The Frequency Domain

DS-GA 1013 / MATH-GA 2824 Mathematical Tools for Data Science

https://cims.nyu.edu/~cfgranda/pages/MTDS_spring20/index.html

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The frequency domain

Sampling

Discrete Fourier transform

Frequency representations in multiple dimensions

Discussion

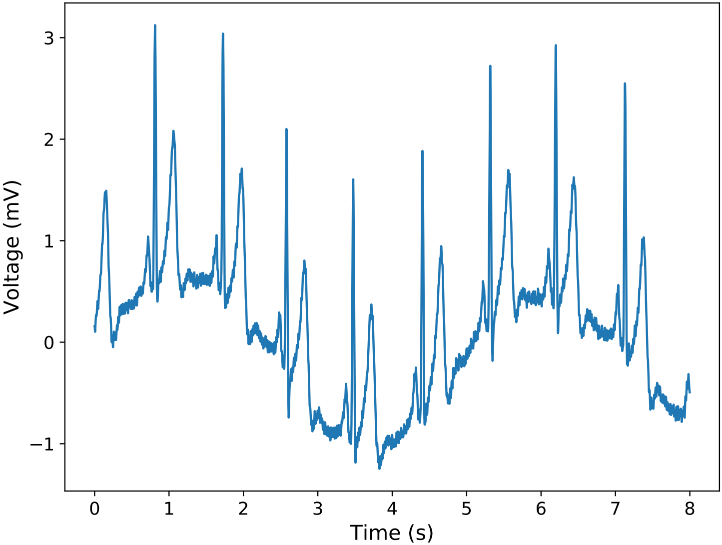
Signal processing

Signal: any structured object of interest (images, audio, video, etc.)

Modeled as function of space, time, etc.

Finding adequate representations is crucial to process signals effectively

Electrocardiogram



Signals as functions

We model signals as square-integrable functions on an interval $[a, b] \subset \mathbb{R}$

Inner product:

$$\langle x, y \rangle := \int_a^b x(t)\,\overline{y(t)}\,\mathrm{d}t$$

Goal: Find basis functions to represent periodic signals

Sinusoids

Sinusoidal function:

$$a \cos(2\pi f t + \theta)$$

- Amplitude: a
- Frequency: f
- Time index: t (periodic with period $1/f$)
- Phase: θ

Problem

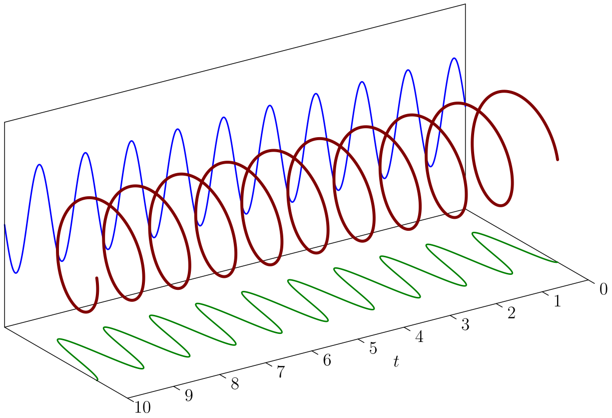
Is this a reasonable basis?

Complex sinusoid

The complex sinusoid with frequency $f \in \mathbb{R}$ is given by

$$\exp(i2\pi ft) := \cos(2\pi ft) + i\sin(2\pi ft)$$

Complex sinusoid



Complex sinusoid

We can express any real sinusoid in terms of complex sinusoids

$$\begin{aligned}\cos(2\pi ft + \theta) &= \frac{\exp(i2\pi ft + i\theta) + \exp(-i2\pi ft - i\theta)}{2} \\ &= \frac{\exp(i\theta)}{2}\exp(i2\pi ft) + \frac{\exp(-i\theta)}{2}\exp(-i2\pi ft)\end{aligned}$$

The phase is encoded in the complex amplitude!

Linear subspace spanned by $\exp(i2\pi ft)$ and $\exp(-i2\pi ft)$ contains all real sinusoids with frequency f

If we add two sinusoids with frequency f the result is a sinusoid with frequency f

Orthogonality of complex sinusoids

The family of complex sinusoids with integer frequencies

$$\phi_k(t) := \exp\left(\frac{i2\pi kt}{T}\right), \qquad k \in \mathbb{Z},$$

is an orthogonal set on $[a, a+T]$, where $a, T \in \mathbb{R}$ and $T > 0$

Proof

$$\begin{aligned}
\langle \phi_k, \phi_j \rangle &= \int_a^{a+T} \phi_k(t)\,\overline{\phi_j(t)}\,\mathrm{d}t \\
&= \int_a^{a+T} \exp\left(\frac{i2\pi(k-j)t}{T}\right)\,\mathrm{d}t \\
&= \frac{T}{i2\pi(k-j)}\left(\exp\left(\frac{i2\pi(k-j)(a+T)}{T}\right) - \exp\left(\frac{i2\pi(k-j)a}{T}\right)\right) \\
&= 0
\end{aligned}$$

Fourier series

The Fourier series coefficients of $x \in \mathcal{L}_2 [a, a+T]$, $a, T \in \mathbb{R}$, $T > 0$, are

$$\hat{x}[k] := \langle x, \phi_k \rangle = \int_a^{a+T} x(t) \exp\left(-\frac{i2\pi kt}{T}\right) \mathrm{d}t.$$

The Fourier series of order k_c is defined as

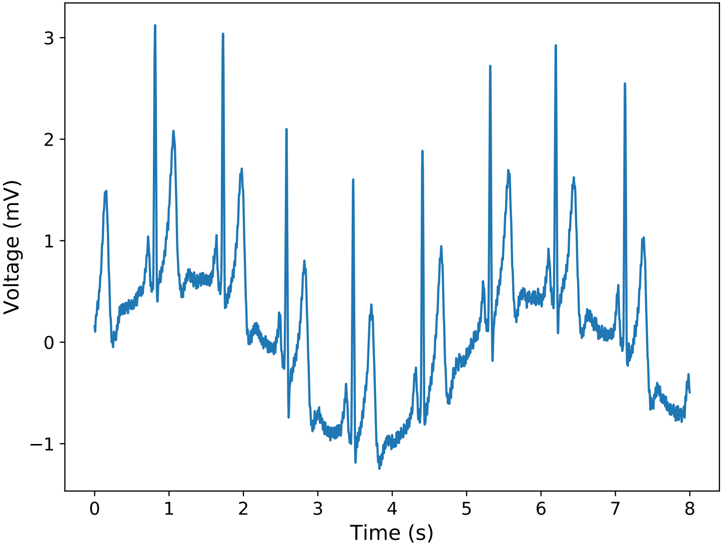
$$\mathcal{F}_{k_c} \{x\} := \frac{1}{T} \sum_{k=-k_c}^{k_c} \hat{x}[k]\phi_k$$

The Fourier series of x is $\lim_{k_c \to \infty} \mathcal{F}_{k_c} \{x\}$

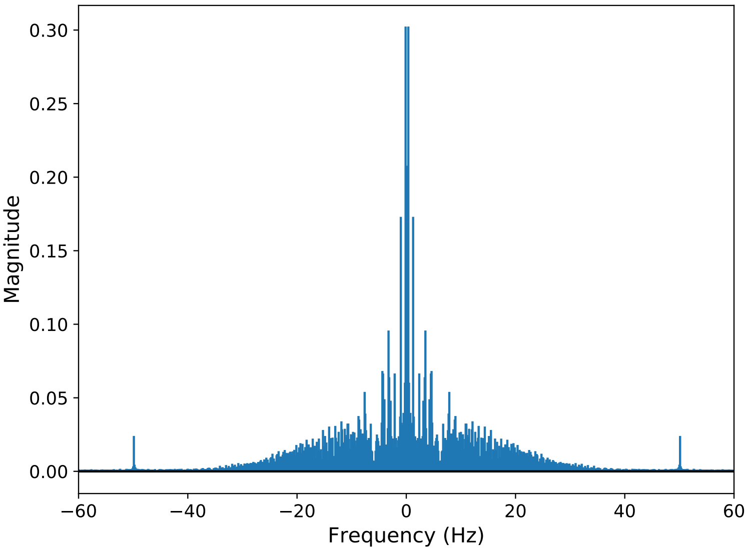
Fourier series as a projection

$$\begin{aligned}\mathcal{P}_{\text{span}(\{\phi_{-k_c},\phi_{-k_c+1},\ldots,\phi_{k_c}\})}\, x &= \sum_{k=-k_c}^{k_c} \left\langle x, \frac{1}{\sqrt{T}}\phi_k \right\rangle \frac{1}{\sqrt{T}}\phi_k \\ &= \mathcal{F}_{k_c}\left\{x\right\}\end{aligned}$$

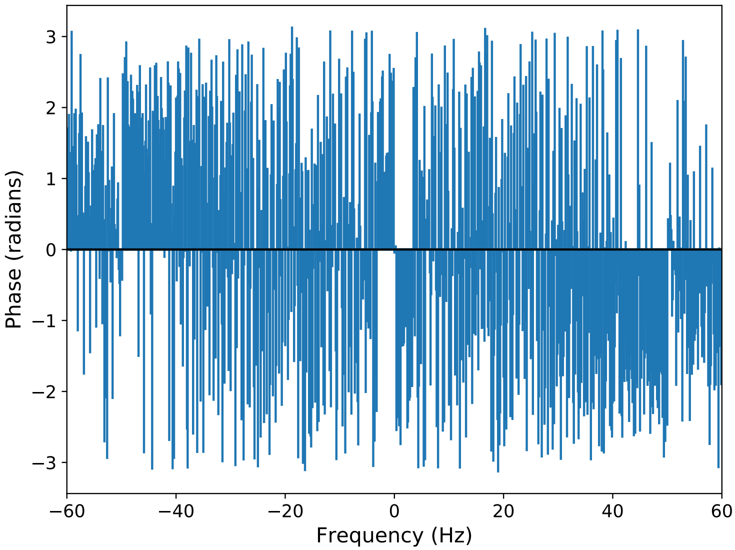
Electrocardiogram



Electrocardiogram: Fourier coefficients (magnitude)



Electrocardiogram: Fourier coefficients (phase)

