Homework 9

Solutions

1. (Real discrete sinusoids)

(a)

$$\begin{aligned}
2\cos(x+y) &= \exp(i(x+y)) + \exp(-i(x+y)) && (1)\\
&= \exp(ix)\exp(iy) + \exp(-ix)\exp(-iy) && (2)\\
&= \cos(x)\cos(y) - \sin(x)\sin(y) + i\cos(x)\sin(x) + i\sin(x)\cos(x) && (3)\\
&+ \cos(-x)\cos(-y) - \sin(-x)\sin(-y) + i\cos(-x)\sin(-x) + i\sin(-x)\cos(-x) \\
&= 2\cos(x)\cos(y) - 2\sin(x)\sin(y). && (4)
\end{aligned}$$

Similarly,

$$\begin{aligned}
2i\sin(x+y) &= \exp(i(x+y)) - \exp(-i(x+y)) && (5)\\
&= \exp(ix)\exp(iy) - \exp(-ix)\exp(-iy) && (6)\\
&= \cos(x)\cos(y) - \sin(x)\sin(y) + i\cos(x)\sin(x) + i\sin(x)\cos(x) && (7)\\
&- \cos(-x)\cos(-y) + \sin(-x)\sin(-y) - i\cos(-x)\sin(-x) - i\sin(-x)\cos(-x) \\
&= 2i\cos(x)\sin(y) + 2i\sin(x)\cos(y). && (8)
\end{aligned}$$

(b) By the previous question

$$\begin{aligned}
\cos(x)\cos(y) &= \frac{\cos(x+y)+\cos(x-y)}{2}, && (9)\\
\sin(x)\sin(y) &= \frac{\cos(x-y)-\cos(x+y)}{2}, && (10)\\
\cos(x)\sin(y) &= \frac{\sin(x+y)-\sin(x-y)}{2}. && (11)
\end{aligned}$$

As a result,

$$\begin{aligned}
\sum_{j=0}^{N-1} \cos\left(\frac{2\pi k_1 j}{N}\right)\cos\left(\frac{2\pi k_2 j}{N}\right) &= \frac{1}{2}\sum_{j=0}^{N-1} \cos\left(\frac{2\pi(k_1+k_2)j}{N}\right) + \cos\left(\frac{2\pi(k_1-k_2)j}{N}\right), && (12)\\
\sum_{j=0}^{N-1} \sin\left(\frac{2\pi k_1 j}{N}\right)\sin\left(\frac{2\pi k_2 j}{N}\right) &= \frac{1}{2}\sum_{j=0}^{N-1} \cos\left(\frac{2\pi(k_1-k_2)j}{N}\right) - \cos\left(\frac{2\pi(k_1+k_2)j}{N}\right), && (13)\\
\sum_{j=0}^{N-1} \cos\left(\frac{2\pi k_1 j}{N}\right)\sin\left(\frac{2\pi k_2 j}{N}\right) &= \frac{1}{2}\sum_{j=0}^{N-1} \sin\left(\frac{2\pi(k_1+k_2)j}{N}\right) + \sin\left(\frac{2\pi(k_1-k_2)j}{N}\right). && (14)
\end{aligned}$$

We have

$$\exp\left(\frac{2\pi(k_1+k_2)j}{N}\right) = \cos\left(\frac{2\pi kj}{N}\right) + i\sin\left(\frac{2\pi kj}{N}\right) \qquad (15)$$

and

$$\sum_{j=0}^{N-1} \exp\left(\frac{2\pi k j}{N}\right) = 0 \tag{16}$$

unless $k \neq 0 \mod N$ (see Lemma 3.2 in the notes on Fourier). This implies that all the sums above are zero unless $k_1 = k_2 \mod N$, which in turn implies that c_0, c_1, ..., $c_{\frac{N-1}{2}}$, s_1, ..., $s_{\frac{N-1}{2}}$ are al orthogonal. When $k_1 = k_2$

$$\sum_{j=0}^{N-1} \cos^2\left(\frac{2\pi k_1 j}{N}\right) = \frac{1}{2}\sum_{j=0}^{N-1} \cos\left(\frac{2\pi 2 k_1 j}{N}\right) + 1 \tag{17}$$

$$= \frac{N}{2}, \tag{18}$$

$$\sum_{j=0}^{N-1} \sin^2\left(\frac{2\pi k_1 j}{N}\right) = \frac{1}{2}\sum_{j=0}^{N-1} 1 - \cos\left(\frac{2\pi (k_1 + k_2) j}{N}\right) \tag{19}$$

$$= \frac{N}{2}. \tag{20}$$

This shows that the vectors have ℓ_2 norm equal to 1.

