Optimization-Based Data Analysis

Recitation 6

1. Let $A \in \mathbb{R}^{m\times n}$ under what conditions does it have a left inverse, i.e., a matrix $B \in \mathbb{R}^{n\times m}$ with $BA = I$. If it has a left inverse, how many does it have?

   Solution. A has a left inverse if and only if it has full column rank. In those cases, it has 1 left inverse if it is square, and infinitely many otherwise. This might be easier to see if we look at the transposed equation $A^T B^T = I$.

2. For vectors in $\mathbb{C}^n$, is the convolution associative? That is, do we have $(\vec{x} * \vec{y}) * \vec{z} = \vec{x} * (\vec{y} * \vec{z})$?

   Solution. Yes. Taking the DFT, we see this reduces to the fact that multiplication is associative.

3. Let $\vec{x}, \vec{y} \in \mathbb{C}^n$ and define $\vec{z} \in \mathbb{C}^n$ by $\vec{z}[k] = \vec{x}[k]\vec{y}[k]$. What is $\vec{Z}$?

   Solution. Below we show $\vec{Z} = \vec{X} * \vec{Y}$.

$$\begin{aligned}
\vec{Z}[k] &= \sum_{j=0}^{n-1} e^{-2\pi ijk/n}\vec{x}[j]\vec{y}[j] \\
&= \frac{1}{n}\sum_{j=0}^{n-1}\sum_{l=0}^{n-1} e^{-2\pi ijk/n}e^{2\pi ijl/n}\vec{x}[j]\vec{Y}[l] \\
&= \frac{1}{n}\sum_{l=0}^{n-1}\vec{Y}[l]\sum_{j=0}^{n-1} e^{-2\pi ij(k-l)/n}\vec{x}[j] \\
&= \frac{1}{n}\sum_{l=0}^{n-1}\vec{Y}[l]\vec{X}[k-l] \\
&= \frac{1}{n}(\vec{Y} * \vec{X})[k] \\
&= \frac{1}{n}(\vec{X} * \vec{Y})[k]
\end{aligned}$$

4. Let $\vec{x}, \vec{y}, \vec{w} \in \mathbb{C}^n$ and define $\vec{z} \in \mathbb{C}^n$ by $\vec{z}[k] = \vec{x}[k]\vec{y}[k]\vec{w}[k]$. What is $\vec{Z}$?

   Solution. Let $\vec{v}[k] = \vec{x}[k]\vec{y}[k]$ so that

$$\vec{Z}[k] = \frac{1}{n}(\vec{V} * \vec{W})[k] = \frac{1}{n^2}((\vec{X} * \vec{Y}) * \vec{W})[k].$$

5. Suppose we plot the magnitude of the spectrum of a signal $\vec{x} \in \mathbb{C}^n$. Then we (circularly) time shift it, and plot the magnitude of the spectrum again. How will the two plots differ?

   Solution. The two plots will be the same. Time shifting multiplies the frequencies by a factor with modulus 1. Thus the magnitudes are unchanged.

6. We are given 10 (uniformly spaced) samples $\vec{x} \in \mathbb{R}^{10}$ of a real signal. We then use the DFT to compute $\vec{X}$.

   (a) Is there anything we can say about what the plot of the magnitudes of the DFT coefficients will look like?

   (b) What frequencies do the DFT coefficients correspond to if the samples are taken over

      i. 1 second?

      ii. 10 seconds?

      iii. 1/2 second?

   Solution.

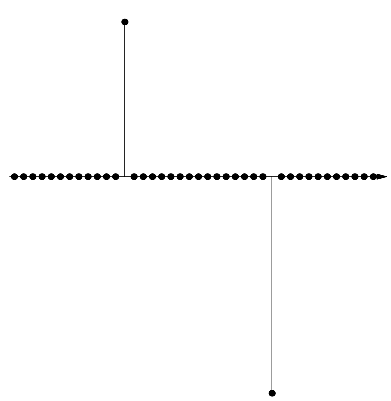
   (a) It will be symmetric (almost). Firstly, note that $\vec{X}[-k] = \overline{\vec{X}[k]}$ by a proof similar to the homework. The only issue is that 10 is even, so we need to think about $\vec{X}[5]$. Since the indices are cyclic modulo 10, we see that $\vec{X}[5]$ and $\vec{X}[-5]$ correspond to the same entry, so we could choose either to have the value.

