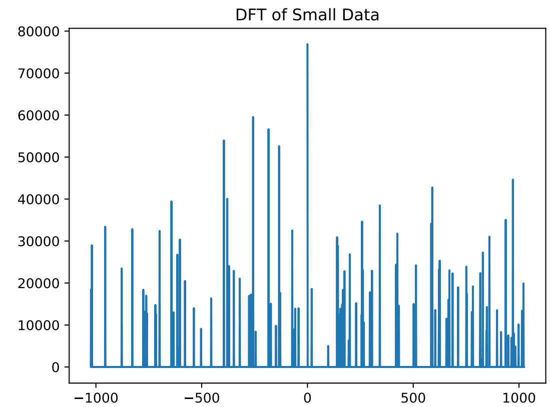
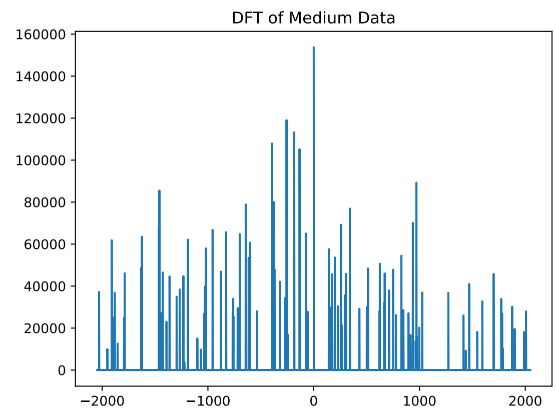
Homework 8

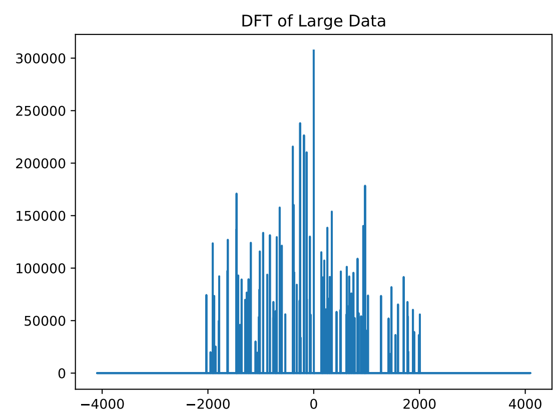
Solutions

1. (Aliasing)

   (a) Below are the three plots.







   Note that the large values are roughly 4 times the small, and the medium values are roughly 2 times the small.

   (b) By the sampling theorem, we can use the DFT values for $N = 8193$ and divide by N. This yields $|a_0| = 37.5$, $|a_{-257}| = 29.0603$ and $|a_{-184}| = 27.6457$.

(c) By the assumption that $k_c \leq 4096$ we have

$$\hat{x}_{[8193]}[k] = 8193 a_k$$

for $k = -4096, \ldots, 4096$ where we treat the indices on $\hat{x}_{[8193]}$ as periodic with period 8193. By the aliasing formula we have

$$\hat{x}_{[2049]}[k] = 2049 \sum_{\substack{m \in \mathbb{Z} \\ m \equiv k \pmod{2049}}} a_m,$$

where $a_m = 0$ for $|m| > 4096$. Thus we have

$$\begin{aligned}
\hat{x}_{[2049]}[3] &= 2049(a_3 + a_{-2046} + a_{2052} + a_{-4095}) && (1)\\
&= \frac{2049}{8193}\big(\hat{x}_{[8193]}[3] + \hat{x}_{[8193]}[-2046] + \hat{x}_{[8193]}[2052] + \hat{x}_{[8193]}[-4095]\big), && (2)\\
&= \frac{2049}{8193}\big(\hat{x}_{[8193]}[3] + \hat{x}_{[8193]}[6147] + \hat{x}_{[8193]}[2052] + \hat{x}_{[8193]}[4098]\big), && (3)
\end{aligned}$$

where we cyclically indexed $\hat{x}_{[8193]}$ mod 8193.

(d) False. Having $a_{2049 \cdot 4097 \cdot 8193} \neq 0$ is consistent with all of the plots.

