Recitation 8

DS-GA 1013 Mathematical Tools for Data Science

1. Let $x : \mathbb{R} \to \mathbb{C}$ be periodic with period 1 and square integrable on $[0, 1)$. For each of the following functions, give a formula for the Fourier coefficients in terms of the Fourier coefficients of x.

   1. $y_1 : \mathbb{R} \to \mathbb{C}$ with $y_1(t) = x(t - s)$ for some fixed $s \in \mathbb{R}$.
   2. $y_2 : \mathbb{R} \to \mathbb{C}$ with $y_2(t) = e^{2\pi ipt}x(t)$ for some fixed $p \in \mathbb{Z}$.
   3. $y_3 : \mathbb{R} \to \mathbb{C}$ with $y_3(t) = x(-t)$
   4. $y_4 : \mathbb{R} \to \mathbb{C}$ with $y_3(t) = x^*(t)$

2. Let $x(t)$ be a square wave, i.e $x(t) = 1 for 0 \leq t \leq 0.5 and -1 for 0.5 < t \leq 1$. What are the Fourier series coefficients for $x(t)$? Simulation. Discontinuity.

3. Let n be a positive integer and define $f : \mathbb{Z}^2 \to \mathbb{C}$ by

$$f(j, k) = e^{2\pi ijk/N}.$$

   1. Show that f is periodic with period N in both arguments. That is, show that

$$f(j + pN, k + qN) = f(j, k)$$

      for all $j, k, p, q \in \mathbb{Z}$.
   2. Let $\vec{\varphi}_j = (1, e^{2\pi ij/N}, \ldots, e^{2\pi i(N-1)j/N})^T \in \mathbb{C}^N$ for $j \in \mathbb{Z}$. When does $\vec{\varphi}_j = \vec{\varphi}_k$?

4. A matrix $A \in \mathbb{C}^{n \times n}$ is called unitary if $A^*A = I$.

   1. Prove that unitary matrices preserve inner products:

$$\langle x, y \rangle = \langle Ax, Ay \rangle$$

      for all $x, y \in \mathbb{C}^n$.
   2. Is the matrix $\widetilde{F}_{[N]} \in \mathbb{C}^{N \times N}, N = 2k_c + 1$ unitary?

$$\widetilde{F}_{[N]} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \exp\left(\frac{i2\pi(-k_c)}{N}\right) & \exp\left(\frac{i2\pi(-k_c+1)}{N}\right) & \cdots & \exp\left(\frac{i2\pi k_c}{N}\right) \\ \cdots & \cdots & \cdots & \cdots \\ \exp\left(\frac{i2\pi(-k_c)j}{N}\right) & \exp\left(\frac{i2\pi(-k_c+1)j}{N}\right) & \cdots & \exp\left(\frac{i2\pi k_c j}{N}\right) \\ \cdots & \cdots & \cdots & \cdots \\ \exp\left(\frac{i2\pi(-k_c)(N-1)}{N}\right) & \exp\left(\frac{i2\pi(-k_c+1)(N-1)}{N}\right) & \cdots & \exp\left(\frac{i2\pi k_c(N-1)}{N}\right) \end{bmatrix}$$

5. There is a signal x given by

$$x(t) = \sum_{k=-k_c}^{k_c} a_k e^{2\pi ikt},$$

   where k_c is known. Suppose we are given n samples

$$x(t_1), x(t_2), \ldots, x(t_n)$$

   where $0 \leq t_1 < t_2 < \cdots < t_n < 1$ need not be uniformly spaced.

1. Under what conditions can we exactly recover the a_k values, and how would this be done?
2. Suppose n is large, but the samples are corrupted by noise. Give a method for estimating the a_k values.