   (b) i. Frequencies are in units of Hz (cycles per second).

      ii. Frequencies are in units of 0.1 Hz. That is, coefficient $\vec{X}[1]$ corresponds to 1 cycle in 10 seconds, or 1/10 of a cycle per second.

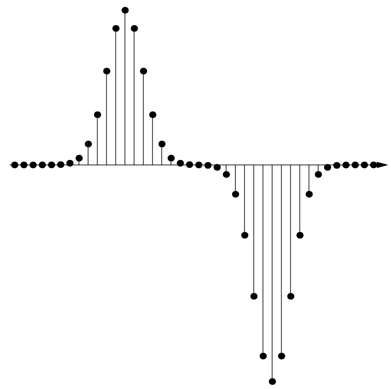
      iii. Frequencies are in units of 2 Hz. That is, coefficient $\vec{X}[1]$ corresponds to 1 cycle in 0.5 seconds, or 2 cycles per second.

7. Suppose we are given the following vector $\vec{x} \in \mathbb{R}^{40}$:

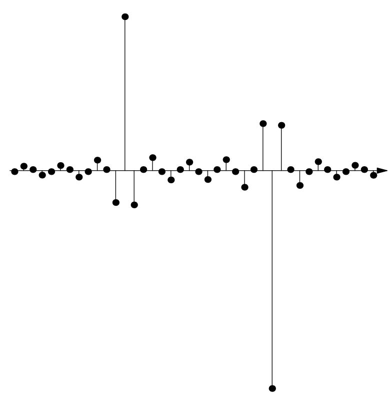


What does it look like when we convolve it with a Gaussian or a Dirichlet kernel?

Solution. Gaussian:



Dirichlet:



8. Let $\omega_C = 262$, $\omega_E = 330$, $\omega_G = 392$ denote the (rounded) frequencies of the (middle) C, E and G notes (in Hz). An audio signal corresponding to a note is just a sine wave of the given frequency. We sample at a rate of 44100 samples per second.

   (a) What will the FFT of a signal for the C-note look like?

   (b) What will the FFT of a signal for the C,E,G-chord look like?

   (c) Suppose we play C,E,G,E,C. What will the FFT of that signal look like?

   (d) Now suppose we filter out only the first E in the sequence above (by setting the signal to zero for the other notes). What will the FFT look like now?

   (e) Now suppose we convolve the signal with a Gaussian centered at the second E. What will the FFT look like now?

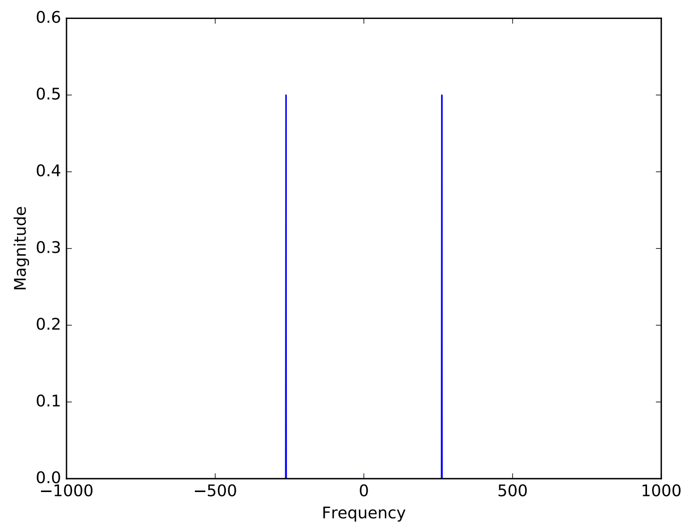
   (f) Suppose we are playing a single C-note again, but we extract the first 400 samples. What will the FFT look like? [Note $44100/262 \approx 168$.]

(g) Suppose we are playing a single C-note again, but we extract the first 30 samples. What will the FFT look like?