2. (Justification of the FFT)

(a) Fixing $j < N$ we have

$$(F_{[N]})_{j,2k+1} = e^{-2\pi i j(2k+1)/N} = e^{-2\pi i j(2k)/N} e^{-2\pi i j/N} = e^{-2\pi i j/N}(F_{[N]})_{j,2k}.$$

(b) Fixing $j < N/2$ we have

$$\begin{aligned}
(F_{[N]})_{j+N/2,2k} = e^{-2\pi i(j+N/2)(2k)/N} &= e^{-2\pi i j(2k)/N} e^{-2\pi i (N/2)(2k)/N}\\
&= e^{-2\pi i j(2k)/N} e^{-2\pi i k}\\
&= e^{-2\pi i j(2k)/N} = (F_{[N]})_{j,2k}.
\end{aligned}$$

(c) Fixing $j < N/2$ we have

$$(F_{[N]})_{j,2k} = e^{-2\pi i j(2k)/N} = e^{-2\pi i jk/(N/2)} = (F_{[N/2]})_{j,k}.$$

3. (Properties of the DFT)

(a) Recall that

$$F^*_{[N]}[j,k] = e^{2\pi i jk/N} = F_{[N]}[-j,k]$$

where we index cyclically (mod N). Let P denote the permutation matrix such that $(Px)[k] = x[-k]$ for all $k = 0, \ldots, N-1$ (where again we index cyclically). Then we have

$$(PF_{[N]})[j,k] = F_{[N]}[-j,k] = F^*_{[N]}[j,k]$$

as required. Furthermore, note that

$$(P^2 x)[k] = (Px)[-k] = x[k]$$

for any x, so $P^2 = I$ as required.

(b) Recall that $F_{[N]}[j,:] = \psi_j^*$ and $F_{[N]}[:,k] = \overline{\psi_k}$. Thus we have

$$\operatorname{trace}(\psi_j^* X \overline{\psi_k}) = \operatorname{trace}(X \overline{\psi_k} \psi_j^*) = \langle X, \psi_j \psi_k^T \rangle_F = \hat{X}[j,k].$$

(c)

$$\overline{\hat{x}[-j,-k]} = \overline{\int_0^1 \int_0^1 x(s,t) e^{2\pi i j s} e^{2\pi i k t} \, ds \, dt} = \int_0^1 \int_0^1 x(s,t) e^{-2\pi i j s} e^{-2\pi i k t} \, ds \, dt = \hat{x}[j,k].$$

4. (Undersampling in MRI)

(a) Let z denote an arbitrary vector whose even DFT coefficients equal y. Recall that $\frac{1}{N} F_{[N]}^* F_{[N]} = I$, which implies

$$\begin{aligned} ||v||_2^2 &= v^* v && (4) \\ &= \frac{1}{N} v^* F_{[N]}^* F_{[N]} v && (5) \\ &= \frac{1}{N} \left|\left| F_{[N]} v \right|\right|_2^2 && (6) \\ &= \frac{1}{N} ||y||_2^2 + \frac{1}{N} ||\hat{v}_{\text{odd}}||_2^2 . && (7) \end{aligned}$$

This can be minimized by setting the odd Fourier coefficients to zero. This yields the consistent vector with the smallest ℓ_2 norm. To reconstruct it, we just need to build a vector $\hat{x}$ where the even entries are equal to y and the odd entries are set to zero, and then compute $\frac{1}{N} F_{[N]}^* \hat{x}$.

(b) Figure 1 shows the results.

(c) The DFT matrix is symmetric. By (b) and (c) in Problem 1, we can write the even-indexed rows of $F_{[N]}$ as $\begin{bmatrix} F_{[N/2]} & F_{[N/2]} \end{bmatrix}$. We have

$$y = \begin{bmatrix} F_{[N/2]} & F_{[N/2]} \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix}, \tag{8}$$

where we denote the first half of the entries of x by x_A and the second half by x_B. Now if we interleave y with zeros and apply $F_{[N]}^*$, this is equivalent to multiplying y by the even columns of $F_{[N]}$, which are equal to $\begin{bmatrix} F_{[N/2]} & F_{[N/2]} \end{bmatrix}^*$. We therefore have

$$\begin{aligned} x_{\text{est}} &= \frac{1}{N} \begin{bmatrix} F_{[N/2]} & F_{[N/2]} \end{bmatrix}^* \begin{bmatrix} F_{[N/2]} & F_{[N/2]} \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} && (9) \\ &= \frac{1}{2} \begin{bmatrix} \frac{2}{N} F_{[N/2]}^* \left(F_{[N/2]} x_A + F_{[N/2]} x_B \right) \\ \frac{2}{N} F_{[N/2]}^* \left(F_{[N/2]} x_A + F_{[N/2]} x_B \right) \end{bmatrix} && (10) \\ &= \frac{1}{2} \begin{bmatrix} x_A + x_B \\ x_A + x_B \end{bmatrix}. && (11) \end{aligned}$$

