Recitation 11

DS-GA 1013 Mathematical Tools for Data Science

1. Is convolution associative? That is, is $(\vec{x} * \vec{y}) * \vec{z} = \vec{x} * (\vec{y} * \vec{z})$ for all $\vec{x}, \vec{y}, \vec{z} \in \mathbb{C}^n$?

2.
    1. Suppose we plot the magnitude of the spectrum of a signal $\vec{x} \in \mathbb{C}^n$. Then we (circularly) time shift it, and plot the magnitude of the spectrum again. How will the two plots differ?
    2. Let's try to answer this stack overflow question: a) Why do we have a large intensity value in the middle for magnitude only reconstruction? b) Why does the phase only reconstruction look like the way it looks?

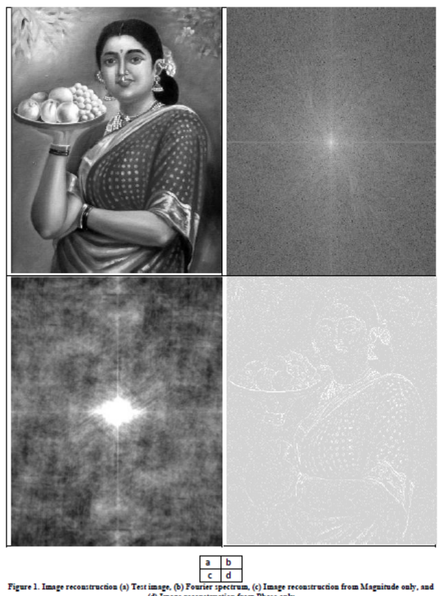


Figure 1. Image reconstruction (a) Test image, (b) Fourier spectrum, (c) Image reconstruction from Magnitude only, and (d) Image reconstruction from Phase only

    3. Suppose we have two images I_1 and I_2 and their DFTs $\hat{I}_1$ and $\hat{I}_2$. We create R_1 by superimposing the phase of $\hat{I}_2$ on the magnitude of $\hat{I}_1$ and R_2 by superimposing the phase of $\hat{I}_1$ on the magnitude of $\hat{I}_2$. How will R_1 and R_2 look like?

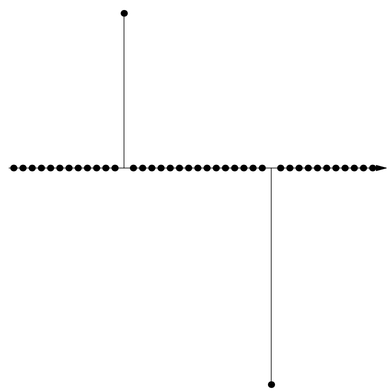
3. Let $\vec{1} \in \mathbb{C}^n$ denote the vector that is all ones. What is $\vec{x} * \vec{1}$ for $\vec{x} \in \mathbb{C}^n$? Can we deconvolve to get $\vec{x}$?

4.
    1. In the homework we blurred the signal by convolving with a Gaussian filter. Explain why this has a blurring effect, and what this suggests about its Fourier transform.
    2. Consider a 1-dimensional sequence that we want to "blur" or smooth by computing a moving average. We will replace entry $\vec{x}[j]$ with the following average:

$$\widetilde{\vec{x}}[j] := \frac{1}{2w+1} \sum_{k=-w}^{w} \vec{x}[j+k].$$

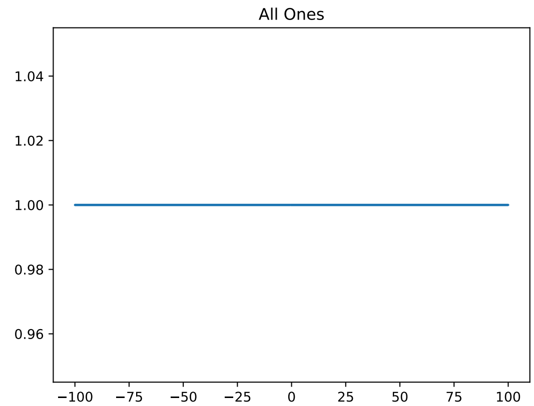
    How do you represent this as a convolution, and what do you expect the Fourier coefficients of the convolution filter to look like?

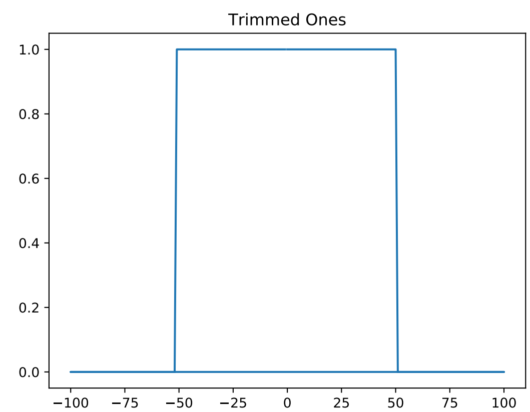
5. Suppose we are given the following vector $\vec{x} \in \mathbb{R}^{40}$:

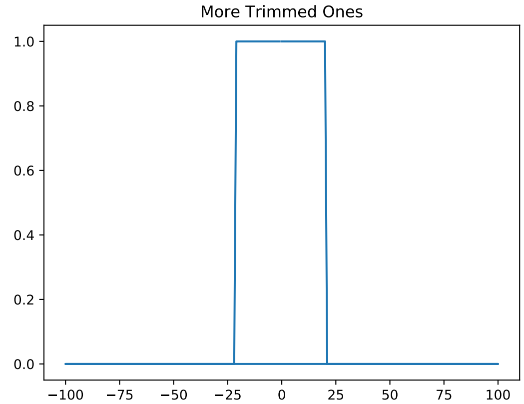


What does it look like when we convolve it with a sampled Gaussian or a Discrete sinc (Dirichlet) ?

