Homework 7

Solutions

1. (Fourier coefficients and smoothness)

   (a) For $k \neq 0$ we have

$$\hat{x}[k] = \int_0^1 e^{-2\pi i k t} x(t)\, dt \tag{1}$$

$$= \frac{1}{-2\pi i k} \left[e^{-2\pi i k t} x(t) \right]_0^1 - \frac{1}{-2\pi i k} \int_0^1 e^{-2\pi i k t} x'(t)\, dt \tag{2}$$

$$= \frac{1}{2\pi i k} \int_0^1 e^{-2\pi i k t} x'(t)\, dt \qquad \text{(Periodicity).} \tag{3}$$

   If we define C_1 by

$$\left| \frac{1}{2\pi i} \int_0^1 e^{-2\pi i k t} x'(t)\, dt \right| \leq \frac{1}{2\pi} \int_0^1 |x'(t)|\, dt =: C_1 \tag{4}$$

   the result holds.

   (b) Continuing we have

$$\hat{x}[k] = \frac{1}{2\pi i k} \int_0^1 e^{-2\pi i k t} x'(t)\, dt \tag{5}$$

$$= \frac{1}{-(2\pi i k)^2} \left[e^{-2\pi i k t} x'(t) \right]_0^1 - \frac{1}{-(2\pi i k)^2} \int_0^1 e^{-2\pi i k t} x''(t)\, dt \tag{6}$$

$$= \frac{1}{(2\pi i k)^2} \int_0^1 e^{-2\pi i k t} x''(t)\, dt \qquad \text{(Periodicity).} \tag{7}$$

   If we define C_2 by

$$\frac{1}{4\pi^2} \int_0^1 |x''(t)|\, dt =: C_2 \tag{8}$$

   the result holds.

2. (Sampling a sum of sinusoids)

   (a) The signal is bandlimited with cut-off frequency $\max(|k_1|, |k_2|)$. By the Sampling Theorem $N \geq 2\max(|k_1|, |k_2|) + 1$.

   (b)

$$x_N = \begin{bmatrix} \phi_{k_1} & \phi_{k_2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}. \tag{9}$$

(c) ϕ_{k_1} and ϕ_{k_2} are orthogonal unless $k_2 = k_1 \mod N$. We can recover the amplitudes as long as $k_2 \neq k_1 \mod N$. This means that N can be much smaller than $2\max(|k_1|, |k_2|) + 1$. Take $k_1 = 1$ and $k_2 = 2$, then $N = 2$ works, whereas $2\max(|k_1|, |k_2|) + 1 = 5$.

(d) You need to know the frequencies before hand.

3. (Sampling theorem for bandpass signals)

(a) The signal is bandlimited with cut-off frequency k_2. By the Sampling Theorem $N \geq 2k_2 + 1$.

(b) By Lemma 2.10 in the lecture notes on frequency representations, the recovered coefficient corresponding to each frequency k is equal to the sum of coefficients from the original signal corresponding to a frequency k' such that $k = k' \mod N$. If $N \geq 2\tilde{k}_c+1$, there can be at most one such frequency for every value of k because the nonzero coefficients are restricted to be in a contiguous interval of length $2\tilde{k}_c+1 \leq N$.

(c) Let

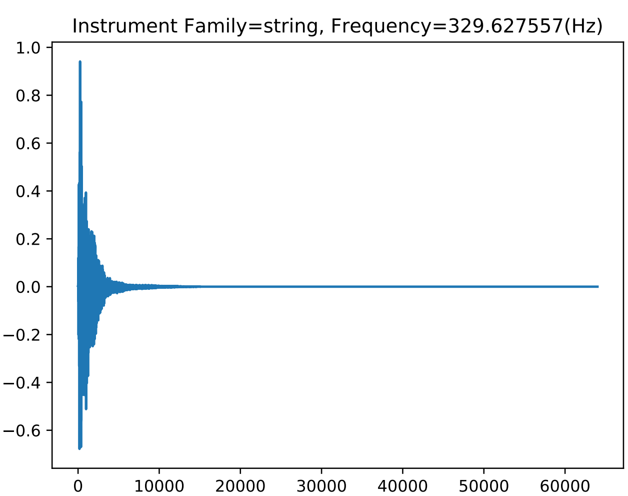
$$\hat{y}[k] := \frac{1}{N}(\widetilde{F}^*_{[N]}\vec{x}_{[N]})[k], \quad -\tilde{k}_c \leq k \leq \tilde{k}_c, \tag{10}$$

