Homework 7

Due April 12 at 11 pm

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1. (Fourier coefficients and smoothness) Let $x : \mathbb{R} \to \mathbb{C}$ be periodic with period 1 and let $\hat{x}[k]$ denote the kth Fourier coefficient of x, for $k \in \mathbb{Z}$ (computed on any interval of length 1).

   (a) Suppose x is continuously differentiable. Prove that for $k \neq 0$ we have

$$|\hat{x}[k]| \leq \frac{C_1}{|k|}$$

for some $C_1 \geq 0$ that depends on x (but not on k). [Hint: Integration by parts. Also note that

$$\left| \int_0^1 f(t)\, dt \right| \leq \int_0^1 |f(t)|\, dt < \infty$$

if f is continuous on $[0, 1]$.]

WLOG we consider the interval [0,1] since x is periodic with period 1 and, we have for $k \neq 0$:

$$\begin{aligned}
\hat{x}[k] &= \int_0^1 x(t) \exp\left(-i2\pi kt\right)\, \mathrm{d}t \text{ (by parts with } u = x(t) \text{, and } v = \frac{-1}{i2\pi k} e^{-i2\pi kt}) \\
&= \frac{-1}{i2\pi k} [x(t) e^{-i2\pi kt}]_0^1 + \frac{1}{i2\pi k} \int_0^1 x'(t) \exp\left(-i2\pi kt\right)\, \mathrm{d}t \\
&= \frac{x(0) - x(1)}{i2\pi k} + \frac{1}{i2\pi k} \int_0^1 x'(t) \exp\left(-i2\pi kt\right)\, \mathrm{d}t \\
&= \frac{1}{i2\pi k} \int_0^1 x'(t) \exp\left(-i2\pi kt\right)\, \mathrm{d}t \quad \text{since period is 1}
\end{aligned}$$

x is continuously differentiable on $[0, 1]$ so:

$$\left| \int_0^1 x'(t)\, dt \right| \leq \int_0^1 |x'(t)|\, dt < \infty$$

Let $M = \int_0^1 |x'(t)|\, dt$, using the previous expression of $\hat{x}[k]$, we can now determine an upper bound:

$$
\begin{aligned}
|\hat{x}[k]| &= |\frac{1}{i2\pi k} \int_0^1 x'(t) \exp\left(-i2\pi kt\right)\, \mathrm{d}t| \\
&= |\frac{1}{i2\pi k}|| \int_0^1 x'(t) \exp\left(-i2\pi kt\right)\, \mathrm{d}t| \\
&\leq |\frac{1}{2\pi k}| \int_0^1 |x'(t) \exp\left(-i2\pi kt\right)| \, \mathrm{d}t \\
&= |\frac{1}{2\pi k}| \int_0^1 |x'(t)|| \exp\left(-i2\pi kt\right)| \, \mathrm{d}t \\
&= |\frac{1}{2\pi k}| \int_0^1 |x'(t)| \, \mathrm{d}t \\
&\leq |\frac{1}{2\pi k}| M
\end{aligned}
$$

So $|\hat{x}[k]| \leq \frac{C_1}{|k|}$ with $C_1 = \frac{M}{2\pi}$.

(b) Suppose x is twice continuously differentiable. Prove that for $k \neq 0$ we have

$$|\hat{x}[k]| \leq \frac{C_2}{|k|^2}$$

for some $C_2 \geq 0$ that depends on x (but not on k).
Let $\hat{x}'[k] = \int_0^1 x'(t) \exp\left(-i2\pi kt\right)\, \mathrm{d}t$, using part a, we can write that

$$\hat{x}'[k] = \frac{x'(0) - x'(1)}{i2\pi k} + \frac{1}{i2\pi k} \int_0^1 x''(t) \exp\left(-i2\pi kt\right)\, \mathrm{d}t$$

x is now twice continuously differentiable:

$$\left| \int_0^1 x''(t)\, dt \right| \leq \int_0^1 |x''(t)|\, dt = M_2 < \infty$$

And since x' is continuous on $[0, 1]$, it is bounded, let $M_1 = \max |x(t)|, t \in [0, 1]$ then

$$
\begin{aligned}
|\hat{x}'[k]| &\leq \frac{|x'(0)| + |x'(1)|}{|2\pi k|} + \frac{1}{|2\pi k|} M_2 \\
&= \frac{2\ M_1 + M_2}{|2\pi k|} \\
|\hat{x}[k]| &= |\frac{1}{i2\pi k}|| \int_0^1 x'(t) \exp\left(-i2\pi kt\right)\, \mathrm{d}t| \quad \text{from part a} \\
&\leq \frac{C_2}{|k^2|} \quad \text{where } C_2 = \frac{2M_1 + M2}{4\pi^2}
\end{aligned}
$$

2. (Sampling a sum of sinusoids) We are interested in a signal x belonging to the unit interval $[0, 1]$ of the form

$$x(t) := a_1 \exp(i2\pi k_1 t) + a_2 \exp(i2\pi k_2 t), \tag{1}$$

where the amplitudes a_1 and a_2 are complex numbers, and the frequencies k_1 and k_2 are known integers. We sample the signal at N equispaced locations 0, $1/N$, $2/N$, ..., $(N-1)/N$, for some positive integer N.