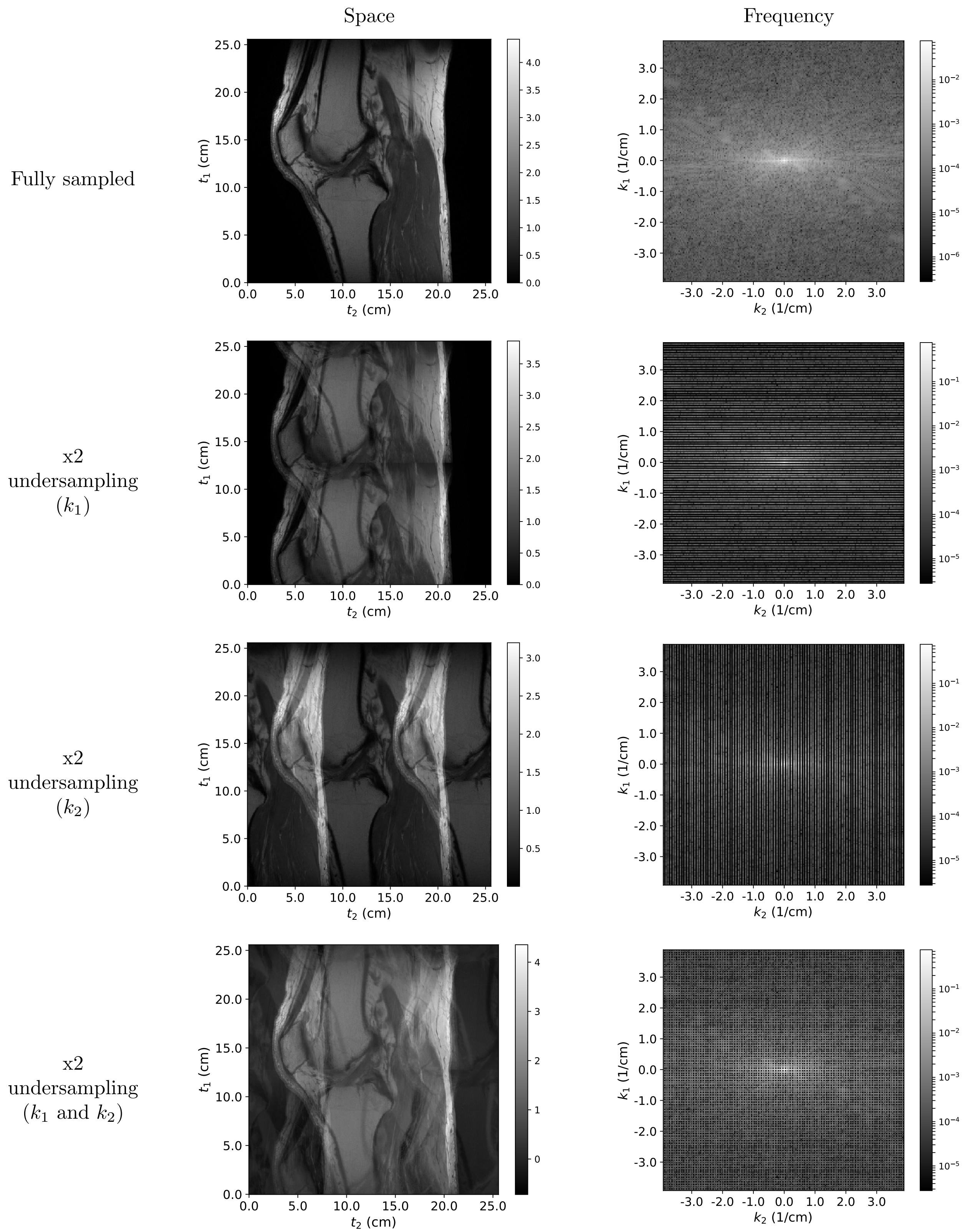


Figure 1: Reconstructed images.