DS-GA.1013 Mathematical Tools for Data Science :
Homework Assignment 0
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1. Projections

   (a) False Consider $\boldsymbol{b_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\boldsymbol{b_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, they form a basis of $\mathbf{R}^2$. When using the definition $\mathcal{P}_{\mathcal{S}} x = \sum_{i=1}^{n} \langle x, b_i \rangle b_i$ we would expect that $\mathcal{P}_{\mathcal{S}} b_1 = b_1$. However $\mathcal{P}_{\mathcal{S}} b_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \neq b_1$.

   (b) True Let $S^{\perp} = \{x | \langle x, y \rangle = 0, \forall y \in S\}$ a subspace of an inner product space X, then $S^{\perp\perp} = \{x | \langle x, y \rangle = 0, \forall y \in S^{\perp}\}$. The inner product being symmetric, $S \subseteq S^{\perp\perp}$. Since for any vector $x \in X$, we have $x = y + z$ where $y \in S, z \in S^{\perp}$, using Gram-schmidt orthonormalization process, we can find a basis of S and $S^{\perp}$ which express any vector of X as a linear combination of these two basis and combining these two basis together forms a new basis for X so $\dim X = \dim S + \dim S^{\perp}$. If $\dim X = n$ and $\dim S = m$ then $\dim S^{\perp} = n - m$. Similarly $\dim S^{\perp\perp} = n - (n - m) = m$ so $\dim S^{\perp\perp} = \dim S$, so $S^{\perp\perp} \subseteq S$ and since the dimension of a space or subspace is the cardinality of its basis, thus $S = S^{\perp\perp}$.

   (c) True consider $\boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, we want $\boldsymbol{w} = \begin{bmatrix} \frac{\sum_{i=1,n} v_i}{n} \\ \vdots \\ \frac{\sum_{i=1,n} v_i}{n} \end{bmatrix}$. The orthogonal projection of $\boldsymbol{v}$ onto the vector $\boldsymbol{b}$ is defined as $\frac{v.b}{\|b\|^2}$, take $b = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$.

2. Eigen decomposition Rewriting the problem in a matrix form:

$$\begin{pmatrix} d_{n+1} \\ w_{n+1} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 5 & -3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} d_n \\ w_n \end{pmatrix}$$

   Let $A = \frac{1}{4} \begin{pmatrix} 5 & -3 \\ 1 & 1 \end{pmatrix}$, $v_{n+1} = \begin{pmatrix} d_{n+1} \\ w_{n+1} \end{pmatrix}$, $v_0 = \begin{pmatrix} d_0 \\ w_0 \end{pmatrix}$ then $v_{n+1} = A v_n = A^n v_0$. We are looking to find the eigen decomposition so we can understand the behavior of v_n as $n \to \infty$. $\det(A - \lambda I) = \frac{1}{2}(2\lambda^2 - 3\lambda + 1)$, we find for eigenvalues $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = 1$ with corresponding eigenvectors

   $w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Since A is diagonalizable the vectors $\{w_1, w_2\}$ forms a basis of $\mathbf{R}^2$ and we can express v_0 in this basis as $v_0 = \alpha w_1 + \beta w_2$ for some $\alpha, \beta \in \mathbf{R}$, thus $v_{n+1} = \alpha A^n w_1 + \beta A^n w_2 = \alpha \lambda_1^n w_1 + \beta \lambda_2^n w_2 = \alpha(\frac{1}{2^n}) w_1 + \beta w_2$. Then taking the $n \to \infty$, the first term goes to zero and $v_{n+1} \sim \beta w_2$. So asymptotically $\frac{d_{n+1}}{w_{n+1}} \sim 3$ which verifies the initial condition: $w_0 < d_0$.