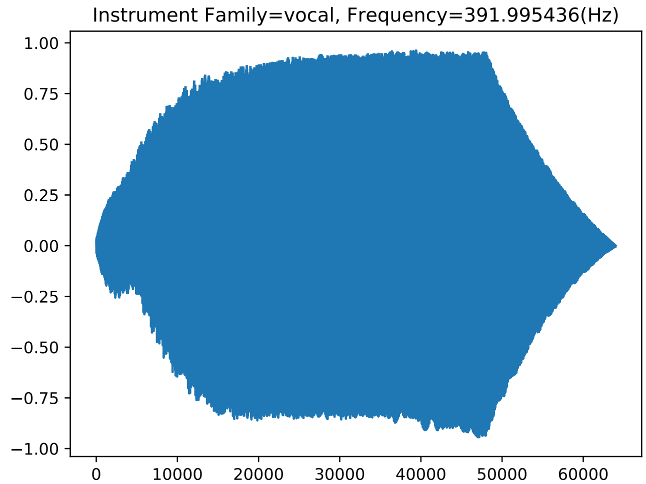
where $\vec{x}_{[N]}$ is the vector of samples. By Lemma 3.5 $\hat{y}[k-mN] = \hat{x}[k]$ for $k_1 < k \leq k_2$, so we set

$$x_{\text{rec}} := \sum_{k=k_1+1}^{k_2} \hat{y}[k-mN] \exp(i2\pi kt). \tag{11}$$

4. (Frequency analysis of musical notes)

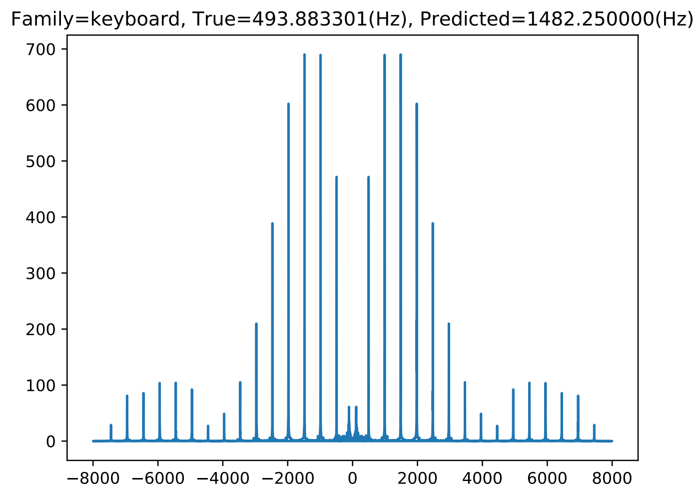
(a) The plots are given below.

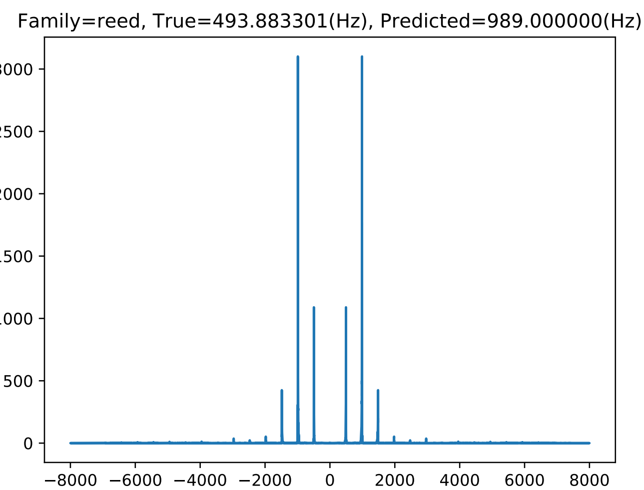




(b) i. Approximately 0.721.

ii. The plots are given below.





iii. Vocal.

(c) i. 0.996

ii. The plots are given below.