Homework 1

Solutions

1. (Rotation) Let $u_1, \ldots, u_n$ be the eigenvectors of A and $\lambda_1, \ldots, \lambda_n$ its eigenvalues. We have

$$x^T A x = x^T \sum_{i=1}^{n} \lambda_i u_i^T x u_i \tag{1}$$

$$= \sum_{i=1}^{n} \lambda_i (u_i^T x)^2. \tag{2}$$

Let us express x in the basis of eigenvectors, $x = \sum_{i=1}^{n} \alpha_i u_i$. Then $x^T A x = 0$ if

$$\sum_{i=1}^{n} \lambda_i \alpha_i^2 = 0. \tag{3}$$

If at least one of the eigenvalues λ_k is positive and at least one of the eigenvalues λ_l is negative, then we can set $\alpha_k = 1$, $\alpha_l = -\lambda_k/\lambda_l$, and all other coefficients to zero to make the whole sum equal zero. This implies that x and Ax are orthogonal. Otherwise, it is impossible to make the sum equal zero with setting all coefficients to zero because all terms are positive or negative.

2. (Matrix decomposition)

(a) The inner product equals

$$\operatorname{tr}\left(A^T B\right) = \sum_{i=1}^{n} (A^T B)_{ii} \tag{4}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} A_{ji} B_{ji} \tag{5}$$

$$= \operatorname{vec}(A)^T \operatorname{vec}(B), \tag{6}$$

i.e. the dot product between the vectorized matrices. Symmetry, linearity and positive-definiteness then follow from symmetry, linearity and positive-definiteness of the dot product.

(b) By the previous result

$$\operatorname{tr}\left(A^T B\right) = \operatorname{vec}(A)^T \operatorname{vec}(B) \tag{7}$$

$$= \operatorname{vec}(B)^T \operatorname{vec}(A) \tag{8}$$

$$= \operatorname{tr}\left(B A^T\right). \tag{9}$$

(c) For $i \neq j$ u_i and u_j are orthogonal so

$$\operatorname{tr}\left(u_i u_i^T u_j u_j^T\right) = 0. \tag{10}$$

The norm of $u_i u_i^T$ for $i = 1, \ldots, n$ equals one because

$$\begin{aligned} \left|\left|u_i u_i^T\right|\right|_{\mathrm{F}} &= \operatorname{tr}\left(u_i u_i^T u_i u_i^T\right) && (11)\\ &= \operatorname{tr}\left(u_i u_i^T\right) && (12)\\ &= \operatorname{tr}\left(u_i^T u_i\right) && (13)\\ &= 1, && (14) \end{aligned}$$

where the penultimate step follows from the result in the previous question.

(d) The projection equals

$$\begin{aligned} \left\langle A, u_i u_i^T \right\rangle u_i u_i^T &= \left\langle \sum_{j=1}^{n} \lambda_j u_j u_j^T, u_i u_i^T \right\rangle u_i u_i^T && (15)\\ &= \sum_{j=1}^{n} \left\langle \lambda_j u_j u_j^T, u_i u_i^T \right\rangle u_i u_i^T && (16)\\ &= \lambda_i u_i u_i^T, && (17) \end{aligned}$$

where the last step follows from the result in the previous question.

(e) The matrix $A' := A - \lambda_1 u_1 u_1^T$ is the component of A that is orthogonal to the matrix $u_1 u_1^T$.

3. (Quadratic forms)

(a)

$$g_v(t) = v^T A v t^2 \tag{18}$$

is a second-order polynomial, i.e. a quadratic function.

(b) By basic calculus $g_v''(t) = 2v^T A v$.

(c) By the spectral theorem the vectors u_1 and u_n that maximize and minimize $v^T A v$ on the unit circle are the eigenvectors corresponding to the maximum and minimum eigenvalues. The maximum and minimum curvature is equal to the maximum and minimum eigenvalue respectively.