2. (PCA of stationary vector)

(a) Recall that $\cos(-t) = \cos(t)$ and $\sin(t) = -\sin(-t)$ for any $t \in \mathbb{R}$, so

$$\cos\left(\frac{2\pi k j}{N}\right) = \cos\left(\frac{2\pi k (N-j)}{N}\right) \tag{21}$$

$$= \cos\left(\frac{2\pi (N-k) j}{N}\right), \tag{22}$$

$$\sin\left(\frac{2\pi k j}{N}\right) = -\sin\left(\frac{2\pi k (N-j)}{N}\right) \tag{23}$$

$$= -\sin\left(\frac{2\pi (N-k) j}{N}\right). \tag{24}$$

In addition, by definition of the autocovariance

$$a_{\tilde{x}}[j] = \mathrm{ac}_{\tilde{x}}(j \bmod N) \tag{25}$$

$$= \mathrm{E}\left(\tilde{x}[j]\tilde{x}[0]\right) \tag{26}$$

$$= \mathrm{E}\left(\tilde{x}[0]\tilde{x}[j]\right) \tag{27}$$

$$= \mathrm{ac}_{\tilde{x}}(-j \bmod N) \tag{28}$$

$$= \mathrm{ac}_{\tilde{x}}(N - j \bmod N) \tag{29}$$

$$= a_{\tilde{x}}[N-j]. \tag{30}$$

We have

$$\hat{a}_{\tilde{x}}[k] = \sum_{j=0}^{N-1} a_{\tilde{x}}[j] \exp\left(\frac{-i 2\pi k j}{N}\right), \tag{31}$$

so the real part satisfies

$$
\begin{aligned}
\operatorname{Re}\left(\hat{a}_{\tilde{x}}[k]\right) &= \sum_{j=0}^{N-1} a_{\tilde{x}}[j] \cos\left(\frac{-2\pi kj}{N}\right) && (32)\\
&= \sum_{j=0}^{N-1} a_{\tilde{x}}[j] \cos\left(\frac{-2\pi (N-k)j}{N}\right) && (33)\\
&= \operatorname{Re}\left(\hat{a}_{\tilde{x}}[N-k]\right), && (34)
\end{aligned}
$$

and the imaginary part is zero,

$$
\begin{aligned}
\operatorname{Im}\left(\hat{a}_{\tilde{x}}[k]\right) &= \sum_{j=0}^{N-1} a_{\tilde{x}}[j] \sin\left(\frac{-2\pi kj}{N}\right) && (35)\\
&= \sum_{j=1}^{\frac{N-1}{2}} -a_{\tilde{x}}[j] \sin\left(\frac{2\pi kj}{N}\right) - a_{\tilde{x}}[N-j] \sin\left(\frac{2\pi k(N-j)}{N}\right) && (36)\\
&= \sum_{j=1}^{\frac{N-1}{2}} -a_{\tilde{x}}[j] \sin\left(\frac{2\pi kj}{N}\right) + a_{\tilde{x}}[N-j] \sin\left(\frac{2\pi kj}{N}\right) && (37)\\
&= 0. && (38)
\end{aligned}
$$

(b) This is a direct consequence of expanding out the product in Corollary 4.4,

$$
\begin{aligned}
\Sigma_{\tilde{x}}[j_2, j_1] &= \frac{1}{N} \sum_{k=0}^{N-1} \hat{a}_{\tilde{x}}[k] \exp\left(-\frac{2\pi k j_1)}{N}\right) \exp\left(\frac{2\pi k j_2}{N}\right) && (39)\\
&= \frac{1}{N} \sum_{k=0}^{N-1} \hat{a}_{\tilde{x}}[k] \exp\left(\frac{2\pi k (j_2 - j_1)}{N}\right). && (40)
\end{aligned}
$$