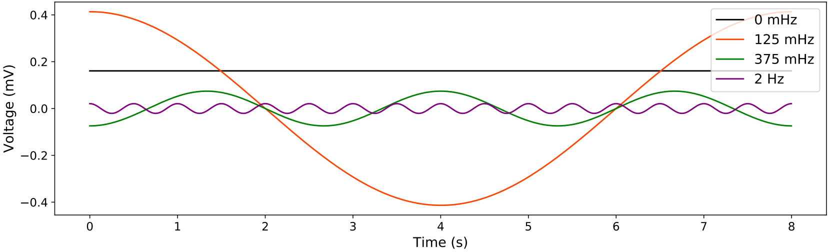


Convergence of Fourier series

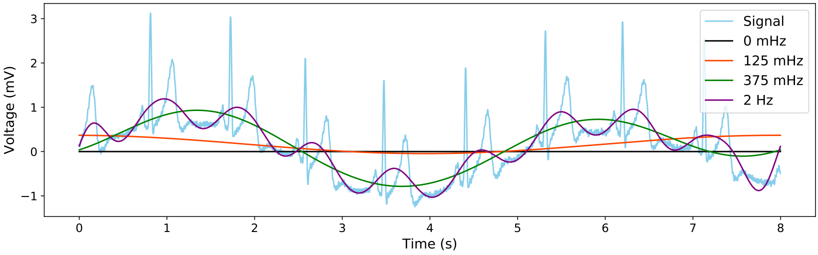
For any function $x \in \mathcal{L}_2[0, T)$, where $a, T \in \mathbb{R}$ and $T > 0$,

$$\lim_{k \to \infty} ||x - \mathcal{F}_k \{x\}||_{\mathcal{L}_2} = 0$$

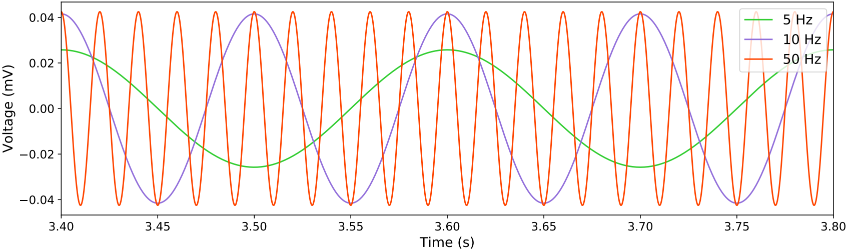
Electrocardiogram: Fourier components



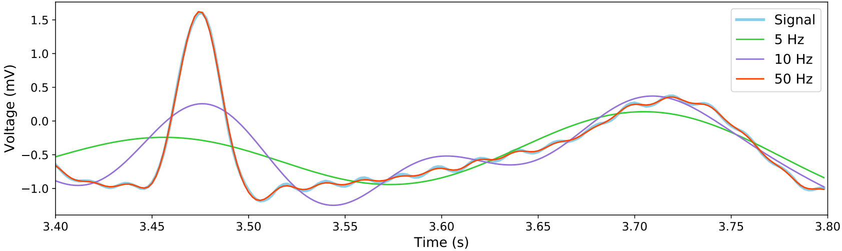
Electrocardiogram: Fourier series



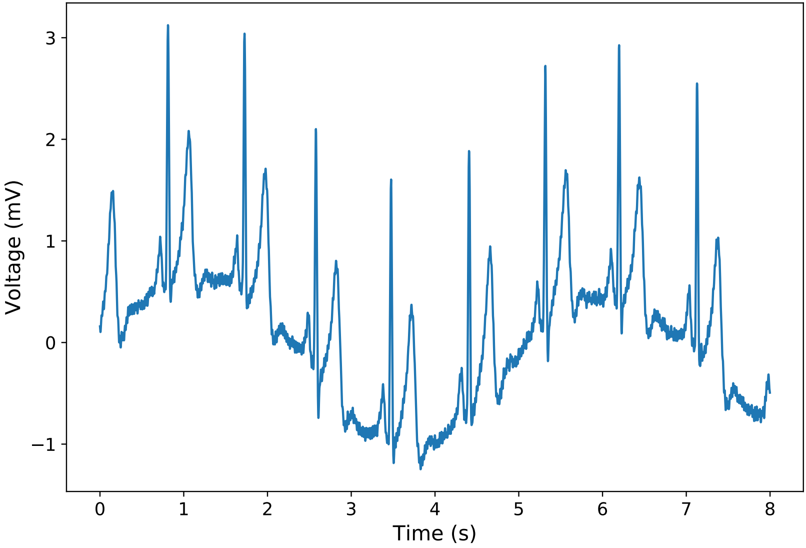
Electrocardiogram: Fourier components



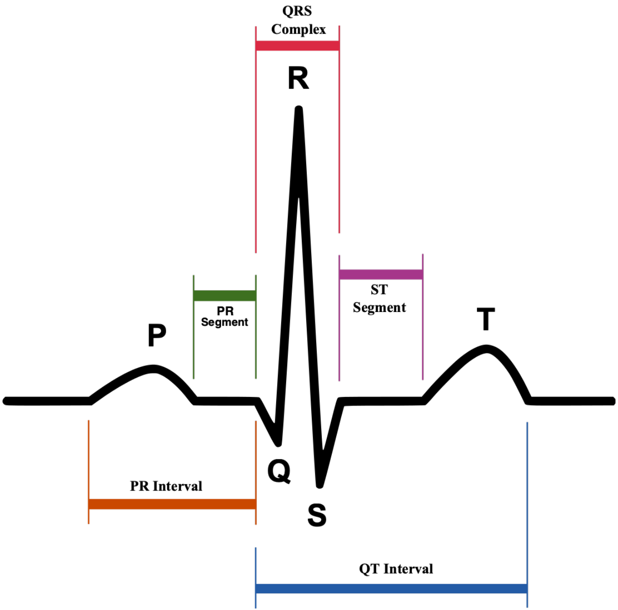
Electrocardiogram: Fourier series



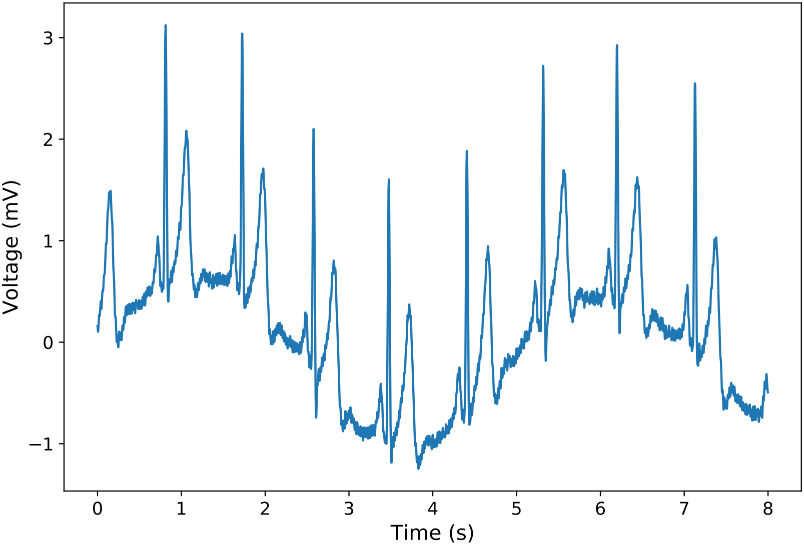
Electrocardiogram data



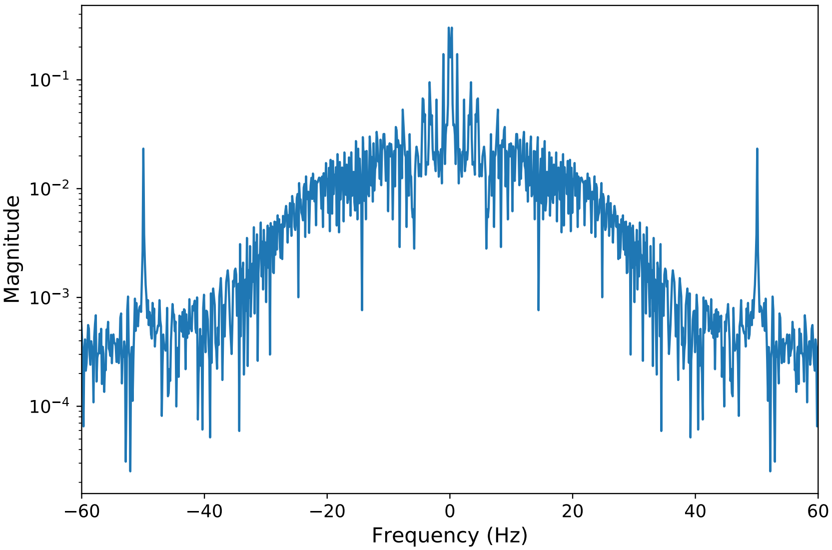
Electrocardiogram features



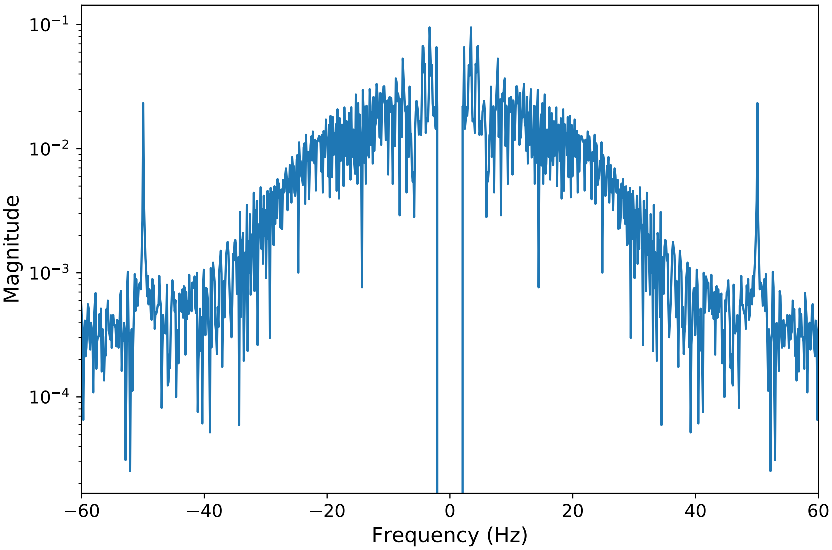
Problem: Baseline wandering



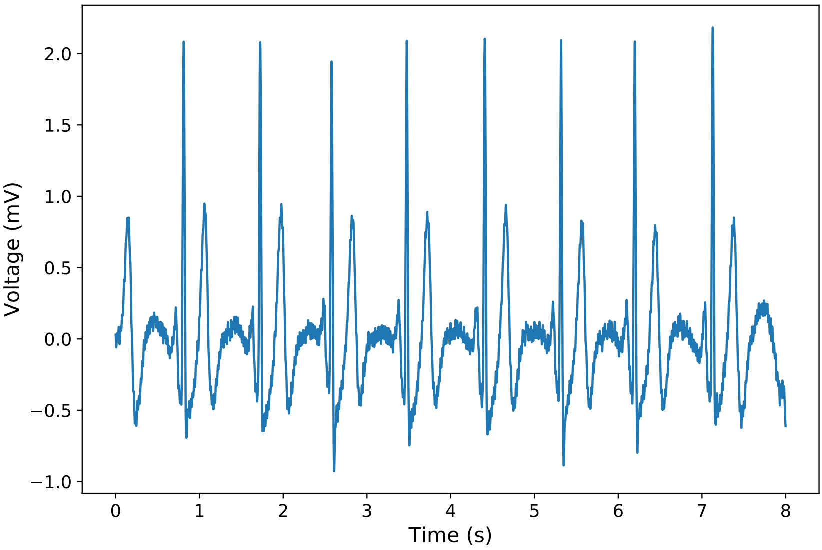
Electrocardiogram: Fourier coefficients (magnitude)