4. (Projected gradient ascent)

(a) For any x, the function

$$\begin{aligned} f(y) &:= ||x - y||_2^2 && (19)\\ &= y^T y - 2x^T y + x^T x && (20) \end{aligned}$$

is a second order polynomial and therefore continuous, so by the same argument as in Lemma 5.1 it achieves a maximum on the unit sphere by the extreme value theorem. Now, in order for a point z to be the maximum, the gradient of f has to be orthogonal to the tangent plane to the unit circle, or equivalently, it must be aligned with z. This implies that there exists a constant α such that

$$\begin{align}
\nabla f(z) &= 2\,(z - x) \tag{21}\\
&= \alpha z, \tag{22}
\end{align}$$

Note that $\alpha = 2$ would imply $x = 0$, which is equidistant to all points in the unit sphere so they are all projections. Assuming $\alpha \neq 2$, we have

$$z = \frac{2}{2-\alpha} x, \tag{23}$$

so z is collinear with x. If $x \neq 0$ there are only two such vectors on the unit circle, $x/\left|\left|x\right|\right|_2$ and $-x/\left|\left|x\right|\right|_2$. The distance between x and the former is $\left|\left|\left|x\right|\right|_2 - 1\right|$. The distance between x and the latter is $\left|\left|x\right|\right|_2 + 1$. We conclude that if $x \neq 0$, its projection equals $x/\left|\left|x\right|\right|_2$.

(b) To find the largest eigenvalue we maximize the quadratic function $f(x) := x^T A x$ on the unit sphere. The gradient ascent iterations are

$$\begin{align}
x^{[k]} &= \mathcal{P}_{\mathcal{S}}(x^{[k-1]} + \alpha \nabla f(x^{[k-1]})) \tag{24}\\
&= \mathcal{P}_{\mathcal{S}}(x^{[k-1]} + 2\alpha A x^{[k-1]}) \tag{25}\\
&= \frac{(I + 2\alpha A) x^{[k-1]}}{\left|\left|(I + 2\alpha A) x^{[k-1]}\right|\right|_2}. \tag{26}
\end{align}$$

(c) We have

$$\begin{align}
(I + 2\alpha A) x^{[k-1]} &= \sum_{i=1}^{n} \beta_i^{[k-1]} (1 + 2\alpha\lambda_i) u_i, \tag{27}\\
\left|\left|(I + 2\alpha A) x^{[k-1]}\right|\right|_2^2 &= \sum_{i=1}^{n} \left(\beta_i^{[k-1]} (1 + 2\alpha\lambda_i)\right)^2 \tag{28}
\end{align}$$

so that

$$\begin{align}
x^{[k]} &= \frac{(I + 2\alpha A) x^{[k-1]}}{\left|\left|(I + 2\alpha A) x^{[k-1]}\right|\right|_2} \tag{29}\\
&= \sum_{i=1}^{n} \frac{\beta_i^{[k-1]} (1 + 2\alpha\lambda_i)}{\sum_{i=1}^{n} \left(\beta_i^{[k-1]} (1 + 2\alpha\lambda_i)\right)^2} u_i. \tag{30}
\end{align}$$

Therefore

$$\beta_i^{[k]} = \frac{\beta_i^{[k-1]} (1 + 2\alpha\lambda_i)}{\sum_{i=1}^{n} \left(\beta_i^{[k-1]} (1 + 2\alpha\lambda_i)\right)^2} \tag{31}$$

and

$$\frac{\beta_1^{[k]}}{\beta_i^{[k]}} = \frac{\beta_1^{[k-1]}(1+2\alpha\lambda_1)}{\beta_i^{[k-1]}(1+2\alpha\lambda_i)} \tag{32}$$

$$= \left(\frac{1+2\alpha\lambda_1}{1+2\alpha\lambda_i}\right)^k \frac{\beta_1^{[0]}}{\beta_i^{[0]}}. \tag{33}$$

If $\beta_1^{[0]}$ is nonzero, then the ratio of $\frac{|\beta_1^{[k]}|}{|\beta_i^{[k]}|}$ grows with k as long as $|1+2\alpha\lambda_1|$ is larger than all other $|1+2\alpha\lambda_i|$, so $\beta_1^{[k]}$ is eventually much larger than the other coefficients and $x^{[k]}$ converges to u_1.