(c)

$$
\begin{aligned}
\Sigma_{\tilde{x}}[j_2, j_1] &= \frac{1}{N} \sum_{k=0}^{N-1} \hat{a}_{\tilde{x}}[k] \exp\left(\frac{i2\pi k (j_2 - j_1)}{N}\right) && (41)\\
&= \frac{\hat{a}_{\tilde{x}}[0]}{N} + \frac{1}{N} \sum_{k=1}^{\frac{N-1}{2}} \hat{a}_{\tilde{x}}[k] \exp\left(\frac{i2\pi k (j_2 - j_1)}{N}\right) + \hat{a}_{\tilde{x}}[N-k] \exp\left(\frac{i2\pi (N-k) (j_2 - j_1)}{N}\right)\\
&= \frac{\hat{a}_{\tilde{x}}[0]}{N} + \frac{1}{N} \sum_{k=1}^{\frac{N-1}{2}} \hat{a}_{\tilde{x}}[k] \left(\exp\left(\frac{i2\pi k (j_2 - j_1)}{N}\right) + \exp\left(-\frac{i2\pi k (j_2 - j_1)}{N}\right)\right) && (42)\\
&= \hat{a}_{\tilde{x}}[0] + \frac{2}{N} \sum_{k=1}^{\frac{N-1}{2}} \hat{a}_{\tilde{x}}[k] \cos\left(\frac{2\pi k (j_2 - j_1)}{N}\right) && (43)\\
&= \frac{\hat{a}_{\tilde{x}}[0]}{N} + \frac{2}{N} \sum_{k=1}^{\frac{N-1}{2}} \hat{a}_{\tilde{x}}[k] \left(\cos\left(\frac{2\pi k j_1}{N}\right) \cos\left(\frac{2\pi k j_2}{N}\right) + \sin\left(\frac{2\pi k j_1}{N}\right) \sin\left(\frac{2\pi k j_2}{N}\right)\right).
\end{aligned}
$$

This yields an eigendecomposition where $\lambda_1 = \hat{a}_{\tilde{x}}[0]$, and $\lambda_k = \lambda_{N-k} = \hat{a}_{\tilde{x}}[k]$ for $k = 1, \ldots, N-1$. The corresponding eigenvectors are

$$u_1 = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix}, \tag{44}$$

$$u_k = \sqrt{\frac{2}{N}} \begin{bmatrix} 1 \\ \cos\left(\frac{2\pi k}{N}\right) \\ \cdots \\ \cos\left(\frac{2\pi k(N-1))}{N}\right) \end{bmatrix}, \quad 1 \leq k \leq \frac{N-1}{2}, \tag{45}$$

$$u_k = \sqrt{\frac{2}{N}} \begin{bmatrix} 1 \\ \sin\left(\frac{2\pi k}{N}\right) \\ \cdots \\ \sin\left(\frac{2\pi k(N-1))}{N}\right) \end{bmatrix}, \quad \frac{N-1}{2} + 1 \leq k \leq N-1. \tag{46}$$

By Problem 1, these vectors are orthonormal.

3. (Discrete filter) If $x * y = x$ for all bandlimited vectors with cut-off frequency k_c, that means $\hat{x} \circ \hat{y} = \hat{x}$, so for $|k| \leq |k|$

$$\hat{x}[k]\hat{y}[k] = \hat{x}[k] \tag{47}$$

for any $\hat{x}[k]$. This is achieved by any vector such that $\hat{y} = 1$ for $|k| \leq |k|$. Recall that F $\frac{1}{N}F^*F = I$ where F is the DFT matrix, so

$$\begin{aligned} ||y||_2^2 &= y^*y && (48) \\ &= \frac{1}{N} y^*F^*Fy && (49) \\ &= \frac{1}{N} ||\hat{y}||_2^2 . && (50) \end{aligned}$$