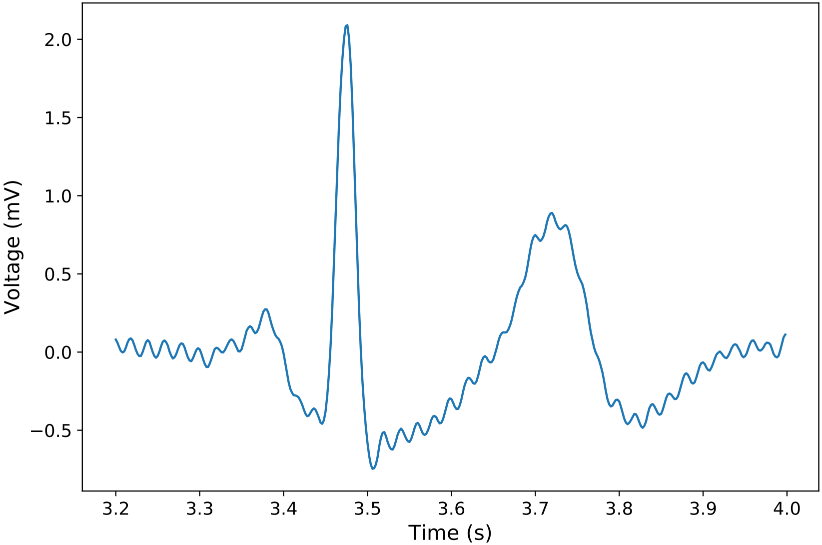
Filtered electrocardiogram



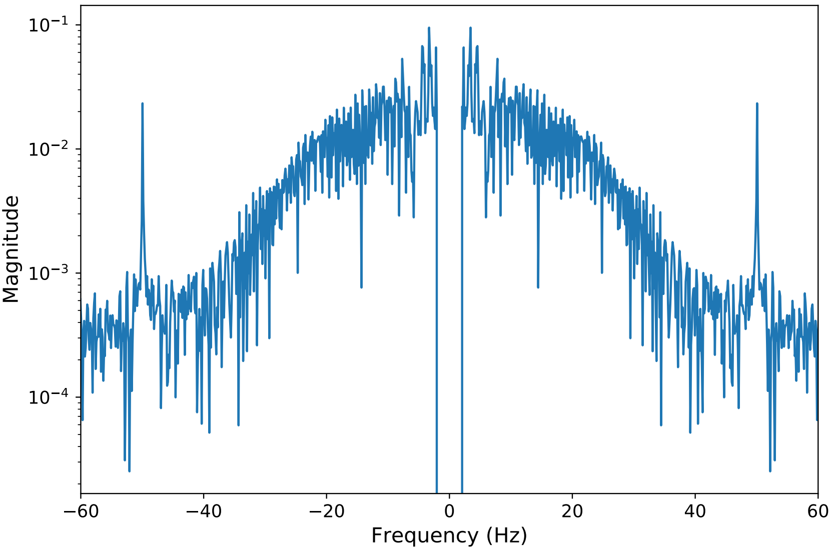
Filtered electrocardiogram



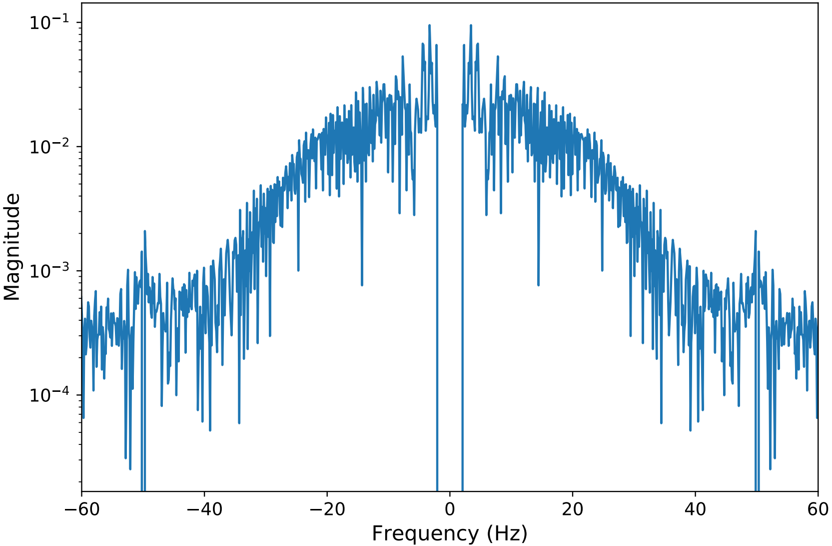
Problem: Interference



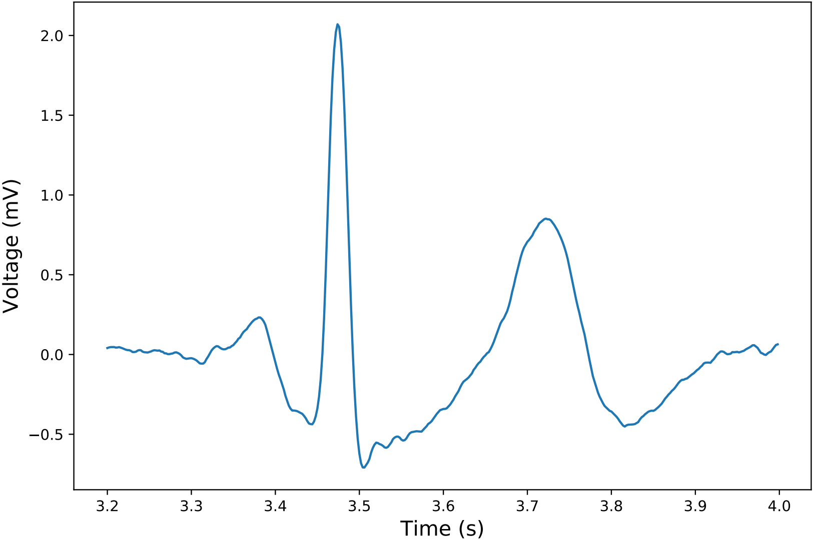
Fourier coefficients (magnitude)



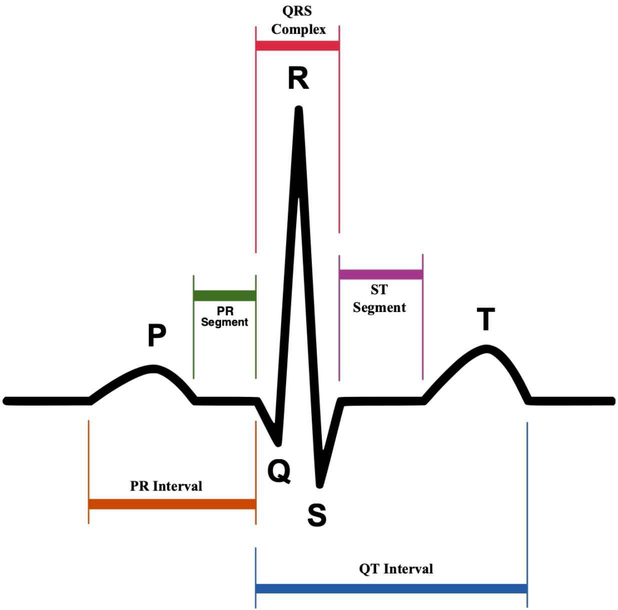
Filtered electrocardiogram



Filtered electrocardiogram



Electrocardiogram features



The frequency domain

Sampling

Discrete Fourier transform

Frequency representations in multiple dimensions

Sampling

Signals are often model continuous objects

Challenge: How to measure them so that they can stored/processed

A common way is sampling their values at specific locations

Sampling a complex sinusoid

Complex sinusoid ϕ_k in $[0, T)$

Samples at $\frac{jT}{N}$, $j \in \{0, 1, \ldots, N-1\}$

$$\begin{aligned}
\phi_k\left(\frac{j}{N}\right) &= \exp\left(\frac{i2\pi kj}{N}\right) \\
&= \exp\left(\frac{i2\pi(k+pN)j}{N}\right) \qquad \text{for any integer } p \\
&= \phi_{k+pN}\left(\frac{j}{N}\right)
\end{aligned}$$

Sampling a complex sinusoid

Indistinguishable frequencies: $\ldots,\ k-2N,\ k-N,\ k,\ k+N,\ k+2N,\ \ldots$

$N := 2k_c + 1$, how many between $-k_c$ and k_c?

All frequencies between $-k_c$ and k_c are distinguishable

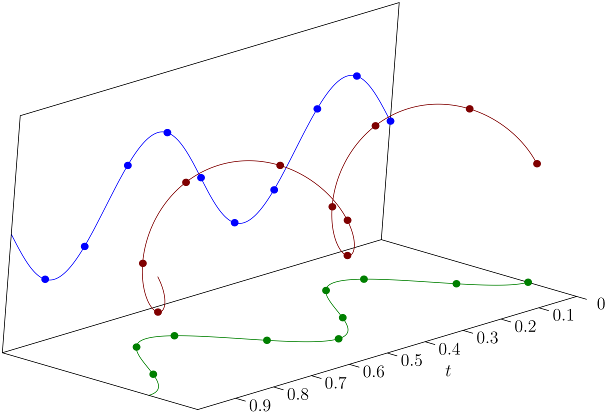
Discrete complex sinusoids

The discrete complex sinusoid $\psi_k \in \mathbb{C}^N$ with frequency k is

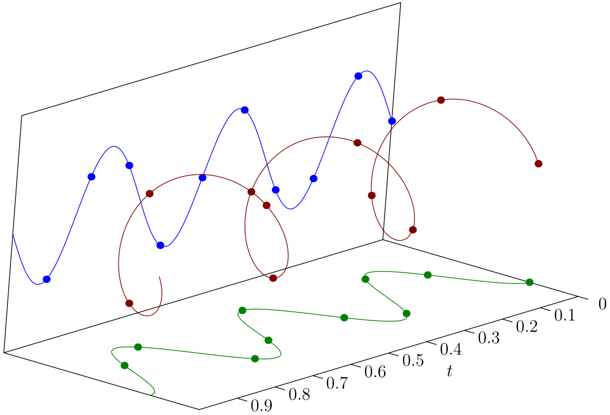
$$\psi_k[j] := \exp\left(\frac{i2\pi kj}{N}\right), \qquad 0 \leq j, k \leq N-1$$

Complex sinusoids scaled by $1/\sqrt{N}$ form an orthonormal basis of $\mathbb{C}^N$

ψ_2 (N=10)



ψ_3 (N=10)



Orthogonality

$$\begin{aligned}
\langle \psi_k, \psi_l \rangle &= \sum_{j=0}^{N-1} \psi_k[j] \, \overline{\psi_l[j]} \\
&= \sum_{j=0}^{N-1} \exp\left(\frac{i2\pi (k-l) j}{N} \right) \\
&= \frac{1 - \exp\left(\frac{i2\pi(k-l)N}{N} \right)}{1 - \exp\left(\frac{i2\pi(k-l)}{N} \right)} \\
&= 0 \qquad \text{if } k \neq l
\end{aligned}$$

Bandlimited signals

A bandlimited signal cut-off frequency k_c is equal to its Fourier series of order k_c

$$x(t) = \sum_{k=-k_c}^{k_c} \hat{x}[k] \exp\left(\frac{i2\pi kt}{T}\right)$$

Bandlimited signals have a finite representation ($2k_c + 1$ coefficients)

Sampling a bandlimited signal on a uniform grid

Bandimited signal x measured at N equispaced points in interval T

Samples: $x\left(\frac{0}{N}\right)$, $x\left(\frac{T}{N}\right)$, $x\left(\frac{2T}{N}\right)$, $\ldots$, $x\left(\frac{(N-1)T}{N}\right)$

Using Fourier series representation

$$
\begin{aligned}
x\left(\frac{jT}{N}\right) &= \sum_{k=-k_c}^{k_c} \hat{x}_k \exp\left(\frac{i2\pi k jT}{NT}\right) \\
&= \sum_{k=-k_c}^{k_c} \hat{x}_k \exp\left(\frac{i2\pi kj}{N}\right)
\end{aligned}
$$

In matrix form

$$\begin{bmatrix} x\left(\frac{0}{N}\right) \\ x\left(\frac{T}{N}\right) \\ \cdots \\ x\left(\frac{jT}{N}\right) \\ \cdots \\ x\left(T - \frac{T}{N}\right) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \exp\left(\frac{i2\pi(-k_c)}{N}\right) & \exp\left(\frac{i2\pi(-k_c+1)}{N}\right) & \cdots & \exp\left(\frac{i2\pi k_c}{N}\right) \\ \cdots & \cdots & \cdots & \cdots \\ \exp\left(\frac{i2\pi(-k_c)j}{N}\right) & \exp\left(\frac{i2\pi(-k_c+1)j}{N}\right) & \cdots & \exp\left(\frac{i2\pi k_c j}{N}\right) \\ \cdots & \cdots & \cdots & \cdots \\ \exp\left(\frac{i2\pi(-k_c)(N-1)}{N}\right) & \exp\left(\frac{i2\pi(-k_c+1)(N-1)}{N}\right) & \cdots & \exp\left(\frac{i2\pi k_c(N-1)}{N}\right) \end{bmatrix} \begin{bmatrix} \hat{x}[-k_c] \\ \hat{x}[-k_c+1] \\ \cdots \\ \hat{x}[k_c] \end{bmatrix}$$

$$x_{[N]} = \widetilde{F}_{[N]} \hat{x}_{[k_c]}$$

Nyquist-Shannon-Kotelnikov sampling theorem

Any bandlimited signal $x \in \mathcal{L}_2[0, T)$, where $T > 0$, with cut-off frequency k_c can be recovered exactly from N uniformly spaced samples $x(0), x(T/N), \ldots, x(T - T/N)$ as long as

$$N \geq 2k_c + 1,$$

where $2k_c + 1$ is known as the Nyquist rate

Recovery

$$\hat{x}_{[k_c]} = \frac{1}{N} \widetilde{F}^*_{[N]} x_{[N]}$$

$$\widetilde{F}_{[N]} := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \exp\left(\frac{i2\pi(-k_c)}{N}\right) & \exp\left(\frac{i2\pi(-k_c+1)}{N}\right) & \cdots & \exp\left(\frac{i2\pi k_c}{N}\right) \\ \cdots & \cdots & \cdots & \cdots \\ \exp\left(\frac{i2\pi(-k_c)(N-1)}{N}\right) & \exp\left(\frac{i2\pi(-k_c+1)(N-1)}{N}\right) & \cdots & \exp\left(\frac{i2\pi k_c(N-1)}{N}\right) \end{bmatrix}$$

Proof

For $-k_c \leq k \leq -1$ and $0 \leq j \leq N-1$,

$$\exp\left(\frac{i2\pi kj}{N}\right) = \exp\left(\frac{i2\pi\left(N+k\right)j}{N}\right)$$

$$\widetilde{F}_{[N]} = \begin{bmatrix} \psi_{N-k_c} & \cdots & \psi_{N-1} & \psi_0 & \cdots & \psi_{k_c} \end{bmatrix}$$

$\widetilde{F}_{[N]}$ is orthogonal!

