Optimization-Based Data Analysis

Recitation 5

1. Consider a function $f(t) = \sum_{k=-k_c}^{k_c} a_k e^{2\pi ikt}$ with $a_k \in \mathbb{C}$.

   (a) For fixed j, show how to compute a_j by integration.

   (b) Suppose you are given N samples $\vec{x} = [f(0), f(1/N), \ldots, f((N-1)/N)]^T$. How would you use these to recover the a_k-values, and how many samples do you need?

   (c) Suppose you had $2N$ samples $\vec{y} = [f(0), \ldots, f((N-1)/N), \ldots, f((2N-1)/N)]^T$. How would you use these to recover the a_j-values, and how many samples do you need?

   Solution.

   (a) Note that

   $$\int_{-1/2}^{1/2} f(t)e^{-2\pi ijt}\,dt = \int_{-1/2}^{1/2} \sum_{k=-k_c}^{k_c} a_k e^{2\pi ikt} e^{-2\pi ijt}\,dt = \sum_{k=-k_c}^{k_c} \int_{-1/2}^{1/2} a_k e^{2\pi ikt} e^{-2\pi ijt}\,dt = a_j,$$

   as all of the integrals are zero when $j \neq k$.

   (b) $a_k = \frac{1}{N}\vec{X}[k]$ if $N \geq 2k_c + 1$.

   (c) We still require $N \geq 2k_c + 1$. Either only use the first half of the data (the rest is just a copy), or compute

   $$a_k = \frac{1}{2N}\vec{Y}[2k].$$

   It turns that $\vec{Y}[2j+1] = 0$ for all j since the data repeats.

2. True or False: A matrix $M \in \mathbb{C}^{n\times n}$ is circulant (each row is obtained by rotating the first row) if and only if it is diagonalized by $\frac{1}{\sqrt{n}}\vec{h}_k^{[n]}$ for $k = 0, \ldots, n-1$.

   Solution. A matrix is circulant if and only if it is a matrix of a (circular) convolution operator. As we know, convolution operators correspond to multiplication in frequency space. This is equivalent to the statement that the matrix is diagonalized by the Fourier basis vectors.

3. Prove that (circular) convolutions are commutative: $\vec{x} * \vec{y} = \vec{y} * \vec{x}$ for $\vec{x}, \vec{y} \in \mathbb{C}^n$.

   Solution. Compute DFT. Both sides are just the pointwise multiplication of the DFT coefficients, which is commutative.

4. Let $\vec{1} \in \mathbb{C}^n$ denote the vector that is all ones. What is $\vec{x} * \vec{1}$ for $\vec{x} \in \mathbb{C}^n$? Can we deconvolve to get $\vec{x}$?

Solution. $\vec{x} * \vec{1}$ is the vector with every entry equal to $\vec{1}^T\vec{x}$. We cannot deconvolve, since we have only learned the sum of the entries. Alternatively, note that all but one of the Fourier coefficients of $\vec{1}$ are 0.

5. Let $f : [-1/2, 1/2] \to \mathbb{C}$ be defined by $f(t) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi ikt}$, where $a_k \in \mathbb{C}$. Suppose we obtain N samples $\vec{x} = [f(0), \ldots, f((N-1)/N)]^T$. What are the N coefficients we obtain by computing the DFT $\vec{X}$?

   Solution. First note that

$$\begin{aligned}
\vec{x}[l] &= f(l/N) \\
&= \sum_{k=-\infty}^{\infty} a_k e^{2\pi ikl/N} \\
&= \sum_{k=0}^{N-1} \sum_{m\in\mathbb{Z}} a_{k+Nm} e^{2\pi i(k+Nm)l/N} \\
&= \sum_{k=0}^{N-1} \left(\sum_{m\in\mathbb{Z}} a_{k+Nm} \right) e^{2\pi ikl/N} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \vec{X}[k] e^{2\pi ikl/N}.
\end{aligned}$$

   Thus $\vec{X}[k] = N \sum_{m\in\mathbb{Z}} a_{k+Nm}$. This shows we cannot recover each Fourier coefficient, but only the sum of all Fourier coefficients for a given remainder modulo N. Stated briefly, N samples gives us the ability to resolve the Fourier coefficients into sums modulo N.