(a) What value of N is required by the Sampling Theorem to guarantee that we can reconstruct x from the samples?

By theorem 3.4, if the number of samples N is larger than $2\max(k_1, k_2) + 1$ then the Fourier coefficients a_1, a_2 and hence the signal x can be reconstructed.

(b) Write a system of equations in matrix form mapping the amplitudes a_1 and a_2 to the samples x_N.

$$\begin{bmatrix} x(\frac{0}{N}) \\ x(\frac{1}{N}) \\ \cdots \\ x(\frac{j}{N}) \\ \cdots \\ x(\frac{N-1}{N}) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \exp(\frac{i2\pi k_1 1}{N}) & \exp(\frac{i2\pi k_2 1}{N}) \\ \cdots & \cdots \\ \exp(\frac{i2\pi k_1 j}{N}) & \exp(\frac{i2\pi k_2 j}{N}) \\ \cdots & \cdots \\ \exp(\frac{i2\pi k_1 (N-1)}{N}) & \exp(\frac{i2\pi k_2 (N-1)}{N}) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

(c) Under what condition on N, k_1 and k_2 can we recover the amplitudes from the samples by solving the system of equations? Can N be smaller than the value dictated by the Sampling Theorem? If yes, give an example. If not, explain why.
It is a system of N equations of two unknowns a_1 and a_2 which has a solution if $N >= 2$ assuming the matrix is at least of column rank 2 which is the case if $k_1 \neq k_2$. If N is smaller that the value dictated by the Sampling Theorem, we may encounter the issue of aliasing for a sinusoid with frequency m such that $m = \max(k_1, k_2) \mod N$, the two signals from the samples will look the same.

(d) What is the limitation of this approach, which could make it unrealistic?
If we have too many equations, the cost of solving this system using a method like least square will be very costly (in practice with OLS we are looking for a solution of the form $[a_0 a_1]^T = (AA^T)^{-1}A$ where A is the two columns matrix defined in part b, this will requires a time proportional at least to $\mathcal{O}(N^3)$, making large problem signal reconstruction intractable. In addition solving this system with the matrix A can be prone to machine errors not only when N is large but also when $k_1 \approx k_2$.

3. (Sampling theorem for bandpass signals) Bandpass signals are signals that have nonzero Fourier coefficients only in a fixed band of the frequency domain. We are interested in sampling a bandpass signal x belonging to the unit interval $[0, 1]$ that has nonzero Fourier-series coefficients between k_1 and k_2, inclusive, where k_1 and k_2 are known positive integers such that $k_2 > k_1$.

   (a) We sample the signal at N equispaced locations 0, $1/N$, $2/N$, $\ldots$, $(N-1)/N$. What value of N is required by the Sampling Theorem to guarantee that we can reconstruct x from the samples?

   (b) Assume that $k_2 := k_1 + 2\tilde{k}_c$, where $\tilde{k}_c$ is a positive integer. For any $N \geq 2\tilde{k}_c + 1$ it is possible to recover the signal from the samples. Explain why (you don't need to derive any explicit expressions).

   (c) Assume that $k_2 := k_1 + 2\tilde{k}_c$, $N \geq 2\tilde{k}_c + 1$, and $mN = k_1 + \tilde{k}_c$ for some integer m. Explain precisely how to recover x from the samples in this case.

4. (Frequency analysis of musical notes) In this exercise you will use the code and data in the `musicdata` folder. Make sure you have the python packages sklearn, pandas, sounddevice, and soundfile installed. The skeleton code for you to work with is given in `analysis.py` which uses tools given in `music_tools.py`. The data used here comes from the NSynth dataset.

   (a) Plot the audio signals for the first signal in the training set, and the first vocal signal in the training set (i.e., the first signal whose `instrument_family_str` field is 'vocal' in the dataframe). In the titles of your two plots, include the `instrument_family_str` and the frequency (in Hz). We recommend you also use `play_signal` to hear what the signals sound like.

   (b) For each signal in the test set, compute the (strictly positive) frequency with the largest amplitude (in absolute value), and convert it to a pitch number (using the tools in `music_tools`). This will be our predicted pitch.

      i. Report what overall fraction of the signals in the test set you accurately predict using this method (i.e., your overall accuracy).

      ii. For the first two signals you misclassify (in the order they occur in the test set), give plots of their absolute DFT coefficients (use `np.fft.fft` and make one plot per signal). In the title of your plots, include the `instrument_family_str`, the true frequency, and the predicted frequency (in Hz). Make sure to plot the coefficients on an axis centered at 0 by using fftfreq with the correct arguments.

      iii. What is the instrument family for which the method got the highest fraction of incorrect predictions (i.e., number incorrect divided by number of examples from that family)?

   (c) Use the `LogisticRegression` class in sklearn to fit a pitch classifier on the training set using the absolute DFT coefficients as the features. Use the default parameters but set `multi_class` to 'multinomial' and `solver` to 'lbfgs'. Note: We will use the negative frequencies as well for convenience, even though they have the same magnitudes as the positive (the L_2 regularization will take care of it for us).

      i. Report your score on the test set as computed by the model.

      ii. Give 3 plots of the model coefficients for pitches 60, 65, and 72. Make sure to plot the coefficients on an axis centered at 0 by using fftfreq with the correct arguments (because the coefficients correspond to frequencies).