Audio

Range of frequencies that human beings can hear is from 20 Hz to 20 kHz

At what frequency should we sample (at least)?

Typical rates used in practice: 44.1 kHz (CD), 48 kHz, 88.2 kHz, 96 kHz

Sampling a real sinusoid

Consider a real sinusoid with frequency equal to 4 Hz

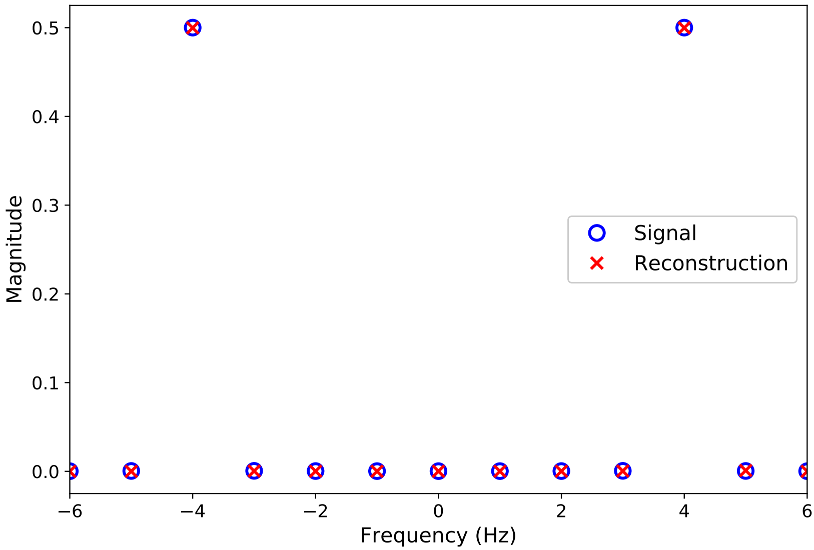
$$\begin{aligned} x(t) &:= \cos(8\pi t) \\ &= 0.5 \exp(-i2\pi 4t) + 0.5 \exp(i2\pi 4t) \end{aligned}$$

measured over one second, i.e. $T = 1$ s

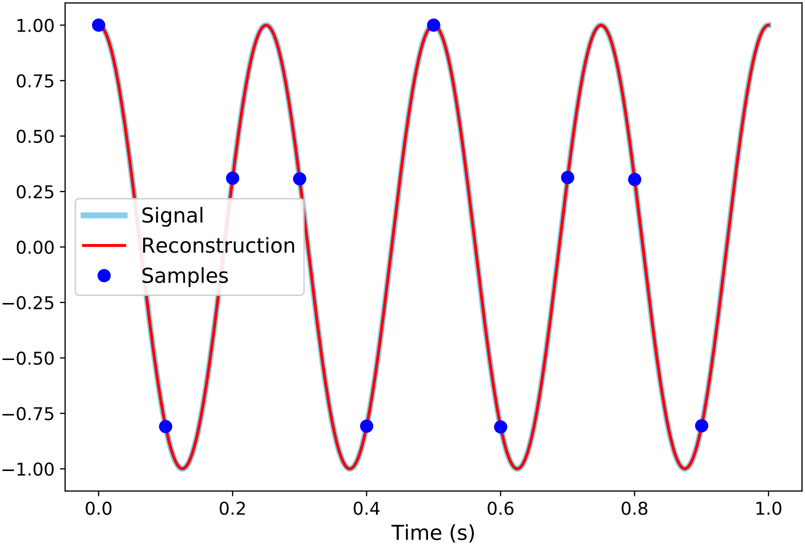
k_c? 4 Hz

Nyquist rate? 9 Hz

Recovered Fourier coefficients ($N = 10$)



Recovered signal ($N = 10$)



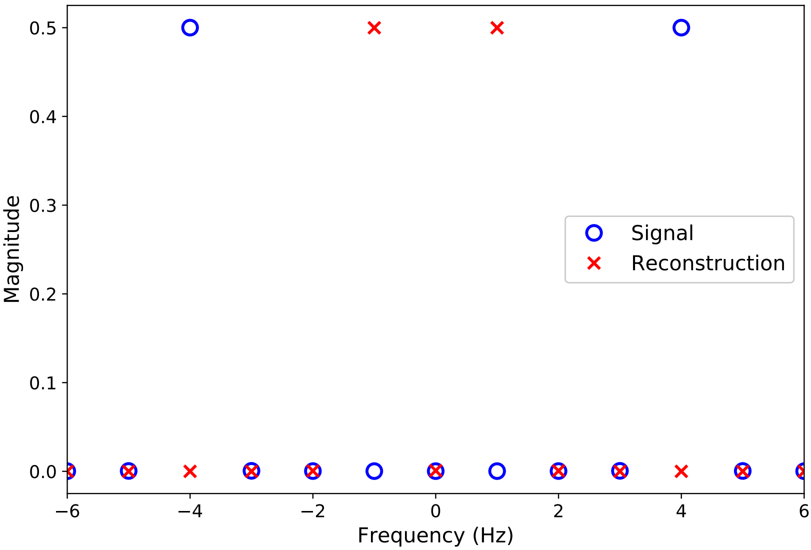
Sampling a real sinusoid

$$x(t) := \cos(8\pi t) = 0.5\exp(-i2\pi 4t) + 0.5\exp(i2\pi 4t)$$

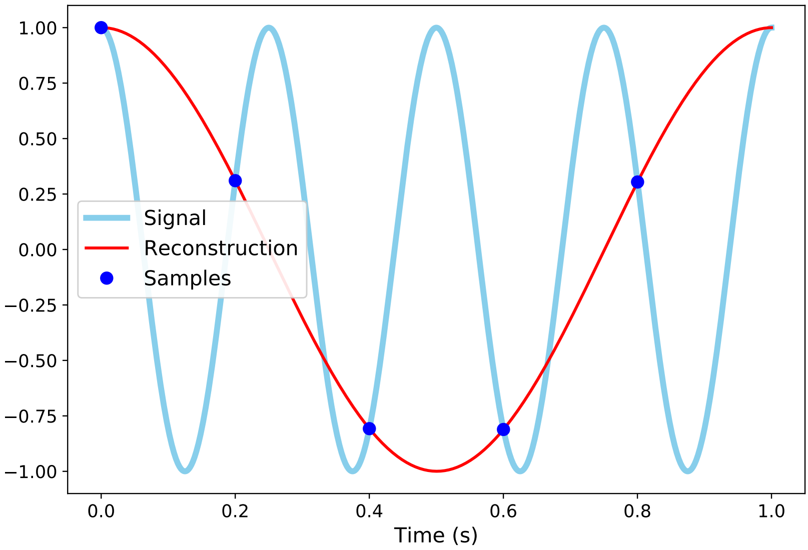
$$N = 5 \quad (\text{as if } k_c = 2)$$

$$\begin{aligned}
\hat{x}^{\text{rec}}[k] &= \sum_{\{(m-k) \bmod 5 = 0\}} \hat{x}[m] \\
\hat{x}^{\text{rec}}[-2] &= 0 \\
\hat{x}^{\text{rec}}[-1] &= \hat{x}^{\text{rec}}[4] = 0.5 \\
\hat{x}^{\text{rec}}[0] &= 0 \\
\hat{x}^{\text{rec}}[1] &= \hat{x}^{\text{rec}}[-4] = 0.5 \\
\hat{x}^{\text{rec}}[2] &= 0
\end{aligned}$$

Recovered Fourier coefficients ($N = 5$)



Recovered signal ($N = 5$)



Aliasing

Show videos

What happens if we sample too slowly?

Let x be a signal that is with cut-off frequency k_{true}

We measure $x_{[N]}$, N samples of x at 0, T/N, $2T/N$, $\ldots T - T/N$

What happens if we recover the signal assuming it is bandlimited with cut-off freq k_{samp}, $N = 2k_{\text{samp}} + 1$, but actually $k_{\text{true}} > k_{\text{samp}}$?

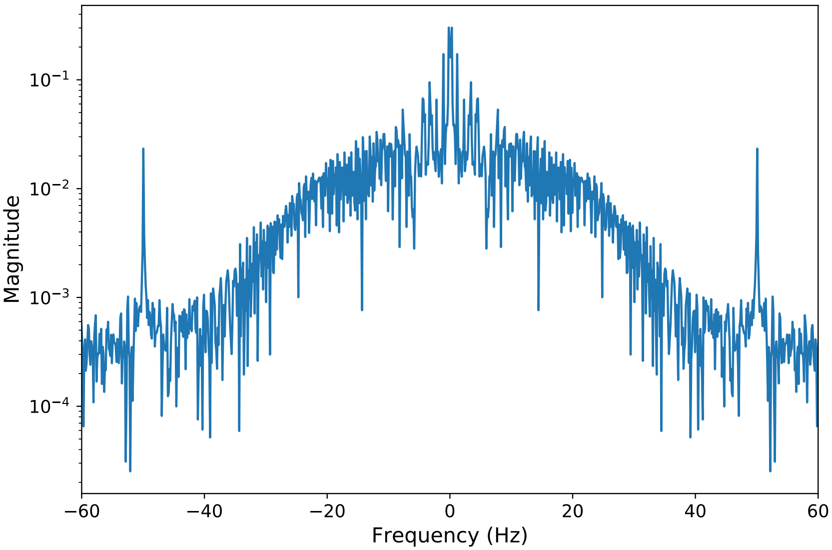
$$\begin{aligned}\hat{x}^{\text{rec}}[k] &:= \frac{1}{N}(\widetilde{F}^{*}_{[N]} x_{[N]})[k] \\ &= \sum_{\{(m-k) \bmod N = 0\}} \hat{x}[m]\end{aligned}$$

This is called aliasing

Proof

$$\begin{aligned}
\frac{1}{N}(\widetilde{F}_{[N]}^{*} x_{[N]})[k] &= \frac{1}{N}\sum_{j=0}^{N-1} \exp\left(-\frac{i2\pi kj}{N}\right) \sum_{m=-k_{\text{true}}}^{k_{\text{true}}} \hat{x}[m] \exp\left(\frac{i2\pi mj}{N}\right) \\
&= \frac{1}{N}\left\langle \psi_k, \sum_{(m-k) \bmod N = 0} \hat{x}[m] \psi_k \right\rangle \\
&= \sum_{\{(m-k) \bmod N = 0\}} \hat{x}[m]
\end{aligned}$$