Setting any of the Fourier coefficients corresponding to $|k| > k_c$ to nonzero values just increases the ℓ_2 norm. Therefore $\hat{y}[k]$ is equal to one if $|k| \leq k_c$ and zero otherwise. To derive y we compute the inverse DFT:

$$\begin{aligned} y[j] &= \frac{1}{N} \sum_{k=-k_c}^{k_c} \exp\left(\frac{i2\pi kj}{N}\right) && (51) \\ &= \frac{\exp\left(-\frac{i2\pi k_c j}{N}\right) - \exp\left(\frac{i2\pi(k_c+1)j}{N}\right)}{N\left(1 - \exp\left(\frac{i2\pi j}{N}\right)\right)} && (52) \\ &= \frac{\exp\left(\frac{i\pi j}{N}\right)\left(\exp\left(-\frac{i2\pi(k_c+1/2)j}{N}\right) - \exp\left(\frac{i2\pi(k_c+1/2)j}{N}\right)\right)}{N\exp\left(\frac{i\pi j}{N}\right)\left(\exp\left(-\frac{i\pi j}{N}\right) - \exp\left(\frac{i\pi j}{N}\right)\right)} && (53) \\ &= \frac{\sin\left(\frac{i2\pi(k_c+1/2)j}{N}\right)}{N\sin\left(\frac{i\pi j}{N}\right)}. && (54) \end{aligned}$$

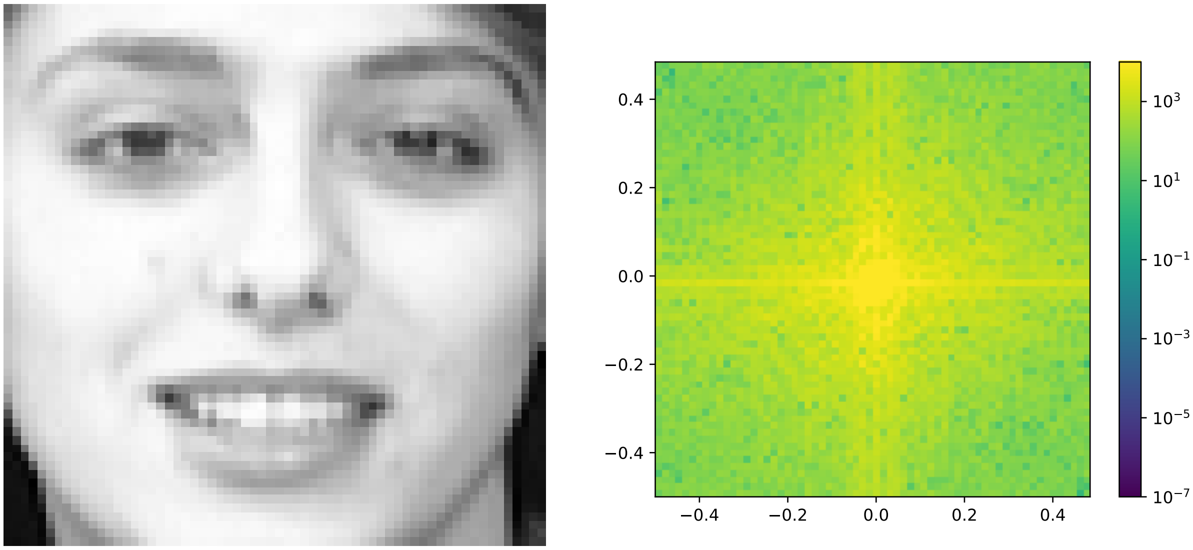

Figure 1: Result of deconvolving blur image.

4. (Deconvolution)

(a) Since convolution is multiplication in the Fourier domain, we can recover the true image by dividing the FFT of the blurred image with that of the FFT of blur kernel. Results are shown in figure 1

(b) The result is catastrophic, as you can see in Figure 2. The reason is obvious when we look at this in the frequency domain (Figure 3). The high-end of the spectrum of the noisy blurred image is dominated by the noise. The deconvolution filter amplifies this high-frequency noise drowning out the actual image! In order to avoid this effect while deconvolving it is necessary to take into account the ratio between the noise level and the signal level at each frequency. Wiener filtering is a principled way of doing this if we can have a prior estimate of the spectral statistics of the signal and the noise as we will see in next question.

