Optimization-Based Data Analysis

Recitation 2

1. For a fixed matrix $A \in \mathbb{R}^{m\times n}$ arrange the following values in order: $\|A\|_F$, $\|A\|$, $\|A\|_*$.

2. Let $A \in \mathbb{R}^{m\times m}$ with SVD $A = USV^T$.

   (a) Assuming A is invertible, give the SVD for A^{-1}.

   (b) Give the SVD for A^T.

   (c) What is the relationship between the SVD of a symmetric matrix and the diagonal factorization given by the spectral theorem?

   (d) Give the SVD for $A = \mathcal{P}_{\mathcal{S}}$, the orthogonal projection onto the subspace $\mathcal{S} \subseteq \mathbb{R}^m$.

3. Suppose you are given a dataset $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^m$ as columns of a matrix $X \in \mathbb{R}^n$. Your goal is to reduce the dimensionality of the data using PCA.

   (a) Suppose you want the resulting reduced vectors to be in $\mathbb{R}^k$. Explain how to obtain this using PCA.

   (b) How do you determine an appropriate k?

   (c) How do you determine the amount of sample variance in the first principal direction?

4. Let $A \in \mathbb{R}^{m\times n}$. Find maximizers $\vec{x} \in \mathbb{R}^m, \vec{y} \in \mathbb{R}^n$ solving

$$\begin{aligned} \text{maximize} \quad & \vec{x}^T A \vec{y} \\ \text{subject to} \quad & \|\vec{x}\|_2 = 1, \\ & \|\vec{y}\|_2 = 1. \end{aligned}$$

   Also give the maximum value obtained.

5. In the following, assume every day is equally likely to be a birthday, and that there are no leap years.

   (a) What is the probability, in terms of n, that at least 2 people in a room of n people have the same birthday?

   (b) Give an upper bound, in terms of n and k, that at least k people in a room of n people have the same birthday.

6. If $X \sim \mathcal{N}(0,1)$ then we say that $X^2 \sim \chi_1^2$ (called a chi-squared distribution with 1 degree of freedom). Give the pdf, mean, and variance of the χ_1^2 distribution.

7. What is the joint pdf of a random vector $X \sim \mathcal{N}(0, I)$ taking values in $\mathbb{R}^n$?

8. Let $A = \begin{bmatrix} 4 & -1 \\ 4 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Suppose $X \sim \mathcal{N}(0, I)$ takes values in $\mathbb{R}^2$, and let $Y = AX + \vec{b}$.

   (a) What is the distribution of Y?

   (b) What are the marginal distributions of the components of Y?

   (c) Are the components of Y independent?

   (d) What do the contour lines of the joint pdf Y look like?

9. Let $X_1, \ldots, X_n$ be i.i.d. random variables taking the values $-1, 0, +1$ with probabilities 1/3 each. Let X denote the random vector in $\mathbb{R}^n$ having X_i as its ith coordinate.

   (a) Compute $E[\|X\|_2^2]$.

   (b) Compute $E[\|X\|_\infty]$.

   (c) Compute the covariance matrix of X.

10. Let X be a random vector taking values in $\mathbb{R}^n$ with mean $\vec{\mu} \in \mathbb{R}^n$ and covariance matrix $\Sigma \in \mathbb{R}^{n \times n}$. If $A \in \mathbb{R}^{m \times n}$ and $\vec{b} \in \mathbb{R}^n$ what are the mean and covariance matrix of $AX + \vec{b}$?