6. What are the signals corresponding to the following DFT plots?

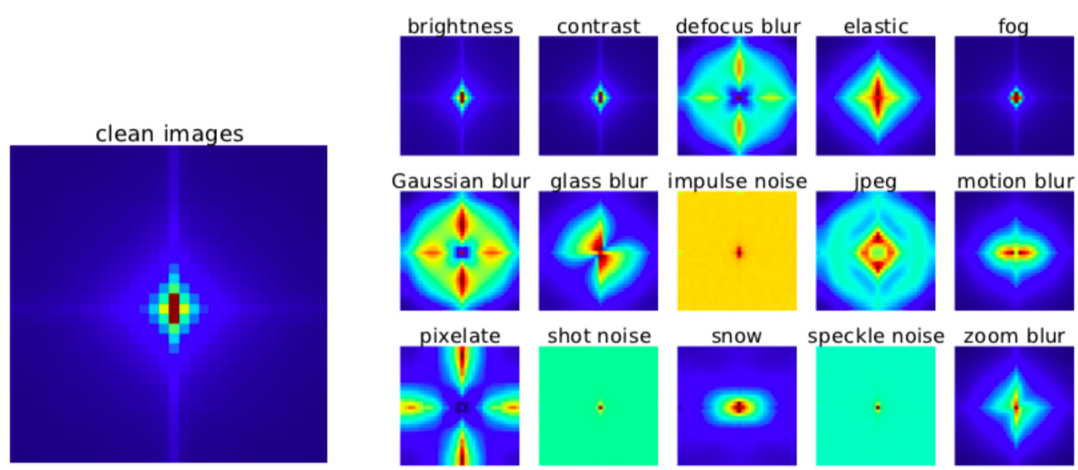






7. (Robustness to perturbations and data augmentation) We train a deep network to do image classification. We're interesting in understanding the robustness properties of the network. Particularly, we would like to know, what kind of perturbations in the frequency domain is this network robust to. (Disclaimer: a lot of this question is hand-wavy arguments)

    1. We perturb the image with Fourier basis vectors (of different frequency) with a fixed L2 norm. Which frequencies do you expect the network to be more robust to?
    2. Based on your answer to above, do you expect the network to be robust to the addition of Gaussian noise to the input?
    3. Gaussian data augmentation is a common trick in deep learning where the training data is augmented with input examples with Gaussian noise. How do you expect this to change your answer in your part(a)?
    4. The figure below shows the spectrum of different kind of corruptions found in a dataset called CIFAR10-C (We are plotting the spectrum of $C(X) - X$ where $C(X)$ is the corrupted version of X). When trained with Gaussian data augmentation, how do you expect robustness to change on speckle noise shot noise, defocus blur, Gaussian blur, fog and contrast?



8. For $\vec{x}, \vec{y} \in \mathbb{C}^n$ the (circular) convolution $\vec{x} * \vec{y} \in \mathbb{C}^n$ is defined by

$$(\vec{x} * \vec{y})[k] = \sum_{i=0}^{n-1} \vec{x}[i]\vec{y}[k-i]$$

where we treat negative indices as cyclic mod n (i.e., $v[-j] = v[-j+n]$).

   1. What is the DFT of $\vec{x} * \vec{y}$?
   2. What is the DFT of $\vec{x} \circ \vec{y}$ where $\circ$ denotes element-wise product?

9. Let $\omega_C = 262$, $\omega_E = 330$, $\omega_G = 392$ denote the (rounded) frequencies of the (middle) C, E and G notes (in Hz). An audio signal corresponding to a note is just a sine wave of the given frequency. We sample at a rate of 44100 samples per second.

   1. What will the FFT of a signal for the C-note look like?
   2. What will the FFT of a signal for the C,E,G-chord look like?
   3. Suppose we play C,E,G,E,C. What will the FFT of that signal look like?
   4. Now suppose we filter out only the first E in the sequence above (by setting the signal to zero for the other notes). What will the FFT look like now?
   5. Now suppose we convolve the signal with a Gaussian centered at the second E. What will the FFT look like now?
   6. Suppose we are playing a single C-note again, but we extract the first 400 samples. What will the FFT look like? [Note $44100/262 \approx 168$.]
   7. Suppose we are playing a single C-note again, but we extract the first 30 samples. What will the FFT look like?
   8. In what situation would we experience aliasing?