The condition $|1+2\alpha\lambda_1|$ is larger than all other $|1+2\alpha\lambda_i|$ can be analyzed in the following three cases:

i. **All** $\lambda_i > 0$: if all $\lambda_i > 0$ and $\lambda_1 > \lambda_i$ for all $i \neq 1$ then $1+2\alpha\lambda_1 > 1+2\alpha\lambda_i > 0$ for all $i \neq 1$ and the algorithm converges for all values of $\alpha(> 0)$. Taking the limit when $\alpha \to \infty$ we have

$$\lim_{\alpha\to\infty} \frac{\beta_1^{[k]}}{\beta_i^{[k]}} = \left(\frac{\lambda_1}{\lambda_i}\right)^k \frac{\beta_1^{[0]}}{\beta_i^{[0]}}, \tag{34}$$

so the algorithm will still converge to λ_1 (or u_1) when α is arbitrarily large.

ii. **All** $\lambda_i < 0$: if all $\lambda_i >< 0$ then we have $|\lambda_1| < |\lambda_i|$ for all $i \neq 1$. The algorithm converges when

$$|1+2\alpha\lambda_1| > |1+2\alpha\lambda_i| \tag{35}$$

$$(1+2\alpha\lambda_1)^2 > (1+2\alpha\lambda_i)^2 \tag{36}$$

$$(2+2\alpha(\lambda_i+\lambda_1))(2\alpha(\lambda_1-\lambda_i)) > 0 \tag{37}$$

Note that $(\lambda_1-\lambda_i) > 0$ which implies

$$2+2\alpha(\lambda_i+\lambda_1) > 0 \tag{38}$$

$$\alpha < \frac{2}{|\lambda_i|+|\lambda_1|} \tag{39}$$

So, we need $\alpha < \frac{2}{|\lambda_n|+|\lambda_1|}$.

Taking the limit when $\alpha \to \infty$ we should converge to λ_n (or u_n) here.

iii. $\lambda_n < 0$ **and** $\lambda_1 > 0$: The algorithm converges when

$$|1+2\alpha\lambda_1| > |1+2\alpha\lambda_i| \tag{40}$$

$$(1+2\alpha\lambda_1)^2 > (1+2\alpha\lambda_i)^2 \tag{41}$$

$$(2+2\alpha(\lambda_i+\lambda_1))(2\alpha(\lambda_1-\lambda_i)) > 0 \tag{42}$$

If $|\lambda_1| < |\lambda_n|$, then $(\lambda_1-\lambda_i) > 0$ for all $i \neq 1$which implies

$$2+2\alpha(\lambda_i+\lambda_1) > 0 \tag{43}$$

$$\alpha < \frac{1}{|\lambda_i|+|\lambda_1|} \tag{44}$$

So, we need $\alpha < \frac{1}{|\lambda_n|+|\lambda_1|}$.
and in this case, when $\alpha \to \infty$ we should converge to λ_n (or u_n) here.
$(\lambda_1 - \lambda_i) > 0$ for all $i \neq 1$ So we need

$$2 + 2\alpha(\lambda_i + \lambda_1) > 0 \tag{45}$$
$$1 + \alpha(\lambda_i + \lambda_1) > 0 \tag{46}$$

if $|\lambda_1| > |\lambda_i|$ for all $i \neq 1$ i.e $|\lambda_1| > |\lambda_n|$ then this equation is satisfied for all $\alpha > 0$. And in this case, when $\alpha \to \infty$ we should converge to λ_1 (or u_1) here.
if $|\lambda_1| < |\lambda_n|$ then we have

$$1 + \alpha(\lambda_1 - |\lambda_n|) > 0 \tag{47}$$
$$\alpha < \frac{1}{(-\lambda_1 + |\lambda_n|)} \tag{48}$$

So, we need $\alpha < \frac{1}{|\lambda_n|-\lambda_1}$.
And in this case, when $\alpha \to \infty$ we should converge to λ_n (or u_n) here.