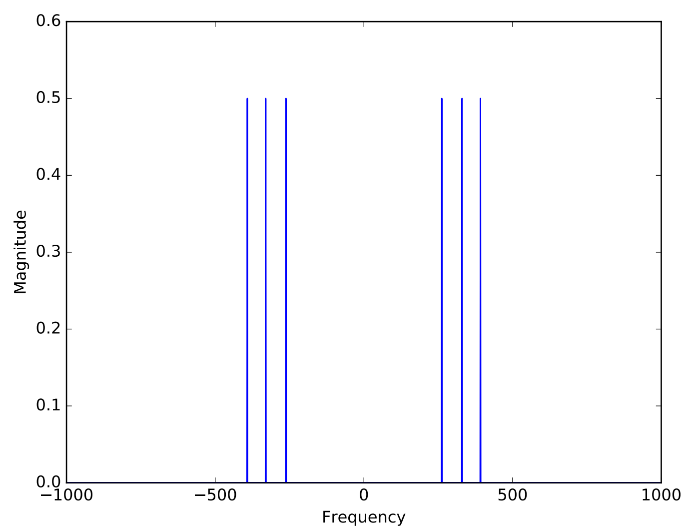
(h) In what situation would we experience aliasing?

Solution. We have scaled all FFT plots below by 1/num samples so that we see the actual amplitudes. In the cases where we window or extract, we divide by the number of remaining samples.

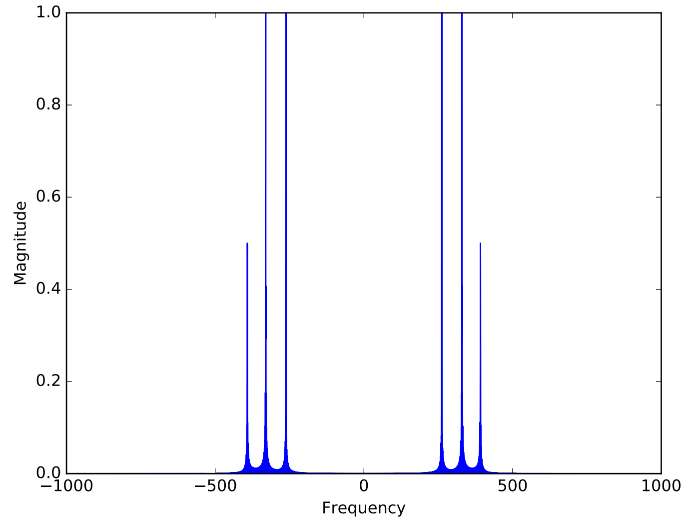
(a) Shown below.



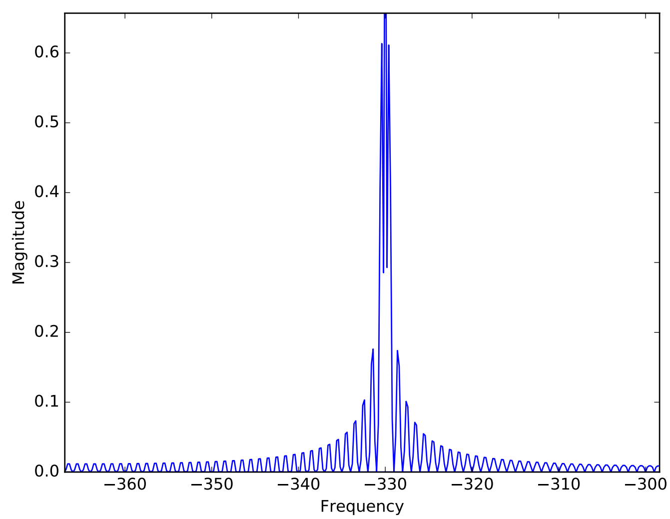
(b) Shown below.



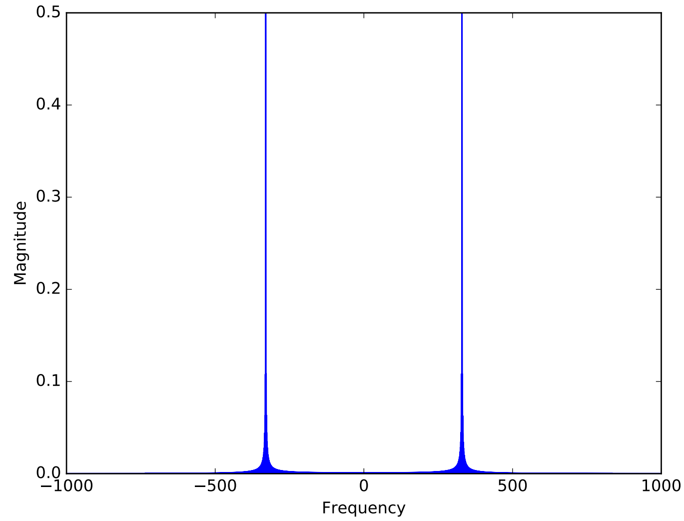
(c) Shown below. You can see the effects of the Dirichlet kernels thickening the spikes below. Also note that 2 of the spikes have half the magnitude of the others, since the G note is played only once.