6. Suppose $f(0) = 5$, $f(1/5) = -2$ and $f(1/3) = 3$. Find a sum of translated Dirichlet kernels that has the same values of f at those 3 points.

   Solution. Let $n = 7$ and use

$$\frac{5}{15} d_7(t) - \frac{2}{15} d_7(t - 1/5) + \frac{3}{15} d_7(t - 1/3).$$

   By the time-translation property, this can be represented using band $k_c = 7$ with

$$d_7(t) = \sum_{k=-7}^{7} e^{2\pi ikt}.$$

   Recall that $d_n(t) = \frac{\sin((2n+1)\pi t)}{\sin(\pi t)}$ has zeros at $j/(2n+1)$ for $k \neq 0$. The finer the resolution at which interpolation must occur, the larger the necessary bandwidth.

7. Recall that the 2-dimensional DFT $\hat{M} \in \mathbb{C}^{m\times n}$ of a matrix $M \in \mathbb{C}^{m\times n}$ is defined by

$$\hat{M}[k_1, k_2] = \langle M, \vec{h}_{k_1}^{[m]}(\vec{h}_{k_2}^{[n]})^T \rangle.$$

Prove that $\hat{M} = W^{[m]} M W^{[n]}$ where $W^{[m]}$ is the DFT matrix defined by

$$W^{[m]} = \begin{bmatrix} | & | & \cdots & | \\ \vec{h}_0^{[m]} & \vec{h}_1^{[m]} & \cdots & \vec{h}_{m-1}^{[m]} \\ | & | & \cdots & | \end{bmatrix}^*.$$

Solution. Recall that

$$\begin{aligned} \hat{M}[k_1, k_2] &= \langle M, \vec{h}_{k_1}^{[m]}(\vec{h}_{k_2}^{[n]})^T \rangle \\ &= \operatorname{tr}\left(M(\vec{h}_{k_1}^{[m]}(\vec{h}_{k_2}^{[n]})^T)^* \right) \\ &= \operatorname{tr}\left(M\overline{\vec{h}_{k_2}^{[n]}}(\vec{h}_{k_1}^{[m]})^* \right) \\ &= \operatorname{tr}\left((\vec{h}_{k_1}^{[m]})^* M \overline{\vec{h}_{k_2}^{[n]}} \right) \\ &= W_{k_1,:} M W_{:,k_2}. \end{aligned}$$

Thus $\hat{M} = WMW$ as required.

8. (a) In class we blurred images by convolving with a Gaussian filter. Explain why this has a blurring effect, and what this suggests about its Fourier transform.

   (b) Consider a 1-dimensional sequence that we want to "blur" or smooth by computing a moving average. We will replace entry $\vec{x}[j]$ with the following average:

   $$\widetilde{\vec{x}}[j] := \frac{1}{2w+1} \sum_{k=-w}^{w} \vec{x}[j+k].$$

   How do you represent this as a convolution, and what do you expect the Fourier coefficients of the convolution filter to look like?

Solution.

(a) The Gaussian averages nearby values causing the blur. We expect this to remove high frequency (i.e., quickly varying) components, as it does.

(b) The convolution filter is $1/(2w+1)$ in a window of width w centered at 0. That is, the filter $\vec{k}$ is given by $\vec{k}[j] = 1/(2w+1)$ for $|j| \leq w$ and 0 otherwise (as usual, we interpret negative indices as wrapping around for a finite vector $\vec{k}$). The Fourier coefficients $\vec{K}$ should look like a Dirichlet kernel, converging toward a single spike as w grows (i.e., the high frequency components are decaying to zero).