(d) If using the **older** version of support code

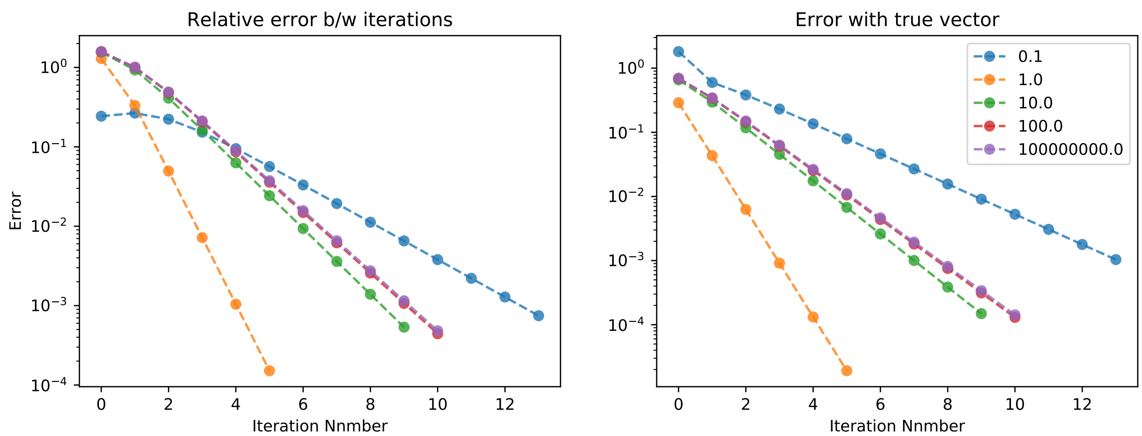

Figure 1: 2×2 matrix

(e) If using the new version of support code