(c) From theorem 5.3 in notes: Let $\tilde{x}$ and $\tilde{y}$ be N-dimensional zero-mean random vectors that are jointly stationary, and let $\tilde{x}_F$ and $\tilde{y}_F$ denote the DFT coefficients of $\tilde{x}$ and $\tilde{y}$ respectively. The linear estimate of $\tilde{y}$ given $\tilde{x}$ that minimizes MSE can be computed by convolving $\tilde{x}$ with the Wiener filter w, which is defined as having DFT coefficients equal to

$$\hat{w}[k] := \frac{\text{Cov}(\tilde{x}_F[k], \tilde{y}_F[k])}{\text{Var}(\tilde{x}_F[k])}, \quad 0 \leq k \leq N-1, \tag{55}$$

Here,

$$\begin{aligned}\text{Cov}(\tilde{x}_F[k], \tilde{y}_F[k]) &= \text{Cov}(b_F[k]\tilde{y}_F[k] + \tilde{z}_F[k], \tilde{y}_F[k]) \\ &= \text{Cov}(b_F[k]\tilde{y}_F[k], \tilde{y}_F[k]) + \text{Cov}(\tilde{z}_F[k], \tilde{y}_F[k]) \\ &= b_F[k]\text{Var}(\tilde{y}_F[k])\end{aligned}$$

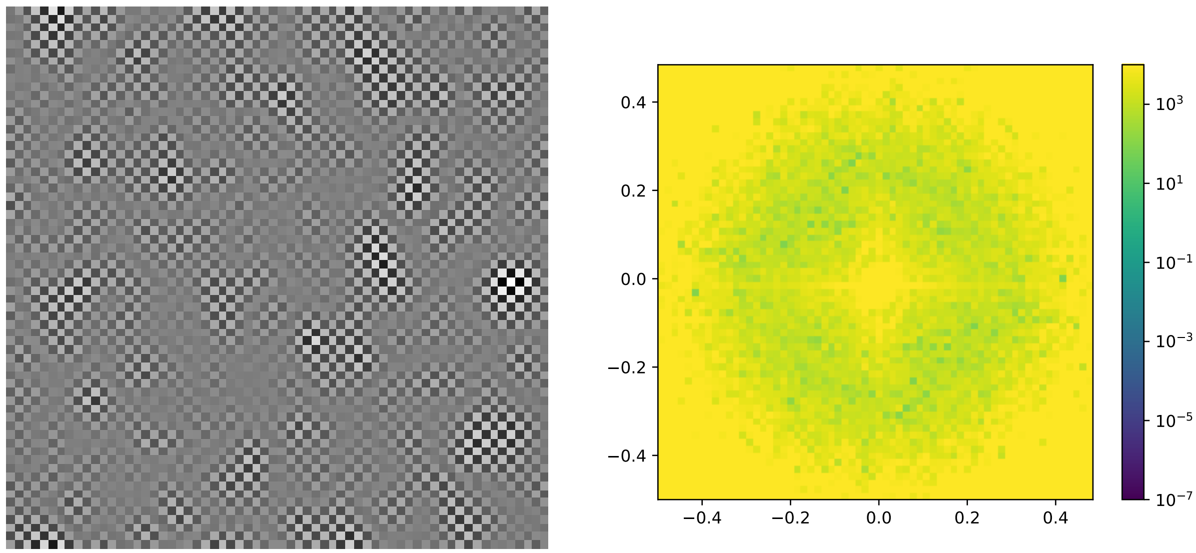

Figure 2: Result of deconvolving blured noisy image.