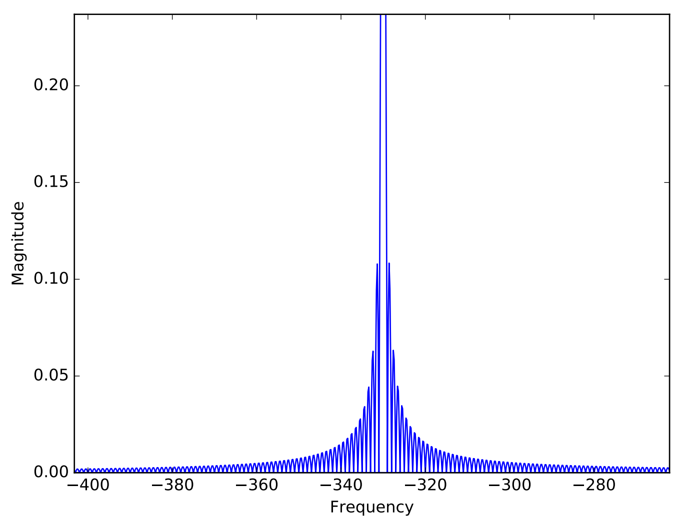
To understand this plot, note that the original signal can be thought of as a sum of 5 windowed notes. Below we zoom in on one of the spikes so you can see the effect of the Dirichlet convolution.



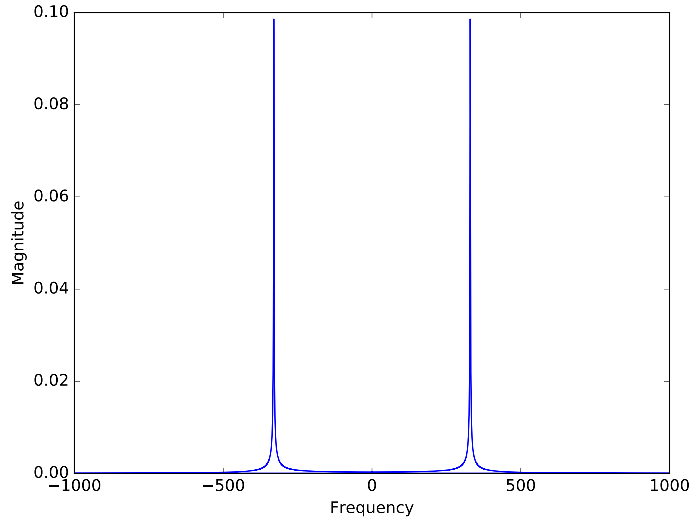
(d) Shown below. It is interesting to note that the plot below is obtained from the previous plot by convolving with a Dirichlet kernel. That kernel exactly cancels out the effects of 4 other notes due to its phase.



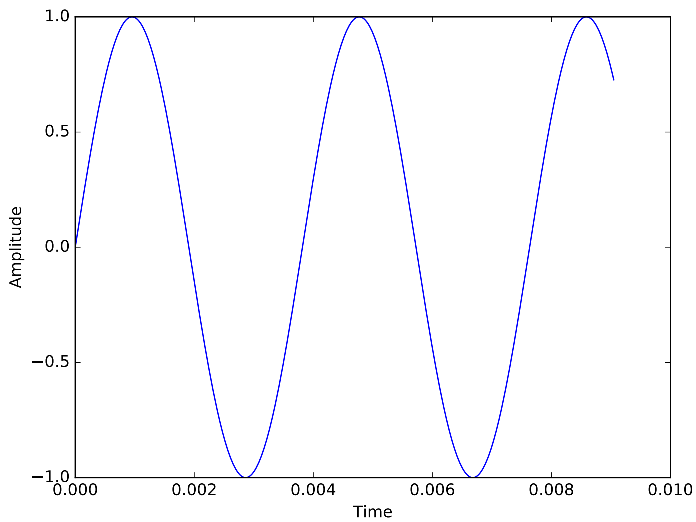
Below we zoom in so you can see the effect of the Dirichlet convolution.