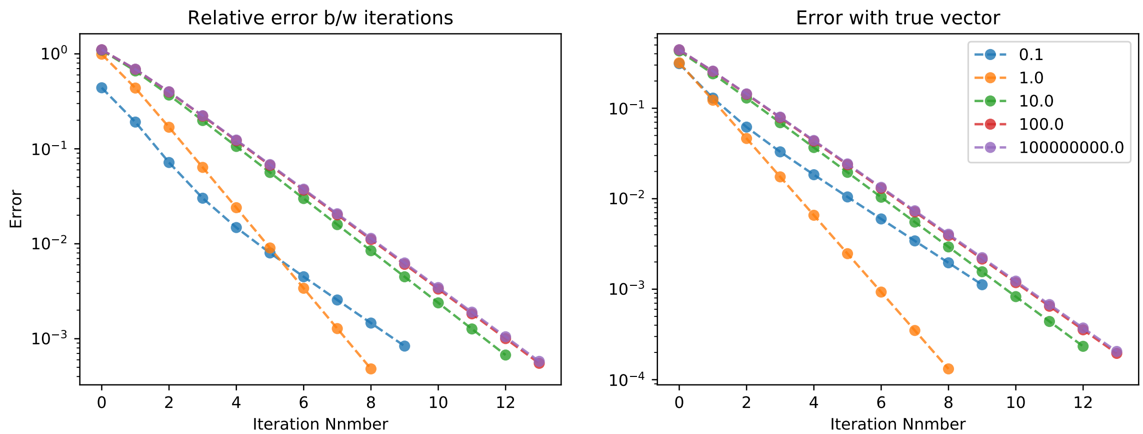

Figure 2: 3×3 matrix

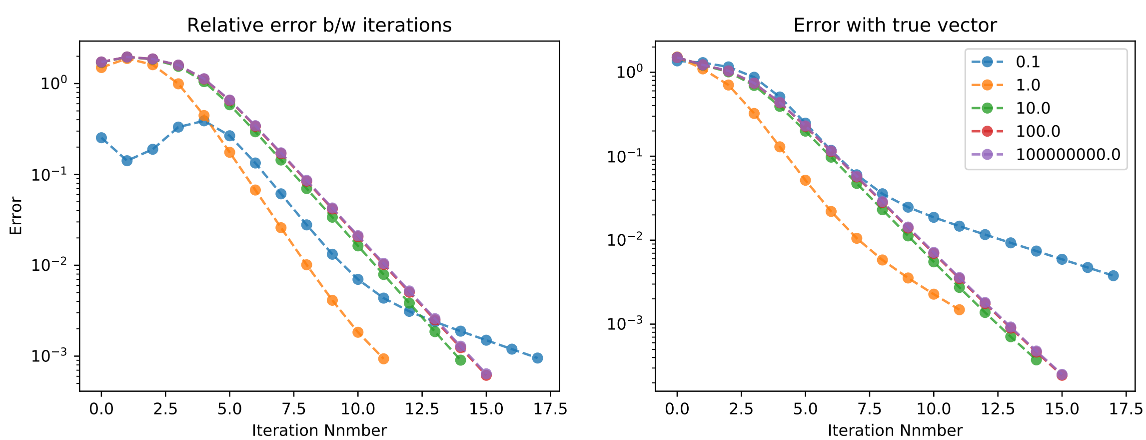

Figure 3: 4×4 matrix

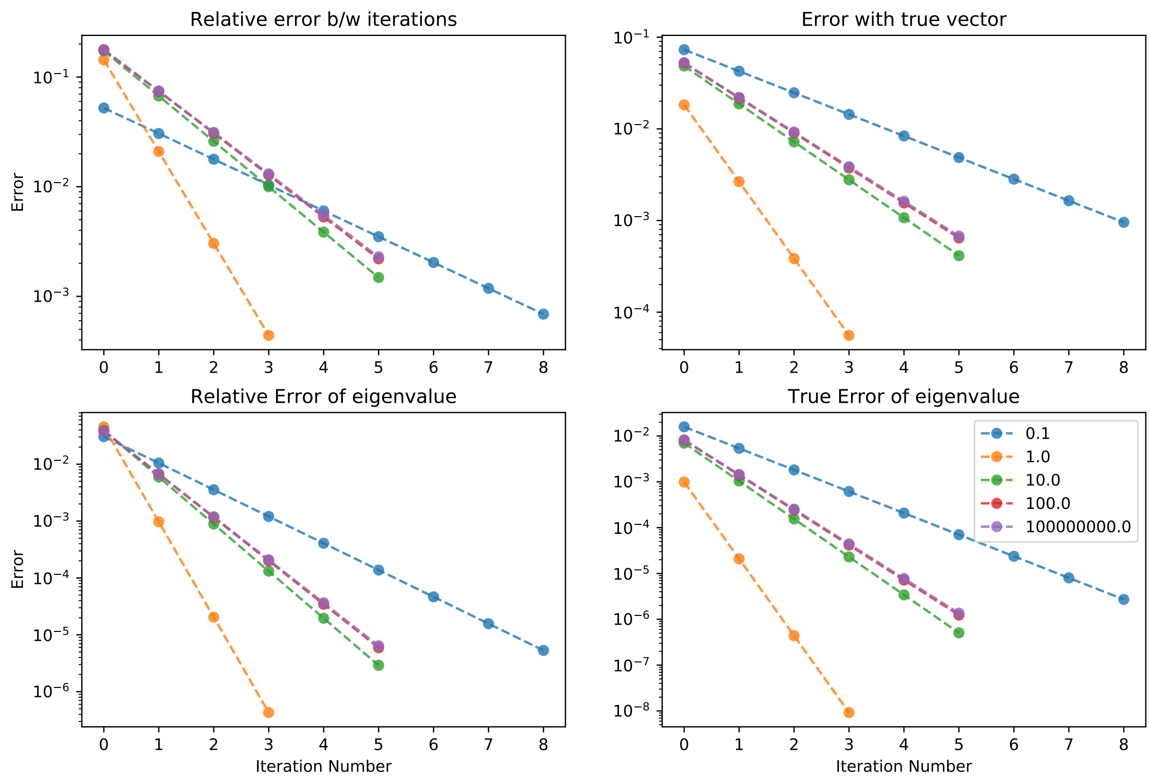