3. Function approximation

(a) Using Gram-Schmidt orthonormalization process, we find

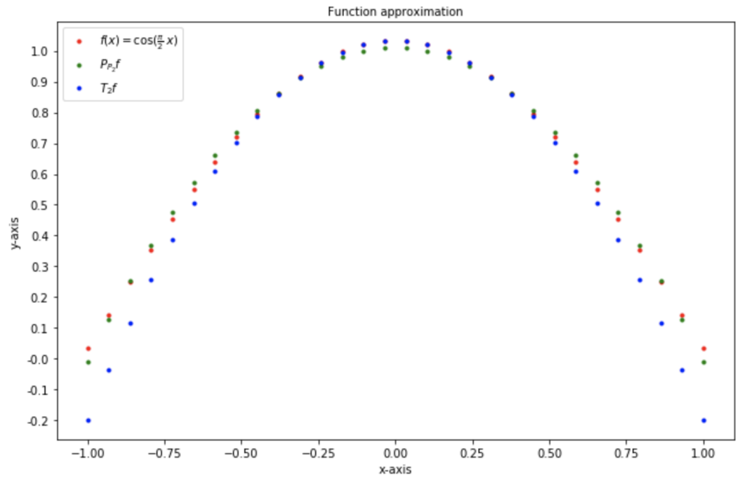
$$\begin{aligned}
v_1 &= 1 \\
v_2 &= x - \langle x, 1\rangle \frac{1}{\langle 1, 1\rangle} \\
&= x \\
v_3 &= x^2 - \langle x^2, v_2\rangle \frac{v_2}{\langle v_2, v_2\rangle} - \langle x^2, v_1\rangle \frac{v_1}{\langle v_1, v_1\rangle} \\
&= x^2 - \frac{1}{3}
\end{aligned}$$

Then we normalize each of these vectors to obtain:

$$\begin{aligned}
w_1 &= \frac{v_1}{\|v_1\|} = \frac{\sqrt{2}}{2} \\
w_2 &= \frac{v_2}{\|v_2\|} = \sqrt{\frac{3}{2}}\; x \\
w_3 &= \frac{v_3}{\|v_3\|} = \sqrt{\frac{45}{8}}\; (x^2 - \frac{1}{3})
\end{aligned}$$

(b) The projection of $f(x) = \cos(\frac{\pi}{2}\; x)$ in the orthonormal basis $\{w_1, w_2, w_3\}$ is: $\sum_{i=1,3} \langle f, w_i\rangle w_i$, where:

$$\begin{aligned}
\langle f, w_1\rangle &= \int_{-1}^{1} \cos(\frac{\pi}{2}\; x)\; \frac{\sqrt{2}}{2}\; dx \\
&= \frac{4}{\pi\sqrt{2}} \sim 0.9 \\
\langle f, w_2\rangle &= \int_{-1}^{1} \cos(\frac{\pi}{2}\; x)\; \frac{\sqrt{3}}{2}\; x\; dx \\
&= 0 \\
\langle f, w_3\rangle &= \int_{-1}^{1} \cos(\frac{\pi}{2}\; x) \sqrt{\frac{45}{8}}\; (x^2 - \frac{1}{3})\; dx \\
&= 2\sqrt{10}\frac{\pi^2 - 12}{\pi^3} \sim -0.43
\end{aligned}$$



(c)

(d) $\mathcal{P}_{P2}f$is the orthogonal projection of $f(x)$ over the subspace of polynomials of degree 2: $\{w_1, w_2, w_3\}$, like the Taylor expansion $\mathcal{T}_2 f$, The difference is that the Taylor is a polynomial expansion of f at 0. So in a neighborhood of 0, there is almost no differences between f and $\mathcal{T}_2 f$, but as we move away the approximation given by $\mathcal{T}_2 f$ is worst than $\mathcal{P}_{P2}f$.

