Recitation 8

DS-GA 1013 Mathematical Tools for Data Science

1. Let $x : \mathbb{R} \to \mathbb{C}$ be periodic with period 1 and square integrable on $[0, 1)$. For each of the following functions, give a formula for the Fourier coefficients in terms of the Fourier coefficients of x.

   1. $y_1 : \mathbb{R} \to \mathbb{C}$ with $y_1(t) = x(t - s)$ for some fixed $s \in \mathbb{R}$.
   2. $y_2 : \mathbb{R} \to \mathbb{C}$ with $y_2(t) = e^{2\pi ipt}x(t)$ for some fixed $p \in \mathbb{Z}$.
   3. $y_3 : \mathbb{R} \to \mathbb{C}$ with $y_3(t) = x(-t)$
   4. $y_4 : \mathbb{R} \to \mathbb{C}$ with $y_3(t) = x^*(t)$

Solution:

1. Note that

$$\begin{aligned}\widehat{y_1}[k] &= \int_0^1 x(t-s)e^{-2\pi ikt}\,dt \\ &= \int_{-s}^{1-s} x(u)e^{-2\pi ik(u+s)}\,du \\ &= e^{-2\pi iks}\hat{x}[k].\end{aligned}$$

2. Note that

$$\begin{aligned}\widehat{y_2}[k] &= \int_0^1 e^{2\pi ipt}x(t)e^{-2\pi ikt}\,dt \\ &= \int_0^1 x(t)e^{-2\pi it(k-p)} \\ &= \hat{x}[k-p].\end{aligned}$$

3. Note that

$$\begin{aligned}\widehat{y_3}[k] &= \int_0^1 x(-t)e^{-2\pi ikt}\,dt \\ &= -\int_0^{-} 1x(t)e^{-2\pi it(-k)} \\ &= \hat{x}[-k].\end{aligned}$$

4. Note that

$$\begin{aligned}\widehat{y_4}[k] &= \int_0^1 x^*(t)e^{-2\pi ikt}\,dt \\ &= (\int_0^1 x(t)e^{-2\pi it(-k)}dt)^* \\ &= \hat{x}^*[-k].\end{aligned}$$

2. Let $x(t)$ be a square wave, i.e $x(t) = 1 for 0 \leq t \leq 0.5 and -1 for 0.5 < t \leq 1$. What are the Fourier series coefficients for $x(t)$? Simulation. Discontinuity.

Solution:

$$\begin{aligned}
\widehat{x}[k] &= \int_0^1 x(t)e^{-2\pi ikt}\,dt \\
&= \int_0^{0.5} e^{-2\pi ikt}\,dt - \int_{0.5}^1 e^{-2\pi ikt}\,dt \\
&= (e^{-\pi ik} - 1) - (e^{-2\pi ik} - e^{\pi ik}))\frac{1}{-2\pi ik} \\
&= \frac{(1-(-1)^k)}{i\pi k}
\end{aligned}$$

if k is even $\widehat{x}[k] = 0$ and if k is odd $\widehat{x}[k] = \frac{2}{i\pi k} = -\frac{2i}{\pi k}$
Note that the coefficients are purely imaginary, therefore the cosine term will be zero (HW6).

3. Let n be a positive integer and define $f : \mathbb{Z}^2 \to \mathbb{C}$ by

$$f(j,k) = e^{2\pi ijk/N}.$$

1. Show that f is periodic with period N in both arguments. That is, show that

$$f(j+pN, k+qN) = f(j,k)$$

for all $j,k,p,q \in \mathbb{Z}$.

2. Let $\vec{\varphi}_j = (1, e^{2\pi ij/N}, \ldots, e^{2\pi i(N-1)j/N})^T \in \mathbb{C}^N$ for $j \in \mathbb{Z}$. When does $\vec{\varphi}_j = \vec{\varphi}_k$?

Solution:

Solution.

1. Note that

$$e^{2\pi i(j+pN)(k+qN)/N} = e^{2\pi ij(k+qN)/N}e^{2\pi ip(k+qN)} = e^{2\pi ijk/N}e^{2\pi ijq} = e^{2\pi ijk/N}.$$

2. Iff $j \equiv k \pmod{N}$. This allows us to abuse notation when indexing our sampled signals and discrete Fourier coefficients using negative indices.

4. A matrix $A \in \mathbb{C}^{n\times n}$ is called unitary if $A^*A = I$.

1. Prove that unitary matrices preserve inner products:

$$\langle x, y\rangle = \langle Ax, Ay\rangle$$

for all $x, y \in \mathbb{C}^n$.

2. Is the matrix $\widetilde{F}_{[N]} \in \mathbb{C}^{N\times N}, N = 2k_c + 1$ unitary?

$$\widetilde{F}_{[N]} = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
\exp\left(\frac{i2\pi(-k_c)}{N}\right) & \exp\left(\frac{i2\pi(-k_c+1)}{N}\right) & \cdots & \exp\left(\frac{i2\pi k_c}{N}\right) \\
\cdots & \cdots & \cdots & \cdots \\
\exp\left(\frac{i2\pi(-k_c)j}{N}\right) & \exp\left(\frac{i2\pi(-k_c+1)j}{N}\right) & \cdots & \exp\left(\frac{i2\pi k_c j}{N}\right) \\
\cdots & \cdots & \cdots & \cdots \\
\exp\left(\frac{i2\pi(-k_c)(N-1)}{N}\right) & \exp\left(\frac{i2\pi(-k_c+1)(N-1)}{N}\right) & \cdots & \exp\left(\frac{i2\pi k_c(N-1)}{N}\right)
\end{bmatrix}$$

Solution:

Solution.

1. Note that
$$\langle Ax, Ay\rangle = (Ay)^*(Ax) = y^* A^* A x = y^* x = \langle x, y\rangle.$$

2. No, but $\frac{1}{\sqrt{N}}\widetilde{F}_{[N]}$ is.

5. There is a signal x given by
$$x(t) = \sum_{k=-k_c}^{k_c} a_k e^{2\pi i k t},$$
where k_c is known. Suppose we are given n samples
$$x(t_1), x(t_2), \ldots, x(t_n)$$
where $0 \le t_1 < t_2 < \cdots < t_n < 1$ need not be uniformly spaced.

   1. Under what conditions can we exactly recover the a_k values, and how would this be done?
   2. Suppose n is large, but the samples are corrupted by noise. Give a method for estimating the a_k values.

Solution:

Solution.

1. The $x(t_i)$ values are given by the system
$$\begin{bmatrix} e^{2\pi i(-k_c)t_1} & e^{2\pi i(-k_c+1)t_1} & \cdots & e^{2\pi i(k_c)t_1} \\ e^{2\pi i(-k_c)t_2} & e^{2\pi i(-k_c+1)t_2} & \cdots & e^{2\pi i(k_c)t_2} \\ \vdots & \vdots & \ddots & \vdots \\ e^{2\pi i(-k_c)t_n} & e^{2\pi i(-k_c+1)t_n} & \cdots & e^{2\pi i(k_c)t_n} \end{bmatrix} \begin{bmatrix} a_{-k_c} \\ a_{-k_c+1} \\ \vdots \\ a_{k_c} \end{bmatrix} = \begin{bmatrix} x(t_1) \\ x(t_2) \\ \vdots \\ x(t_n) \end{bmatrix}.$$
This has a unique solution when $n \ge 2k_c + 1$ since the corresponding matrix is a Vandermonde matrix, and thus has full rank. That said, it becomes ill-conditioned as the t_i become close.

2. Apply least squares regression to the above overdetermined linear system.