Electrocardiogram: Fourier coefficients (magnitude)



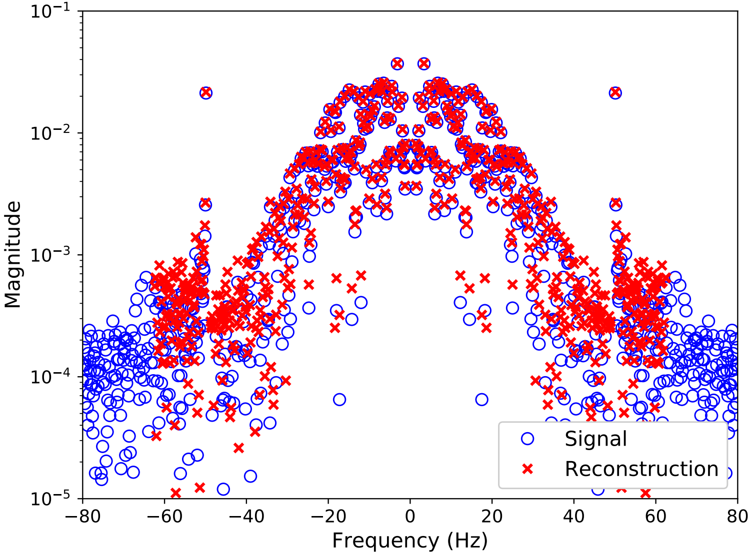
Sampling an electrocardiogram

Signal is approximately bandlimited at 50 Hz

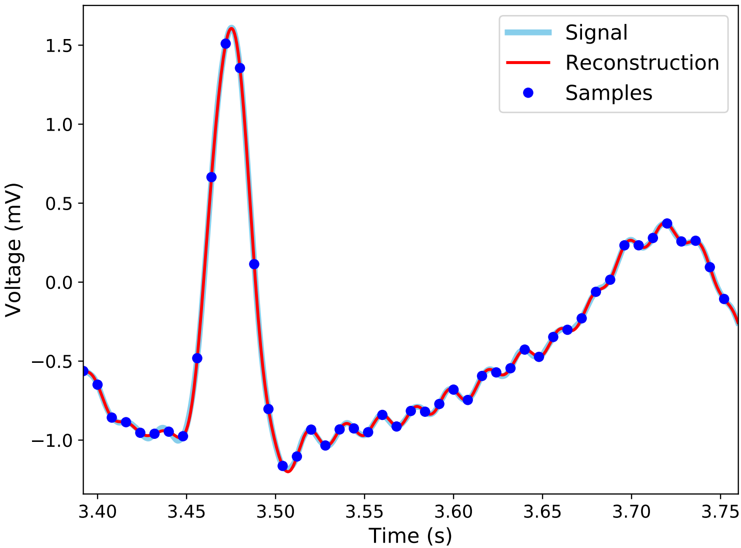
$T = 8$ s, so $k_c = 50/(1/T) = 400$

To avoid aliasing $N \geq 801$

Recovered Fourier coefficients (N=1,000)



Recovered signal (N=1,000)



Sampling an electrocardiogram

Signal is approximately bandlimited at 50 Hz

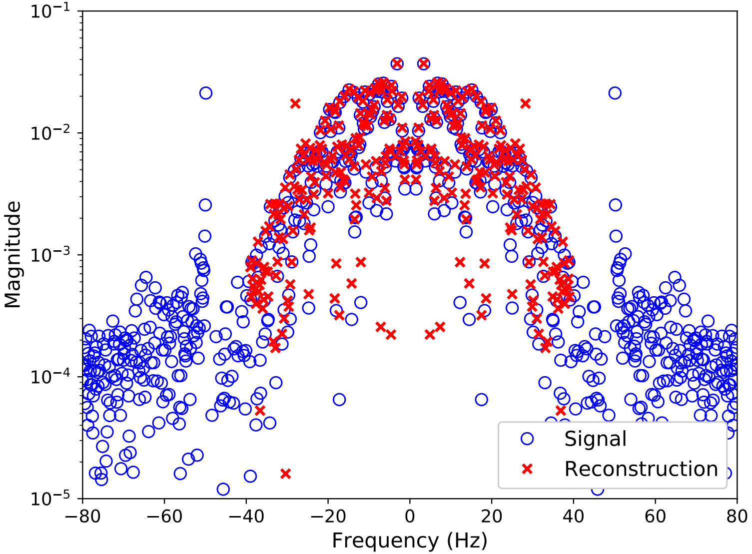
$T = 8$ s, so $k_c = 50/(1/T) = 400$

$N = 625$

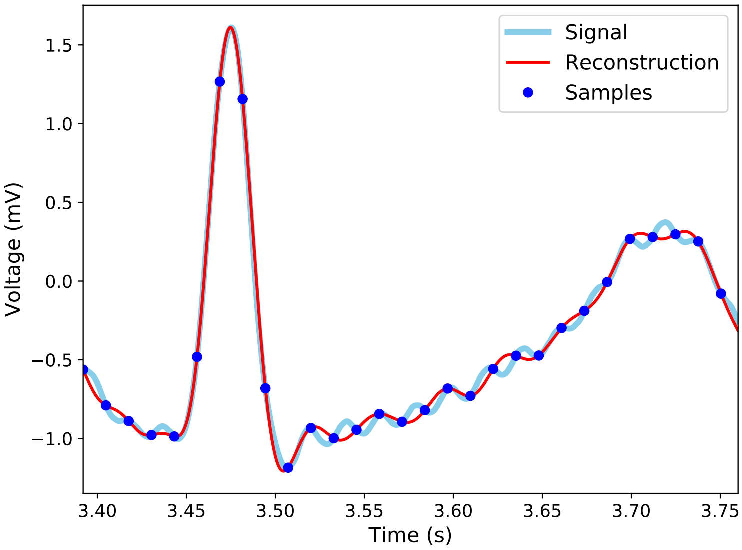
$$\hat{x}^{\text{rec}}[k] = \sum_{\{(m-k) \bmod 625 = 0\}} \hat{x}[m]$$

Component at $m = \pm 400$ (50 Hz) shows up at ± 225 (28.1 Hz)

Recovered Fourier coefficients ($N = 625$)



Recovered signal ($N = 625$)



The frequency domain

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Discrete Fourier transform

Frequency representations in multiple dimensions

Discrete complex sinusoids

The discrete complex sinusoid $\psi_k \in \mathbb{C}^N$ with frequency k is

$$\psi_k[j] := \exp\left(\frac{i 2\pi k j}{N}\right), \qquad 0 \leq j, k \leq N-1$$

Discrete complex sinusoids scaled by $1/\sqrt{N}$: orthonormal basis of $\mathbb{C}^N$

Discrete Fourier transform

The discrete Fourier transform (DFT) of $x \in \mathbb{C}^N$ is

$$\hat{x} := \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \exp\left(-\frac{i2\pi}{N}\right) & \exp\left(-\frac{i2\pi 2}{N}\right) & \cdots & \exp\left(-\frac{i2\pi(N-1)}{N}\right) \\ 1 & \exp\left(-\frac{i2\pi 2}{N}\right) & \exp\left(-\frac{i2\pi 4}{N}\right) & \cdots & \exp\left(-\frac{i2\pi 2(N-1)}{N}\right) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \exp\left(-\frac{i2\pi(N-1)}{N}\right) & \exp\left(-\frac{i2\pi 2(N-1)}{N}\right) & \cdots & \exp\left(-\frac{i2\pi(N-1)^2}{N}\right) \end{bmatrix} x$$

$$= F_{[N]} x$$

$$\hat{x}[k] = \langle x, \psi_k \rangle, \qquad 0 \le k \le N-1$$

Inverse discrete Fourier transform

The inverse DFT of a vector $\hat{y} \in \mathbb{C}^N$ equals

$$\vec{y} = \frac{1}{N} F_{[N]}^{*} \hat{y}$$

It inverts the DFT

Interpretation in terms of bandlimited signals

If $x \in \mathbb{C}^N$ contains samples of a bandlimited signal such that $2k_c + 1 \leq N$ the DFT contains the Fourier series coefficients of the function

$$\hat{x}_{[k_c]} = \frac{1}{N} \widetilde{F}_{[N]}^{*} x_{[N]}$$

$$\widetilde{F}_{[N]} := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \exp\left(\frac{i2\pi(-k_c)}{N}\right) & \exp\left(\frac{i2\pi(-k_c+1)}{N}\right) & \cdots & \exp\left(\frac{i2\pi k_c}{N}\right) \\ \cdots & \cdots & \cdots & \cdots \\ \exp\left(\frac{i2\pi(-k_c)(N-1)}{N}\right) & \exp\left(\frac{i2\pi(-k_c+1)(N-1)}{N}\right) & \cdots & \exp\left(\frac{i2\pi k_c(N-1)}{N}\right) \end{bmatrix}$$

Rows of $\widetilde{F}_{[N]}$ equal rows of $F_{[N]}$ in a different order!

Complexity of computing the DFT

Complexity of multiplying $N \times N$ matrix with N-dim. vector is N^2

Very slow!

We can exploit the structure of the matrix to do much better

Fast Fourier transform

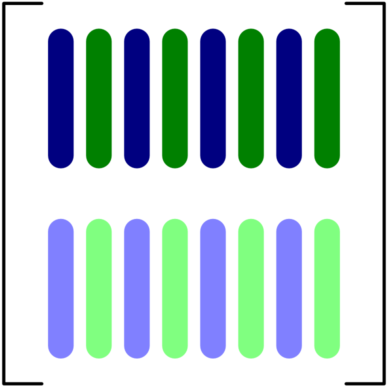
The most important numerical algorithm of our lifetime (G. Strang)

Main insight:

Action of N-order DFT matrix on vector can be decomposed into action of $N/2$-order DFT submatrices on subvectors

DFT

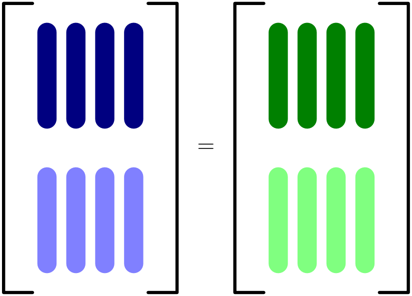
$$\begin{bmatrix} \hat{x}[0] \\ \hat{x}[1] \\ \hat{x}[2] \\ \hat{x}[3] \\ \\ \hat{x}[4] \\ \hat{x}[5] \\ \hat{x}[6] \\ \hat{x}[7] \end{bmatrix} =$$



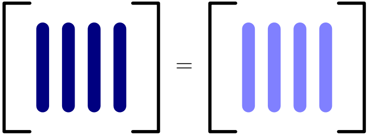
$$\begin{bmatrix} \vec{x}[0] \\ \vec{x}[1] \\ \vec{x}[2] \\ \vec{x}[3] \\ \vec{x}[4] \\ \vec{x}[5] \\ \vec{x}[6] \\ \vec{x}[7] \end{bmatrix}$$

DFT

$$\begin{bmatrix} e^{-2\pi i(0)/8} & & & & & & & \\ & e^{-2\pi i(1)/8} & & & & & & \\ & & e^{-2\pi i(2)/8} & & & & & \\ & & & e^{-2\pi i(3)/8} & & & & \\ & & & & e^{-2\pi i(4)/8} & & & \\ & & & & & e^{-2\pi i(5)/8} & & \\ & & & & & & e^{-2\pi i(6)/8} & \\ & & & & & & & e^{-2\pi i(7)/8} \end{bmatrix}$$



