Recitation 9

DS-GA 1013 Mathematical Tools for Data Science

A $*$ is born [1]

Let $x[n]$ and $y[n]$ with $n \in \mathbb{Z}$ be two discrete time signals (essentially just think of them as two array of numbers). The convolution between them is a new signal y defined as $y := x * h$ given by

$$y[k] := \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{1}$$

1. Consider two sequence $x[n] = [3, 1, 2]$ and $h[n] = [3, 2, 1]$. The indexing of both the sequences starts from $n = 0$. Find the convolution $x * h$ by using equation 1. Start by filling in the following table:

k	-3	-2	-1	0	1	2	3	4	5	6	$\sum_{k=-\infty}^{\infty} x[k]h[n-k]$
$x[k]$											
$h[-1-k]$											
$h[-k]$											
$h[1-k]$											
$h[2-k]$											
$h[3-k]$											
$h[4-k]$											
$h[5-k]$											
$h[6-k]$											

2. Describe $h[-k]$ and $h[n-k]$ for a given n.

Solution: $h[-k]$ is time reversed signal and $h[n-k]$ is time reversed signal shifted to the right by n.

3. Let $x[n]$ have a width of w_x, that is it takes indices $n = 0, \ldots, w_x - 1$ and $h[n]$ have a width of w_h, i.e it takes indices $n = 0, \ldots, w_h - 1$. What are the indices that the signal $y = x * h$ takes and what is the width of the signal?

Solution: $w_y = w_x + w_h - 2$

4. Are the following properties true?

[1] Title shamelessly lifted off from Osgood's book on Fourier transform.

1. $x * h = h * x$
2. $x * (h_1 + h_2) = x * h_1 + x * h_2$
3. $x * (\alpha h) = (\alpha x) * h = \alpha(x * h)$ for some $\alpha \in \mathbb{R}$

Solution: All of these properties are true.

5. Is convolution is a linear operation? If yes, represent represent the operation in question 1 as $y = Cx$ where y and x are vectors and C is a matrix that will depend on h.

Solution: Yes, it's a linear operation. The table in question 1 is already in the form of a matrix.

6. Consider that the signals x and h in question 1 are periodic with period $N = 3$, that is $x[n + kN] = x[n]$ for any $k \in \mathbb{Z}$. Considering that x and h are periodic with $N = 4$ fill the following table. The convolution between x and h considering that they're periodic is called circular convolution.

k	-3	-2	-1	0	1	2	3	4	5	6	$\sum_{k=-\infty}^{\infty} x[k]h[n-k]$
$x[k]$											
$h[-1-k]$											
$h[-k]$											
$h[1-k]$											
$h[2-k]$											
$h[3-k]$											
$h[4-k]$											
$h[5-k]$											
$h[6-k]$											

7. Is the result of circular convolution periodic?

Solution: yes

8. Write the matrix for circulation convolution matrix like you did in the question before.

9. The analogue of equation 1 for the case when the signals are continuous is equation 2. For two continuous signals f and g, the convolution between them is defined as

$$(f * g)(t) := \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau. \tag{2}$$

Let $f(t) = 1$ for $-1 \leq t \leq 1$. Find $f * f$. Use a compute to see what happens if you keep convolving f with itself $f * f * f * \ldots$.

Solution: It should look like a triangle. Cascaded convolution will make it look like a gaussian - central limit theorem.

2D Convolutions

if x and h are discrete 2 dimensional signals, the convolution between them is defined as

$$x(n_1, n_2) * *h(n_1, n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} h(k_1, k_2)x(n_1 - k_1, n_2 - k_2)$$

10. Describe the action of the following filters to an image:

1. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 2. $\begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ 3. $\begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ 4. $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ 5. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ 6. $\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$

It might be helpful to sum the entries of the following filters when thinking about them:

7. $\begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ 8. $\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ 9. $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ 10. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 5 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ 11. $\begin{bmatrix} -1 & -1 & -1 \\ -1 & 9 & -1 \\ -1 & -1 & -1 \end{bmatrix}$

12. $\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ 13. $\frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$