Figure 4: 2×2 matrix

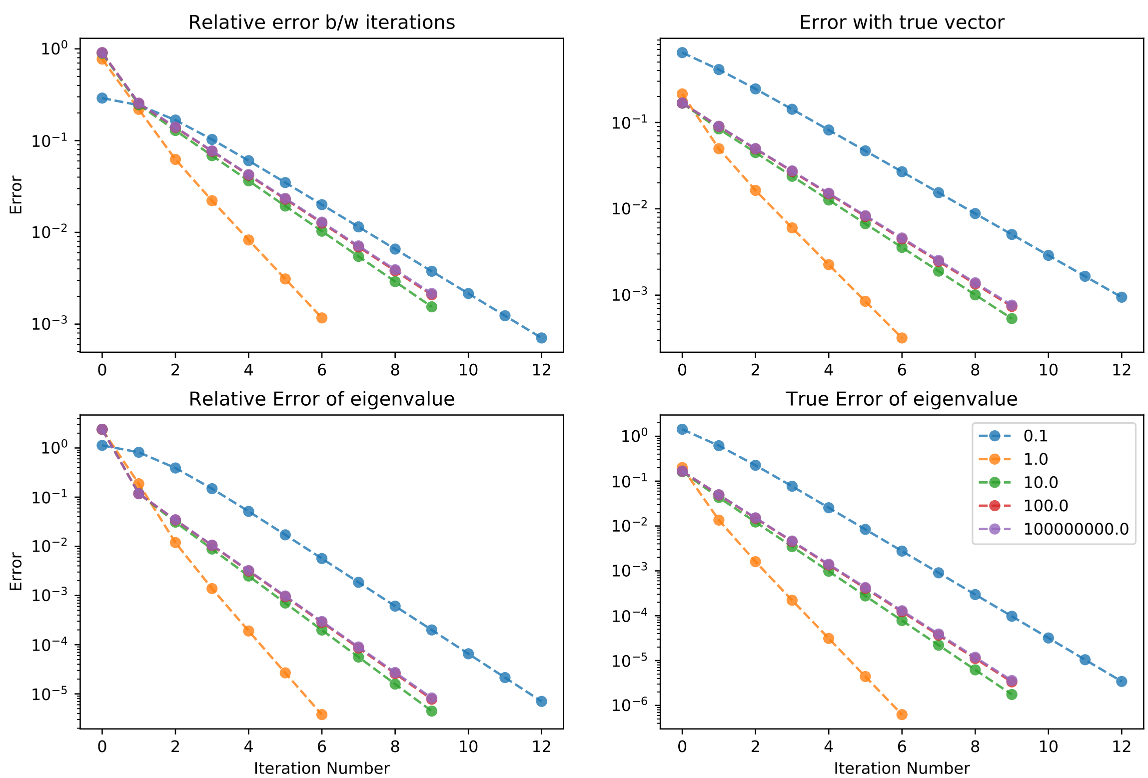

Figure 5: 3×3 matrix

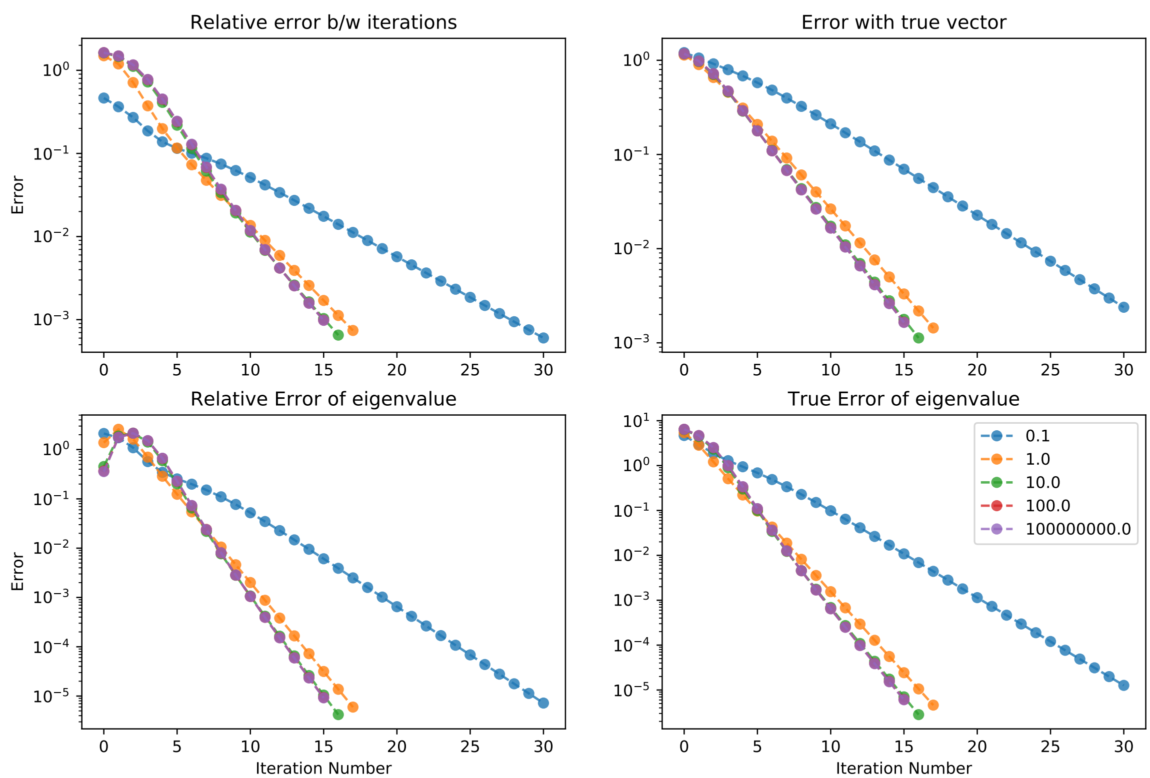

Figure 6: 4×4 matrix