(c + di) = ac - bd + (ad + bc)i$
- $|a + bi|^2 = a^2 + b^2 = (a + bi)(a - bi) = (a + bi)(\overline{a + bi})$
- $|zw| = |z||w|$, $|z + w| \le |z| + |w|$
- $e^{a+bi} = e^a(cos(b) + i\sin(b))$, $e^z e^w = e^{z+w}$
- $|e^{a+bi}| = e^a$
- $z = \overline{z}$ if and only if $z \in \mathbb{R}$
- $z + \overline{z} = 2\,\mathrm{Re}(z)$ and $z - \overline{z} = 2i\,\mathrm{Im}(z)$
- $\langle\vec{x}, \vec{y}\rangle = \overline{\langle\vec{y}, \vec{x}\rangle}$
- $\langle c\vec{x}, \vec{y}\rangle = c\langle\vec{x}, \vec{y}\rangle$ and $\langle\vec{x}, c\vec{y}\rangle = \overline{c}\langle\vec{x}, \vec{y}\rangle$
- $\|\vec{x}\|^2 = \langle\vec{x}, \vec{x}\rangle$
- For $\vec{x} \in \mathbb{C}^n$, $\vec{x}^* := \overline{(\vec{x})}^T$
- For $A \in \mathbb{C}^{m\times n}$, $A^* = \overline{A}^T$
- For $\vec{x}, \vec{y} \in \mathbb{C}^n$, $\langle\vec{x}, \vec{y}\rangle = \vec{y}^*\vec{x} = \sum_{i=1}^n \overline{\vec{y}[i]}\vec{x}[i]$

7. Compute $z = 1 + e^{2\pi i/n} + e^{4\pi i/n} + \cdots + e^{2(n-1)\pi i/n}$ where $n \ge 1$. Explain your answer geometrically.

Solution: If $n = 1$ then $z = n$. Otherwise $e^{2\pi i/n}z = z$ with $e^{2\pi i/n} \ne 1$ so $z = 0$.

Another way to see this is:

$$\sum_{n=0}^{N-1} e^{2\pi ik/N} = \frac{1 - e^{2\pi iN/N}}{1 - e^{2\pi i/N}} = \frac{1 - e^{2\pi i}}{1 - e^{2\pi i/N}} = 0$$

This answer also makes complete sense geometrically. The general complex exponential $re^{i\theta}$ can be thought of as a vector in the complex plane, of length r and at an angle θ counterclockwise from the real axis. Thus $w = e^{2\pi i/n}$ is a vector of length 1 at an angle of $2\pi/n$. Similarly, $w^k = e^{2\pi ik/n}$ has length 1 and is at an angle of $2\pi k/n$. The points are distinct and equidistantly spaced $2\pi/n$ radians apart around the unit circle. Consider the n points equally spaced around the unit circle as vertices of a regular n -gon, and the $e^{2\pi ik/n}$ as vectors from 0 to the vertices. The (vector) sum of the points is then the perimeter of the polygon. Viewed as a closed loop, the vector sum is the zero vector.

8. For $z, z_1, z_2 \in \mathbb{C}$, if
$$\left| z - \left(\frac{z_1 + z_2}{2} \right) \right| = \frac{|z_1 - z_2|}{2}$$
then show that
$$|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$$

Solution: Notice the geometry. z lies in a cirlce with z_1 and z_2 being end points of a diameter. Result follows from geometry.

9. Show that $|r_1 e^{it} - r_2 e^{is}| \geq |r_1 - r_2|$ for all $r_1, r_2 > 0$ and $t, s \in \mathbb{R}$.

Solution: By the triangle inequality
$$|r_1 e^{it} - r_2 e^{is}| + |r_2 e^{is}| \geq |r_1 e^{it}| \implies |r_1 e^{it} - r_2 e^{is}| \geq |r_1| - |r_2| = r_1 - r_2.$$
The same argument holds with the two terms flipped giving the result.

10. Prove that Cauchy-Schwarz holds in a complex inner product space:
$$|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \|\vec{y}\|.$$

Solution: Since
$$\langle a\vec{x}, \vec{y} \rangle = a \langle \vec{x}, \vec{y} \rangle \quad \text{and} \quad \langle \vec{x}, a\vec{y} \rangle = \bar{a} \langle \vec{x}, \vec{y} \rangle$$
we see that proving the inequality with $\|\vec{x}\| = \|\vec{y}\| = 1$ is sufficient. Let $c \in \mathbb{C}$ with $|c| = 1$ and observe that
$$\begin{aligned} 0 &\leq \|c\vec{x} - \vec{y}\|^2 \\ &= \langle c\vec{x} - \vec{y}, c\vec{x} - \vec{y} \rangle \\ &= \|c\vec{x}\|^2 + \|\vec{y}\|^2 - \langle c\vec{x}, \vec{y} \rangle - \langle \vec{y}, c\vec{x} \rangle \\ &= 2 - 2\,\mathrm{Re}(c \langle \vec{x}, \vec{y} \rangle). \end{aligned}$$
Let c be given by
$$c = \frac{\overline{\langle \vec{x}, \vec{y} \rangle}}{|\langle \vec{x}, \vec{y} \rangle|}$$
so that the above gives
$$|\langle \vec{x}, \vec{y} \rangle| \leq 1$$
completing the proof.