Homework 11

Due May 10 at 11 pm

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1. (Lasso and ℓ_0) The file `X.txt` contains a 50×300 matrix X, and the file `y.txt` contains the 50×1 vector y. Each line of each file represents a row of the corresponding matrix, and the values on each line are space-delimited.

   (a) Consider the lasso problem

   $$\min_{\beta} \frac{1}{2n}\|X\beta - y\|^2 + \lambda\|\beta\|_1$$

   where $\lambda > 0$ is a parameter and $n = 50$. Construct a (semilogx) plot that draws a separate path for each coefficient value as a function of λ. Include values of λ between 0.01 and 2 (you can include more if you want), and make your values spaced evenly on the log axis (e.g., `np.geomspace`). You can solve the lasso problem using whatever code/library you want.

   (b) Determine the minimizer of

   $$\begin{array}{ll} \text{minimize} & \|\beta\|_0 \\ \text{subject to} & X\beta = y. \end{array}$$

   Assume that the minimizer has small ℓ_0 norm, i.e $\ell_0 \leq 2$. Explain your strategy and justify that it finds the minimizer. Report the nonzero coefficients of the minimizer, and their values. Remember that two floating point values may be different for numerical reasons even if they represent the same value.

   (c) Will your strategy in (b) always find the optimal minimizer of any least-squares problem with ℓ_0 regularization?

2. (Proximal operator) The proximal operator of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as

   $$\text{prox}_f (y) := \arg\min_{x} f(x) + \frac{1}{2}\|x - y\|_2^2 . \tag{1}$$

   (a) Derive the proximal operator of the squared ℓ_2 norm weighted by a constant $\alpha > 0$, i.e. $f(x) = \alpha\|x\|_2^2$.

   (b) Prove that the proximal operator of the ℓ_1 norm weighted by a constant $\alpha > 0$ is a soft-thresholding operator,

   $$\text{prox}_{\alpha\|\cdot\|_1} (y) = \mathcal{S}_\alpha (y), \tag{2}$$

   where

   $$\mathcal{S}_\alpha (y)[i] := \begin{cases} y[i] - \text{sign}(y[i])\,\alpha & \text{if } |y[i]| \geq \alpha, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

(c) Prove that if $X \in \mathbb{R}^{p \times n}$ has orthonormal rows ($p \leq n$) and $y \in \mathbb{R}^n$, then for any function f

$$\arg\min_{\beta} \frac{1}{2} \left|\left|y - X^T \beta\right|\right|_2^2 + f(\beta) = \arg\min_{\beta} \frac{1}{2} \left|\left|Xy - \beta\right|\right|_2^2 + f(\beta). \tag{4}$$

(d) Use the answers to the previous questions to compare the ridge-regression and lasso estimators for a regression problem where the features are orthonormal.

3. (Proximal gradient method)

(a) The first-order approximation to a function $f : \mathbb{R}^p \to \mathbb{R}$ at $x \in \mathbb{R}^p$ equals

$$f(x) + \nabla f(x)^T (y - x). \tag{5}$$

We want to minimize this first-order approximation locally. To this end we fix a real constant $\alpha > 0$ and augment the approximation with an ℓ_2-norm term that keeps us close to x,

$$f_x(y) := f(x) + \nabla f(x)^T (y - x) + \frac{1}{2\,\alpha} \left|\left|y - x\right|\right|_2^2. \tag{6}$$

Prove that the minimum of f_x is the gradient descent update $x - \alpha \nabla f(x)$.

(b) Inspired by the previous question, how would you modify gradient descent to minimize a function of the form

$$h(x) = f_1(x) + f_2(x), \tag{7}$$

where f_1 is differentiable, and f_2 is nondifferentiable but has a proximal operator that is easy to compute?

(c) Show that a vector x^* is a solution to

$$\text{minimize} \quad f_1(x) + f_2(x), \tag{8}$$

where f_1 is differentiable, and f_2 is nondifferentiable, if and only if it is a fixed point of the iteration you proposed in the previous question for any $\alpha > 0$.

4. (Iterative shrinkage-thresholding algorithm)

(a) What is the proximal gradient update corresponding to the lasso problem defined below? Your answer will involve a hyperparameter which we will call as α.

$$\frac{1}{2} \left|\left|y - X\beta\right|\right|_2^2 + \lambda |\beta|_1$$

(b) How would you check whether you have reached an optimum? How would you modify this to take into account possible numerical inaccuracies?

(c) Implement the method and apply it to the problem in `pgd_lasso-question.ipynb`. You have to fill in blocks of code corresponds to the proximal gradient update step and termination condition. Report all the generated plots.