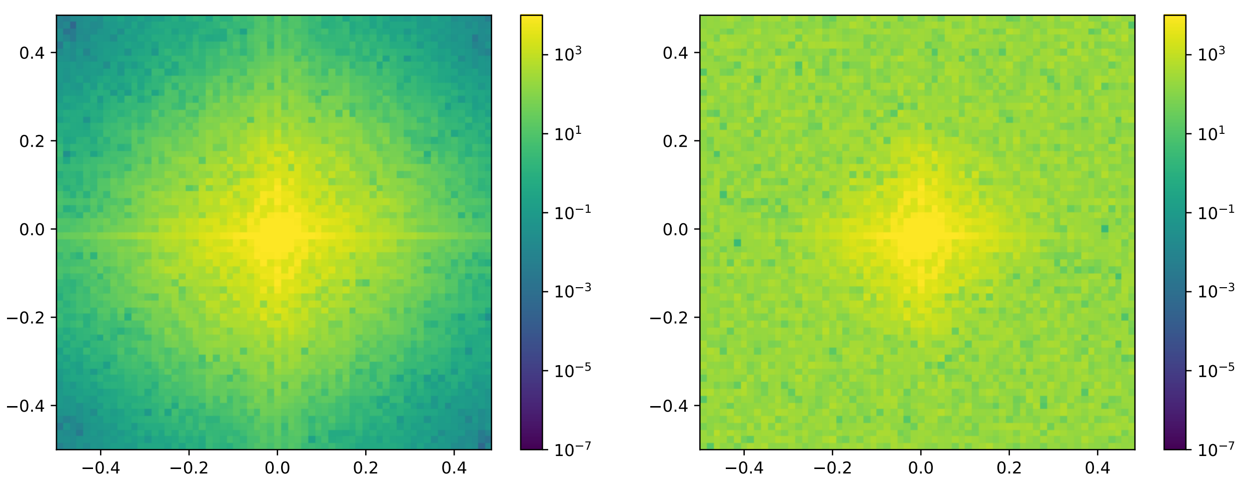

Figure 3: The FFT of blurred image in part (a) (left) and the noisy blurred image in part (b) (right). The high-end of the spectrum of the noisy blurred image is dominated by the noise.

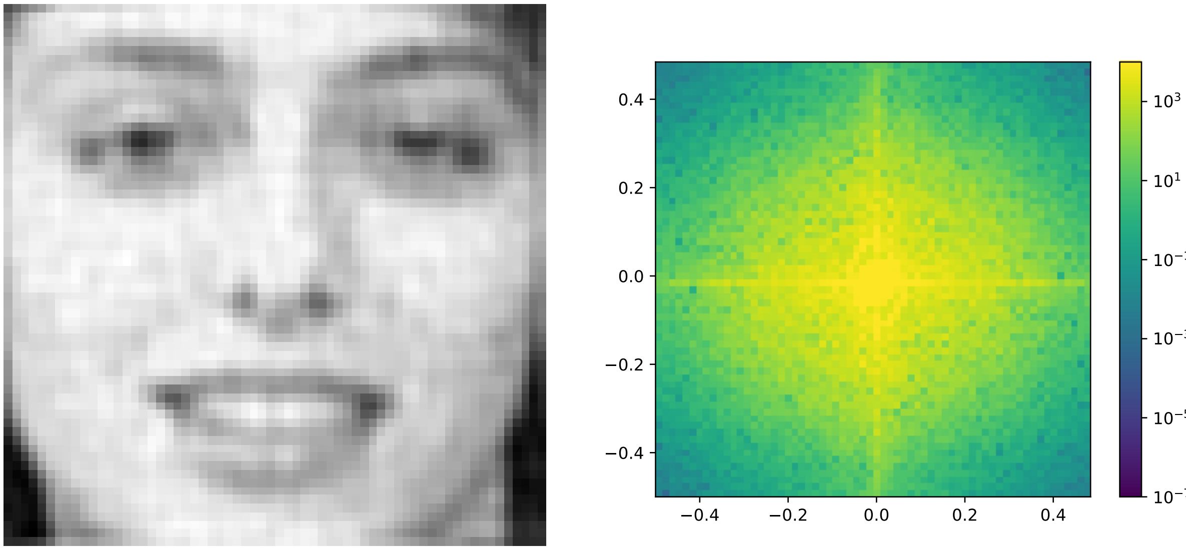

Figure 4: Results of Wiener deconvolution

Last step uses the same reasoning as that of example 5.5.

$$\begin{aligned}
\mathrm{Var}(\tilde{x}_F[k]) &= \mathrm{Var}(b_F[k]\tilde{y}_F[k]) + \mathrm{Var}(\tilde{z}_F[k]) \\
&= |b_F[k]|^2 \mathrm{Var}(\tilde{y}_F[k]) + N\sigma^2
\end{aligned}$$

(d) Here, the signals are not mean zero. So we should take care of that. Results are shown in Figure 4.