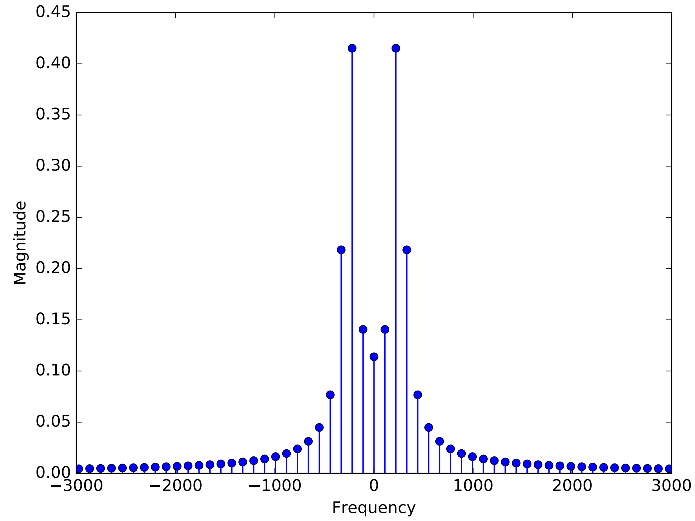
(e) Shown below.



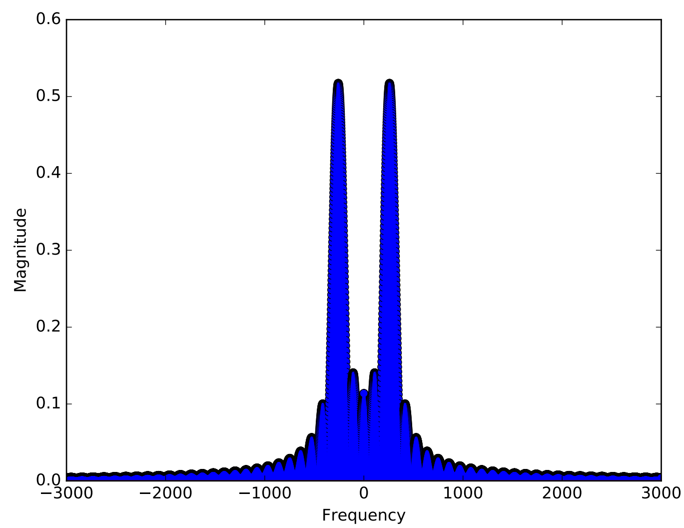
(f) We first cut out the first 400 samples and treat it as a vector in $\mathbb{R}^{400}$:



This gives the following frequencies:



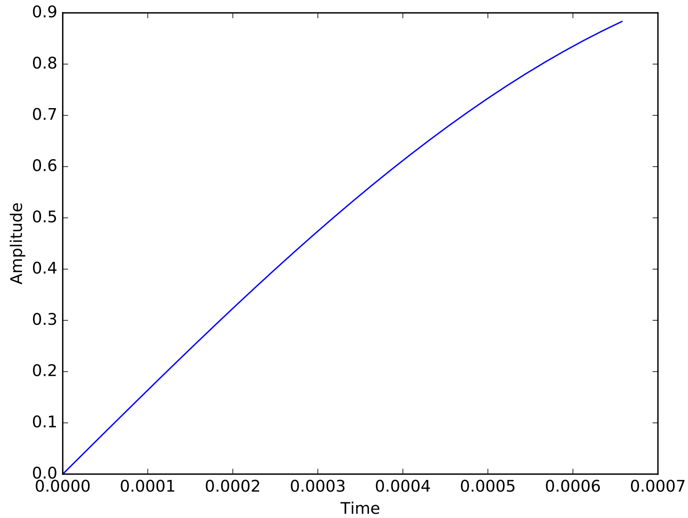
If instead we extract the first 400 samples using a window (zeroing out all other samples) we get:



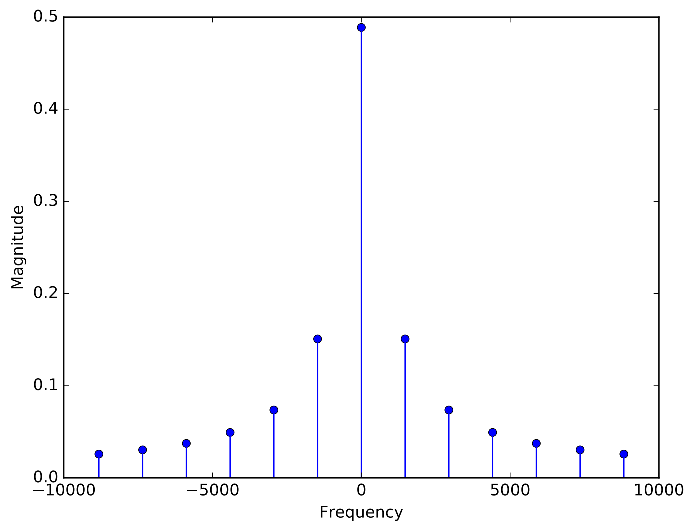
This is also the effect of a convolution with a Dirichlet kernel, but since our window is small (400 compared to the 44100 before), the resulting Dirichlet kernel is very fat (the central lobe has a large width).

Note that the extracted version is a "subsampled" version of this. [Aside: This can be seen by looking at the DFT of a spaced comb, and noting that the extracted version is the periodization of the windowed version. That is, the extracted version is obtained from the windowed version by convolving with the spaced comb.]

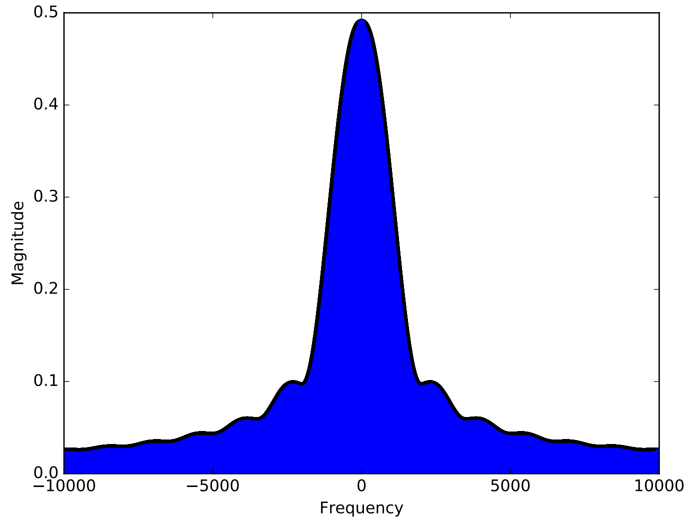
(g) We first cut out the first 30 samples and treat it as a vector in $\mathbb{R}^{30}$:



This gives the following frequencies:



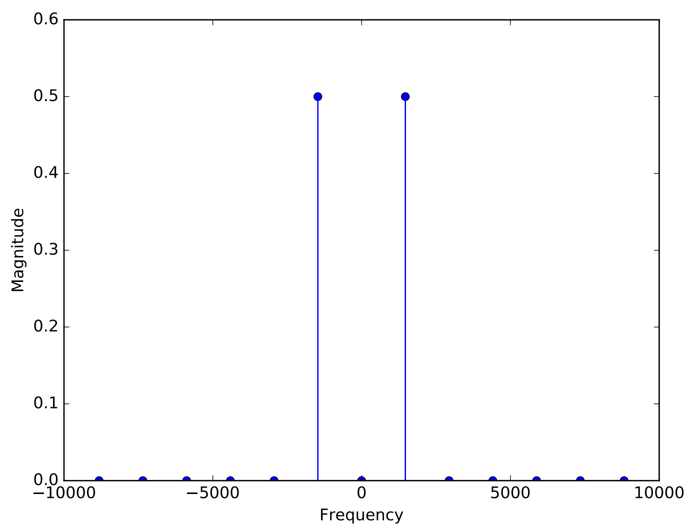
If instead we extract the first 30 samples using a window (zeroing out all other samples) we get:

