

## Homework 7

Due April 12 at 11 pm

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1. (Fourier coefficients and smoothness) Let  $x : \mathbb{R} \rightarrow \mathbb{C}$  be periodic with period 1 and let  $\hat{x}[k]$  denote the  $k$ th Fourier coefficient of  $x$ , for  $k \in \mathbb{Z}$  (computed on any interval of length 1).

(a) Suppose  $x$  is continuously differentiable. Prove that for  $k \neq 0$  we have

$$|\hat{x}[k]| \leq \frac{C_1}{|k|}$$

for some  $C_1 \geq 0$  that depends on  $x$  (but not on  $k$ ). [Hint: Integration by parts. Also note that

$$\left| \int_0^1 f(t) dt \right| \leq \int_0^1 |f(t)| dt < \infty$$

if  $f$  is continuous on  $[0, 1]$ .]

(b) Suppose  $x$  is twice continuously differentiable. Prove that for  $k \neq 0$  we have

$$|\hat{x}[k]| \leq \frac{C_2}{|k|^2}$$

for some  $C_2 \geq 0$  that depends on  $x$  (but not on  $k$ ).

2. (Sampling a sum of sinusoids) We are interested in a signal  $x$  belonging to the unit interval  $[0, 1]$  of the form

$$x(t) := a_1 \exp(i2\pi k_1 t) + a_2 \exp(i2\pi k_2 t), \quad (1)$$

where the amplitudes  $a_1$  and  $a_2$  are complex numbers, and the frequencies  $k_1$  and  $k_2$  are known integers. We sample the signal at  $N$  equispaced locations  $0, 1/N, 2/N, \dots, (N-1)/N$ , for some positive integer  $N$ .

- (a) What value of  $N$  is required by the Sampling Theorem to guarantee that we can reconstruct  $x$  from the samples?
- (b) Write a system of equations in matrix form mapping the amplitudes  $a_1$  and  $a_2$  to the samples  $x_N$ .
- (c) Under what condition on  $N$ ,  $k_1$  and  $k_2$  can we recover the amplitudes from the samples by solving the system of equations? Can  $N$  be smaller than the value dictated by the Sampling Theorem? If yes, give an example. If not, explain why.
- (d) What is the limitation of this approach, which could make it unrealistic?
3. (Sampling theorem for bandpass signals) Bandpass signals are signals that have nonzero Fourier coefficients only in a fixed band of the frequency domain. We are interested in sampling a bandpass signal  $x$  belonging to the unit interval  $[0, 1]$  that has nonzero Fourier-series coefficients between  $k_1$  and  $k_2$ , inclusive, where  $k_1$  and  $k_2$  are known positive integers such that  $k_2 > k_1$ .

- (a) We sample the signal at  $N$  equispaced locations  $0, 1/N, 2/N, \dots, (N-1)/N$ . What value of  $N$  is required by the Sampling Theorem to guarantee that we can reconstruct  $x$  from the samples?
  - (b) Assume that  $k_2 := k_1 + 2\tilde{k}_c$ , where  $\tilde{k}_c$  is a positive integer. For any  $N \geq 2\tilde{k}_c + 1$  it is possible to recover the signal from the samples. Explain why (you don't need to derive any explicit expressions).
  - (c) Assume that  $k_2 := k_1 + 2\tilde{k}_c$ ,  $N \geq 2\tilde{k}_c + 1$ , and  $mN = k_1 + \tilde{k}_c$  for some integer  $m$ . Explain precisely how to recover  $x$  from the samples in this case.
4. (Frequency analysis of musical notes) In this exercise you will use the code and data in the `musicdata` folder. Make sure you have the python packages `sklearn`, `pandas`, `sounddevice`, and `soundfile` installed. The skeleton code for you to work with is given in `analysis.py` which uses tools given in `music_tools.py`. The data used here comes from the NSynth dataset.
- (a) Plot the audio signals for the first signal in the training set, and the first vocal signal in the training set (i.e., the first signal whose `instrument_family_str` field is 'vocal' in the dataframe). In the titles of your two plots, include the `instrument_family_str` and the frequency (in Hz). We recommend you also use `play_signal` to hear what the signals sound like.
  - (b) For each signal in the test set, compute the (strictly positive) frequency with the largest amplitude (in absolute value), and convert it to a pitch number (using the tools in `music_tools`). This will be our predicted pitch.
    - i. Report what overall fraction of the signals in the test set you accurately predict using this method (i.e., your overall accuracy).
    - ii. For the first two signals you misclassify (in the order they occur in the test set), give plots of their absolute DFT coefficients (use `np.fft.fft` and make one plot per signal). In the title of your plots, include the `instrument_family_str`, the true frequency, and the predicted frequency (in Hz). Make sure to plot the coefficients on an axis centered at 0 by using `fftfreq` with the correct arguments.
    - iii. What is the instrument family for which the method got the highest fraction of incorrect predictions (i.e., number incorrect divided by number of examples from that family)?
  - (c) Use the `LogisticRegression` class in `sklearn` to fit a pitch classifier on the training set using the absolute DFT coefficients as the features. Use the default parameters but set `multi_class` to 'multinomial' and `solver` to 'lbfgs'. Note: We will use the negative frequencies as well for convenience, even though they have the same magnitudes as the positive (the  $L_2$  regularization will take care of it for us).
    - i. Report your score on the test set as computed by the model.
    - ii. Give 3 plots of the model coefficients for pitches 60, 65, and 72. Make sure to plot the coefficients on an axis centered at 0 by using `fftfreq` with the correct arguments (because the coefficients correspond to frequencies).