DFT



DFT

$$\begin{bmatrix} \hat{x}[0] \\ \hat{x}[1] \\ \hat{x}[2] \\ \hat{x}[3] \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} \vec{x}[0] \\ \vec{x}[2] \\ \vec{x}[4] \\ \vec{x}[6] \end{bmatrix} + \begin{bmatrix} e^{-2\pi i(0)/8} & & & \\ & e^{-2\pi i(1)/8} & & \\ & & e^{-2\pi i(2)/8} & \\ & & & e^{-2\pi i(3)/8} \end{bmatrix} \begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} \vec{x}[1] \\ \vec{x}[3] \\ \vec{x}[5] \\ \vec{x}[7] \end{bmatrix}$$

$$\begin{bmatrix} \hat{x}[4] \\ \hat{x}[5] \\ \hat{x}[6] \\ \hat{x}[7] \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} \vec{x}[0] \\ \vec{x}[2] \\ \vec{x}[4] \\ \vec{x}[6] \end{bmatrix} + \begin{bmatrix} e^{-2\pi i(4)/8} & & & \\ & e^{-2\pi i(5)/8} & & \\ & & e^{-2\pi i(6)/8} & \\ & & & e^{-2\pi i(7)/8} \end{bmatrix} \begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} \vec{x}[1] \\ \vec{x}[3] \\ \vec{x}[5] \\ \vec{x}[7] \end{bmatrix}$$

DFT

Let $F_{[N]}$ denote the $N \times N$ DFT matrix, where N is even.
For $k = 0, 1, \ldots, N/2 - 1$, and any vector $x \in \mathbb{C}^N$

$$F_{[N]}x\,[k] = F_{[N/2]}x_{\text{even}}\,[k] + \exp\left(-\frac{i2\pi k}{N}\right) F_{[N/2]}x_{\text{odd}}\,[k]\,,$$

$$F_{[N]}x\,[k + N/2] = F_{[N/2]}x_{\text{even}}\,[k] - \exp\left(-\frac{i2\pi k}{N}\right) F_{[N/2]}x_{\text{odd}}\,[k]\,,$$

where x_{even} and x_{odd} contain the even and odd entries of $\vec{x}$ respectively.

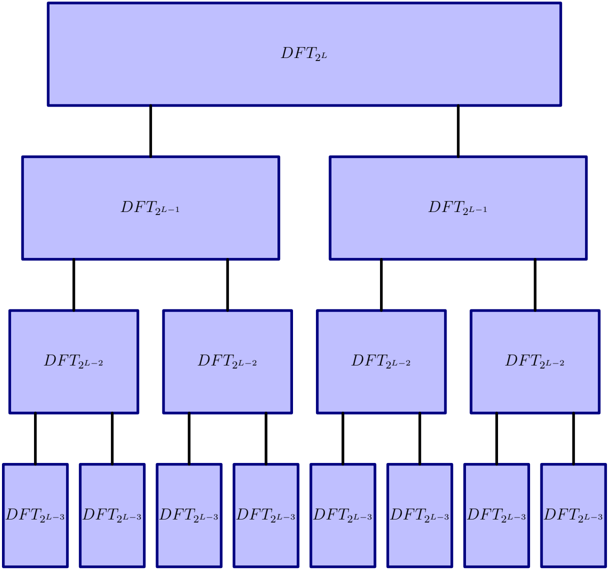
Cooley-Tukey Fast Fourier transform

1. Compute $F_{[N/2]}x_{\text{even}}$.
2. Compute $F_{[N/2]}x_{\text{odd}}$.
3. For $k = 0, 1, \ldots, N/2 - 1$ set

$$F_{[N]}x\left[k\right] := F_{[N/2]}x_{\text{even}}\left[k\right] + \exp\left(-\frac{i2\pi k}{N}\right)F_{[N/2]}x_{\text{odd}}\left[k\right],$$

$$F_{[N]}x\left[k + N/2\right] := F_{[N/2]}x_{\text{even}}\left[k\right] - \exp\left(-\frac{i2\pi k}{N}\right)F_{[N/2]}x_{\text{odd}}\left[k\right].$$

Complexity


DFT_{2^L}
$DFT_{2^{L-1}}$
$DFT_{2^{L-1}}$
$DFT_{2^{L-2}}$
$DFT_{2^{L-2}}$
$DFT_{2^{L-2}}$
$DFT_{2^{L-2}}$
$DFT_{2^{L-3}}$
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$DFT_{2^{L-3}}$
$DFT_{2^{L-3}}$
$DFT_{2^{L-3}}$
$DFT_{2^{L-3}}$

Complexity

Assume $N = 2^L$

$L = \log_2 N$ levels

At level $l \in \{1, \ldots, L\}$ there are 2^l nodes

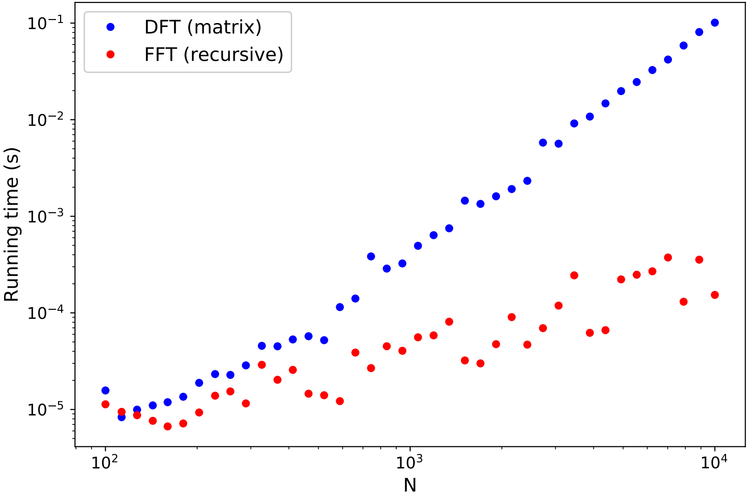
At each node, scale a vector of dim 2^{L-l} and add to another vector

Complexity at each node: 2^{L-l}

Complexity at each level: $2^{L-l} 2^l = 2^L = N$

Complexity is $O(N \log N)$!

In practice



DFT (matrix)
FFT (recursive)
Running time (s)
N
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{2}
10^{3}
10^{4}

Multidimensional signals

Square-integrable functions defined on a hyperrectangle
$\mathcal{I} := [a_1, b_1] \times \ldots \times [a_p, b_p] \subset \mathbb{R}^p$

Inner product:

$$\langle x, y \rangle := \int_{\mathcal{I}} x(t) \overline{y(t)} \, \mathrm{d}t.$$

Goal: Extension of frequency representations to multidimensional signals

Multidimensional sinusoid

$$a \cos\left(2\pi\langle f, t \rangle + \theta\right).$$

The frequency and time indices are now d-dimensional

Periodic with period $1/\left|\left|f\right|\right|_2$ in direction of f

For any integer m

$$\begin{aligned} a \cos\left(i2\pi \left\langle f, t + \frac{m}{\left|\left|f\right|\right|_2}\frac{f}{\left|\left|f\right|\right|_2}\right\rangle + \theta\right) &= a \cos\left(i2\pi\langle f, t\rangle + i2\pi m + \theta\right) \\ &= a \cos\left(i2\pi\langle f, t \rangle + \theta\right) \end{aligned}$$

Multidimensional complex sinusoids

Complex sinusoid with frequency $f \in R^d$:

$$\exp(i2\pi\langle f, t\rangle) := \cos(2\pi\langle f, t\rangle) + i\sin(2\pi\langle f, t\rangle).$$

$$\cos\left(i2\pi\langle f, t\rangle + \theta\right) = \frac{\exp(i\theta)}{2}\exp(i2\pi\langle f, t\rangle) + \frac{\exp(-i\theta)}{2}\exp(-i2\pi\langle f, t\rangle)$$

Multidimensional complex sinusoids

Can be expressed as product of 1D complex sinusoids

$$\exp(i2\pi\langle f, t\rangle) := \exp\left(i2\pi\sum_{j=1}^{d} f[j]t[j]\right)$$
$$= \prod_{j=1}^{d}\exp(i2\pi f[j]t[j])$$

From now on $d = 2$: $t[1] = t_1$, $t[2] = t_2$

Orthogonality of multidimensional complex sinusoids

The family of complex sinusoids with integer frequencies

$$\phi^{2D}_{k_1,k_2}(t_1,t_2) := \exp\left(\frac{i2\pi k_1 t_1}{T}\right)\exp\left(\frac{i2\pi k_2 t_2}{T}\right), \qquad k_1, k_2 \in \mathbb{Z},$$

is an orthogonal set of functions on any interval of the form $[a, a+T] \times [b, b+T]$, $a, b, T \in \mathbb{R}$ and $T > 0$

Proof

We have

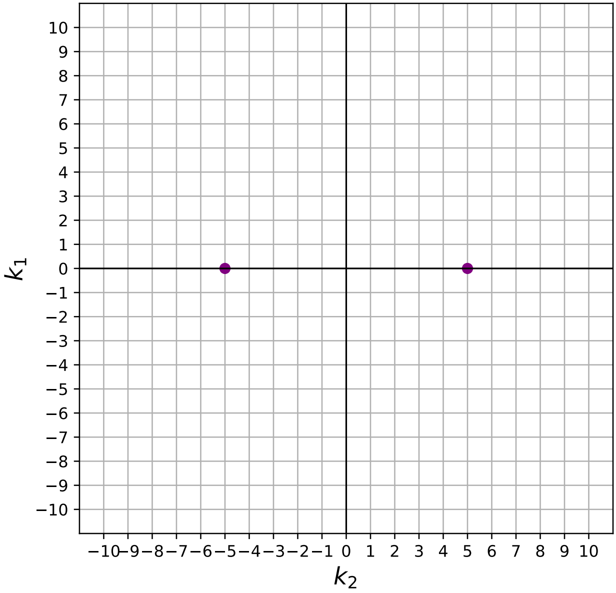
$$\phi^{\text{2D}}_{k_1,k_2}(t_1,t_2) = \phi_{k_1}(t_1)\,\phi_{k_2}(t_2),$$

so that

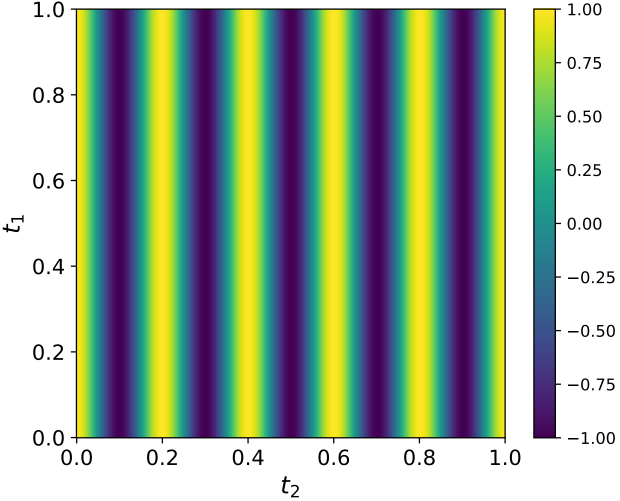
$$\begin{aligned}\left\langle \phi^{\text{2D}}_{k_1,k_2}, \phi^{\text{2D}}_{j_1,j_2} \right\rangle &= \int_{t_1=a}^{a+T} \int_{t_2=b}^{b+T} \phi_{k_1}(t_1)\,\phi_{k_2}(t_2)\,\overline{\phi_{j_1}(t_1)\,\phi_{j_2}(t_2)}\,\mathrm{d}t_1\,\mathrm{d}t_2 \\ &= \langle \phi_{k_1}, \phi_{j_1} \rangle \langle \phi_{k_2}, \phi_{j_2} \rangle \\ &= 0\end{aligned}$$