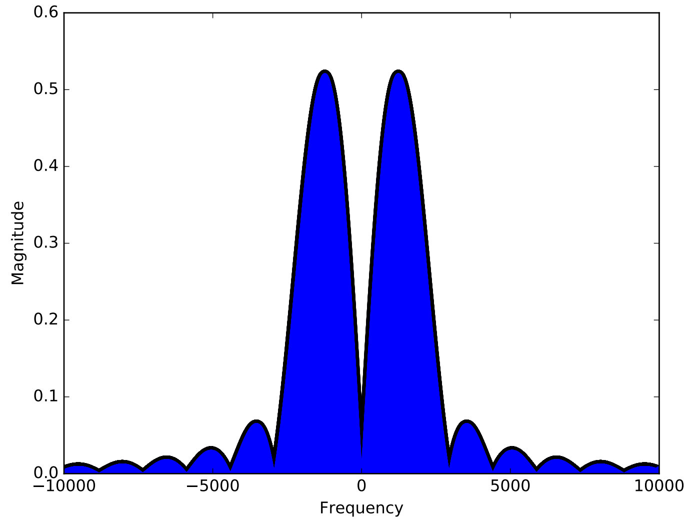


Since our window is so small, we are unable to detect "low" frequency signals that do not oscillate on the time frame of 30 samples. The smallest frequency we can see a full cycle of completes a cycle in $30/44100 = .00068$ seconds for a frequency of 1470 Hz. When you take the inner product of a slowly oscillating signal with much faster ones, you approximately get zero. This is why we only see zero.

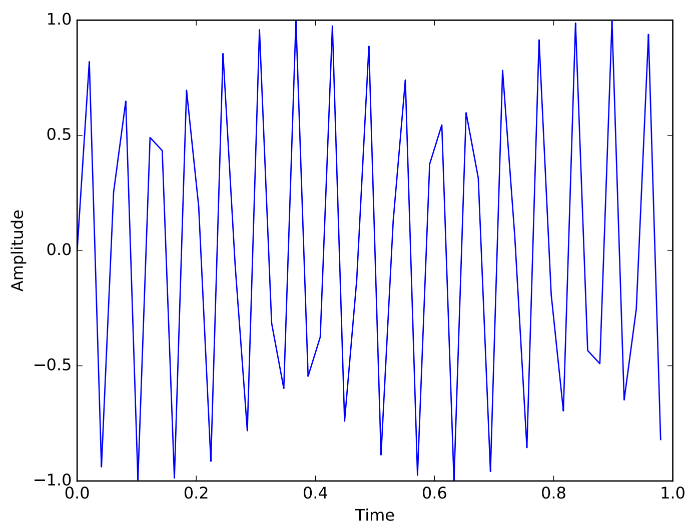
As an alternative way to understand this, the resulting Dirichlet kernel we convolve with in frequency-space has a central lobe that is so wide that it "swallows" the frequency information of the low frequencies (the first zeros of the kernel occur at approximately 1470 Hz).

Below we show the same plots (first extracted, then windowed), but for a signal with frequency 1470 Hz to illustrate what we described above.

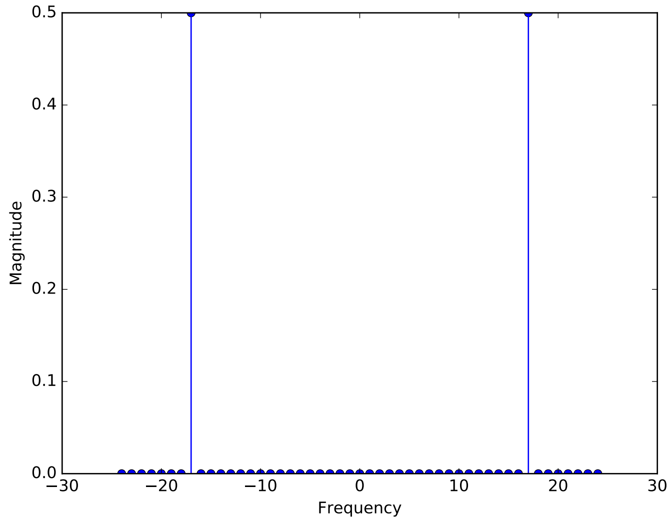




(h) To observe aliasing we would need to regularly subsample our signal over the full time interval. This is a different phenomenon from cutting out an excerpt. If we aggressively undersample, we will not be able to differentiate high frequency signals from low frequency signals. Below we draw a curve through our 49 samples of the data:



As you can see, it appears to be oscillating much more slowly than it should be. Below we compute the Fourier coefficients.



We exactly see two spikes at 17 which is 262 modulo 49. This also shows why our plot of the signal above isn't perfectly sinusoidal. It is because 17 and 49 are relatively prime, so our cycles don't line up well with the subsampling.