Homework 0

Due February 9 at 11 pm

1. (Projections) Are the following statements true or false? Prove that they are true or provide a counterexample.

   (a) The projection of a vector on a subspace $\mathcal{S}$ is equal to

   $$\mathcal{P}_{\mathcal{S}}\, x = \sum_{i=1}^{n} \langle x, b_i \rangle\, b_i$$

   for any basis $b_1, \ldots, b_d$ of $\mathcal{S}$.

   (b) The orthogonal complement of the orthogonal complement of a subspace $\mathcal{S} \subseteq \mathbb{R}^n$ is $\mathcal{S}$.

   (c) Replacing each entry of a vector in $\mathbb{R}^n$ by the average of all its entries is equivalent to projecting the vector onto a subspace.

2. (Eigendecomposition) The populations of deer and wolfs in Yellowstone are well approximated by

   $$d_{n+1} = \frac{5}{4} d_n - \frac{3}{4} w_n, \tag{1}$$

   $$w_{n+1} = \frac{1}{4} d_n + \frac{1}{4} w_n, \qquad n = 0, 1, 2, \ldots, \tag{2}$$

   where d_n and w_n denote the number of deer and wolfs in year n. Assuming that there are more deer than wolfs to start with ($w_0 < d_0$), what is the proportion between the numbers of deer and wolfs as $n \to \infty$?

3. (Function approximation) In this problem we will work in the real inner product space $L^2[-1,1]$ given by

   $$L^2[-1,1] = \left\{ f : [-1,1] \to \mathbb{R} \;\middle|\; \int_{-1}^{1} f(x)^2\, dx < \infty \right\}.$$

   On this space, the inner product is given by

   $$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx.$$

   In the following exercises, you may use a computer to perform the integral calculations.

   (a) The functions $\{1, x, x^2\}$ form a basis for the 3-dimensional subspace P_2 of $L^2[-1,1]$ consisting of the polynomials of degree at most 2. Give the orthonormal basis for P_2 obtained by applying Gram-Schmidt to this set of functions.

   (b) Compute the orthogonal projection of $f(x) = \cos(\pi x/2)$ onto P_2.

(c) Plot $f(x) = \cos(\pi x/2)$, $\mathcal{P}_{P_2} f$, and $T_2 f$ on the same axis. Here $\mathcal{P}_{P_2} f$ is the projection computed in the previous part, and $T_2 f$ is the quadratic Taylor polynomial for f centered at $x = 0$:

$$T_2 f(x) = f(0) + f'(0)x + \frac{f''(0)}{2} x^2.$$

Include this plot in your submitted homework document.

(d) The plot from the previous part shows that $\mathcal{P}_{P_2} f$ is a better approximation than $T_2 f$ over most of $[-1, 1]$. Explain why this is the case.

4. (Scalar linear estimation)

(a) Let $\tilde{x}$ be a random variable with mean $\mu_{\tilde{x}}$ and variance $\sigma_{\tilde{x}}^2$, and $\tilde{y}$ a random variable with mean $\mu_{\tilde{y}}$ and variance $\sigma_{\tilde{y}}^2$. The correlation coefficient between them is $\rho_{\tilde{x},\tilde{y}}$. What values of a, $b \in \mathbb{R}$ minimize the mean square error $\mathrm{E}[(a\tilde{x} + b - \tilde{y})^2]$? Express your answer in terms of $\mu_{\tilde{x}}$, $\sigma_{\tilde{x}}$, $\mu_{\tilde{y}}$, $\sigma_{\tilde{y}}$, and $\rho_{\tilde{x},\tilde{y}}$.

(b) Let $\tilde{x} = \tilde{y}\tilde{z}$, where $\tilde{y}$ has mean $\mu_{\tilde{y}}$ and variance $\sigma_{\tilde{y}}^2$, and $\tilde{z}$ has mean zero and variance $\sigma_{\tilde{z}}^2$. If $\tilde{y}$ and $\tilde{z}$ are independent, what is the best linear estimate of $\tilde{y}$ given $\tilde{x}$?

(c) Assume $\tilde{y}$ is positive with probability one. Can you think of a zero-mean random variable $\tilde{z}$ such that $\tilde{y}$ can be estimated perfectly from $\tilde{x}$ in the previous question?

5. (Gradients) Recall that the entries of the gradient of a function are equal to its partial derivatives. Use this fact to:

(a) Compute the gradient of $f(x) = bx$ where $b \in \mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$.

(b) Compute the gradient of $f(x) = x^T A x$ where $A \in \mathbb{R}^{d \times d}$ and $f : \mathbb{R}^d \to \mathbb{R}$.