as long as $j_1 \neq k_1$ or $j_2 \neq k_2$

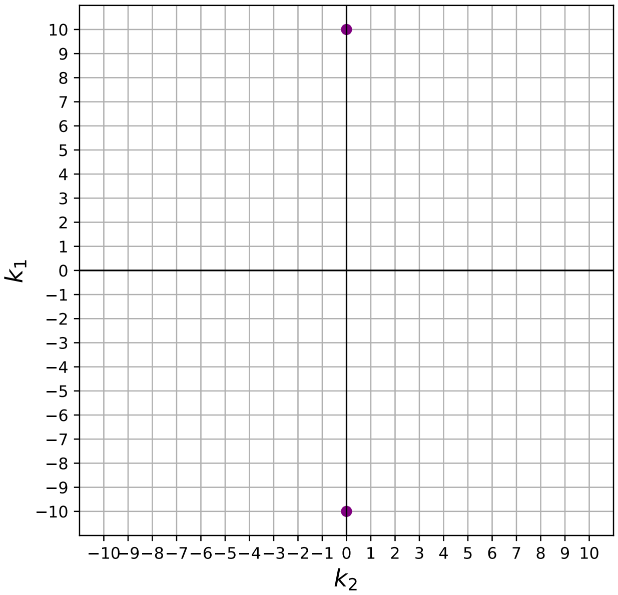
$\phi^{2D}_{0,5} + \phi^{2D}_{0,-5}$



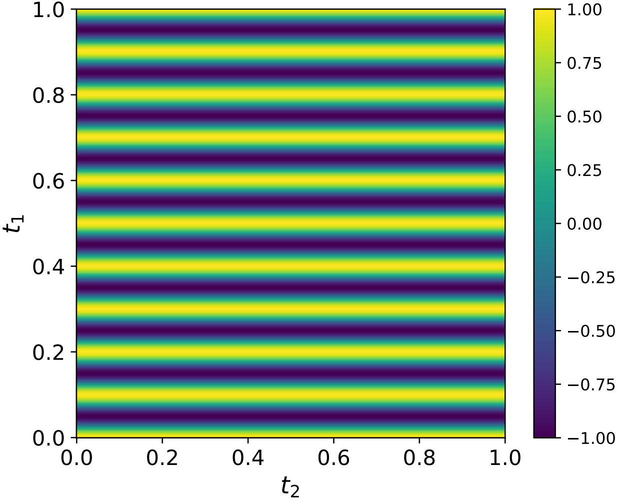
$\phi^{\text{2D}}_{0,5} + \phi^{\text{2D}}_{0,-5}$



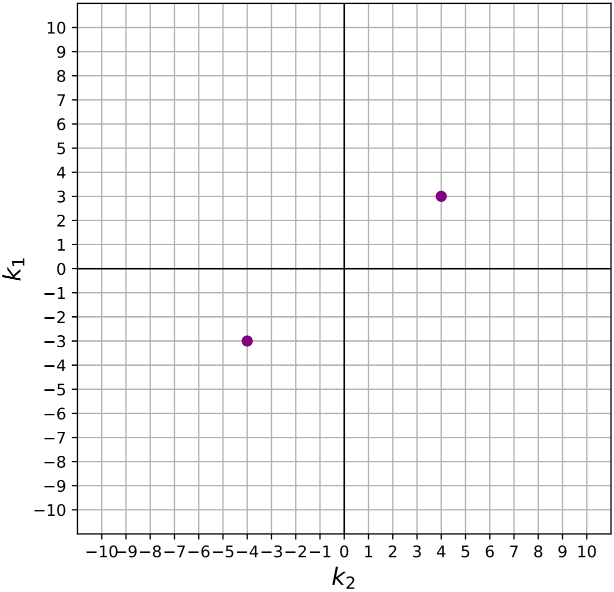
$$\phi^{2D}_{10,0} + \phi^{2D}_{-10,0}$$



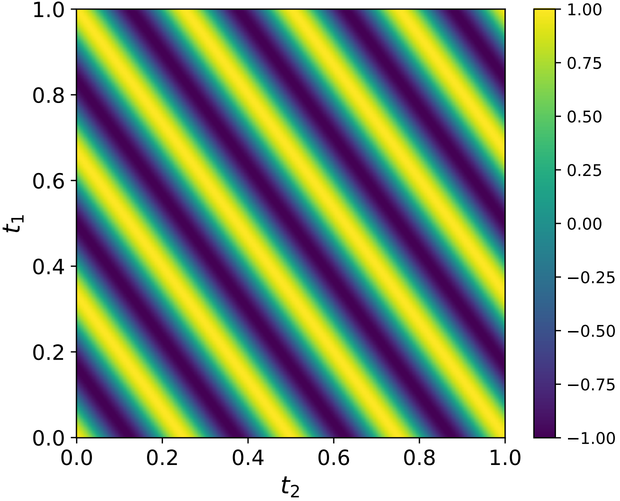
$\phi^{2D}_{10,0} + \phi^{2D}_{-10,0}$



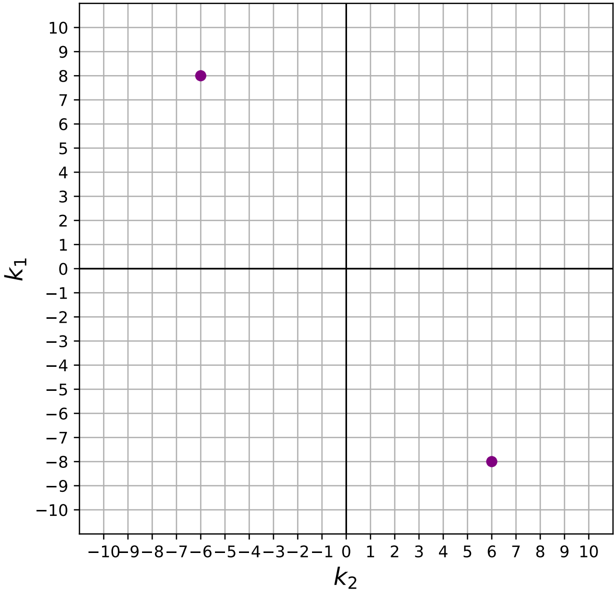
$\phi^{\mathrm{2D}}_{3,4} + \phi^{\mathrm{2D}}_{-3,-4}$



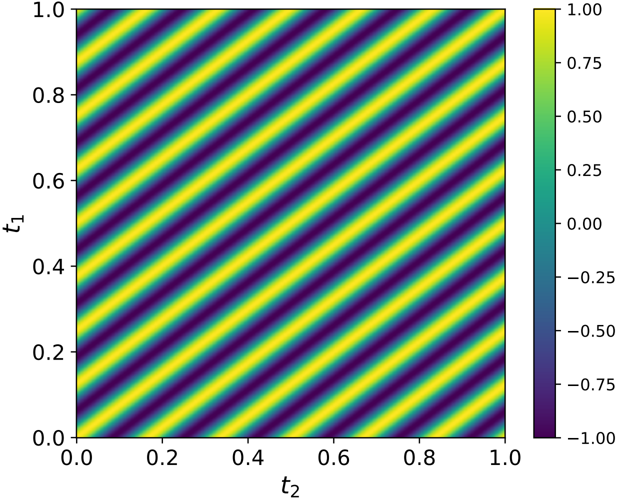
$\phi^{\mathrm{2D}}_{3,4} + \phi^{\mathrm{2D}}_{-3,-4}$



$\phi^{\text{2D}}_{8,-6} + \phi^{\text{2D}}_{-8,6}$



$\phi^{\mathrm{2D}}_{8,-6} + \phi^{\mathrm{2D}}_{-8,6}$



2D Fourier series

The Fourier series coefficients of a function $x \in \mathcal{L}_2\left[a, a+T\right]$ for any $a, T \in \mathbb{R},\ T > 0$, are given by

$$\hat{x}[k_1, k_2] := \left\langle x, \phi^{\text{2D}}_{k_1,k_2} \right\rangle$$
$$= \int_{t_1=a}^{a+T} \int_{t_2=b}^{b+T} x(t_1, t_2) \exp\left(-\frac{i 2\pi k_1 t_1}{T}\right) \exp\left(-\frac{i 2\pi k_2 t_2}{T}\right) \,\mathrm{d}t_1 \,\mathrm{d}t_2$$

The Fourier series of order $k_{c,1},\ k_{c,2}$ is defined as

$$\mathcal{F}_{k_{c,1},k_{c,2}}\left\{x\right\} := \frac{1}{T} \sum_{k_1=-k_{c,1}}^{k_{c,1}} \sum_{k_2=-k_{c,2}}^{k_{c,2}} \hat{x}[k_1, k_2] \phi^{\text{2D}}_{k_1,k_2}.$$

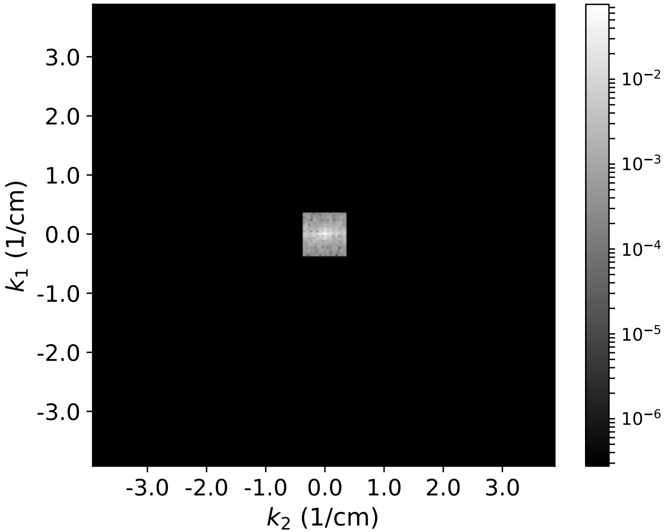
Magnetic resonance imaging

Non-invasive medical-imaging technique

Measures response of atomic nuclei in biological tissues to high-frequency radio waves when placed in a strong magnetic field

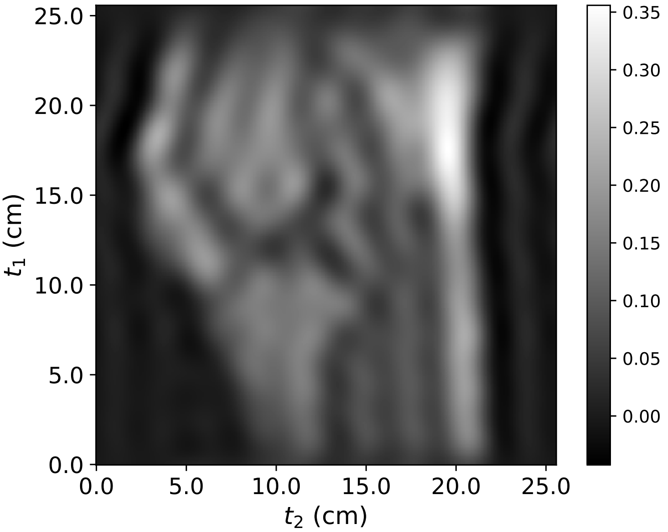
Radio waves adjusted so that each measurement equals 2D Fourier coefficients of proton density of hydrogen atoms in a region of interest

Data

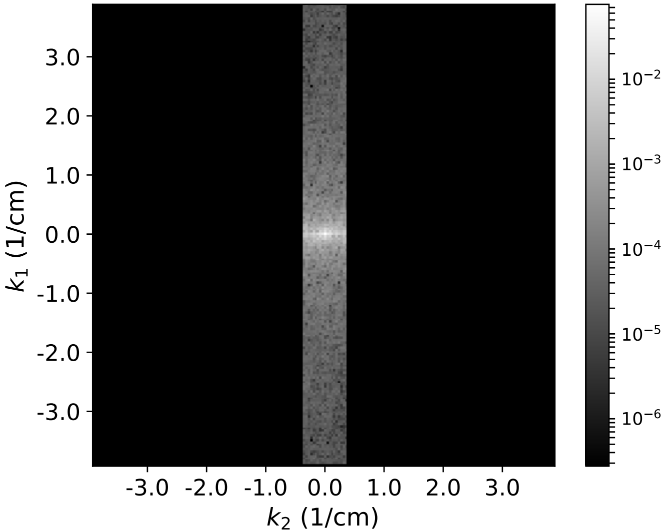


k_1 (1/cm)
3.0
2.0
1.0
0.0
-1.0
-2.0
-3.0
-3.0 -2.0 -1.0 0.0 1.0 2.0 3.0
k_2 (1/cm)
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}