4. Scalar linear approximation

(a) First we write $\mathrm{E}[(ax + b - y)^2] = \mathrm{E}[((ax - y) - (-b))^2]$, we know that the best mean-squared error mimimizer of a random variable is its mean so $-b = \mathrm{E}[ax - y] = a\,\mathrm{E}[x] - \mathrm{E}[y] = a\mu_x - \mu_y$. Substituting b in the expression we want to minimize gives us:

$$\begin{aligned}
\mathrm{E}[(ax + b - y)^2] &= \mathrm{E}[(ax - y - (a\mu_x - \mu_y))^2] \\
&= \mathrm{E}[\{a(\mu_x - x) - (y - \mu_y)\}^2] \\
&= a^2\,\mathrm{E}[(x - \mu_x)^2] + \mathrm{E}[(y - \mu_y)^2] - 2a\,\mathrm{E}[(x - \mu_x)(y - \mu_y)] \\
&= a^2\sigma_x^2 + \sigma_y^2 - 2\,a\,\mathrm{Cov}(x, y)
\end{aligned}$$

Let $f(a) = a^2\sigma_x^2 + \sigma_y^2 - 2\,a\,\mathrm{Cov}(x, y)$, then $f'(a) = 2(\sigma_x^2 a - \mathrm{Cov}(x, y))$ and $f''(a) = 2\sigma_x^2$. The function is strictly convex, and its second derivative is positive, thus its minimizer is $a = \frac{\mathrm{Cov}(x,y)}{\sigma_x^2} = \rho_{x,y}\,\frac{\sigma_y}{\sigma_x}$.

(b) Applying the result from the previous question, the best linear estimate of y given x is $y = \rho_{x,y}\,\frac{\sigma_y}{\sigma_x}(x - \mu_x) + \mu_y$. Notice that $\mathrm{Var}(x) = \mathrm{Var}(y\ z) = \mathrm{E}\left[y^2\ z^2\right] - \mathrm{E}\left[y\ z\right]^2 = \mathrm{E}\left[y^2\right]\mathrm{E}\left[z^2\right] - \mathrm{E}\left[y\right]^2 E[z]^2 = (\sigma_y^2 + \mu_y^2)\sigma_z^2$ where we have used that a and z are independent and z has zero-mean. And $\mathrm{E}\left[x\right] = \mathrm{E}[y\ \ z] = \mathrm{E}[y]\ .\ 0 = 0$. Thus the best linear estimate of y given x is: $\rho_{x,y}\frac{\sigma_y}{\sigma_z\sqrt{\sigma_y^2 + \mu_y^2}}\ x + \mu_y$.

(c) If in the expression above of y given x, z is normally distributed with $\sigma_z = 1$ then y is perfectly estimated from x.

5. Gradients

   (a) Compute the gradient of $f(x) = b^T x$ where $b \in \mathbf{R}^d$ and $f : \mathbf{R}^d \to \mathbf{R}$. $\frac{\partial f(x)}{x_j} = \sum_i b_i \frac{\partial x_i}{\partial x_j} = b_i$, thus $\nabla f(x) = b$.

   (b) Compute the gradient of $f(x) = x^T A x$ where $A \in \mathbf{R}^{d \times s}$ and $f : \mathbf{R}^d \to \mathbf{R}$. $f(x) = x^T A x = \sum_{i=1}^d \sum_{j=1}^d a_{ij} x_i x_j$, then

$$\begin{aligned}
\frac{\partial f}{\partial x_k} &= \sum_{i=1}^d \sum_{j=1}^d a_{ij} \frac{\partial x_i x_j}{x_k} \\
&= \sum_{i=1}^d \sum_{j=1}^d a_{ij} (x_j \delta_{ik} + x_i \delta_{jk}) \\
&= \sum_{i=1}^d \sum_{j=1}^d a_{ij} x_j \delta_{ik} + \sum_{i=1}^d \sum_{j=1}^d a_{ij} x_i \delta_{jk} \\
&= \sum_{j=1}^d a_{kj} x_j + \sum_{i=1}^d a_{ik} x_i \\
&= (Ax)_k + (Ax)_k^T
\end{aligned}$$

   thus $\nabla f(x) = (A + A^T)x$.