Recovered image

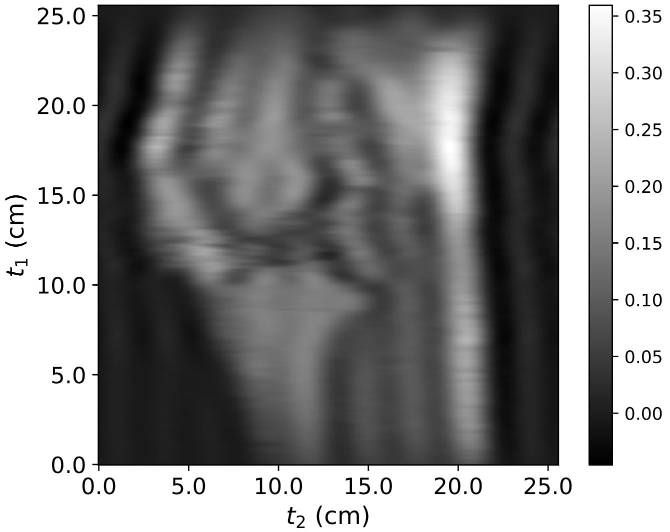


Data

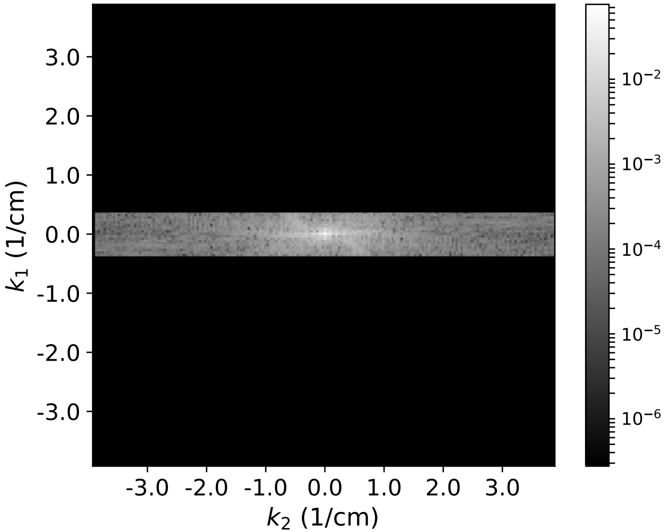


3.0
2.0
1.0
0.0
-1.0
-2.0
-3.0
k_1 (1/cm)
-3.0 -2.0 -1.0 0.0 1.0 2.0 3.0
k_2 (1/cm)
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}

Recovered image

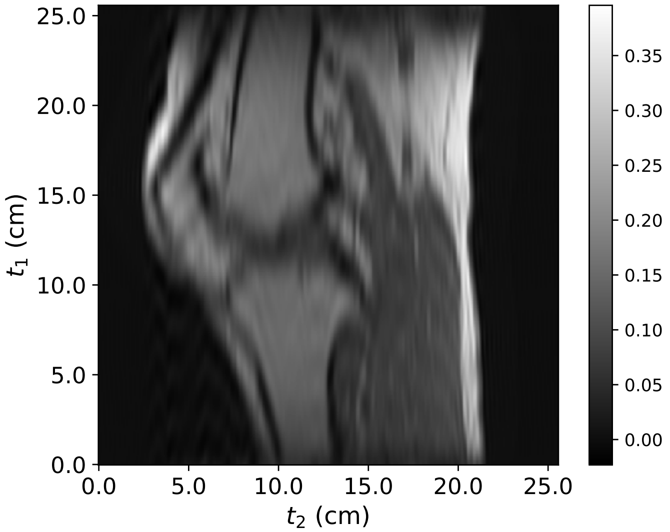


Data

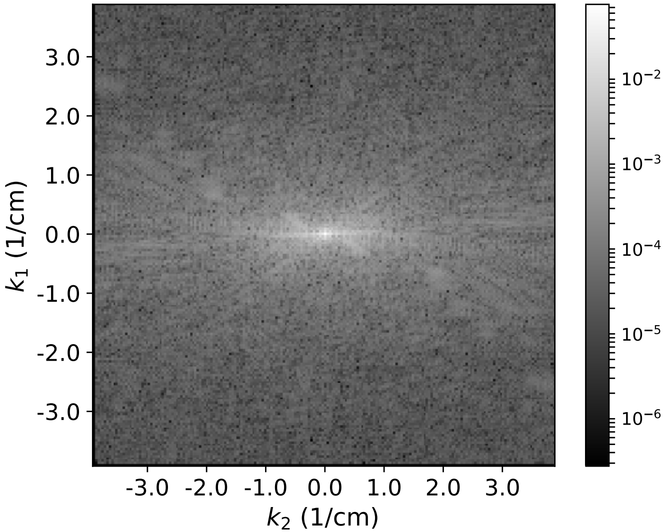


3.0
2.0
1.0
0.0
-1.0
-2.0
-3.0
k_1 (1/cm)
-3.0 -2.0 -1.0 0.0 1.0 2.0 3.0
k_2 (1/cm)
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}

Recovered image

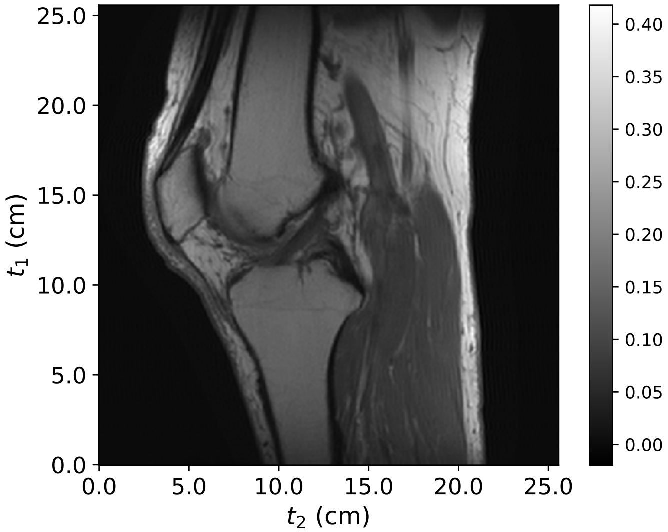


Data



3.0
2.0
1.0
0.0
-1.0
-2.0
-3.0
k_1 (1/cm)
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
k_2 (1/cm)
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}

Recovered image



Bandlimited signal

A signal defined on the 2D rectangle $[a, a + T] \times [b, b + T]$, where $a, b, T \in \mathbb{R}$ and $T > 0$ is bandlimited with a cut-off frequency k_c if it is equal to its Fourier series representation of order k_c, i.e.

$$x(t_1, t_2) = \sum_{k_1=-k_c}^{k_c} \sum_{k_2=-k_c}^{k_c} \hat{x}[k_1, k_2] \exp\left(\frac{i2\pi k_1 t_1}{T}\right) \exp\left(\frac{i2\pi k_2 t_2}{T}\right)$$

Equispaced grid

$$X_{[N]} := \begin{bmatrix} x\left(\frac{0}{N}, \frac{0}{N}\right) & x\left(\frac{0}{N}, \frac{T}{N}\right) & \cdots & x\left(\frac{0}{N}, T - \frac{T}{N}\right) \\ x\left(\frac{T}{N}, \frac{0}{N}\right) & x\left(\frac{T}{N}, \frac{T}{N}\right) & \cdots & x\left(\frac{T}{N}, T - \frac{T}{N}\right) \\ \cdots & \cdots & \cdots & \cdots \\ x\left(T - \frac{T}{N}, \frac{0}{N}\right) & x\left(T - \frac{T}{N}, \frac{T}{N}\right) & \cdots & x\left(T - \frac{T}{N}, T - \frac{T}{N}\right) \end{bmatrix}.$$

Nyquist-Shannon-Kotelnikov sampling theorem

Any bandlimited signal $x \in \mathcal{L}_2[0, T)^2$, where $T > 0$, with cut-off frequency k_c can be recovered from N^2 uniformly spaced samples if

$$N \geq 2k_c + 1,$$

where $2k_c + 1$ is known as the *Nyquist rate*

Nyquist-Shannon-Kotelnikov sampling theorem

We have

$$X_{[N]} = \widetilde{F}_{[N]} \widehat{X}_{[k_c]} \widetilde{F}_{[N]}^T,$$

$$\widehat{X}_{[k_c]} := \begin{bmatrix} \hat{x}_{-k_c,-k_c} & \hat{x}_{-k_c,-k_c+1} & \cdots & \hat{x}_{-k_c,k_c} \\ \hat{x}_{-k_c+1,-k_c} & \hat{x}_{-k_c+1,-k_c+1} & \cdots & \hat{x}_{-k_c+1,k_c} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{x}_{k_c,-k_c} & \hat{x}_{k_c,-k_c+1} & \cdots & \hat{x}_{k_c,k_c} \end{bmatrix}$$

So that

$$\widehat{X}_{[k_c]} = \frac{1}{N^2} \widetilde{F}_{[N]}^* X_{[N]} \left(\widetilde{F}_{[N]}^* \right)^T$$

2D discrete signals

We represent 2D signals as matrices belonging to the vector space of $\mathbb{C}^{N\times N}$ matrices endowed with the standard inner product

$$\langle A, B\rangle := \operatorname{tr}\left(A^* B\right), \quad A, B \in \mathbb{C}^{N\times N}.$$

Equivalent to dot product between vectorized matrices

Discrete complex sinusoids

The discrete complex sinusoid $\Phi_{k_1,k_2} \in \mathbb{C}^{N\times N}$ with integer frequencies k_1 and k_2 is defined as

$$\Phi_{k_1,k_2}[j_1, j_2] := \exp\left(\frac{i2\pi k_1 j_1}{N}\right) \exp\left(\frac{i2\pi k_2 j_2}{N}\right), \qquad 0 \leq j_1, j_2 \leq N-1,$$

Equivalently

$$\Phi_{k_1,k_2} = \psi_{k_1} \psi_{k_2}^T.$$

The discrete complex exponentials $\frac{1}{N}\Phi_{k_1,k_2}$, $0 \leq k_1, k_2 \leq N-1$, form an orthonormal basis of $\mathbb{C}^{N\times N}$

Proof

$$
\begin{aligned}
\langle \Phi_{k_1,k_2}, \Phi_{l_1,l_2} \rangle &= \operatorname{tr}\left((\Phi_{l_1,l_2})^* \, \Phi_{k_1,k_2} \right) \\
&= (\psi_{k_1})^* \psi_{l_1} (\psi_{k_2})^* \psi_{l_2}
\end{aligned}
$$

2D discrete Fourier transform

The discrete Fourier transform (DFT) of a 2D array $X \in \mathbb{C}^{N \times N}$ is

$$\widehat{X}\,[k_1, k_2] := \langle X, \Phi_{k_1,k_2} \rangle\,, \qquad 0 \leq k_1, k_2 \leq N-1,$$

or equivalently

$$\widehat{X} := F_{[N]} X\, F_{[N]},$$

where $F_{[N]}$ is the 1D DFT matrix

Inverse 2D discrete Fourier transform

The inverse DFT of a 2D array $\widehat{Y} \in \mathbb{C}^{N \times N}$ equals

$$Y = \frac{1}{N^2} F_{[N]}^{*} \widehat{Y} F_{[N]}^{*}$$

It inverts the 2D DFT

2D discrete Fourier transform

Can be interpreted as Fourier series of samples (as in 1D)

Complexity $O(N^2 \log N)$ instead of $O(N^3)$ (FFT)